1)

a) $x \mid N, \theta \sim$ Binomial $\qquad N = 50$

prior: $\theta \sim U(0,1)$ $\qquad$ uniform

Likelihood:

$$p(x \mid N, \theta) = \binom{N}{x} \theta^{x} (1-\theta)^{N-x}$$

$$\Rightarrow P(x \mid N, \theta) \propto \theta^{x} (1-\theta)^{N-x}$$

prior $\propto c$

$\Rightarrow$ Posterior:

$$p(\theta \mid x) \propto \theta^{x} (1-\theta)^{N-x} \cdot 1$$

$$\propto \theta^{x+1-1} (1-\theta)^{N+1-x-1}$$

$$\propto \text{Beta}(x+1, N+1-x)$$

$$\propto \underline{\text{Beta}(x+1, 51-x)}$$

d)

Hamiltonian Monte Carlo & Variational Inference

Hamiltonian Monte Carlo is one of the MCMC method in which extra momentum parameters is added, so that Instead of using only the parameter distribution, we use a joint distribution of the parameter and its momentum.

$$p(\theta, \phi \mid y) = p(\theta \mid y)\, p(\phi)$$

Hamilton's equations which are a system of differential equations are used to walk through the distribution.

In Variational Inference, the idea is focused on optimization. In this technique we are trying to solve the optimization problem over a range of distributions $q_i(\theta_i)$ Instead of just approximating one $q(\theta)$ for the posterior distribution we try to find the optimum $q \in Q$ which is the most similar to $P(\theta|y)$ (posterior). The similarity can be obtained by minimizing the distance between P & q. which Kullback-Leibler distance is one method to compute the distances between P & q.

4) a unequal gender balance → Females are more common than males

- 20 random caught $\begin{cases} 16 \to \text{female} \to s : \text{success} \\ 4 \to \text{male} \to f : \text{failure} \end{cases}$

a) observations → Bernoulli (θ) → $y_1, \dots y_{20} \sim Bern(\theta)$

prior: $\theta \sim Beta(2,2)$

new fish: $\tilde{y}$ → $\tilde{y} \in \{0,1\}$ $\begin{cases} 1: \text{female} \\ 0: \text{male} \end{cases}$

$$P(\tilde{y}=1 | y_{1:20}) = \int \underbrace{p(\tilde{y}|\theta)}_{\text{Likelihood of new fish}} \underbrace{p(\theta|y_{1:20})}_{\text{Posterior of } \theta \text{ given the previous data}} d\theta$$

$$P(\tilde{y}|\theta) = \theta^{\tilde{y}} (1-\theta)^{1-\tilde{y}}$$

$$P(\theta|y_{1:20}) \propto \underbrace{\theta^{s}(1-\theta)^{f}}_{\text{Likelihood}} \overbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}^{\text{Prior}}$$

$$\propto \theta^{16}(1-\theta)^{4}\theta^{2-1}(1-\theta)^{2-1}$$

$$\propto \theta^{18-1}(1-\theta)^{6-1} \propto \underbrace{Beta(18,6)}_{\text{posterior}(\theta|y_1 \dots y_{20})}$$

$$p(\tilde{y}|y_{1:20}) = \int \theta^{\tilde{y}}(1-\theta)^{1-\tilde{y}} \frac{\Gamma(18+6)}{\Gamma(18)\Gamma(6)} \theta^{18-1}(1-\theta)^{6-1}$$

$$= \frac{\Gamma(18+6)}{\Gamma(18)\Gamma(6)} \int \theta^{\tilde{y}+18-1}(1-\theta)^{7-\tilde{y}-1} d\theta$$

= ~~18~~ →

4a)
~~4a~~ Continue

$$= \frac{\Gamma(18+6)}{\Gamma(18)\Gamma(6)} \times \frac{\Gamma(\tilde{y}+18)\Gamma(7-\tilde{y})}{\Gamma(\tilde{y}+18+7-\tilde{y})} \underbrace{\int \frac{\Gamma(\tilde{y}+18+7-\tilde{y})}{\Gamma(\tilde{y}+18)\Gamma(7-\tilde{y})} \theta^{\tilde{y}+18-1} (1-\theta)^{7-\tilde{y}-1} d\theta}$$

This is density function of $Beta(\tilde{y}+18, 7-\tilde{y})$ which equals to $\underline{1}$

$$\Rightarrow P(\tilde{y} \mid y_{1:20}) = \frac{\Gamma(18+6)}{\Gamma(18)\Gamma(6)} \times \frac{\Gamma(\tilde{y}+18)\Gamma(7-\tilde{y})}{\Gamma(18+6+1)}$$

Using this formula: $\Gamma(x+1) = x\Gamma(x)$

$$\hookrightarrow \Rightarrow P(\tilde{y} \mid y_{1:20}) = \frac{\cancel{\Gamma(18+6)}\Gamma(\tilde{y}+18)\Gamma(7-\tilde{y})}{\Gamma(18)\Gamma(6)\cancel{\Gamma(18+6)}(18+6)}$$

$$\Rightarrow \boxed{P(\tilde{y} \mid y_{1:20}) = \frac{\Gamma(\tilde{y}+18)\Gamma(7-\tilde{y})}{24 \times \Gamma(18)\Gamma(6)}}$$

$$\tilde{y} = 1 \Rightarrow P(\tilde{y}=1 \mid y_{1:20}) = \frac{\overset{18\Gamma(18)}{\cancel{\Gamma(18+1)}}\cancel{\Gamma(7-1)}}{24\Gamma(18)\cancel{\Gamma(6)}} = \frac{18}{24} = 0.75$$

The probability that the new fish is female is 0.75

4) b

	Ave. length	Ave. weight
F	14	300
M	12	280

Measurments $\overset{iid}{\sim}$ Normal (θ)

θ:

μ_{FL} : female length $\quad \sigma_{FL} = 2$

μ_{ML} : male length $\quad \sigma_{ML} = 2$

μ_{FW} : Female Weight $\quad \sigma_{FW} = 50$

μ_{MW} : Male weight $\quad \sigma_{MW} = 50$

Prior: uniform

If we assume : y : length measurments :

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)\, p(\theta|y)\, d\theta$$

Assuming the uniform prior for the length :

$$\theta | y \sim N(\bar{y}, \frac{\sigma^2}{n})$$ → Posterior of length parameter given the prior & data

$$\tilde{y} | \theta \sim N(\theta, \sigma^2)$$

$$\Rightarrow \tilde{y} | y \sim N(\bar{y}, \sigma^2 + \frac{\sigma^2}{n})$$

$$\Rightarrow \boxed{\tilde{y} | y \sim N(12, 4 + \frac{4}{20})}$$

4. c)

$P(\tilde{y} \mid y, L, W)$ → predictive probability ~~that~~ of a new fish given length & weight

$\underbrace{\text{Length \& weight}}_{X}$ are ~~features~~ features and Class $= \{1, 0\}$ are the class labels.

Naive Bayes : $P(c \mid X) \propto P(X \mid c) P(c)$

In a) $n = 20$ $\begin{cases} n_{female} = 16 & \{1\} \\ n_{male} = 4 & \{0\} \end{cases}$, $\underline{X^*}$ $\begin{cases} \text{Length} = 10 \\ \text{weight} = 20 \end{cases}$

new feature

$$P(\tilde{y} \mid x^*, y) \propto P(x^* \mid \tilde{y}, y) P(\tilde{y} \mid y)$$

$$\propto P(x_1^* \mid \tilde{y}, y) P(x_2^* \mid \tilde{y}, y) P(\tilde{y} \mid y)$$

In Naive Bayes we assume the features are independent ⟹

$$P(x^* \mid \tilde{y}, y) = P(x_1^* \mid \tilde{y}, y) P(x_2^* \mid \tilde{y}, y)$$

$P(\tilde{y} \mid y)$ → Predictive distribution for a new fish given the data which was computed in 4a

$$\rightarrow \boxed{P(\tilde{y} \mid y) = \frac{\Gamma(\tilde{y}+18)\,\Gamma(7-\tilde{y})}{24\,\Gamma(18)\,\Gamma(6)}}$$

$P(\tilde{y} = 1 \mid y) = 0.75$

$P(\tilde{y} = 0 \mid y) = 0.25$

4 c) Continue!

$P(x_1^* \mid \tilde{y}, y)$:

X_1 : length

x_2 : weight

$X_1 \mid y = \text{female}(1) \sim N(\mu_{FL}, 4)$

$X_1 \mid y = \text{male}(0) \sim N(\mu_{ML}, 4)$

$X_2 \mid y = \text{female}(1) \sim N(\mu_{FW}, 50^2)$

$X_2 \mid y = \text{male}(0) \sim N(\mu_{MW}, 50^2)$

we assumed a uniform prior for all μ

$p(\mu) \propto 1$

$P(x_1^* \mid \tilde{y}, y)$ is actually predictive distribution of new value for X_1 (length) given the new fish and previous fish

$\Rightarrow P(X_*^* \mid \tilde{y}, y) = N(\bar{X}_*, \sigma^2 + \frac{\sigma^2}{n})$

$\Rightarrow P(X_1^* \mid \tilde{y}=1, y) = N(14, 4 + \frac{4}{20})$

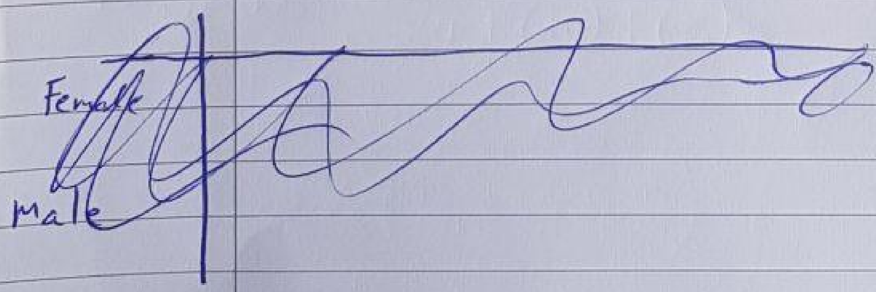
$P(x_1^* \mid \tilde{y}=0, y) = N(12, 4 + \frac{4}{20})$

length

$P(x_2^* \mid \tilde{y}=1, y) = N(300, 50^2 + \frac{50^2}{20})$

$P(x_2^* \mid \tilde{y}=0, y) = N(280, 50^2 + \frac{50^2}{20})$

weight



4c Continue 2

$$\Rightarrow p(\tilde{y}=1 \mid x^*, y) = 0.75 \times \overbrace{N\left(14, 4\left(1+\frac{1}{20}\right)\right)}^{1} \times \overbrace{N\left(300, 50^2\left(1+\frac{1}{20}\right)\right)}^{2}$$

and

$$p(\tilde{y}=0 \mid x^*, y) = 0.25 \times \underbrace{N\left(12, 4\left(1+\frac{1}{20}\right)\right)}_{3} \times \underbrace{N\left(280, 50^2\left(1+\frac{1}{20}\right)\right)}_{4}$$

1, 2, 3, 4 are computed for new measurements of length (x_1^*) and weight (x_2^*)

10 cm 250 g

$\tilde{y}$	unnormalized prob.	Normalized
(female) 1	0.000105	0.35
(male) 0	0.000198	0.65

$\Rightarrow$ The probability that the new fish is female is <u>0.35</u>

given the previous data and length = 10 cm

weight = 250 g