

Bayesian Statistics I

Lecture 10 - Bayesian Model Comparison

Mattias Villani

Department of Statistics
Stockholm University

Department of Computer and Information Science
Linköping University



- Bayesian model comparison
- Marginal likelihood
- Log Predictive Score

Using likelihood for model comparison

- Consider two models for the data $\mathbf{y} = (y_1, \dots, y_n)$: M_1 and M_2 .
- Let $p_i(\mathbf{y}|\theta_i)$ denote the data density under model M_i .
- If know θ_1 and θ_2 , the **likelihood ratio** is useful

$$\frac{p_1(\mathbf{y}|\theta_1)}{p_2(\mathbf{y}|\theta_2)}.$$

- The **likelihood ratio** with **ML estimates** plugged in:

$$\frac{p_1(\mathbf{y}|\hat{\theta}_1)}{p_2(\mathbf{y}|\hat{\theta}_2)}.$$

- Bigger models always win in estimated likelihood ratio.
- **Hypothesis tests** are problematic for non-nested models. End results are not very useful for analysis.

Bayesian model comparison

- Just use your priors $p_1(\theta_1)$ och $p_2(\theta_2)$.
- The **marginal likelihood** for model M_k with parameters θ_k

$$p_k(y) = \int p_k(y|\theta_k)p_k(\theta_k)d\theta_k.$$

- θ_k is 'removed' by the prior. **Not a silver bullet.** **Priors matter!**
- The **Bayes factor**

$$B_{12}(y) = \frac{p_1(y)}{p_2(y)}.$$

- **Posterior model probabilities**

$$\underbrace{\Pr(M_k|\mathbf{y})}_{\text{posterior model prob.}} \propto \underbrace{p(\mathbf{y}|M_k)}_{\text{marginal likelihood}} \cdot \underbrace{\Pr(M_k)}_{\text{prior model prob.}}$$

Bayesian hypothesis testing - Bernoulli

- **Hypothesis testing** is just a special case of model selection:

$$M_0 : x_1, \dots, x_n \stackrel{iid}{\sim} \text{Bernoulli}(\theta_0)$$

$$M_1 : x_1, \dots, x_n \stackrel{iid}{\sim} \text{Bernoulli}(\theta), \theta \sim \text{Beta}(\alpha, \beta)$$

$$p(x_1, \dots, x_n | M_0) = \theta_0^s (1 - \theta_0)^f,$$

$$\begin{aligned} p(x_1, \dots, x_n | M_1) &= \int_0^1 \theta^s (1 - \theta)^f B(\alpha, \beta)^{-1} \theta^{\alpha-1} (1 - \theta)^{\beta-1} d\theta \\ &= B(\alpha + s, \beta + f) / B(\alpha, \beta), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

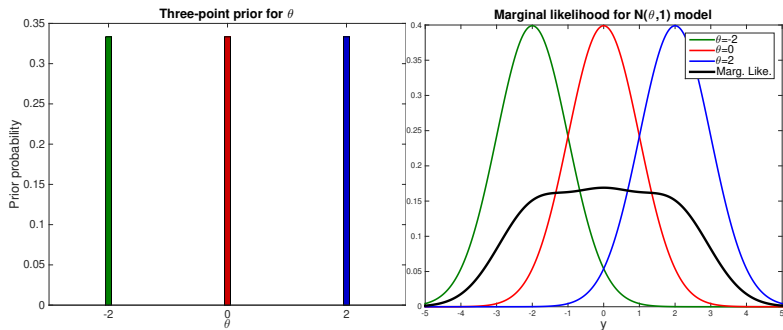
- **Posterior model probabilities**

$$Pr(M_k | x_1, \dots, x_n) \propto p(x_1, \dots, x_n | M_k) Pr(M_k), \text{ for } k = 0, 1.$$

- The **Bayes factor**

$$BF(M_0; M_1) = \frac{p(x_1, \dots, x_n | H_0)}{p(x_1, \dots, x_n | H_1)} = \frac{\theta_0^s (1 - \theta_0)^f B(\alpha, \beta)}{B(\alpha + s, \beta + f)}.$$

Priors matter



Example: Geometric vs Poisson

- Model 1 - **Geometric** with Beta prior:

- ▶ $y_1, \dots, y_n | \theta_1 \sim \text{Geo}(\theta_1)$
- ▶ $\theta_1 \sim \text{Beta}(\alpha_1, \beta_1)$

- Model 2 - **Poisson** with Gamma prior:

- ▶ $y_1, \dots, y_n | \theta_2 \sim \text{Poisson}(\theta_2)$
- ▶ $\theta_2 \sim \text{Gamma}(\alpha_2, \beta_2)$

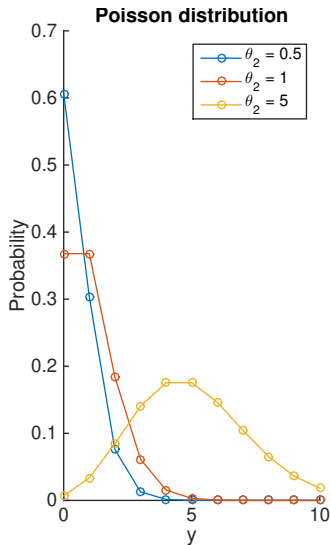
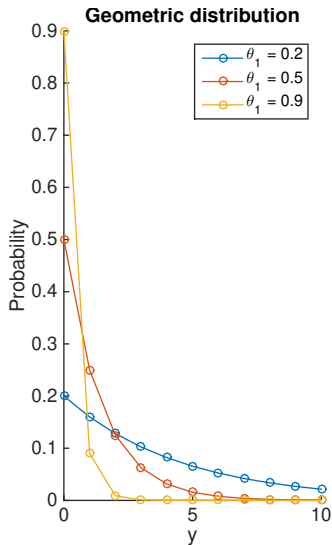
- **Marginal likelihood** for M_1

$$\begin{aligned} p_1(y_1, \dots, y_n) &= \int p_1(y_1, \dots, y_n | \theta_1) p(\theta_1) d\theta_1 \\ &= \frac{\Gamma(\alpha_1 + \beta_1)}{\Gamma(\alpha_1) \Gamma(\beta_1)} \frac{\Gamma(n + \alpha_1) \Gamma(n\bar{y} + \beta_1)}{\Gamma(n + n\bar{y} + \alpha_1 + \beta_1)} \end{aligned}$$

- **Marginal likelihood** for M_2

$$p_2(y_1, \dots, y_n) = \frac{\Gamma(n\bar{y} + \alpha_2) \beta_2^{\alpha_2}}{\Gamma(\alpha_2) (n + \beta_2)^{n\bar{y} + \alpha_2}} \frac{1}{\prod_{i=1}^n y_i!}$$

Geometric and Poisson



Geometric vs Poisson

- Priors match prior predictive means:

$$E(y_i|M_1) = E(y_i|M_2) \iff \alpha_1\alpha_2 = \beta_1\beta_2$$

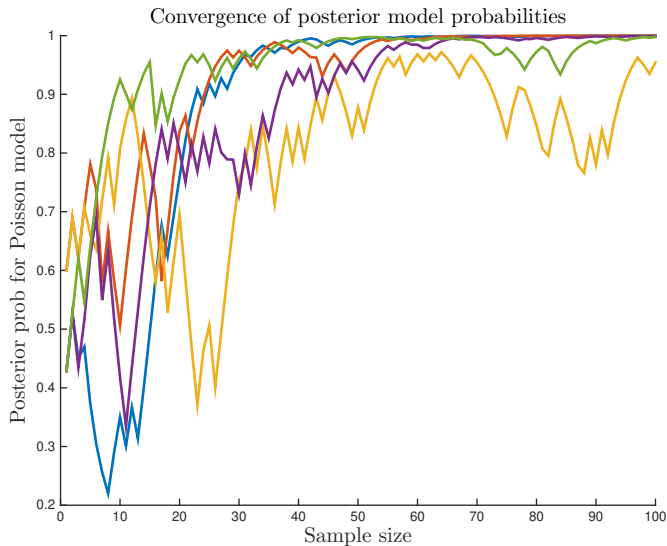
- Data: $y_1 = 0, y_2 = 0$.

	$\alpha_1 = 1, \beta_1 = 2$ $\alpha_2 = 2, \beta_2 = 1$	$\alpha_1 = 10, \beta_1 = 20$ $\alpha_2 = 20, \beta_2 = 10$	$\alpha_1 = 100, \beta_1 = 200$ $\alpha_2 = 200, \beta_2 = 100$
BF_{12}	1.5	4.54	5.87
$\Pr(M_1 \mathbf{y})$	0.6	0.82	0.85
$\Pr(M_2 \mathbf{y})$	0.4	0.18	0.15

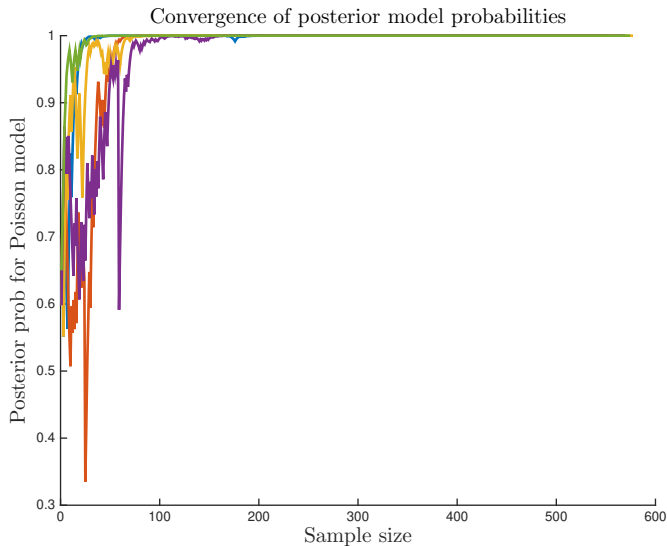
- Data: $y_1 = 3, y_2 = 3$.

	$\alpha_1 = 1, \beta_1 = 2$ $\alpha_2 = 2, \beta_2 = 1$	$\alpha_1 = 10, \beta_1 = 20$ $\alpha_2 = 20, \beta_2 = 10$	$\alpha_1 = 100, \beta_1 = 200$ $\alpha_2 = 200, \beta_2 = 100$
BF_{12}	0.26	0.29	0.30
$\Pr(M_1 \mathbf{y})$	0.21	0.22	0.23
$\Pr(M_2 \mathbf{y})$	0.79	0.78	0.77

Geometric vs Poisson for Pois(1) data



Geometric vs Poisson for $\text{Pois}(1)$ data



Model choice in multivariate time series

■ Multivariate time series

$$\mathbf{x}_t = \alpha\beta'\mathbf{z}_t + \Phi_1\mathbf{x}_{t-1} + \dots\Phi_k\mathbf{x}_{t-k} + \Psi_1 + \Psi_2t + \Psi_3t^2 + \varepsilon_t$$

■ Need to choose:

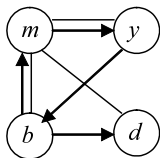
- ▶ **Lag length**, ($k = 1, 2, \dots, 4$)
- ▶ **Trend model** ($s = 1, 2, \dots, 5$)
- ▶ **Long-run (cointegration) relations** ($r = 0, 1, 2, 3, 4$).

THE MOST PROBABLE (k, r, s) COMBINATIONS IN THE DANISH MONETARY DATA.

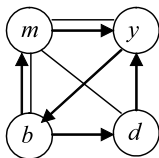
k	1	1	1	1	1	1	1	1	0	1
r	3	3	2	4	2	1	2	3	4	3
s	3	2	2	2	3	3	4	4	4	5
$p(k, r, s y, x, z)$.106	.093	.091	.060	.059	.055	.054	.049	.040	.038

Graphical models for multivariate time series

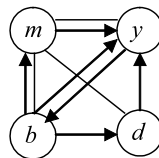
- **Graphical models** for multivariate time series.
- Zero-restrictions on the effect from time series i on time series j , for all lags. (**Granger Causality**).
- Zero-restrictions on inverse covariance matrix of the errors. Contemporaneous conditional independence.



$$p(G|\mathbf{X}) = 0.0033$$



$$p(G|\mathbf{X}) = 0.0028$$



$$p(G|\mathbf{X}) = 0.0025$$

Properties of Bayesian model comparison

- Coherence of pair-wise comparisons

$$B_{12} = B_{13} \cdot B_{32}$$

- **Consistency** when true model is in $\mathcal{M} = \{M_1, \dots, M_K\}$


$$\Pr(M = M_{TRUE} | \mathbf{y}) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty$$

- “KL-consistency” when $M_{TRUE} \notin \mathcal{M}$

$$\Pr(M = M^* | \mathbf{y}) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty,$$

M^* minimizes **KL divergence** between $p_M(\mathbf{y})$ and $p_{TRUE}(\mathbf{y})$.

- Smaller models always win when priors are very vague.

- **Improper priors** cannot be used for model comparison. 

Marginal likelihood measures out-of-sample predictive performance

- The **marginal likelihood** can be **decomposed** as

$$p(y_1, \dots, y_n) = p(y_1)p(y_2|y_1) \cdots p(y_n|y_1, y_2, \dots, y_{n-1})$$

- Assume that y_i is independent of y_1, \dots, y_{i-1} conditional on θ :

$$p(y_i|y_1, \dots, y_{i-1}) = \int p(y_i|\theta)p(\theta|y_1, \dots, y_{i-1})d\theta$$

- **Prediction of y_1** is based on the prior of θ . Sensitive to prior.
- **Prediction of y_n** uses almost all the data to infer θ . Not sensitive to prior when n is not small.

Normal example

- **Model:** $y_1, \dots, y_n | \theta \sim N(\theta, \sigma^2)$ with σ^2 known.
- **Prior:** $\theta \sim N(0, \kappa^2 \sigma^2)$.
- **Intermediate posterior** at time $i - 1$

$$\theta | y_1, \dots, y_{i-1} \sim N \left[w_i(\kappa) \cdot \bar{y}_{i-1}, \frac{\sigma^2}{i - 1 + \kappa^{-2}} \right]$$

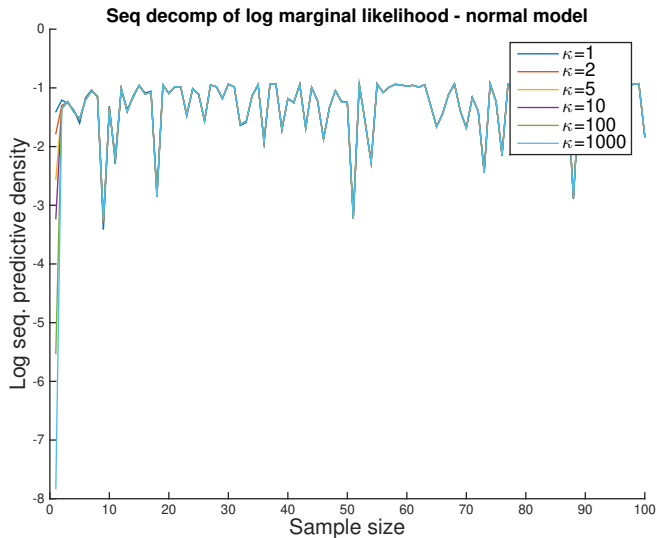
where $w_i(\kappa) = \frac{i-1}{i-1+\kappa^{-2}}$.

- **Intermediate predictive density** at time $i - 1$

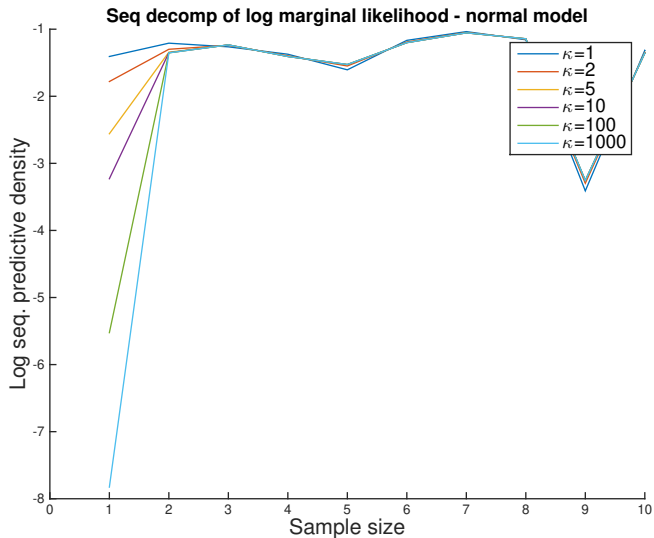
$$y_i | y_1, \dots, y_{i-1} \sim N \left[w_i(\kappa) \cdot \bar{y}_{i-1}, \sigma^2 \left(1 + \frac{1}{i - 1 + \kappa^{-2}} \right) \right]$$

- For $i = 1$, $y_1 \sim N \left[0, \sigma^2 \left(1 + \frac{1}{\kappa^{-2}} \right) \right]$ can be very sensitive to κ .
- For large i : $y_i | y_1, \dots, y_{i-1} \overset{\text{approx}}{\sim} N(\bar{y}_{i-1}, \sigma^2)$, not sensitive to κ .

First observation is sensitive to κ



First observation is sensitive to κ - zoomed



Log Predictive Score - LPS

- Reduce sensitivity to the prior: sacrifice n^* observations to train the prior into a posterior.
- **Predictive (Density) Score (PS)**. Decompose $p(y_1, \dots, y_n)$ as
$$\underbrace{p(y_1)p(y_2|y_1) \cdots p(y_{n^*}|y_{1:(n^*-1)})}_{\text{training}} \underbrace{p(y_{n^*+1}|y_{1:n^*}) \cdots p(y_n|y_{1:(n-1)})}_{\text{test}}$$
- Usually report on log scale: **Log Predictive Score (LPS)**.
- Time-series: obvious which data are used for training.
- Cross-sectional data: training-test split by **cross-validation**:

Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Fold 1	Fold 2	Fold 3	Fold 4	Fold 5
Fold 1	Fold 2	Fold 3	Fold 4	Fold 5

And hey! ... let's be careful out there

- Be especially **careful** with Bayesian model comparison when
 - ▶ The **compared models** are
 - very different in structure
 - severely misspecified
 - very complicated (black boxes).
 - ▶ The **priors** for the parameters in the models are
 - not carefully elicited
 - only weakly informative
 - not matched across models.
 - ▶ The **data**
 - has outliers (in all models)
 - has a multivariate response.