Solution to computer exam in Bayesian learning

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First load all the data into memory by running the R-file given at the exam

```r
rm(list=ls())
source("ExamData.R")
set.seed(1)
```

Problem 1

1a

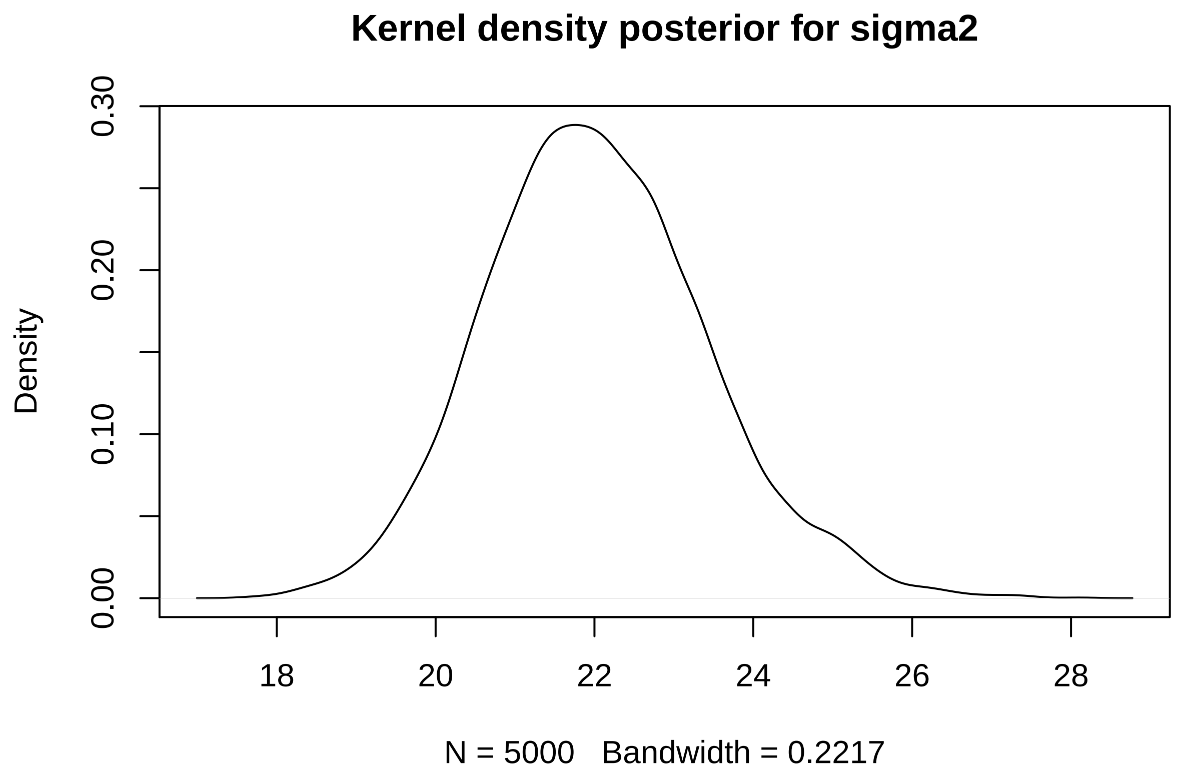
```r
# Prior
nCovs = dim(X)[2]
mu_0 = rep(0,nCovs)
Omega_0 = (1/100)*diag(nCovs)
v_0 = 1
sigma2_0 = 5^2
BostonRes <-  BayesLinReg(y, X, mu_0, Omega_0, v_0, sigma2_0, nIter = 5000)
Bmean = colMeans(BostonRes$betaSample)
Bq025 = apply(BostonRes$betaSample,2,quantile,.025)
Bq975 = apply(BostonRes$betaSample,2,quantile,.975)
print(data.frame(round(cbind(Bmean,Bq025,Bq975),3)),row.names=covNames)
```

```
##              Bmean   Bq025   Bq975
## intercept  36.043  26.128  45.877
## crim       -0.108  -0.172  -0.047
## zn          0.047   0.020   0.073
## indus       0.019  -0.101   0.140
## chas        2.714   1.070   4.348
## nox       -17.469 -25.041 -10.039
## rm          3.822   3.005   4.638
## age         0.001  -0.025   0.027
## dis        -1.473  -1.853  -1.086
## rad         0.304   0.181   0.428
## tax        -0.012  -0.019  -0.005
## ptratio    -0.943  -1.199  -0.684
## black       0.009   0.004   0.015
## lstat      -0.525  -0.624  -0.419
```

With 95% probability, a higher nox by 1 part per 10 million is associated with a house price that is between 10 and 25 thousand dollars lower.

1b

```r
sigma2Sample = BostonRes$sigma2Sample
dS = density(sigma2Sample)
par(mfrow=c(1,1))
plot(dS,main="Kernel density posterior for sigma2")
```



```r
dn=sort(dS$y/sum(dS$y),index.return=TRUE);
dnn = cumsum(dn$x)
HPDind = sort(dn$ix[dnn > .1])
HPDCI = c(min(dS$x[HPDind]),max(dS$x[HPDind]))
cat("Posterior mode: ",dS$x[which.max(dS$y)])
```

```
## Posterior mode:  21.76603
```

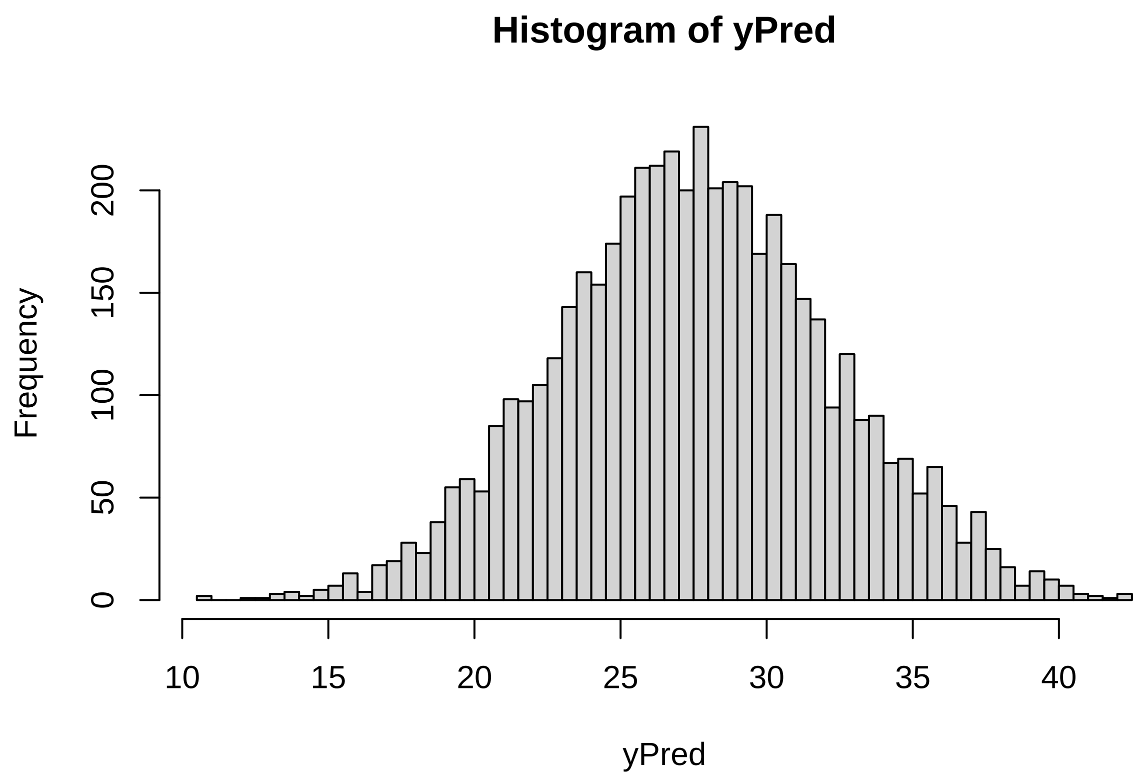
```r
cat("Highest posterior density interval: ",HPDCI)
```

```
## Highest posterior density interval:  19.66839 24.27857
```

1c

What we need here is the predictive distribution for the price given the explanatory variables. We can obtain the predictive distribution by simulation.

```r
nSim <- dim(BostonRes$betaSample)[1] # One predictive draw for each posterior parameter draw
yPred <- matrix(0,nSim,1)
for (i in 1:nSim){
  yPred[i] <- XNewHouse%*%BostonRes$betaSample[i,] + rnorm(n = 1, mean = 0, sd = sqrt(BostonRes$sigma2Sa
}
par(mfrow=c(1,1))
hist(yPred,50)
```



```r
sum(yPred>=20)/nSim
```

```
## [1] 0.9438
```

Probability of getting $20000 is quite large (0.94), so the construction project is probably a good idea.

Problem 2

2a

On paper.

2b

```r
x <- c(35,14,4,10,2)
n <- 5
alpha <- 1
beta <- 1
nSim <- 10000
theta <- rbeta(nSim,alpha+n,beta+sum(x))
xPred <- rgeom(nSim,theta)
quantile(xPred,0.95)
```

```
## 95%
##  44
```

The 95% upper bound of the predictive distribution is roughly 44 years.

Problem 3

3a

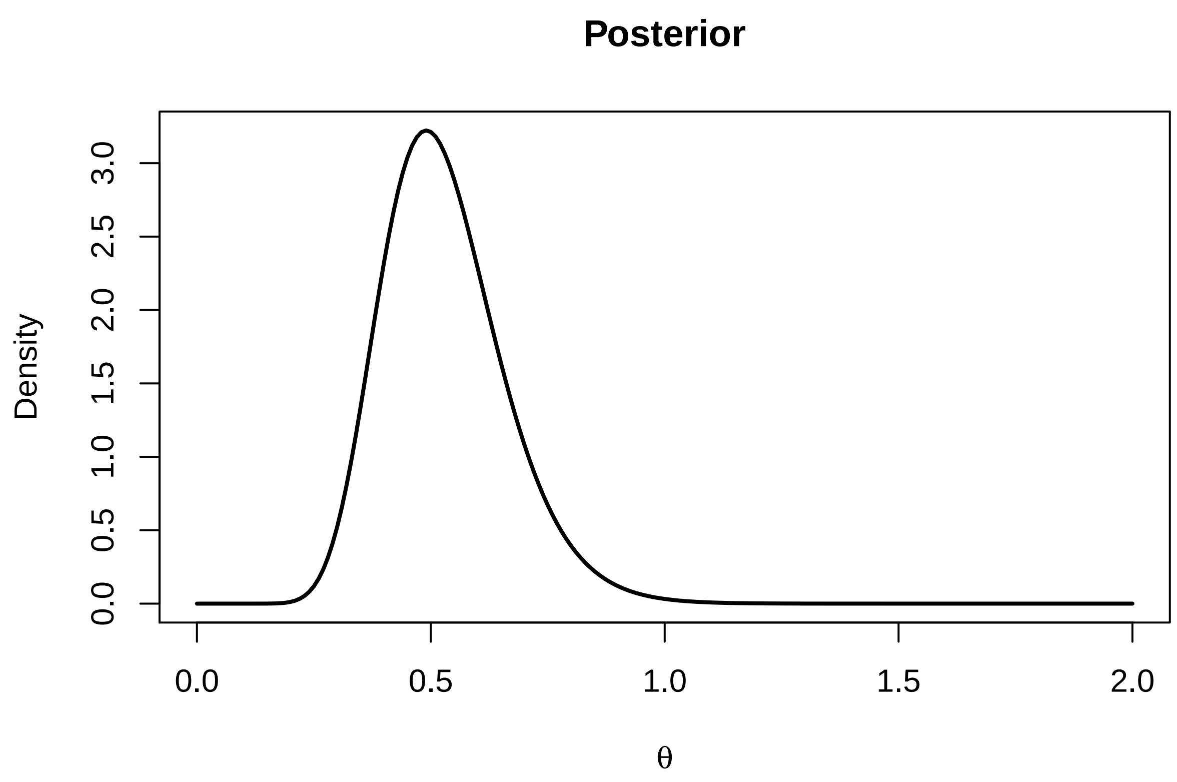
On paper.

3b

On paper.

3c

```r
# Alternative own implementation of t pdf
# logdt <- function(y,theta){
#   lgamma(3) - lgamma(2.5) - .5*log(5*pi) -3*log(1+1/5*(y-log(theta))^2)
# }
# Log posterior
logpost <- function(x, y, theta){
  logdens <- sum(dexp(x,rate=theta,log=TRUE)) + sum(dt(y-log(theta),df=5,log=TRUE)) + dgamma(theta,shap
  return(logdens)
}
# Plot the posterior over a grid of theta values
par(mfrow=c(1,1))
gridWidth <- .01
thetaGrid <- seq(0, 2, by = gridWidth)
logPostGrid <- rep(0,length(thetaGrid))
count <- 0
for (theta in thetaGrid){
  count <- count + 1
  logPostGrid[count] <- logpost(xData,yData,theta)
}
# Taking exp() and normalizing so the density integrates to one. Note the (1/gridWidth) factor.
postGrid <- (1/gridWidth)*exp(logPostGrid)/sum(exp(logPostGrid))
plot(thetaGrid, postGrid, type = "l", lwd = 2,main="Posterior",
     ylab = "Density", xlab = expression(theta))
```



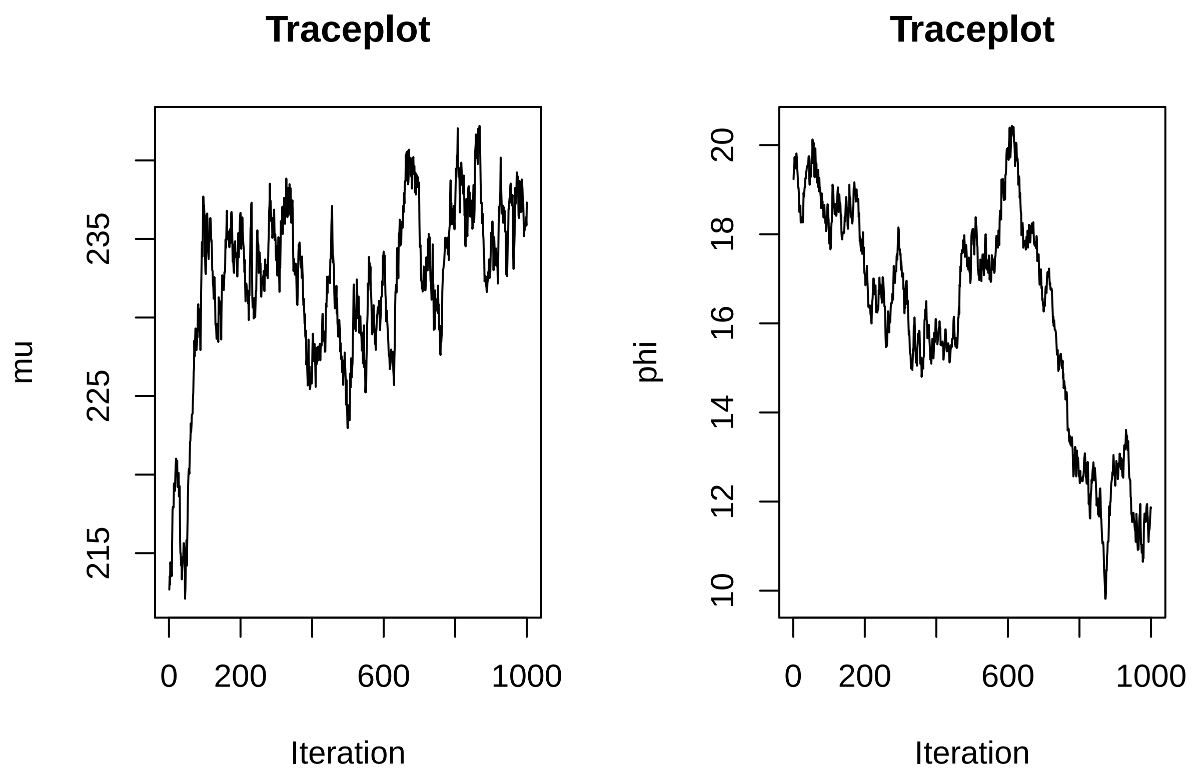
Problem 4

4a

```
x = incidents$incidents
# plot(x)
# Own definition of neg-bin log pdf
logdNegBin <- function(x,mu,phi){
  lchoose(x+phi-1,x) + x*log(mu/(mu+phi)) + phi*log(phi/(mu+phi))
}
logPostNegBin <- function(param, x){
  theta1 = param[1]
  theta2 = param[2]
  logPost = sum(logdNegBin(x, theta1, theta2))  - 2*log(theta2)
  return(logPost)
}
# Code that could be used for normal approximation
# initVal = c(200,20)
# optRes <- optim(par = initVal, fn  = logPostNegBin, gr = NULL, x, method = c("L-BFGS-B"),
#                  lower = c(0.0001,0.0001), upper = c(Inf,Inf), control = list(fnscale = -1), hessian =
# postMean <- optRes$par # This is the mean vector
# postCov <- -solve(optRes$hessian) # This is posterior covariance matrix
library(mvtnorm)
Metropolis <- function(c,niter,warmup,initVal,Sigma,logPostFunc,x) {
```

```r
  theta <- initVal
  thetamat <- matrix(0,length(theta),warmup+niter)
  thetamat[,1] <- theta
  accprobvec <- rep(0,warmup+niter)

  for(i in 2:(warmup+niter)) {
    thetaProp <- t(rmvnorm(1,theta,c*Sigma))
    thetaProp[thetaProp <= 0] = 1e-6
    accprob <- exp(logPostFunc(thetaProp,x) - logPostFunc(theta,x))
    accprobvec[i] <- min(accprob,1)
    if(runif(1) < accprob) {
      theta <- thetaProp
    }
    thetamat[,i] <- theta
  }
  return(list(thetamat=thetamat,accprobvec=accprobvec))
}
c <- .1
niter <- 1000
warmup <- 50
initVal = c(200,20)
Sigma = c*diag(c(100,5))
mp <- Metropolis(c,niter,warmup,initVal,Sigma,logPostNegBin,x)
# Optionally compute posterior mean and variance
# rowMeans(mp$thetamat[,warmup+1:niter])
# apply(mp$thetamat,1,var)
par(mfrow=c(1,2))
plot(mp$thetamat[1,warmup+1:niter], type="l",main="Traceplot",xlab="Iteration",ylab="mu")
plot(mp$thetamat[2,warmup+1:niter], type="l",main="Traceplot",xlab="Iteration",ylab="phi")
```



```r
mean(mp$accprobvec)
```

```
## [1] 0.9333279
```

The traceplots show that the parameters move slowly in the posterior space which means that the sampler is not efficient. This can also be seen in that the acceptance rate is high (around 90%) and should optimally be around 25-30% for random walk Metropolis. The efficiency could be improved by increasing the value of c to make more distant proposals.

4b

Function for proposing from the independent multivariate gamma distribution

```r
rmvgamma <- function(theta,c){
  return(c(rgamma(1,shape=c*theta[1],rate=c),rgamma(1,shape=c*theta[2],rate=c)))
}
```

Function to compute the log proposal density ratio for the Metropoli-Hastings acceptance probability

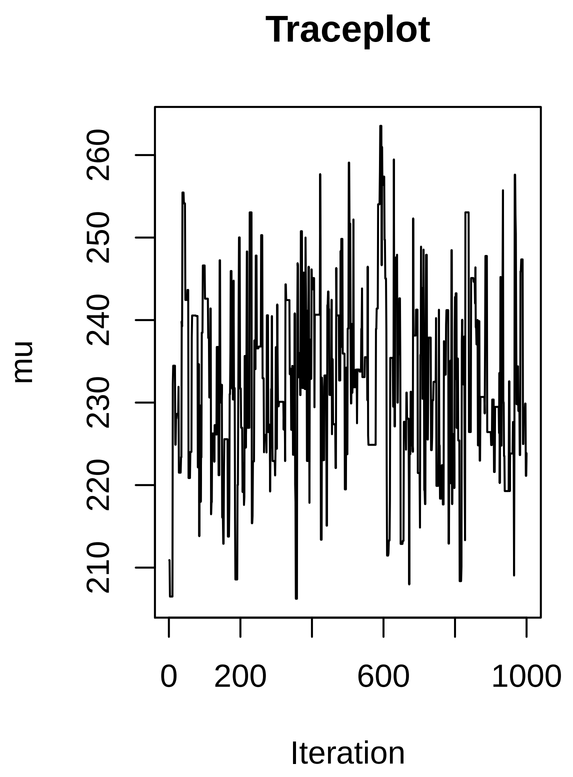
```r
ldmvgamma <- function(theta,thetaProp,c){
  ldens = dgamma(theta[1],shape=c*thetaProp[1],rate=c,log=TRUE) +
          dgamma(theta[2],shape=c*thetaProp[2],rate=c,log=TRUE) -
          dgamma(thetaProp[1],shape=c*theta[1],rate=c,log=TRUE) -
          dgamma(thetaProp[2],shape=c*theta[2],rate=c,log=TRUE)
  return(ldens)
}
```

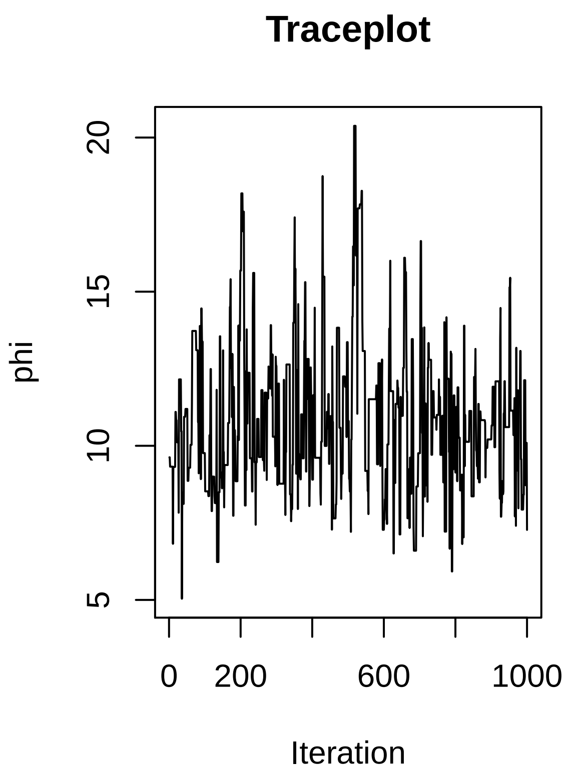
```r
c = .8
MetropolisHastings <- function(c,niter,warmup,initVal,logPostFunc,x) {

  theta <- initVal
  thetamat <- matrix(0,length(theta),warmup+niter)
  thetamat[,1] <- theta
  accprobvec <- rep(0,warmup+niter)

  for(i in 2:(warmup+niter)) {
    thetaProp <- rmvgamma(theta,c)
    accprob <- exp(logPostFunc(thetaProp,x) - logPostFunc(theta,x) + ldmvgamma(theta,thetaProp,c))
    accprobvec[i] <- min(accprob,1)
    if(runif(1) < accprob) {
      theta <- thetaProp
    }
    thetamat[,i] <- theta
  }

  return(list(thetamat=thetamat,accprobvec=accprobvec))
}
mp <- MetropolisHastings(c,niter,warmup,initVal,logPostNegBin,x)
# Optionally compute posterior mean and variance
# rowMeans(mp$thetamat[,warmup+1:niter])
# apply(mp$thetamat,1,var)
par(mfrow=c(1,2))
plot(mp$thetamat[1,warmup+1:niter], type="l",main="Traceplot",xlab="Iteration",ylab="mu")
plot(mp$thetamat[2,warmup+1:niter], type="l",main="Traceplot",xlab="Iteration",ylab="phi")
```





```
mean(mp$accprobvec)
```

```
## [1] 0.3591745
```

The sampler seems to rapidly explore the posterior distribution so it is quite efficient.