Q.1/a

$p(\theta|N,x) \propto p(x|N,\theta)\, p(\theta|N)$

$$\propto \binom{N}{x} \theta^{x} (1-\theta)^{N-x} \cdot 1$$

~~This also a conjugate Beta(1,1)~~

$$\propto \theta^{\overbrace{x+1-1}^{\alpha}} (1-\theta)^{\overbrace{51-x-1}^{\beta}}$$

$\therefore\ \theta|x \sim \text{Beta}(x+1, 51-x)$

Q.2/d

- Posterior of β can either approximated by Normal Approximation
- or by laplace approximation

- Normal Approximation we take the posterior mode as the mean and the reciprocal of the second derivative at the posterior mode as the variance

$$\beta|x \overset{\text{approx}}{\sim} \mathcal{N}(\hat{\beta}, J_x^{-1}(\hat{\beta}))$$

and with large number of observation we get better approx.

- In Laplace Approximation we also need the posterior mode and the reciprocal of the second derivative at the mode and the log of the posterior can be given as bellow:

$$\ln p(\beta|x) = \ln p(x|\hat{\beta}) + \ln p(\hat{\beta}) + \frac{1}{2}\ln|J_{\hat{\beta}}^{-1}| + \frac{5}{2}\ln(2\pi)$$

Where $\hat{\beta}$ is the posterior mode of β

Q4 / a

$$P(X_{21} \mid X_{1:20}) = \int P(X_{21} \mid \theta) P(\theta \mid X_{1:20}) \, d\theta$$

$$= \int \theta^{X_{21}} (1-\theta)^{1-X_{21}} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\Gamma(\beta+f)} \theta^{\alpha+s-1} (1-\theta)^{\beta+f-1} \, d\theta$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\Gamma(\beta+f)} \frac{\Gamma(X_{21}+\alpha+s)\Gamma(1-X_{21}+\beta+f)}{\Gamma(1+\alpha+\beta+n)}$$

$$= \frac{\Gamma(X_{21}+\alpha+s)\Gamma(1-X_{21}+\beta+f)}{\Gamma(\alpha+s)\Gamma(\beta+f)(\alpha+\beta+n)}$$

$$\Rightarrow P(X_{21} = \text{male} \mid X_{1:20}) = \frac{\Gamma(1+\alpha+s)}{\Gamma(\alpha+s)(\alpha+\beta+n)} = \frac{(\alpha+s)\Gamma(\alpha+s)}{\Gamma(\alpha+s)(\alpha+\beta+n)} = \frac{\alpha+s}{\alpha+\beta+n}$$

$$= \frac{2+4}{2+2+20} = \frac{6}{24} = 0.25$$

$$\Rightarrow P(X_{21} = \text{female} \mid X_{1:20}) = 1 - 0.25 = 0.75$$

b) $P(L \mid M, W) \propto P(M, W \mid L) P(L)$

$$= P(M \mid L) P(W \mid L) P(L)$$

the conditional posterior for $M \mid L \sim N(\bar{M}_L, \sigma_m^2(1+\frac{1}{n}))$

" " " , $W \mid L \sim N(\bar{W}_L, \sigma_w^2(1+\frac{1}{n}))$

$$P(L) \sim \frac{n_{mL}+1}{n+2} = \frac{4+1}{20+2} = \frac{5}{22}$$

$$P(L \mid M, W) = \phi(L_{n+1}, \mu = 12, \sigma^2 = 4) \cdot \phi(W_{n+1}, \mu = 280, \sigma^2 = 2500) \cdot \frac{5}{22}$$

where ϕ is the pdf of normal dist.