

Problem 2a)

Model: $X_1, \dots, X_n | \theta \stackrel{\text{iid}}{\sim} N(\theta, \sigma^2)$ σ^2 known

Prior: $P(\theta) \propto \text{constant}$

Posterior: $p(\theta | X_1, \dots, X_n) \propto p(X_1, \dots, X_n | \theta) \cdot P(\theta)$
 $= p(X_1, \dots, X_n | \theta)$ [prior is constant]

$$= \prod_{i=1}^n p(X_i | \theta) \quad [\text{Independence}]$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(X_i - \theta)^2\right)$$

$$\propto \prod_{i=1}^n \exp\left(-\frac{1}{2\sigma^2}(X_i - \theta)^2\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \theta)^2\right)$$

$$\begin{aligned} \text{Now, } \sum_{i=1}^n (X_i - \theta)^2 &= \sum_{i=1}^n ((X_i - \bar{X}) - (\theta - \bar{X}))^2 \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (\theta - \bar{X})^2 - 2(\theta - \bar{X}) \sum_{i=1}^n (X_i - \bar{X}) \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\theta - \bar{X})^2 - 2(\theta - \bar{X}) \underbrace{\left(\sum_{i=1}^n X_i - n\bar{X}\right)}_0 \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\theta - \bar{X})^2 \end{aligned}$$

$$\begin{aligned}
\text{So, } P(\theta | x_1, \dots, x_n) &\propto \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n (x_i - \bar{x})^2 - n(\theta - \bar{x})^2\right)\right) \\
&= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x})^2\right) \cdot \exp\left(-\frac{1}{2\sigma^2}n(\theta - \bar{x})^2\right) \\
&\propto \exp\left(-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right) \\
&\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)
\end{aligned}$$

Problem 2b)

prior: $\theta \sim N(\mu_0, \tau_0^2)$

posterior: $p(\theta | x_1, \dots, x_n) \propto p(x_1, \dots, x_n | \theta) \cdot p(\theta)$

$$\begin{aligned}
&\propto \exp\left(-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right) \cdot \exp\left(-\frac{1}{2\tau_0^2}(\theta - \mu_0)^2\right) \\
&= \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma^2/n}(\theta - \bar{x})^2 + \frac{1}{\tau_0^2}(\theta - \mu_0)^2\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{n}{\sigma^2}(\theta^2 + \bar{x}^2 - 2\theta\bar{x}) + \frac{1}{\tau_0^2}(\theta^2 + \mu_0^2 - 2\theta\mu_0) \\
&= \text{const} + \theta^2\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right) - 2\left(\frac{n}{\sigma^2}\bar{x} + \mu_0\right)\theta
\end{aligned}$$

↑ Not important. It will end up in the normalization constant.

we want this to be of the form

$$\frac{1}{\tau_n^2} (\theta - \mu_n)^2 \quad \text{since then the posterior will be } N(\mu_n, \tau_n^2)$$

this is achieved if

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2} \quad \left(\text{since this is the coefficient on } \theta^2 \text{ above} \right)$$

and

$$\frac{\mu_n}{\tau_n^2} = \frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau_0^2} \quad \left(\text{since this is the coefficient on } -2\theta \text{ above} \right)$$

$$\begin{aligned} \text{So, } \mu_n &= \tau_n^2 \left(\frac{n}{\sigma^2} \bar{x} + \frac{1}{\tau_0^2} \mu_0 \right) \\ &= \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}} \left(\frac{n}{\sigma^2} \bar{x} + \frac{1}{\tau_0^2} \mu_0 \right) \end{aligned}$$

$$= w\bar{x} + (1-w)\mu_0$$

$$\text{with } w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}$$

Phew!!!

Problem 4b)

Model: $x_1, \dots, x_n \sim \text{Exp}(\theta)$

Prior: $\theta \sim \text{Gamma}(\alpha, \beta)$

Posterior: $p(\theta | x_1, \dots, x_n) \propto p(x_1, \dots, x_n | \theta) \cdot p(\theta)$

$$\propto \prod_{i=1}^n \underbrace{\theta e^{-\theta x_i}}_{\text{Exponential distribution}} \cdot \underbrace{\theta^{\alpha-1} e^{-\beta \theta}}_{\text{Proportional to Gamma}(\alpha, \beta) \text{ density}}$$

$$\propto \theta^n e^{-\theta \sum_{i=1}^n x_i} \theta^{\alpha-1} e^{-\beta \theta}$$

$$= \theta^{n+\alpha-1} e^{-(\sum_{i=1}^n x_i + \beta)\theta}$$

$$\propto \text{Gamma}\left(\alpha+n, \beta + \sum_{i=1}^n x_i\right)$$