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2.2

$$A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$$

a. $5A = 5\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$

b. $BA = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$

c. $A'B' = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 & -2 \\ -3 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$

d. $C'B = [5 \ -4 \ 2]\begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} = [12 \ -7]$

e. AB is not defined because number of columns in A does not equal the number of rows in B

2.7 $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

a.

$|A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right| = 0$

$(9-\lambda)(6-\lambda) - 4 = 0 \Rightarrow 50 - 15\lambda + \lambda^2 = 0 \Rightarrow (\lambda - 5)(\lambda - 10) = 0$

$\lambda_1 = 10, \ \lambda_2 = 5$

$Ae = \lambda e \Rightarrow \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 10\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$\left.\begin{aligned} 9e_1 - 2e_2 &= 10e_1 \\ -2e_1 + 6e_2 &= 10e_2 \end{aligned}\right\} \rightarrow \quad e_1 = -2e_2 \Rightarrow e_1 = -2, \ e_2 = 1$

$\vec{e}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$$\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 5\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\left.\begin{aligned} 9e_1 - 2e_2 &= 5e_1 \\ -2e_1 + 6e_2 &= 5e_2 \end{aligned}\right\} \longrightarrow e_2 = 2e_1 \rightarrow e_1 = 1, e_2 = 2$$

$$\vec{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

b. $A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$

$$= 10\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} + 5\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

c. $A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} \frac{6}{50} & \frac{2}{50} \\ \frac{2}{50} & \frac{9}{50} \end{bmatrix}$

d. $|A^{-1} - \lambda I| = \left|\begin{bmatrix} \frac{6}{50} - \lambda & \frac{2}{50} \\ \frac{2}{50} & \frac{9}{50} - \lambda \end{bmatrix}\right| \Rightarrow (\frac{6}{50} - \lambda)(\frac{9}{50} - \lambda) - \frac{4}{2500} = 0$

$\Rightarrow \lambda^2 - \frac{15}{50}\lambda + \frac{50}{2500} = 0 \Rightarrow (\lambda - \frac{1}{5})(\lambda - \frac{1}{10}) = 0$

$\lambda_1 = \frac{1}{5}, \lambda_2 = \frac{1}{10}$

$A e = \lambda e \Rightarrow \begin{bmatrix} \frac{6}{50} & \frac{2}{50} \\ \frac{2}{50} & \frac{9}{50} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$$\left.\begin{aligned} \frac{6}{50}e_1 + \frac{2}{50}e_2 &= \frac{1}{5}e_1 \\ \frac{2}{50}e_1 + \frac{9}{50}e_2 &= \frac{1}{5}e_2 \end{aligned}\right\} \rightarrow e_2 = 2e_1 \Rightarrow e_1 = 1, e_2 = 2$$

$$\vec{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{50} & \frac{2}{50} \\ \frac{2}{50} & \frac{9}{50} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} \frac{6}{50} e_1 + \frac{2}{50} e_2 = \frac{1}{10} e_1 \\ \frac{2}{50} e_1 + \frac{9}{50} e_2 = \frac{1}{10} e_2 \end{aligned}\right\} \rightarrow e_1 = -2e_2 \Rightarrow e_1 = -2,\ e_2 = 1$$

$$\vec{e}_2 = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

4.2

a. $$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} * \exp\left\{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]\right\}$$

$$= \frac{1}{2\pi\sqrt{2\left(\frac{3}{4}\right)}} * \exp\left\{-\frac{1}{2} * \frac{4}{3}\left[\left(\frac{x_1}{\sqrt{2}}\right)^2 + \left(\frac{x_2-2}{1}\right)^2 - \frac{x_1}{\sqrt{2}} * \frac{x_2-2}{1}\right]\right\}$$

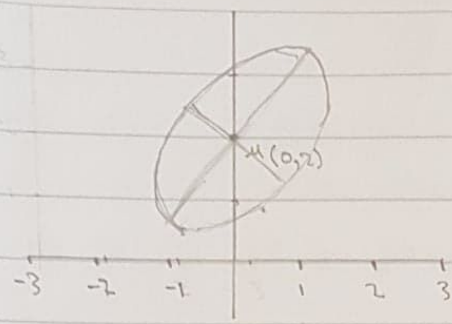
$$= \frac{1}{2\pi\sqrt{1.5}} * \exp\left\{-\frac{1}{2}\left[\frac{2}{3}x_1^2 + \frac{4}{3}(x_2^2-2) - \frac{4}{3\sqrt{2}} x_1 (x_2-2)\right]\right\}$$

b. $$(x-\mu)'\Sigma^{-1}(x-\mu) = \frac{1}{1-\rho_{12}^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]$$

$$= \frac{4}{3}\left[\frac{1}{2}x_1^2 + (x_2-2)^2 - \frac{1}{\sqrt{2}} x_1 (x_2-2)\right]$$

$$= \frac{2}{3}x_1^2 + \frac{4}{3}x_2^2 + \frac{8}{3\sqrt{2}}x_1 - \frac{16}{3}x_2 - \frac{4}{3\sqrt{2}}x_1x_2 + \frac{16}{3}$$

c. $(x-\mu)'\Sigma^{-1}(x-\mu) \le \chi^2_{.05}$ $\rightarrow$ $c^2 = \chi^2_{.05} = 1.39$ at 2 degrees of freedom



4.3

a. X_1 and X_2 are not independent since $\sigma_{12} = -2 \neq 0$

b. X_2 and X_3 are independent since $\sigma_{23} = 0$

c. $X = \begin{bmatrix} X_1 \\ X_2 \\ \hline X_3 \end{bmatrix} \rightarrow \Sigma = \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$

we see that $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and X_3 have covarince matrix $\Sigma_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
therefore they are independent

d. $AX = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_3 \\ \frac{1}{2}X_1 + \frac{1}{2}X_2 \end{bmatrix}$

$A\Sigma A' = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{3}{2} \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ $\therefore$ X_3 and $\frac{X_1 + X_2}{2}$ are independent since σ_{12} of AX is 0

e. $AX = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_2 \\ -\frac{5}{2}X_1 + X_2 - X_3 \end{bmatrix}$

$$A\Sigma A' = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & \frac{5}{2} \\ 5 & 10 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & \frac{23}{4} \end{bmatrix}$$

X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent since σ_{12} of $AX = 10 \neq 0$

4.4

a. $$AX = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 3X_1 - 2X_2 + X_3$$

$$A\mu = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 13$$

$$A\Sigma A' = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 9$$

$3X_1 - 2X_2 + X_3$ is distributed as $N_1(\mu = 13, \sigma^2 = 9)$

b. Let $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$AX = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_2 - a \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \end{bmatrix}$$

$$A\Sigma A' = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix} \begin{bmatrix} 1 & 1 - a_1 - a_2 \\ 3 & 3 - a_1 - 2a_2 \\ 2 & 2 - a_1 - 2a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3-a_1-2a_2 \\ 3-a_1-2a_2 & -a_1+a_1^2+a_1a_2+3-a_1-2a_2-2a_2+a_1a_2+2a_2^2 \end{bmatrix}$$

for X_2 and $X_2 - a\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ to be independent $3-a_1-2a_2$ is set to 0

$\rightarrow a_1 = 3-2a_2 \quad \rightarrow a = \begin{bmatrix} 3-2a_2 \\ a_2 \end{bmatrix}$

4.5

a. $\text{mean} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$

$\Sigma_{12} = \rho_{12}\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}$

$$\begin{aligned}\text{mean} &= \mu_1 + \rho_{12}\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}^{-1}(x_2-\mu_2) \\ &= 0 + 0.5\sqrt{2}\sqrt{1}^{-1}(x_2-2) \\ &= 0.707\,x_2 - 1.414\end{aligned}$$

$$\begin{aligned}\text{covariance} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ &= \Sigma_{11} - \rho_{12}\sqrt{\Sigma_{11}}\cancel{\sqrt{\Sigma_{22}}}\,\cancel{\Sigma_{22}^{-1}}\,\rho_{12}\sqrt{\Sigma_{11}}\cancel{\sqrt{\Sigma_{22}}} \\ &= 2 - 0.5 \times 2 \times 0.5 \\ &= 1.5\end{aligned}$$

The conditional distribution of X_1 given $X_2 = x_2$ is $N(0.707x_2 - 1.414, 1.5)$

b. $$\begin{aligned}\text{mean} &= \mu_2 + \Sigma_{12}\Sigma_{11}^{-1}(x_1-\mu_1) + \Sigma_{23}\Sigma_{33}^{-1}(x_3-\mu_3) \\ &= 1 + (-2)(1)(x_1+3) + 0 \times \tfrac{1}{2}(x_3-4) \\ &= -2x_1 - 5\end{aligned}$$

$$\begin{aligned}\text{covariance} &= \Sigma_{22} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{21} - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32} \\ &= 5 - (-2)(1)(-2) - (0)(\tfrac{1}{2})(0) \\ &= 1\end{aligned}$$

The conditional distribution of X_2 given $X_1 = x_1$ and $X_3 = x_3$ is $N(-x_1 - 5, 1)$

c. mean $= \mu_3 + \Sigma_{13}\Sigma_{11}^{-1}(x_1-\mu_1) + \Sigma_{23}\Sigma_{22}^{-1}(x_2-\mu_2)$

$= 1 + (1)(1)^{-1}(x_1-2) + (2)(3)^{-1}(x_2+3)$

$= x_1 + \frac{2}{3}x_2 + 1$

covariance $= \Sigma_{33} - \Sigma_{13}\Sigma_{11}^{-1}\Sigma_{31} - \Sigma_{23}\Sigma_{22}^{-1}\Sigma_{32}$

$= 2 - (1)(1)^{-1}(1) - (2)(3)^{-1}(2)$

$= 2 - 1 - \frac{4}{3} = -\frac{1}{3}$

The conditional distribution of X_3 given $X_1 = x_1$ and $X_2 = x_2$ is $N(x_1 + \frac{2}{3}x_2 + 1, -\frac{1}{3})$

4.21

a. $\bar{X}$ is distributed $N_4(\mu, \frac{1}{60}\Sigma)$

b. $(X_1-\mu)'\Sigma^{-1}(X_1-\mu)$ is distributed χ^2_4

c. $n(\bar{X}-\mu)'\Sigma^{-1}(\bar{X}-\mu)$ is distributed χ^2_4

d. $n(\bar{X}-\mu)'S^{-1}(\bar{X}-\mu)$ is approximately distributed as χ^2_4, since $n-p$ is relatively large.

4.22

a. $\bar{X}$ is distributed as $N(\mu, \frac{1}{75}\Sigma)$

b. $n(\bar{X}-\mu)'S^{-1}(\bar{X}-\mu)$ is approximately distributed as χ^2_p