TENTAMEN (EXAMINATION)

Tentamensdatum/*Examination date:* 2020-01-14
(åå-mm-dd/*yy-mm-dd*)

AID-nummer
AID number

Ifylles av student: 3 0 3 2
Completed by student

Ifylles av vakt: 3 0 3 2
Completed by supervisor

Utbildningskod/*Education code:* 732A97 **Modul/*Module:*** TENT

Kursnamn/*Course title:* Multivariate Statistical Methods

Institution/*Department:* IDA

Jag intygar att varken mobil eller något annat otillåtet hjälpmedel finns tillgängligt under tentamen.
I confirm that no mobile or other non-permitted aids are available during the examination. ☑

Inlämnat: antal lösblad 10 **tentamensformulär** ☐
Enclosed: number of sheets ***exam booklet***

Markera behandlade uppgifter med X/*Mark tasks attempted with an X*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X här/*here*	X	X	X	X											
Erhållna poäng *Points obtained*	5	4½	3½	3½											
	16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
X här/*here*															
Erhållna poäng *Points obtained*															

Anvisningar/*Instructions*

1. Skriv AID-nummer, datum, utb.kod, modul på varje blad som lämnas in/*Write AID number, date, edu.code and module on every sheet that is handed in*
2. På varje papper får högst en uppgift lösas om inget annat anges/ *Maximum one task per sheet unless otherwise instructed*
3. Skriv endast på pappets ena sida om inget annat anges/ *Use only one side of each sheet unless otherwise instructed*
4. Numrera de papper som lämnas in/*Number every sheet that is handed in*
5. Använd inte röd penna/*Do not use a red pen/pencil*

Sen inlämning ☐
Late hand in

Klockslag ______
Time

Orsak ______
Reason

______

Σ Poäng/*Points:* 16½ Betyg/*Grade:* B

Examinator/*Examiner:* K. Burb

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1) $p \geq 2 \qquad \alpha \geq 0$

5

$$\Sigma = (1-\alpha) I + \alpha \vec{1}_p \vec{1}_p^T \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{p \times p}$$

$$(1-\alpha) I = (1-\alpha) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{p \times p} = \begin{bmatrix} 1-\alpha & 0 & \cdots & 0 \\ 0 & 1-\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1-\alpha \end{bmatrix}_{p \times p} \quad (1)$$

$$\alpha \vec{1}_p \vec{1}_p^T = \alpha \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{p \times 1} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1 \times p} = \begin{bmatrix} \alpha & \alpha & \cdots & \alpha \\ \alpha & \alpha & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & \alpha \end{bmatrix}_{p \times p} \quad (2)$$

$$(1), (2) \Rightarrow \begin{bmatrix} 1-\alpha & 0 & \cdots & 0 \\ 0 & 1-\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\alpha \end{bmatrix} + \begin{bmatrix} \alpha & \alpha & \cdots & \alpha \\ \alpha & \alpha & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & \alpha \end{bmatrix} = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \alpha \\ \alpha & \cdots & \alpha & 1 \end{bmatrix}$$

$$\Rightarrow \Sigma = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{bmatrix} \Rightarrow \boxed{\Sigma = \Sigma^T \Rightarrow \Sigma \text{ is symmetric}}$$

$$\vec{X} \Sigma \vec{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} \begin{bmatrix} x_1 + \alpha x_2 + \alpha x_3 + \cdots + \alpha x_p \\ \alpha x_1 + x_2 + \alpha x_3 + \cdots + \alpha x_p \\ \alpha x_1 + \alpha x_2 + x_3 + \cdots + \alpha x_p \\ \vdots \\ \alpha x_1 + \alpha x_2 + \alpha x_3 + \cdots \end{bmatrix}$$

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1) Continue:

$$= x_1^2 + \alpha x_1(x_2 + x_3 + \dots + x_p) + x_2^2 + \alpha x_2(x_1 + x_3 + \dots + x_p) + x_3^2 + \alpha x_3(x_1 + x_2 + x_4 + \dots + x_p)$$

$$+ \dots + x_p^2 + \alpha x_p(x_1 + \dots + x_{p-1})$$

$$= (x_1^2 + x_2^2 + \dots + x_p^2) + \alpha \Big[(x_1 x_2 + x_1 x_3 + \dots + x_1 x_p) + (x_2 x_1 + x_2 x_3 + \dots + x_2 x_p)$$

$$+ (x_3 x_1 + x_3 x_2 + x_3 x_4 + \dots x_3 x_p) + \dots +$$

$$(x_p x_1 + x_p x_2 + \dots + x_p x_{p-1}) \Big] \Rightarrow$$

$$\vec{x} \Sigma \vec{x} = (x_1^2 + x_2^2 + \dots + x_p^2) + \alpha \Big[2x_1 x_2 + 2x_1 x_3 + 2x_1 x_4 \dots + 2x_1 x_p + 2x_2 x_3 + 2x_2 x_4 + \dots + 2x_2 x_p$$

$$2x_3 x_4 + 2x_3 x_5 + \dots + 2x_3 x_p + 2x_4 x_5 + \dots + 2x_4 x_p +$$

$$2x_5 x_6 + \dots + 2x_5 x_p + \dots + 2x_{p-1} x_p \Big]$$

If $\alpha = 0 \Rightarrow$ The second term is zero and the first term is always positive (1)

If $\alpha = 1 \Rightarrow \vec{x} \Sigma \vec{x} = (x_1^2 + \dots + x_p^2 + 2x_1 x_2 + 2x_1 x_3 + \dots 2x_{p-1} x_p) = (x_1 + x_2 + \dots + x_p)^2$ (2)

if $\alpha > 1 \Rightarrow \vec{x} \Sigma \vec{x} > (x_1 + x_2 + \dots + x_p)^2 \geq 0 \Rightarrow \vec{x} \Sigma \vec{x} \geq 0$ (3)

? why here it would fail with negative x_i

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1) $\{2x_1x_2 + 2x_1x_3 + 2x_1x_4 + \cdots + 2x_{p-1}x_p\} = \left[(x_1+x_2+\cdots+x_p)^2 - (x_1^2+x_2^2+\cdots x_p^2)\right]$

$\Rightarrow \vec{X}\Sigma\vec{X} = (x_1^2+x_2^2+\cdots+x_p^2) + \alpha\left[(x_1+x_2+\cdots+x_p)^2 - (x_1^2+x_2^2+\cdots+x_p^2)\right]$

$= \alpha(x_1+x_2+\cdots+x_p)^2 + (x_1^2+x_2^2+\cdots+x_p^2)(1-\alpha)$

if $0<\alpha<1 \Rightarrow 1-\alpha>0 \Rightarrow$ This term is always positive for $\vec{x}\neq 0$ (4)

and

if $\vec{x}=0 \Rightarrow \vec{X}\Sigma\vec{X}=0$ (5)

from (1), (2), (3), (4) and (5) we conclude that

$\vec{X}\Sigma\vec{X} \geq 0 \Rightarrow \Sigma$ is positive-semi-definite.

and already proved that Σ is symmetric $\Rightarrow \Sigma$ is

symmetric-positive-semi definite

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2) $\vec{X} \sim N(\vec{\mu}, \Sigma)$, G : an orthogonal matrix

4 1/2

$\vec{\mu} = \vec{0}$, $\Sigma = \sigma^2 I$ $\qquad G\vec{X} = ?$ $\qquad G^T = G^{-1}$ orthogonal

$E(G\vec{X}) = G\,E(\vec{X}) = G\vec{\mu} = G\vec{0} = \vec{0}$ (1)

$Var(G\vec{X}) = G\Sigma G^T = G(\sigma^2 I)G^T$

because G is an orthogonal matrix $\Rightarrow G^T = G^{-1} \Rightarrow GG^T = GG^{-1} = I$

$\Rightarrow G\sigma^2 I G^T = G\sigma^2 I G^{-1} = I\sigma^2 I = \sigma^2 I = \Sigma$ (2)

$\Rightarrow G\vec{X} \sim N(\vec{0}, \Sigma)$ $\rightarrow$ Same distribution as $\vec{X}$

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3) $n = 25$ $\bar{x} = \begin{bmatrix} 185.72 \\ 183.84 \end{bmatrix}$ $S = \begin{bmatrix} 91.481 & 66.875 \\ 66.875 & 96.775 \end{bmatrix}$

3½

$\Sigma = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$ $S^{-1} = \begin{bmatrix} 0.022 & -0.015 \\ -0.015 & 0.021 \end{bmatrix}$

$\alpha = 0.05$

a) $H_0: \mu = \mu_0 = \begin{bmatrix} 182 \\ 182 \end{bmatrix}$

1

In this case we have one sample taken from one population, the sample size ($n=25$) is small for two variables, ~~and S is not equal to Σ~~. So we form the test statistics as follow:

We know Σ!

$$T^2 = (\bar{x} - \vec{\mu})^T \left(\frac{1}{n} S\right)^{-1} (\bar{x} - \vec{\mu}) \sim c^2 = \frac{(n-1)P}{n-P} F_{P, n-P}$$

$$\bar{x} - \vec{\mu} = \begin{bmatrix} 185.72 \\ 183.84 \end{bmatrix} - \begin{bmatrix} 182 \\ 182 \end{bmatrix} = \begin{bmatrix} 3.72 \\ 1.84 \end{bmatrix}$$ $n = 25$, $P = 2$

$$T^2 = 25 \begin{bmatrix} 3.72 & 1.84 \end{bmatrix} \begin{bmatrix} 0.022 & -0.015 \\ -0.015 & 0.021 \end{bmatrix} \begin{bmatrix} 3.72 \\ 1.84 \end{bmatrix}$$

$$= 25 \times \left[0.022(3.72)^2 + 0.021(1.84)^2 - 2(0.015)(3.72)(1.84)\right]$$

$$= 25(0.30 + 0.071 - 0.205) = \boxed{4.15} \rightarrow T^2$$

$T^2 < c^2 \Rightarrow$

$$c^2 = \frac{(24)2}{23} \underbrace{F_{2,23}(0.05)}_{\text{From Table}} = 2.087 \times 3.42 = 7.14$$

~~⇒ We reject H₀ and conclude that μ₀ = [182 182] is not a plausible vector for mean population~~

$T^2 = 4.15 < c^2 = 7.14 \Rightarrow$ We do not reject H_0, and conclude that $\mu_0 = \begin{bmatrix} 182 \\ 182 \end{bmatrix}$ is a plausible mean vector for the population.

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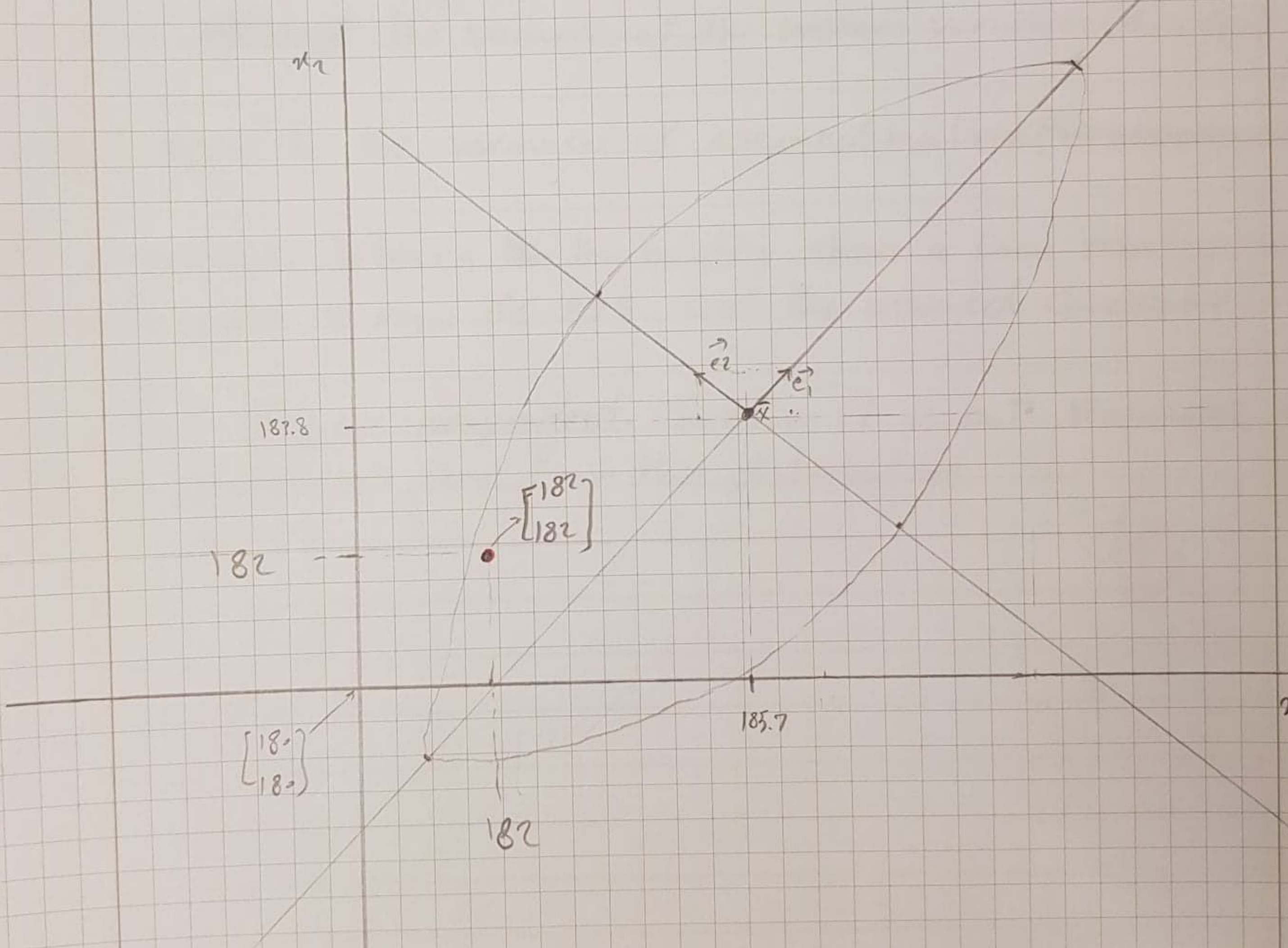
3-b) eigen values of S: $\lambda_1 = 161.055$, $\lambda_2 = 27.201$

eigen vectors of S $\vec{e}_1 = \begin{bmatrix} 0.693 \\ 0.721 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} -0.721 \\ 0.693 \end{bmatrix}$

The ellipse is centered at $\bar{x} = \begin{bmatrix} 185.72 \\ 183.84 \end{bmatrix}$

half-length of major axis $= \sqrt{c^2}\sqrt{\frac{\lambda_1}{n}} = \sqrt{7.14}\sqrt{\frac{161.055}{25}} = 6.78$ in the direction of $\vec{e}_1$

" " minor axis $= \sqrt{c^2}\sqrt{\frac{\lambda_2}{n}} = \sqrt{7.14}\sqrt{\frac{27.201}{25}} = 2.79$ " " " " $\vec{e}_2$



As it is shown $\begin{bmatrix} 182 \\ 182 \end{bmatrix}$ lies inside the ellipse

why ?

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3-d) $\Sigma_1 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$ according to this assumed Covariance ma
the Covariance betwee two variables is

i.e. $\sigma_{12} = \sigma_{21} = 0$, So we can conclude that the t
variables are independent. But,

The sample Covariance = $\begin{bmatrix} 91.481 & 66.875 \\ 66.875 & 96.775 \end{bmatrix}$ → according to th

matrix, ~~[illegible]~~ the variances of the ~~vectors~~ variables (σ_{11}, σ_{22}

almost equal to the variances of assumed matrix(~~[illegible]~~

The Covariance between ~~the~~ two variables shows a linear relation
dependency, as $\sigma_{12} = \sigma_{21} = 66.875$. and the assumed Covarianc

shows that they are independent. There fore it is not reasonabl
that the measurements Come from this distribution.

1

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8

4) $\Sigma_2 = \begin{bmatrix} 100 & 50 \\ 50 & 100 \end{bmatrix}$ $\quad \Sigma_2^{-1} = \begin{bmatrix} 1/75 & -1/150 \\ -1/150 & 1/75 \end{bmatrix}$ $\quad \vec{\bar{X}} - \vec{\mu}_o = \begin{bmatrix} 3.72 \\ 1.84 \end{bmatrix}$

a) 3

$$T^2 = n(\bar{X} - \vec{\mu}_o)' \begin{bmatrix} 1/75 & -1/150 \\ -1/150 & 1/75 \end{bmatrix} (\bar{X} - \mu_o) = 25 (3.72 \quad 1.84) \begin{bmatrix} 1/75 & -1/150 \\ -1/150 & 1/75 \end{bmatrix} \begin{bmatrix} 3.72 \\ 1.84 \end{bmatrix}$$

$$= 25 \left[\frac{1}{75}(3.72)^2 + \frac{1}{75}(1.84)^2 - \frac{2}{150}(3.72)(1.84) \right] = 25 [0.184 + 0.045 - 0.091]$$

$\Rightarrow T^2 = 3.45$, $c^2 = 7.14$ $\Rightarrow T^2 < c^2$

$\Rightarrow$ we do not reject H_0, and we conclude that ~~assuming~~ given the assumed covariance matrix, $\mu_o = \begin{bmatrix} 182 \\ 182 \end{bmatrix}$ is a plausible ~~vector~~ vector for $\vec{\mu}$ (mean vector of population)

* The sample size is small, so ~~we choose to~~ it is reasonable to use F distribution.

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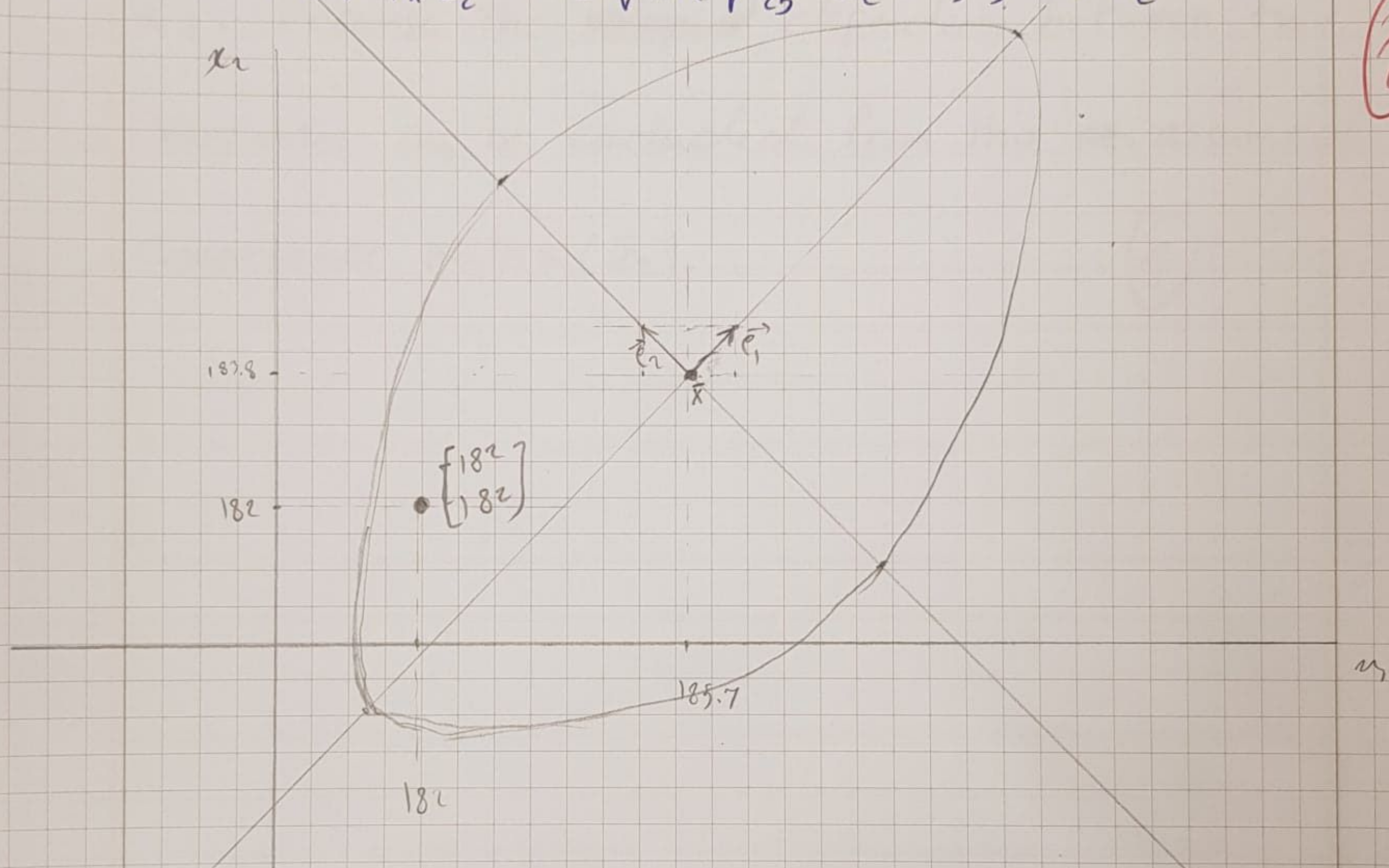
4-b) $\bar{x} = \begin{bmatrix} 185.72 \\ 183.84 \end{bmatrix}$ $\lambda_1 = 150$ $\vec{e}_1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$ $\lambda_2 = 50$ $\vec{e}_2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$ $c^2 = 7.14$

major axis: $\pm c\sqrt{\frac{\lambda_1}{n}}\,\vec{e}_1 = \pm\sqrt{7.14}\sqrt{\frac{150}{25}}\,\vec{e}_1 = \pm 6.54\,\vec{e}_1$

minor axis: $\pm c\sqrt{\lambda_2/n}\,\vec{e}_2 = \pm\sqrt{7.14}\sqrt{\frac{50}{25}}\,\vec{e}_2 = \pm 3.78\,\vec{e}_2$



As it is shown, $\begin{bmatrix} 182 \\ 182 \end{bmatrix}$ lies inside the ellipse

Why?

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4-d In new assumed matrix, Σ_2, the covariance between two variables is 5.0 ($\sigma_{12} = \sigma_{21} = 5.0$), and so it assumes the two variables are ~~correlated~~ not independent. Having considered, S, Σ_2 seems more reasonable than Σ_1. Because the ~~independent~~ dependency between two variables can be concluded from this ~~...~~ assumed covariance matrix (Σ_2).

1