SCIA - 10TH SCANDINAVIAN CONFERENCE ON IMAGE ANALYSIS, LAPPEENRANTA, FINLAND, PP. 439-446, JUNE 1997.

Critical Motion Sequences and Conjugacy of Ambiguous Euclidean Reconstructions

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Abstract

This paper deals with *critical motion sequences*, i.e. sequences of camera motions that lead to inherent ambiguities in uncalibrated Euclidean reconstruction or self-calibration. Concretely, we focus on how to deal with ambiguous reconstructions. We show that the ambiguous Euclidean reconstructions from a critical motion sequence are conjugated in a special sense. We discuss how these conjugacies may be used to identify all ambiguous Euclidean reconstructions, even if there are discrete solutions or disjoint families of solutions.

1 Introduction

1.1 Uncalibrated Vision and Self-Calibration

One of the major goals of computer vision is the recovery of spatial information about the environment. Classical approaches assume that the cameras are *calibrated* beforehand, but a great interest in *uncalibrated* vision and on-line calibration has arisen during the last couple of years. A key result is that even with completely uncalibrated cameras, spatial information – *projective structure* – can be obtained: the scene can be reconstructed up to an unknown projective transformation [6, 8]. Furthermore, a moving camera can *self-calibrate*, i.e. the calibration parameters can be estimated solely from feature correspondences between several images [16][1]. This allows the projective ambiguity in the reconstruction to be reduced to a Euclidean one (up to a similarity transformation), and we speak of *monocular uncalibrated Euclidean reconstruction*.

1.2 Critical Motion Sequences

It is known that some types of camera motion prevent self-calibration, i.e. the calibration parameters can not be determined uniquely. Accordingly, Euclidean structure can not be obtained, although reconstruction at some level between projective and Euclidean is generally possible. For example, from pure translations, only affine structure can be obtained [17], while general planar motions of the camera allow a Euclidean reconstruction up to a scale ambiguity in one direction [1].

These ambiguities are inherent in that they can not be resolved by any algorithm without additional knowledge. Sequences of camera motions that imply such ambiguities will be referred to as *critical motion sequences*. By "sequences" we mean that not only the motion between two frames, but that over the complete sequence of frames, is critical. Another type of inherent ambiguity in structure recovery is caused by *critical surfaces*: if all observed features lie on a special surface (certain ruled quadrics) and the cameras have a special position with respect to that surface, then the structure can not be recovered uniquely [12, 13, 15]. Contrary to critical surfaces, critical motion sequences lead to ambiguities for **any** scene!

[1] Approaches that assume special camera motions, are sometimes included in self-calibration (e.g. [1, 2, 4, 5, 10]). However, we prefer to call this *calibration from motion constraints*.

Critical motion sequences have already become established in practical works on self-calibration through the development of algorithms specially designed for certain types of critical motions [1, 10, 17]. However, if applied to other motion sequences, they will fail. Conversely, algorithms developed for general camera motion [9, 11, 14, 18, 21, 22] will, if applied to critical sequences, hopefully find one of the ambiguous solutions, but this will generally not be the correct one.

In a recent paper [20], we derive a complete characterization of critical motion sequences, which is independent of the number of frames and of the algorithm used. We have shown that the problem of critical motion sequences is important, since many image sequences used for object modeling are indeed critical.

In this paper, we reveal the existence of a "conjugacy" between the ambiguous solutions of Euclidean reconstruction from a critical image sequence. We discuss how, if only one ambiguous Euclidean reconstruction is known, this conjugacy may allow to determine **all** other possible Euclidean reconstructions of the scene. This is especially useful if the set of ambiguous Euclidean reconstructions is partitioned into disjoint solution families or even discrete solutions.

1.3 Basic Idea

Now we sketch the basic idea behind the derivation and further consideration of the critical motion sequences.

Euclidean reconstruction is equivalent to the determination of the absolute conic Ω [7] (see also 2.4). This can only be based upon the special properties which distinguish it from all other conics in 3-space. The main property, and usually the only one used in existing algorithms, is that the projection of Ω is invariant under camera motions provided the intrinsic parameters do not change. Its image ω can thus be determined as the "fixed conic of a sequence" [1]. Furthermore, Ω is a proper virtual conic (see 2.3), and for perfect perspective projection, its images must also be proper virtual conics. Besides these properties (or equivalent ones) there is no means to determine Ω from monocular uncalibrated image sequences.

Hence, the problem of monocular uncalibrated Euclidean reconstruction fails to have a unique solution exactly when there is at least one other conic besides Ω with the same properties, i.e. a proper virtual conic Ω' that is projected onto some proper virtual conic ω' in all frames of the sequence.

1.4 Structure of the Paper

In section 2 we provide the theoretical background of this paper. Basic definitions are settled in section 3. In section 4 we discuss the partitioning of the ambiguous Euclidean reconstructions. The conjugacy between ambiguous Euclidean reconstructions is derived in section 5 and proved in Appendix A. Conclusions are drawn in section 6.

2 Background

The definitions in this section are mainly taken from [3] and [19]. Some of the results for general quadrics are presented only for central conics.

2.1 Notation

We refer to the *plane at infinity* as the *ideal plane* and denote it by Π_∞. $\mathcal{P}^n$ is the n-dimensional projective space and $\sim$ means equality up to a scalar factor. We use the abbreviation *PVC* for proper virtual conics (see 2.3).

2.2 Pinhole Camera Model

We use the projective pinhole camera model where a camera is represented by a *projection center* $\mathbf{O}$ and a *retinal plane*. The projection $\mathbf{q}$ of a 3D point $\mathbf{Q}$ is the intersection of the line $\langle\mathbf{O},\mathbf{Q}\rangle$, with the retinal plane. This projection can be represented by a 3×4 *projection matrix* $\mathbf{P}$ such that $\mathbf{q}\sim\mathbf{PQ}$, where $\mathbf{Q}$ and $\mathbf{q}$ are represented by homogeneous coordinates. The *optical axis* is the line through the projection center perpendicular to the retinal plane. We consider only the case of perfect perspective projection, i.e. the projection center does not lie on Π_∞.

With regard to physical cameras, the projection matrix can be decomposed into a *calibration matrix* $\mathbf{A}$ and a *pose matrix* $\mathbf{T}$. The pose matrix represents the position and orientation of the camera in some absolute coordinate frame. The calibration matrix describes the invertible affine transformation from the *canonical projection* to pixel coordinates. For the pinhole model, the calibration matrix is determined by 5 *intrinsic parameters*: the focal length, measured in horizontal and vertical pixels, the two coordinates of the principal point, and the skew angle between the pixel axes.

2.3 Quadrics and Conics

A *quadric* in $\mathcal{P}^n$ is a set of points satisfying a homogeneous quadratic equation. Each quadric can be represented by a symmetric $(n+1)\times(n+1)$ matrix.

A *virtual quadric* is a quadric with no real point and a *proper quadric* is a quadric whose matrix has a non-zero determinant. *Conics* are planar quadrics; we will not distinguish between a conic and its matrix. A conic in $\mathcal{P}^3$, or *3D conic*, is defined by its *supporting plane* and the conic's equation in that plane.

2.4 Absolute Quadric and Absolute Conic

The *absolute quadric* of $\mathcal{P}^n$ is defined by the equations $x_1^2+\cdots+x_n^2 = x_{n+1} = 0$. The *absolute conic* Ω is the absolute quadric of $\mathcal{P}^3$ and the absolute quadric of the projective plane $\mathcal{P}^2$ consists of two conjugate virtual points known as the *circular points*.

The absolute quadric of $\mathcal{P}^n$ is a virtual quadric in the ideal hyperplane whose position uniquely defines the Euclidean structure of the considered space, e.g. knowing the absolute conic is equivalent to knowing the Euclidean structure of 3-space.

The calibration of a camera is equivalent to determining the image ω of Ω, respectively its dual ω^* [16, 7]. From the relation $\omega^* \sim \mathbf{AA^T}$, the calibration matrix A can be uniquely recovered by Cholesky decomposition.

3 Basic Definitions

We define a *motion sequence* S of m camera positions as $S = (\mathbf{R_i}, \mathbf{t_i})_{\mathbf{i=1}}^{\mathbf{m}}$, where $(\mathbf{R_i}, \mathbf{t_i})$ are the rotational and translational components of the ith camera pose. Note that any two frames in a motion sequence are relied by a rigid transformation (rotation + translation). In the following, we sometimes talk of rigid motion sequences, in order to distinguish them from 'projective motion', i.e. the set of projection matrices of a projective reconstruction.

We note that the question of whether a given motion sequence is critical is independent of the cameras intrinsic parameters: a conic has the same image in a set of views taken by the same camera, exactly if it has the same image in the corresponding canonical projections[2]. It is thus sufficient to consider only the pose of the camera.

Definition 1. Let S be a motion sequence and $\mathbf{P_i}$ the canonical projection for the ith frame. Let $\mathbf{P_i}(\Phi)$ be the image of the 3D conic Φ.

The motion sequence S is **critical** if there exists a proper virtual conic Φ, *distinct from* Ω, that projects to the same proper virtual conic ϕ in all frames of S: $\phi \sim \mathbf{P_i}(\Phi)$ for $i = 1, \ldots, m$.

Such PVC Φ will be referred to as **potential absolute conics** and we say that the motion sequence S is **critical with respect to** Φ. ■

From the reflections in section 1.3 it follows that Euclidean reconstruction from an uncalibrated monocular image sequence is ambiguous exactly when the underlying camera motion is a critical motion sequence.

On the basis of Definition 1, we derived all types of critical motion sequences [20]. In the same paper, we also discuss motion sequences which are critical for affine reconstruction or self-calibration.

4 Potential Absolute Conics

We briefly discuss, how the set of potential absolute conics for a critical motion sequence can look like. There may exist potential absolute conics only on the ideal plane, or exclusively not on the ideal plane. The potential absolute conics may form distinct linear families and there may even exist isolated ones. A linear family of potential absolute conics may contain both conics on and not on the ideal plane. In Figure 1, two examples are illustrated.

Consider now an arbitrary self-calibration algorithm which tries to recover Euclidean structure and motion. Without additional knowledge, e.g. on the scene or the calibration parameters, the ambiguity due to a critical motion sequence can not be resolved by any means. We may hope however, that the algorithm finds at least one of the ambiguous

[2] There is an invertible affine transformation between the canonical projection and the image plane (2.2).

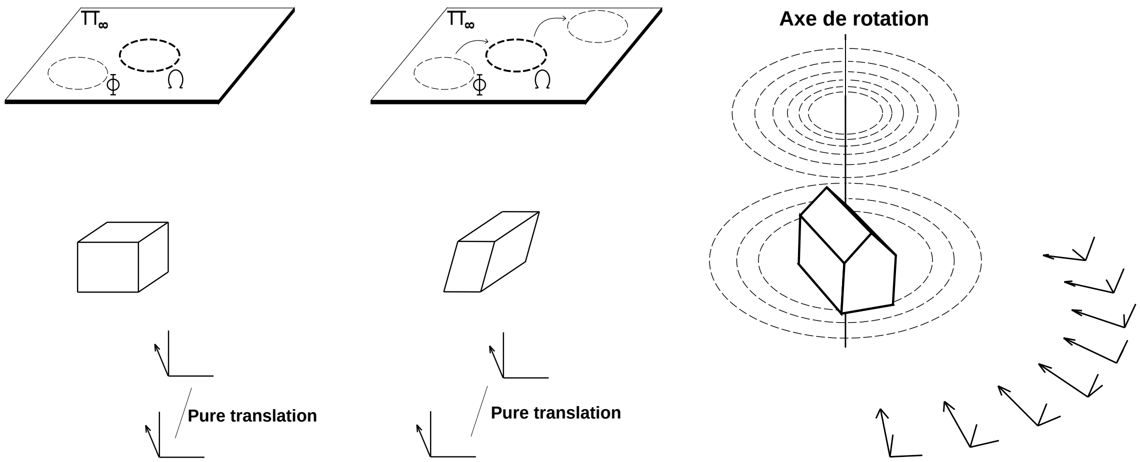

Figure 1. Examples of potential absolute conics. (a) *The figure on the left shows a cube, observed by a translating camera. Due to the translational motion, all conics on Π_∞ are potential absolute conics.* **(b)** *In the middle, the effect of choosing a 'wrong' potential absolute conics is shown: Φ is transformed to Ω. This is done by an affine transformation, which affects the reconstructed scene, i.e. the cube becomes a parallelepiped. However, the transformed projection matrices are still relied by pure translations.* **(c)** *Some potential absolute conics for an orbital motion. The shown motion is similar to the one of the image sequence in Figure 2.*

solutions, i.e. detects one of the potential absolute conics. A first issue a general self-calibration algorithm should provide is to recognize if the solution is ambiguous or not. Second, the other potential solutions should be identified if possible. One way to do so would be the analysis of correlations of the unknown parameters, after their estimation. Highly correlated parameters may indicate that the solution found is not unique, but member of a whole family of ambiguous solutions. Such an approach fails however, if the solution is either discrete (an isolated potential absolute conic), or if distinct solution families exist.

In the following we discuss a way, how this may be overcome. We prove the existence of conjugacies between ambiguous solutions of the Euclidean reconstruction problem and show how these might allow to identify all ambiguous solutions, knowing only one of them.

5 Conjugacy of Ambiguous Euclidean Reconstructions

In this section, we investigate on links that exist between the ambiguous Euclidean reconstructions from a critical motion sequence.

Let S be a motion sequence critical with respect to a set $\mathcal{A}$ of potential absolute conics. Let $\mathbf{P_i^E}$ be the canonical projection matrices of the frames in S (*cf.* Definition 1). We suppose that a projective reconstruction of the scene is already obtained[3], i.e. we dispose of projection matrices $\mathbf{P_i^P}$ which differ from the 'true' ones by an unknown projective transformation $\mathbf{T}$: $\mathbf{P_i^P} = \mathbf{P_i^E}\mathbf{T^{-1}}$. Analogously, 3D scene components are reconstructed up to the same transformation, for example a 3D point is reconstructed as $\mathbf{Q_j^P} = \mathbf{T}\mathbf{Q_i^E}$. The projective reconstruction can be upgraded to Euclidean, if the absolute conic can be identified. Let $\Omega^\mathbf{P}$ be the absolute conic in the projective reconstruction. To turn the projective reconstruction into Euclidean, a projective transformation $\mathbf{T^P}$ must be determined which maps $\Omega^\mathbf{P}$ on the canonical form Ω of the absolute conic (see 2.4). $\mathbf{T^P}$ is not uniquely defined since even after adding any Euclidean transformation Ω is still the image of $\Omega^\mathbf{P}$. Applying $\mathbf{T^P}$ on the projection matrices and 3D scene features results in a Euclidean reconstruction.

The question is now, what happens if $\Omega^\mathbf{P}$ is **not** the correct choice of absolute conic, but one of the potential absolute conics in $\mathcal{A}$? We still can determine transformations $\mathbf{T'}$ mapping $\Omega^\mathbf{P}$ on Ω[4], but transforming the projective reconstruction by $\mathbf{T'}$ will no longer result in a Euclidean reconstruction (see Figure 1 (b)). That means, that no Euclidean transformation $\mathbf{T}$ exists with $\mathbf{P_i^P}\mathbf{T'^{-1}} = \mathbf{P_i^E}\mathbf{T^{-1}}$ for all i.

[3] Projective reconstruction is always possible unless all cameras are located at a same point.
[4] Any PVC can be transformed to any other PVC by a projective transformation.

However, we can show (Lemma 2 in Appendix A) that the motion between the transformed projection matrices $\mathbf{P}_i^{\mathbf{P}}\mathbf{T}'^{-1}$ is **Euclidean** (see the example in Figure 2), which is not the case for the $\mathbf{P}_i^{\mathbf{P}}$! At first sight, this might be surprising, however this was to be expected since a non-Euclidean motion would allow to discard Ω' from the set of potential absolute conics, i.e. there would exist other means of identifying the absolute conic than those mentioned in section 1.3.

What are the consequences of this observation ? The set S' of projection matrices $\mathbf{P}_i^{\mathbf{P}}\mathbf{T}'^{-1}$ can be considered as a rigid motion sequence (up to a uniform scaling), according to the definition given in section 3. Furthermore, S' is a critical motion sequence: S and S' are relied by a projective transformation ($\mathbf{T}$, followed by $\mathbf{T}'$) and thus the potential absolute conics of S are, after this projective transformation, potential absolute conics of S'.

In summary, ambiguous Euclidean reconstructions from a critical motion sequence are conjugated in the following way: the absolute conic of any ambiguous reconstruction is a potential absolute conic of any other ambiguous reconstruction.

Thus, identifying the potential absolute conics of S' is equivalent to identifying those of S and thus to determining all ambiguous Euclidean reconstructions from the sequence S ! In the following subsection we discuss how to determine the potential absolute conics of a rigid motion sequence.

Figure 2. Example of a critical image sequence. The images are obtained while rotating about the same axis. (a)-(c) *Three images of the 6-image sequence. A self-calibration was done from point correspondences of the images [21]. As expected, the result is not a Euclidean structure of the scene, since for example the estimated aspect ratio is about 2, while the true one is about 1.5. However, the recovered motion sequence is, as the original one, an orbital motion, i.e. it consists of rotations about a fixed axis. This is illustrated in* **(d)** *where a top view of the recovered projection centers shows that they lie very close to a circle.*

5.1 Determining the Potential Absolute Conics of a Rigid Motion Sequence

The potential absolute conics of a sequence of known rigid motions can be found by inspection of the rotational and translational components of the motions. This is relatively straightforward, due to the derivation of critical motion sequences reported in [20]. The membership of the rigid sequence to a certain class of critical motion sequences can be checked by hypothesis verification and then the potential absolute conics can be determined directly or as the solutions of elementary geometric problems. This process is quite easy if more than 4 camera positions are given, but might still be a delicate mathematical problem with less views.

With more than 4 camera positions, the supporting planes of potential absolute conics are either finitely many (3) or form a pencil of parallel planes. In both cases, the determination of the supporting planes is straightforward, since they must be equidistant to the projection centers of all cameras [20]. The centers and axes of potential absolute conics are also easily obtained. Two parameters are remaining which are the lengths of the conics axes. These are determined through the relative rotations between the frames. Due to lack of space, we can not give more details here.

6 Conclusion

In this paper, we have discussed the problem of ambiguous solutions of the Euclidean reconstruction problem for uncalibrated image sequences. The major result is that the ambiguous solutions are not fully Euclidean reconstructions, but that the recovered inter-frame motion is Euclidean. We have shown how this might be used to determine, from only one of the ambiguous solutions, all of them. This allows to detect if the reconstruction problem is ambiguous at all and if so, to determine the degree and type of ambiguity. Providing such information is essential for general purpose self-calibration systems, because not labeling a wrong, since ambiguous solution, as such, will in general devalue any subsequent processing.

Work in progress is mainly concerned with the investigation of approaches for the implementation of a "stratified" reconstruction system, i.e. a system which automatically detects ambiguities and provides the correct geometric level of reconstruction, e.g. affine reconstruction when the camera is translating.

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A Inter-Frame Motions in Ambiguous Reconstructions

Lemma 2. *Let S be a motion sequence critical with respect to a 3D conic Φ, and $\mathbf{P_i^E}$ be the canonical projection matrices of the frames in S. Let $\mathbf{T}$ be any projective transformation mapping Φ to Ω and $\mathbf{P_i^P} = \mathbf{P_i^E}\mathbf{T^{-1}}$ be the by $\mathbf{T}$ transformed projection matrices.*
There exists a Euclidean transformation between any pair of $\mathbf{P_i^P}$.

Proof. Let the canonical projection matrices $\mathbf{P_i^E}$ be given as $\mathbf{P_i^E} = (\ \mathbf{R_i} \ \ | -\mathbf{R_i t_i})$. We distinguish two cases for Φ: Φ lies on Π_∞ or it does not.

Case 1: Φ lies on Π_∞.

Let $\mathbf{C}$ be Φ's matrix in the ideal plane. From S being a critical sequence with respect to Φ it follows that for any i, j we have $\mathbf{P_i^E}(\Phi) \sim \mathbf{P_j^E}(\Phi)$, i.e.

$$\mathbf{R_i C R_i^T} \sim \mathbf{R_j C R_j^T}$$

from which follows that

$$\mathbf{R_i C^{-1} R_i^T} \sim \mathbf{R_j C^{-1} R_j^T} \quad . \tag{1}$$

Let $\mathbf{T}$ be a point transformation which maps Φ on Ω. Since both Φ and Ω lie on the ideal plane, $\mathbf{T}$ must be an affine transformation and is thus of the form

$$\mathbf{T} \sim \begin{pmatrix} \mathbf{A} & \mathbf{v} \\ \mathbf{0^T} & 1 \end{pmatrix} \quad .$$

The restriction of $\mathbf{T}$ on ideal points is expressed by the submatrix $\mathbf{A}$. Since $\mathbf{T}$ maps Φ on Ω the following constraint holds for $\mathbf{A}$: $\mathbf{A^{-T} C A^{-1}} \sim \mathbf{I_3}$. From this it follows that

$$\mathbf{A^{-1} A^{-T}} \sim \mathbf{C^{-1}} \tag{2}$$

and

$$\mathbf{A C^{-1} A^T} \sim \mathbf{I_3} \quad . \tag{3}$$

Now we consider the transformed projection matrices $\mathbf{P_i^P}$. The inverse of $\mathbf{T}$ is given by

$$\mathbf{T^{-1}} \sim \begin{pmatrix} \mathbf{A^{-1}} & -\mathbf{A^{-1} v} \\ \mathbf{0^T} & 1 \end{pmatrix}$$

and we obtain the transformed projection matrices as

$$\mathbf{P_i^P} = \mathbf{P_i^E T^{-1}} \sim \begin{pmatrix} \mathbf{R_i A^{-1}} & -\mathbf{R_i}(\mathbf{A^{-1} v} + \mathbf{t_i}) \end{pmatrix} \quad .$$

The motion between any two $\mathbf{P_i^P}$ and $\mathbf{P_j^P}$ is Euclidean exactly if there exist a rotation matrix $\mathbf{R_{ij}}$ and a 3-vector $\mathbf{t}$ such that

$$\mathbf{P_i^P} \sim \mathbf{P_j^P} \begin{pmatrix} \mathbf{R_{ij}} & \mathbf{t} \\ \mathbf{0^T} & 1 \end{pmatrix} \quad .$$

The first 3 columns of this matrix equation are:

$$\mathbf{R_i A^{-1}} \sim \mathbf{R_j A^{-1} R_{ij}} \quad .$$

Thus, $\mathbf{R_{ij}}$ exists exactly if $\mathbf{M} = \mathbf{A R_j^T R_i A^{-1}}$ is orthogonal. It is easy to see that, if $\mathbf{R_{ij}}$ exists then we also always find an appropriate $\mathbf{t}$.

$\mathbf{M}$ is orthogonal exactly if $\mathbf{M M^T} \sim \mathbf{I_3}$. We proove this in the following:

$$\begin{aligned} \mathbf{M M^T} &= \mathbf{A R_j^T R_i A^{-1} A^{-T} R_i^T R_j A^T} \\ &\overset{(2)}{\sim} \mathbf{A R_j^T R_i C^{-1} R_i^T R_j A^T} \\ &\overset{(1)}{\sim} \mathbf{A R_j^T R_j C^{-1} R_j^T R_j A^T} \\ &= \mathbf{A C^{-1} A^T} \\ &\overset{(3)}{\sim} \mathbf{I_3} \end{aligned}$$

Case 2: Φ does not lies on Π_∞. This case is slightly more complicated than Case 1, but can be treated analogously. Without loss of generality let the supporting plane of Φ be the plane $\Pi \equiv (\mathbf{z} = \mathbf{0})$ and let $\mathbf{C}$ be Φ's matrix in Π. The restriction of the projection $\mathbf{P_i^E}$ on points of Π is the transformation $\mathbf{R_i M_i}$ with

$$\mathbf{M_i} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & -\mathbf{t_i} \\ 0 & 0 & \end{pmatrix} \quad .$$

From S being a critical sequence with respect to Φ it follows that for any i, j we have $\mathbf{P_i^E}(\Phi) \sim \mathbf{P_j^E}(\Phi)$, i.e.

$$\mathbf{R_i M_i^{-T} C M_i^{-1} R_i^T} \sim \mathbf{R_j M_j^{-T} C M_j^{-1} R_j^T}$$

from which follows that

$$\mathbf{R_i M_i C^{-1} M_i^T R_i^T} \sim \mathbf{R_j M_j C^{-1} M_j^T R_j^T} \quad . \tag{4}$$

Let $\mathbf{T}$ be a point transformation which maps Φ on Ω. $\mathbf{T}$ maps Π on the ideal plane. Thus, we have the following equation for the dual transformation of $\mathbf{T}$:

$$\mathbf{T^{-T}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

from which follows that $\mathbf{T}$ is of the form

$$\mathbf{T} \sim \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad .$$

Let $\mathbf{A}$ be the matrix of the first, second and fourth 3-subcolumns of $\mathbf{T}$ and $\mathbf{v}$ be the third 3-subcolumn, i.e.:

$$\mathbf{T} \sim \begin{pmatrix} \mathbf{A_{\bullet 1}} & \mathbf{A_{\bullet 2}} & \mathbf{v} & \mathbf{A_{\bullet 3}} \\ & & & \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The restriction of $\mathbf{T}$ on points on Π is expressed by the submatrix $\mathbf{A}$. Since $\mathbf{T}$ maps Φ on Ω, Equations (2) and (3) from Case 1 hold also here.
Now we consider the transformed projection matrices $\mathbf{P_i^P}$. The inverse of $\mathbf{T}$ is given by

$$\mathbf{T^{-1}} \sim \begin{pmatrix} & \mathbf{B_{1\bullet}} & & \mathbf{s_1} \\ & \mathbf{B_{2\bullet}} & & \mathbf{s_2} \\ 0 & 0 & 0 & 1 \\ & \mathbf{B_{3\bullet}} & & \mathbf{s_3} \end{pmatrix}$$

where $\mathbf{B} = \mathbf{A^{-1}}$ and $\mathbf{s} = -\mathbf{A^{-1}v}$. The transformed projection matrices are (the fourth column is not of interest in the following)

$$\mathbf{P_i^P} = \mathbf{P_i^E T^{-1}} \sim \begin{pmatrix} & \times \\ \mathbf{R_i M_i A^{-1}} & \times \\ & \times \end{pmatrix} \quad .$$

The motion between any two $\mathbf{P_i^P}$ and $\mathbf{P_j^P}$ is Euclidean exactly if there exist a rotation matrix $\mathbf{R_{ij}}$ and a 3-vector $\mathbf{t}$ such that

$$\mathbf{P_i^P} \sim \mathbf{P_j^P} \begin{pmatrix} \mathbf{R_{ij}} & \mathbf{t} \\ & \\ \mathbf{0^T} & 1 \end{pmatrix} \quad .$$

The first 3 columns of this matrix equation are:

$$\mathbf{R_i M_i A^{-1}} \sim \mathbf{R_j M_j A^{-1} R_{ij}} \ .$$

Thus, $\mathbf{R_{ij}}$ exists exactly if $\mathbf{M} = \mathbf{A M_j^{-1} R_j^T R_i M_i A^{-1}}$ is orthogonal. It is easy to see that, if $\mathbf{R_{ij}}$ exists then we also always find an appropriate $\mathbf{t}$.
We proove now the orthogonality of $\mathbf{M}$:

$$\begin{aligned}
\mathbf{M M^T} &= \mathbf{A M_j^{-1} R_j^T R_i M_i A^{-1} A^{-T} M_i^T R_i^T R_j M_j^{-T} A^T} \\
&\overset{(2)}{\sim} \mathbf{A M_j^{-1} R_j^T R_i M_i C^{-1} M_i^T R_i^T R_j M_j^{-T} A^T} \\
&\overset{(4)}{\sim} \mathbf{A M_j^{-1} R_j^T R_j M_j C^{-1} M_j^T R_j^T R_j M_j^{-T} A^T} \\
&= \mathbf{A C^{-1} A^T} \\
&\overset{(3)}{\sim} \mathbf{I_3}
\end{aligned}$$

□

The statement of Lemma 2 can be inversed, as we show in the following Lemma.

Lemma 3. *Let S be a motion sequence and $\mathbf{P_i^E}$ be the canonical projection matrices of the frames in S. Let $\mathbf{P_i^P}$ be the projection matrices of a projective reconstruction of S, i.e. $\mathbf{P_i^P} \sim \mathbf{P_i^E T^{-1}}$ for all i and a non singular projective transformation $\mathbf{T}$.*
The motion between any two $\mathbf{P_i^P}$ is Euclidean exactly if the “absolute conic” of the projective reconstruction is a by $\mathbf{T}$ transformed potential absolute conic of the original motion sequence S.

Proof. One direction of this equivalence is already given by Lemma 2. We now prove the other direction.
... □