1.1 Approximating Area

1 *The Area Problem*

We first begin by trying to solve the area problem, that is, find the area beneath a curve of some function that we will call $f(x)$.

Let $f(x)$ be a continuous and non-negative function over the closed interval $[a, b]$.

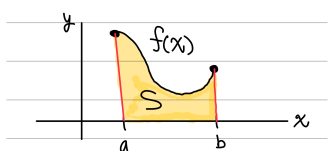

Figure 1: We want to approximate the area of region S

Before finding the exact area of region S, we first approximate the area by dividing the region into many small rectangles, since the area of rectangle is easy to calculate; base $\times$ height.

To do this, we begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\dfrac{b-a}{n}$. We then select equally spaced points, $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a$ and $x_n = b$ and $x_i - x_{i-1} = \dfrac{b-a}{n}$ for $i = 1, 2, 3, \ldots, n$.

We denote each width as Δx, so $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$.

Definition: A set of points $P = \{x_i\}$ for $i = 0, 1, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a partition a partition of $[a, b]$. If all the subintervals have the same width, then the set of points form a regular partition of $[a, b]$.

Rule: Left Endpoint Approximation

On each subinterval, $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval.

So, the area of one rectangle under the curve using the left endpoint is $f(x_{i-1}) \cdot \Delta x$. So, if we let L_n represent the left endpoint approximation, $A \approx L_n = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \ldots f(x_{n-1}) \cdot \Delta x$ or more conveniently, $A \approx L_n = \sum\limits_{i=1}^{n} f(x_{i-1}) \Delta x$.

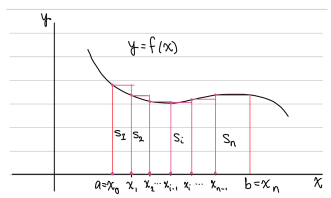

Figure 2: Left Endpoint Approximation

Rule: Right Endpoint Approximation

Similarly to the approximation of the left endpoint, on each subinterval $[x_{i-1}, x_i$ for $i = 1, 2, \ldots, n$, construct a rectangle of width Δx and height equal to $f(x_i)$. Let's define R_n for right endpoint of n subintervals. And so $A \approx R_n = \sum\limits_{i=1}^{n} f(x_i) \cdot \Delta x$.

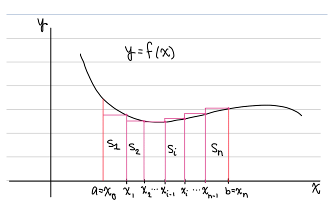

Figure 3: Right Endpoint Approximation

Forming Riemann Sums

So far we have been constructing rectangles to approximate the area under a curve $y = f(x)$ on the interval $[a, b]$ via the left or right endpoint. In reality, there is no need to restrict evaluating the function at either the left or right endpoint. We can take any sample point, we will call x_i^*, such that $x_i^* \in [x_{i-1}, x_i]$. Now the height of the rectangle is $f(x_i^*)$. So we can now approximate the area under a curve at any point which is of the form, $A \approx \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$.

Definition: Let $f(x)$ be defined on the closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval, $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann Sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x.$$

Now this begs the question, How do we determine the exact area under a curve? Well, as we construct more rectangles, the approximation of the area gets better, so why don't we let n approach ∞?

If we construct infinitely many rectangles, the approximation of the area will keep getting better. Therefore, we return to limits to help us out.

Definition:
Let $f(x)$ be a non-negative and continuous function defined on the closed interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann Sum for $f(x)$. Therefore, we define the area under the curve, $y = f(x)$ on $[a, b]$ to be,

$$A = \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i^*)\Delta x \right].$$

Upper and Lower Sums

If we want an overestimate of the area under the curve, we can choose some $\{x_i^*\}$ such that for $i = 1, \ldots, n$, $f(x_i^* \geq f(x))$ for all $x \in [x_{i-1}, x_i$. In this way, the Riemann sum is called an upper sum.

Similarly, if we want an underestimate of the area under the curve, we can choose some $\{x_i^*\}$ such that for $i = 1, \ldots, n$, $f(x_i^*) \leq f(x)$ for all $x \in [x_{i-1}, x_i]$. This way, the Riemann sum is called a lower sum.

Extra Notes:

- When choosing x_i^* for approximating area with right endpoints, choose $x_i^* = x_i = a + i \cdot \Delta x$
- When choosing x_i^* for approximating area with left endpoints, choose $x_i^* = x_{i-1} = a + (i-1) \cdot \Delta x$
- If f is increasing/decreasing over $[a, b]$, right endpoints overestimate/underestimate
  - the function value at the right endpoint is larger than the function value at the left endpoint for each subinterval. This causes the Riemann Sum to overestimate
  - the function value at the right endpoint is smaller than theh function value at the left endpoint for each subinterval. This causes the Riemann sum to underestimate
- If f is increasing/decreasing over $[a, b]$, left endpoints underestimate/overestimate
  - the function value at the left endpoint is smaller than the function value at the right endpoint for each subinterval. This causes the Riemann sum to underestimate
  - the function value at the left endpoint is larger than the function value at the right endpoint for each subinterval. This causes the Riemann sum to overestimate

1.1 *Practice Questions*

Let L_n denote the left-endpoint sum using nsubintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

1. R_8 for $x^2 - 2x + 1$ on $[0, 2]$
2. L_8 for $x^2 - 2x + 1$ on $[0, 2]$
3. Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations of A with subintervals n using left and right endpoints, respectively.

(a) How are A, L_n, R_n related?

(b) Show that:

$$R_n - L_n = \frac{b-a}{n}\left[f(b) - f(a)\right]$$

4. Find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

   $f(x) = \dfrac{2x}{x^2+1}$, $1 \leq x \leq 3$

1.2 *Solutions to Practice Questions*

1. Solution.

 Define $f(x) = x^2 - 2x + 1 = (x-1)^2$ on the closed interval $[a, b] = [0, 2]$. Now the width of each rectangle is going to be $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$. Since we are looking for $R_8, n = 8$ and so $\Delta x = \dfrac{1}{4}$. Now we have the following subintervals,

 $[0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1], [1, 1.25], [1.5, 1.75], [1.75, 2]$.

 $= [x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_5, x_6], [x_7, x_8]$.

 Forming the Riemann Sum,

 $$R_8 = \sum_{i=1}^{8} \left[f(x_i) \cdot \frac{1}{4} \right]$$
 $$R_8 = \frac{1}{4} \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8) \right)$$
 $$R_8 = \frac{1}{4} \left(\frac{20}{16} + \frac{1}{2} + 1 \right)$$
 $$R_8 = \frac{5}{16} + \frac{1}{8} + \frac{1}{4} \text{ after distributing } \frac{1}{4} \text{ to each term}$$
 $$R_8 = \frac{9}{16} \text{ as needed.}$$

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2. Similar Solution to question 1. but with left endpoints.

 Answer: $\dfrac{11}{16}$

3.(a) Solution.

 Since f is an increasing and continuous function over the closed interval $[a, b]$, we can conclude that right endpoint approximation will result in an over estimate while a left endpoint approximation will result in an under estimate of the area. Thus, we have the following relationship, $L_n \leq A \leq R_n$.

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3.(b) *Proof.* Suppose f is an increasing and continuous function over the closed interval $[a, b]$. We want to show that $R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$. So let's recall the definition of right and left endpoint approximations.

$R_n = \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

$L_n = \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

Now let's consider $R_n - L_n$.

$\implies R_n - L_n = \sum\limits_{i=1}^{n} f(x_i^*)\Delta x - \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

$\implies R_n - L_n = \Delta x \left[\sum\limits_{i=1}^{n} f(x_i) - \sum\limits_{i=1}^{n} f(x_{i-1})\right]$ since Δx is just a constant and $x_i^* = x_i$, $x_i^* = x_{i-1}$ for right endpoints and left endpoints respectively.

$\implies R_n - L_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n) - (f(x_0) + f(x_1) + \ldots f(x_{n-1})]$ after expanding the summations.

$\implies R_n - L_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n) - f(x_0) - f(x_1) - \cdots - f(x_{n-1})]$

$\implies R_n - L_n = \frac{b-a}{n}[f(x_n) - f(x_0)]$ after collecting like terms

Now recall that $x_0 = a$ and $x_n = b$.

Therefore we have that $R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$ as required. $\square$

4. Solution.

Using the limit definition of the area under the curve of f, $A = \lim\limits_{n\to\infty}\left[\sum\limits_{i=1}^{n} f(x_i^*)\Delta x\right]$.
Now we just need to find Δx, x_i^*, and $f(x_i^*)$.

$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

$x_i^* = x_i = 1 + i \cdot \Delta x = 1 + \frac{2i}{n} = \frac{n+2i}{n}$ (more common to use right endpoint than left endpoint so computation is easier)

$$f(x_i^*) = \frac{2\left(\frac{n+2i}{n}\right)}{\left(\frac{n+2i}{n}\right)^2 + 1}$$

Therefore, $A = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{2\left(\dfrac{n+2i}{n}\right)}{\left(\dfrac{n+2i}{n}\right)^2 + 1} \cdot \frac{2}{n} \right]$ as needed.

$\blacksquare$