1.3 The Fundamental Theorem of Calculus

1 The Mean Value Theorem for Integrals

The Mean Value Theorem for integrals state that for a continuous function on a closed interval let's say $[a, b]$, we know that we can find some point, $c \in [a, b]$ such that $f(c)$ is equivalent to the average value of the function over $[a, b]$.

Recall that the average value of a continuous function f over the closed interval $[a, b]$ is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

Theorem 1.3

If $f(x)$ is continuous over the closed interval $[a, b]$, the there exists at least one point $c \in [a, b]$ such that,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

1.1 Mean Value Theorem for Integrals Practice

1. Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

2. Given $\int_0^3 \left(2x^2 - 1\right)\, dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

2 Fundamental Theorem of Calculus Part 1

Theorem 1.4
If $f(x)$ is continuous over the closed interval $[a, b]$, and the function $F(x)$ is defined by
$$F(x) = \int_a^x f(t)\, dt,$$
then $F'(x) = f(x)$ over $[a, b]$.

- This theorem establishes the fact that any integrable function has an antiderivative
- This theorem also implies that differentiation and integration are inverse operations

Now think about what happens if the bounds on the integral were functions, let's say $g(x)$ and $h(x)$.
If f is a continuous function with antiderivative F, and g and h are differentiable functions, then
$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(s)\, ds = \frac{d}{dx} [F(h(x)) - F(g(x))].$$
Now observe that when we take the derivative of $F(h(x)) - F(g(x))$, we need to apply the chain rule. And so,
$\frac{d}{dx} \int_{g(x)}^{h(x)} f(s)\, ds = F'(h(x))h'(x) - F'(g(x))g'(x)$
$\implies \frac{d}{dx} \int_{g(x)}^{h(x)} f(s)\, ds = f(h(x))h'(x) - f(g(x))g'(x).$

2.1 Fundamental Theorem of Calculus Part 1 Practice

1. Use the Fundamental Theorem of Calculus Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2 + 4}\, dx$.
2. Find $\frac{d}{dx} \int_1^{x^4} \sec(t)\, dt$

3 Fundamental Theorem of Calculus Part 2

Theorem 1.5
If $f(x)$ is continuous over $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$
Alternatively, $\int_a^b f(x)\,dx = F(x)\big|_a^b$ is a common notation to denote the expression $F(b) - F(a)$.

- Part 2 is extremely useful because it gives us an easier way to evaluate integrals rather than using Riemann Sums

3.1 Fundamental Theorem of Calculus Part 2 Practice

1. What is wrong with the following calculation?
   $\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg|_{-1}^{3} = \frac{-1}{3} - 1 = \frac{-4}{3}$

2. Find the derivative of the function.
   $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$

4 Solutions to Practice Questions

1.1.1 ***Solution:***

To find the average value, recall that $f_{ave} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$ where f is continuous over the interval $[a, b]$. So since $f(x) = \dfrac{x}{2}$ is indeed continuous over $[0, 6]$ we can find the average value.

$\implies f_{ave} = \dfrac{1}{6}\int_0^6 \dfrac{x}{2}\,dx$

$\implies f_{ave} = \dfrac{1}{12}\int_0^6 x\,dx$

$\implies f_{ave} = \dfrac{1}{12} \cdot \left.\dfrac{x^2}{2}\right|_0^6$

$\implies f_{ave} = \dfrac{18}{12} = \dfrac{3}{2} = 1.5.$

Now we must find some $c \in [0, 6]$ such that $f(c) = f_{ave}$.

So, $f(c) = \dfrac{c}{2} = \dfrac{3}{2}$. By some simple algebra, we see that $c = 3 \in [0, 6]$ as needed.

$\blacksquare$

1.1.2 We are given that $\int_0^3 \left(2x^2 - 1\right)\, dx = 15$.

$\implies f_{ave} = \dfrac{15}{3-0} = 5$. Now we must find some $c \in [0, 3]$ such that $f(c) = f_{ave}$. So $f(c) = 2c^2 - 1 = 5$.

$\implies c = \pm\sqrt{3}$. But since $-\sqrt{3} \notin [0, 3], c$ must then be $c = \sqrt{3}$ as needed.

$\blacksquare$

2.1.1 ***Solution:***

By the Fundamental Theorem of Calculus Part 1, $g'(r) = \sqrt{r^2 + 4}$.

$\blacksquare$

2.1.2 ***Solution:***

Let $F(x) = \int_1^{x^4} \sec(t)\, dt$. We want to find $F'(x)$. But notice that we can't directly apply part 1 of the fundamental theorem of calculus. So let $u(x)$ be some function such that $u(x) = x^4$. Now we have the following integral,

$F(x) = \int_1^{u(x)} \sec(t)\, dt$. Now applying the fundamental theorem of calculus part 1, $F'(x) = \sec(u(x))\,\dfrac{du}{dx}$.

$\implies F'(x) = 4x^3 \sec\left(x^4\right)$ as needed.

$\blacksquare$

3.1.1 ***Solution:***

Notice that on the closed interval $[-1,3], f(x) = \dfrac{1}{x^2}$ is not continuous since the function has an infinite discontinuity when $x = 0 \in [-1,3]$. Thus, the fundamental theorem of calculus cannot be applied.

$\blacksquare$

3.1.2 ***Solution:***

We can rewrite g as the following, $g(x) = \int_{2x}^{0} \dfrac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \dfrac{u^2-1}{u^2+1}\,dx$.

$\implies g(x) = -\int_{0}^{2x} \dfrac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \dfrac{u^2-1}{u^2+1}\,du$

$\implies g(x) = \int_{0}^{3x} \dfrac{u^2-1}{u^2+1}\,du - \int_{0}^{2x} \dfrac{u^2-1}{u^2+1}\,du$. Now let's let $t(x) = 2x$ and $s(x) = 3x$.

$\implies g(x) = \int_{0}^{s(x)} \dfrac{u^2-1}{u^2+1}\,du - \int_{0}^{t(x)} \dfrac{u^2-1}{u^2+1}\,du$.

Applying the fundamental theorem of calculus part 1,

$g'(x) = \dfrac{[s(x)]^2-1}{[s(x)]^2+1}\dfrac{ds}{dx} + \dfrac{[t(x)]^2-1}{[t(x)]^2+1}\dfrac{dt}{dx}$.

$\implies g'(x) = 3 \cdot \dfrac{9x^2-1}{9x^2+1} + 2 \cdot \dfrac{4x^2-1}{4x^2+1}$

$\blacksquare$