1.4 Integration Formulas and Net Change Theorem

1 Basic Integration Formulas

- Just like how derivatives have the power rule, integration also has a similar rule:
  Suppose we have the indefinite integral, $\int x^r \, dx$ where $r \in \mathbb{R} \setminus \{-1\}$. The antiderivative of $\int x^r \, dx$ is $\dfrac{x^{r+1}}{r+1} + C$.

  Some common Integrals:

- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$
- $\int \sin x \, dx = \cos x + C$
- $\int \cos x \, dx = -\sin x + C$

1.1 Indefinite Integrals

A function F is an antiderivative of the function f if $F'(x) = f(x)$.

So given a function f, the indefinite integral of f, denoted as $\int f(x) \, dx$ is the most general antiderivative f.

If F is an antiderivative of f, then $\int f(x) \, dx = F(x) + C$, where C is any constant. Due to the extra constant, the indefinite integral returns a family of functions while the definite integral, $\int_a^b f(x) \, dx$ returns an actual value which is the area under the curve on the interval $[a, b]$.

Example:

$\int x^2\,dx = \dfrac{x^3}{3} + C$ since $\dfrac{d}{dx}\left(\dfrac{x^3}{3} + C\right) = x^2$

$\int_0^1 x^2\,dx = \left[\dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{3}$ which represents the total area under the curve of $y = x^2$ on the interval $[0,1]$.

1.2 Practice Questions

1. Evaluate $\int \dfrac{\cos(x)}{\sin^2(x)}\,dx$
2. Evaluate $\int_0^3 \left(x^3 - 6x\right)\,dx$

2 The Net Change Theorem

Theorem 1.6
The new value of a changing quantity equals the initial value plus the integral of the rate of change:
$F(b) = F(a) + \int_a^b F'(x)\,dx$

or

$\int_a^b F'(x)\,dx = F(b) - F(a)$

2.1 Practice Questions

1. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

2. Recall that the marginal revenue function $R'(x)$ is the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)\,dx$ represent?

3. Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

2.2 Integrating Odd and Even Functions

Rule: Integrating Odd and Even Functions

Suppose f is a continuous function such that $f(x) = f(-x)$.
Then, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Suppose f is a continuous function such that $f(-x) = -f(x)$.
Then, $\int_{-a}^{a} f(x)\,dx = 0$.

3 Solutions

1.2.1 $\int \frac{\cos(x)}{\sin^2 x}\,dx = \int \frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}\,dx = \int \cot(x)\csc(x)\,dx = -\csc(x) + C$

1.2.2 $\int_0^3 \left(x^3 - 6x\right)\,dx = \left.\frac{x^4}{4} - 3x^2\right]_0^3 = \frac{81}{4} - 27 = \frac{-27}{4}$

2.2.1(a) Recall the Net Change Theorem: $\int_a^b F'(x)\,dx = F(b) - F(a)$. Also recall that $v(t) = s'(t)$, so $\int v(t)\,dt = \int s'(t)\,dt = s(t)$.

So to find the displacement of $v(t)$ where $t \in [1,4]$, we need to evaluate the following integral,
$\int_1^4 \left(t^2 - t - 6\right)\,dt$.

Thus, the displacement is, $\int_1^4 \left(t^2 - t - 6\right)\,dt = \left.\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = \frac{-9}{2} = -4.5$.

Therefore, the particle traveled 4.5m towards the left.

(b) To find the distance traveled, we need to evaluate the following integral, $\int_1^4 |t^2 - t - 6|\,dt = \int_1^4 |(t-3)(t+2)|\,dt$.

Now observe that $v(t) \leq 0$ on the interval $[1,3]$ and $v(t) \geq 0$ on $[3,4]$. Thus, the the distance traveled is,

$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt = -\int_1^3 v(t)\,dt + \int_3^4 v(t)\,dt = \frac{22}{3} + \frac{17}{6} = \frac{61}{6}$.

Therefore, the particle has traveled $\frac{61}{6}$m over the time interval $[1,4]$.

2.1.2 $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$ by the net change theorem. And so, $R(5000) - R(1000)$ calculates the total change in revenue as the number of units sold increases from $x = 1000$ to $x = 5000$.

2.1.3 Since $P(s)$ represents the perimeter of a square with side length s, $P(s) = 4s \implies P'(s) = 4$. So we can express the increase in the perimeter as the following integral,

$\int_2^4 P(s)\,ds = \int_2^4 4\,ds. \implies \int_2^4 4\,ds = (4s)\,|_2^4 = 4(4) - 4(2) = 16 - 8 = 8$.
Therefore, the increase in the perimeter as its side length increases from 2 units to 4 units is 8 units.