CS402: Computer Graphics

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Necessary Linear Algebra #1

1 Vectors

1.1 Vector Spaces

Real numbers can be added, and so can pairs of real numbers. If $\mathbb{R}^2$ is the set of all ordered pairs (α, β) of real numbers , then it is natural to define the sum of two elements in $\mathbb{R}^2$ as follows:

$$(\alpha, \beta) + (\gamma, \sigma) = (\alpha + \gamma, \beta + \sigma).$$

The result is that $\mathbb{R}^2$ becomes an abelian group. There is also scalar multiplication, in which we define multiplying a scalar (a real number) by a vector as follows:

$$\alpha(\beta, \gamma) = (\alpha\beta, \alpha\gamma).$$

The result of this multiplication is a structure consisting of three parts: an abelian group, namely $\mathbb{R}^2$, a field, namely $\mathbb{R}$ and way of multiplying the elements of the group by the elements of the field.

The vector space over a field F (of elements called scalars) is an additive (commutative) group V (of elements called vectors), together with an operation that assigns to each scalar α and each vector x a product αx, which is again a vector. The operation is called scalar multiplication, and it follows that:

- The vector distributive law

  $$(\alpha + \beta)x = \alpha x + \beta x$$

  where $\alpha \, and \, \beta$ are scalars and x is a vector.

- The scalar distributive law

  $$\alpha(x + y) = \alpha x + \alpha y$$

  where $x \, and \, y$ are vectors and α is a scalar.

- The scalar identity law

$$1x = x$$

for every vector x.

- Commutative law

$$x + y = y + x$$

where x and y are vectors.

- Associative law

$$z + (x + y) = (z + x) + y$$

where x,y and z are vectors.

- Additive Inverses

$$x + y = 0\, and\, y + x = 0$$

where $x = y^{-1}$, x and y are inverses.

2 Linear Combinations

For an example take the vector space over $\mathbb{R}^2$ of all ordered pairs of real numbers, such as

$$\begin{cases} (1, 1), \\ (0, \pi^2), \\ (\frac{1}{2}, \sqrt{2}), \\ (0, -200), \\ (\frac{1}{\sqrt{5}}, -\sqrt{5}) \end{cases} \tag{1}$$

An example, different from $\mathbb{R}^2$, in which we will throw away most of the pairs in $\mathbb{R}^2$, typical among the ones we'll keep are:

$$\begin{cases} (0, 0), \\ (-\frac{1}{2}, 1), \\ (\sqrt{5}, -2\sqrt{5}), \\ (\frac{1}{\sqrt{2}}, -\sqrt{2}) \end{cases} \tag{2}$$

Are these four enough to form a pattern? The pattern in the second vector space is that the y element is always -2 times the x element. Indeed; $0 \cdot (-2) = 0$, $-\frac{1}{2} \cdot (-2) = 1$ and so on.

The important aspect about vectors, is not what they look like, but what we can do with them, namely add them, and multiply them by scalars. More generally, if $x = (\alpha_1, \alpha_2)$ and $y = (\beta_1, \beta_2)$ are vectors in $\mathbb{R}^2$, and if γ and δ are real numbers, then it's possible to form

$$\gamma x + \delta y = (\gamma\alpha_1, \gamma\alpha_2) + (\delta\beta_1, \delta\beta_2) = (\gamma\alpha_1 + \delta\beta_1, \gamma\alpha_2 + \delta\beta_2)$$

which is a vector in the same space called a **linear combination** of the given vectors x and y.

An example would be:

$$3(4,0) - 2(0,5) = (12,-10)$$

In which, the vector (12,-10) is a linear combination of the two vectors (4,0) and (0,5).

Another example;

$$(7,\pi) = 7(1,0) + \pi(0,1)$$

Again, indeed, $(7,\pi)$ is a linear combination of both vectors (1,0) and (0,1)

3 Subspaces

Suppose that V and W are two vector spaces that have identical definitions vector addition and scalar multiplication, and that W is a subset of V, W $\subseteq$ V . Then W is a subspace of V.

Theorem: Suppose that V is a vector space and W is a subset of V, W $\subseteq$ V. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- W is nonempty, W $\neq \phi$.
- If $x \in W$ and $y \in W$ then $x + y \in W$
- If $\alpha \in \mathbb{C}$ and $x \in W$, then $\alpha x \in W$.

4 Spans

In section 3 we introduced linear combinations of only two vectors, but does it make sense to have a linear combination of more than two vectors? Sure. If, for instance, x, y and z are three vectors in $\mathbb{R}^3$, or, in $\mathbb{R}^2$, for that matter, and if α, β, γ are scalars, then the vector

$$\alpha x + \beta y + \gamma z$$

is a linear combination of the three vectors, and it is also in $\mathbb{R}^3$, or, in $\mathbb{R}^2$, whichever it is we chose.

Formally, the linear span (also called the linear hull or just span) of a set of vectors in a vector space is the intersection of all subspaces containing that set. The linear span of a set of vectors is therefore a vector space. Spans can be generalized to matroids and modules.

Given a vector space V over a field K, the span of a set S of vectors (not necessarily finite) is defined to be the intersection W of all subspaces of V that contain S. W is referred to as the subspace spanned by S, or by the vectors in S. Conversely, S is called a spanning set of W, and we say that S spans W.

Alternatively, the span of S may be defined as the set of all finite linear combinations of elements of S, which follows from the above definition.

$$Span(S) = \{\sum_{i=1}^{k} \lambda_i v_i | k \in \mathbb{N}, v_i \in S, \lambda_i \in K\}$$

Theorem 1: The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

This theorem is so well known that at times it is referred to as the definition of span of a set.

Theorem 2: Every spanning set S of a vector space V must contain at least as many elements as any linearly independent set of vectors from V.

Theorem 3: Let V be a finite-dimensional vector space. Any set of vectors that spans V can be reduced to a basis for V by discarding vectors if necessary (i.e. if there are linearly dependent vectors in the set). If the axiom of choice holds, this is true without the assumption that V has finite dimension.

5 Total Set

Is there a subset E of a vector space V such that the only subspace of V that includes E is V itself? Why not? An example in $\mathbb{R}^1$is the singleton {x} of any non-zero x; an example in $\mathbb{R}^2$ is E = {(1,0),(0,1)}.

If the only subspace of V that includes E is V, then, the intersection of all the subspaces of V containing E is only V.

6 Dependence

The three vectors

$$x=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad y=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad z=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a total set for $\mathbb{R}^2$; in fact x and y are sufficient enough on their own, and z is superfluous (unnecessary), since it's already a combination of the first two. There is a simple doctrine to follow; adjoining extra vectors to a total set leaves it total. The new vectors do no harm, but provide no new information.

Again, as we've mentioned, z is the sum of the vectors x and y, and that makes every linear combination of x,y and z a linear combination of only x and y, since z is already a linear combination of x and y.

For example, let

$$x=\begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad y=\begin{bmatrix} 2 \\ 8 \end{bmatrix} \qquad z=\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

It's not so very clear that this is a total set, is it? And it's not clear that z is superfluous, right? But it's true.

One way to verify;

$$x - \frac{7}{4}y + \frac{7}{6}z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \textit{ and } \frac{1}{7}x + \frac{7}{6}y - \frac{2}{21}z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

But, that's not so very satisfying. Since every vector in $\mathbb{R}^2$ is already a linear combination of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

As for superfluity; since

$$4x - 5y + 2z = 0$$

it follows that z is superfluous in the sense that z is a linear combination of x and y ($z = \frac{5}{2}y - 2x$), and following the same logic, x is also superfluous, and so is y. If any of x, y and z is omitted from {x,y,z}, what's left is a total set.

7 Basis Vectors of the Plane

The unit vectors (i.e, whose length is one) on the x and y-axis are also called the standard basis vectors of the plane. The collection of all scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the first coordinate axis, and so it follows that the collection of all scalar multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the second coordinate axis.

And it follows that any vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is represented by

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

8 Dot Product

The dot product or scalar product is an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors) and returns a single number. Sometimes it is called inner product in the context of Euclidean space, or rarely projection product for emphasizing the geometric significance.

Algebraically, the dot product is the sum of the products of the corresponding entries of the two sequences of numbers. Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them. It's used to normalize vectors (i.e, generate vectors whose length is one; the unit vectors), and it's also used to measure the angle between two vectors.

`Rule␣9.1:`

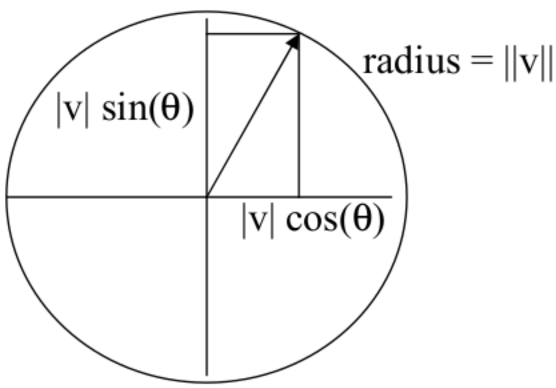
$$[a\, b\, c\, d] \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = ax + by + cz + dw$$

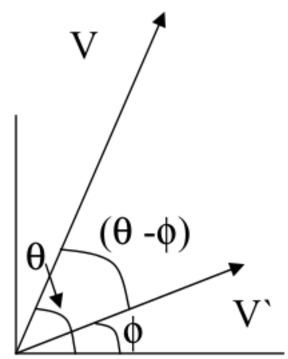
The result is a scalar, and vector, of course. Many CPUs support the dot product of floats in a MAC vector instruction: Multiply and accumulate, that takes two 1-D arrays as operands (this instruction exists because the dot product is computed so often in so many different applications)

- The dot product of a vector by itself, $(v \cdot v)$ is the square of its length.
- We define the norm of a vector (i.e, its length) to be $||v|| = \sqrt{v \cdot v}$
- To make an arbitrary vector v into a unit vector, i.e. to "normalize" it, divide by the length (norm) of v, which is denoted ||v||. Note that if v = 0, then its unit vector is undefined. So in general (with the 0 exception) we have $= \frac{1}{||v||} v$

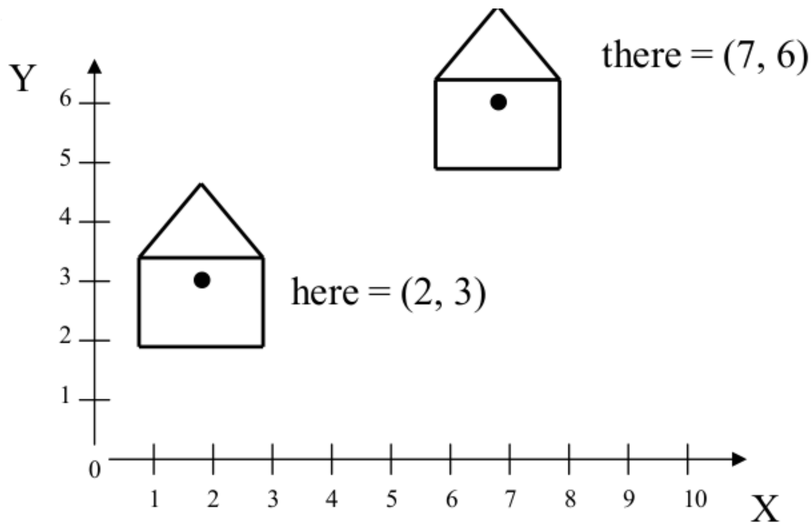
8.1 Finding the Angle Between Two Vectors

- The dot product of two non-zero vectors is the product of their lengths and and the cosine of the angle between them: v · v' = ||v|| ||v'||cos ($\theta - \phi$)





9 Translating Objects



Facing a problem, in which you need to move an object in a 2-dimensional space, *or* $\mathbb{R}^2$, from the position (2,3) to (7,6). Tough call, what do you do?

Just kidding, you just add 5 in the x-axis and 3 in the y-axis, so you basically add the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to the vector you already have $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and then you get:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

10 Matrices

A matrix can be easily visualized as a vector of vectors. Specifically (n) rows of (m) dimensional vectors. Intuitively, it's a 2D array of numbers. Matrices are used to represent transformations which act on points, vectors and other matrices.

10.1 Translation

The first transformation we know thus far is a component-wise addition of vectors

$v' = v + t$ where $v' = \begin{bmatrix} x \\ y \end{bmatrix}, v' = \begin{bmatrix} x' \\ y' \end{bmatrix}, t = \begin{bmatrix} dx \\ dy \end{bmatrix}$

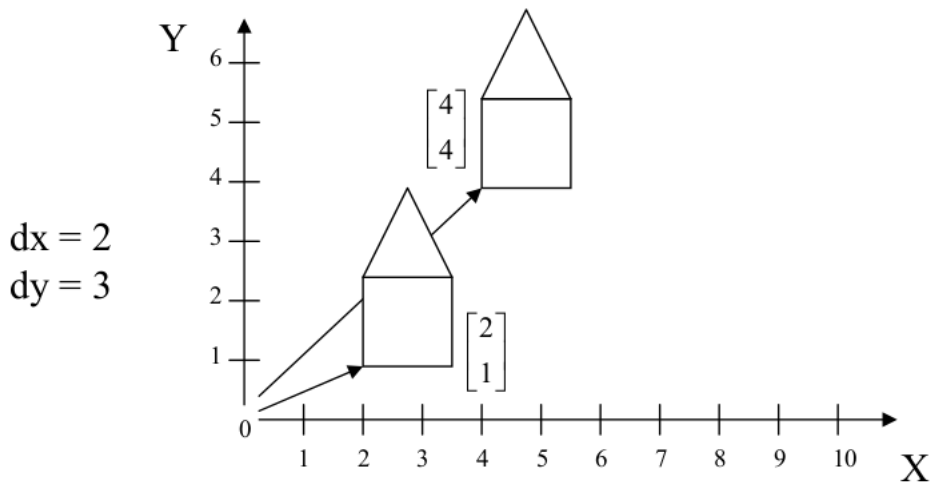
and

$x' = x + dx$

$y' = y + dy$

To move polygons: just translate nodes (vectors) and then redraw lines between them.

The features of this approach is that it preserves both lengths and and angles of the shapes being translated.



10.2 Scaling

Component wise scalar multiplication of vectors

v' = Sv where $v = \begin{bmatrix} x \\ y \end{bmatrix}, v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

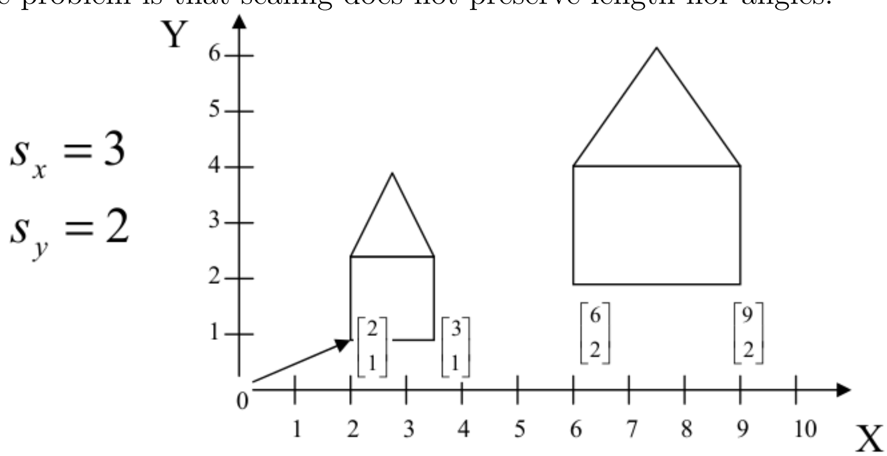
and S=$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$

and

x' = $s_x x$

y' = $s_y y$

The problem is that scaling does not preserve length nor angles.



10.3 Rotation

Rotation of vectors through an angle θ

v'= R_θv where v = $\begin{bmatrix} x \\ y \end{bmatrix}, v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

and $R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$
and
$x' = xcos\theta - ysin\theta$
$y' = xsin\theta + ycos\theta$
And rotation does preserve angles and lengths as well.