Lecture 2: Algorithm Analysis

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Data Structures and Algorithms

- A famous quote: Program = Algorithm + Data Structure

- Algorithm
  - Outline, the essence of a computational procedure, step-by-step instructions

- Program – an implementation of an algorithm in some programming language

- Data structure
  - **Organization** of data needed to solve the problem

Algorithm Specification

- Criteria
  - input: zero or more quantities that are externally supplied
  - output: at least one quantity is produced
  - definiteness: clear and unambiguous
  - finiteness: terminate after a finite number of steps

- Representation
  - A natural language, like English or Chinese.
  - A graphic, like flowcharts.
  - A computer language, like C.

Algorithm Analysis

- Analysis:
  - How to predict an algorithm's performance
  - How well an algorithm scales up
  - How to compare different algorithms for a problem

- Data Structures
  - How to efficiently store, access, manage data
  - Data structures effect algorithm's performance

Example Algorithms

- Two algorithms for computing the Factorial
- Which one is better?

```
int factorial (int n) {
   if (n <= 1)    return 1;
   else   return n * factorial(n-1);
}
```

```
int factorial (int n) {
   if (n<=1)    return 1;
   else {
     fact = 1;
     for (k=2; k<=n; k++)
         fact *= k;
     return fact;
   }
}
```

How to Measure Algorithm Performance?

- What metric should be used to judge algorithms?
  - Length of the program (lines of code)
  - Ease of programming (bugs, maintenance)
  - Memory required
  - Running time

- Running time is the dominant standard
  - Quantifiable and easy to compare
  - Often the critical bottleneck

Running Time

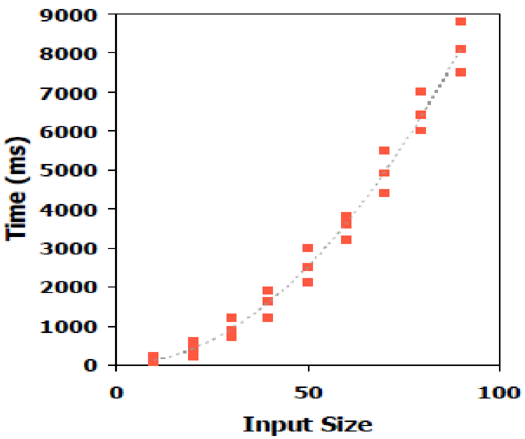
- The running time of an algorithm varies with the input and typically grows with the input size.

- Average case difficult to determine.

- In most of computer science we focus on the *worst* case running time.
  - Easier to analyze.
  - Crucial to many applications: what would happen if an autopilot algorithm ran drastically slower for some unforeseen, untested inputs?

How to measure running time?

- Experimentally
  - Write a program implementing the algorithm
  - Run the program with inputs of varying size
  - Measure the actual running times and plot the results



Why not?

- You have to implement the algorithm which isn't always doable!
- Your inputs may not entirely test the algorithm.
- The running time depends on the particular computer's hardware and software speed.

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation.
- Take into account all possible inputs.
- Evaluate speed of an algorithm independent of the hardware or software environment.
- By inspecting pseudocode, we can determine the number of statements executed by an algorithm as a function of the input size.

Elementary Operations

- Algorithmic "time" is measured in elementary operations:
  - Math (+, -, *, /, max, min, log, sin, cos, abs, ...)
  - Comparisons (==, >, <=, ...)
  - Function calls and value returns
  - Variable assignment
  - Variable increment or decrement
  - Array allocation
  - Creating a new object

- In practice, all of these operations take different amounts of time.
- For the purpose of algorithm analysis, we assume each of these operations takes the same time: "1 operation"

Best/Worst/Average Case

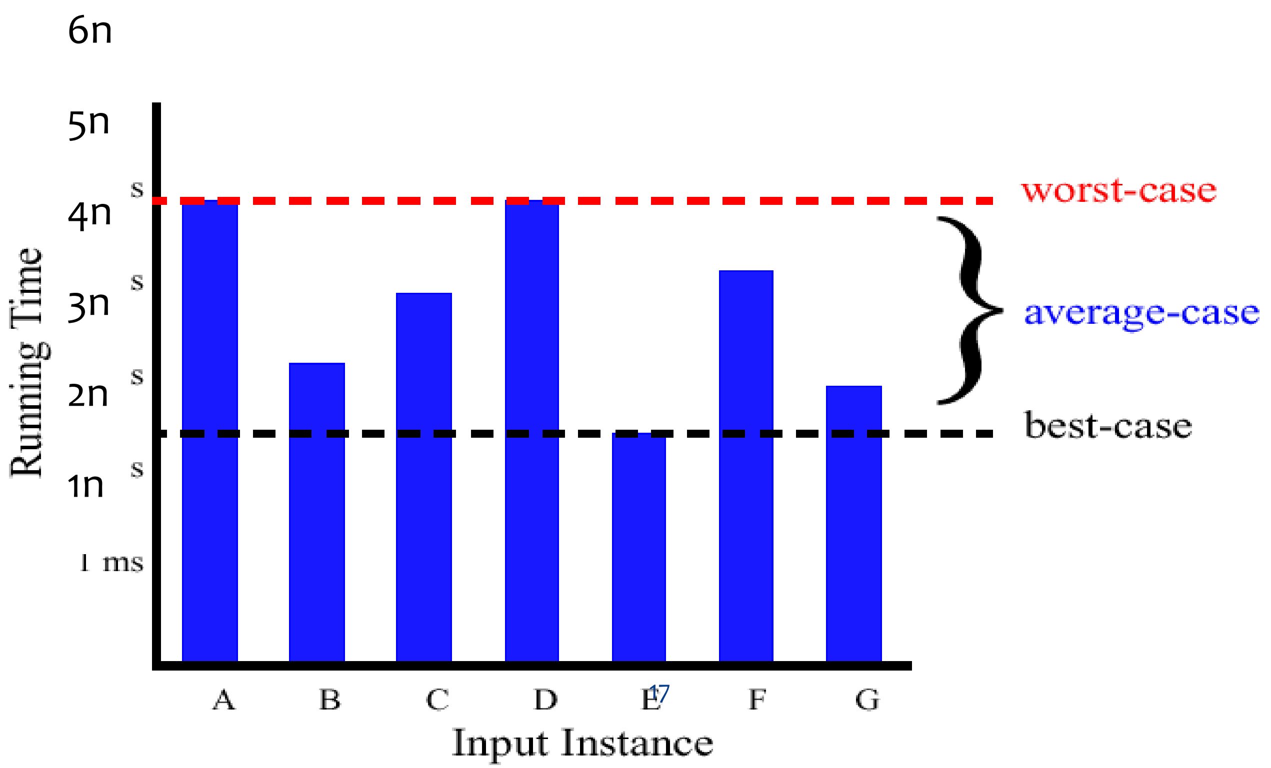
- **Best case:** elements already sorted $\rightarrow$ t_j*=1*, running time = *f(n)*, i.e., *linear* time.

- **Worst case:** elements are sorted in inverse order $\rightarrow$ $t_j=j$, running time = $f(n^2)$, i.e., *quadratic* time

- **Average case:** $t_j=j/2$, running time = $f(n^2)$, i.e., *quadratic* time

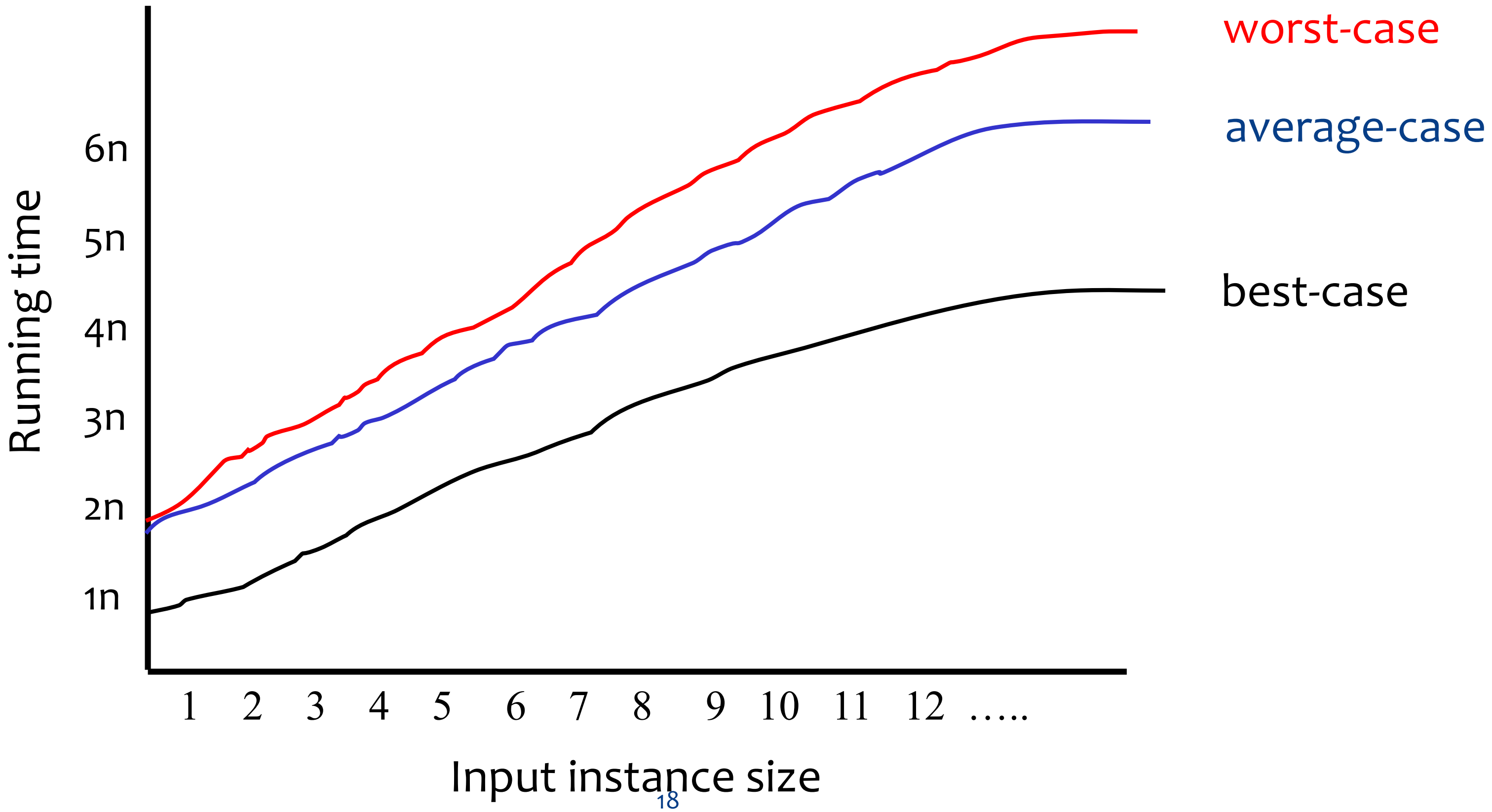
Best/Worst/Average Case (2)

- For a specific size of input *n*, investigate running times for different input instances:



Best/Worst/Average Case (3)

- For inputs of all sizes:



Best/Worst/Average Case (4)

- **Worst case** is usually used:
  - It is an upper-bound and in certain application domains (e.g., air traffic control, surgery) knowing the **worst-case** time complexity is of crucial importance
  - For some algorithms **worst case** occurs fairly often
  - The **average case** is often as bad as the **worst case**
  - Finding the **average case** can be very difficult

Big O Notation

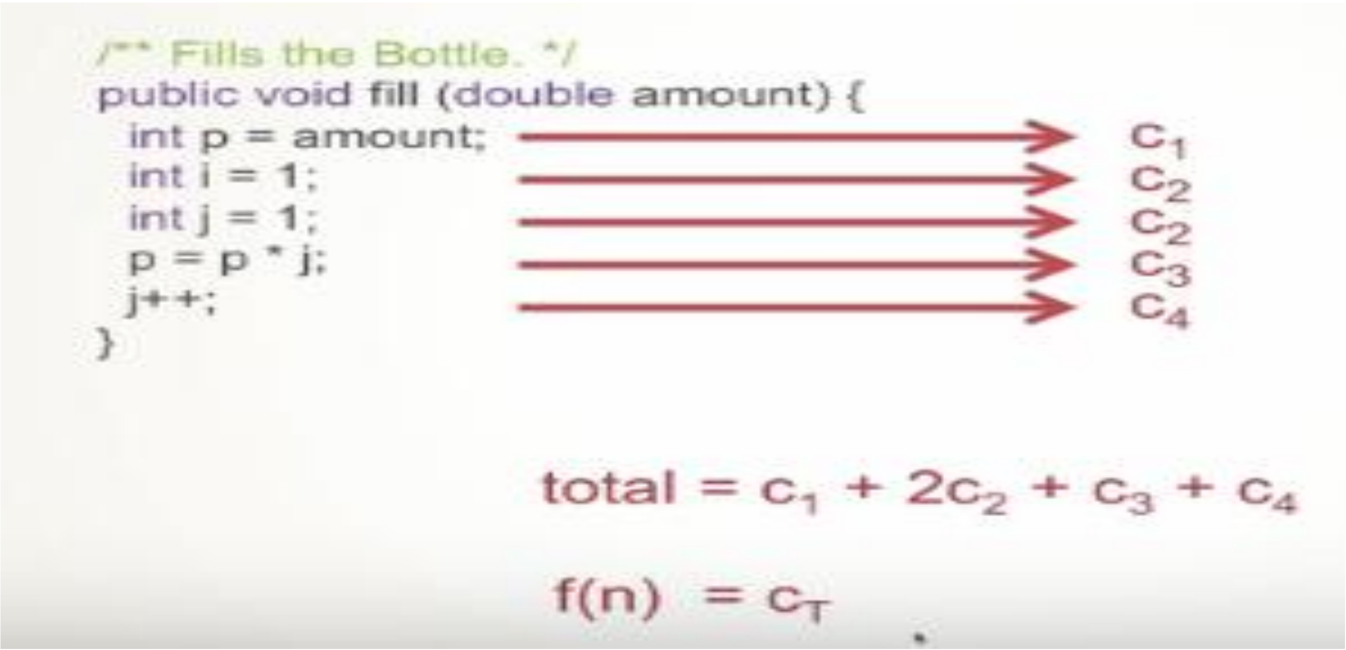
- Big O notation is used in Computer Science to describe the performance or complexity of an algorithm.

- It is the formal method of expressing the upper bound of an algorithm's running time. It's a measure of the longest amount of time it could possibly take for the algorithm to complete.

- Big O specifically describes the **worst-case** scenario, and can be used to describe the execution time required or the space used (e.g. in memory or on disk) by an algorithm.

Constant Running Time

- **O(1)**
  - O(1) describes an algorithm that will always execute in the same time (or space) regardless of the size of the input data set.



Linear Running Time

- **O(N)**
  - O(N) describes an algorithm whose performance will grow linearly and in direct proportion to the size of the input data set.

```
/** Fills the Bottle. */
public void fill (double amount) {
  int p = amount;        ->  c1
  int i = 1;             ->  c2
  int j = 1;             ->  c2
  while (i < n) {        ->  c3 x n
    p = p * j;           ->  c4 x n
    i++;                 ->  c5 x n
  }
  j++;                   ->  c4
}
```

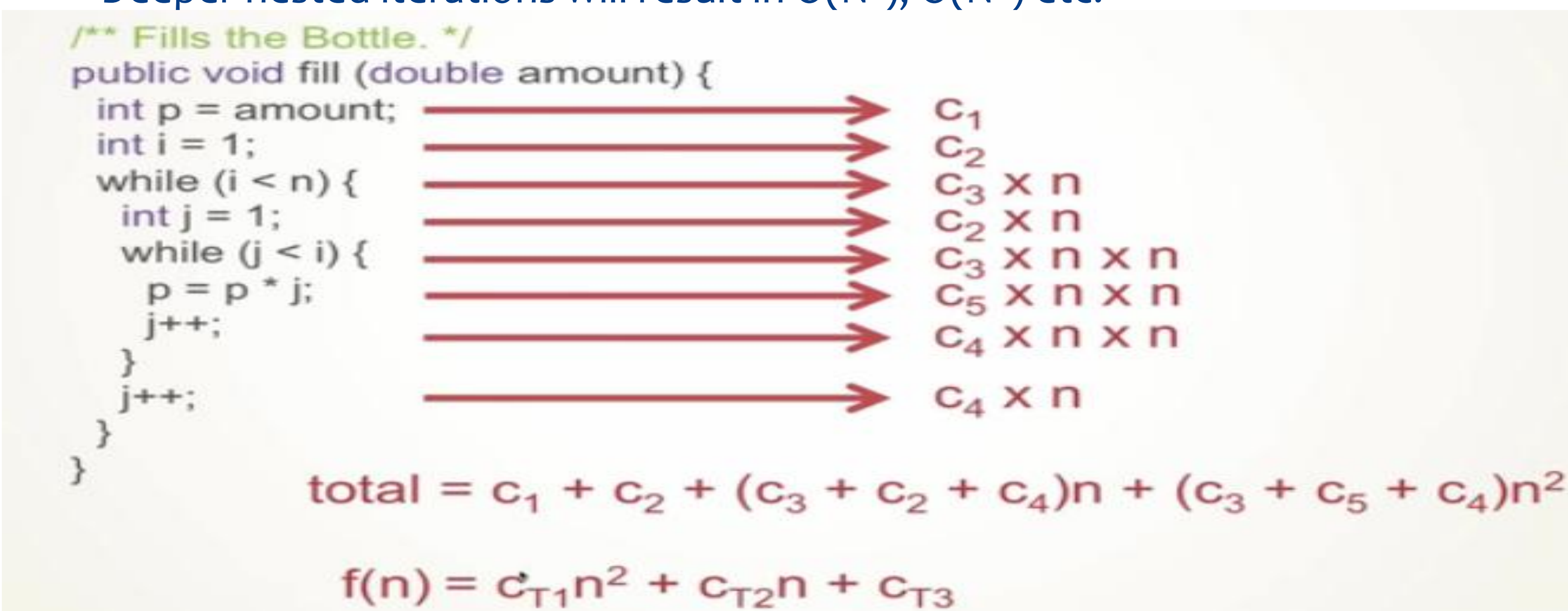
$$\text{total} = c_1 + 2c_2 + (c_3 + c_4 + c_5)n + c_4$$

$$f(n) = c_{T1}n + c_{T2}$$

Quadratic Running Time

O(N²)

- O(N²) represents an algorithm whose performance is directly proportional to the square of the size of the input data set.
- This is common with algorithms that involve nested iterations over the data set. Deeper nested iterations will result in O(N³), O(N⁴) etc.



Logarithms O(log N)

- **Logarithms O(log N)**
  - The iterative halving of data sets described in the binary search example produces a growth curve that peaks at the beginning and slowly flattens out as the size of the data sets increase.

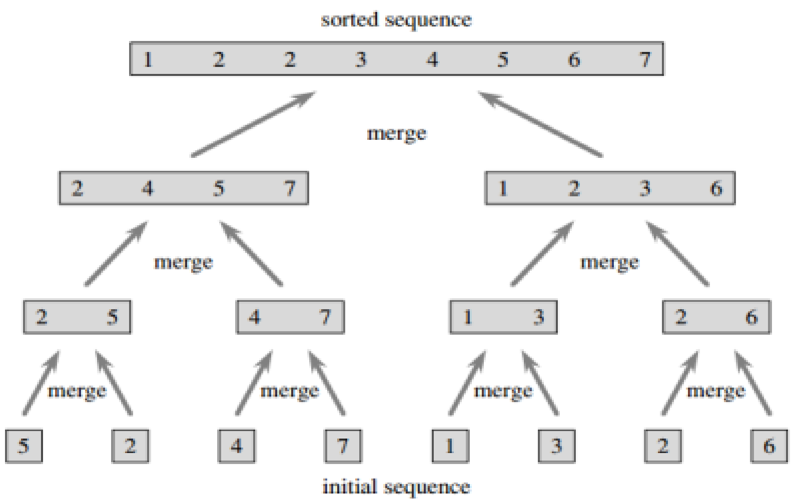
$O(2^N)$

- **$O(2^N)$**
  - $O(2^N)$ denotes an algorithm whose growth will double with each additional element in the input data set. The execution time of an $O(2^N)$ function will quickly become very large.

Recurrence Relation

- When an algorithm contains a recursive call to itself, its running time can often be described by a recurrence equation or recurrence, which describes the overall running time on a problem of size n in terms of the running time on smaller inputs.
- Example: (Merge Sort)

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \, , \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \, . \end{cases}$$



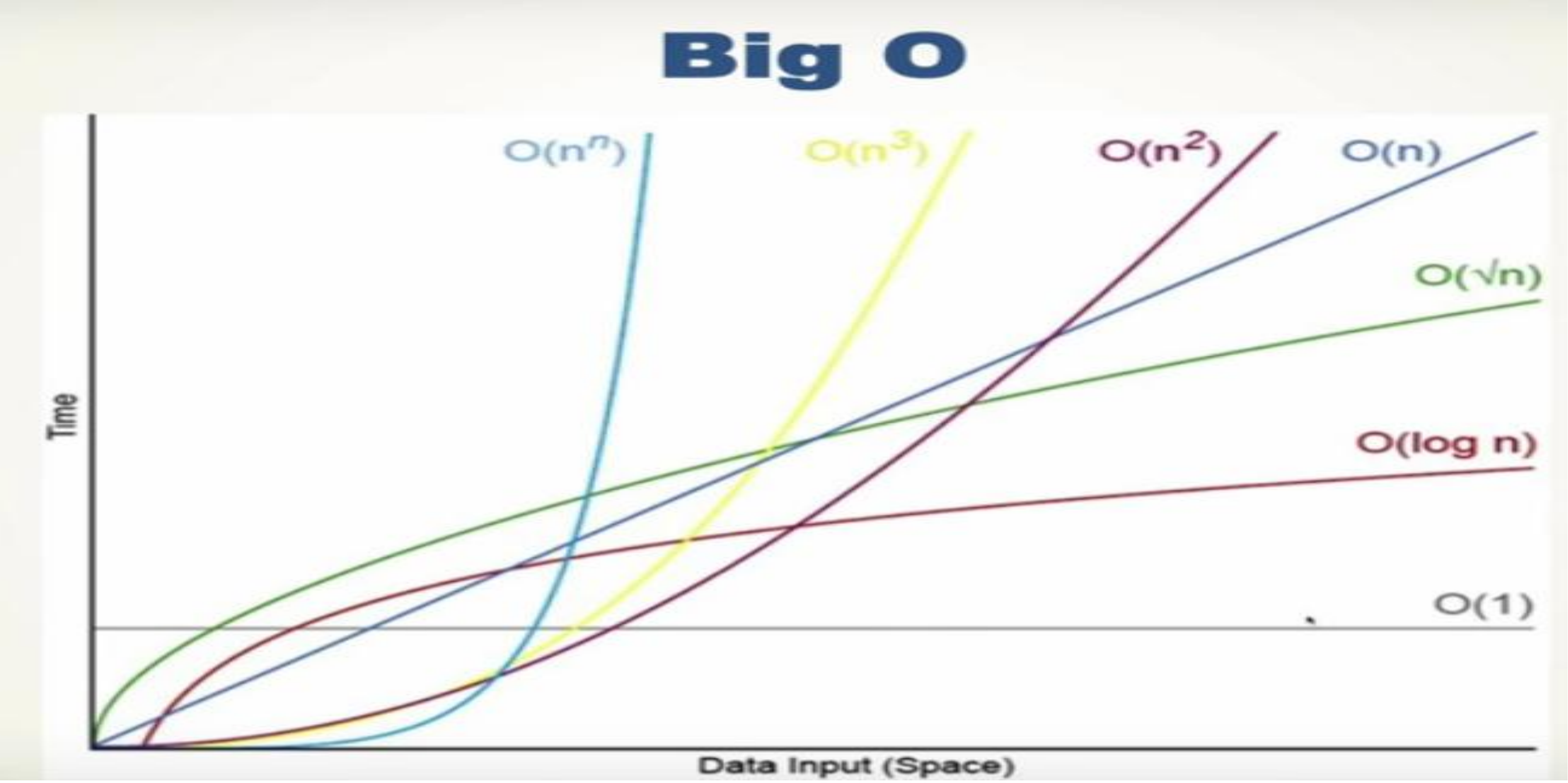
The operation of merge sort on the array $A = \langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$. The lengths of the sorted sequences being merged increase as the algorithm progresses from bottom to top.

Both Figures are taken from Introduction to algorithms Cormen, Leiserson, Rivest and Stein

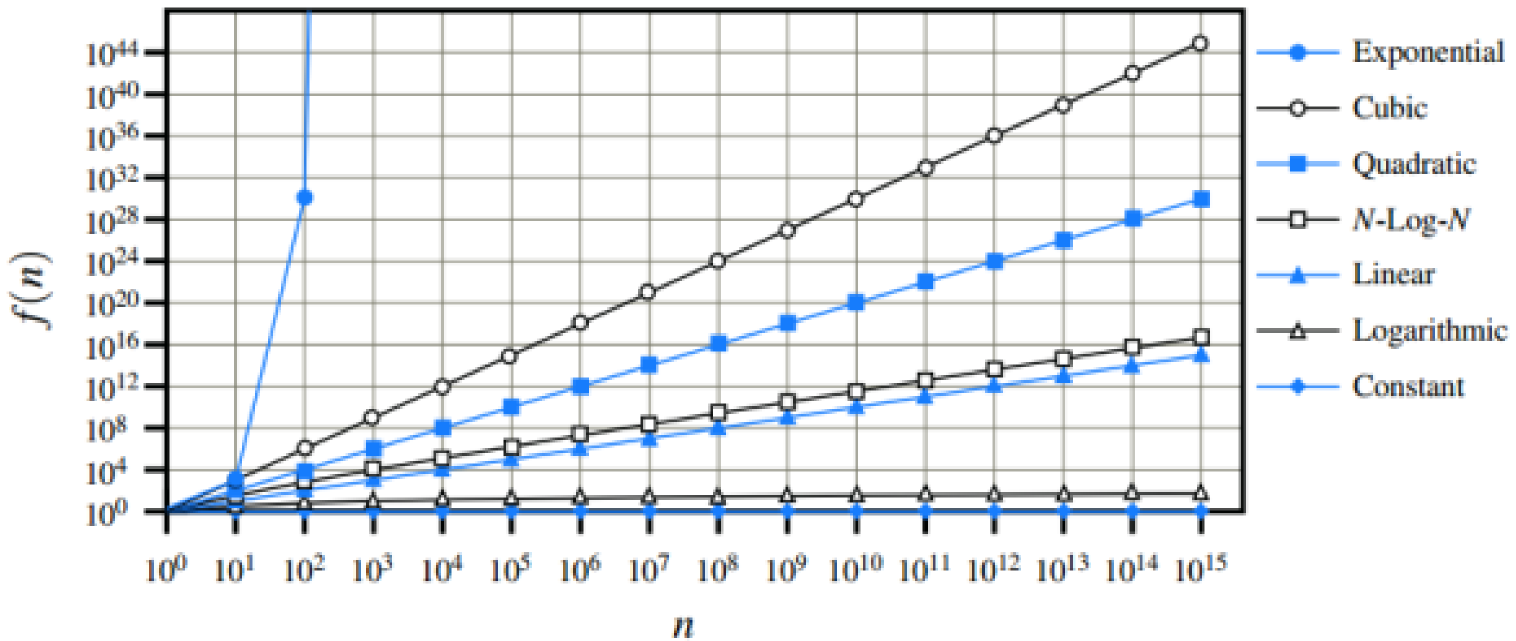
Common functions for analysis of algorithms

#	Name	Function and Description	Example
1	Constant	f(n) =c, for some fixed constant c	Array Indexing
2	Logarithm	f(n) =$\log_b n$, for some constant b>1	Binary Search
3	linear	f(n) =n	Linked List indexing
4	N-Log-N	f(n) =nlogn	Merge Sort
5	Quadratic	f(n) =n^2	Insertion Sort
6	Cubic	f(n) =n^3	
7	Polynomials	f(n) =$a_0+a_1n+a_2n^2+a_3n^3+\cdots+a_dn^d$	
8	Exponential	f(n) =b^n where b is a positive constant, called the base, and the argument n is the exponent	TSP (Travelling Sales Person)

Comparison



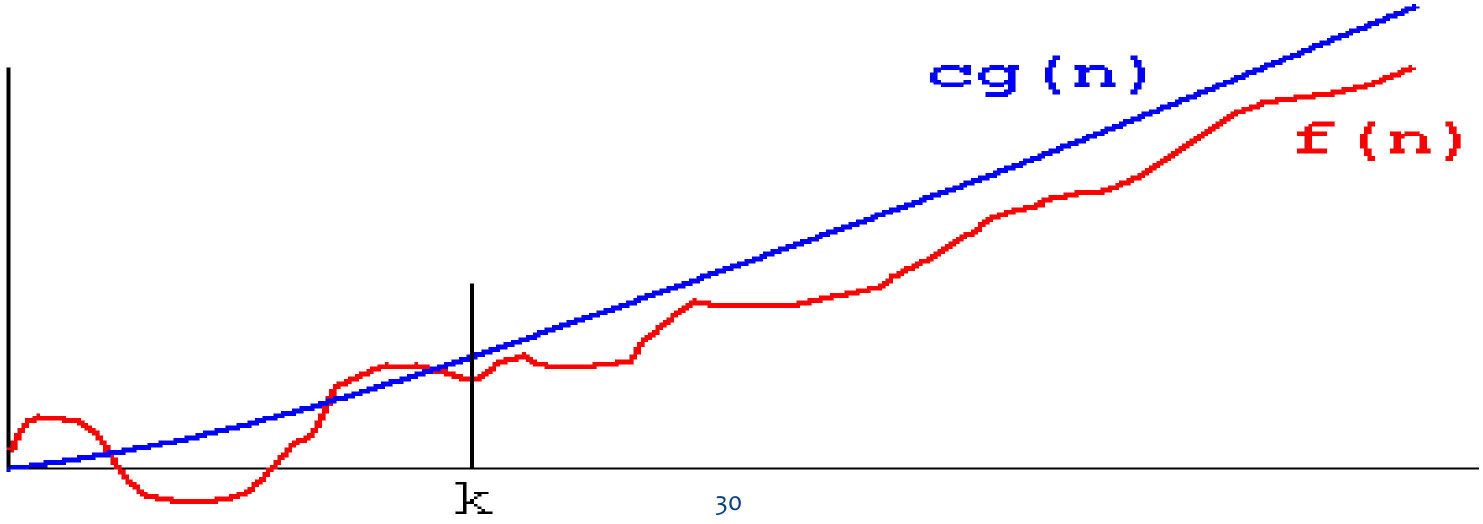
Comparison



Lg N < N < N log N < N^2 < N^3 < b^N

Comparison

- Let f(n) and g(n) be two functions on positive integers. We say f(n) is O(g(n)) if there exist two positive constants c and k such that **f(n) <= cg(n)** for all n >= k.



Proving Big-Oh: Example

Example 1: Prove that running time $T(n) = n^3 + 20n + 1$ is $O(n^3)$

Proof: by the Big-Oh definition, $T(n)$ is $O(n^3)$ if $T(n) \le c \cdot n^3$ for some $n \ge n_0$. Let us check this condition: if $n^3 + 20n + 1 \le c \cdot n^3$ then $1 + \frac{20}{n^2} + \frac{1}{n^3} \le c$. Therefore, the Big-Oh condition holds for $n \ge n_0 = 1$ and $c \ge 22$ $(= 1 + 20 + 1)$. Larger values of n_0 result in smaller factors c (e.g., for $n_0 = 10$ $c \ge 1.201$ and so on) but in any case the above statement is valid.

Proving Big-Oh: Example

- $f(n) = 10n + 5$ and $g(n) = n$
  $f(n)$ is $O(g(n))$
- To show $f(n)$ is $O(g(n))$ we must show constants c and k such that $f(n) <= cg(n)$ for all $n >=k$
- or: $10n+5 <= cn$ for all $n >= k$
- Try $c = 15$. Then we need to show: $10n + 5 <= 15n$.
- Solving for n we get: $5 <= 5n$ or $1 <= n$.
- So $f(n) = 10+5 <= 15g(n)$ for all $n >= 1$. ($c = 15$, $k = 1$).
- Therefore we have shown $f(n)$ is $O(g(n))$.

Proving Big-Oh: Example

Show that $f(n) = n^2 + 2n + 1$ is $O(n^2)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^2 + 2n + 1}{n^2} < \frac{n^2 + 2n^2 + n^2}{n^2} = 4$$

Choose $C = 4$. Note that $2n < 2n^2$ and $1 < n^2$.

Thus, $n^2 + 2n + 1$ is $O(n^2)$ because
$n^2 + 2n + 1 \leq 4n^2$ whenever $n > 1$.

Proving Big-Oh: Example

Show that $f(n) = 3n + 7$ is $O(n)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{3n+7}{n} < \frac{3n+7n}{n} = \frac{10n}{n} = 10$$

Choose $C = 10$. Note that $7 < 7n$.

Thus, $3n + 7$ is $O(n)$ because $3n + 7 \leq 10n$ whenever $n > 1$.

Proving Big-Oh: Example

Show that $f(n) = (n+1)^3$ is $O(n^3)$.

Choose $k = 1$.

Assuming $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{(n+1)^3}{n^3} < \frac{(n+n)^3}{n^3} = \frac{8n^3}{n^3} = 8$$

Choose $C = 8$. Note that $n+1 < n+n$ and $(n+n)^3 = (2n)^3 = 8n^3$. Thus, $(n+1)^3$ is $O(n^3)$ because $(n+1)^3 \leq 8n^3$ whenever $n > 1$.

Proving Not Big-Oh: Example

Show that $f(n) = n^2 - 2n + 1$ is not $O(n)$.

Assume $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^2 - 2n + 1}{n} > \frac{n^2 - 2n}{n} = n - 2$$

$n > C + 2$ implies $n - 2 > C$ and $f(n) > Cn$.

So choosing $n > 1$, $n > k$, and $n > C + 2$ implies $n > k \wedge f(n) > Cn$.

- "Decrease" numerator to "simplify" fraction.