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Math 371
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Homework #8: Runge-Kutta method **April 9, 2020**

Q1
Show that the forth-order Runge-Kutta method when applied to the differential equation $y' = \lambda y$, can be written in the form

$$w_{i+1} = (1 + (h\lambda) + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4)w_i$$

Answer:
$K_1 = h\lambda w_i$
$K_2 = h\lambda(w_i + 1/2\ K_1) = h\lambda w_i + 1/2\ h^2\lambda^2 w_i$
$K_3 = h\lambda(w_i + 1/2\ K_2) = h\lambda w_i + 1/2\ h^2\lambda^2 w_i + 1/4\ h^3\lambda^3 w_i$
$K_4 = h\lambda w_i + h^2\lambda^2 w_i + 1/2\ h^3\lambda^3 w_i + 1/4\ h^4\lambda^4 w_i$

Let $A = h\lambda w_i,\ B = h^2\lambda^2 w_i,\ C = h^3\lambda^3 w_i,$ and $D = h^4\lambda^4 w_i$, Hence we have:

$$\begin{aligned} w_{i+1} &= w_i + 1/6\ (A + 2(A + 1/2\ B) + 2(A + 1/2\ B + 1/4\ C) + (A + B + 1/2\ C + 1/4\ D)) \\ &= w_i + 1/6\ (6A + 3B + C + 1/4\ D) \\ &= (w_i + A + 1/2\ B + 1/6\ C + 1/24\ D) \\ &= w_i(1 + h\lambda + 1/2\ h^2\lambda^2 + 1/6\ h^3\lambda^3 + 1/24\ h^4\lambda^4) \quad \square \end{aligned}$$

Q2
Use Runge-Kutta method of order four first with $h = 0.1$ and then with $h = 0.2$ to obtain approximations to the solution of the initial-value problem

$$y' = -100ty^2,\ 0 \le t \le 2,\ y(0) = 2$$

The exact solution is $y = 2/(1 + 100t^2)$ Compare the stability for the different step sizes.

Answer:

$$\begin{aligned} &K_1 = 0.1f(0, 2) = 0.1(0) = 0 \\ &K_2 = 0.1f(0 + 0.05, 2 + 0/2) = 0.1(-100 \cdot 0.05 \cdot 2^2) = -2 \\ &K_3 = 0.1f(0 + 0.05, 2 - 2/2) = 0.1(-100 \cdot 0.05 \cdot -1) = -0.5 \\ &K_4 = 0.1f(0.1, 2) = 0.1(-100 \cdot 0.1 \cdot 1.5^2) = -2.25 \\ &w_1 = 2 + 1/6(0 + 2(-2) + 2(-0.5) - 9/4) = 0.79167 \end{aligned}$$

Continuing this way for all w_i, we get:

t	w	y	Error
0	2	2	0
0.1	0.79167	1	-0.20833
0.2	0.36518	0.4	-0.034824
0.3	0.19169	0.2	-0.0083111
0.4	0.11495	0.11765	-0.0026969
0.5	0.07582	0.076923	-0.0011026
0.6	0.053525	0.054054	-0.00052911
0.7	0.039716	0.04	-0.00028442
0.8	0.030603	0.030769	-0.0001662
0.9	0.024287	0.02439	-0.00010353
1	0.019734	0.019802	$-6.7814 \times 10^{-}05$
1.1	0.016347	0.016393	$-4.6261 \times 10^{-}05$
1.2	0.01376	0.013793	$-3.2633 \times 10^{-}05$
1.3	0.011741	0.011765	$-2.3676 \times 10^{-}05$
1.4	0.010135	0.010152	$-1.7592 \times 10^{-}05$
1.5	0.0088362	0.0088496	$-1.3344 \times 10^{-}05$
1.6	0.0077718	0.0077821	$-1.0304 \times 10^{-}05$
1.7	0.0068885	0.0068966	$-8.0829 \times 10^{-}06$
1.8	0.0061474	0.0061538	$-6.4293 \times 10^{-}06$
1.9	0.0055197	0.0055249	$-5.1779 \times 10^{-}06$
2	0.0049833	0.0049875	$-4.2167 \times 10^{-}06$

We can notice the error is decreasing as t increases.
the algorithm is unstable since for a small change in h from 0.1 to 0.2 caused a significant error.

t	w	y	Error
0	2	2	0
0.2	-27.3333	0.4	-27.7333
0.4	$-1.116954817416216 \times 10^{+}29$	0.11765	$-1.116954817416216 \times 10^{+}29$
0.6	∞	0.054054	∞
0.8	∞	0.030769	∞
1	∞	0.019802	∞
1.2	∞	0.013793	∞
1.4	∞	0.010152	∞
1.6	∞	0.0077821	∞
1.8	∞	0.0061538	∞
2	∞	0.0049875	∞