
بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

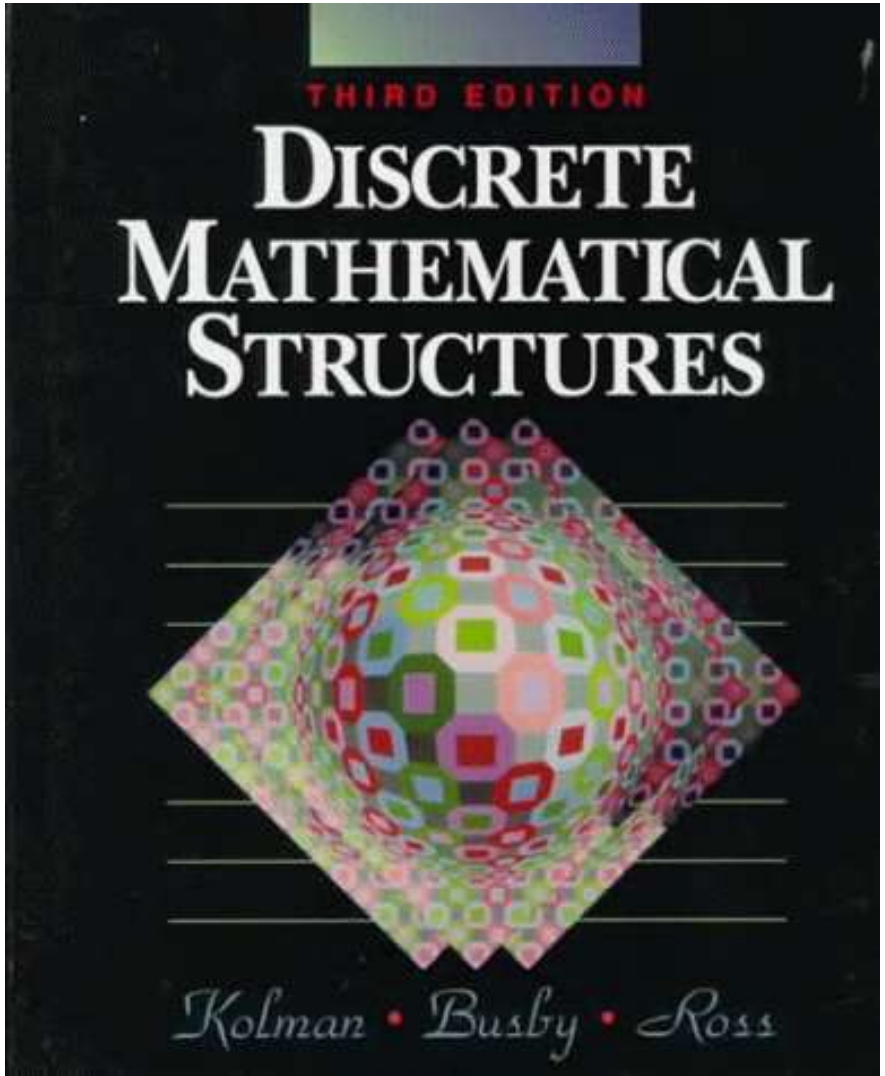
CS103
Discrete Structures

Zagazig university
Faculty of computers and informatics
Department of Computer Science

Dr. Abduallah Gamal

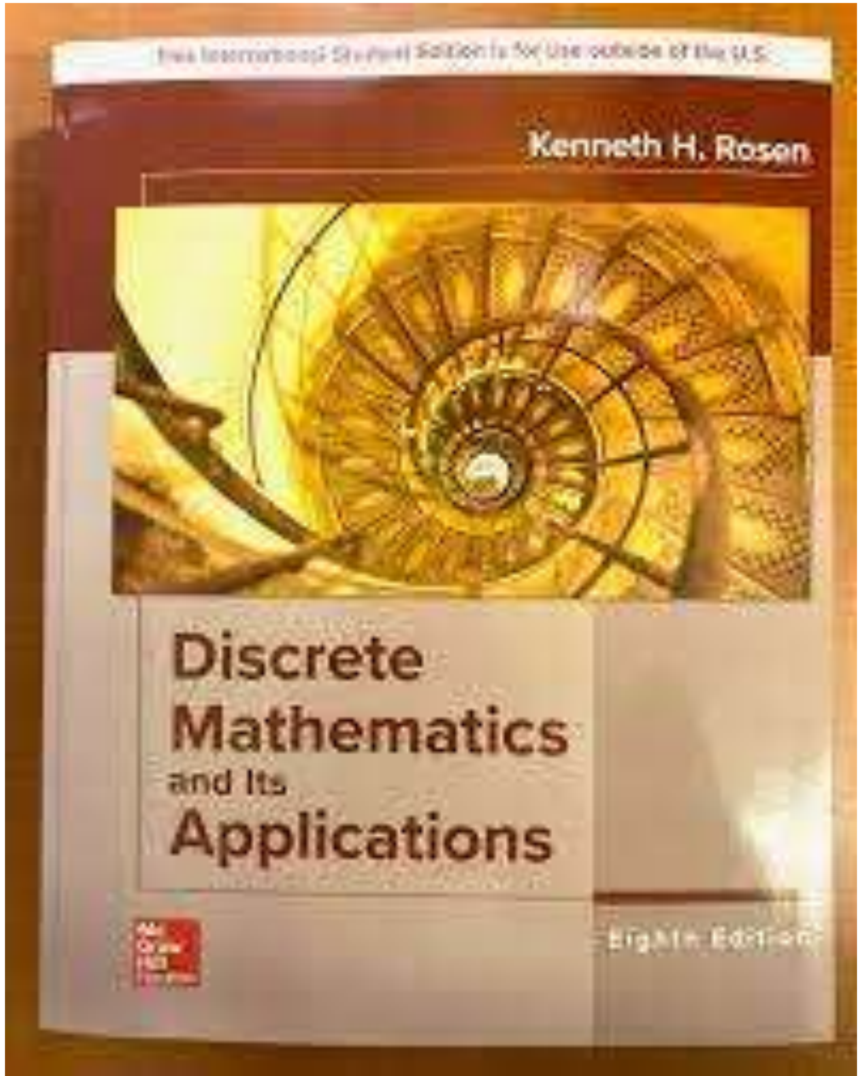
Lectures Reference

- Discrete Mathematical Structures



Lectures Reference

- Discrete Mathematics and its Applications



Course Objectives

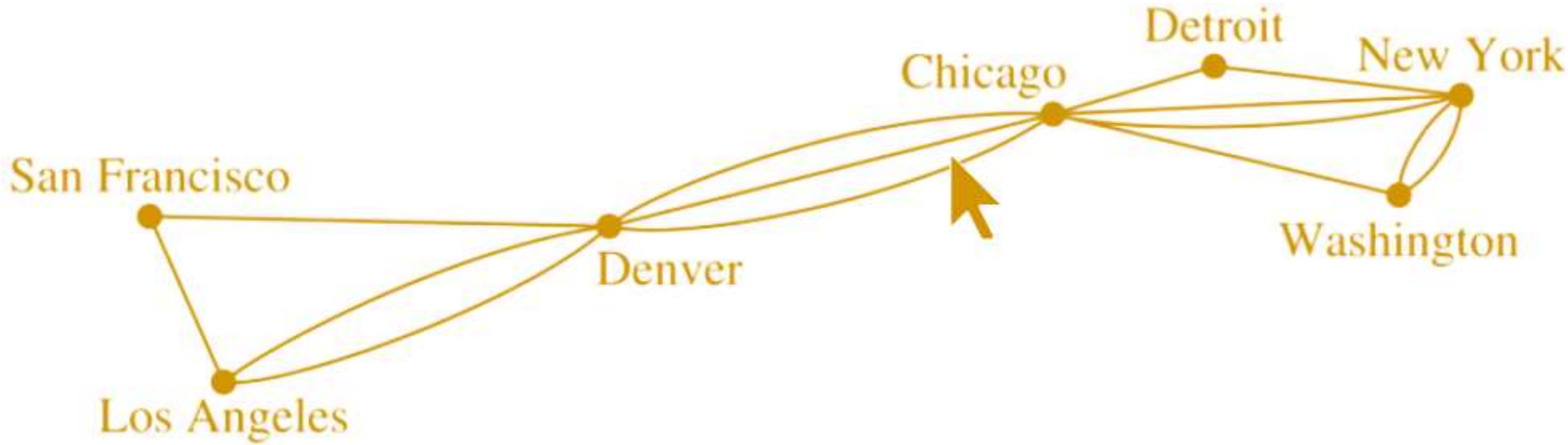
- Developing rigorous treatment.
- Developing mathematical foundations to CS.
- Learn how to think mathematically.
- Grasp the basic logical and reasoning mechanisms of mathematical thought.
- Acquire logic and proof as the basics for abstract thinking.
- Improve problem solving skills.
- Grasp the basic elements of induction, recursion, combination and discrete structures.

DM is a Gateway Course

Topics in discrete mathematics will be important in many courses that you will take in the future:

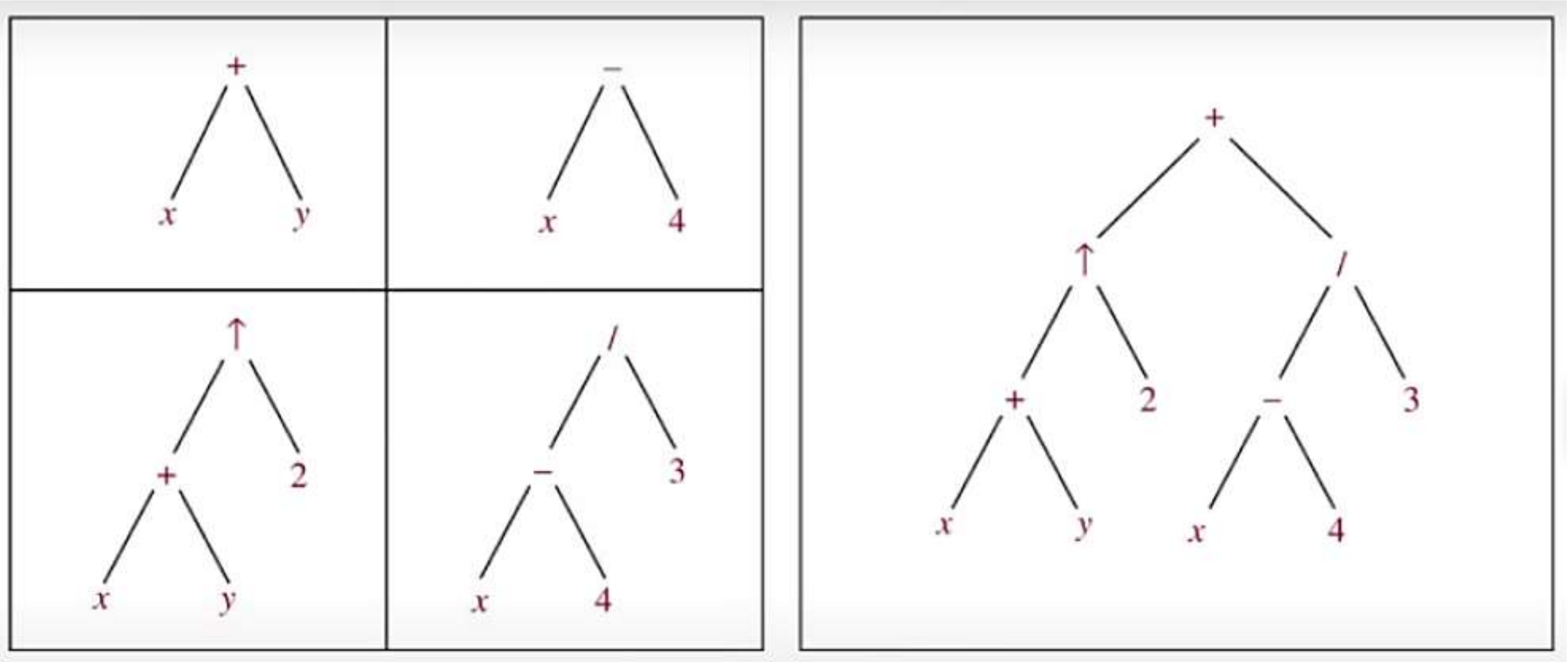
- **Computer Science:** Computer Architecture, Data Structures, Algorithms, Programming Languages, Compilers, Computer Security, Databases, Artificial Intelligence, Networking, Graphics, Game Design, Theory of Computation, ……
- **Mathematics:** Logic, Set Theory, Probability, Number Theory, Abstract Algebra, Combinatorics, Graph Theory, Game Theory, Network Optimization, …
- **Other Disciplines:** You may find concepts learned here useful in courses in philosophy, economics, linguistics, and other departments.

Graphs

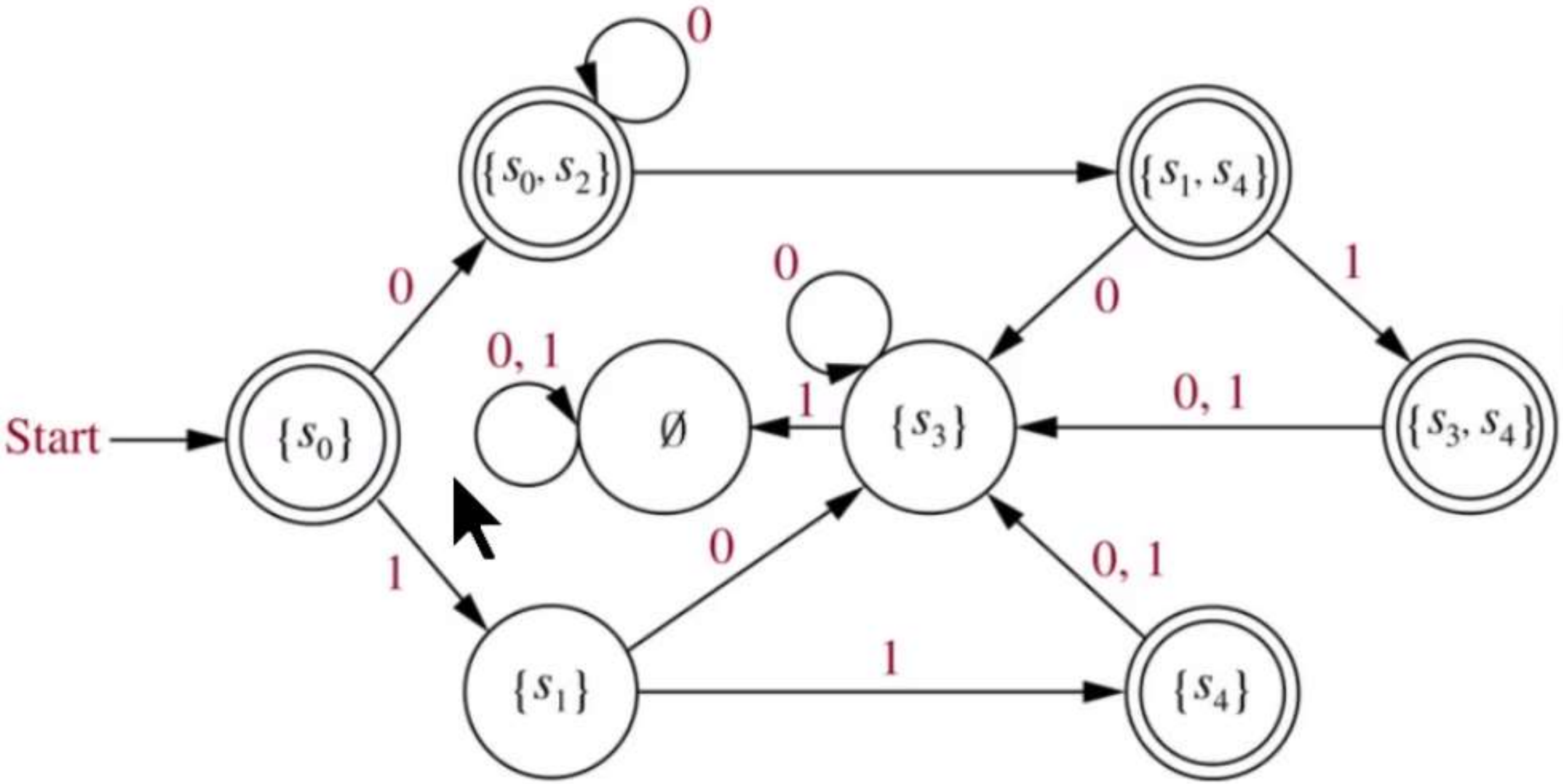


Detroit
New York
Chicago
San Francisco
Denver
Washington
Los Angeles

Trees

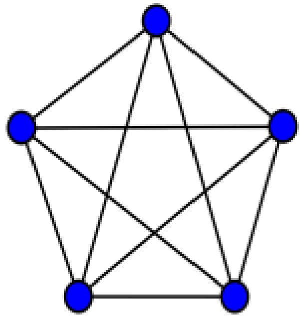


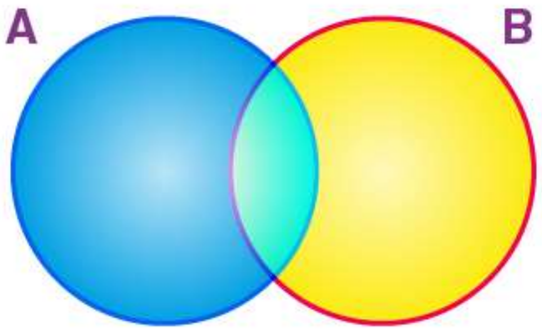
Modelling Computation

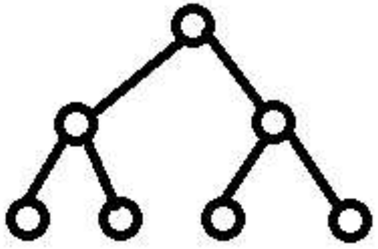


Discrete mathematics

- **Discrete mathematics** is a part of mathematics devoted to the study of discrete objects, such as:
  - ✓Set theory
  - ✓Logic
  - ✓Relations and digraphs
  - ✓Probability theory
  - ✓Graph theory
  - ✓Trees
- The course of discrete mathematics provides the mathematical background needed for all subsequent courses in computer science.







Course Syllabus

- ❑ Set theory
- ❑ Logic
- ❑ Relations and digraphs
- ❑ Probability theory
- ❑ Graph theory
- ❑ Trees

Set theory

- **Set**: is a collection of well-defined objects and each member of the set is called an element of the set.

- The set is denoted with capital letters: A, B, C, X, Y, … .
- The elements is denoted with small letters: a, b, c, z, … .

  - ✓If x is an element of a set A, we denoted it as $x \in A$.
  - ✓If x is not an element of a set A, we denoted it as $x \notin A$.

Set theory

- **Cardinality of a set** is the number of elements in the set. Given a set S, then the cardinality of this set is denoted by #(S) or |S| or n(S).

- **Example**: consider the set $A=\{1,2,3,5,7\}$, then the cardinality of a set is $|A|=5$.

Types of set

1. **Null set (empty set)**: is a set that has no element, it is denoted by Φ = {}.

2. | Φ |=0. |{}|=0. |{Φ}|=1. |{Φ, a, b}|=3

Is an empty box Φ the same thing as a box with an empty box inside {Φ}?

1. **Finite set** is a set of a finite numer of elements.
2. **Infinite set** is a set of infinite number elements.

Example: A={1,2,3,4,5} is a finite set.

A={x|x ∈ Z} is an infinite set.

(Another way to describe a set is to use **set builder** notation).

Set

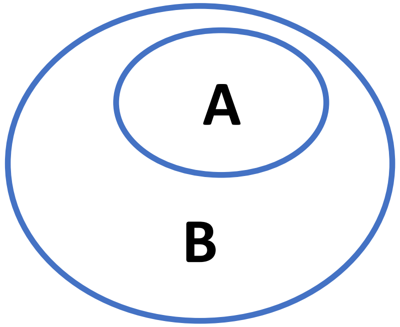
Note that:

- Z^+={x|x is a positive integer}
  Z^+={1, 2, 3, 4, …}
- Z^-={x|x is a negative integer}
  Z^-={-1, -2, -3, -4, …}
- **N**={x|x is a positive integer or zero}
  N={0,1, 2, 3, …}
- **Z**={x|x is an integer}. {……,-3,-2,-1,0,1,2,3,….}
- **Q**={x|x is a rational number}
  It is written as $\frac{a}{b}$, b$\neq$ 0 and a, b are integer numbers.
- R={x|x is a real number}

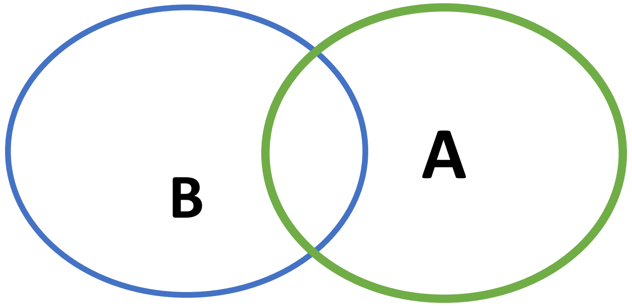
Subset

❑ Subset

A set A is called a subset of B if every element of A is an element of B, denoted by **A⊆B**. If A is not a subset of B, written as **A⊈B**.



A⊆B



Venn diagrams

A⊈B

We use the notation $A \subseteq B$ to indicate that A is a subset of the set B .

$$\boxed{A \subseteq B \;\leftrightarrow\; \forall x(x \in A \rightarrow x \in B)}$$

Note that:

For any set A, we have Φ ⊆A.

Proper Subset

The set A is a subset of the set B but that $A \neq B$,
we write $A \subset B$
and say that A is a **proper subset** of B.

$$\boxed{A \subset B \leftrightarrow \left(\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)\right)}$$

Subset and Proper Subset

- Consider A={1, 3, 5, 7, 9, 11 }
  B={3, 5, 7}
  C={3,5,7}
  then B⊂ A
  B⊆ C

Note that:

A= B if and only if A ⊆ B and B ⊆ A.

Example: if A={1, 2, 3} and B={x|x is a positive integer and $x^2 < 12$} then A=B.

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$?

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

Subset and Proper Subset

- Examples:

A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? True

A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$?

Subset and Proper Subset

- Examples:

A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? True

A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? True

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$?

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? False

Subset and Proper Subset

- Examples:

$A = \{3, 9\}, B = \{5, 9, 1, 3\},$	$A \subseteq B$?	True
$A = \{3, 3, 3, 9\}, B = \{5, 9, 1, 3\},$	$A \subseteq B$?	True
$A = \{1, 2, 3\}, B = \{2, 3, 4\},$	$A \subseteq B$?	False
$B=\{2, 3\}, A=\{1, 2, 3, 4, 5, 6\}$	$B \subset A$?	

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$,	$A \subseteq B$?	True
$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$,	$A \subseteq B$?	True
$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$,	$A \subseteq B$?	False
$B=\{2, 3\}$, $A=\{1, 2, 3, 4, 5, 6\}$	$B \subset A$?	True

Subset and Proper Subset

- Examples:

$A = \{3, 9\}, B = \{5, 9, 1, 3\},\quad A \subseteq B$? True

$A = \{3, 3, 3, 9\}, B = \{5, 9, 1, 3\},\quad A \subseteq B$? True

$A = \{1, 2, 3\}, B = \{2, 3, 4\},\quad A \subseteq B$? False

$B=\{2, 3\}, A=$ $\{1, 2, 3, 4, 5, 6\}$ **$B \subset A$** ? True

$B=\{2, 3\}, A=$ $\{1, 2, 3, 4, 5, 6\}$ **$B \subset B$** ?

Subset and Proper Subset

- Examples:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? True

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? False

$B=\{2, 3\}$, $A=\{1, 2, 3, 4, 5, 6\}$ $B \subset A$? True

$B=\{2, 3\}$, $A=\{1, 2, 3, 4, 5, 6\}$ $B \subset B$? False

$B=\{2, 3\}$, $A=\{3, 2\}$ $B \subseteq A$? True

$B= \{1, 2, 3, 4, 5,6, 7\}$, $A=\{1, 2, 3, 4, 5, 6\}$ $B \supset A$? True

Cont.

If A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$, if A and B are equal sets.

- The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements.
- $\{1, 3, 3, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

Subset True/False

- Let $B=\{3,\{2\},\{3\}\}$ which is true?
- $2 \in B$
- $\{2\} \in B$.
- $\{2\} \subseteq B$.
- $\{\{2\}\} \subseteq B$.
- $\{3\} \in B$.
- $\{3\} \subseteq B$.
- $\{\{3\}\} \subseteq B$.
- $\{\{3\}\} \subset B$

- $\{a, b\} \subseteq \{a, b, c\}$. True
- $\{c, d\} \subseteq \{c, d\}$. True
- $\{a\} \subseteq \{\{a\}\}$ False
- $\Phi \subseteq \{x, y, z\}$ True
- $Z^+ \subseteq Z$. true

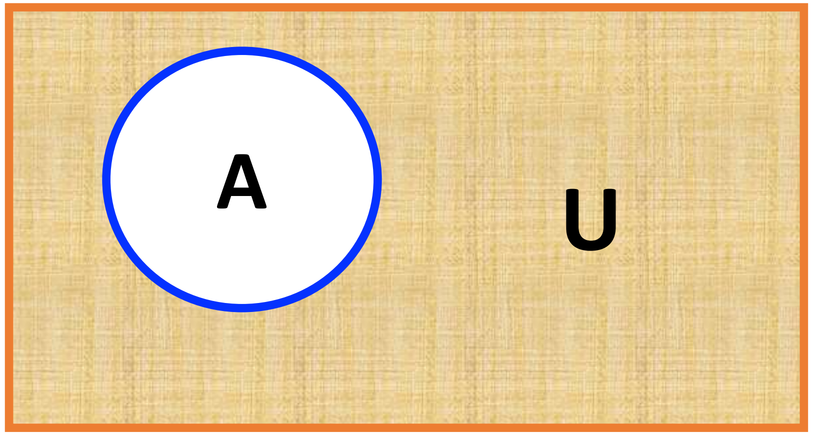
Subset True/False

- Let $B=\{3,\{2\},\{3\}\}$ which is true?
- $2 \in B$ **False**
- $\{2\} \in B$. **True**
- $\{2\} \subseteq B$. False
- $\{\{2\}\} \subseteq B$. **True**
- $\{3\} \in B$. **True**
- $\{3\} \subseteq B$. True
- $\{\{3\}\} \subseteq B$. **True.**
- $\{\{3\}\} \subset B$. **True.**

- $\{a, b\} \subseteq \{a, b, c\}$. True
- $\{c, d\} \subseteq \{c, d\}$. True
- $\{a\} \subseteq \{\{a\}\}$ False
- $\Phi \subseteq \{x, y, z\}$ True
- $Z^+ \subseteq Z$. true

Universal set

- **Universal set (U)** is a set that contains all the elements of interest.
- **Example**: if we are dealing with integer sets, the universal set is the set of all integers.



Power set

- **Power set** is a set consisting of all subsets of a given set A and is denoted by P(A).
- **Example**: let A={1, 2, 3} **then**

  P(A)=P({1, 2, 3})={{}, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}}
- The set of n elements has a power set of 2^n elements.
- $|P(A)|=2^3=8$.
- $P(\Phi)=\{\{\}\}$. or $P(\{\})=\{\Phi\}$
- $P(\{\Phi\})=\{\Phi,\{\Phi\}\}$

Power set

- **Determine if the following sets are power sets of some unknown set.**
- $\{\Phi\}$. $2^0=1$. $p(\Phi)=\{\Phi\}$
- $\{\Phi,\{a\},\{\Phi,a\}\}$. $3=2^n$. Can't find n then not power set
- $\{\Phi,\{a\},\{\Phi,a\},\{\Phi\}\}$. is power set of set $\{\Phi, a\}$
- $\{\Phi,\{a\}\}$ $2=2^1$. $P(\{a\})=\{\Phi,\{a\}\}$

Questions

1. Let $A = \{1, 2, 4, a, b, c\}$. Identify each of the following as true or false.

(a) $2 \in A$ (b) $3 \in A$ (c) $c \notin A$
(d) $\varnothing \in A$ (e) $\{\,\} \notin A$ (f) $A \in A$

Answer:

a. true
b. False
c. false
d. False
e. True
f. false

The ordered n-tuple

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, … , and a_n as its nth element.

In particular, ordered 2-tuples are called ordered pairs (e.g., the ordered pairs (a, b))

Cartesian Products

Let A and B be sets.

The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Cartesian Products - Example

Let $A = \{1,2\}$, and $B = \{a, b, c\}$

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$|A \times B| = |A| * |B| = 2 * 3 = 6$$

Find $B \times A$?

The Cartesian product of more than two sets.

The Cartesian product of the sets $A_1, A_2, \dots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \dots, n$. In other words,

$$\boxed{\begin{gathered} A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}. \end{gathered}}$$

Example

$A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$

$$\boxed{\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \end{aligned}}$$

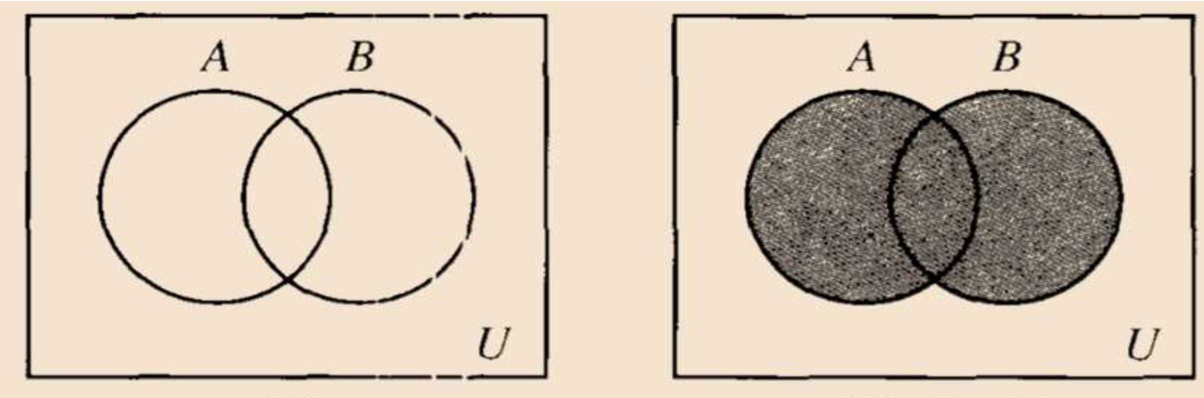
Operations on sets

- **Union** (A∪B): $A \cup B = \{x \mid x \in A \vee x \in B\}$

The union of two sets A and B is a third set containing all the elements of A or B and denoted by A∪B.

- **Example**: Let A ={1, 2, 3} and B={2, 3, 5} **then** A∪B={1, 2, 3, 5}

- A∪B =B∪A
- A∪(B∪C)= (A∪B)∪C.
- ϕ ∪A=A
- A ⊂ (A ∪ B)
- B⊂ (A ∪ B)
- If A ⊂ B and C⊂ D then (A∪C) ⊂ (B∪D)



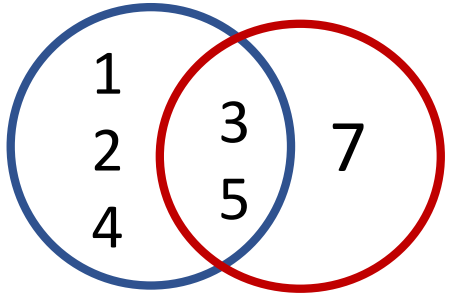
(a) (b) $A \cup B$

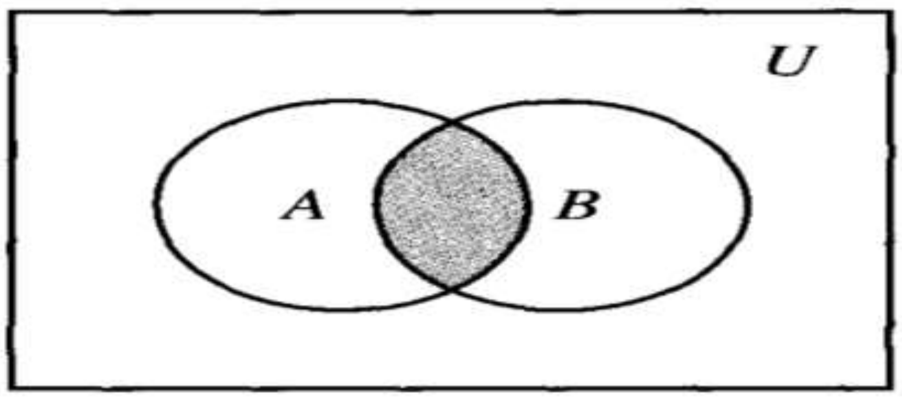
Operations on sets

❑ **Intersection (**A∩B**)** $A \cap B = \{x \mid x \in A \wedge x \in B\}$

The intersection of two sets A and B is a third set containing the common elements of A and B or elements that belong to both A and B

❑ **Example**: If A={1,2,3,4,5} and B={3, 5, 7} then A∩B={3,5}.

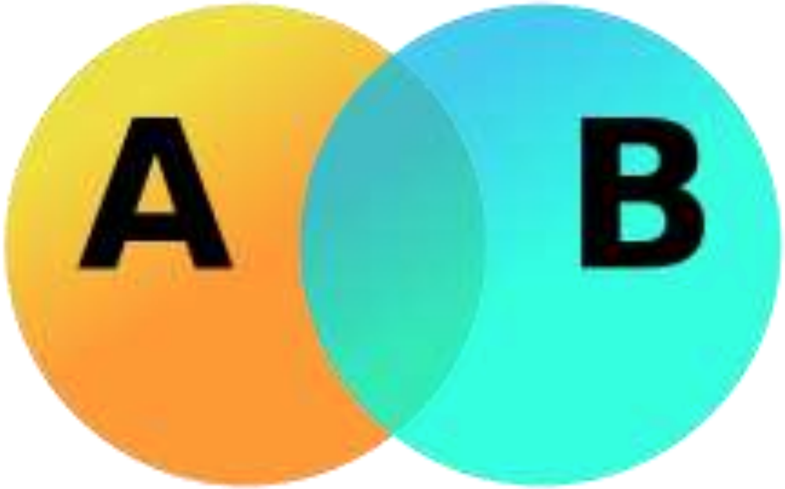




(b) $A \cap B$

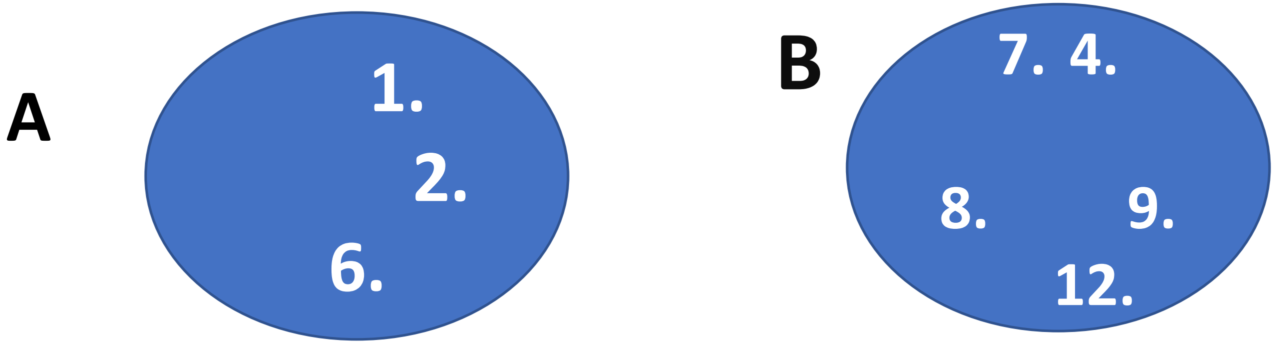
Operations on sets

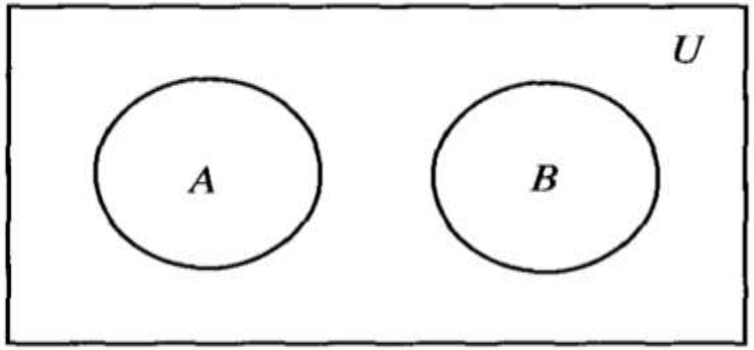
- **Note that:**
- $A \cap B = B \cap A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $\phi \cap A = \phi$
- $(A \cap B) \subset A$
- $(A \cap B) \subset B$
- If $A \subset B$ and $C \subset D$ then $(A \cap C) \subseteq (B \cap D)$
- $A \subset B$ iff $A \cap B = A$



Disjoint sets

- If $A=\{1, 2, 3\}$ and $B=\{5, 6, 7, 8\}$ then $A \cap B=\phi$ i.e. A and B are disjoint sets.





- The two sets are disjoint if they have not any common elements.

- If $A=\{1, 2, 3\}$ and $B=\{5, 6, 7, 8\}$ then $A \cap B=\phi$ i.e. A and B are disjoint sets.

Operations on sets

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

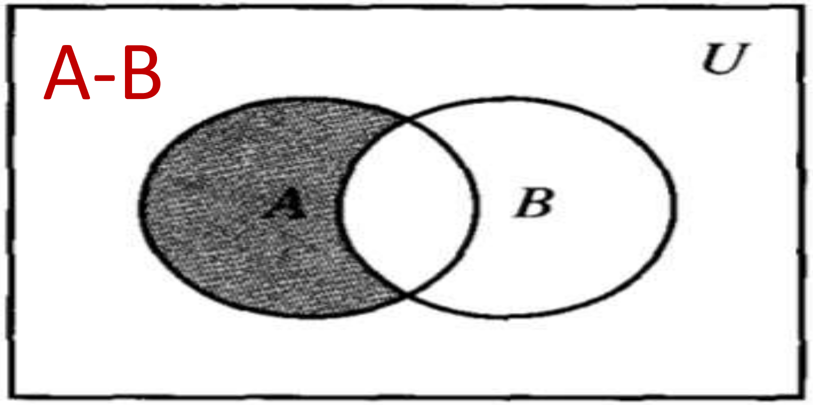
- The **difference** (A-B) of A and B is a third set containing all of the elements of A which doesn’t exist in B.

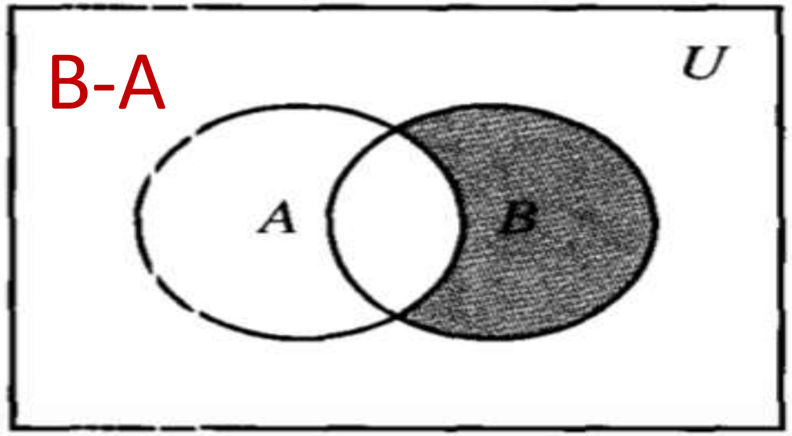
- **Example**: Let A ={1, 3, 4, 5, 6}

  B={3, 4, 7} then

  A-B={1, 5, 6}

  B-A={7}

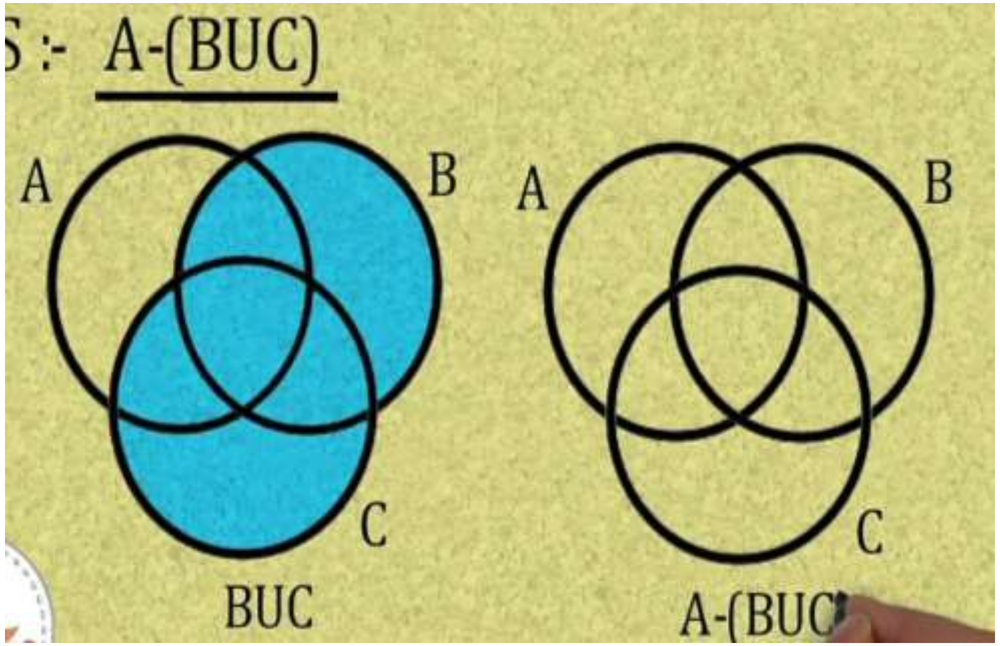




Operations on sets

Note that:

- ✓ $A-B \neq B-A$
- ✓ $A-B \subseteq A$
- ✓ $A-(B \cup C)=(A-B) \cap (A-C)$
- ✓ $A-(B \cap C)=(A-B) \cup (A-C)$
- ✓ $A \subset B$ iff $A-B= \phi$



Operations on sets

- The **symmetric difference** ($A \oplus B$) between A and B can be defined as: $A \oplus B = (A-B) \cup (B-A)$.

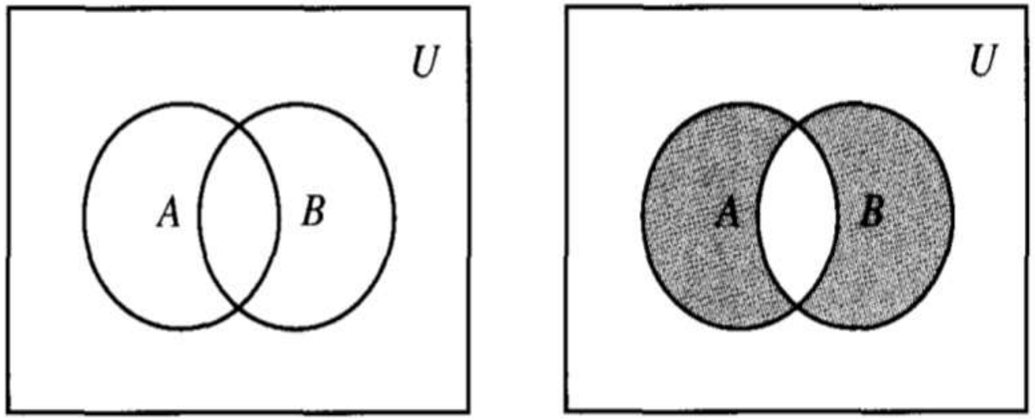
  $B \oplus A = (A-B) \cup (B-A)$. $\quad A \oplus B \equiv B \oplus A$

- **Example**: Let A ={1, 2, 4} and B={2, 4, 5} then

  A-B={1}

  B-A={5}

Then $A \oplus B = \{1, 5\}$ and also $B \oplus A = \{5, 1\}$



(a) (b) $A \oplus B$

Operations on sets

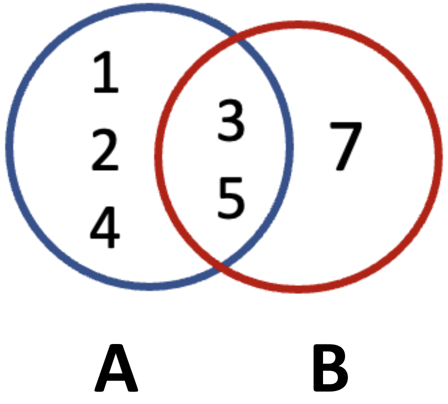
- The **complement of set A** ($\bar{A}$ or $\grave{A}$) $\quad \bar{A} = \{x \in U \mid x \notin A\}$
- **Example**: If U={1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and A={3, 4, 5, 7} then $\bar{A}$ ={1, 2, 6, 8, 9, 10}.
- **Note that:**
  - $\bar{\bar{A}} = A$
  - $\bar{\phi}$ =U
  - $\bar{U} = \phi$
  - A∪ $\bar{A}$ = U
  - A∩ $\bar{A} = \phi$
  - A∪A=A
  - A ∩ A=A
  - A∪U=U
  - A ∩U=A
  - A∪ ϕ=A
  - A ∩ ϕ= ϕ

De Morgan's law

- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$
- If A and B are finite sets then |A ∪ B|=|A|+|B|-|A ∩ B|

Example: |A ∪ B|=|A|+|B|-|A ∩ B|

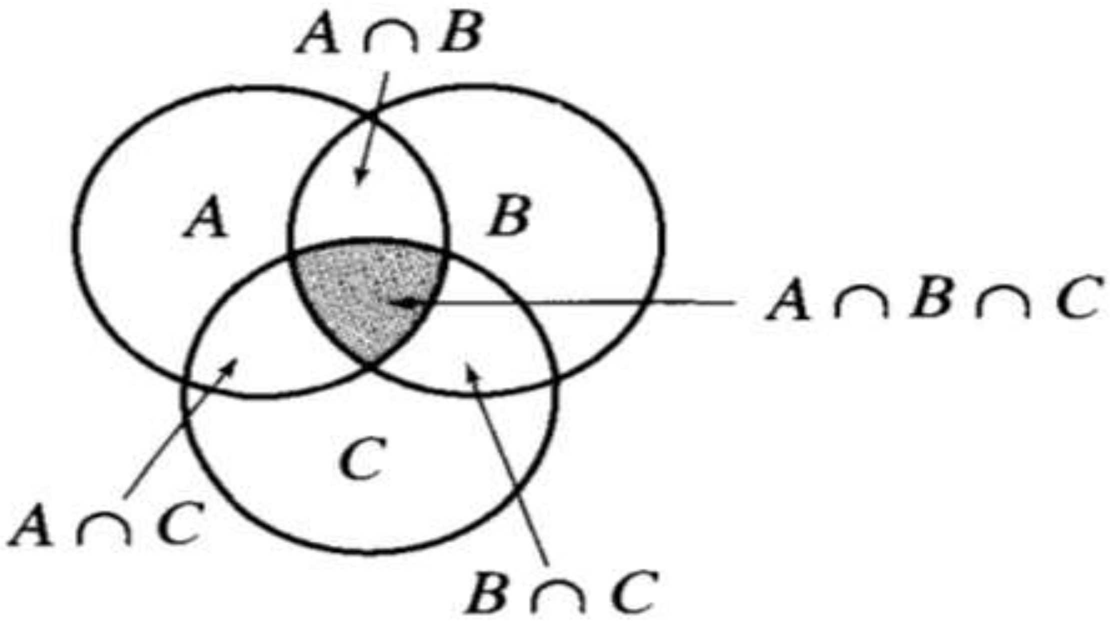
= 5 + 3 - 2 = 6



- If A and B are disjoint sets then:
  - ✓ A ∩B= ϕ
  - ✓ |A ∩ B|=0
  - ✓ |A ∪ B|=|A|+|B|

- If A, B and C are finite sets then

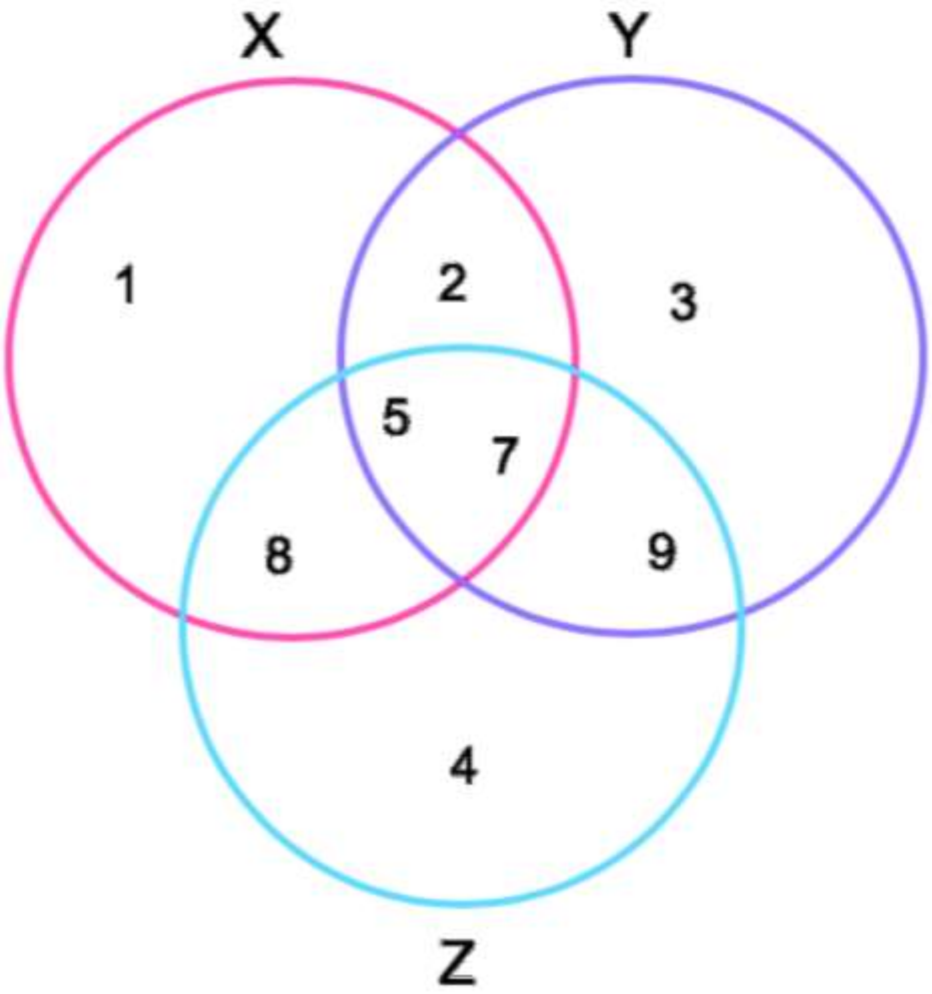
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Example

- $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |X \cap Z| + |X \cap Y \cap Z|$

  $= 5+5+5-3-3-3+2=8$



Set Identities (1/8)

TABLE Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Set Identities (2/8)

TABLE Set Identities.	
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities (3/8)

Example1

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Set Identities (4/8)

Example1 – Answer

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Set Identities (5/8)

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Set Identities (6/8)

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set Identities (7/8)

Example2

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Set Identities (8/8)

Example2 – Answer

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Matrices

- A is called square matrix if $[A]_{nxn}$
- **Zero matrix** is a matrix that has zero entries $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- A.B$\neq B.A$
- We can perform matrix multiplication when the number of the columns of the first matrix must be equal to the number of rows in the second matrix.
- Identity matrix(I)= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrices

- If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then A^T= $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Note that:

- $(A^T)^T$=A
- $(A+B)^T=A^T+B^T$
- $(AB)^T=B^T.A^T$
- The matrix is symmetric if A^T=A.

Symmetric

$$\boxed{A^T = A}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

Boolean matrix operations

- **Boolean matrix** is a matrix with entries zero or one.
- Let A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
- Compute A∨B and A∧B.

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Boolean matrix operations

- Boolean matrix means mxn matrix with entries zero or one.

- Let A = $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

- Compute Boolean product of A and B.

A⊙B=

$$\begin{bmatrix} (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) & (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) & (0\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) \\ (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 0) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Boolean product of two matrices

$$A \odot B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Problem 2. Let A be a 3×3 zero-one matrix. Let I be a 3×3 identity matrix. Show that $A \odot I = I \odot A = A$.

Solution. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then

$$A \odot I = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (a \wedge 1) \vee (b \wedge 0) \vee (c \wedge 0) & (a \wedge 0) \vee (b \wedge 1) \vee (c \wedge 0) & (a \wedge 0) \vee (b \wedge 0) \vee (c \wedge 1) \\ (d \wedge 1) \vee (e \wedge 0) \vee (f \wedge 0) & (d \wedge 0) \vee (e \wedge 1) \vee (f \wedge 0) & (d \wedge 0) \vee (e \wedge 0) \vee (f \wedge 1) \\ (g \wedge 1) \vee (h \wedge 0) \vee (i \wedge 0) & (g \wedge 0) \vee (h \wedge 1) \vee (i \wedge 0) & (g \wedge 0) \vee (h \wedge 0) \vee (i \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} a \vee 0 \vee 0 & 0 \vee b \vee 0 & 0 \vee 0 \vee c \\ d \vee 0 \vee 0 & 0 \vee e \vee 0 & 0 \vee 0 \vee f \\ g \vee 0 \vee 0 & 0 \vee h \vee 0 & 0 \vee 0 \vee i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A;$$

Q9Compute $A \vee B$, $A \wedge B$ and $A \odot B$ for the given matrices A and B

1. (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

A: (a) $A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$; $A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; $A \odot B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(b) $A \vee B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $A \wedge B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $A \odot B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

b. Suppose $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Find

(a) $A \wedge B$. (b) $A \vee B$. (c) $A \odot B$.

- **Q8:** a. Suppose $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$
- Find :
- (a) $A \wedge B$. (b) $A \vee B$. (c) $A \odot B$.

Sequence

- **Sequence** is a list of objects arranged in a definited order.
  - ✓**Finite sequence** 1, 3, 5, 7
  - ✓**Infinite sequence.** 3, 8, 13, 17, …

Sequence vs. set

- A **set** has no order and no duplicated elements.
- A **sequence** has a specific order and elements may be duplicated.
- 1, 2, 3, 2, 2, 3, 1 is a sequence but not a set.
- The sequence is 1, 2, 3, 2, 2, 3, 1 and 2, 3, 1, 2 is made of the set {1, 2, 3}.
- In strings: Sequences can be made up of characters. Ex. W, a, k, e, ,u, p
- The string Wake up is made of the sequence W, a, k, e, ,u, p and the set {w, a, k, e, , u, p}

Questions

In Exercises 1 through 4, give the set corresponding to the sequence.

1. 2, 1, 2, 1, 2, 1, 2, 1

2. 0, 2, 4, 6, 8, 10, . . .

3. *aabbccddee* $\cdots$ *zz*

4. *abbcccdddd*

1. {1,2}
2. {x|x is an even number}
3. {a,b,c,d,e,…,z}
4. {a,b,c,d}

Formula

Formula is used to describe a sequence:

1. Recursive formula:

- ❑Every recursive formula must include a starting place.
- ❑The next term is determined from a previous value.
- ✓Sequence: 3, 8, 13, 18, …

  recursive formula: $a_1=3$. $a_n=a_{n-1}+5$. $1<n<\infty$
- ✓Sequence: 1, 2, 3, 5, 8, …

  recursive formula: $a_1=1$. $a_2=2$. $a_n=a_{n-1}+a_{n-2}$. $1<n<\infty$

Formula

2. Explicit formula

- Describes a term using only its position number.

✓ Sequence: 2, 4, 6, 8

Explicit formula: $a_n=2n$. $1 \leq n \leq 4$

Recursive formula: $a_1=2$. $a_n=a_{n-1}+2$. $1 \leq n \leq 4$

✓ Sequence: 1, 3, 5. 7

Explicit formula: $a_n=2n-1$. $1 \leq n \leq 4$

Recursive formula: $a_1=1$. $a_n=a_{n-1}+2$. $1 \leq n \leq 4$

Questions

In Exercises 11 through 16, write a formula for the nth term of the sequence. Identify your formula as recursive or explicit.

11. $1, 3, 5, 7, \ldots$

12. $0, 3, 8, 15, 24, 35, \ldots$

13. $1, -1, 1, -1, 1, -1, \ldots$

14. $0, 2, 0, 2, 0, 2, \ldots$

15. $1, 4, 7, 10, 13, 16$

16. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

17. Write an explicit formula for the sequence 2, 5, 8, 11, 14, 17, ...

Answer

11. $a_n = a_{n-1} + 2, a_1 = 1$ recursive.

13. $c_n = (-1)^{n+1}$ explicit.

15. $e_n = e_{n-1} + 3, e_1 = 1$ recursive.

17. $a_n = 2 + 3(n - 1)$.

Examples

- 1, -1, 1,-1,1,…

Explicit formula: $a_n=(-1)^{n+1}$

Recursive formula: $a_1=1$, $a_n=-a_{n-1}$

- 1, 1/2, 1/3, 1/4, ….

Explicit formula: $a_n= 1/n$.

Recursive formula: $a_1=1$, $a_n=\left(\frac{1}{a_{n-1}}+1\right)^{-1}$

- 1,1/3,1/5,….

Explicit: $a_n=1/(2n-1)$

Recursive: $a_1=1$, $a_n=(a_{n-1}+2)^{-1}$. **or** $a_n=1/(a_{n-1}+2)$

Questions

- **5, 9, 13, 17, 21, …**

  $a_n=4n+1$

  $a_1=5,\ a_n=a_{n-1}+4$

- **3, 3, 3, 3, …**

  $a_1=3,\ a_n=a_{n-1}$

- **15, 20, 25, 30, 35, …**

  $a_n=5(n+2)$

- **1, 0.9, 0.8, 0.7, 0.6, …**

  $a_n = 1 - \frac{n-1}{10}$

Questions

- **1, 1/3, 1/5, 1/7, 1/9, …**
  - $a_n=1/(2n-1)$
  - $a_1=1,\ a_n=1/(a_{n-1}+2)$
- **2, 0, 2, 0, …**
  - $a_n=1+(-1)^{n+1}$

Representation of set on computer (Characteristic function)

- A **set (A)** is always defined in a computer program with respect to an underlying universal set (U).
- The set A is represented in a computer as a string of binary digits $b_1 b_2 \ldots b_n$ where n is the cardinality of the universal set.

  $b_i = 1$ if i^{th} element of U is in A
  $b_i = 0$ if i^{th} element of U is not in A

- For example, if U = {1, 2, 3,4, 5, 6, 7, 8, 9, 10} then the representation of M = {1, 2, 5, 8} is given by string 1100100100.

Characteristic function

The characteristic function of set A can be defined as:

$$F_A = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

- **Note that:** $F_{A \cap B} = F_{A.}\ F_B$

  $F_{A \cup B} = F_A + F_B - F_{A.}\ F_B$

  $F_{A \oplus B} = F_A + F_B - 2F_A.F_B$

- It represents a set in a computer, the elements of the set must be arranged in a sequence.

Sequence

- Let U={1, 2, 3, 4, 5, 6}

  A={1, 2}

  B={2, 4, 6}

  C={4, 5, 6}

- F_A=1,1,0,0,0,0, F_B=0,1,0,1,0,1, F_C=0,0,0,1,1,1
- $F_{A\cap B}= F_A . F_B$=0,1,0,0,0,0
- $F_{A\cup B} = F_A + F_B - F_A.F_B$= 1,2,0,1,0,1 - 0,1,0,0,0,0= 1,1,0,1,0,1
- $F_{A\oplus B}= F_A + F_B - 2F_A.F_B$= 1,2,0,1,0,1- 2(0,1,0,0,0,0)

  = 1,2,0,1,0,1- 0,2,0,0,0,0 =1,0,0,1,0,1
- Another method: $F_{A\oplus B}$=(A-B)100000 ∪ (B-A)000101=100101