Mathematics CS1003

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Finding Inverses Using a Matrix of Cofactors I

Recall that:

Definition:

A square matrix A of order n is said to be **invertible** iff there exists an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n$$

A^{-1} is called the **inverse** of A.

Remarks:

- Not all $n \times n$ matrices have inverses.
- The inverse, if it exists, is unique.

Finding Inverses Using a Matrix of Cofactors II

Properties of Inverse Matrices

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$ (the inverse of a product is the product of the inverses in reverse order.)
- $(A^T)^{-1} = (A^{-1})^T$
- $\det A^{-1} = \dfrac{1}{\det A}$

Finding Inverses Using a Matrix of Cofactors III

We know how to find the inverse of a 2×2 matrix

Definition:

Let A be a square matrix of order 2. then A^{-1} is defined as

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

We now want to generalise this definition to square matrices of arbitrary order. To do this we need to the concept of a Minor (which we have met before) and a new, concept, the matrix of Cofactors.

Finding Inverses Using a Matrix of Cofactors IV

Definition:

The minor M_{ij} of a $(n \times n)$ matrix A is the determinant of the submatrix of A formed by deleting the $i-th$ row and $j-th$ column of A.

Definition:

The matrix of cofactors of A is the square matrix of order n, $\tilde{A}$, such that

$$(\tilde{A})_{ij} = \tilde{a}_{ij} = (-1)^{i+j} M_{i,j}$$

Finding Inverses Using a Matrix of Cofactors V

Rather than working out $(-1)^{i+j}$ each time, it can be easier to work with a matrix of signs of the appropriate dimensions e.g. for a 3×3 matrix we use the following matrix of signs:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

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In simple terms, to form the matrix of cofactors (which we call $\tilde{A}$), you need to replace each element in matrix A by its corresponding minor multiplied by the corresponding element from the matrix of signs.

Definition:

Let A be a square matrix of order n. If A is invertible, A^{-1} is defined as

$$A^{-1} = \frac{1}{\det A} \left(\tilde{A}\right)^T$$

Finding Inverses Using a Matrix of Cofactors VII

Example:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -3 \\ 4 & -2 & 0 \end{pmatrix}$$

$$M_{1,1} = \det \begin{pmatrix} 5 & -3 \\ -2 & 0 \end{pmatrix} \text{ so } M_{1,1} = -6$$

$$M_{1,2} = \det \begin{pmatrix} 0 & -3 \\ 4 & 0 \end{pmatrix} = 12$$

$$M_{1,3} = \det \begin{pmatrix} 0 & 5 \\ 4 & -2 \end{pmatrix} = -20$$

etc.

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Thus

$$\hat{A} = \begin{pmatrix} -6 & -12 & -20 \\ 2 & 4 & 10 \\ -1 & 3 & 5 \end{pmatrix}$$

The determinant of A is:

$$\begin{aligned} \det A &= (-1)^{1+1}(1)\begin{vmatrix} 5 & -3 \\ -2 & 0 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} 0 & -3 \\ 4 & 0 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} 0 & 5 \\ 4 & -2 \end{vmatrix} \\ &= 1(-6) - 2(12) - 1(-20) \\ &= -10 \end{aligned}$$

We get

$$A^{-1} = \frac{1}{-10}\begin{pmatrix} -6 & -12 & -20 \\ 2 & 4 & 10 \\ -1 & 3 & 5 \end{pmatrix}^T = \frac{1}{-10}\begin{pmatrix} -6 & 2 & -1 \\ -12 & 4 & 3 \\ -20 & 10 & 5 \end{pmatrix}$$