CS1003 Tutorial Sheet – Linear Systems, Matrices and Eigenvalues

I. Matrices

1. If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 1 \\ 2 & 8 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix},$$

then show that

(i) $(AB)C = A(BC)$,

(ii) $A(B + C) = AB + AC$.

2. If

$$A = \begin{pmatrix} 4 & -10 \\ -2 & 6 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix},$$

show that

$$AB = BA = I$$

where I is the 2×2 **identity matrix**.

3. If

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 2 & 1 & 1 \end{pmatrix},$$

(i) Calculate the matrix products AB and BA.

(ii) Is $AB = BA$? If not give a reason why not.

4. Find a, b, a_1 and b_1 if

(i)

$$\begin{pmatrix} a & b \\ a_1 & b_1 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ a_1 & b_1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ -1 & 4 \end{pmatrix}$$

II. Solving Linear Systems

1. In the following question, find the unique solution of the given linear system:

(i)

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -2x_2 - x_1 + 3x_3 &= 1 \\ 4x_3 - 7x_2 + 3x_1 &= 10. \end{aligned}$$

(ii)

$$\begin{aligned} 2x + 4y - 6z &= 14 \\ x - y + 2z &= 0 \\ y - 2z &= 2. \end{aligned}$$

(iii)

$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= 3a - 2c \\ -x_1 + x_3 &= -c \\ x_1 + 2x_2 + 2x_3 &= 7a + b, \end{aligned}$$

where a, b, c are real constants.

III. Linear Systems with Infinitely many or no solutions

1. In the following question, solve the given linear system (provided that a solution exists):

(i)

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= 5. \end{aligned}$$

(ii)

$$\begin{aligned} x + y + 3z - w &= 1 \\ 2x - y + z + w &= 2 \\ -x + 4y + 3z + 2w &= 4. \end{aligned}$$

(iii)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 10 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 + 2x_5 &= 9 \\ 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 - x_2 - x_3 + x_4 + x_5 &= 5. \end{aligned}$$

2. Recall that a linear system is **consistent** if it is **not inconsistent**. Now, find the conditions that b_1, b_2, b_3, b_4 must satisfy for the following system to be consistent.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= b_1 \\ x_1 + 5x_2 + x_3 - 2x_4 &= b_2 \\ -x_1 + 2x_2 + 2x_3 - 3x_4 &= b_3 \\ 3x_1 + x_2 - 3x_3 + 4x_4 &= b_4. \end{aligned}$$

3. (i) For what value of k does the following linear system have **no solution**?

$$\begin{aligned} -2x_3 + 7x_5 &= 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 + kx_5 &= -1. \end{aligned}$$

(ii) If $k = -5$, solve the system.

4. A system of three linear equations in three unknowns has been reduced to the form shown below

$$\begin{pmatrix} 1 & 3 & 4 & -2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & t^2 - t - 6 & t + 2 \end{pmatrix}$$

Give a value of t (one value in each case) for which the system of equations has:

(a) no solution;

(b) an infinite number of solutions;

(c) a unique solution;

5. Find a value of c for which the following system of equations is consistent:

$$\begin{aligned} x - y + 2z &= 1 \\ 2x + 3y + 4z &= 7 \\ 4x - 7y + 5z &= c \\ 8x - 4y + 6z &= 2 \end{aligned}$$

Find the general solution of the system when c has this particular value.

6. Solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$

(a) Using Gaussian Elimination

(b) Calculating the appropriate inverse matrix.

IV. Matrix Inverses

1. Find the **inverses** of each of the following matrices:

(i)

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

(ii)

$$B = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}.$$

(iii)

$$C = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 6 & 4 \\ 3 & 1 & 4 \end{pmatrix}.$$

V. Inverses, Transposes and Determinants

1. Find the **inverse** of each of the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & a & 1 & 0 \\ a^3 & a^2 & a & 1 \end{pmatrix} \quad \text{for } a \neq 0, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 7 & 6 \end{pmatrix}.$$

2. Using the answer from question 1 above, find the solution to the linear system

$$\begin{aligned} y + 2z &= 1 \\ 2x + 4y + 3z &= 2 \\ 3x + 7y + 6z &= 0. \end{aligned}$$

3. Calculate the **determinant** of each of the following matrices:

$$A = \begin{pmatrix} -2 & 1 & -4 \\ 1 & 1 & 2 \\ 3 & 1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 6 & 6 \end{pmatrix}.$$

Which of these matrices is **invertible**?

4. If

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ -3 & 2 \\ 5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix},$$

show that

$$(ABC)^T = C^T B^T A^T.$$

Note: This result is true for any three matrices in general, and can be extended to the transpose of the product of arbitrarily many matrices.

5.

(i) If A, B, C are invertible, show that

$$(ABC)C^{-1}B^{-1}A^{-1} = I,$$

and hence that

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

(ii) If A and B are invertible matrices, simplify

$$\left(A^{-1}B\right)^{-1}BA(ABA)^{-1}BA(BA^2)^{-1}.$$

6. Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = (1-r)^3.$$

7. Show that

$$A = \begin{pmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{pmatrix}$$

has an inverse if and only if $abc \neq 0$ and find A^{-1} in that case.

VI. Eigenvalues and Eigenvectors

1. Find the **eigenvalues** and **eigenvectors** of each of the following matrices:

(i)

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

(ii)

$$B = \begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}.$$

(iii)

$$C = \begin{pmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}.$$

(iv)

$$D = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}.$$

(v)

$$E = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

(vi)

$$F = \begin{pmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$