CS1003

TAYLOR POLYNOMIALS

INTRODUCTION

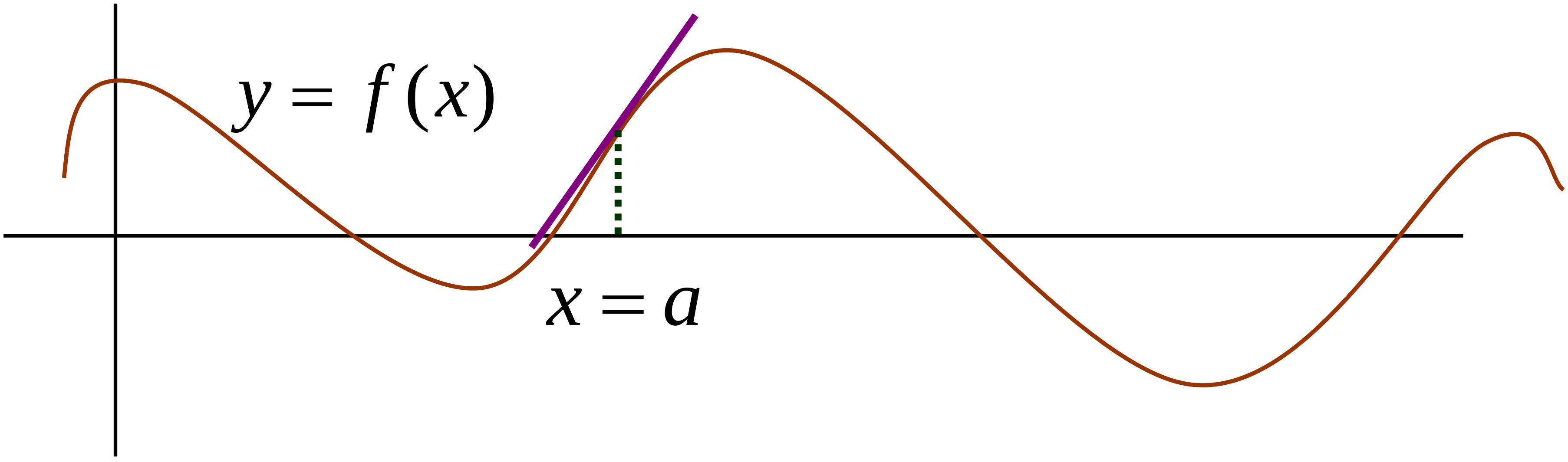
If We Could…

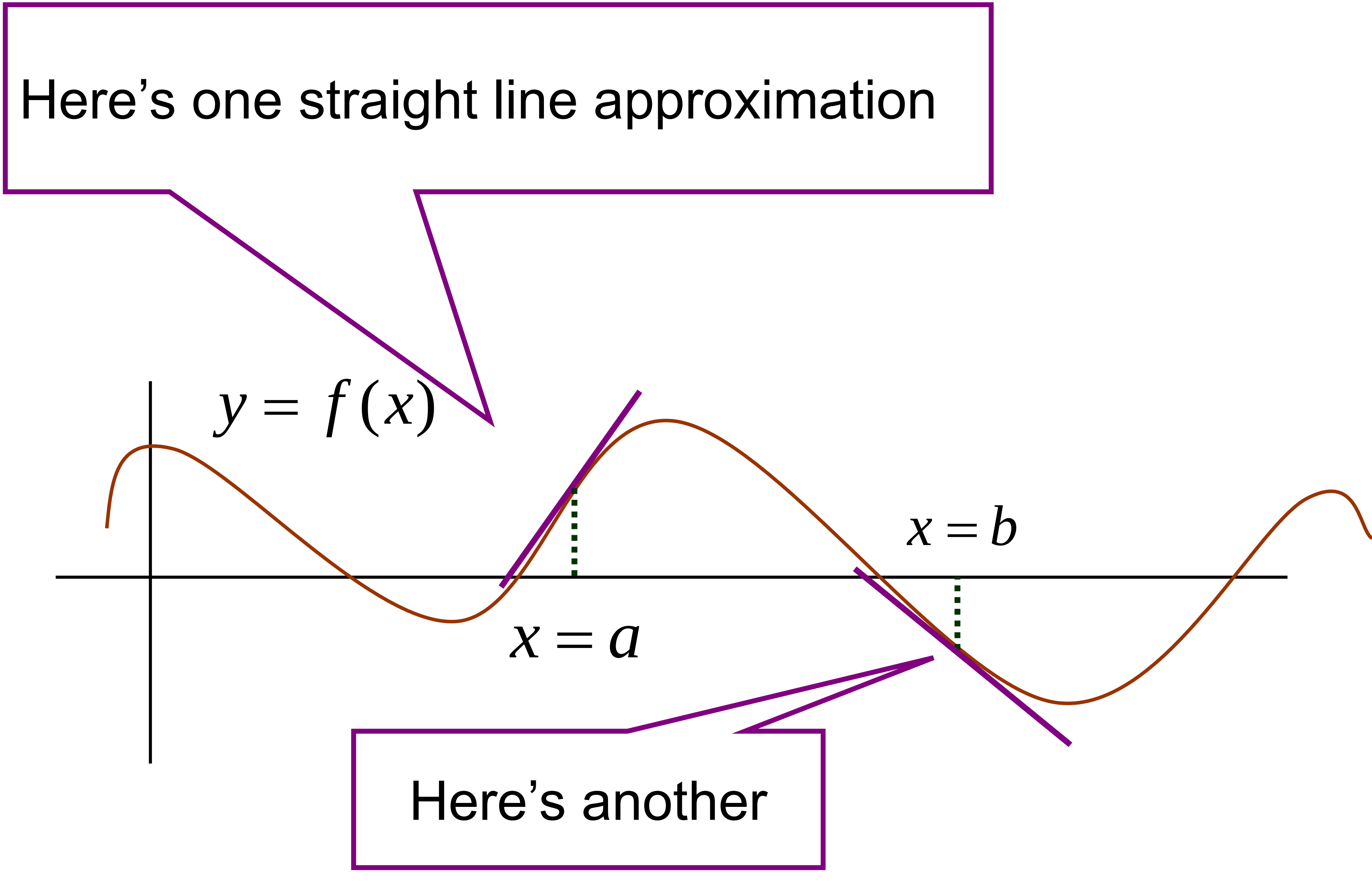
….approximate functions with polynomials, where would we begin?

Begin with the simplest polynomial.
The simplest polynomial is ***linear***.

Begin With a Line

What's the best straight line approximation
to a function at a point?




Here’s one straight line approximation
$y = f(x)$
$x = b$
$x = a$
Here’s another

Linear Approximation at a Point

What is the equation for the tangent to the graph of

$$f(x) \text{ at } x = a$$

First differentiate

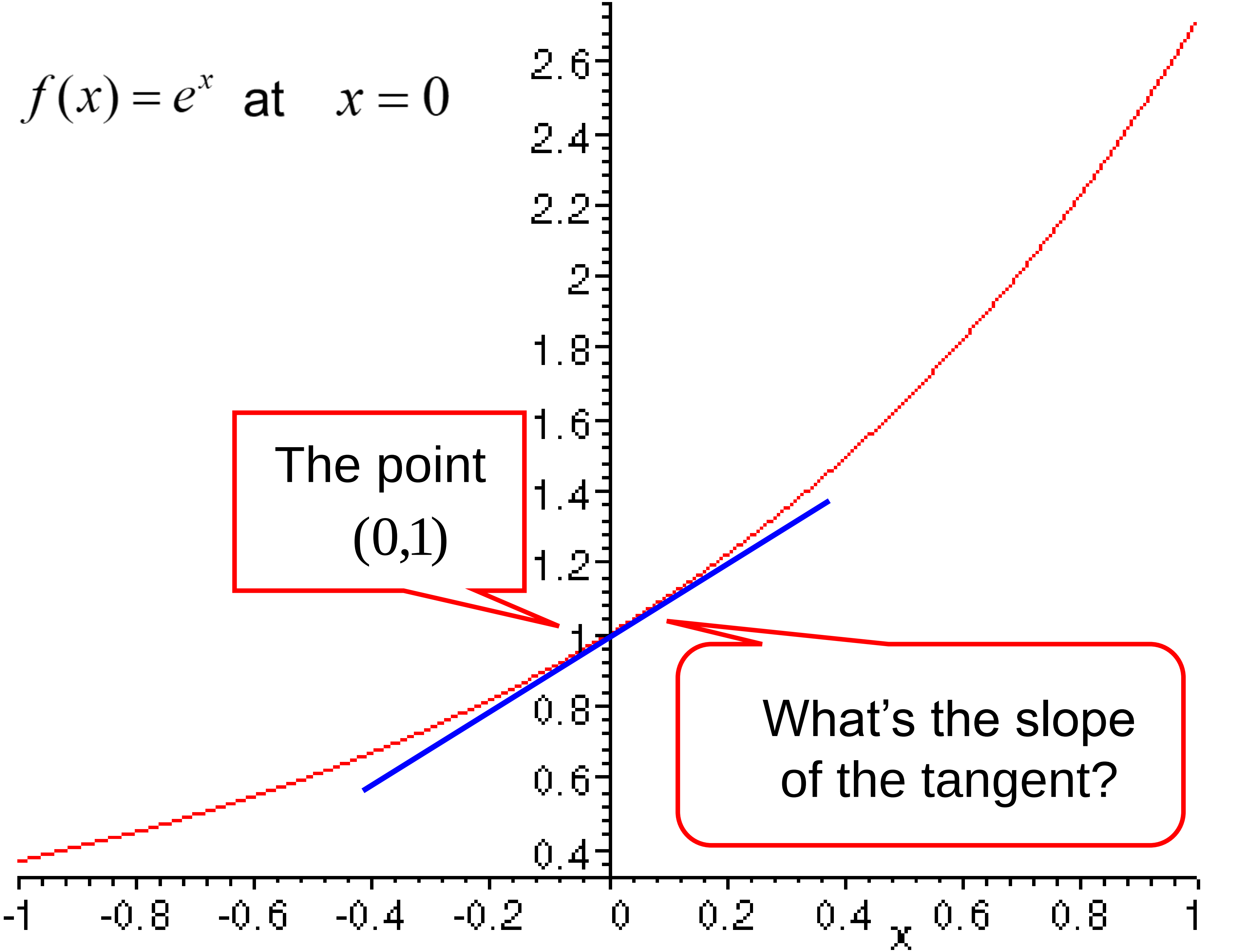
Then evaluate the derivative at the point -- that's the ***slope*** of the line, *m*

Remember a ***point on the line*** $y = mx + c$ is the point on the curve $f(x)$ at $x = a$

An Example

The best straight line approximation to

$f(x) = e^x$ at $x = 0$


$f(x) = e^x$ at $x = 0$
The point (0,1)
What's the slope of the tangent?
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
x
0.6
0.8
1

The slope of $f(x) = e^x$ at $x = 0$

The slope of the tangent is the slope of the curve.

Since $f'(x) = e^x$ and $f'(0) = e^0 = 1$

the slope of the tangent is 1.

The Equation for the Line

$f(x) = e^x$ at $x = 0$

The slope of the line is 1.

The point (0, 1) is on the line.

$y = x + c$

so $1 = 0 + c$ so that $c = 1$

thus $y = x + 1$

The General Procedure

The best straight line approximation to a differentiable function *f(x)* at a point *x = a is*

$$y = f'(a)(x-a) + f(a)$$

Example

$$f(x) = \sin x \qquad f'(x) = \cos(x)$$

Find the linear Taylor polynomial for $f(x)$ about x $= \pi/3$

$$y = f'(a)(x-a) + f(a)$$

$$y = f'(\pi/3)(x-\pi/3) + f(\pi/3)$$

$$y = \frac{1}{2}(x-\pi/3) + \frac{\sqrt{3}}{2}$$

The Tangent Line

This is the only line with the same first derivative as the function, passing through the designated point.

The second derivative will give us our degree two approximation.

The only parabola with the same ***first and second*** derivatives as the function, passing through the designated point.

Example

For $f(x) = e^x$ at $x = 0$

find the parabola $p(x) = ax^2 + bx + c$ with

$$p(0) = f(0) = e^0 = 1$$

$$p'(0) = f'(0) = e^0 = 1$$

$$p''(0) = f''(0) = e^0 = 1$$

How?

$$f(x) = e^x \quad x = 0$$

$$p(0) = f(0) = e^0 = 1$$

$$p'(0) = f'(0) = e^0 = 1$$

$$p''(0) = f''(0) = e^0 = 1$$

Use $p(x) = ax^2 + bx + c$

$$p'(x) = 2ax + b$$

$$p''(x) = 2a$$

This gives us

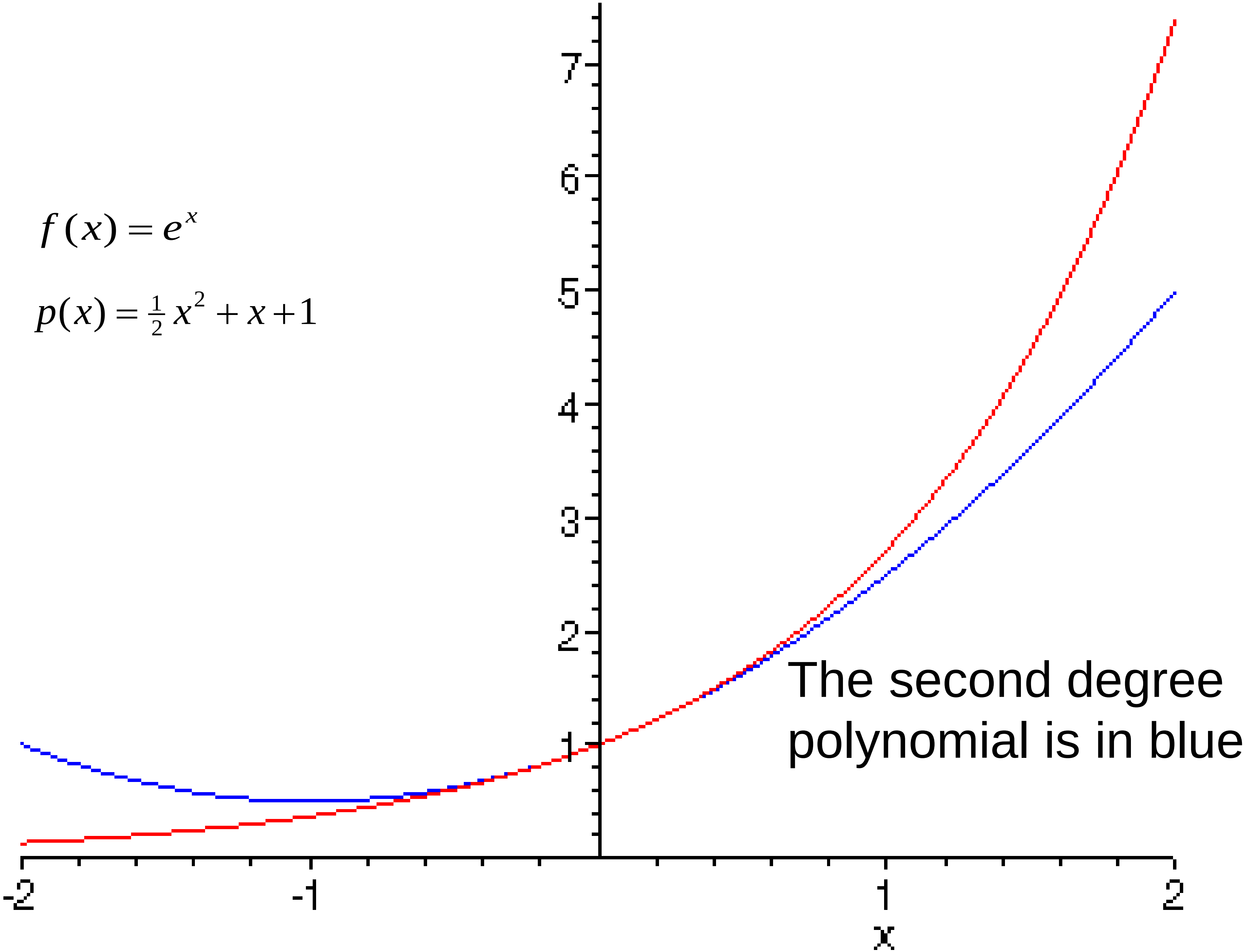
$$p(0) = a \times 0^2 + b \times 0 + c = c \quad = \quad f(0) = 1$$

$$p'(0) = 2a \times 0 + b = b \quad = \quad f'(0) = 1$$

$p''(0) = 2a = f''(0) = 1$ so that $a = 1/2$

Thus $p(x) = \frac{1}{2}x^2 + x + 1$

is the best second degree polynomial approximation to $f(x) = e^x$ at $x = 0$


$f(x) = e^x$
$p(x) = \frac{1}{2}x^2 + x + 1$
The second degree
polynomial is in blue
7
6
5
4
3
2
1
-2
-1
1
2
x

To get better approximations, use higher degree polynomials.

The Taylor polynomial of degree n

for a function $f(x)$ which is

n times differentiable at a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(If we take the approximation about 0, then a = 0)

The Taylor polynomials are

the polynomials with ***the same value***

and ***the same*** first, second,... , *n*th ***derivatives***

as the given function, at the given point.

The derivatives of a function determine its contours.

Functions with the same derivatives have the same shape

Find the Taylor polynomial about zero of degree 4 for $h(x) = \ln\sqrt{3+x}$

Re-write and differentiating repeatedly

Evaluate at $x = 0$

$$h(x) = \frac{1}{2}\ln(3+x) \qquad h(0) = \frac{1}{2}\ln 3$$

$$h'(x) = \frac{1}{2}\left(\frac{1}{3+x}\right) = \frac{1}{2}(3+x)^{-1} \qquad h'(0) = \frac{1}{6}$$

$$h''(x) = \frac{-1}{2}(3+x)^{-2} \qquad h''(0) = \frac{-1}{18}$$

$$h^{(3)}(x) = (3+x)^{-3} \qquad h^{(3)}(0) = \frac{1}{27}$$

$$h^{(4)}(x) = -3(3+x)^{-4} \qquad h^{(4)}(0) = \frac{-1}{27}$$

Hence,

$$p_4(x) = \frac{1}{2}\ln 3 + \frac{1}{6}x + \frac{-1}{18}\frac{x^2}{2!} + \frac{1}{27}\frac{x^3}{3!} + \frac{-1}{27}\frac{x^4}{4!}$$

$$p_4(x) = \ln\sqrt{3} + \frac{x}{6} - \frac{x^2}{36} + \frac{x^3}{162} - \frac{x^4}{648}$$