Mathematics CS1003

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Michaelmas Term

Linear Algebra – Some Basics I

Definition:

A linear function f is a mathematical function in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction. A linear equation is of the form $f(x, y, \cdots) = 0$ with f linear.

Tell me if those following functions or equations are linear. If not, do you know what they are called ?

A.

1. $f(x) = 3x - 4$
2. $f(x) = 3x^2 - 4$
3. $f(x) = \frac{3}{x} - 4$
4. $f(x, y) = 3x + y - 2$

B.

1. $f(x, y) = 3xy$
2. $x^4 + 3x^2 - 4 = 0$
3. $\frac{3}{x} - 4 = 0$
4. $3x + y - 2 = 0$

Linear Algebra – Some Basics II

Definition:

The root (or zero) of a function f is a value that makes the function equal to zero.

In geometric terms, a root of a function is where the graph of the function crosses the x-axis.

Definition:

An equation is written in the form:

$$f(x) = 0$$

A solution to an equation is a value for x that make the equation true.

For example:

The equation $3x - 4 = 0$ has solution $x = \frac{4}{3}$.

Linear Algebra – Some Basics III

Solving Linear equations:

- Find x such that: $3x = 4$.

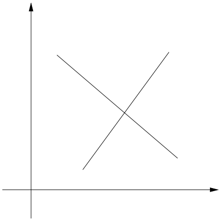
- Find (x, y) such that:

$$\begin{cases} x - 2y - 3 = 0 \\ 3x + y = -1 \end{cases}$$

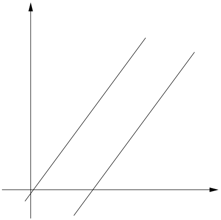
Linear Systems in Two Unknowns I

- One equation: $ax + by = c$ represents a line in $\mathbb{R}^2$. There are infinitely many solutions of this equation (one for each point on the line). (Here a, b and c are real numbers, and a and b are not both zero).
- Two equations: Solutions of linear equations in two variables correspond to intersection points of lines.

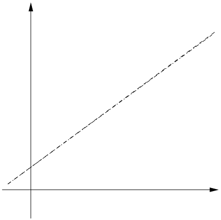
Three ways in which lines can intersect:



Intersect at unique point



Parallel lines - no intersection point



Lines coincide

Linear Systems in Two Unknowns II

- Three possibilities for solutions of a Linear System in two unknowns:
  - (1) A **unique** solution
  - (0) **No** solution
  - (∞) **Infinitely many** solutions

  We call a linear system with no solutions **INCONSISTENT** or **INCOMPATIBLE**.

  We call a linear system with infinitely many solutions **INDETERMINATE**.

- Solve:
  1. $$\begin{cases} x + y = 1 \\ 2x - y = 2 \end{cases}$$
  2. $$\begin{cases} 2x - y = -1 \\ 2x - y = 0 \end{cases}$$
  3. $$\begin{cases} y - x = 2 \\ 3y - 3x = 6 \end{cases}$$

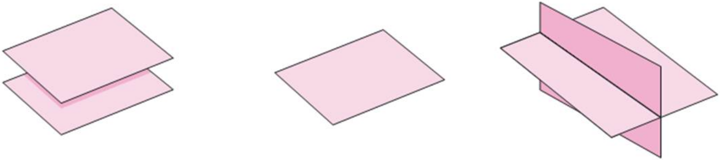
Linear Systems in Two Unknowns III

Simultaneous linear equations in two unknowns are solved with the following steps:

1. Write both equations in the form $ax + by = k$
2. Make the coefficients of one of the variables the same in both equations.
3. Add or substract (depending on signs) on equation from the other to form a new equation in one variable.
4. Solve the new equation obtained in step 3.
5. Put the value obtained in step 4 into one of the given equations to find the corresponding value of the other variables.

Linear Systems in Three Unknowns I

- One equation: $ax + by + cz = d$ represents a line in $\mathbb{R}^3$. There are infinitely many solutions of this equation (one for each point in the plane). (Here a, b, $c$and d are real numbers, and a, b and c are not both zero).
- Two equations: Solutions of two simultaneous linear equations in three unknown variables correspond to intersection points of two planes in $\mathbb{R}^3$.

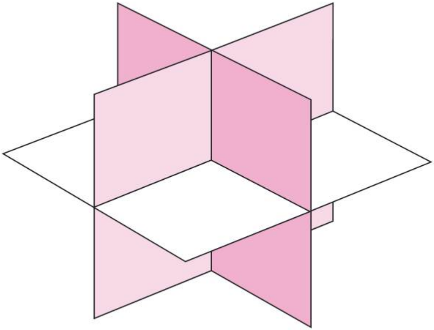


The first image above corresponds to no solution while the second two images correspond to an infinite number of solutions (where the planes either coincide (lie on top of each other) or the planes intersect in a line.

Linear Systems in Three Unknowns II

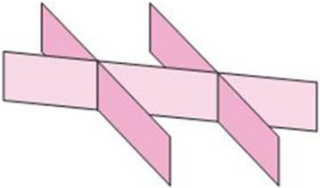
Three equations:

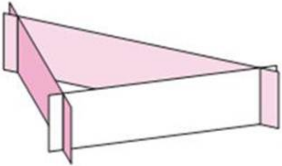
- A unique solution when the three planes meet at one point.



Linear Systems in Three Unknowns III

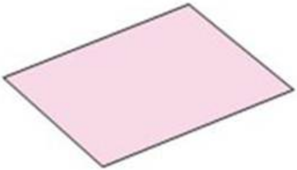
- No solution — when the planes are parallel or when the three planes form a triangular prism.

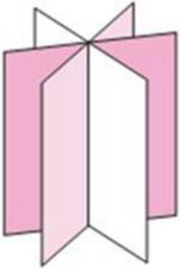




Linear Systems in Three Unknowns IV

- Infinitely many solutions, when the planes intersect either in a plane or in a line.





Matrices – Some Terminology I

Definition:

A matrix is a rectangular table of (real or complex) numbers. The order of a matrix gives the number of rows and columns it has. A matrix with m rows and n columns has order $m \times n$.

Write down the order of the following matrices:

1. $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$
3. $C = \begin{pmatrix} -5 & 8 \end{pmatrix}$

Matrices – Some Terminology II

Definition:

A square matrix has the same number of rows and columns (order of the form $(m \times m)$). For instance, $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is not square, but $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$ is.

Definition:

The transpose of a matrix M, written M^T is obtained by interchanging the rows and columns. For example,

$$\textit{If } M = \begin{pmatrix} -1 & 5 \\ 3 & -2 \end{pmatrix}, \textit{ then } M^T = \begin{pmatrix} -1 & 3 \\ 5 & -2 \end{pmatrix}$$

Matrices – Some Terminology III

Compute

1. $(M^T)^T$ with $M = \begin{pmatrix} -1 & 5 \\ 3 & -2 \end{pmatrix}$
2. M^T with

$$M = \begin{pmatrix} 8 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

Matrix Addition I

Definitions:

- Matrices can only be added, or substracted, if they are of the same order. Simply add or substract corresponding elements. This gives a matrix of the same order.
- To multiply a matrix by a scalar (a number), multiply each element of the matrix by the number.

Example

$$2\begin{pmatrix} 8 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 4 & 8 \\ 6 & 2 & 4 \end{pmatrix}.$$

Matrix Addition II

Example

$$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 & 1 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 0 \\ 6 & 1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 4 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}$$ is **not** defined.

With $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} -5 & 8 \end{pmatrix}$, compute (if you can):

1. $A + B$
2. $A + C^T$
3. $3B$

Matrix Addition III

Definitions:

- Matrix addition is commutative. Let A and B be two matrices of the same order then:

$$A + B = B + A$$

- Matrix addition is associative. Let A, B and C be three matrices of the same order then:

$$A + (B + C) = (A + B) + C$$

Matrix Multiplication I

Multiply each row of the first matrix by each column of the second matrix.

Two matrices can only be multiplied if the number of columns in the first matrix equals the number of rows in the second.

Memory aid: **Row by Column**

Example

$$\begin{pmatrix} 1 & 1 \\ \boxed{0 \quad 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} & 2 \\ 5 & 7 & & 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & \boxed{9} & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

The entry "9" in the new matrix was found by calculating $(0 \times 8) + (1 \times 9)$ to get $0 + 9 = 9$.

Matrix Multiplication II

Let

$$A = \begin{pmatrix} 2 & 1 & -5 \\ 2 & 8 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -8 & 4 & 0 \\ 6 & 5 & 1 & 2 \\ 2 & 9 & 0 & 0 \end{pmatrix}$$

1. Find AB and BA, if possible.
2. Is the multiplication of matrices commutative?

The Determinant of a Matrix I

Definition:

The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc = \det(A)$.

If $\det(A) = 0$, A is called a singular matrix.

Compute the determinant of the matrices:

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
2. $B = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$

The Identity Matrix & Inverse Matrices I

Definition:

A square matrix with 1s on the main diagonal and 0s elsewhere is called the identity matrix and denoted by I.

Note the main diagonal runs from the top left corner to the bottom right.

Definition:

If A is a square matrix, and there exists a matrix B so that

$$AB = BA = I,$$

we say that A is invertible, and B is called an inverse of A. It can be shown that an invertible matrix has exactly one inverse, so we refer to the inverse of an invertible matrix.

Show that if A is not square, it cannot have an inverse.

The Identity Matrix & Inverse Matrices II

- If A is invertible, we denote the inverse of A by A^{-1}, so

$$A^{-1}A = AA^{-1} = I.$$

- A^{-1} plays the rôle of the reciprocal or multiplicative inverse in ordinary multiplication, since

$$aa^{-1} = a^{-1}a = 1,$$

  for non–zero $a \in \mathbb{R}$.

The Identity Matrix & Inverse Matrices III

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1. Exchange the elements of the main diagonal.
2. Change the signs of the elements of the other diagonal.
3. Multiply by $\frac{1}{\det(A)}$.

Prove the above result (i.e. compute AA^{-1} and $A^{-1}A$, and check they are equal to I).

Working with Matrices I

We can:

- Add, subtract and multiply matrices.
- Form an identity matrix and a zero matrix (i.e. a matrix where all entries are 0).
- Find an inverse of a matrix using the determinant.

So we have all the tools we need to carry out calculations using matrices:

Let $M = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 7 & 10 \\ 21 & 23 \end{pmatrix}$.

1. Compute $M^{-1}A$.

2. If $MB = 2M + A$, express B in matrix form.

Working with Matrices II

If $M = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$,

1. Compute $(MN)^{-1}$.
2. Compute $N^{-1}M^{-1}$
3. Any comment on $(MN)^{-1}$ and $N^{-1}M^{-1}$?

Working with Matrices III

Let's come back to solving a system of linear equations:

$$\begin{cases} x - 2y - 3 = 0 \\ 3x + y = -1 \end{cases}$$

Solve this system by matrix methods.

Working with Matrices IV

Definitions:

A vector (or column vector) is a matrix with a single column. A matrix with a single row is a row vector. The entries of a vector are its components.

Solving simultaneous equations of the form:

$$A\mathbf{x} = \mathbf{b}$$

with A a square matrix and $\mathbf{x}$ unknown vector and $\mathbf{b}$ vector of constants, is given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

when A is invertible.

Working with Matrices V

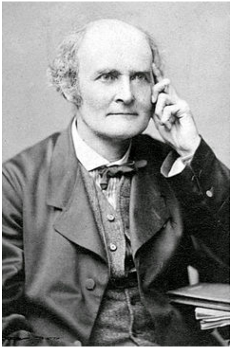
Solve

$$\begin{cases} 2x + 3y = 4 \\ 10x + 4y = 9 \end{cases}$$

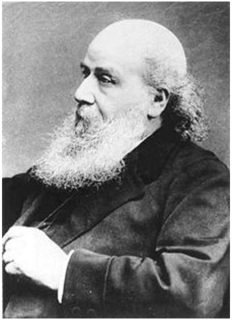
Working with Matrices VI

How old are matrices?

- Matrices were invented by the British mathematician Arthur Cayley (1821-1895).
- Cayley presented a paper giving the rule for matrix operations and the conditions under which a matrix has an inverse to the Royal Society in 1858.
- Cayley's friend, James Joseph Sylvester (1814-1897), was the person who first used the term "matrix" in 1850.



Arthur Cayley (1821-1895)



James Joseph Sylvester (1814-1897)