Mathematics CS1003

Dr. Meriel Huggard

Room 1.10 Lloyd Institute
School of Computer Science and Statistics
Trinity College Dublin, IRELAND
http://mymodule.tcd.ie
Meriel.Huggard@tcd.ie

Michaelmas Term

Determinants. I

Definition:

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant of A is defined as:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

- How to compute the determinant of $(n \times n)$ matrices ?

Determinants. II

Informal Definition:

The minor M_{ij} of a $(n \times n)$ matrix A is the determinant of the submatrix of A formed by deleting the $i-th$ row and $j-th$ column of A.

Formal Definition:

The minor M_{ij} of a $(n \times n)$ matrix A is the determinant of the submatrix of A formed by deleting the $i-th$ row and $j-th$ column of A:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{pmatrix} \quad M_{ij} = \begin{vmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

Determinants. III

The determinant of A is then defined as:

$$\begin{aligned} \det(A) &= a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+n}a_{1n}M_{1n} \\ &= \textstyle\sum_{j=1}^{n}(-1)^{1+j}a_{1j}M_{1j} \end{aligned}$$

and more generally:

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

or

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

Determinants. IV

Example of a 3×3 matrix:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Since you know how to compute a determinant for a (2×2) matrix, we have

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinants. V

Compute the determinant of

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 3 & 6 & 5 \\ -4 & 0 & 7 \end{pmatrix}.$$

Determinants. VI

The transpose of a matrix A is the matrix given by exchanging rows and columns of A. We denote the transpose of A by A^T.

Properties:

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $\det(A) = \det(A^T)$
4. If $\det(A) \neq 0$ then A is invertible.

Show that:

1. $(ABC)^T = C^T B^T A^T$
2. $(ABCD)^T = D^T C^T B^T A^T$

Determinants. VII

You have a similar property for invertible matrices:

Properties:

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$

Show that:

1. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
2. $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$