MAT120: Integral Calculus and Differential Equations
BRAC University
Assignment 2

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1 Evaluate the following indefinite integral by using a trigonometric substitution or otherwise

$$\int \frac{8}{4+9x^2}\,dx$$

$$Let,$$

$$u = \frac{3x}{2} \qquad \therefore\ x = \frac{2u}{3}$$

$$\Rightarrow \frac{du}{dx} = \frac{3}{2}$$

$$\therefore dx = \frac{2}{3}du$$

$$Now,$$

$$\begin{aligned}
\int \frac{8}{4+9x^2} &= \int \frac{8}{4+9x^2}\,\frac{2}{3}du \\
&= \int \frac{16}{3(4+9x^2)}du \\
&= \int \frac{16}{3(4+9\ \frac{4u^2}{9})}du \\
&= \int \frac{16}{3(4+4u^2)}du \\
&= \int \frac{16}{12(1+u^2)}du \\
&= \frac{16}{12}\int \frac{1}{1+u^2}du \\
&= \frac{4}{3}\,tan^{-1}(u) + c && \left[\because\ \int \frac{1}{1+x^2} = tan^{-1}(x) + c\right] \\
&= \frac{4}{3}\,tan^{-1}\left(\frac{3x}{2}\right) + c && \left[\because u = \frac{3x}{2}\right]
\end{aligned}$$

[ANSWER]

2 Evaluate the following indefinite integral by using a trigonometric substitution or otherwise

$$\int \frac{1}{\sqrt{4-9x^2}}dx$$

Solution

$$\begin{aligned}
&Let, \\
&x = \frac{3u}{2} \qquad\qquad \therefore\ u = \frac{3w}{2} \\
&\therefore dx = \frac{2}{3}du
\end{aligned}$$

$$Now,$$

$$\begin{aligned}
\int \frac{1}{\sqrt{2-9x^2}}dx &= \int \frac{1}{\sqrt{4-9\left(\frac{2}{3}u\right)^2}}\ \frac{2}{3}du \\
&= \frac{2}{3}\int \frac{1}{\sqrt{4-9\ \frac{4}{9}u^2}}du \\
&= \frac{2}{3}\int \frac{1}{\sqrt{4-4u^2}}du \\
&= \frac{2}{3}\int \frac{1}{2\sqrt{1-u^2}}du \\
&= \frac{2}{6}\int \frac{1}{\sqrt{1-u^2}}du \\
&= \frac{1}{3}\ sin^{-1}(u) + c \qquad \left[\because\ \int \frac{1}{\sqrt{1-x^2}}dx\ = sin^{-1}(x) + c\right] \\
&= \frac{1}{3}\ sin^{-1}\left(\frac{3x}{2}\right) + c
\end{aligned}$$

[ANSWER]

3 Integrate the following with the help of Gamma functions or otherwise

$$\int_0^\infty x^5 e^{-\frac{x^2}{5}} dx$$

Solution

$$\begin{aligned}
&Let, \\
&u = x^2 \\
&\Rightarrow \frac{du}{dx} = 2x \\
&\therefore dx = \frac{du}{2x}
\end{aligned}
\qquad
\begin{aligned}
\therefore u^2 &= x^4 \\
w &= \frac{u}{5} \therefore u^2 = 25w^2 \\
\Rightarrow \frac{dw}{du} &= \frac{1}{5} \\
\therefore du &= 5dw
\end{aligned}$$

$$Now,$$

$$\begin{aligned}
\int_0^\infty x^5 e^{-\frac{x^2}{5}} dx &= \int_0^\infty x^5 e^{-\frac{u}{5}} \frac{du}{2x} \\
&= \frac{1}{2} \int_0^\infty x^4 e^{-\frac{u}{5}} du \\
&= \frac{1}{2} \int_0^\infty u^2 e^{-\frac{u}{5}} du \\
&= \frac{1}{2} \int_0^\infty 25w^2 e^{-w} 5dw \\
&= \frac{25}{2} 5 \int_0^\infty w^2 e^{-w} dw \qquad \left[\because \int_0^\infty x^{n-1} e^{-x} dx\right] \\
&= \frac{125}{2} \Gamma(3) \\
&= \frac{125}{2} 2! \\
&= 125
\end{aligned}$$

[ANSWER]

4 Evaluate the following with the help of trigonometric form of Beta functions or otherwise

$$\int_0^{\frac{3\pi}{2}} sin^6\left(\frac{x}{3}\right)cos^4\left(\frac{x}{3}\right)$$

Solution

$$we\ know\ the\ trigonometric\ beta\ function,$$

$$\beta(x,y) = \int_0^{\frac{\pi}{2}} 2sin^{2x-1}(t)cos^{2y-1}(t)dt$$

$$Let,$$

$$z = 3x \qquad \therefore x = \frac{z}{3}$$

$$\Rightarrow \frac{dz}{dx} = 3$$

$$\therefore dx = 3dz$$

$$z \longrightarrow 0,\ x = 0$$

$$z \longrightarrow \frac{3\pi}{2},\ x = \frac{\pi}{2}$$

$$Now,$$

$$\begin{aligned}
\int_0^{\frac{3\pi}{2}} sin^6\left(\frac{x}{3}\right)cos^4\left(\frac{x}{3}\right) &= \int_0^{\frac{\pi}{2}} sin^6\left(\frac{z}{3}\,\frac{1}{3}\right)cos^4\left(\frac{z}{3}\,\frac{1}{3}\right)3dz \\
&= \frac{1}{3}\,\frac{1}{3}\int_0^{\frac{\pi}{2}} sin^6\left(\frac{z}{3}\right)cos^4\left(\frac{z}{3}\right)3dz \\
&= \frac{1}{9}\int_0^{\frac{\pi}{2}} sin^6(z)cos^4(z)dz \\
&= \frac{1}{9}\,\frac{\Gamma\left(\frac{6+1}{2}\right)\ \Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{6+4+2}{2}\right)} \\
&= \frac{1}{9}\,\frac{\Gamma\left(\frac{7}{2}\right)\ \Gamma\left(\frac{5}{2}\right)}{2\,\Gamma(6)} \\
&= \frac{1}{18}\,\frac{\Gamma\left(\frac{7}{2}\right)\ \Gamma\left(\frac{5}{2}\right)}{\Gamma(6)}
\end{aligned}$$

[ANSWER]

5 Evaluate the following with the help of Beta functions or otherwise

$$\int_0^{3^{\frac{2}{3}}} x^{\frac{7}{2}}\left(3 - x^{\frac{3}{2}}\right)^4 dx$$

Solution

$$\begin{aligned} &Let, \\ x &= 3y \\ \therefore dx &= 3dy \\ x \longrightarrow 0,\ y &= 0 \\ x \longrightarrow 3^{\frac{2}{3}},\ y &= 1 \end{aligned}$$

$$\begin{aligned} u &= y^{\frac{3}{2}} \ \therefore u^2 = y^3 \\ \Rightarrow \frac{du}{dy} &= \frac{3}{2}y^{\frac{1}{2}} \\ \therefore dy &= \frac{2du}{3y^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} &Now, \\ \int_0^{3^{\frac{2}{3}}} x^{\frac{7}{2}}\left(3 - x^{\frac{3}{2}}\right)^4 dx &= \int_0^1 (3y)^{\frac{7}{2}}\left(3 - (3y)^{\frac{3}{2}}\right)^4 3dy \\ &= 3^{\frac{7}{2}}\ 3\int_0^1 y^{\frac{7}{2}}\ 3^4(1 - y^{\frac{3}{2}})^4 dy \\ &= 3^{\frac{7}{2}}\ 3\ 3^4\int_0^1 y^{\frac{7}{2}}\ (1 - y^{\frac{3}{2}})^4 dy \\ &= 3^{\frac{17}{2}}\int_0^1 y^{\frac{7}{2}}\ (1-u)^4\frac{2du}{3y^{\frac{1}{2}}} \\ &= 3^{\frac{17}{2}}\ \frac{2}{3}\int_0^1 u^2\ (1-u)^4 du \qquad \left[\because \int_0^1 t^{x-1}(1-t)^{y-1}dt\right] \end{aligned}$$

$$\begin{aligned} &So, \\ x - 1 &= 2 \qquad \therefore x = 3 \\ y - 1 &= 3 \qquad \therefore y = 5 \end{aligned}$$

$$\begin{aligned} \therefore 3^{\frac{17}{2}}\ \frac{2}{3}\int_0^1 u^2\ (1-u)^4 du &= 3^{\frac{17}{2}}\ \frac{2}{3}\ \beta(3,5) \\ &= 3^{\frac{17}{2}}\ \frac{2}{3}\ \frac{\Gamma(3)\ \Gamma(5)}{\Gamma(3+5)} \qquad \left[\because \beta(x,y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)}\right] \\ &= 3^{\frac{17}{2}}\ \frac{2}{3}\ \frac{2!\ 4!}{7!} \end{aligned}$$

[ANSWER]