MAT120: Integral Calculus and Differential Equations
BRAC University
Assignment 1

Name - Ishraq Ahmed Esha
ID - 19301261
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1 Evaluate the following integral by interpreting it as area, or otherwise:

$$\int_{-1}^{5} |x-3|$$

Solution

$$x-3 = \begin{cases} x-3;\ x-3 \geq 0;\ x \geq 3 \\ -(x+3);\ x-3 \leq 0;\ x \leq 3 \end{cases}$$

$$\begin{aligned}
\int_{-1}^{5} |x-3| &= \int_{-1}^{3} (-x+3)\,dx\ + \int_{3}^{5} (x-3)\,dx \\
&= \int_{-1}^{3} -x\,dx\ + \int_{-1}^{3} 3\,dx\ + \int_{3}^{5} x\,dx\ - \int_{3}^{5} 3\,dx \\
&= \left[\frac{-x^2}{2}\right]_{-1}^{3}\ +\ 3x\ +\ \left[\frac{x^2}{2}\right]_{3}^{5}\ -\ 3x \\
&= \left(\frac{-9}{2} + 9 + \frac{7}{2}\right)\ +\ \left(\frac{25}{2} - 15 + \frac{9}{2}\right) \\
&= 8 + 2 \\
&= 10 \qquad \text{[ANSWER]}
\end{aligned}$$

2 Evaluate the following indefinite integral with a substitution, or otherwise:

$$\int \frac{x}{\sqrt{4-x^2}} \qquad\qquad where (4 \geq x^2)$$

Solution

$$\begin{aligned}
&Let, \\
u &= 4 - x^2 \\
\frac{du}{dx} &= -2x \\
du &= -2x\ dx \\
-2x &= du \\
x &= \frac{-1}{2}\ du \\
&Now, \\
\int \frac{x}{\sqrt{4-x^2}}\ dx &= \int -\frac{1}{2\sqrt{u}}\ du \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{u}} + c \\
&= -\sqrt{u} + c \\
&= -\sqrt{4-x^2} + c \qquad\qquad [\text{ANSWER}]
\end{aligned}$$

3 (a) If $I_n = \int x^{n-1}e^x \, dx$, by using integration by parts, prove that

$$I_n = \int x^{n-1}e^x - (n-1)I_{n-1}$$

Solution

$$\begin{aligned} &Here, \\ f(x) &= x^{n-1} & g(x) &= e^x \\ f'(x) &= \frac{d}{dx}(x^{n-1}) & G(x) &= \int e^x \, dx \\ &= (n-1)x^{n-2} & &= e^x \end{aligned}$$

$$\begin{aligned} &Now, \\ \int x^{n-1}e^x \, dx \; &= x^{n-1}e^x - \int (n-1)x^{n-2}e^x \, dx \\ &= x^{n-1}e^x - \; (n-1)\int x^{n-2}e^x \, dx \\ &= x^{n-1}e^x - \; (n-1)I_{n-1} \end{aligned}$$

$\therefore I_n = \int x^{n-1}e^x - (n-1)I_{n-1}$ [PROVED]

3 (b) Evaluate $I_1 = \int e^x\ dx$ and use the reduction formula in part (a) to calculate I_3
Solution

$$\begin{aligned}
& Given, \\
I_n &= \int x^{n-1} e^x\ dx \\
I_1 &= \int x^{1-1} e^x\ dx \\
I_1 &= \int x^0 e^x\ dx \\
I_1 &= \int e^x\ dx \\
\therefore I_1 &= \int e^x\ dx & [\text{ANSWER}]
\end{aligned}$$

$$\begin{aligned}
& Given \\
I_n &= \int x^{n-1} e^x\ dx \\
I_3 &= \int x^{3-1} e^x\ dx \\
I_3 &= \int x^2 e^x\ dx \\
\therefore I_3 &= \int x^2 e^x\ dx
\end{aligned}$$

$Now\ using\ Reduction\ Formula,$

$$\begin{aligned}
f(x) &= x^2 & g(x) &= e^x \\
f'(x) &= \frac{d}{dx}(x^2) & G(x) &= \int e^x\ dx \\
&= 2x & &= e^x
\end{aligned}$$

$$\begin{aligned}
& So, \\
I_3 = \int x^2 e^x\ dx &= x^2 e^x - \int 2x e^x\ dx \\
&= x^2 e^x - 2x e^x - 2e^x \quad \left[\because \int uv\ dx = u \int v dx - \int \frac{du}{dx} \int v dx\ dx\right]
\end{aligned}$$

$$\therefore I_3 = x^2 e^x - 2x e^x - 2e^x \qquad [\text{ANSWER}]$$

4 Evaluate the following integral by decomposing it into partial fractions, or otherwise:

$$\int \frac{2x+7}{x^2-16x+55}$$

Solution

$$\int \frac{2x+7}{x^2-16x+55} \qquad = x^2-16x+55$$

$$= \int \frac{2x+7}{(x-11)(x-5)} \qquad = x^2-11x-5x+55$$

$$= x(x-11)-5(x-11)$$
$$= (x-11)(x-5)$$

$$Let,$$

$$\int \frac{2x+7}{(x-11)(x-5)} = \frac{A}{(x-11)} + \frac{B}{(x-5)}$$

$$2x+7 = A(x-5)+B(x-11)$$

$$Now,$$

$$x=5; \quad 17=-6B \quad \therefore B=-\frac{17}{6}$$

$$x=1; \quad 29=6A \quad \therefore A=\frac{29}{6}$$

$$So,$$

$$\int \frac{2x+7}{x^2-16x+55} = \frac{29}{6}\int \frac{1}{(x-11)} + -\frac{17}{6}\int \frac{1}{(x-5)}$$
$$= \frac{29}{6}ln(x-11) - \frac{17}{6}ln(x-5)+c \qquad \text{[ANSWER]}$$

5 Evaluate the following improper integral with the help of Gamma functions, or otherwise:

$$\int_0^\infty (3t)^7 e^{(-3t)}\, dt$$

Solution

$$\begin{aligned}
&Let,\\
3t &= u\\
\Rightarrow t &= \frac{u}{3}\\
\Rightarrow \frac{d}{dt}(3t) &= \frac{d}{dt}(u)\\
\Rightarrow 3 &= \frac{du}{dt}\\
\therefore dt &= \frac{du}{3}
\end{aligned}$$

$$\begin{aligned}
&Now,\\
\int_0^\infty (3t)^7 e^{(-3t)}\, dt &= \int_0^\infty u^7 e^{-u}\, dt\\
&= \int_0^\infty u^7 e^{-u} \frac{du}{3}\\
&= \frac{1}{3} \int_0^\infty u^{8-1} e^{-u} du\\
&= \frac{1}{3}\, \Gamma(8) \qquad \left[\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx\right]\\
&= \frac{1}{3}\, (7!)\\
&= 1680 \qquad [\text{ANSWER}]
\end{aligned}$$