Application Of Newton's Law Of Cooling

By
Ahmed Meshref Ramadan
Wenseslaus Raphael Mwita
Goodluck Caiser Malata

Case Scenario:
Paul is a young Nigerian man that is a huge fan of soup. Every weekend he makes soup for the whole week and stores it in the refrigerator before he sleeps. His refrigerator can not hold anything with a temperature warmer than 20 C. Since Paul is very smart, he discovered that he could cool the pot of soup by using a sink full of water at a constant temperature of 5 C and constantly stirring. That would bring down the temperature of the soup to 60 degrees C in 10 minutes. Since Paul is very particular about his sleeping time, he would like to know how long will it take for the soup to be ready so he can put it in the fridge and sleep on time.

Introduction:
From the above case, we can use the concept of differential equations to help Paul solve his problem. Differential equations are used to describe physical phenomena around us. Different case scenarios like population growth, the decay of a bacterial or spread of a virus can be solved using differential equations.

In the end, we get a functional equation which is a solution to the differential equations instead of the actual value. We can apply the functional equation as many times as possible to obtain different levels of differential equations. The main goal in solving a differential equation is to find the function so that we can understand the behavior of a specific input at all times instead of just one output value.

We can say a function is a solution to a differential equation if, we plug it's derivative into the equation we get the equation is satisfied.

Note: The solution to a differential equation is always a function. This is not like a solution to an algebraic equation which is a number or a set of numbers.

An example of a differential equation is $dy/dx = f(x)$. This is the same as $f'(x) = f(x)$

Going back to Paul's problem.
Using **Newton's law of cooling** that states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings:

Let

$T(t)$ = Temperature of the soup at time t (in min).

$T(0) = T_o$ = Initial Temperature of the soup =100 deg.

T_a = Surrounding temperature = 5 deg.

So, The rate of change of the temperature of the soup dT/dt is proportional to the difference between the temperature of the soup T and the surrounding (ambient) temperature Ta which means:

$$dT/dt \textit{ is proportional to } T - Ta$$

The previous equation can be translated to:

$$dT/dt = -k(T - Ta)$$

Note:
1. K is the cooling co-efficient (positive constant).
2. The negative sign (-) at the beginning of the equation means that the soup is **cooling down** which means that the change in the temperature of the soup is decreasing because we guarantee that it is higher than the temperature of the surrounding at the beginning of the experiment.

Objective:
Finding an equation of $T(t)$ to get the temperature of the soup at any time during the experiment knowing that k and Ta are constants.

Solution:
Using the separation technique to solve the Differential equation:

$$dT/(T - Ta) = -kdt$$

By integrating both sides:

$$ln(T - Ta) = -kt$$

Applying e for both sides:

$$T - Ta = e^{-kt}.c$$

$$T(t) = e^{-kt}c + Ta$$

Using the initial case to get the value of c and k:
At $t = 0$; $T(t) = 100$

$$T(t) = e^{0}c + 5$$
$$100 - 5 = c$$
$$c = 95$$

We also know that after 10 minutes, the soup cools to 60 degrees; $t = 10$ and $T(t) = 60$

$$60 = 95e^{-10k} + 5$$
$$55/95 = e^{-10k}$$
$$k = 0.054 \text{ per minute}$$

Final Equation:

$$T(t) = 95e^{-0.054t} + 5$$

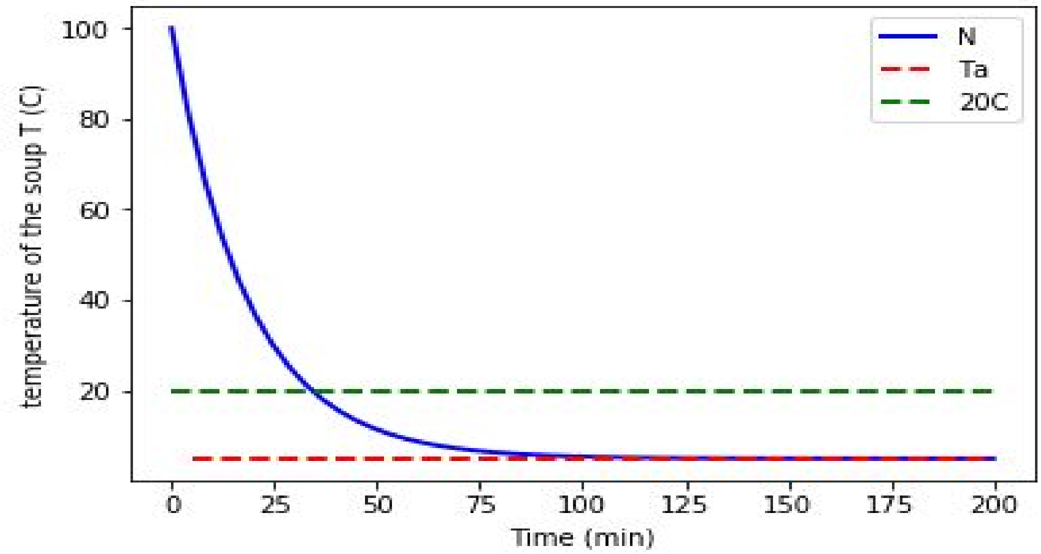
But:
Paul is interested in the amount of time it will take him to cool the soup to 20 degrees C so he can put the soup in the refrigerator and sleep as he is very particular about his sleeping time. From the equation above $T(t) = 20\ degrees\ C$, since he will need to wait until it reaches 20 degrees C.

$$20 = 95e^{-0.054t} + 5$$

$$15/95 = e^{-0.054t}$$

When you solve, $t = 34.2\ minutes$

The following graph shows the cooling of the soup over time:



Code

As the graph shows the temperature of the soup started at 100 C and then it decreased quickly during the first 30 minutes reaching almost 24 C, then it started to decrease slowly till it reached the temperature of the surrounding after 100 minutes.

So it will take Paul 34.2 minutes to cool the soup to 20 degrees every time he boils soup if the temperature of the water used is 5 degrees and if he also boiled the soup to 100 degrees C.

Conclusion:

Now, Paul can use the above equation to know how long it will take him to cool the soup to 20 degrees before he can go to sleep. This will help him plan out his time well and know what time he should start cooking, cooling the soup and even go to bed. Not only that but also if he didn't boil the soup to 100 degrees C or if the cooling water was not at 5 degrees C, he can still know how long it will take him to cool the soup before putting in the refrigerator.