# Numerical Analysis

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# 1 Introduction

**Matrix:** Rectangular array of numbers, symbols or expressions arranged in rows and column

**Ex:** $2x3$ matrix would look like this

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

The number of elements in the matrix is $2 \times 3 = 6$

**Remark:** Matrices are used in different mathematical operations such as addition, multiplication, and finding determinants and inverses.

**General form for the matrix** $m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

**Matrix Applications:**

1. Computer Graphics
   - Scaling and rotation images
   - 3D
2. Machine Learning:
   - Features of data points in algorithms, Such as linear regression, Neural networks
3. Economics
   - Building an economy system
   - Analyze the strategies
4. Single processing
   - Can be used in Image processing
5. Control systems
   - Finding the relationship between inputs/outputs in databases
6. Statistics
   - Regression analysis

## 2 Matrices Types

1. **Row Matrix:** Have only one row

   **EX:** $A = 1 \times 3$

$$A = \begin{bmatrix} 4 & 0 & \frac{1}{3} \end{bmatrix}$$

2. **Column Matrix:** Have only one column

   **EX:** $A = 3 \times 1$

$$A = \begin{bmatrix} 5 \\ 33 \\ 0.4 \end{bmatrix}$$

3. **Square Matrix:** Have an equal number of rows and columns

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} 5 & -1 & 2 \\ 33 & 3 & 20 \\ 0.4 & -3 & 0 \end{bmatrix}$$

   **EX:** $B = 1 \times 1$

$$B = \begin{bmatrix} 3.14 \end{bmatrix}$$

4. **Diagonal Matrix:** All non-diagonal elements are zeros

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{8} \end{bmatrix}$$

5. **Scalar Matrix:** Square matrix with all non-diagonal elements are zero and all diagonal elements are equal

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

6. **Unit Matrix(Identity):** Type of diagonal matrix, its all diagonal elements equal to 1

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

7. **Null(Zero) Matrix:** All elements equal to 0

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8. **Symmetric Matrix:** If the matrix $A$ is equal to $A^\top$
   Where
   $A^\top$: is the transpose matrix from $A$, Generated by switching rows and columns

   **EX:** Check if $A = A^\top$

$$A = \begin{bmatrix} -1 & 2 & 10 \\ 30 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} \rightarrow A^\top = \begin{bmatrix} -1 & 30 & -3 \\ 2 & 0 & -2 \\ 10 & 1 & -5 \end{bmatrix}$$

   $\therefore A \neq A^\top$
   $\therefore A$ Is not a symmetric matrix

# 3 Addition And Subtraction

Two matrices can be added or subtracted if and only if the have the same order which is mean the same numbers of rows and columns

**Ex:** Find $A + B$ and $A - B$ Where

$$A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix} B = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$$

**Sol:**

$$A + B = \begin{bmatrix} 2+3 & 0+6 \\ -5+4 & 6+1 \end{bmatrix} \rightarrow A + B = \begin{bmatrix} 5 & 6 \\ -1 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-3 & 0-6 \\ -5-4 & 6-1 \end{bmatrix} \rightarrow A - B = \begin{bmatrix} -1 & -6 \\ -9 & 5 \end{bmatrix}$$

**Ex:** Find $A + B$ Where

$$A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix} B = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$$

**Sol:** $A + B$ Is not defined since they have different orders

**Properties Of Addition**:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0$ (zero matrix) $= A$
4. $A - B \neq B - A$
5. $(A - B) - C = A - (B - C)$
6. $A^{\top^{\top}} = A$
7. $aA^{\top} = aA^{\top}$ Where $a$ is a number
8. $(A + B)^{\top} = B^{\top} + A^{\top}$

**H.W(1):** Prove that all the properties of matrix addition are satisfied

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & -10 \\ 3 & 5 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -5 & -3 \\ -2 & 1 & -0 \\ 4 & -7 & 10 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 6 & -1 \\ 20 & 0 & 10 \\ -10 & 3 & -30 \end{bmatrix}$$

# 4 Matrix multiplication

To defined the multiplication of two matrices $AB$, If and only if the number of columns in $A$ is equal to the number of rows in $B$

**Ex:** Find $AB$ Where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

**Sol:**

$$AB = \begin{bmatrix} 1\times 2 + 2\times 3 & 1\times 1 + 2\times 0 & 1\times 4 + 2\times 5 \\ 3\times 2 + 4\times 3 & 3\times 1 + 4\times 0 & 3\times 4 + 4\times 5 \\ 0\times 2 + 1\times 3 & 0\times 1 + 1\times 0 & 0\times 4 + 1\times 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 8 & 1 & 14 \\ 18 & 3 & 32 \\ 3 & 0 & 5 \end{bmatrix}$$

**Properties of multiplication**:

1. $A(B + C) = AB + AC$
2. $(B + C)A = BA + CA$
3. $AB \neq BA$
4. $AB(C) = A(BC) = ABC$
5. $AI = IA = A$
6. $AA^{-1} = I$
   Where $A^{-1}$: is the inverse of $A$
7. $(AB)^{\top} = B^{\top}A^{\top}$

## 5 Determinant of matrix

Single number can be found only for $2 \times 2$ matrices, Will be denoted by $det(A)$ or $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$det(A) = |A| = a_{11} \times a_{22} - a_{12} \times a_{21}$$

**Ex:** Find Determinant of $A$ Where

$$A = \begin{bmatrix} 6 & 4 \\ 7 & 9 \end{bmatrix} \Rightarrow |A| = 6 \times 9 - 4 \times 7 \Rightarrow |A| = 26$$

For a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{12} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Ex:** Find $|A|$
Where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

**Sol:**

$$A = 1 \times \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 0 \\ -2 & 3 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$$

$$|A| = 1(-1 \times 3 - 0 \times 0) - 2(0 \times 3 - 0 \times -2) + 4(0 \times 0 - 1 \times -2) \rightarrow |A| = 5$$

# 6 Cofactor of a matrix

$$let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can find the Cofactor of $A$ by

$$C_{ij} = (-1)^{i+j} A_{ij}$$

$$C_{11} = -1^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{11} = a_{22}a_{33} - a_{23}a_{32}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{12} = -\big(a_{21}a_{33} - a_{23}a_{31}\big)$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow C_{13} = a_{21}a_{32} - a_{22}a_{31}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{21} = -\big(a_{12}a_{33} - a_{13}a_{32}\big)$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{22} = a_{11}a_{33} - a_{13}a_{31}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow C_{23} = -\big(a_{11}a_{32} - a_{12}a_{31}\big)$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \Rightarrow C_{31} = a_{12}a_{23} - a_{13}a_{22}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \Rightarrow C_{32} = -\big(a_{11}a_{23} - a_{13}a_{21}\big)$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow C_{33} = a_{11}a_{22} - a_{12}a_{21}$$

**Ex(2):** Find the cofactor for the following

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 5 & -4 \end{vmatrix} \Rightarrow C_{11} = 0 \cdot (-4) - 3 \cdot 5 = -15$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} \Rightarrow C_{12} = -\big(1 \cdot (-4) - 3 \cdot 2\big) = -(-4-6) = 10$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} \Rightarrow C_{13} = 1 \cdot 5 - 0 \cdot 2 = 5$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix} \Rightarrow C_{21} = -\big(4 \cdot (-4) - (-1) \cdot 5\big) = -(-16 + 5) = 11$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} \Rightarrow C_{22} = 3 \cdot (-4) - (-1) \cdot 2 = -12 + 2 = -10$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} \Rightarrow C_{23} = -\big(3 \cdot 5 - 4 \cdot 2\big) = -(15 - 8) = -7$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 4 & -1 \\ 0 & 3 \end{vmatrix} \Rightarrow C_{31} = 4 \cdot 3 - (-1) \cdot 0 = 12$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \Rightarrow C_{32} = -\big(3 \cdot 3 - (-1) \cdot 1\big) = -(9 + 1) = -10$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} \Rightarrow C_{33} = 3 \cdot 0 - 4 \cdot 1 = -4$$

$$C = \begin{bmatrix} -15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$$

**Remark:** if $A$ is a matrix of $2 \times 2$ you can find the cofactor like this

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

**Ex:** Find cofactor for $A$ Where

$$A = \begin{bmatrix} -2 & -4 \\ 10 & 5 \end{bmatrix}$$

**Sol:**

$$A = \begin{vmatrix} -2 & -4 \\ 10 & 5 \end{vmatrix} \quad \Rightarrow \quad C = \begin{vmatrix} 5 & -10 \\ 4 & -2 \end{vmatrix}$$

## 7 Adjoint Matrix

Adj($A$) Can be found $adj(A) = C^\top$ Where $C$ is the cofactor of $A$

**Ex(3):** Find adjoint for $A$ from the example number 2

$$C = \begin{bmatrix} 15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$$

**Sol:**

$$adj(A) = C^\top = \begin{bmatrix} -15 & 11 & 12 \\ 10 & -10 & -10 \\ 5 & -7 & -4 \end{bmatrix}$$

## 8 Inverse Matrix ($A^{-1}$)

You can find the inverse from this form $A^{-1} = \frac{adj(A)}{|A|}$, Where $|A| \neq 0$

If $|A| = 0$ Then $A$ doesn't have inverse

**Ex:** Find $A^{-1}$ from Ex(3)

$$adj(A) = C^\top = \begin{bmatrix} -15 & 11 & 12 \\ 10 & -10 & -10 \\ 5 & -7 & -4 \end{bmatrix}$$

**Sol:**

After Finding the cofactor it will equal $|A| = -10$

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} \frac{15}{10} & -\frac{11}{10} & -\frac{12}{10} \\ -\frac{10}{10} & \frac{10}{10} & \frac{10}{10} \\ -\frac{5}{10} & \frac{7}{10} & \frac{4}{10} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1.5 & -1.1 & -1.2 \\ -1 & 1 & 1 \\ -0.5 & 0.7 & 0.4 \end{bmatrix}$$

## 9 Lower & Upper Triangular Matrix

**Lower Triangular Matrix** All the elements above the main diagonal are zeros

**Ex:** $3 \times 3$ Lower Triangular matrix

$$A = \begin{bmatrix} a_{11} & \boxed{0} & \boxed{0} \\ a_{21} & a_{22} & \boxed{0} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

**Upper Triangular Matrix** All the elements below the main diagonal are zeros

**Ex:** $3 \times 3$ Upper Triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{0} & a_{22} & a_{23} \\ \boxed{0} & \boxed{0} & a_{33} \end{bmatrix}$$

**Ex:** Transfer $A$ to upper triangular matrix, Where

$$A = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

**Sol:**
$R_1 = R2$, Swapping two rows

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

Apply $R_3 = 2R_3 - 3R_1$ to the above matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & -7 & -13 \end{bmatrix}$$

Apply $R_3 = 7R_2 + 4R_3$ to the above matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & -13 \end{bmatrix}$$

**H.W(2):** Make it lower triangular matrix

## 10 Systems of linear equations

Linear system of equations can be written as

$$\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\
a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\
&\vdots \\
a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m
\end{aligned}$$

This system can be transfer into $AX = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

To solve this system of linear equations, We can use the following methods

1. **Gauss Elimination Method** To use this method we need to transfer the matrix $A$ into a lower or upper Triangular matrix using row operations

   **Ex:** Solve the following system of the linear equations using Gauss method, Where

$$\begin{aligned}
4y + 2z &= 1 \\
2x + 3y + 5z &= 0 \\
3x + y + z &= 11
\end{aligned}$$

**Sol:**

$$AX = b, \quad AX = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 2 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 1 & 1 & 11 \end{array}\right]$$

$$R_1 = R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right]$$

$$R_3 = 2R_3 - 3R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & -7 & -13 & 22 \end{array}\right]$$

$$R_3 = 4R_3 - 7R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -38 & 95 \end{array}\right]$$

$$\begin{aligned} -38z = 95 \quad & \Rightarrow z = -\frac{95}{38} = -2.5 \\ 4y + 2z = 1 \quad & \Rightarrow 4y + 2(-2.5) = 1 \\ & \Rightarrow 4y - 5 = 1 \\ & \Rightarrow 4y = 6 \Rightarrow y = \frac{3}{2} = 1.5 \\ 2x + 3y + 5z = 0 \quad & \Rightarrow 2x + 3(1.5) + 5(-2.5) = 0 \\ & \Rightarrow 2x + 4.5 - 12.5 = 0 \\ & \Rightarrow 2x - 8 = 0 \\ & \Rightarrow x = 4 \end{aligned}$$

2. **Gauss-Jordan Elimination Method**
   We have to transfer the system to $JX = T$

$$J = \begin{bmatrix} a_{11} & 0 & 0 & \dots \\ 0 & a_{22} & 0 & \dots \\ 0 & 0 & a_{33} & \vdots \\ 0 & 0 & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$

   The solution will be

$$t_1, t_2, t_3, \dots t_n$$

**Ex:** Solve the following system of linear equations using Gauss-Jordan

$$x + 2y + 3z = 9$$
$$2x + 3y + z = 8$$
$$3x + y + 2z = 7$$

**Sol:**

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 3 & 1 & 8 \\ 3 & 1 & 2 & 7 \end{array}\right]$$

$$R_2 = R_2 - 2R_1,\ R_3 = R_3 - 3R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -1 & -5 & -10 \\ 0 & -5 & -7 & -20 \end{array}\right]$$

$$R_2 = -R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 5 & 10 \\ 0 & -5 & -7 & -20 \end{array}\right]$$

$$R_3 = R_3 + 5R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$R_1 = R_1 - 2R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -11 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$R_1 = 18R_1 + 7R_3,\ R_2 = 18R_2 - 5R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 18 & 0 & 0 & 12 \\ 0 & 18 & 0 & 30 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$18x = 12 \quad \Rightarrow x = \frac{12}{18} = 0.6$$

$$18y = 30 \quad \Rightarrow y = \frac{30}{18} = 1.667$$

$$18z = 30 \quad \Rightarrow z = \frac{30}{18} = 1.667$$

3. **General Form For Cramer's Rule** For a system with linear equations with $n$ variables of the form

$$\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\
a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\
&\vdots \\
a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m
\end{aligned}$$

the solution will be

$$x_i = \frac{D_i}{D}$$

Where

$D$: is the Determinant of the matrix $A$ generated from the system $AX = b$
$D_i$: Determinant of the matrix formed by replacing the $i-th$ column of $A$ with the constant vector $b$

**Ex:** Solve using Cramer's rule

$$\begin{aligned}
x + 2y + z &= 4 \\
2x + y + 3z &= 10 \\
3x + 2y + 2z &= 12
\end{aligned}$$

**Sol:**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 10 \\ 12 \end{bmatrix}$$

$$D = |A| = det(A) = 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 7$$

Now Replacing first column with the constants vector $b$

$$\begin{aligned}
D_1 &= \begin{vmatrix} 4 & 2 & 1 \\ 10 & 1 & 3 \\ 12 & 2 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 10 & 3 \\ 12 & 2 \end{vmatrix} + 1\begin{vmatrix} 10 & 1 \\ 12 & 2 \end{vmatrix} \\
&= 4\big(1 \times 2 - 3 \times 2\big) - 2\big(10 \times 2 - 3 \times 12\big) + 1\big(10 \times 2 - 1 \times 12\big) \\
&= 24
\end{aligned}$$

$$\begin{aligned}
D_2 &= \begin{vmatrix} 1 & 4 & 1 \\ 2 & 10 & 3 \\ 3 & 12 & 2 \end{vmatrix} = 1\begin{vmatrix} 10 & 3 \\ 12 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 10 \\ 3 & 12 \end{vmatrix} \\
&= 1\big(10 \times 2 - 3 \times 12\big) - 4\big(2 \times 2 - 3 \times 3\big) + 1\big(2 \times 12 - 10 \times 3\big) \\
&= -2
\end{aligned}$$

$$D_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 10 \\ 3 & 2 & 12 \end{vmatrix} = 1\begin{vmatrix} 1 & 10 \\ 2 & 12 \end{vmatrix} - 2\begin{vmatrix} 2 & 10 \\ 3 & 12 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$
$$= 1\big(1 \times 12 - 10 \times 2\big) - 2\big(2 \times 12 - 10 \times 3\big) + 4\big(2 \times 2 - 1 \times 3\big)$$
$$= 8$$

$$x = \frac{D_1}{D} \Rightarrow x = \frac{24}{7} \Rightarrow x \approx 3.4$$
$$y = \frac{D_2}{D} \Rightarrow y = -\frac{2}{7} \Rightarrow y \approx 0.2$$
$$z = \frac{D_3}{D} \Rightarrow z = \frac{8}{7} \Rightarrow z \approx 1.1$$

4. **Inverse Matrix Method**

   For system of equation written in a matrix form $AX = b$

   If $A \neq 0$, Then the solution of the system will be $X = A^{-1}b$, which comes from the following

   $AX = b \Rightarrow A^{-1}AX = A^{-1} \Rightarrow IX = A^{-1}b \Rightarrow \boxed{X = A^{-1}b}$

   Where is a matrix with all it's diagonal elements are ones

$$I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   **Ex(1):** solve the system using inverse method

$$2x + 3y = 8$$
$$x + 4y = 7$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

$$AX = b \Rightarrow X = A^{-1}b$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$$

$$Adj(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \times \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

$$x = \frac{4}{5} \times 8 - \frac{3}{5} \times 7 \Rightarrow x = \frac{32}{5} - \frac{21}{5} \Rightarrow x = \frac{11}{5} = 2.2$$

$$y = -\frac{1}{5} \times 8 + \frac{2}{5} \times 7 \Rightarrow y = -\frac{8}{5} + \frac{14}{5} \Rightarrow y = \frac{6}{5} \Rightarrow y = 1.2$$

**Ex(2):**

$$x + y + z = 6$$
$$2y + 5z = -4$$
$$2x + 5y - z = 27$$

**Sol:**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

$$AX = b \Rightarrow X = A^{-1}b, \text{ Where } A^{-1} = \frac{adj(A)}{|A|}$$

$$adj(A) = C^{\top}$$

$$C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{vmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix}2 & 5\\ 5 & -1\end{vmatrix} = 1\big[2\times(-1) - 5\times 5\big] = -27$$
$$C_{12} = (-1)^{1+2}\begin{vmatrix}0 & 5\\ 2 & -1\end{vmatrix} = -1\big[0\times(-1) - 5\times 2\big] = 10$$
$$C_{13} = (-1)^{1+3}\begin{vmatrix}0 & 2\\ 2 & 5\end{vmatrix} = 1\big[0\times 5 - 2\times 2\big] = -4$$
$$C_{21} = (-1)^{2+1}\begin{vmatrix}1 & 1\\ 5 & -1\end{vmatrix} = -1\big[1\times(-1) - 1\times 5\big] = 6$$
$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix} = 1\big[1\times(-1) - 1\times 2\big] = -3$$
$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 1\\ 2 & 5\end{vmatrix} = -1\big[1\times 5 - 1\times 2\big] = -3$$
$$C_{31} = (-1)^{3+1}\begin{vmatrix}1 & 1\\ 2 & 5\end{vmatrix} = 1\big[1\times 5 - 1\times 2\big] = 3$$
$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\ 0 & 5\end{vmatrix} = -1\big[1\times 5 - 1\times 0\big] = -5$$
$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix} = 1\big[1\times 2 - 1\times 0\big] = 2$$
$$C = \begin{bmatrix}-27 & 10 & -4\\ 6 & -3 & -3\\ 3 & -5 & 2\end{bmatrix}$$

$$\operatorname{adj}(A) = C^{\top} = \begin{bmatrix}-27 & 6 & 3\\ 10 & -3 & -5\\ -4 & -3 & 2\end{bmatrix}$$

Finding the cofactor

$$\begin{aligned}|A| &= 1\begin{vmatrix}2 & 5\\ 5 & -1\end{vmatrix} - 1\begin{vmatrix}0 & 5\\ 2 & -1\end{vmatrix} + 1\begin{vmatrix}0 & 2\\ 2 & 5\end{vmatrix}\\ |A| &= 1(2\times -1 - 5\times 5) - 1(0\times -1 - 5\times 2)\\ &\quad + 1(0\times 5 - 2\times 2)\\ |A| &= -21\end{aligned}$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \begin{bmatrix}\frac{-27}{-21} & \frac{6}{-21} & \frac{3}{-21}\\ \frac{10}{-21} & \frac{-3}{-21} & \frac{-5}{-21}\\ \frac{-4}{-21} & \frac{-3}{-21} & \frac{2}{-21}\end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix}\frac{9}{7} & -\frac{2}{7} & -\frac{1}{7}\\ -\frac{10}{21} & \frac{1}{7} & \frac{5}{21}\\ \frac{4}{21} & \frac{1}{7} & -\frac{2}{21}\end{bmatrix}$$

$$X = A^{-1}b$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{10}{21} & \frac{1}{7} & \frac{5}{21} \\ \frac{4}{21} & \frac{1}{7} & -\frac{2}{21} \end{bmatrix} \times \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

$$x = \frac{9}{7} \times 6 + \frac{2}{7} \times 4 - \frac{1}{7} \times 27 \Rightarrow x = 5$$

$$y = -\frac{10}{21} \times 6 - \frac{1}{7} \times 4 + \frac{5}{21} \times 27 \Rightarrow y = 3$$

$$z = \frac{4}{21} \times 6 - \frac{1}{7} \times 4 - \frac{2}{21} \times 27 \Rightarrow z = -2$$

5. **Lower/Upper Decomposition Method**

   In this method we will write $A = LU$, Where

   $L$: Lower triangular matrix with (1)s on the main diagonal

   $U$: Upper triangular Matrix

   Then

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

   Which will produce the following simplification

$$A = LU$$

$$LU = \begin{bmatrix} 1 \times u_{11} + 0 \times 0 + 0 \times 0 & 1 \times u_{12} + 0 \times u_{22} + 0 \times 0 & 1 \times u_{13} + 0 \times u_{23} + 0 \times u_{33} \\ l_{21}u_{11} + 1 \times 0 + 0 \times 0 & l_{21}u_{12} + 1 \times u_{22} + 0 \times 0 & l_{21}u_{13} + 1 \times u_{23} + 0 \times u_{33} \\ l_{31}u_{11} + l_{32} \times 0 + 1 \times 0 & l_{31}u_{12} + l_{32}u_{22} + 1 \times 0 & l_{31}u_{13} + l_{32}u_{23} + 1 \times u_{33} \end{bmatrix}$$

   Simplifies this

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU$$

$$A = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

This will produce this equations

$$u_{11} = a_{11}$$
$$u_{12} = a_{12}$$
$$u_{13} = a_{13}$$
$$l_{21}u_{11} = a_{21}$$
$$l_{21}u_{12} + u_{22} = a_{22}$$
$$l_{21}u_{13} + u_{23} = a_{23}$$
$$l_{31}u_{11} = a_{31}$$
$$l_{31}u_{12} + l_{32}u_{22} = a_{32}$$
$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

**Ex:**

$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + 7y + 5z &= 2 \\ 6x + 18y + 19z &= 3 \end{aligned}$$

**Sol:**

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 6 & 18 & 19 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} u_{11} &= a_{11} = 2 \\ u_{12} &= a_{12} = 3 \\ u_{13} &= a_{13} = 1 \end{aligned}$$

$$\begin{aligned} l_{21}u_{11} &= a_{21} \\ l_{21} \times 2 &= 4 \\ l_{21} &= 2 \end{aligned}$$

$$\begin{aligned} l_{21}u_{12} + u_{22} &= a_{22} \\ 2 \times 3 + u_{22} &= 7 \\ 6 + u_{22} &= 7 \\ u_{22} &= 7 - 6 \\ u_{22} &= 1 \end{aligned}$$

$$\begin{aligned} l_{21}u_{13} + u_{23} &= a_{23} \\ 2 \times 1 + u_{23} &= 5 \\ 2 + u_{23} &= 5 \\ u_{23} &= 5 - 2 \\ u_{23} &= 3 \end{aligned}$$

$$
\begin{aligned}
l_{31}u_{11} &= a_{31} \\
l_{31}2 &= 6 \\
l_{31} &= 3
\end{aligned}
$$

$$
\begin{aligned}
l_{31}u_{12} + l_{32}u_{22} &= a_{32} \\
3 \times 3 + l_{32}1 &= 18 \\
9 + l_{32} &= 18 \\
l_{32} &= 18 - 9 \\
l_{32} &= 9
\end{aligned}
$$

$$
\begin{aligned}
l_{31}u_{13} + l_{32}u_{23} + u_{33} &= a_{33} \\
3 \times 1 + 9 \times 3 + u_{33} &= 19 \\
3 + 27 + u_{33} &= 19 \\
30 + u_{33} &= 19 \\
u_{33} &= 19 - 30 \\
u_{33} &= -11
\end{aligned}
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix}
$$

$$
Ly = b
$$

$$
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

$$
1 \times y_1 + 0 \times y_2 + 0 \times y_3 = 1
$$

$$
\begin{aligned}
2 \times y_1 + 1 \times y_2 + 0 \times y_3 &= 2 \\
2 \times 1 + y_2 &= 2 \\
2 + y_2 &= 2 \\
y2 &= 2 - 2 \\
y_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
3 \times y_1 + 9 \times y_2 + 1 \times y_3 &= 3 \\
3 \times 1 + 9 \times 0 + y_3 &= 3 \\
3 + 0 + y_3 &= 3 \\
y_3 &= 3 - 3 \\
y_3 &= 0
\end{aligned}
$$

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$ux = y$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From the third row

$$\begin{aligned} 0 \times x + 0 \times y - 11 \times z &= 0 \\ -11z &= 0 \\ z &= 0 \end{aligned}$$

The second row

$$\begin{aligned} 0 \times x + 1 \times y + 3 \times z &= 0 \\ 0 + y + 3(0) &= 0 \\ y &= 0 \end{aligned}$$

The first row

$$\begin{aligned} 2 \times x + 3 \times y + 1 \times z &= 1 \\ 2x + 3 \times 0 + 1 \times 0 &= 1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

The past 5 methods are called numerical methods to solve a system of linear equations

**Direct Methods** : this methods are called iterative methods

1. **Jacobi Method**

   Generale form:

   $$x_i^{(k+1)} = \frac{1}{a_{ii}}\big(b_i - \sum_{i \neq j}^{n} a_{ij} x_j^k\big)$$

   used to solve $AX = b$, Where

   $A$: square matrix

   $X$: Vector of unknowns

   $b$: Vector of knowns

   $k$: is a counter starts from 0

Where $i = 1$

$$x_i^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - \sum_{i\neq j}^{3} a_{ij}x_j^k\right)$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - \left[a_{12}x_2^k + a_{13}x_3^k\right]\right)$$

$i = 2$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - \left[a_{21}x_1^k + a_{23}x_3^k\right]\right)$$

$i = 3$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - \left[a_{31}x_1^k + a_{32}x_3^k\right]\right)$$

Method Steps

(a) Start with initial values: $x^0 = \left[x_1^0, x_2^0, \ldots, x_n^0\right]$

(b) Compute new values $x_i$: $x_i^{(k+1)} = \left(x_1^{k+1}, x_2^{k+1}, \ldots, x_n^{k+1}\right)$

(c) Repeat until the values converge between iterations, This mean

$$\left|x_i^{(k+1)} - x_i^k\right| < \epsilon$$

**Ex:** solve the following equations

$$\begin{aligned} 10x_1 + 2x_2 + x_3 &= 9 \\ x_1 + 5x_2 + x_3 &= 7 \\ 2x_1 + 3x_2 + 10x_3 &= 15 \end{aligned}$$

**Sol:**

$$A = \begin{bmatrix} 10 & 2 & 1 \\ 1 & 5 & 1 \\ 2 & 3 & 10 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 7 \\ 15 \end{bmatrix}$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^k - a_{23}x_3^k\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^k - a_{32}x_2^k\right)$$

$$x_1^{(k+1)} = \frac{1}{10}\big(9 - 2x_2^k - x_3^k\big)$$

$$x_2^{(k+1)} = \frac{1}{5}\big(7 - x_1^k - x_3^k\big)$$

$$x_3^{(k+1)} = \frac{1}{10}\big(15 - 2x_1^k - 3x_2^k\big)$$

$$k = 0 \Rightarrow \big(x_1^0 = 0, \quad x_2^0 = 0, \quad x_3^0 = 0\big)$$

$$x_1^1 = \frac{1}{10}\big(9 - 2x_2^0 - x_3^0\big) \Rightarrow x_1^1 = \frac{1}{10}\big(9 - 2 \times 0 - 0\big) \Rightarrow x_1^1 = \frac{9}{10} = 0.9$$

$$x_2^1 = \frac{1}{5}\big(7 - x_1^0 - x_3^0\big) \Rightarrow x_2^1 = \frac{1}{5}\big(7 - 0 - 0\big) \Rightarrow x_2^1 = \frac{7}{5} = 1.4$$

$$x_3^1 = \frac{1}{10}\big(15 - 2x_1^0 - 3x_2^0\big) \Rightarrow x_3^1 = \frac{1}{10}\big(15 - 2 \times 0 - 3 \times 0\big) \Rightarrow x_3^1 = \frac{15}{10} = 1.5$$

$$\big|x_1^1 - x_1^0\big| \Rightarrow |0.9 - 0| = 0.9 > \epsilon$$

$$k = 1 \Rightarrow \big(x_1^1 = 0.9, \quad x_2^1 = 1.4, \quad x_3^1 = 1.5\big)$$

$$x_1^2 = \frac{1}{10}\big(9 - 2x_2^1 - x_3^1\big) \Rightarrow x_1^2 = \frac{1}{10}\big(9 - 2 \times 1.4 - 1.5\big) \Rightarrow x_1^2 = 0.47$$

$$x_2^2 = \frac{1}{5}\big(7 - x_1^1 - x_3^1\big) \Rightarrow x_2^2 = \frac{1}{5}\big(7 - 0.9 - 1.5\big) \Rightarrow x_2^2 = 0.92$$

$$x_3^2 = \frac{1}{10}\big(15 - 2x_1^1 - 3x_2^1\big) \Rightarrow x_3^2 = \frac{1}{10}\big(15 - 2 \times 0.9 - 3 \times 1.4\big) \Rightarrow x_3^2 = 0.9$$

$$\big|x_1^2 - x_1^1\big| \Rightarrow |0.47 - 0.9| = 0.43 > \epsilon$$

$$k = 2 \Rightarrow \big(x_1^2 = 0.47, \quad x_2^2 = 0.92, \quad x_3^2 = 0.9\big)$$

$$x_1^3 = \frac{1}{10}\big(9 - 2x_2^2 - x_3^2\big) \Rightarrow x_1^3 = \frac{1}{10}\big(9 - 2 \times 0.92 - 0.9\big) \Rightarrow x_1^3 = 0.626$$

$$x_2^3 = \frac{1}{5}\big(7 - x_1^2 - x_3^2\big) \Rightarrow x_2^3 = \frac{1}{5}\big(7 - 0.47 - 0.9\big) \Rightarrow x_2^3 = 1.126$$

$$x_3^3 = \frac{1}{10}\big(15 - 2x_1^2 - 3x_2^2\big) \Rightarrow x_3^3 = \frac{1}{10}\big(15 - 2 \times 0.47 - 3 \times 0.92\big) \Rightarrow x_3^3 = 1.13$$

$$\big|x_1^3 - x_1^2\big| \Rightarrow |0.626 - 0.47| = 0.156 < \epsilon$$

$$\big|x_2^3 - x_2^2\big| \Rightarrow |1.126 - 0.92| = 0.206 < \epsilon$$

$$\big|x_3^3 - x_3^2\big| \Rightarrow |1.13 - 0.9| = 0.23 < \epsilon$$

So the solution set: $x_1^3 = 0.626, \quad x_2^3 = 1.126, \quad x_3^3 = 1.13$

2. **Gauss-Seidal Iteration Formula**

Suppose you have a system of 3 equations

(a) $a_{11}x_1 + a_{12}x_1 + a_{13}x_1 = b_1$
(b) $a_{21}x_2 + a_{22}x_2 + a_{23}x_2 = b_2$
(c) $a_{31}x_3 + a_{32}x_3 + a_{33}x_3 = b_3$

To solve this system, use this form which is refers to the general form

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\big(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^k\big)$$

Where:
$i : 1, 2, 3, \ldots$
$k : 0, 1, 2, \ldots$
$n$ : number of equations
It can be written as the following
$i = 1$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\big(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\big)$$

$i = 2$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\big(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\big)$$

$i = 3$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\big(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\big)$$

Method Steps

(a) Choose initial values $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$

(b) Apply the iteration using the Formula

(c) Check if $max\big(\left|x_i^{(k+1)} - x_i^{(k)}\right|\big) < \epsilon$

**Ex:**

$4x_1 - x_2 + x_3 = 7$

$-2x_1 + 5x_2 - x_3 = -8$

$x_1 + x_2 + 3x_3 = 6$

$x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0$

$\epsilon = 0.01$

**Sol:**

$$\begin{bmatrix} 4 & -1 & 1 \\ -2 & 5 & -1 \\ 1 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} 7 \\ -8 \\ 6 \end{bmatrix}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\big(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^k\big)$$

$k = 0$
$i = 1$

$$x_1^{(1)} = \frac{1}{a_{11}}\big(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\big)$$

$$x_1^{(1)} = \frac{1}{4}\big(7 - (-1 \times 0) - (1 \times 0)\big) = \frac{7}{4} \Rightarrow x_1^{(1)} = 1.75$$

$i = 2$

$$x_2^{(1)} = \frac{1}{a_{22}}\big(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\big)$$

$$x_2^{(1)} = \frac{1}{5}\big( - 8 - (-2 \times 1.75) - (-1 \times 0)\big) = -\frac{9}{10} \Rightarrow x_2^{(1)} = -0.9$$

$i = 3$

$$x_3^{(1)} = \frac{1}{a_{33}}\big(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\big)$$

$$x_3^{(1)} = \frac{1}{3}\big(6 - (1 \times 1.75) - (1 \times -0.9)\big) = \frac{5.15}{3} \Rightarrow x_3^{(1)} = 1.71$$

$$x_1^{(1)} = 1.75, x_2^{(1)} = -0.9, x_3^{(1)} = 1.71$$

$$\left|x_1^{(1)} - x_1^{(0)}\right| = |1.75 - 0| = 1.75 > \epsilon$$

$k = 1$
$i = 1$

$$x_1^{(2)} = \frac{1}{a_{11}}\big(b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\big)$$

$$x_1^{(2)} = \frac{1}{4}\big(7 - (-1 \times -0.9) - (1 \times 1.71)\big) \Rightarrow x_1^{(2)} = 1.0975$$

$i = 2$

$$x_2^{(2)} = \frac{1}{a_{22}}\big(b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}\big)$$

$$x_2^{(2)} = \frac{1}{5}\big( - 8 - (-2 \times 1.0975) - (-1 \times 1.71)\big) \Rightarrow x_2^{(2)} = -0.819$$

$i = 3$

$$x_3^{(2)} = \frac{1}{a_{33}}\big(b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}\big)$$

$$x_3^{(2)} = \frac{1}{3}\big(6 - (1 \times 1.0975) - (1 \times -0.819)\big) \Rightarrow x_3^{(2)} = 1.9071$$

$$x_1^{(2)} = 1.0975, x_2^{(2)} = -0.819, x_3^{(2)} = 1.9071$$

$$\left|x_1^{(2)} - x_1^{(1)}\right| = |1.0975 - 1.75| = 0.6525 > \epsilon$$

$k = 2$
$i = 1$

$$x_1^{(3)} = \frac{1}{a_{11}}\big(b_1 - a_{12}x_2^{(2)} - a_{13}x_3^{(2)}\big)$$

$$x_1^{(3)} = \frac{1}{4}\big(7 - (-1 \times -0.819) - (1 \times 1.9071)\big) \Rightarrow x_1^{(3)} = 1.0684$$

$i = 2$

$$x_2^{(3)} = \frac{1}{a_{22}}\big(b_2 - a_{21}x_1^{(3)} - a_{23}x_3^{(2)}\big)$$

$$x_2^{(3)} = \frac{1}{5}\big( - 8 - (-2 \times 1.0684) - (-1 \times 1.9071)\big) \Rightarrow x_2^{(3)} = -0.79122$$

$i = 3$

$$x_3^{(3)} = \frac{1}{a_{33}}\big(b_3 - a_{31}x_1^{(3)} - a_{32}x_2^{(3)}\big)$$

$$x_3^{(3)} = \frac{1}{3}\big(6 - (1 \times 1.0684) - (1 \times -0.79122)\big) \Rightarrow x_3^{(3)} = 1.9076$$

$$x_1^{(3)} = 1.0684, x_2^{(3)} = -0.79122, x_3^{(3)} = 1.9076$$

$$\left|x_1^{(3)} - x_1^{(2)}\right| = |1.0684 - 1.0975| = 0.0291 > \epsilon$$

$k=3$
$i=1$

$$x_1^{(4)} = \frac{1}{a_{11}}\big(b_1 - a_{12}x_2^{(3)} - a_{13}x_3^{(3)}\big)$$

$$x_1^{(4)} = \frac{1}{4}\big(7 - (-1 \times -0.79122) - (1 \times 1.9076)\big) \Rightarrow x_1^{(4)} = 1.0752$$

$i=2$

$$x_2^{(4)} = \frac{1}{a_{22}}\big(b_2 - a_{21}x_1^{(4)} - a_{23}x_3^{(3)}\big)$$

$$x_2^{(4)} = \frac{1}{5}\big(-8 - (-2 \times 1.0752) - (-1 \times 1.9076)\big) \Rightarrow x_2^{(4)} = -0.7884$$

$i=3$

$$x_3^{(4)} = \frac{1}{a_{33}}\big(b_3 - a_{31}x_1^{(4)} - a_{32}x_2^{(4)}\big)$$

$$x_3^{(4)} = \frac{1}{3}\big(6 - (1 \times 1.0752) - (1 \times -0.7884)\big) \Rightarrow x_3^{(4)} = 1.9044$$

$$x_1^{(4)} = 1.0752, x_2^{(4)} = -0.7884, x_3^{(4)} = 1.9044$$

$$\left|x_1^{(4)} - x_1^{(3)}\right| = |1.0752 - 1.0684| = 0.0068 < \epsilon$$

$$\left|x_2^{(4)} - x_2^{(3)}\right| = |-0.7884 - (-0.79122)| = 0.00282 < \epsilon$$

$$\left|x_3^{(4)} - x_3^{(3)}\right| = |1.9044 - 1.9076| = 0.0032 < \epsilon$$

So the solution set is: $x_1^{(4)} = 1.0752, x_2^{(4)} = -0.7884, x_3^{(4)} = 1.9044$

3. textbfNewton-Raphson method

   The general form of the method is

$$x_(n+1) = x_n - \frac{f(x_n)}{f'(x_n)}$$

   Where:

   $n = 0, 1, , 2, 3, \ldots$

   $f(x) = 0$ at $[a, b]$

   $x_0$: Defined the value $x_(n+1)$ is a root of the function $f(x) = 0$ if and only if it satisfies the following

   $\big|x_(n+1) - x_n\big| \leq \epsilon$ , Where $\epsilon$ is a small value

**Remark:** If $x_0$ is not defined it can be calculated using the formula Below

$x_0 = \frac{a+b}{2}$

**EX:** using Newton-Raphson method to find the root of the following function within $[0, 1]$

$x_0 = 0, x = e^{-x}, \epsilon = 0.005$

**Sol:**

$$f(x) = 0, x = e^{-x} \xrightarrow{\text{move } e^{-x}} f(x) = x - e^{-x}$$

Now find derivative of $f(x)$

$$f'(x) = 1 + e^{-x}$$

$$x_(n + 1) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{f(0)}{f'(0)} \Rightarrow x_1 = 0 - \frac{0 - e^0}{1 + e^0} \Rightarrow x_1 = \frac{1}{2} = 0.5$$

$$|x_{n+1} - x_n| \Rightarrow |x_1 - x_0| \Rightarrow |0.5 - 0| = 0.5 > \epsilon$$

$n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)} \Rightarrow x_2 = 0.5 - \frac{0.5 - e^{-0.5}}{1 + e^{-0.5}} \Rightarrow x_2 = 0.566$$

$$|x_2 - x_1| \Rightarrow |0.566 - 0.5| = 0.066 > \epsilon$$

$n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.566 - \frac{f(0.566)}{f'(0.566)} \Rightarrow x_3 = 0.566 - \frac{0.566 - e^{-0.566}}{1 + e^{-0.566}} \Rightarrow x_3 = 0.567$$

$$|x_3 - x_2| \Rightarrow |0.567 - 0.566| = 0.001 < \epsilon$$

$x_3 = 0.567$ is the root of $f(x) = x - e^{-x}$

**Newton's** method to find the $n_{th}$ roots To find $\sqrt{B}$ , we can follow the function below

(a) $x = \sqrt{B} \Rightarrow x^2 = B \Rightarrow x^2 - B = 0 \Rightarrow \boxed{f(x) = x^2 - B}$

(b) Now $x = \pm\sqrt{B}$

(c) $f'(x) = 2x$

(d) $x = g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow x - \frac{x^2-B}{2x} = \frac{x^2+B}{2x} \Rightarrow \frac{x+\frac{B}{x}}{2}$

(e) Now we can use the formula: $r_{(n+1)} = \frac{r_n+\frac{B}{r_n}}{2}$

**EX:** Find $\sqrt{3}$ with $r_0 = 1$

**Sol:**

$$r_{(n+1)} = \frac{r_n + \frac{B}{r_n}}{2}, \quad B = 3$$

$n = 0$

$$r_1 = \frac{r_0 + \frac{B}{r_0}}{2}$$

$$r_1 = \frac{1 + \frac{3}{1}}{2} \Rightarrow r_1 = \frac{4}{2} \Rightarrow r_1 = 2$$

$n = 1$

$$r_2 = \frac{r_1 + \frac{B}{r_1}}{2}$$

$$r_2 = \frac{2 + \frac{3}{2}}{2} \Rightarrow r_2 = \frac{7}{4} \Rightarrow r_2 = 1.75$$

$n = 2$

$$r_3 = \frac{r_2 + \frac{B}{r_2}}{2}$$

$$r_3 = \frac{1.75 + \frac{3}{1.75}}{2} \Rightarrow r_3 = \frac{97}{56} \Rightarrow r_3 = 1.732142857$$

$n = 3$

$$r_4 = \frac{r_3 + \frac{B}{r_3}}{2}$$

$$r_4 = \frac{1.732142857 + \frac{3}{1.732142857}}{2} \Rightarrow r_4 = 1.73205$$

$n = 4$

$$r_5 = \frac{r_4 + \frac{B}{r_4}}{2}$$

$$r_5 = \frac{1.73205 + \frac{3}{1.73205}}{2} \Rightarrow r_5 = 1.73205$$

We will stop if our last two results matches

| $n$ | $r_{n+1}$ |
|---|---|
| 0 | 2 |
| 1 | 1.75 |
| 2 | 1.732142857 |
| 3 | 1.73205 |
| 4 | 1.73205 |

4. **Fixed point method**

   To find the root of an equation using this method, we can follow the following steps:

   (a) Putting the function $f(x)$ in the form of $x = g(x)$

   (b) Make sure $|g'(x_0)| < 1$ , If not we need to chose another form of the $g(x)$

   (c) Find $x_{n+1}$ , using the formula $x_{n+1} = g(x_n), \quad n = 0, 1, 2, \dots$

   (d) The value $x_{x+1}$ is a root of the function $f(x) = 0$ if and only if $|x_{n+1} - x_n| \leq \epsilon$ , Where $\epsilon$ is a small value

   **EX:** Use fixed point method to find the root of $f(x) = x^2 - x - 3 = 0$ within $[1, 2]$ and $\epsilon = 0.01$

   **Sol:**

$$x_0 = \frac{a+b}{2} \Rightarrow x_0 = \frac{1+2}{2} \Rightarrow x_0 = \frac{3}{2} \Rightarrow x_0 = 1.5$$

$$f(x) = x^2 - x - 3 = 0$$

$$x^2 - x - 3 = 0 \xrightarrow{\text{move } x} x = x^2 - 3$$

$$x = g(x), \quad g'(x) = 2x$$

$$|g'(x_0)| \Rightarrow |g'(1.5)| \Rightarrow |2 \times 1.5| = 3 > 1$$

   We need to find another form of $g(x)$

$$x^2 - x - 3 = 0 \Rightarrow x^2 = x + 3 \Rightarrow x = \sqrt{x+3} = g(x)$$

$$g'(x) = \frac{1}{2\sqrt{x+3}}$$

$$|g'(x_0)| \Rightarrow \left|\frac{1}{2\sqrt{1.5+3}}\right| = 0.23 < 1$$

$$x_{n+1} = g(x_n)$$

$n = 0$

$$x_1 = g(x_0)$$

$$x_1 = g(1.5) = \sqrt{1.5 + 3} = \sqrt{4.5} = 2.12 \Rightarrow x_1 = 2.12$$

$$|x_1 - x_0| = |2.12 - 1.5| = 0.62 > \epsilon$$

$n = 1$

$$x_2 = g(x_1)$$

$$x_2 = g(2.12) = \sqrt{2.12 + 3} = \sqrt{5.12} = 2.26 \Rightarrow x_2 = 2.26$$

$$|x_2 - x_1| = |2.26 - 2.12| = 0.14 > \epsilon$$

$n = 2$

$$x_3 = g(x_2)$$

$$x_3 = g(2.26) = \sqrt{2.26 + 3} = \sqrt{5.26} = 2.29 \Rightarrow x_3 = 2.29$$

$$|x_3 - x_2| = |2.29 - 2.26| = 0.03 > \epsilon$$

$n = 3$

$$x_4 = g(x_3)$$

$$x_4 = g(2.29) = \sqrt{2.29 + 3} = \sqrt{5.29} = 2.3 \Rightarrow x_4 = 2.3$$

$$|x_4 - x_2| = |2.3 - 2.29| = 0.01 = \epsilon$$

So $x_4 = 2.3$ is the root of $f(x)$

**Interpolation theory:** Used to find the missing data (values) of the existing data based on the relationships between the data (values)

**Lagrang interpolation method:** In a series of x-values, This method can be used to find the value of $f(x)$ , this is one of the simplest numerical methods that can be show by the following data

| $x$ | $x_0$ | $x_1$ | $\dots$ | $x_n$ |
|---|---|---|---|---|
| $f(x)$ | $f(x_0)$ | $f(x_1)$ | $\dots$ | $f(x_n)$ |

The Lagrang method for the above data will be

$$f(x^*) = f(x_0)L_0x^* + f(x_1)L_1x^* + f(x_2)L_2x^*$$

Where:
$x^*$: The value we want to find
$L_ix^* = \frac{(x^*-x_1)(x^*-x_2)}{(x_0-x_1)(x_0-x_2)} \times \frac{(x^*-x_0)(x^*-x_2)}{(x_1-x_0)(x_1-x_2)} \times \frac{(x^*-x_0)(x^*-x_1)}{(x_2-x_0)(x_2-x_1)}$
So the general form is

$$f(x^*) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{x^* - x_i}{x_j - x_i}$$

**EX:** By Lagrang formula, find $f(3)$ and $f(5)$ using the following set

| $x$ | 0 | 1 | 2 | 4 |
|---|---|---|---|---|
| $f(x)$ | 1 | 1 | 2 | 5 |

**Sol**

$$f(x^*) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{x^* - x_i}{x_j - x_i}$$

$n = 3$ because we have 4 values in the table

$$\begin{aligned} f(x^*) =& f(x_0)\frac{(x^*-x_1)(x^*-x_2)(x^*-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \quad j=0, i=1,2,3 \\ &+f(x_1)\frac{(x^*-x_0)(x^*-x_2)(x^*-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \quad j=1, i=0,2,3 \\ &+f(x_2)\frac{(x^*-x_0)(x^*-x_1)(x^*-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \quad j=2, i=0,1,3 \\ &+f(x_3)\frac{(x^*-x_0)(x^*-x_1)(x^*-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \quad j=3, i=0,1,2 \end{aligned}$$

$$\begin{aligned}
f(3) = & f(0)\frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} \\
& + f(1)\frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} \\
& + f(2)\frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} \\
& + f(4)\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}
\end{aligned}$$

$$f(3) = 1 \times \frac{-2}{-8} + 1 \times \frac{-3}{3} + 2 \times \frac{-6}{-4} + 5 \times \frac{6}{24} = 3 \Rightarrow f(3) = 3$$

$$\begin{aligned}
f(5) = & f(0)\frac{(5-1)(5-2)(5-4)}{(0-1)(0-2)(0-4)} \\
& + f(1)\frac{(5-0)(5-2)(5-4)}{(1-0)(1-2)(1-4)} \\
& + f(2)\frac{(5-0)(5-1)(5-4)}{(2-0)(2-1)(2-4)} \\
& + f(4)\frac{(5-0)(5-1)(5-2)}{(4-0)(4-1)(4-2)}
\end{aligned}$$

$$f(5) = 1 \times \frac{12}{-8} + 1 \times \frac{15}{3} + 2 \times \frac{20}{-4} + 5 \times \frac{60}{24} = 6 \Rightarrow f(5) = 6$$

**Inverse Lagrang:** used to find $x$ from $y^*$ as follows

$$x^* = \sum_{j=0}^{n} x_j \prod_{\substack{i=0 \\ i \neq j}}^{n}$$

Lets say $n = 2$

$$x_* = x_0 \frac{(y^* - y_1)(y^* - y_2)}{(y_0 - y_1)(y_0 - y_2)} + x_1 \frac{(y^* - y_0)(y^* - y_2)}{(y_1 - y_0)(y_1 - y_2)} + x_2 \frac{(y^* - y_0)(y^* - y_1)}{(y_2 - y_0)(y_2 - y_1)}$$

**EX:** Find the value of $x^*$ when $y^* = 0.2703$ for the following data

| | | | |
|---|---|---|---|
| $x$ | 1.6 | 2.9 | 4.8 |
| $y$ | 0.625 | 0.3448 | 0.2083 |

**Sol:**

$$x^* = \sum_{j=0}^{n} x_j \prod_{\substack{i=0 \\ i \neq j}}^{n}$$

$n = 2$

$$x_* = x_0 \frac{(y^* - y_1)(y^* - y_2)}{(y_0 - y_1)(y_0 - y_2)} + x_1 \frac{(y^* - y_0)(y^* - y_2)}{(y_1 - y_0)(y_1 - y_2)} + x_2 \frac{(y^* - y_0)(y^* - y_1)}{(y_2 - y_0)(y_2 - y_1)}$$

$$
\begin{aligned}
x_* =& 1.6\frac{(0.2703-0.3448)(0.2703-0.2083)}{(0.625-0.3448)(0.625-0.2083)}\\
&+2.9\frac{(0.2703-0.625)(0.2703-0.2083)}{(0.3448-0.625)(0.3448-0.2083)}\\
&+4.8\frac{(0.2703-0.625)(0.2703-0.3448)}{(0.2083-0.625)(0.2083-0.3448)}
\end{aligned}
$$

$$
x_* = 1.6\frac{-0.004619}{0.11675934} + 2.9\frac{-0.0219914}{-0.0382473} + 4.8\frac{0.02642515}{0.05687955} \Rightarrow x^* = 3.834131631
$$