# Numerical Analysis

Ahmed Mohamed

November 22, 2025

# 1 Introduction

**Matrix:** Rectangular array of numbers, symbols or expressions arranged in rows and column

**Ex:** $2x3$ matrix would look like this

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

The number of elements in the matrix is $2 \times 3 = 6$

**Remark:** Matrices are used in different mathematical operations such as addition, multiplication, and finding determinants and inverses.

**General form for the matrix** $m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

**Matrix Applications:**

1. Computer Graphics
   - Scaling and rotation images
   - 3D
2. Machine Learning:
   - Features of data points in algorithms, Such as linear regression, Neural networks
3. Economics
   - Building an economy system
   - Analyze the strategies
4. Single processing
   - Can be used in Image processing
5. Control systems
   - Finding the relationship between inputs/outputs in databases
6. Statistics
   - Regression analysis

## 2 Matrices Types

1. **Row Matrix:** Have only one row

   **EX:** $A = 1 \times 3$

   $$A = \begin{bmatrix} 4 & 0 & \frac{1}{3} \end{bmatrix}$$

2. **Column Matrix:** Have only one column

   **EX:** $A = 3 \times 1$

   $$A = \begin{bmatrix} 5 \\ 33 \\ 0.4 \end{bmatrix}$$

3. **Square Matrix:** Have an equal number of rows and columns

   **EX:** $A = 3 \times 3$

   $$A = \begin{bmatrix} 5 & -1 & 2 \\ 33 & 3 & 20 \\ 0.4 & -3 & 0 \end{bmatrix}$$

   **EX:** $B = 1 \times 1$

   $$B = \begin{bmatrix} 3.14 \end{bmatrix}$$

4. **Diagonal Matrix:** All non-diagonal elements are zeros

   **EX:** $A = 3 \times 3$

   $$A = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{8} \end{bmatrix}$$

5. **Scalar Matrix:** Square matrix with all non-diagonal elements are zero and all diagonal elements are equal

   **EX:** $A = 3 \times 3$

   $$A = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

6. **Unit Matrix(Identity):** Type of diagonal matrix, its all diagonal elements equal to 1

   **EX:** $A = 3 \times 3$

   $$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

7. **Null(Zero) Matrix:** All elements equal to 0

   **EX:** $A = 3 \times 3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8. **Symmetric Matrix:** If the matrix $A$ is equal to $A^\top$
   Where
   $A^\top$: is the transpose matrix from $A$, Generated by switching rows and columns

   **EX:** Check if $A = A^\top$

$$A = \begin{bmatrix} -1 & 2 & 10 \\ 30 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} \rightarrow A^\top = \begin{bmatrix} -1 & 30 & -3 \\ 2 & 0 & -2 \\ 10 & 1 & -5 \end{bmatrix}$$

   $\therefore A \neq A^\top$
   $\therefore A$ Is not a symmetric matrix

# 3 Addition And Subtraction

Two matrices can be added or subtracted if and only if the have the same order which is mean the same numbers of rows and columns

**Ex:** Find $A + B$ and $A - B$ Where

$$A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix} B = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$$

**Sol:**

$$A + B = \begin{bmatrix} 2+3 & 0+6 \\ -5+4 & 6+1 \end{bmatrix} \rightarrow A + B = \begin{bmatrix} 5 & 6 \\ -1 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-3 & 0-6 \\ -5-4 & 6-1 \end{bmatrix} \rightarrow A - B = \begin{bmatrix} -1 & -6 \\ -9 & 5 \end{bmatrix}$$

**Ex:** Find $A + B$ Where

$$A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix} B = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$$

**Sol:** $A + B$ Is not defined since they have different orders

**Properties Of Addition**:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0$ (zero matrix) $= A$
4. $A - B \neq B - A$
5. $(A - B) - C = A - (B - C)$
6. $A^{\top^{\top}} = A$
7. $aA^{\top} = aA^{\top}$ Where $a$ is a number
8. $(A + B)^{\top} = B^{\top} + A^{\top}$

**H.W(1):** Prove that all the properties of matrix addition are satisfied

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & -10 \\ 3 & 5 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -5 & -3 \\ -2 & 1 & -0 \\ 4 & -7 & 10 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 6 & -1 \\ 20 & 0 & 10 \\ -10 & 3 & -30 \end{bmatrix}$$

# 4 Matrix multiplication

To defined the multiplication of two matrices $AB$, If and only if the number of columns in $A$ is equal to the number of rows in $B$

**Ex:** Find $AB$ Where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

**Sol:**

$$AB = \begin{bmatrix} 1 \times 2 + 2 \times 3 & 1 \times 1 + 2 \times 0 & 1 \times 4 + 2 \times 5 \\ 3 \times 2 + 4 \times 3 & 3 \times 1 + 4 \times 0 & 3 \times 4 + 4 \times 5 \\ 0 \times 2 + 1 \times 3 & 0 \times 1 + 1 \times 0 & 0 \times 4 + 1 \times 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 8 & 1 & 14 \\ 18 & 3 & 32 \\ 3 & 0 & 5 \end{bmatrix}$$

**Properties of multiplication**:

1. $A(B + C) = AB + AC$
2. $(B + C)A = BA + CA$
3. $AB \neq BA$
4. $AB(C) = A(BC) = ABC$
5. $AI = IA = A$
6. $AA^{-1} = I$
   Where $A^{-1}$: is the inverse of $A$
7. $(AB)^{\top} = B^{\top}A^{\top}$

# 5 Determinant of matrix

Single number can be found only for $2 \times 2$ matrices, Will be denoted by $det(A)$ or $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$det(A) = |A| = a_{11} \times a_{22} - a_{12} \times a_{21}$$

**Ex:** Find Determinant of $A$ Where

$$A = \begin{bmatrix} 6 & 4 \\ 7 & 9 \end{bmatrix} \Rightarrow |A| = 6 \times 9 - 4 \times 7 \Rightarrow |A| = 26$$

For a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{12} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Ex:** Find $|A|$
Where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

**Sol:**

$$A = 1 \times \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 0 \\ -2 & 3 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$$

$$|A| = 1(-1 \times 3 - 0 \times 0) - 2(0 \times 3 - 0 \times -2) + 4(0 \times 0 - 1 \times -2) \rightarrow |A| = 5$$

# 6 Cofactor of a matrix

$$let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can find the Cofactor of $A$ by

$$C_{ij} = (-1)^{i+j} A_{ij}$$

$$C_{11} = -1^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{11} = a_{22}a_{33} - a_{23}a_{32}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{12} = -\big(a_{21}a_{33} - a_{23}a_{31}\big)$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow C_{13} = a_{21}a_{32} - a_{22}a_{31}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{21} = -\big(a_{12}a_{33} - a_{13}a_{32}\big)$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{22} = a_{11}a_{33} - a_{13}a_{31}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow C_{23} = -\big(a_{11}a_{32} - a_{12}a_{31}\big)$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \Rightarrow C_{31} = a_{12}a_{23} - a_{13}a_{22}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \Rightarrow C_{32} = -\big(a_{11}a_{23} - a_{13}a_{21}\big)$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow C_{33} = a_{11}a_{22} - a_{12}a_{21}$$

**Ex(2):** Find the cofactor for the following

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 5 & -4 \end{vmatrix} \Rightarrow C_{11} = 0 \cdot (-4) - 3 \cdot 5 = -15$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} \Rightarrow C_{12} = -\big(1 \cdot (-4) - 3 \cdot 2\big) = -(-4-6) = 10$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} \Rightarrow C_{13} = 1 \cdot 5 - 0 \cdot 2 = 5$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix} \Rightarrow C_{21} = -\big(4 \cdot (-4) - (-1) \cdot 5\big) = -(-16 + 5) = 11$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} \Rightarrow C_{22} = 3 \cdot (-4) - (-1) \cdot 2 = -12 + 2 = -10$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} \Rightarrow C_{23} = -\big(3 \cdot 5 - 4 \cdot 2\big) = -(15 - 8) = -7$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 4 & -1 \\ 0 & 3 \end{vmatrix} \Rightarrow C_{31} = 4 \cdot 3 - (-1) \cdot 0 = 12$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \Rightarrow C_{32} = -\big(3 \cdot 3 - (-1) \cdot 1\big) = -(9 + 1) = -10$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} \Rightarrow C_{33} = 3 \cdot 0 - 4 \cdot 1 = -4$$

$$C = \begin{bmatrix} -15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$$

**Remark:** if $A$ is a matrix of $2 \times 2$ you can find the cofactor like this

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

**Ex:** Find cofactor for $A$ Where

$$A = \begin{bmatrix} -2 & -4 \\ 10 & 5 \end{bmatrix}$$

**Sol:**

$$A = \begin{vmatrix} -2 & -4 \\ 10 & 5 \end{vmatrix} \quad \Rightarrow \quad C = \begin{vmatrix} 5 & -10 \\ 4 & -2 \end{vmatrix}$$

## 7 Adjoint Matrix

Adj($A$) Can be found $adj(A) = C^{\top}$ Where $C$ is the cofactor of $A$

**Ex(3):** Find adjoint for $A$ from the example number 2

$$C = \begin{bmatrix} 15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$$

**Sol:**

$$adj(A) = C^{\top} = \begin{bmatrix} -15 & 11 & 12 \\ 10 & -10 & -10 \\ 5 & -7 & -4 \end{bmatrix}$$

## 8 Inverse Matrix ($A^{-1}$)

You can find the inverse from this form $A^{-1} = \frac{adj(A)}{|A|}$, Where $|A| \neq 0$

If $|A| = 0$ Then $A$ doesn't have inverse

**Ex:** Find $A^{-1}$ from Ex(3)

$$adj(A) = C^{\top} = \begin{bmatrix} -15 & 11 & 12 \\ 10 & -10 & -10 \\ 5 & -7 & -4 \end{bmatrix}$$

**Sol:**

After Finding the cofactor it will equal $|A| = -10$

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} \frac{15}{10} & -\frac{11}{10} & -\frac{12}{10} \\ \\ -\frac{10}{10} & \frac{10}{10} & \frac{10}{10} \\ \\ -\frac{5}{10} & \frac{7}{10} & \frac{4}{10} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1.5 & -1.1 & -1.2 \\ -1 & 1 & 1 \\ -0.5 & 0.7 & 0.4 \end{bmatrix}$$

# 9 Lower & Upper Triangular Matrix

**Lower Triangular Matrix** All the elements above the main diagonal are zeros

**Ex:** $3 \times 3$ Lower Triangular matrix

$$A = \begin{bmatrix} a_{11} & \boxed{0} & \boxed{0} \\ a_{21} & a_{22} & \boxed{0} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

**Upper Triangular Matrix** All the elements below the main diagonal are zeros

**Ex:** $3 \times 3$ Upper Triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{0} & a_{22} & a_{23} \\ \boxed{0} & \boxed{0} & a_{33} \end{bmatrix}$$

**Ex:** Transfer $A$ to upper triangular matrix, Where

$$A = \begin{array}{c} \begin{bmatrix} 0 & 4 & 2 \\ 2 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

**Sol:**
$R_1 = R2$, Swapping two rows

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

Apply $R_3 = 2R_3 - 3R_1$ to the above matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & -7 & -13 \end{bmatrix}$$

Apply $R_3 = 7R_2 + 4R_3$ to the above matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & -13 \end{bmatrix}$$

**H.W(2):** Make it lower triangular matrix

## 10 Systems of linear equations

Linear system of equations can be written as

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\ &\vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m \end{aligned}$$

This system can be transfer into $AX = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

To solve this system of linear equations, We can use the following methods

1. **Gauss Elimination Method** To use this method we need to transfer the matrix $A$ into a lower or upper Triangular matrix using row operations

   **Ex:** Solve the following system of the linear equations using Gauss method, Where

$$\begin{aligned} 4y + 2z &= 1 \\ 2x + 3y + 5z &= 0 \\ 3x + y + z &= 11 \end{aligned}$$

**Sol:**

$$AX = b, \quad AX = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 2 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 1 & 1 & 11 \end{array}\right]$$

$$R_1 = R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 3 & 1 & 1 & 11 \end{array}\right]$$

$$R_3 = 2R_3 - 3R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & -7 & -13 & 22 \end{array}\right]$$

$$R_3 = 4R_3 - 7R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -38 & 95 \end{array}\right]$$

$$\begin{aligned}
-38z = 95 \quad & \Rightarrow z = -\frac{95}{38} = -2.5 \\
4y + 2z = 1 \quad & \Rightarrow 4y + 2(-2.5) = 1 \\
& \Rightarrow 4y - 5 = 1 \\
& \Rightarrow 4y = 6 \Rightarrow y = \frac{3}{2} = 1.5 \\
2x + 3y + 5z = 0 \quad & \Rightarrow 2x + 3(1.5) + 5(-2.5) = 0 \\
& \Rightarrow 2x + 4.5 - 12.5 = 0 \\
& \Rightarrow 2x - 8 = 0 \\
& \Rightarrow x = 4
\end{aligned}$$

2. **Gauss-Jordan Elimination Method**
   We have to transfer the system to $JX = T$

$$J = \begin{bmatrix} a_{11} & 0 & 0 & \dots \\ 0 & a_{22} & 0 & \dots \\ 0 & 0 & a_{33} & \vdots \\ 0 & 0 & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$

   The solution will be

$$t_1, t_2, t_3, \dots t_n$$

**Ex:** Solve the following system of linear equations using Gauss-Jordan

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x + 3y + z &= 8 \\ 3x + y + 2z &= 7 \end{aligned}$$

**Sol:**

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 3 & 1 & 8 \\ 3 & 1 & 2 & 7 \end{array}\right]$$

$$R_2 = R_2 - 2R_1, \; R_3 = R_3 - 3R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -1 & -5 & -10 \\ 0 & -5 & -7 & -20 \end{array}\right]$$

$$R_2 = -R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 5 & 10 \\ 0 & -5 & -7 & -20 \end{array}\right]$$

$$R_3 = R_3 + 5R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$R_1 = R_1 - 2R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -11 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$R_1 = 18R_1 + 7R_3, \; R_2 = 18R_2 - 5R_3$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 18 & 0 & 0 & 12 \\ 0 & 18 & 0 & 30 \\ 0 & 0 & 18 & 30 \end{array}\right]$$

$$18x = 12 \qquad \Rightarrow x = \frac{12}{18} = 0.6$$

$$18y = 30 \qquad \Rightarrow y = \frac{30}{18} = 1.667$$

$$18z = 30 \qquad \Rightarrow z = \frac{30}{18} = 1.667$$

3. **General Form For Cramer's Rule** For a system with linear equations with $n$ variables of the form

$$\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\
a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\
&\vdots \\
a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m
\end{aligned}$$

the solution will be

$$x_i = \frac{D_i}{D}$$

Where

$D$: is the Determinant of the matrix $A$ generated from the system $AX = b$
$D_i$: Determinant of the matrix formed by replacing the $i - th$ column of $A$ with the constant vector $b$

**Ex:** Solve using Cramer's rule

$$\begin{aligned}
x + 2y + z &= 4 \\
2x + y + 3z &= 10 \\
3x + 2y + 2z &= 12
\end{aligned}$$

**Sol:**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 10 \\ 12 \end{bmatrix}$$

$$D = |A| = det(A) = 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 7$$

Now Replacing first column with the constants vector $b$

$$\begin{aligned}
D_1 = \begin{vmatrix} 4 & 2 & 1 \\ 10 & 1 & 3 \\ 12 & 2 & 2 \end{vmatrix} &= 4\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 10 & 3 \\ 12 & 2 \end{vmatrix} + 1\begin{vmatrix} 10 & 1 \\ 12 & 2 \end{vmatrix} \\
&= 4\big(1 \times 2 - 3 \times 2\big) - 2\big(10 \times 2 - 3 \times 12\big) + 1\big(10 \times 2 - 1 \times 12\big) \\
&= 24
\end{aligned}$$

$$\begin{aligned}
D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 10 & 3 \\ 3 & 12 & 2 \end{vmatrix} &= 1\begin{vmatrix} 10 & 3 \\ 12 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 10 \\ 3 & 12 \end{vmatrix} \\
&= 1\big(10 \times 2 - 3 \times 12\big) - 4\big(2 \times 2 - 3 \times 3\big) + 1\big(2 \times 12 - 10 \times 3\big) \\
&= -2
\end{aligned}$$

$$\begin{aligned}D_3 &= \begin{vmatrix}1 & 2 & 4\\ 2 & 1 & 10\\ 3 & 2 & 12\end{vmatrix} = 1\begin{vmatrix}1 & 10\\ 2 & 12\end{vmatrix} - 2\begin{vmatrix}2 & 10\\ 3 & 12\end{vmatrix} + 4\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\\ &= 1\big(1\times 12 - 10\times 2\big) - 2\big(2\times 12 - 10\times 3\big) + 4\big(2\times 2 - 1\times 3\big)\\ &= 8\end{aligned}$$

$$x = \frac{D_1}{D} \Rightarrow x = \frac{24}{7} \Rightarrow x \approx 3.4$$
$$y = \frac{D_2}{D} \Rightarrow y = -\frac{2}{7} \Rightarrow y \approx 0.2$$
$$z = \frac{D_3}{D} \Rightarrow z = \frac{8}{7} \Rightarrow z \approx 1.1$$

4. **Inverse Matrix Method**

For system of equation written in a matrix form $AX = b$

If $A \neq 0$, Then the solution of the system will be $X = A^{-1}b$, which comes from the following

$AX = b \Rightarrow A^{-1}AX = A^{-1} \Rightarrow IX = A^{-1}b \Rightarrow \boxed{X = A^{-1}b}$

Where is a matrix with all it's diagonal elements are ones

$$I_{3\times 3} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

**Ex(1):** solve the system using inverse method

$$2x + 3y = 8$$
$$x + 4y = 7$$

$$A = \begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix} \qquad b = \begin{bmatrix}8\\ 7\end{bmatrix}$$

$$AX = b \Rightarrow X = A^{-1}b$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$|A| = \begin{vmatrix}2 & 3\\ 1 & 4\end{vmatrix} = 5$$

$$Adj(A) = \begin{bmatrix}4 & -3\\ -1 & 2\end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix}\frac{4}{5} & -\frac{3}{5}\\ -\frac{1}{5} & \frac{2}{5}\end{bmatrix}$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \times \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

$$x = \frac{4}{5} \times 8 - \frac{3}{5} \times 7 \Rightarrow x = \frac{32}{5} - \frac{21}{5} \Rightarrow x = \frac{11}{5} = 2.2$$

$$y = -\frac{1}{5} \times 8 + \frac{2}{5} \times 7 \Rightarrow y = -\frac{8}{5} + \frac{14}{5} \Rightarrow y = \frac{6}{5} \Rightarrow y = 1.2$$

**Ex(2):**

$$x + y + z = 6$$
$$2y + 5z = -4$$
$$2x + 5y - z = 27$$

**Sol:**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

$$AX = b \Rightarrow X = A^{-1}b, \text{ Where } A^{-1} = \frac{adj(A)}{|A|}$$

$$adj(A) = C^{\top}$$

$$C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{vmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix}2 & 5\\5 & -1\end{vmatrix} = 1\big[2\times(-1) - 5\times 5\big] = -27$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}0 & 5\\2 & -1\end{vmatrix} = -1\big[0\times(-1) - 5\times 2\big] = 10$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}0 & 2\\2 & 5\end{vmatrix} = 1\big[0\times 5 - 2\times 2\big] = -4$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}1 & 1\\5 & -1\end{vmatrix} = -1\big[1\times(-1) - 1\times 5\big] = 6$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\2 & -1\end{vmatrix} = 1\big[1\times(-1) - 1\times 2\big] = -3$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 1\\2 & 5\end{vmatrix} = -1\big[1\times 5 - 1\times 2\big] = -3$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}1 & 1\\2 & 5\end{vmatrix} = 1\big[1\times 5 - 1\times 2\big] = 3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\0 & 5\end{vmatrix} = -1\big[1\times 5 - 1\times 0\big] = -5$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 1\\0 & 2\end{vmatrix} = 1\big[1\times 2 - 1\times 0\big] = 2$$

$$C = \begin{bmatrix}-27 & 10 & -4\\6 & -3 & -3\\3 & -5 & 2\end{bmatrix}$$

$$\operatorname{adj}(A) = C^{\top} = \begin{bmatrix}-27 & 6 & 3\\10 & -3 & -5\\-4 & -3 & 2\end{bmatrix}$$

Finding the cofactor

$$\begin{aligned}|A| &= 1\begin{vmatrix}2 & 5\\5 & -1\end{vmatrix} - 1\begin{vmatrix}0 & 5\\2 & -1\end{vmatrix} + 1\begin{vmatrix}0 & 2\\2 & 5\end{vmatrix}\\ |A| &= 1(2\times -1 - 5\times 5) - 1(0\times -1 - 5\times 2)\\ &\quad + 1(0\times 5 - 2\times 2)\\ |A| &= -21\end{aligned}$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \begin{bmatrix}\frac{-27}{-21} & \frac{6}{-21} & \frac{3}{-21}\\ \frac{10}{-21} & \frac{-3}{-21} & \frac{-5}{-21}\\ \frac{-4}{-21} & \frac{-3}{-21} & \frac{2}{-21}\end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix}\frac{9}{7} & -\frac{2}{7} & -\frac{1}{7}\\ -\frac{10}{21} & \frac{1}{7} & \frac{5}{21}\\ \frac{4}{21} & \frac{1}{7} & -\frac{2}{21}\end{bmatrix}$$

$$X = A^{-1}b$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{10}{21} & \frac{1}{7} & \frac{5}{21} \\ \frac{4}{21} & \frac{1}{7} & -\frac{2}{21} \end{bmatrix} \times \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

$$x = \frac{9}{7} \times 6 + \frac{2}{7} \times 4 - \frac{1}{7} \times 27 \Rightarrow x = 5$$

$$y = -\frac{10}{21} \times 6 - \frac{1}{7} \times 4 + \frac{5}{21} \times 27 \Rightarrow y = 3$$

$$z = \frac{4}{21} \times 6 - \frac{1}{7} \times 4 - \frac{2}{21} \times 27 \Rightarrow z = -2$$

5. **Lower/Upper Decomposition Method**

   In this method we will write $A = LU$, Where

   $L$: Lower triangular matrix with (1)s on the main diagonal

   $U$: Upper triangular Matrix

   Then

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

   Which will produce the following simplification

$$A = LU$$

$$LU = \begin{bmatrix} 1 \times u_{11} + 0 \times 0 + 0 \times 0 & 1 \times u_{12} + 0 \times u_{22} + 0 \times 0 & 1 \times u_{13} + 0 \times u_{23} + 0 \times u_{33} \\ l_{21}u_{11} + 1 \times 0 + 0 \times 0 & l_{21}u_{12} + 1 \times u_{22} + 0 \times 0 & l_{21}u_{13} + 1 \times u_{23} + 0 \times u_{33} \\ l_{31}u_{11} + l_{32} \times 0 + 1 \times 0 & l_{31}u_{12} + l_{32}u_{22} + 1 \times 0 & l_{31}u_{13} + l_{32}u_{23} + 1 \times u_{33} \end{bmatrix}$$

   Simplifies this

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU$$

$$A = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

This will produce this equations

$$u_{11} = a_{11}$$
$$u_{12} = a_{12}$$
$$u_{13} = a_{13}$$
$$l_{21}u_{11} = a_{21}$$
$$l_{21}u_{12} + u_{22} = a_{22}$$
$$l_{21}u_{13} + u_{23} = a_{23}$$
$$l_{31}u_{11} = a_{31}$$
$$l_{31}u_{12} + l_{32}u_{22} = a_{32}$$
$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

**Ex:**

$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + 7y + 5z &= 2 \\ 6x + 18y + 19z &= 3 \end{aligned}$$

**Sol:**

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 6 & 18 & 19 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} u_{11} &= a_{11} = 2 \\ u_{12} &= a_{12} = 3 \\ u_{13} &= a_{13} = 1 \end{aligned}$$

$$\begin{aligned} l_{21}u_{11} &= a_{21} \\ l_{21} \times 2 &= 4 \\ l_{21} &= 2 \end{aligned}$$

$$\begin{aligned} l_{21}u_{12} + u_{22} &= a_{22} \\ 2 \times 3 + u_{22} &= 7 \\ 6 + u_{22} &= 7 \\ u_{22} &= 7 - 6 \\ u_{22} &= 1 \end{aligned}$$

$$\begin{aligned} l_{21}u_{13} + u_{23} &= a_{23} \\ 2 \times 1 + u_{23} &= 5 \\ 2 + u_{23} &= 5 \\ u_{23} &= 5 - 2 \\ u_{23} &= 3 \end{aligned}$$

$$
\begin{aligned}
l_{31}u_{11} &= a_{31} \\
l_{31}2 &= 6 \\
l_{31} &= 3
\end{aligned}
$$

$$
\begin{aligned}
l_{31}u_{12} + l_{32}u_{22} &= a_{32} \\
3 \times 3 + l_{32}1 &= 18 \\
9 + l_{32} &= 18 \\
l_{32} &= 18 - 9 \\
l_{32} &= 9
\end{aligned}
$$

$$
\begin{aligned}
l_{31}u_{13} + l_{32}u_{23} + u_{33} &= a_{33} \\
3 \times 1 + 9 \times 3 + u_{33} &= 19 \\
3 + 27 + u_{33} &= 19 \\
30 + u_{33} &= 19 \\
u_{33} &= 19 - 30 \\
u_{33} &= -11
\end{aligned}
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix}
$$

$$
Ly = b
$$

$$
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

$$
1 \times y_1 + 0 \times y_2 + 0 \times y_3 = 1
$$

$$
\begin{aligned}
2 \times y_1 + 1 \times y_2 + 0 \times y_3 &= 2 \\
2 \times 1 + y_2 &= 2 \\
2 + y_2 &= 2 \\
y2 &= 2 - 2 \\
y_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
3 \times y_1 + 9 \times y_2 + 1 \times y_3 &= 3 \\
3 \times 1 + 9 \times 0 + y_3 &= 3 \\
3 + 0 + y_3 &= 3 \\
y_3 &= 3 - 3 \\
y_3 &= 0
\end{aligned}
$$

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$ux = y$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From the third row

$$\begin{aligned} 0 \times x + 0 \times y - 11 \times z &= 0 \\ -11z &= 0 \\ z &= 0 \end{aligned}$$

The second row

$$\begin{aligned} 0 \times x + 1 \times y + 3 \times z &= 0 \\ 0 + y + 3(0) &= 0 \\ y &= 0 \end{aligned}$$

The first row

$$\begin{aligned} 2 \times x + 3 \times y + 1 \times z &= 1 \\ 2x + 3 \times 0 + 1 \times 0 &= 1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$