Computer-based Engineering Mathematics

Systems of linear equations in MATLAB – Poisson's equation

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Example 1 (of an engineering problem with linear system): Poisson's equation

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- Poisson's equation
  - is an elliptic partial differential equation,
  - has become standard problem in treating them numerically
  - has a broad utility in theoretical physics, for example
    - numerical simulation of incompressible friction-affected flow fields ("Navier-Stokes equations")
    - potential field caused by a given electric charge or mass density distribution
  - named after the French mathematician and physicist Siméon Denis Poisson

Poisson's equation - maths point of view (2D)

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- Let $\overline{\Omega} = [0;1] \times [0;1]$ be a two dimensional unit square and
- $\Gamma = \partial\Omega$ is the boundary of Ω.
- We search for a function $u\colon \overline{\Omega} \to \mathbb{R}$, which fulfills the following conditions:

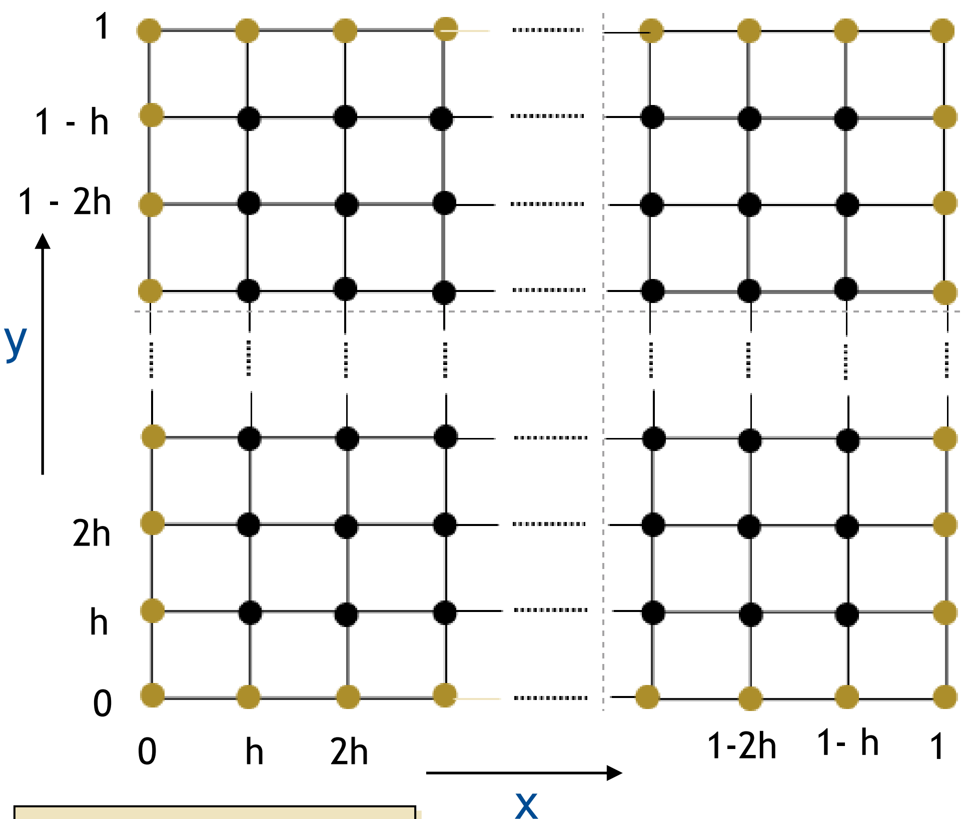
(1.1) $\Delta u = \frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{dy^2} = f$ in Ω (differential equation)

(1.2) $u = g$ on $\Gamma = \partial\Omega$ (Dirichlet boundary condition)

Δ: Laplace-Operator, here: 2D
f and u in general: real or complex-valued functions, here: real-valued
In the Euclidean space, the Laplace operator is often denoted as ∇^2
If $f = 0$, we obtain Laplace's equation.

📖 https://en.wikipedia.org/wiki/Laplace_operator

3D: $\Delta u = \frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{dy^2} + \frac{\partial^2 u}{dz^2}$



Discretization of Ω

with the grid width $h = \frac{1}{N+1}, N \in \mathbb{N}$

$\overline{\Omega} = [0;1] \otimes [0;1]$

$\rightarrow \ \overline{\Omega}_h, \overline{\Omega}_h, \partial\Omega_h = \overline{\Omega}_h \backslash \Omega_h$

$\overline{\Omega}_h = \{(kh, lh): k, l = 0,1,2,\ldots, N+1\}$

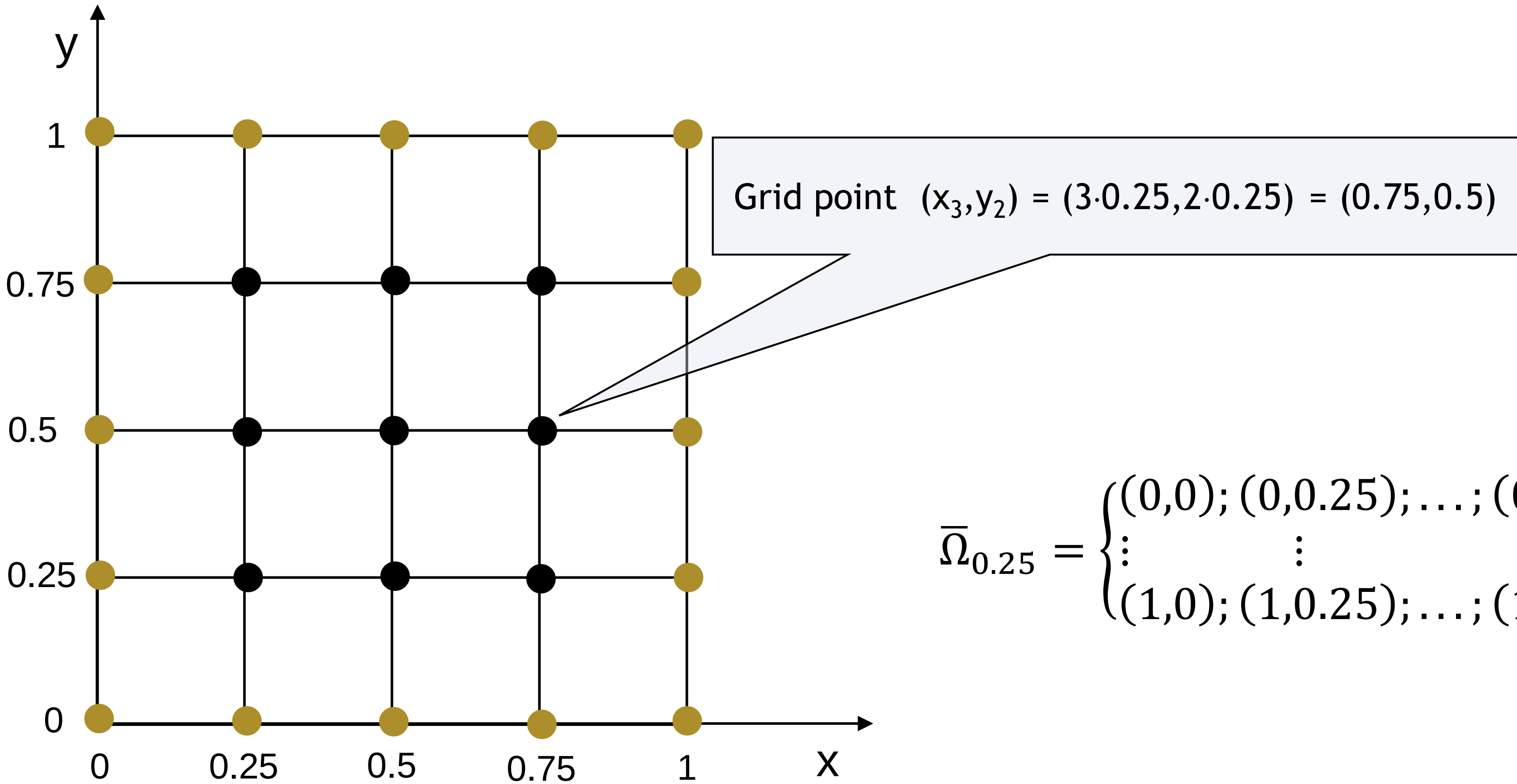
This discretization of the square $\overline{\Omega}$ generates a discrete set $\overline{\Omega}_h$ with $(N+2)^2$ elements.

- Boundary point
- Inner point

Mathematical functions can be (and often are) continuous.
Computers operate on discrete values.

Example: $N = 3$, discretization of Ω

- $N = 3 \Rightarrow$ grid width $h = \frac{1}{3+1} = 0.25$,
- The discrete set $\overline{\Omega}_{0.25}$ has $(3+2)^2 = 25$ elements.



$$\overline{\Omega}_{0.25} = \left\{\begin{matrix} (0,0);(0,0.25);\ldots;(0,1); \\ \vdots \qquad\qquad \vdots \\ (1,0);(1,0.25);\ldots;(1,1) \end{matrix}\right\}$$

Generating the "difference equation" from the differential equation (1.1) - considering x

w. r. t. = with respect to

For a fixed $y \in [0;1]$, we get the Taylor expansion of $u(x,y)$ w. r. t. x:

(1.3) $$u(x+h,y) = u(x,y) + h\frac{\partial u}{\partial x}(x,y) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\xi,y), \xi \in [x; x+h]$$

(1.4) $$u(x-h,y) = u(x,y) - h\frac{\partial u}{\partial x}(x,y) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\theta,y), \theta \in [x-h; x]$$

Addition of both equations (1.3) and (1.4) follows:

(1.5) $$u(x+h,y) + u(x-h,y) = 2u(x,y) + h^2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi,y) + \frac{\partial^4 u}{\partial x^4}(\theta,y)\right)$$

Solving the equation (1.5) for the second partial derivative of u w. r. t. x gives :

(1.6) $$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi,y) + \frac{\partial^4 u}{\partial x^4}(\theta,y)\right)$$

Generating the "difference equation" from the differential equation (1.1) - considering *y*

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Similarly, we get for the second partial derivative w. r. t. *y* :

We do the same as before, but keep x fixed and use y as variable.

(1.7) $$\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{u(x,y+h)+u(x,y-h)-2u(x,y)}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(x,\delta) + \frac{\partial^4 u}{\partial x^4}(x,\mu)\right), \delta,\mu \in [y-h; y+h]$$

Result from previous slide w. r. t. *x* :

(1.6) $$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{u(x+h,y)+u(x-h,y)-2u(x,y)}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi,y) + \frac{\partial^4 u}{\partial x^4}(\theta,y)\right), \xi,\theta \in [y-h; y+h]$$

Next step: use (1.6) and (1.7) in Poisson's equation (1.1)

$$\Delta u(x,y) = \underbrace{\frac{\partial^2 u}{dx^2}}_{(1.6)} + \underbrace{\frac{\partial^2 u}{dy^2}}_{(1.7)}$$

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Generating the "difference equation" from the differential equation (1.1) - considering y

Therefore:

(1.8) $$\Delta u(x,y) = \frac{u(x+h,y)+u(x-h,y)-4u(x,y)+u(x,y+h)+u(x,y-h)}{h^2}$$

$$\underbrace{-\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(x,\delta)+\frac{\partial^4 u}{\partial x^4}(x,\mu)+\frac{\partial^4 u}{\partial x^4}(\xi,y)+\frac{\partial^4 u}{\partial x^4}(\theta,y)\right)}$$

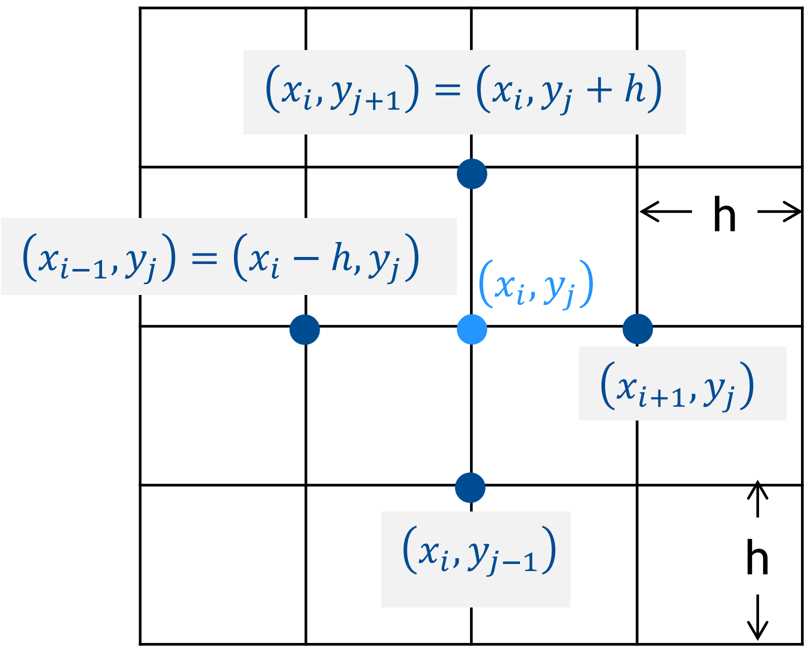
Discretization error

We substitute:

(1.9) $$\Delta_h u(x,y) = \frac{u(x+h,y)+u(x-h,y)-4u(x,y)+u(x,y+h)+u(x,y-h)}{h^2}$$

u on the grid points

On the grid points $(x_i, y_j) \in \overline{\Omega}_h, \;\; i, j = 0,1, \ldots, N+1$ we set $\quad u(x_i, y_j) = u_{i,j} \quad$ (1.10)



We use these notation for the inner grid points with $i, j = 1, \ldots, N$:

$$u(x_i + h, y_j) = u_{i+1,j}$$

$$u(x_i - h, y_j) = u_{i-1,j}$$

$$u(x_i, y_j + h) = u_{i,j+1}$$

$$u(x_i, y_j - h) = u_{i,j-1}$$

and rewrite eq. (1.9) as

$$\Delta_h u(x_i, y_j) = \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2}, \quad i, j = 1, \ldots N \tag{1.11}$$

Discretized Laplace-Operator and discrete boundary conditions

With $f(x_i, y_j) = f_{i,j}$, we get the *difference equation*:

(1.12) $$\Delta_h u(x_i, y_j) = \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2} = f_{i,j}, i, j = 1, \ldots N$$

at the inner grid points of Ω_h for the discretized Laplace-Operator

with the discrete boundary conditions

(1.13) $$\begin{cases} u(0, y_j) = u_{0,j} = g(0, y_j) = g_{0,j}, j = 0, \ldots, N+1 \\ u(1, y_j) = u_{N+1,j} = g(1, y_j) = g_{N+1,j}, j = 0, \ldots, N+1 \\ u(x_i, 0) = u_{i,0} = g(x_i, 0) = g_{i,0}, i = 0, \ldots, N+1 \\ u(x_i, 1) = u_{i,N+1} = g(x_i, 1) = g_{i,N+1}, i = 0, \ldots, N+1 \end{cases}$$

By multiplying both sides with h^2 we can write the equation (1.12) as

(1.14) $$u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1} = h^2 f_{i,j}, i, j = 1, \ldots, N$$

linear system of equations with N^2 equations and unknowns !

Building a linear system with N^2 equations

Boundary points with the indices $(i, 0)$, $(i, N+1)$, $(0, j)$, and $(N+1, j)$ are evaluated with the boundary function g as in (1.13). $u_{i,0}, u_{i,N+1}, u_{0,j}, u_{N+1,j}$ are expressed through g and brought to the right hand side of (1.14). Thus we get :

(1.15) Equations with two boundary points:

$$u_{2,1} + u_{1,2} - 4u_{1,1} = h^2 f_{1,1} - g_{0,1} - g_{1,0}$$
$$u_{N-1,1} + u_{N,2} - 4u_{N,1} = h^2 f_{N,1} - g_{N+1,1} - g_{N,0}$$
$$u_{1,N-1} + u_{2,N} - 4u_{1,N} = h^2 f_{1,N} - g_{0,N} - g_{1,N+1}$$
$$u_{N-1,N} + u_{N,N-1} - 4u_{N,N} = h^2 f_{N,N} - g_{N+1,N} - g_{N,N+1}$$

(1.16) Equations with one boundary point:

$$u_{i+1,1} + u_{i-1,1} - 4u_{i,1} + u_{i,2} = h^2 f_{i,1} - g_{i,0}, i = 2, \ldots, N-1$$
$$u_{i+1,N} + u_{i-1,N} - 4u_{i,N} + u_{i,N-1} = h^2 f_{i,N} - g_{i,N+1}, i = 2, \ldots, N-1$$
$$u_{N-1,j} + u_{N,j+1} - 4u_{N,j} + u_{N,j-1} = h^2 f_{N,j} - g_{N+1,j}, j = 2, \ldots, N-1$$
$$u_{2,j} + u_{1,j+1} - 4u_{1,j} + u_{1,j-1} = h^2 f_{1,j} - g_{0,j}, j = 2, \ldots, N-1$$

(1.17) Equations without boundary point:

$$u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1} = h^2 f_{i,j}, i, j = 2, \ldots, N-1$$

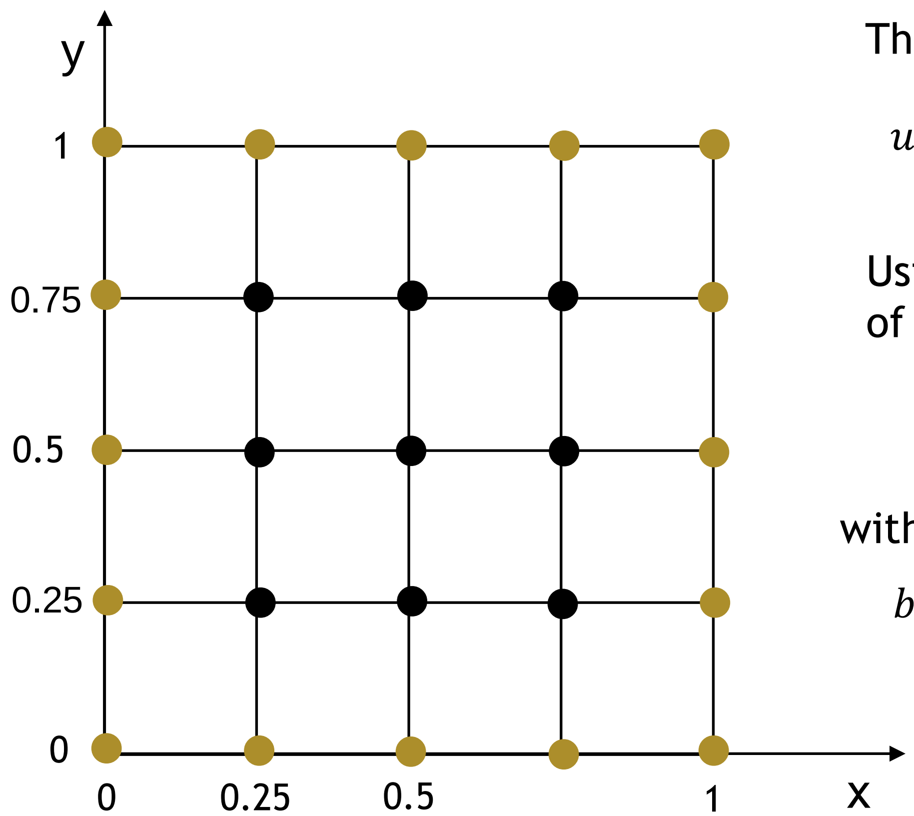
Final step: solution vector u

We arrange the solution vector $\boldsymbol{u}$ of the systems (1.14) as follows:

$$u = (u_{11}, \ldots, u_{1N}, u_{21}, \ldots, u_{2N}, \ldots, u_{N1}, \ldots, u_{NN}) \in \mathbb{R}^{N \times N} \tag{1.18}$$

Example: $N = 3$*, solution vector* u *and linear system of equations*

We already know that for *N* = 3, the grid width is $h = \frac{1}{3+1} = 0.25$



The solution vector has then the components

$$u = (u_{11}, \dots, u_{13}, u_{21}, \dots, u_{23}, u_{31}, \dots, u_{33}) \in \mathbb{R}^9$$

Using this in (1.15) - (1.17) we get the system of equation of the form

$$\boldsymbol{Au = b}$$

with coefficients matrix $\boldsymbol{A}$ and the right hand side

$$b = (b_{11}, \dots, b_{13}, b_{21}, \dots, b_{23}, b_{31}, \dots, b_{33}) \in \mathbb{R}^9$$

Example: $N = 3$, coefficients matrix A and right hand side b

The coefficients matrix $\boldsymbol{A}$ is given as

$$\boldsymbol{A} = \begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

For the right hand side $\boldsymbol{b}$ applies:

$$\boldsymbol{b} = (h^2 f_{1,1} - g_{0,1} - g_{1,0}, h^2 f_{1,2} - g_{0,2}, h^2 f_{1,3} - g_{0,3} - g_{1,4}, h^2 f_{2,1} - g_{2,0}, h^2 f_{2,2}, h^2 f_{2,3} - g_{2,4}, \\ h^2 f_{3,1} - g_{4,1} - g_{3,0}, h^2 f_{3,2} - g_{4,2}, h^2 f_{3,3} - g_{4,3} - g_{3,4}) \tag{1.19}$$

Structure of the coefficients matrix A

- If the solution vector $\boldsymbol{u}$ has the arrangement like in (1.18),
  - then the coefficients matrix $\boldsymbol{A}$ is called a band matrix
  - with half band width N, symmetric and irreducibly diagonally dominant.
- Therefore, the matrix $\boldsymbol{A}$
  - is regular and positively definite.
- The matrix $\boldsymbol{A}$ has always the following form,

$$\boldsymbol{A} = \begin{pmatrix} B & I & & & 0 \\ I & B & \ddots & & \\ & \ddots & \ddots & I & \\ & & I & B & I \\ 0 & & & I & B \end{pmatrix} \quad \text{with} \quad \boldsymbol{B} = \begin{pmatrix} -4 & 1 & & & 0 \\ 1 & -4 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & -4 & 1 \\ 0 & & & 1 & -4 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

where $\boldsymbol{I}$ is the identity matrix in $\mathbb{R}^{N\times N}$

Definition: band matrix

Definition 1.1.1:

(i) A $n \times n$-matrix **A** = (a_{ij}) is known as *band matrix*, if there is a number $m < n$ such that $a_{ij} = 0$ for all index pairs with $|i - j| > m$. The number m is called *band width*. A band matrix has all its non-zero entries on the main diagonal and on sloping lines parallel to it.

(ii) A $n \times n$-matrix **A** is known as *diagonal-dominant*, if $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \; i = 1, \dots, n$

(iii) A $n \times n$-matrix **A** is known as *irreducible* if there exists no permutation matrix **P** such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$
, where A_{11} is a $k \times k$-matrix and A_{22} is a $(m-k) \times (m-k)$-matrix.

Example: $N = 3$, structure of coefficients matrix

We can clearly recognise this pattern by our coefficients matrix in the example with $N = 3$

$$\mathbf{A} = \begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

Summary and implementation in MATLAB

Knowing this structure, we can implement

- problem (1.1) $\Delta u = \frac{\partial^2 u}{dx^2} + \frac{\partial^2 u}{dy^2} = f$ in Ω
- with boundary conditions (1.2) $u = g$ on $\Gamma = \partial\Omega$

We need a MATLAB-function depending on

- the grid width (i.e. on N)
- the function f and
- the boundary function g

Simple case:

- $f = 0$
- constant boundary values, i.e. $g = const.$
- any grid width.

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Questions?

- discussion forum @ Moodle (preferred, everyone can reply)
- e-mail to claudia.weis@uni-due.de (use your @uni-due.de email address!)