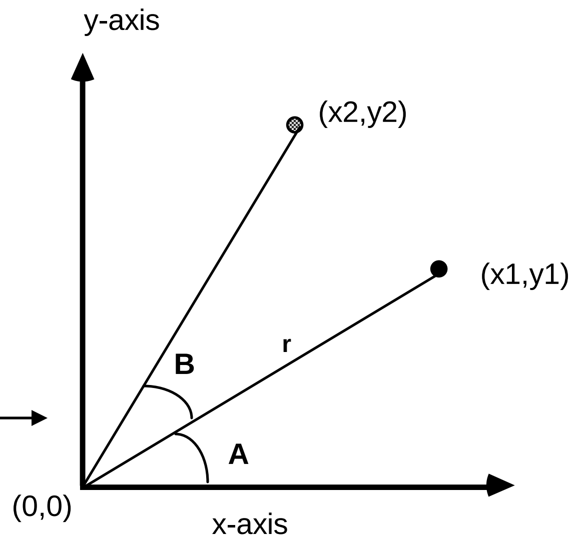
# 2D Transformations

# Rotation About the Origin

To rotate a line or polygon, we must rotate each of its vertices.




To rotate point $(x_1,y_1)$ to point $(x_2,y_2)$ we observe:

*From the illustration* we know that:

$r\sin(A + B) = y_2$ $\quad r\cos(A + B) = x_2$

$r\sin A = y_1$ $\quad r\cos A = x_1$

# Rotation About the Origin

*From the double angle formulas*: $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

*Substituting*: $y_2/r = (y_1/r)\cos B + (x_1/r)\sin B$

*Therefore*: $y_2 = y_1\cos B + x_1\sin B$

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We have

$$\mathbf{x_2 = x_1\cos B - y_1\sin B}$$

$$\mathbf{y_2 = x_1\sin B + y_1\cos B}$$

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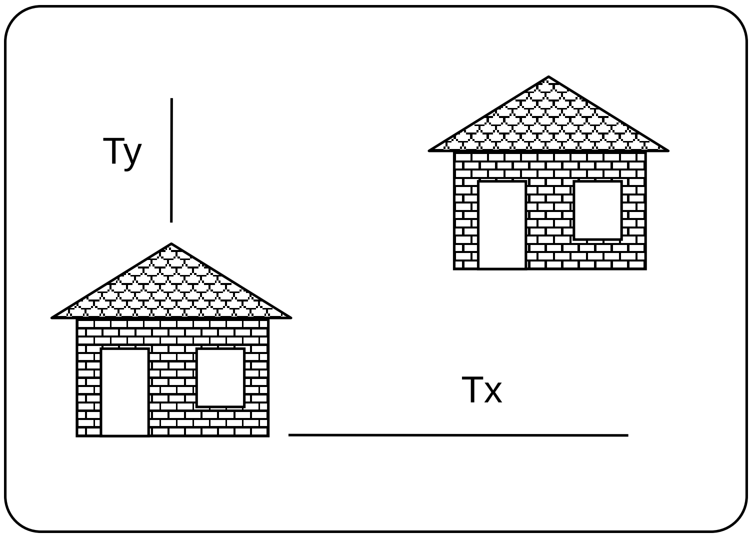
$$\mathbf{P_2 = R \cdot P_1}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

# Translations

Moving an object is called a translation. We translate a point by adding to the x and y coordinates, respectively, the amount the point should be shifted in the x and y directions. We translate an object by translating each vertex in the object.



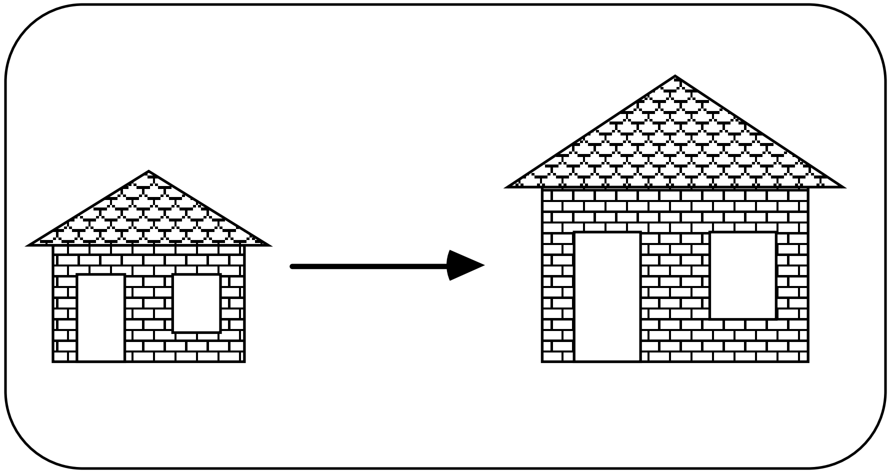

$$\mathbf{P_2 = P_1 + T}$$

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad T = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$P_2 = \begin{pmatrix} x_1 + t_x \\ y_1 + t_y \end{pmatrix}$$

# Scaling

Changing the size of an object is called a scale. We scale an object by scaling the x and y coordinates of each vertex in the object.



$$\mathbf{P_2 = S \cdot P_1}$$

$$S = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \qquad P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} s_x x_1 \\ s_y y_1 \end{pmatrix}$$

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**(modified by Amos Johnson)**

# Homogeneous Coordinates

Although the formulas we have shown are usually the most efficient way to implement programs to do scales, rotations and translations, it is easier to use matrix transformations to represent and manipulate them.

In order to represent a translation as a matrix operation we use 3 x 3 matrices and pad our points to become 1 x 3 matrices.

$$R\text{ø} = \begin{pmatrix} \cos \text{ø} & -\sin \text{ø} & 0 \\ \sin \text{ø} & \cos \text{ø} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Point } P = \begin{matrix} (x) \\ (y) \\ (1) \end{matrix}$$

# Composite Transformations - Scaling

Given our three basic transformations we can create other transformations.

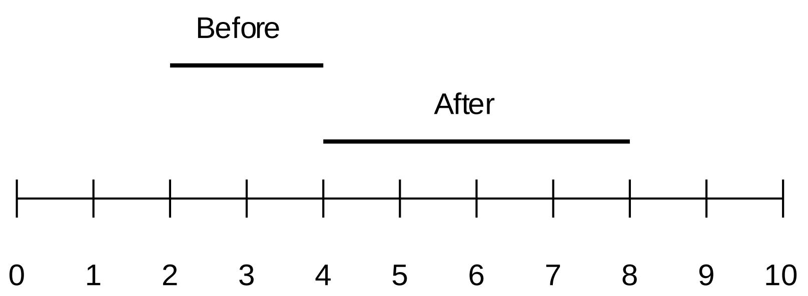
**Scaling with a fixed point**

A problem with the scale transformation is that it also moves the object being scaled.

*Scale a line between (2, 1)  (4,1) to twice its length.*

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$$

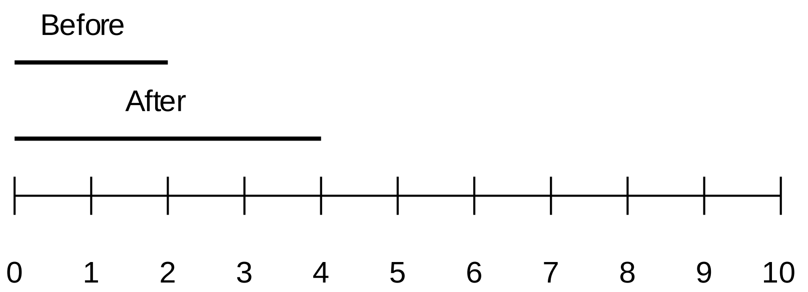


# Composite Transforms - Scaling (cont.)

If we scale a line between (0, 1) (2,1) to twice its length, the left-hand endpoint does not move.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$




(0,0) is known as a ***fixed point*** for the basic scaling transformation. We can used composite transformations to create a scale transformation with different fixed points.

# Fixed Point Scaling

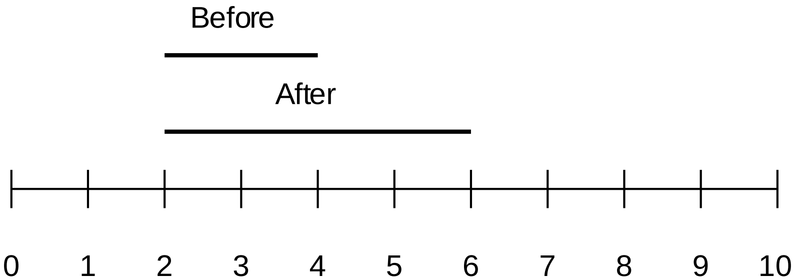
Scale by 2 with fixed point = (2,1)

- Translate the point (2,1) to the origin
- Scale by 2
- Translate origin to point (2,1)

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$$



# More Fixed Point Scaling

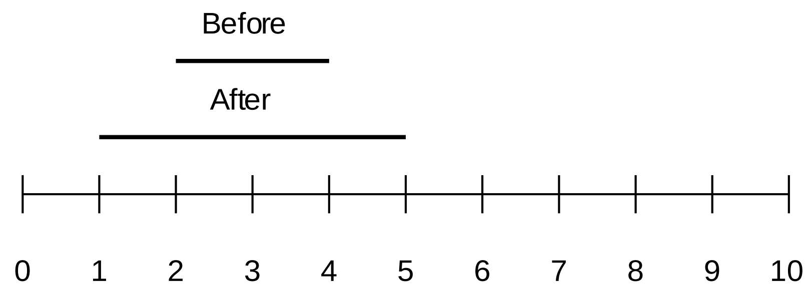
Scale by 2 with fixed point = (3,1)

- Translate the point (3,1) to the origin
- Scale by 2
- Translate origin to point (3,1)

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

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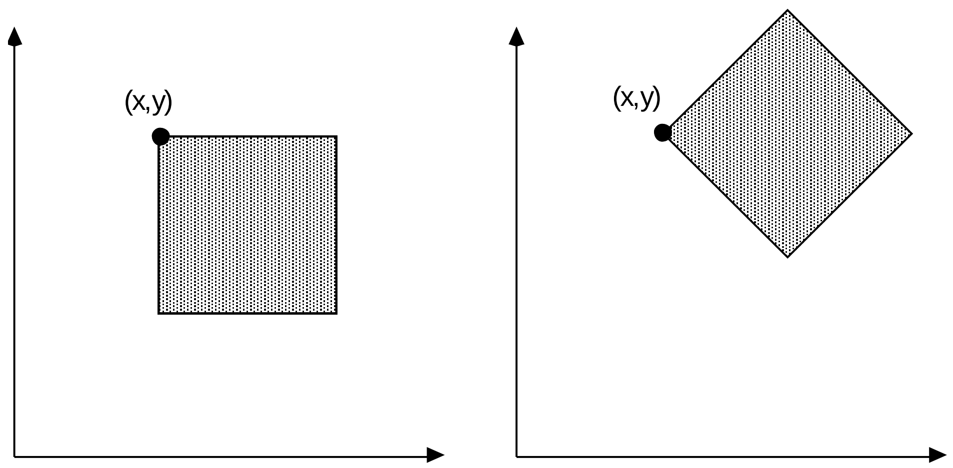
# Rotation about a Fixed Point

Rotation of Ø Degrees About Point (x,y)

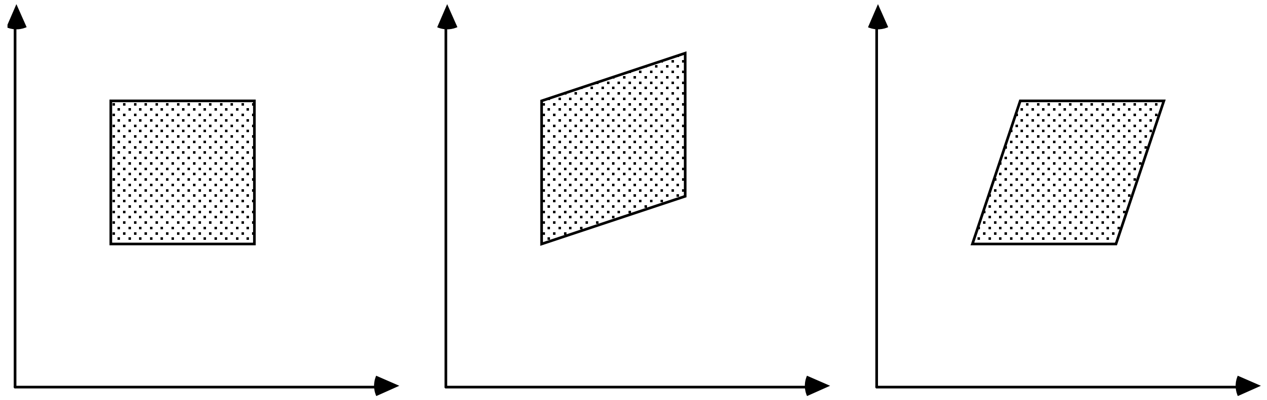
- Translate (x,y) to origin
- Rotate
- Translate origin to (x,y)




$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\text{ø} & -\sin\text{ø} & 0 \\ \sin\text{ø} & \cos\text{ø} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\text{ø} & -\sin\text{ø} & -x\cos\text{ø} + y\sin\text{ø} + x \\ \sin\text{ø} & \cos\text{ø} & -x\sin\text{ø} - y\cos\text{ø} + y \\ 0 & 0 & 1 \end{pmatrix}$$

<u>You rotate the box by rotating each vertex.</u>

# Shears



Original Data    y Shear    x Shear

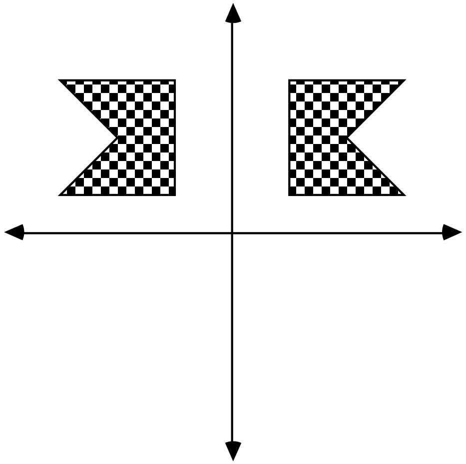
$$\begin{matrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

**GRAPHICS** --> x shear --> ***GRAPHICS***

# Reflections

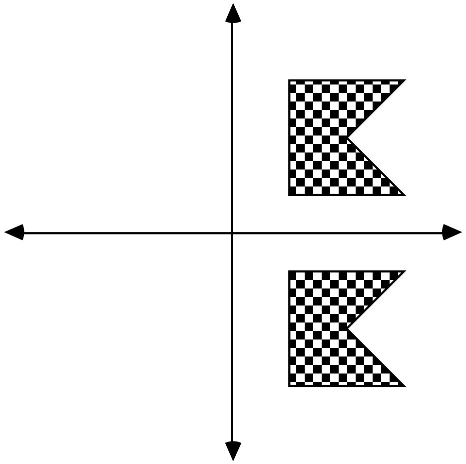
Reflection about the y-axis

$$\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



Reflection about the x-axis

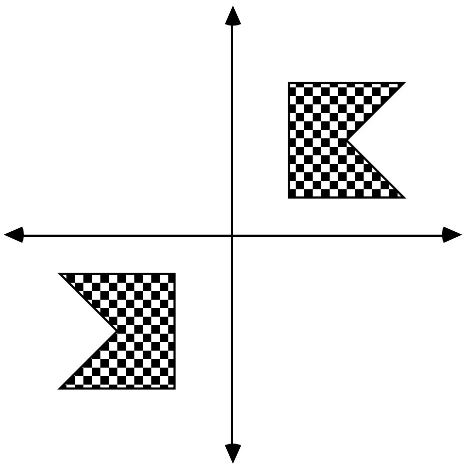
$$\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



# More Reflections

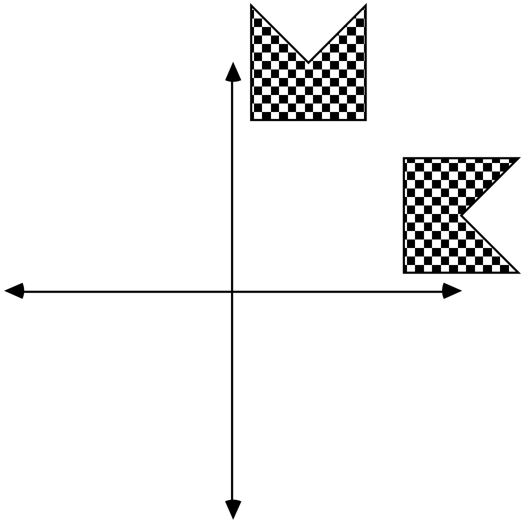
Reflection about the origin

| | | |
|---|---|---|
| -1 | 0 | 0 |
| 0 | -1 | 0 |
| 0 | 0 | 1 |



Reflection about the line y=x

| | | |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 1 |



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