

Real and Complex Analysis

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1 Real Numbers

1.1 Bounded Sets of Numbers

Definition. *The Greatest and Least members of a Set:* If S is a set consisting of finitely many different real numbers, plainly there is one member of the set which is greater than all the others and one member which is less than all the others. Convenient abbreviations for these greatest and least numbers are MAX and MIN.

If now S is a set with infinitely many members, there may or may not be a member of S which is greater than all the others (or one which is less than all the others).

Definition. Let S be a set of real numbers. If there is a number K such that, for every member x of S , $x \leq K$, we say that S is bounded above. K is called an upper bound of S . Similarly, if there is a K such that $x \geq K$ for every $x \in S$, then S is said to be bounded below, and K is a lower bound of S .

If S is bounded both above and below, we say simply that it is bounded. A set which is not bounded is called unbounded.

Theorem 1.1.1. If S is a (non-empty) set of numbers which is bounded above, then of all the upper bounds, there is a least one.

Proof. Divide the real numbers x into two classes L , R by these rules:

Put x in L if there is a member s of S such that $s > x$.

Put x in R if, whatever member s of S is taken, $s \leq x$.

Then every x goes either into L or into R . Moreover, neither L nor R is empty. For, if s is some member of S , then (say) $x = s - 1$ is in L .

And, since S is bounded above, any upper bound K , for which $s \leq K$ for all s , is in R .

Any l of L is less than any r of R . For there is some s which is greater than l , and this s is less than or equal to r .

By Dedekind's axiom, there is a dividing number ξ such that, for every positive ϵ , $\xi - \epsilon$ is in L and $\xi + \epsilon$ is in R . In the Dedekind axiom, ξ itself may belong either to L or to R . We shall prove that, in the present application, ξ belongs to R .

Suppose, if possible, that ξ belongs to L . Then there is a member s of S which $s > \xi$.

The number $\eta = \frac{1}{2}(s + \xi)$ satisfies $s > \eta > \xi$: η is in R since it is greater than the dividing number ξ . So $s \leq \eta$ by the rule for R . This contradicts our earlier inequality $s > \eta$. So ξ belongs to R . We have proved that ξ satisfies

1. $s \leq \xi$ for every s in S
2. $\xi - \epsilon$ being any number less than ξ , there is an s for which $s > \xi - \epsilon$.

1 Real Numbers

The property (1) shows that ξ is an upper bound of S , and (2) that it is the least upper bound. \square

Definition. If, given a set of numbers S , there is a number K such that

1. $s \leq K$ for every s in S
2. for every positive ϵ , there is an s in S for which $s > K - \epsilon$, then we write $K = \sup S$.

Theorem 1.1.1 proved the existence of K when the set S is bounded above.

By reversing inequality signs, we set up an analogous theory of lower bounds and the greatest of the lower bounds (the *infimum*, abbreviated \inf).

Definition. If, given a set of numbers S , there is a number k such that

1. $s \geq k$ for every s in S
2. for every positive ϵ , there is an s in S for which $s < k + \epsilon$, then we write $k = \inf S$.

A set which is bounded below can be proved to have an infimum.

1.1.1 Examples

1. Let S be the rational numbers x for which $0 \leq x \leq \frac{1}{2}$. Then $\frac{1}{2}$ is the least upper bound. It is also the greatest member of S .
2. Let S be the rational numbers x for which $x^2 < 2$. The number $\sqrt{2}$ is the least upper bound.
3. Let S denote the set of all rational numbers r satisfying $r \leq 3$. Show that 3 is the $\sup(S)$. In this case, $3 \in S$, so S includes its least upper bound.

From the given inequality $r \leq 3$, so that part (1) of Definition 1.1 is satisfied. Now whatever the value of $\beta > 0$, $3 > 3 - \beta$ and $3 \in S$ so that part (2) of the definition is satisfied. Hence 3 is the l.u.b for S .

4. Let S denote the set of rational numbers satisfying $x < 6$. Then S has no largest member.

The number 6 is not the largest member of S because it does not belong to S , and indeed no number greater than 6 belongs to S . On the other hand if l is a member of S which is less than 6 then the number $(l + 6)/2$ is a rational number which is greater than l , and less than 6, so that it is a member of S . So l could not be the largest member of S . This demonstrates that S has no largest member.

5. Let S denote the set of all rational numbers whose squares are less than 2. Then S is a bounded set but it has no largest member.

If $x \geq 2$ then $x^2 \geq 4 > 2$, and so $x \notin S$. Hence, the number 2 is an upper bound for S . It is clearly not the least upper bound, as a similar argument demonstrates that

$3/2$, for example, is also an upper bound for S . Similarly, -2 is a lower bound for S , so that S is a bounded set. Suppose that l were the largest member of S . We cannot have $l^2 = 2$, for there is no such rational number, as we have shown. We cannot have $l^2 > 2$ for then $l \notin S$. We shall show that $l^2 < 2$ is also impossible.

If $l^2 < 2$ we shall show that we can find a larger number $l + \alpha$, ($\alpha > 0$) whose square is also less than 2, as follows:

$$\begin{aligned}(l + \alpha)^2 &= l^2 + 2l\alpha + \alpha^2 \\ &< l^2 + 4\alpha + \alpha^2 \quad \text{since } l < 2 \\ &< l^2 + 5\alpha \quad \text{provided } \alpha < l \\ &< 2 \quad \text{provided } \alpha < (2 - l^2)/5\end{aligned}$$

6. Let A and B be sets of numbers which are bounded above, with l.u.b a and b respectively. Let C denote the set of all numbers of the form $x + y$, where $x \in A$ and $y \in B$. Show that the l.u.b of C is $a + b$.

For all $x \in A$ and $y \in B$, $x \leq a$ and $y \leq b$, using part (1) of the definition of l.u.b. Hence, $x + y \leq a + b$, and so $a + b$ is an upper bound for C . We now apply part (2) of the definition of l.u.b to A and to B , with β replaced by $\beta/2$. So for any positive number β , there are numbers $x \in A$ and $y \in B$ satisfying $x > a - \beta/2$ and $y > b - \beta/2$. Thus, $x + y > a + b - \beta$ and so we have shown that there is a number in C exceeding $a + b - \beta$. So part (2) of the definition is satisfied, showing that $a + b$ is the l.u.b for C .

1.2 Axiom of Completeness for the Real Number System

Definition. This states that every non-empty set of real numbers which is bounded above must have a least upper bound in the set of real numbers. See Theorem 1.1.1 for proof.

Proposition 1.2.1. There is a real number satisfying the equation $x^2 = 2$.

Proof. We consider the set of all rational numbers r satisfying $r^2 < 2$. We know that this set is bounded above, and the axiom of completeness says that S must have a least upper bound l . We shall show that $l^2 = 2$ by demonstrating that $l^2 < 2$ and $l^2 > 2$ are both impossible.

In a previous example, we showed that if $l^2 < 2$, then we can find a larger number $l + \alpha$ whose square is also less than 2, implying that l could not be an upper bound for S . Now suppose that $l^2 > 2$. We shall show that there is a number of the form $l - \beta$ where $\beta > 0$ whose square is also greater than 2. This will mean that $l - \beta$ is also an upper bound, so that l could not be the *least* upper bound. Now

$$\begin{aligned}
 (l - \beta)^2 &= l^2 - 2l\beta - \beta^2 \\
 &> l^2 - 4\beta \quad (\text{since } \beta^2 > 0 \text{ and } l < 2) \\
 &> 2 \quad \text{provided } \beta < (l^2 - 2)/4
 \end{aligned}$$

So we have shown that there is a real number whose square is 2, finally legitimising the use of the symbol $\sqrt{2}$ for the number l defined above. \square

Proposition 1.2.2. *There is a unique positive real number satisfying the equation $x^2 = 2$.*

Proof. Existence was established in Proposition 1.2.1. Here we are demonstrating uniqueness. The proof uses the method of contradiction. Suppose that x_1 and x_2 are both positive numbers satisfying $x^2 = 2$. Then $x_1^2 = x_2^2$, and so $x_1^2 - x_2^2 = 0$. Factorizing this gives $(x_1 + x_2)(x_1 - x_2) = 0$. We must therefore have $(x_1 + x_2) = 0$ or $(x_1 - x_2) = 0$. Since x_1 and x_2 are both positive we cannot have $(x_1 + x_2) = 0$. Hence $(x_1 - x_2) = 0$ and so $x_1 = x_2$, proving uniqueness. \square

Proposition 1.2.3. *Given any two positive real numbers a and b , there is a positive integer n satisfying $na \geq b$. This is known as the **Archimedean Property**.*

Proof. As on so many occasions, the proof employs the method of contradiction. So we suppose that the result is false, i.e. that there are two real numbers a and b such that for every positive integer n , $na < b$. This means that the set S of numbers of the form na is bounded above by b . S therefore has a least upper bound l . Now, $l - a < l$ and so there is a number of the set S between $l - a$ and l , i.e. there is a positive integer n_0 satisfying $l - a < n_0 a < l$. Adding a to each side of the left-hand inequality gives $l < (n_0 + 1)a$, giving a number of S greater than l . This is a contradiction. \square

Proposition 1.2.4. *Between any two real numbers there are both rational and irrational numbers.*

Proof. \square

2 Absolute Value, Complex Numbers and Inequalities

2.1 Absolute Value

$$|x| = \max(x, -x)$$

Equivalently:

$$|x| = \sqrt{x^2}$$

$|x - y|$ is the distance between x and y . Example, if $x = -5$ and $y = 4$, then

$$|x - y| = |-5 - 4| = |-9| = 9$$

2.1.1 Example

Fix $x_0 \in \mathbb{R}$ $\epsilon > 0$.

$$\{x \in \mathbb{R} : |x - x_0| < \epsilon\}$$

Algebraically:

$$|x - x_0| < \epsilon \Leftrightarrow \max(x - x_0, x_0 - x) < \epsilon$$

There are two cases:

$$x - x_0 > 0 \Rightarrow \max(x - x_0, x_0 - x) = x - x_0 < \epsilon$$

$$x < x_0 \Rightarrow \max(x - x_0, x_0 - x) = x_0 - x < \epsilon$$

$$\Rightarrow x_0 - \epsilon < x < x_0 + \epsilon$$

2.2 Complex Numbers

Let $z = x + iy$ $x, y \in \mathbb{R}$. $|z|$ is the distance of z to the origin and equals

$$\sqrt{x^2 + y^2}$$

$|z - w|$ is the distance of z to w .

2.3 Triangle Inequality

Theorem 2.3.1.

$$|z - w| \leq |z| + |w|$$

Equivalent to

$$|z + w| \leq |z| + |w|$$

Proof.

$$\begin{aligned} |z + w|^2 &= (z + w)^2 = z^2 + 2zw + w^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2 \\ &\Rightarrow |z + w| \leq |z| + |w| \end{aligned}$$

□

2.4 Reverse Triangle Inequality

Theorem 2.4.1.

$$|z - w| \geq ||z| - |w||$$

Proof.

$$\begin{aligned} |z - w| + |w| &\geq |z - w + w| = |z| \\ |w - z| + |z| &\geq |w - z + z| = |w| \end{aligned}$$

This can be re-written as

$$\begin{aligned} |z - w| &\geq |z - w + w| = |z| - |w| \\ |w - z| &\geq |w - z + z| = |w| - |z| \end{aligned}$$

Now, $|z - w| = |w - z|$ and if $t \geq a$ and $t \geq -a$, then $t \geq |a|$. Using this, we can conclude

$$|z - w| \geq ||z| - |w||$$

□

2.5 Bernoulli Inequality

The following inequality is very important in analysis.

$$(1 + x)^n \geq 1 + nx \quad \text{for } x > -1 \quad \text{and } n \in \mathbb{N}$$

Proof. We use induction on n . If $n = 0$

$$(1+x)^0 \geq 1 + 0(x) \Rightarrow 1 \geq 1$$

is trivially true. Assume it is true for $n = k$ for some k , i.e.

$$(1+x)^k \geq 1 + kx$$

$$(1+x)(1+x)^k \geq (1+x)(1+kx)$$

This is true, since $x > -1$ and $(1+x) > 0$.

$$\begin{aligned} \Rightarrow (1+x)^{k+1} &\geq 1 + x + kx + kx^2 \\ &\geq 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \end{aligned}$$

since $kx^2 > 0$. Therefore, by induction, the statement is true for all $n \in \mathbb{N}$.

□

3 Sequences

A sequence is a countable set of real or complex numbers written down in some order.

$$\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}$$

$$1, -1, 1, -1, \dots, (-1)^{n-1}$$

$$3, 9, 27, \dots, 3^n$$

Definition. A real or complex sequence is a mapping

$$S : \mathbb{N} \rightarrow \mathbb{R} \text{ or } \mathbb{C}$$

S_n is the n^{th} term of the sequence, also used to denote the sequence itself. For example, $S_n = \frac{1}{n}$ designates the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$$

Lets take $S_n = \frac{1}{n^2}$. As n increases the n^{th} term tends to 0. However, $S_n = (-1)^{n-1}$ does not tend to a single number as n increases. $S_n = 3^n$ runs away as n increases and goes to infinity.

3.1 Null Sequences

Definition. A sequence is called a null sequence if for any $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ (depending on ϵ), such that for all $n > N$, we have $|S_n| < \epsilon$.

3.1.1 Example

$S_n = \frac{1}{2^n}$ is a null sequence. Let $\epsilon > 0$ be arbitrary, and take N equal to some natural number such that $2^N > \frac{1}{\epsilon} \Leftrightarrow N > \log(1/\epsilon)$.

Then if $n \geq N \Rightarrow 2^n > 2^N > 1/\epsilon \Rightarrow \frac{1}{2^n} < \epsilon$. But $\frac{1}{2^n} = |\frac{1}{2^n}| = S_n$. So, for all $n > N$, $|S_n| < \epsilon$. For example, take $\epsilon = \frac{1}{2^{1000}}$. We can choose $N = 1001$ or $N = 2000$ etc.

Same argument shows that

$$S_n = \frac{(-1)^n}{2^n}$$

3 Sequences

is a null sequence, since

$$S_n = \left| \frac{(-1)^n}{2^n} \right| = \frac{1}{2^n}$$

3.1.2 Example

$$S_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

is not a null sequence.

Proof. Suppose it were a null sequence. Then apply the definition, with $\epsilon = \frac{1}{2}$. Then there should exist an N such that if $n > N$, then $|S_n| < \frac{1}{2}$. But this gives a contradiction, if we take an $n > N$, which is even (e.g. $n = 2N$). Then $|S_n| = 1 > \frac{1}{2}$. Hence it is not a null sequence. \square

3.2 Sequences Having a Limit (Converging Sequences)

Definition. A sequence S_n tends to a limit L , if for all $\epsilon > 0$, there exists an N (depending on ϵ), such that for all $n > N$,

$$|S_n - L| < \epsilon$$

Remarks:

- S_n tends to L if and only if $S_n - L$ is a null sequence.
- S_n is a null sequence if and only if S_n tends to 0.

3.2.1 Example

$$S_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$S_n - 1 = \frac{1}{n}$$

is a null sequence, so $S_n \rightarrow 1$.

Or, using the definition:

$$|S_n - 1| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{n+1-n}{n} \right| = \frac{1}{n}$$

We want

$$\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$$

So, for any $\epsilon > 0$, let N be any natural number $> \frac{1}{\epsilon}$. Then if

$$n > N \Rightarrow \frac{1}{n} < \frac{1}{N} < \epsilon$$

So,

$$n > N \Rightarrow |S_n - 1| < \epsilon$$

Since $\epsilon > 0$ is arbitrary, this shows that

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

3.3 Properties of Limits

Lemma 3.3.1. *If x_n and y_n are null sequences, then so is $x_n + y_n$.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $x_n \rightarrow 0$, there exists an N_1 such that if $n > N_1$, then $|x_n| < \frac{\epsilon}{2}$.

Since $y_n \rightarrow 0$, there exists N_2 such that if $n > N_2$, then $|y_n| < \frac{\epsilon}{2}$. If $n > \max\{N_1, N_2\}$, then $|x_n + y_n| \leq |x_n| + |y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

So, we have shown that there exists an N such that for all $n > N$,

$$|x_n + y_n| < \epsilon \Rightarrow x_n + y_n \rightarrow 0$$

□

Lemma 3.3.2. *If x_n is a null sequence ($x_n \rightarrow 0$) and y_n is a bounded sequence, then $x_n y_n$ is again a null sequence.*

Proof. y_n is bounded, so there exists a number k such that $|y_n| \leq k$ for all n .

$$|x_n y_n| \leq k |x_n| < \epsilon \Leftrightarrow |x_n| < \frac{\epsilon}{k}$$

Now, $x_n \rightarrow 0$, so applying the definition with ϵ/k instead of ϵ , there exists N such that

$$n > N \Rightarrow |x_n| < \frac{\epsilon}{k}$$

But then, if $n > N$, then

$$|x_n y_n| \leq k |x_n| < k \frac{\epsilon}{k} = \epsilon$$

So, for any $\epsilon > 0$, we can find an N such that

$$|x_n y_n| < \epsilon \Rightarrow x_n y_n \rightarrow 0$$

□

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Lemma 3.3.3. Suppose a sequence t_n has a limit, then t_n is a bounded sequence, i.e., there exists a $k > 0$ such that

$$|t_n| < k \quad \forall n$$

Proof. Suppose $t_n \rightarrow t$. Apply the definition with $\epsilon = 1$.

$$\begin{aligned} \exists N : n > N \Rightarrow |t_n - t| < 1 \\ |t_n| = |t_n - t + t| \leq |t_n - t| + |t| < 1 + |t| \quad \forall n > N \end{aligned}$$

If $n \leq N$, then

$$|t_n| \leq \max(|t_1|, |t_2|, \dots, |t_N|)$$

So

$$|t_n| \leq \max(|t_1|, |t_2|, \dots, |t_N|, 1 + |t|) = k \quad \forall n$$

□

Theorem 3.3.1. Suppose $s_n \rightarrow s$ and $t_n \rightarrow t$. Then

1. $s_n + t_n \rightarrow s + t$
2. $s_n t_n \rightarrow st$
3. $\frac{s_n}{t_n} \rightarrow \frac{s}{t}$ provided $t_n \neq 0$ and $t \neq 0$.

Proof. 1.

$$s_n - s \rightarrow 0 \quad t_n - t \rightarrow 0$$

Apply Lemma 3.3.1 to $x_n = s_n - s$ and $y_n = t_n - t$.

$$(s_n - s) + (t_n - t) = (s_n + t_n) - (s + t) \rightarrow 0$$

Hence $s_n + t_n \rightarrow s + t$.

2.

$$s_n \rightarrow s \quad t_n \rightarrow t$$

We want to show that $s_n t_n \rightarrow st$ or equivalently $s_n t_n - st \rightarrow 0$.

$$s_n t_n - st = (s_n - s)t_n + s(t_n - t)$$

$t_n \rightarrow t$, so t_n is a bounded sequence (Lemma 3.3.3). $s_n \rightarrow s$, so $s_n - s$ is a null sequence. From Lemma 3.3.2, $(s_n - s)t_n$ is a null sequence, i.e., $(s_n - s)t_n \rightarrow 0$.

$t_n \rightarrow t$, so $t_n - t$ is a null sequence. The sequence which is constant and equal to s is bounded. By Lemma 3.3.2, $s(t_n - t) \rightarrow 0$.

Since the sum of a null sequence is a null sequence, we get

$$(s_n - s)t_n + s(t_n - t) \rightarrow 0$$

Hence

$$s_n t_n \rightarrow st$$

3. We will prove this part in several stages.

a) Explain why

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{1}{|b_n b|} |b_n - b|$$

We can write the left hand side as:

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n b} \right| = \frac{1}{|b_n b|} |b_n - b|$$

since $|b - b_n| = |b_n - b|$.

b) Show in two steps that the sequence, $\left\{ \frac{1}{|b_n b|} \right\}$ is bounded.

i. Show that the set $\{b_n\}$ is bounded away from zero, i.e., there is some $\delta > 0$ such that $|b_n| \geq \delta$ for all n .

We know that

$$b_n \rightarrow b \Rightarrow \exists N : |b_n - b| < \epsilon \quad \forall n > N$$

Let $\epsilon = \frac{|b|}{2} > 0$. It follows that

$$|b_n - b| < \frac{|b|}{2}$$

Using the reverse triangle inequality

$$\begin{aligned} |b_n - b| &\geq ||b_n| - |b|| \\ &\geq ||b| - |b_n|| \end{aligned}$$

Using this, we have

$$|b| - |b_n| < \frac{|b|}{2} \Rightarrow |b_n| > \frac{|b|}{2} \quad \forall n > N$$

The remaining set $S = \{|b_1|, |b_2|, \dots, |b_N|\}$ is a finite set of positive numbers, and so has a positive minimum. We can choose our $\delta = \min(S)$ and conclude that

$$|b_n| \geq \delta \quad \forall n, \delta > 0$$

ii. Show that

$$\frac{1}{|b_n b|} < \frac{2}{(\delta |b|)} \quad \forall n$$

$$|b_n| \geq \delta \Rightarrow |b_n b| \geq \delta |b| > \frac{\delta |b|}{2}$$

since $|b| > |b|/2$.

$$\Rightarrow \frac{1}{|b_n b|} < \frac{2}{(\delta |b|)} \quad \forall n$$

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c) We have shown in part (b) above that $\frac{1}{|b_n b|}$ is bounded. Now prove that

$$\frac{1}{b_n} \rightarrow \frac{1}{b}$$

From part (a), we have

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{1}{|b_n b|} |b_n - b|$$

Since $\frac{1}{b_n b}$ is bounded above, say by M , that is, $\frac{1}{b_n b} < M$

$$\Rightarrow \left| \frac{1}{b_n} - \frac{1}{b} \right| < M |b_n - b|$$

Since $b_n \rightarrow b$, $b_n - b$ is a null sequence. By the product rule, $M|b_n - b|$ is also a null sequence and hence,

$$\Rightarrow \left| \frac{1}{b_n} - \frac{1}{b} \right| \rightarrow 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{b}$$

Finally, we can use the product rule again to show that $\frac{s_n}{t_n} \rightarrow \frac{s}{t}$. By part (c) we have shown that $\frac{1}{t_n} \rightarrow \frac{1}{t}$. Now,

$$\frac{s_n}{t_n} = s_n \left(\frac{1}{t_n} \right) \rightarrow \frac{s}{t}$$

□

3.3.1 Example of an application of this theorem

$$S_n = \frac{n^2 + 4n - 3}{2n^2 + 3n + 5} = \frac{1 + 4/n + 3/n^2}{2 + 3/n + 5/n^2}$$

Now, $4/n \rightarrow 0$, $3/n^2 \rightarrow 0$, $3/n \rightarrow 0$, $5/n^2 \rightarrow 0$. Therefore $S_n \rightarrow \frac{1}{2}$.

3.4 Divergence to Infinity

Definition. $S_n \rightarrow \infty$ if, for any given real number $A > 0$, there exists an N (depending on A) such that if $n > N$, then $S_n > A$.

Definition. $S_n \rightarrow -\infty$ if, for any given real number $A > 0$, there exists an N (depending on A) such that if $n > N$, then $S_n < -A$.

3.4.1 Example

Take

$$S_n = n^2$$

This sequence does not have a limit. The general term n^2 becomes bigger than any given number if n grows. We say that

$$\lim_{n \rightarrow \infty} n^2 = \infty \quad \text{or} \quad n^2 \rightarrow \infty$$

$$S_n = n^2 > A \Leftrightarrow n > \sqrt{A}$$

So, we can take for N any natural number $> \sqrt{A}$.

3.4.2 Remark

$$s_n + t_n \rightarrow s + t$$

remains valid, if $s_n \rightarrow \infty$, provided t_n has a finite limit or $t_n \rightarrow \infty$, i.e.

$$s_n \rightarrow \infty, \quad t_n \rightarrow t \in \mathbb{R} \quad \text{or} \quad t_n \rightarrow \infty \Rightarrow s_n + t_n \rightarrow \infty$$

However, this is not true if $t_n \rightarrow -\infty$, because $\infty + (-\infty)$ is not defined.

3.5 Example

$$s_n = n, \quad t_n = -n + a, \quad a \in \mathbb{R}$$

$s_n \rightarrow \infty$ and $t_n \rightarrow -\infty$, whatever the value of a . But note that $s_n + t_n = a$!

3.6 Monotone Sequences

Definition. s_n is an increasing (decreasing) sequence if $s_{n+1} \geq s_n$ ($s_{n+1} \leq s_n$).

Theorem 3.6.1. Monotone and bounded sequences have a limit.

Suppose that s_n is an increasing sequence which is bounded from above, i.e. there exists a $K \in \mathbb{R}$ such that $s_n \leq K$ for all n . Then s_n tends to a limit.

Similarly, if s_n is a decreasing sequence which is bounded from below, i.e. $s_n \geq L$ for all n , for some $L \in \mathbb{R}$, then s_n also tends to a limit.

Proof. 1. Increasing Case:

The set s_n is bounded from above. Put $s = \sup\{s_n\}$. We claim that $s_n \rightarrow s$.

Let $\epsilon > 0$. On the one hand, s is an upper bound, so

$$s_n \leq s \quad \forall n \tag{3.1}$$

3 Sequences

On the other hand, s is the smallest upper bound, i.e. the least upper bound. Hence $s - \epsilon$ is no longer an upper bound. Therefore, there exists at least one N such that

$$s_N > s - \epsilon$$

but the sequence is increasing, so

$$n > N \Rightarrow s_n > s_N > s - \epsilon$$

So, we have shown that

$$s_n > s - \epsilon \quad \forall n > N \quad (3.2)$$

By using Equations 3.1 and 3.2:

$$\begin{aligned} s - \epsilon &< s_n \leq s \\ \epsilon - s &> -s_n \geq -s \\ \epsilon &> s - s_n \geq 0 \Rightarrow 0 \leq s - s_n < \epsilon \\ |s - s_n| &< \epsilon \quad \forall n > N \end{aligned}$$

Hence,

$$s_n \rightarrow s$$

2. Decreasing Case

If s_n is decreasing and bounded from below,

$$s_n \geq A \quad \forall n$$

then $-s_n$ is increasing and bounded from above.

$$-s_n \leq -A \quad \forall n \Rightarrow -s_n \rightarrow -s \Rightarrow s_n \rightarrow s$$

□

3.6.1 Application: Sequences Defined by a Recurrence Relation

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x_{n+1} = f(x_n)$$

x_1 is given (this is a one term recurrence relation).

1. If f is increasing, then the sequence x_n is increasing
2. Next, if we can show that x_n is bounded, then $\lim_{n \rightarrow \infty} x_n$ exists.

How do we find $x = \lim_{n \rightarrow \infty} f(x_n)$? Let $n \rightarrow \infty$ in $x_{n+1} = f(x_n)$. Note that in order for this technique to apply, we need something like the following to apply:

$$\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$$

This works in general if f is continuous. Now:

$$\begin{aligned} x_{n+1} &\rightarrow x \\ f(x_n) &\rightarrow f(x) \end{aligned}$$

3.6.2 Example

Suppose that $x_1 = 1$, and for $n \geq 1$,

$$x_{n+1} = \sqrt{1 + x_n}$$

Thus, x_n begins with: 1, $\sqrt{2}$, $\sqrt{1 + \sqrt{2}}$, $\sqrt{1 + \sqrt{1 + \sqrt{2}}}$, etc. It is certainly not immediately clear what the limit will be if it exists! However, the first few terms do suggest that x_n is increasing. We prove this in general using induction:

$$x_1 = 1 < \sqrt{2} = x_2$$

Now, if $x_{n-1} < x_n$, then

$$x_n = \sqrt{1 + x_{n-1}} < \sqrt{1 + x_n} = x_{n+1}$$

Hence, x_n is increasing, by induction. Also, $x_1 < 2$, and if $x_n < 2$ then

$$x_{n+1} = \sqrt{1 + x_n} < \sqrt{1 + 2} = \sqrt{3} < 2$$

So x_n is bounded above by 2.

Thus, we have shown that x_n is increasing and bounded above, hence $x = \lim_{n \rightarrow \infty} x_n$ exists in \mathbb{R} . To find its value we first need to find

$$\lim_{n \rightarrow \infty} \sqrt{1 + x_n}$$

In fact, this limit is simply $\sqrt{1 + x}$. To prove this, write

$$\left| \sqrt{1 + x_n} - \sqrt{1 + x} \right| = \frac{|1 + x_n - (1 + x)|}{\sqrt{1 + x_n} + \sqrt{1 + x}} < |x_n - x|$$

since x_n and x are positive, so the denominator is greater than 1. But

$$x_n \rightarrow x \Rightarrow |x_n - x| < \epsilon, \quad \epsilon > 0$$

3 Sequences

$$\Rightarrow \left| \sqrt{1+x_n} - \sqrt{1+x} \right| < \epsilon$$

Hence $\sqrt{1+x_n} \rightarrow \sqrt{1+x}$ as claimed.

Now, we have $x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = \sqrt{1+x}$.

$$x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 + \sqrt{5}}{2}$$

since $x > 0$. This number is known as the golden section, which is also the limit of ratios of successive Fibonacci numbers.

3.6.3 Example

Consider the sequence given by

$$x_0 = a + 1, \quad x_{n+1} = x_n \left(1 + \frac{a - x_n^k}{kx_n^k} \right)$$

for a fixed $a > 0$ and fixed $k \in \mathbb{N}$.

Let's try $a = 1, k = 1$.

$$\Rightarrow x_0 = 2 \Rightarrow x_1 = 2 \left(1 + \frac{1-2}{2} \right) = 1$$

So, we claim that $x_{n+1} < x_n$, $x_n > 0$ and $x_n^k > a$ (in order for the sequence to decrease). Let's prove all three claims by using induction. If $n = 0$:

$$x_0 = a + 1 > 0$$

$$x_0^k = (a + 1)^k > a$$

$$\begin{aligned} x_1 &= (a + 1) \left(1 + \frac{a - (a + 1)^k}{k(a + 1)^k} \right) \\ &< (a + 1) < x_0 \end{aligned}$$

since

$$(a + 1)^k > a \Rightarrow a - (a + 1)^k < 0 \Rightarrow \frac{a - (a + 1)^k}{k(a + 1)^k} < 0$$

So all three claims are true for $n = 0$. For the inductive step, assume they hold for some fixed n .

Now, since $x_n > 0$ and $a < (a + 1)^k$

$$\Rightarrow x_{n+1} < x_n$$

What about x_{n+1} ? In order for this to remain > 0 , we need

$$1 + \frac{a - x_n^k}{kx_n^k} > 0$$

$$1 + \frac{a - x_n^k}{kx_n^k} = \frac{kx_n^k + a - x_n^k}{kx_n^k} = \frac{x_n^k(k-1) + a}{kx_n^k} > 0$$

since $k \geq 1$, $x_n > 0$ and $a > 0$. Hence

$$x_{n+1} > 0$$

Finally, we need to verify that $x_{n+1}^k > a$. We will use the Bernoulli inequality with $x = \frac{a - x_n^k}{kx_n^k}$ (which is greater than -1 because $kx_n^k + a - x_n^k > 0$).

$$x_{n+1}^k = x_n^k \left(1 + \frac{a - x_n^k}{kx_n^k} \right)^k > x_n^k \left(1 + k \frac{a - x_n^k}{kx_n^k} \right) > x_n^k \left(\frac{x_n^k + a - x_n^k}{x_n^k} \right) = a$$

Hence, by induction, all three claims are true for all n . So x_n is decreasing and bounded below by 0. **It therefore converges to $\alpha = \inf\{x_n\}$, which is non-negative.** The recursion formula can be written as

$$\begin{aligned} x_{n+1} &= x_n \left(\frac{kx_n^k + a - x_n^k}{kx_n^k} \right) \\ kx_n^k x_{n+1} &= x_n (kx_n^k + a - x_n^k) \\ kx_{n+1} x_n^{k-1} &= (k-1)x_n^k + a \end{aligned}$$

and by the algebra of limits, we can let $n \rightarrow \infty$ on both sides to obtain

$$k\alpha\alpha^{k-1} = (k-1)\alpha^k + a$$

$$k\alpha^k = (k-1)\alpha^k + a = k\alpha^k - \alpha^k + a$$

$$\Rightarrow \alpha^k = a$$

In other words, $\alpha = a^{1/k}$. We have shown, therefore, that every positive real number has a positive k^{th} root for every fixed $k \in \mathbb{N}$. This root is *unique*, since if we have two positive numbers α, β with $\alpha < \beta$, then $0 < \alpha^k < \beta^k$, so α^k and β^k cannot both equal a .

In fact, this is just a version of *Newton's method* for finding approximate roots of equations. In particular, we can approximate roots of

$$f(x) = x^k - a$$

by taking an overestimate x_n of this root, and choosing

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3 Sequences

as the next estimate (where f' denotes the derivative). In the case $f(x) = x^k - a$ we have $f'(x) = kx^{k-1}$, and we have just verified that this procedure does yield a convergent sequence whose limit is the positive root of the equation

$$x^k = a$$

3.6.4 Example

The sequence a_n is defined by

$$a_1 = 1 \quad a_n = \frac{3a_{n-1} + 4}{2a_{n-1} + 3}$$

Show that the sequence has a limit and find the value of the limit.

3.6.5 Example

The sequence a_n is defined by

$$a_1 = 2 \quad a_n = \frac{1}{a_{n-1} + 2}$$

Show that the sequence has a limit and find the value of the limit.

4 Infinite Series

4.1 Examples

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{2^k} &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ \sum_{k=0}^{\infty} 2^k &= 1 + 2 + 4 + 8 + \dots \\ \sum_{k=0}^{\infty} (-1)^k &= 1 - 1 + 1 - 1 + \dots\end{aligned}$$

4.2 Partial Sums of the Series

$$s_n = \sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_n$$

Definition. The series $\sum_{k=1}^{\infty} u_k$ converges if the sequence $s_n = \sum_{k=1}^n u_k$ has a limit: $s_n \rightarrow s$. In this case, we call this limit, s , the sum of the series, and we write:

$$\sum_{k=1}^{\infty} u_k = s$$

4.2.1 Example

Is the series $\sum_{k=0}^{\infty} \frac{1}{2^k}$ convergent?

$$s_n = \sum_{k=0}^n \frac{1}{2^k} = 1 + \frac{1}{2} + \dots + \frac{1}{2^n}$$

Note that the geometric series converges to:

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

4 Infinite Series

We can derive this by noting that:

$$\begin{aligned}S_n &= 1 + r + \dots + r^n \\rS_n &= r + r^2 + \dots + r^n + r^{n+1} \\S_n - rS_n &= 1 - r^{n+1} \\S_n(1 - r) &= 1 - r^{n+1} \\S_n &= \frac{1 - r^{n+1}}{1 - r}\end{aligned}$$

So, $1 + \frac{1}{2} + \dots + \frac{1}{2^n} = \frac{1 - (1/2)^{n+1}}{1 - (1/2)} = \frac{1 - 1/2^{n+1}}{1 - (1/2)}$.

Now, $\frac{1}{2^{n+1}} \rightarrow 0$ as $n \rightarrow \infty$. So $\frac{1 - 1/2^{n+1}}{1 - (1/2)} \rightarrow \frac{1}{1 - (1/2)} = 2$.

Conclusion: $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.

4.2.2 Example

Is the series $\sum_{k=0}^{\infty} 2^k$ convergent?

$$S_n = \sum_{k=0}^n 2^k = \frac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1$$
$$\lim_{n \rightarrow \infty} 2^{n+1} - 1 \rightarrow \infty$$

Therefore, $\sum_{k=0}^{\infty} 2^k$ diverges (does not converge).

4.2.3 Example

Is the series $\sum_{k=0}^{\infty} (-1)^k$ convergent?

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0$$

The sequence of partial sums, S_n does not have a limit, therefore, the series diverges.

4.3 Application

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{n^s} + \dots$$

Theorem 4.3.1. $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converge if $s > 1$ and diverges if $s \leq 1$.

In particular, the series for $s = 1$ (harmonic series):

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

diverges.

Proof. We divide the proof into three separate cases.

1. Case when $s = 1$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$$

The idea we use is this:

$$\frac{1}{3} + \frac{1}{4} > 2 \left(\frac{1}{4} \right) = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4 \left(\frac{1}{8} \right) = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > 8 \left(\frac{1}{16} \right) = \frac{1}{2}$$

Hence

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots \\ & > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \rightarrow \infty \end{aligned}$$

More formally, we group together the $\frac{1}{n}$ terms with n between $2^l + 1$ and 2^{l+1} .

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{l=0}^{\infty} \left(\sum_{n=2^l+1}^{2^{l+1}} \frac{1}{n} \right)$$

We can check that this works, by expanding the summation on the right hand side:

$$l = 0: \sum_{n=2}^2 \frac{1}{n} = \frac{1}{2}$$

$$l = 1: \sum_{n=3}^4 \frac{1}{n} = \frac{1}{3} + \frac{1}{4}$$

$$l = 2 : \sum_{n=5}^8 \frac{1}{n} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$l = 3 : \sum_{n=9}^{16} \frac{1}{n} = \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$$

Let's look at the partial sums:

$$S_{2^{k+1}} = \sum_{n=1}^{2^{k+1}} \frac{1}{n} = 1 + \sum_{l=0}^k \left(\sum_{n=2^l+1}^{2^{l+1}} \frac{1}{n} \right)$$

Let's check that the right hand side works:

$$l = k : \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n}$$

So, now, each of the groups inside the right hand side bracketed summation is \geq (number of terms) \times (smallest term), i.e.

$$\geq 2^l \times \frac{1}{2^{l+1}} = \frac{1}{2}$$

Hence

$$S_{2^{k+1}} \geq 1 + \sum_{l=0}^k \frac{1}{2} = 1 + \frac{k+1}{2}$$

$$S_{2^{k+1}} \geq 1 + \frac{k+1}{2} \rightarrow \infty \text{ as } k \rightarrow \infty$$

So, the sequence of partial sums do not have a finite limit, hence $\sum \frac{1}{n}$ diverges.

2. Case when $s \leq 1$:

$$n^s \leq n \Rightarrow \frac{1}{n^s} \geq \frac{1}{n}$$

So, partial sums

$$\sum_{n=1}^k \frac{1}{n^s} \geq \sum_{n=1}^k \frac{1}{n} \rightarrow \infty \text{ as } k \rightarrow \infty$$

Hence, $\sum \frac{1}{n^s}$ diverges.

3. Case when $s > 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{l=0}^{\infty} \left(\sum_{n=2^l+1}^{2^{l+1}} \frac{1}{n^s} \right)$$

All the terms are greater than or equal to 0. Hence the sequence of partial sums

$$S_k = \sum_{n=1}^k \frac{1}{n^s}$$

is increasing. To show convergence, it is sufficient to show that the partial sums stay bounded. To show that the sequence of partial sums stay bounded, it suffices to show that S_{2^k} stays bounded ($S_k \leq S_{2^k}$).

$$S_{2^k} = \sum_{n=1}^{2^k} \frac{1}{n^s} = 1 + \sum_{l=0}^{k-1} \left(\sum_{n=2^l+1}^{2^{l+1}} \frac{1}{n^s} \right)$$

Now the right hand bracketed summation terms are \leq (number of terms) \times (biggest term):

$$\begin{aligned} &\leq 2^l \frac{1}{(2^l+1)^s} \\ S_{2^k} &\leq 1 + \sum_{l=0}^{k-1} \frac{2^l}{(2^l+1)^s} < 1 + \sum_{l=0}^{k-1} \frac{2^l}{2^{sl}} = 1 + \sum_{l=0}^{k-1} \frac{1}{(2^{s-1})^l} \\ &= 1 + \frac{1 - \frac{1}{(2^{s-1})^k}}{1 - \frac{1}{2^{s-1}}} \end{aligned}$$

Now, $\frac{1}{2^{s-1}} < 1$, since $s > 1$.

$$\leq \frac{1}{1 - \frac{1}{2^{s-1}}} < \infty$$

We have shown that $S_{2^k} \leq \frac{1}{1 - \frac{1}{2^{s-1}}}$ for all k (and independent of k) and hence the partial sums are bounded from above. Since they are increasing, the series converges.

Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

where $\zeta(s)$ is the Riemann Zeta function.

□

5 Continuous Functions

$$f : D \rightarrow \mathbb{R} \text{ (or } \mathbb{C})$$

where $D \subseteq \mathbb{R}$, $x \in D, x \rightarrow f(x) \in \mathbb{R}$. The set D is called the domain of the function.

5.0.1 Examples of Functions

1. $f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2.$

2. $f : (0, 1) \rightarrow \mathbb{R} \quad f(x) = \frac{1}{x(x-1)}$

3. $f : \mathbb{R} \setminus \{0, 1\} \rightarrow \mathbb{R} \quad f(x) = \frac{1}{x(x-1)}$

4. $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

5. $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

6. $f : (0, \infty) \rightarrow \mathbb{R}$

$$x \in \mathbb{Q} \quad x = \frac{p}{q} \quad p, q \in \mathbb{Z} \quad \gcd(p, q) = 1$$

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(3) is different from (2) since domain is different. Graph of f :

$$\{(x, f(x)) : x \in D\} = \{(x, y) \in \mathbb{R}^2 : x \in D, y = f(x)\}$$

We cannot really draw the graphs of (5) and (6), since rational points are dense.

5 Continuous Functions

Definition. Take a function f defined as:

$$f : D \rightarrow \mathbb{R} \quad D \subseteq \mathbb{R} ; \quad a \in D$$

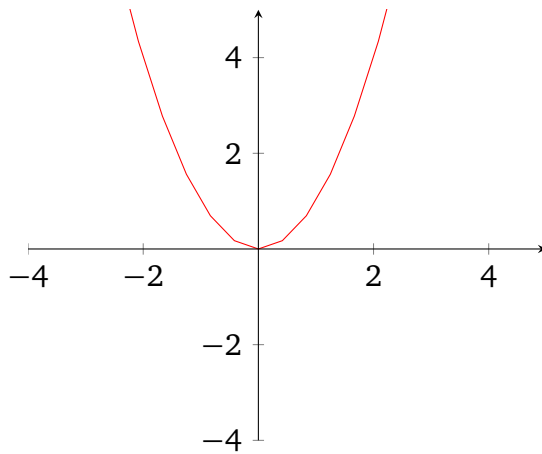
Then $\lim_{x \rightarrow a} f(x) = l \in \mathbb{R}$, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x \in D$ with $0 < |x - a| < \delta$ we have that $|f(x) - l| < \epsilon$. We can also write this as:

$$\lim_{x \rightarrow a} f(x) = l \iff \forall \epsilon > 0, \exists \delta > 0 : \forall x \in D, 0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

Or, in other words, give them an $\epsilon > 0$, and they'll give a δ , such that, $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$.

5.0.2 Example

Prove $\lim_{x \rightarrow 0} x^2 = 0$.



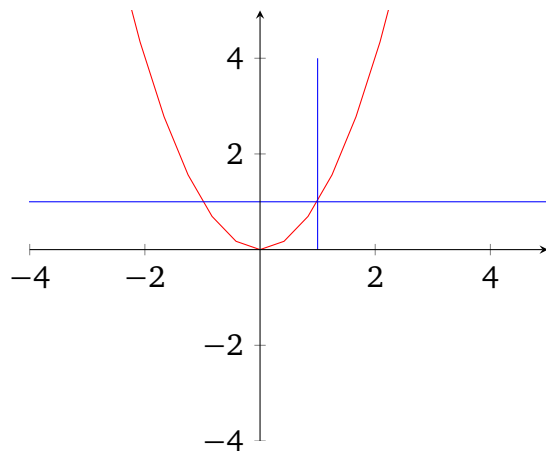
Proof. Here $l = 0$. Take $\epsilon > 0$, we want $|x^2 - 0| < \epsilon = |x^2| = x^2 < \epsilon$.

$$\begin{aligned} x^2 < \epsilon &\iff -\sqrt{\epsilon} < x < \sqrt{\epsilon} \\ &\iff |x| < \sqrt{\epsilon} \end{aligned}$$

So, $\forall \epsilon > 0, \exists \delta$ (namely, $\delta = \sqrt{\epsilon}$), such that $|x| = |x - 0| < \delta \iff |x^2| = |x^2 - 0| < \epsilon$.
So, $\lim_{x \rightarrow 0} x^2 = 0$. □

5.0.3 Example

Prove $\lim_{x \rightarrow 1} x^2 = 1$.



Proof. This proof is a bit more difficult than the previous one. We have to prove:

$$\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - 1| < \delta \Leftrightarrow |x^2 - 1| < \epsilon$$

We want $|x^2 - 1|$ small. $x^2 - 1 = (x - 1)(x + 1)$.

$$|x^2 - 1| = |x - 1||x + 1| \leq |x - 1|(1 + |x|)$$

since, using the triangle inequality, we can write:

$$|x + 1| \leq |x| + 1$$

Now observe that if $|x - 1| < 1 \Rightarrow |x| < 2$, since

$$|x| = |x - 1 + 1|$$

This follows from the triangle inequality:

$$|x - 1 + 1| \leq |x - 1| + |1| = |x - 1| + 1 < 2$$

□

5.0.4 Example

Prove $\lim_{x \rightarrow 1} \frac{3x(x-1)}{(x-1)} = 3$. Or in other words, prove: Given $\epsilon > 0$, I can give a $\delta > 0$ as long as $0 < |x - 1| < \delta$, then $\left| \frac{3x(x-1)}{(x-1)} - 3 \right| < \epsilon$.

Proof. We start where we want to get to:

$$\left| \frac{3x(x-1)}{(x-1)} - 3 \right| < \epsilon$$

As long as $x \neq 1$ (which is OK since we only consider the case of x approaching 1), we can cancel out $(x-1)$:

$$\Leftrightarrow |3x - 3| < \epsilon$$

$$\Leftrightarrow |3(x-1)| < \epsilon$$

$$\Leftrightarrow |3||x-1| < \epsilon$$

$$\Leftrightarrow 3|x-1| < \epsilon$$

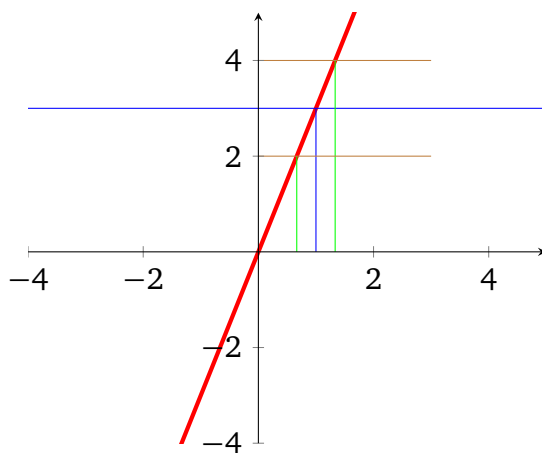
$$\Leftrightarrow |x-1| < \frac{\epsilon}{3}$$

So, we can use $\frac{\epsilon}{3}$ as our δ . For example, if you give me an $\epsilon = 1$, then what this means is that

$$\left| \frac{3x(x-1)}{(x-1)} - 3 \right| < 1 \Leftrightarrow |x-1| < \frac{1}{3}$$

i.e. as long as x is within $\frac{1}{3}$ of 1, then distance between the function and 3 will be less than 1.

□



6 Complex Numbers