Data Analytics
EEE 4774 & 6777

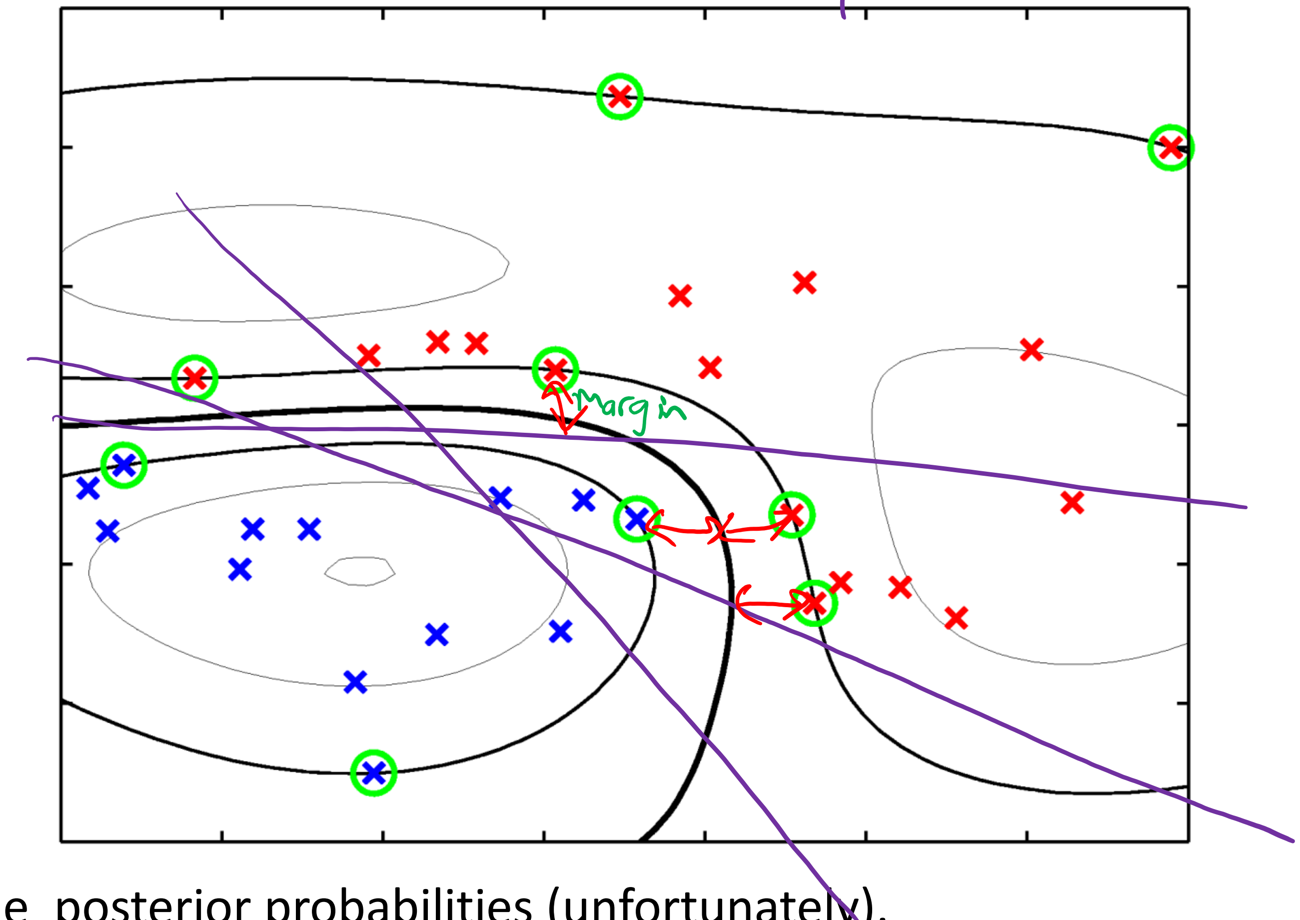
Module 4 - Classification

Support Vector Machine (SVM)

Spring 2022

Support Vector Machine (SVM)

Nonlinear Classification



- Aims at maximizing the margin between decision boundary and data points

- Used for both classification and regression

- Determination of model parameters corresponds to a convex optimization problem, so any local solution is also a global optimum

- SVM makes extensive use of the Lagrange Multipliers concept from the Optimization Theory

- SVM is a decision machine, so does not provide posterior probabilities (unfortunately).

- Relevance Vector Machine (RVM) is based on Bayesian formulation, and provides posterior probabilities.

Support Vector Machine (SVM)

ϕ: Nonlinear func. of x

- $y(\boldsymbol{x}) = \boldsymbol{w}^T \phi(\boldsymbol{x}) + b$: score for x

- Nonlinear fixed feature space mapping $\phi(\boldsymbol{x})$

actual class label (Grand truth)

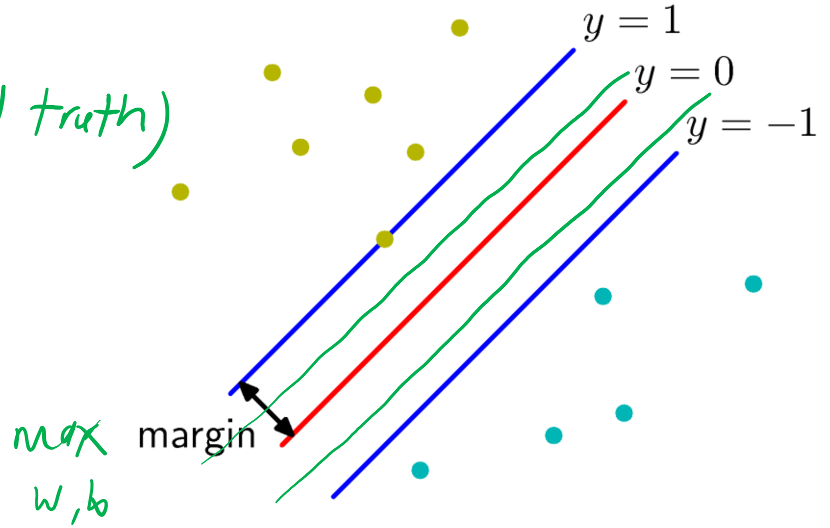
- 2-class model, $t_n \in \{-1, +1\}$

Class label estimate (SVM output)

- $\hat{t}_n = \begin{cases} +1, & y(\boldsymbol{x}) \geq 0 \\ -1, & y(\boldsymbol{x}) < 0 \end{cases}$ Class 1 / Class 2

- If linearly separable, many solutions exist

- SVM: Maximum margin classifier



Correct Class: $t_n = \hat{t}_n = +1$, $t_n = \hat{t}_n = -1$ } $t_n \hat{t}_n = 1$

Support Vector Machine (SVM)

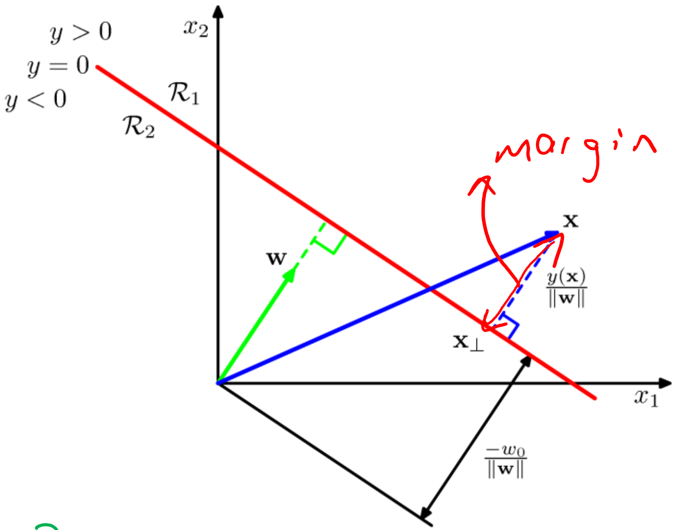
Objective: Find w, b which give the maximum margin

- Correctly classified points: $t_n \overbrace{y(\boldsymbol{x}_n)}^{\hat{t}_n} > 0$

- $\max \frac{t_n y(\boldsymbol{x}_n)}{\|\boldsymbol{w}\|} = \max \frac{t_n(\overbrace{\boldsymbol{w}^T\phi(\boldsymbol{x}_n)+b}^{y(x_n)})}{\|\boldsymbol{w}\|}$

  $w^T w = \|w\|^2$

- $\arg\max_{\boldsymbol{w},b}\left\{\frac{1}{\|\boldsymbol{w}\|}\min_{n}[t_n(\overbrace{\boldsymbol{w}^T\phi(\boldsymbol{x}_n)+b}^{y(x_n)})]\right\}$



$\frac{t_n[\cancel{\kappa} w^T\phi(x_n) + \cancel{\kappa} b_n]}{\|\cancel{\kappa} w\|} = \frac{\cancel{\kappa}\, t_n y(x_n)}{\cancel{\kappa}\, \|w\|}$

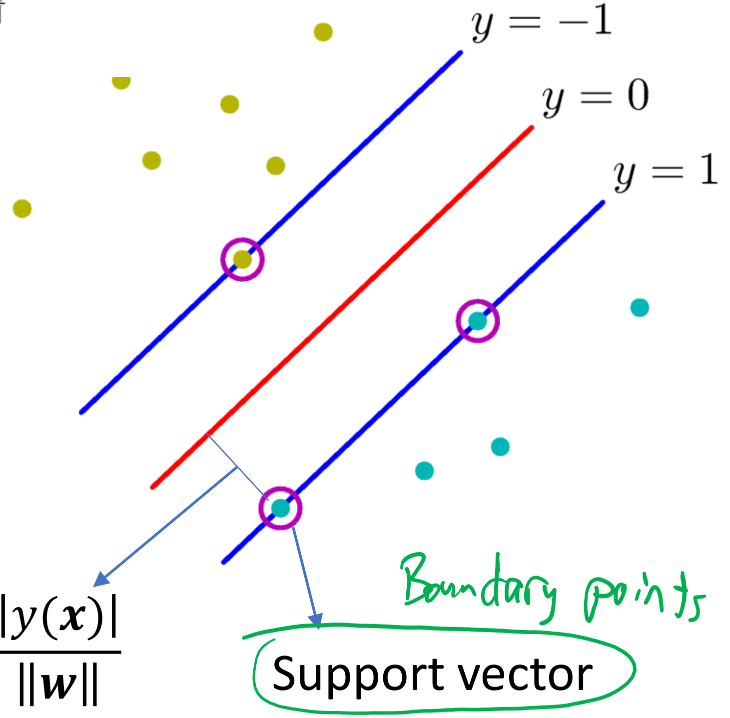
- margin $= \frac{t_n y(\boldsymbol{x}_n)}{\|\boldsymbol{w}\|}$ does not change when $\boldsymbol{w} \to \kappa\boldsymbol{w}$ and b $\to \kappa b$

$t_n y(x_n) = 1$

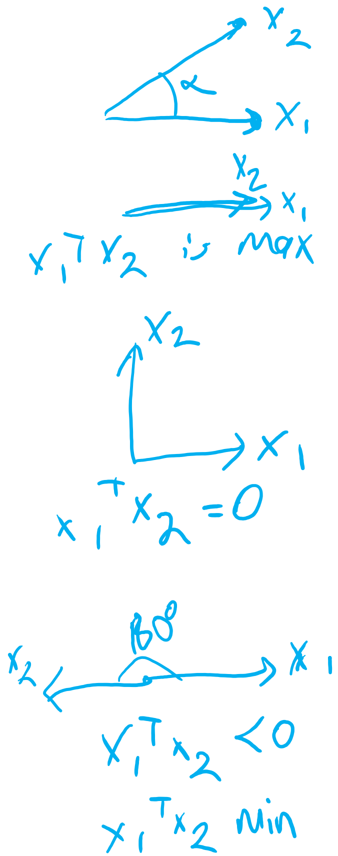
- Choose κ such that $t_n\left(\boldsymbol{w}^T\phi(\boldsymbol{x}_n)+b\right) = 1$ for all

$t_n y(x_n) \geq 1 \quad \forall n$

- $\arg\min_{\boldsymbol{w},b}\|\boldsymbol{w}\|^2$ s.t. $t_n\left(\boldsymbol{w}^T\phi(\boldsymbol{x}_n)+b\right) \geq 1$, $n = 1, \dots, N$



Support Vector Machine (SVM)



Kernel func. : similarity score betw. x and x_n

$k(x, x_n)$

Lagrange multiplier

- Using Lagrange multipliers $a_n \geq 0$ we obtain $\boxed{\boldsymbol{w} = \sum_{n=1}^{N} a_n t_n \, \phi(\boldsymbol{x}_n)}$

$\phi(x_n)^T \phi(x)$

- Hence, $y(\boldsymbol{x}) = \sum_{n=1}^{N} a_n t_n k(\boldsymbol{x}, \boldsymbol{x}_n) + b$

t_n → label for n^{th} training instance

$y(x_n) = w^T \phi(x_n) + b$

$y(x_n) = \sum_{n=1}^{N} a_n t_n \underbrace{\phi(x_n)^T \phi(x_n)} + b$

- Kernel function: $\boxed{k(\boldsymbol{x}_1, \boldsymbol{x}_2) = \phi(\boldsymbol{x}_1)^T \phi(\boldsymbol{x}_2)}$

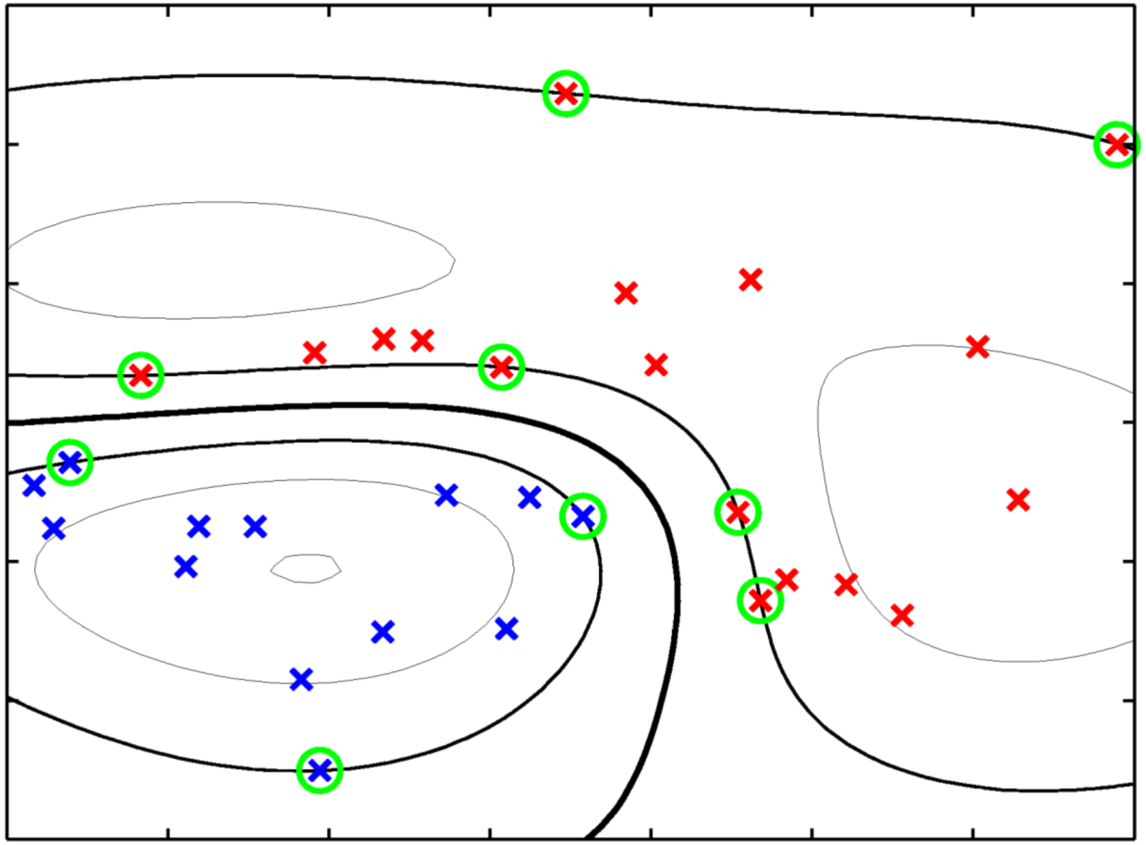
- Stationary kernels: $k(\boldsymbol{x}_1, \boldsymbol{x}_2) = k(\boldsymbol{x}_1 - \boldsymbol{x}_2)$

- Homogeneous kernels (Radial basis functions): $k(\boldsymbol{x}_1, \boldsymbol{x}_2) = k(\underbrace{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|})$

Rbf

Support Vector Machine (SVM)

Kernel Trick: Compute the similarity score $k(\boldsymbol{x}, \boldsymbol{x}_n)$ directly without defining $\phi(\boldsymbol{x})$



- Linear Kernel:

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{x}_1^T \boldsymbol{x}_2 = x_{11}x_{21} + x_{12}x_{22}$$

- Polynomial kernel

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = \left(\boldsymbol{x}_1^T \boldsymbol{x}_2 + r\right)^d$$

e.g., (r=0, d=2) 2nd degree polynomial

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = x_{11}^2 x_{21}^2 + x_{12}^2 x_{22}^2 + 2x_{11}x_{12}\, x_{21}x_{22}$$

(Hidden) $\phi(\boldsymbol{x}_1) = [x_{11}^2, x_{12}^2, \sqrt{2}\, x_{11}x_{12}]^T$

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = \phi(\boldsymbol{x}_1)^T \phi(\boldsymbol{x}_2)$$

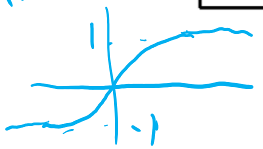
- Rbf (Gaussian kernel)

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = e^{-\gamma \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2} = e^{-\gamma (\boldsymbol{x}_1 - \boldsymbol{x}_2)^T (\boldsymbol{x}_1 - \boldsymbol{x}_2)}$$

- Sigmoid

hyperbolic tangent

$$k(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tanh(\boldsymbol{x}_1^T \boldsymbol{x}_2 + r)$$



Custom Kernels: You can define your own kernel function. Must be a valid kernel function!

Support Vector Machine (SVM)

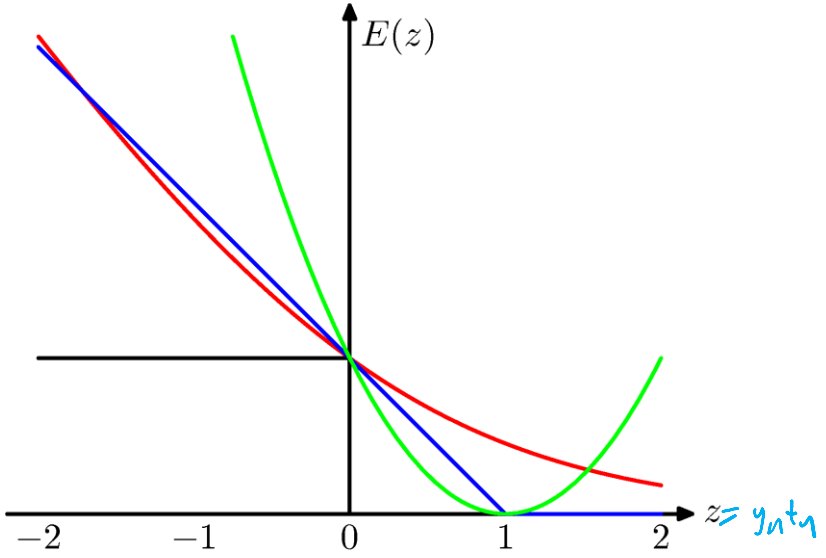
- When written in terms of minimization of a regularized error function, SVM has similarities with Logistic Regression and Perceptron:

  in the figure,
  blue for SVM (also for Perceptron by a shift of 1)
  red for Logistic Regression
  black for Misclassification error
  green for Quadratic error

  < 0 if $y_n t_n > 1 \Rightarrow$ correctly classified training pts.
  $\Rightarrow E(y_n t_n) = 0$

  > 0 if $y_n t_n < 1 \Rightarrow E(y_n, t_n) > 0$

  Hinge Loss: $\mathrm{E}(y_n t_n) = \max\{1 - y_n t_n, 0\}$



- Weighted voting (compare to kNN) with weights coming from the similarity metric $k(\boldsymbol{x}, \boldsymbol{x}_n)$
  kernel func.

- Extends to multiclass problems

  Support Vector Regression

- Used also for anomaly detection (One-Class SVM) and regression (SVR)