

UTP Designs

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1 Design Signature and Core Laws

```
theory utp-des-core
imports UTP.utp
begin
```

In UTP, in order to explicitly record the termination of a program, a subset of alphabetised relations is introduced. These relations are called designs, and their alphabet contains the special boolean observational variable *ok*. It is used to record the start and termination of a program. For more information on designs please see Chapter 3 of the UTP book [3], or the more accessible designs tutorial [2].

1.1 Definitions

Two named theorem sets exist are created to group theorems that, respectively, provide pre-postcondition definitions, and simplify operators to their normal design form.

```
named-theorems ndes and ndes-simp
```

```
alphabet des-vars =
  ok :: bool
```

```
declare des-vars.defs [lens-defs]
```

The two locale interpretations below are a technicality to improve automatic proof support via the predicate and relational tactics. This is to enable the (re-)interpretation of state spaces to remove any occurrences of lens types after the proof tactics *pred-simp* and *rel-simp*, or any of their derivatives have been applied. Eventually, it would be desirable to automate both interpretations as part of a custom outer command for defining alphabets.

```
interpretation des-vars: lens-interp  $\lambda r. (ok_v\ r, more\ r)$ 
apply (unfold-locales)
apply (rule injI)
apply (clarsimp)
done
```

```
interpretation des-vars-rel:
  lens-interp  $\lambda(r, r'). (ok_v\ r, ok_v\ r', more\ r, more\ r')$ 
apply (unfold-locales)
apply (rule injI)
apply (clarsimp)
done
```

```
lemma ok-ord [usubst]:
  $ok <_v $ok'
  by (simp add: var-name-ord-def)
```

```
type-synonym ' $\alpha$  des = ' $\alpha$  des-vars-scheme
type-synonym (' $\alpha$ , ' $\beta$ ) rel-des = (' $\alpha$  des, ' $\beta$  des) urel
type-synonym ' $\alpha$  hrel-des = (' $\alpha$  des) hrel
```

```
translations
  (type) ' $\alpha$  des <= (type) ' $\alpha$  des-vars-scheme
```

$(type) \ ' \alpha \ des \leq (type) \ ' \alpha \ des\text{-}vars\text{-}ext$
 $(type) \ (' \alpha, ' \beta) \ rel\text{-}des \leq (type) \ (' \alpha \ des, ' \beta \ des) \ urel$
 $(type) \ ' \alpha \ hrel\text{-}des \leq (type) \ ' \alpha \ des \ hrel$

notation *des-vars-child-lens* (Σ_D)

lemma *ok-des-bij-lens*: *bij-lens* $(ok +_L \Sigma_D)$

by $(unfold\text{-}locales, simp\text{-}all \text{ add: } ok\text{-}def \ des\text{-}vars\text{-}child\text{-}lens\text{-}def \ lens\text{-}plus\text{-}def \ prod.\text{case}\text{-}eq\text{-}if)$

Define the lens functor for designs

definition *lmap-des-vars* :: $(' \alpha \implies ' \beta) \implies (' \alpha \ des\text{-}vars\text{-}scheme \implies ' \beta \ des\text{-}vars\text{-}scheme) \ (lmap_D)$
where $[lens\text{-}defs]: lmap\text{-}des\text{-}vars = lmap[des\text{-}vars]$

lemma *lmap-des-vars*: *vwb-lens* $f \implies vwb\text{-}lens \ (lmap\text{-}des\text{-}vars \ f)$

by $(unfold\text{-}locales, auto \ simp \text{ add: } lens\text{-}defs)$

lemma *lmap-id*: $lmap_D \ 1_L = 1_L$

by $(simp \text{ add: } lens\text{-}defs \ fun\text{-}eq\text{-}iff)$

lemma *lmap-comp*: $lmap_D \ (f ;_L g) = lmap_D \ f ;_L lmap_D \ g$

by $(simp \text{ add: } lens\text{-}defs \ fun\text{-}eq\text{-}iff)$

The following notations define liftings from non-design predicates into design predicates using alphabet extensions.

abbreviation *lift-desr* $(\lceil \cdot \rceil_D)$

where $\lceil P \rceil_D \equiv P \oplus_P (\Sigma_D \times_L \Sigma_D)$

abbreviation *lift-pre-desr* $(\lceil \cdot \rceil_{D<})$

where $\lceil p \rceil_{D<} \equiv \lceil \lceil p \rceil_{<} \rceil_D$

abbreviation *lift-post-desr* $(\lceil \cdot \rceil_{D>})$

where $\lceil p \rceil_{D>} \equiv \lceil \lceil p \rceil_{>} \rceil_D$

abbreviation *drop-desr* $(\lfloor \cdot \rfloor_D)$

where $\lfloor P \rfloor_D \equiv P \upharpoonright_e (\Sigma_D \times_L \Sigma_D)$

abbreviation *dcond* :: $(' \alpha, ' \beta) \ rel\text{-}des \Rightarrow ' \alpha \ upred \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des$

$((\exists - \triangleleft - \triangleright_D / -) [52,0,53] \ 52)$

where $P \triangleleft b \triangleright_D Q \equiv P \triangleleft \lceil b \rceil_{D<} \triangleright Q$

definition *design*:: $(' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \ (\mathbf{infixl} \vdash_{60})$ **where**

$[upred\text{-}defs]: P \vdash Q = (\$ok \wedge P \Rightarrow \$ok' \wedge Q)$

An rdesign is a design that uses the Isabelle type system to prevent reference to ok in the assumption and commitment.

definition *rdesign*:: $(' \alpha, ' \beta) \ urel \Rightarrow (' \alpha, ' \beta) \ urel \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \ (\mathbf{infixl} \vdash_r \ 60)$ **where**

$[upred\text{-}defs]: (P \vdash_r Q) = \lceil P \rceil_D \vdash \lceil Q \rceil_D$

An ndesign is a normal design, i.e. where the assumption is a condition

definition *ndesign*:: $' \alpha \ cond \Rightarrow (' \alpha, ' \beta) \ urel \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \ (\mathbf{infixl} \vdash_n \ 60)$ **where**

$[upred\text{-}defs]: (p \vdash_n Q) = (\lceil p \rceil_{<} \vdash_r Q)$

definition *skip-d* :: $' \alpha \ hrel\text{-}des \ (II_D)$ **where**

$[upred\text{-}defs]: II_D \equiv (true \vdash_r II)$

definition *bot-d* :: (α, β) *rel-des* (\perp_D) **where**
[upred-defs]: $\perp_D = (\text{false} \vdash \text{false})$

definition *pre-design* :: (α, β) *rel-des* \Rightarrow (α, β) *urel* (*pre_D*) **where**
[upred-defs]: *pre_D*(*P*) = $\lfloor \neg P \llbracket \text{true}, \text{false} / \$ok, \$ok' \rrbracket \rfloor_D$

definition *post-design* :: (α, β) *rel-des* \Rightarrow (α, β) *urel* (*post_D*) **where**
[upred-defs]: *post_D*(*P*) = $\lfloor P \llbracket \text{true}, \text{true} / \$ok, \$ok' \rrbracket \rfloor_D$

syntax

-*ok-f* :: *logic* \Rightarrow *logic* (f [1000] 1000)
-*ok-t* :: *logic* \Rightarrow *logic* (t [1000] 1000)
-*top-d* :: *logic* (\top_D)

translations

$P^f \Rightarrow \text{CONST usubst } (\text{CONST subst-upd } \text{CONST id } (\text{CONST ovar } \text{CONST ok}) \text{ false}) P$
 $P^t \Rightarrow \text{CONST usubst } (\text{CONST subst-upd } \text{CONST id } (\text{CONST ovar } \text{CONST ok}) \text{ true}) P$
 $\top_D \Rightarrow \text{CONST not-upred } (\text{CONST utp-expr.var } (\text{CONST ivar } \text{CONST ok}))$

1.2 Lifting, Unrestriction, and Substitution

lemma *drop-desr-inv* [*simp*]: $\lfloor \lfloor P \rfloor_D \rfloor_D = P$
by (*simp add: prod-mwb-lens*)

lemma *lift-desr-inv*:

fixes *P* :: (α, β) *rel-des*
assumes $\$ok \# P \$ok' \# P$
shows $\lfloor \lfloor P \rfloor_D \rfloor_D = P$

proof –

have *bij-lens* ($\Sigma_D \times_L \Sigma_D +_L (\text{in-var ok} +_L \text{out-var ok}) :: (-, \alpha \text{ des-vars-scheme} \times \beta \text{ des-vars-scheme})$
lens)

(**is** *bij-lens* (?*P*))

proof –

have $?P \approx_L (\text{ok} +_L \Sigma_D) \times_L (\text{ok} +_L \Sigma_D)$ (**is** $?P \approx_L ?Q$)

apply (*simp add: in-var-def out-var-def prod-as-plus*)

apply (*simp add: prod-as-plus [THEN sym]*)

apply (*meson lens-equiv-sym lens-equiv-trans lens-indep-prod lens-plus-comm lens-plus-prod-exchange des-vars-indeps(1)*)

done

moreover have *bij-lens* ?*Q*

by (*simp add: ok-des-bij-lens prod-bij-lens*)

ultimately show ?*thesis*

by (*metis bij-lens-equiv lens-equiv-sym*)

qed

with *assms* **show** ?*thesis*

apply (*rule-tac aext-arestr [of - in-var ok +_L out-var ok]*)

apply (*simp add: prod-mwb-lens*)

apply (*simp*)

apply (*metis alpha-in-var lens-indep-prod lens-indep-sym des-vars-indeps(1) out-var-def prod-as-plus*)

using *unrest-var-comp* **apply** *blast*

done

qed

lemma *unrest-out-des-lift* [*unrest*]: $\text{out } \alpha \# p \Rightarrow \text{out } \alpha \# \lfloor p \rfloor_D$

by (*pred-simp*)

lemma *lift-dist-seq* [*simp*]:
 $\lceil P \rrbracket_D \mathrel{;;} \lceil Q \rrbracket_D = (\lceil P \rrbracket_D \mathrel{;;} \lceil Q \rrbracket_D)$
 by (*rel-auto*)

lemma *lift-des-skip-dr-unit* [*simp*]:
 $(\lceil P \rrbracket_D \mathrel{;;} \lceil II \rrbracket_D) = \lceil P \rrbracket_D$
 $(\lceil II \rrbracket_D \mathrel{;;} \lceil P \rrbracket_D) = \lceil P \rrbracket_D$
 by (*rel-auto*)⁺

lemma *lift-des-skip-dr-unit-unrest*: $\$ok' \# P \implies (P \mathrel{;;} \lceil II \rrbracket_D) = P$
 by (*rel-auto*)

lemma *state-subst-design* [*usubst*]:
 $\lceil \sigma \oplus_s \Sigma_D \rrbracket_s \dagger (P \vdash_r Q) = (\lceil \sigma \rrbracket_s \dagger P) \vdash_r (\lceil \sigma \rrbracket_s \dagger Q)$
 by (*rel-auto*)

lemma *design-subst* [*usubst*]:
 $\llbracket \$ok \# \sigma; \$ok' \# \sigma \rrbracket \implies \sigma \dagger (P \vdash Q) = (\sigma \dagger P) \vdash (\sigma \dagger Q)$
 by (*simp add: design-def usubst*)

lemma *design-msubst* [*usubst*]:
 $(P(x) \vdash Q(x)) \llbracket x \rightarrow v \rrbracket = (P(x) \llbracket x \rightarrow v \rrbracket \vdash Q(x) \llbracket x \rightarrow v \rrbracket)$
 by (*rel-auto*)

lemma *design-ok-false* [*usubst*]: $(P \vdash Q) \llbracket false / \$ok \rrbracket = true$
 by (*simp add: design-def usubst*)

lemma *ok-pre*: $(\$ok \wedge \lceil pre_D(P) \rrbracket_D) = (\$ok \wedge (\neg P^f))$
 by (*pred-auto robust*)

lemma *ok-post*: $(\$ok \wedge \lceil post_D(P) \rrbracket_D) = (\$ok \wedge (P^t))$
 by (*pred-auto robust*)

1.3 Basic Design Laws

lemma *design-export-ok*: $P \vdash Q = (P \vdash (\$ok \wedge Q))$
 by (*rel-auto*)

lemma *design-export-ok'*: $P \vdash Q = (P \vdash (\$ok' \wedge Q))$
 by (*rel-auto*)

lemma *design-export-pre*: $P \vdash (P \wedge Q) = P \vdash Q$
 by (*rel-auto*)

lemma *design-export-spec*: $P \vdash (P \Rightarrow Q) = P \vdash Q$
 by (*rel-auto*)

lemma *design-ok-pre-conj*: $(\$ok \wedge P) \vdash Q = P \vdash Q$
 by (*rel-auto*)

lemma *true-is-design*: $(false \vdash true) = true$
 by (*rel-auto*)

lemma *true-is-rdesign*: $(false \vdash_r true) = true$

```

by (rel-auto)

lemma bot-d-true:  $\perp_D = true$ 
by (rel-auto)

lemma bot-d-ndes-def [ndes-simp]:  $\perp_D = (false \vdash_n true)$ 
by (rel-auto)

lemma design-false-pre:  $(false \vdash P) = true$ 
by (rel-auto)

lemma redesign-false-pre:  $(false \vdash_r P) = true$ 
by (rel-auto)

lemma ndesign-false-pre:  $(false \vdash_n P) = true$ 
by (rel-auto)

lemma ndesign-miracle:  $(true \vdash_n false) = \top_D$ 
by (rel-auto)

lemma top-d-ndes-def [ndes-simp]:  $\top_D = (true \vdash_n false)$ 
by (rel-auto)

lemma skip-d-alt-def:  $II_D = true \vdash II$ 
by (rel-auto)

lemma skip-d-ndes-def [ndes-simp]:  $II_D = true \vdash_n II$ 
by (rel-auto)

lemma design-subst-ok:
   $(P \llbracket true/\$ok \rrbracket \vdash Q \llbracket true/\$ok \rrbracket) = (P \vdash Q)$ 
by (rel-auto)

lemma design-subst-ok-ok':
   $(P \llbracket true/\$ok \rrbracket \vdash Q \llbracket true, true/\$ok, \$ok' \rrbracket) = (P \vdash Q)$ 
proof -
  have  $(P \vdash Q) = ((\$ok \wedge P) \vdash (\$ok \wedge \$ok' \wedge Q))$ 
  by (pred-auto)
  also have  $\dots = ((\$ok \wedge P \llbracket true/\$ok \rrbracket) \vdash (\$ok \wedge (\$ok' \wedge Q \llbracket true/\$ok' \rrbracket) \llbracket true/\$ok \rrbracket))$ 
  by (metis conj-eq-out-var-subst conj-pos-var-subst upred-eq-true utp-pred-laws.inf-commute ok-vwb-lens)
  also have  $\dots = ((\$ok \wedge P \llbracket true/\$ok \rrbracket) \vdash (\$ok \wedge \$ok' \wedge Q \llbracket true, true/\$ok, \$ok' \rrbracket))$ 
  by (simp add: usubst)
  also have  $\dots = (P \llbracket true/\$ok \rrbracket \vdash Q \llbracket true, true/\$ok, \$ok' \rrbracket)$ 
  by (pred-auto)
  finally show ?thesis ..
qed

lemma design-subst-ok-ok':
   $(P \vdash Q \llbracket true/\$ok' \rrbracket) = (P \vdash Q)$ 
proof -
  have  $(P \vdash Q) = (P \vdash (\$ok' \wedge Q))$ 
  by (pred-auto)
  also have  $\dots = (P \vdash (\$ok' \wedge Q \llbracket true/\$ok' \rrbracket))$ 
  by (metis conj-eq-out-var-subst upred-eq-true utp-pred-laws.inf-commute ok-vwb-lens)
  also have  $\dots = (P \vdash Q \llbracket true/\$ok' \rrbracket)$ 

```

by (pred-auto)
 finally show ?thesis ..
 qed

1.4 Sequential Composition Laws

theorem *design-skip-idem* [simp]:

$$(II_D ;; II_D) = II_D$$

by (rel-auto)

theorem *design-composition-subst*:

assumes

$$\$ok' \# P1 \ \$ok \# P2$$

shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) =$

$$(((\neg (\neg P1) ;; true)) \wedge \neg (Q1 \llbracket true/\$ok' \rrbracket ;; (\neg P2))) \vdash (Q1 \llbracket true/\$ok' \rrbracket ;; Q2 \llbracket true/\$ok \rrbracket))$$

proof –

$$\text{have } ((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (\exists \ ok_0 \cdot ((P1 \vdash Q1) \llbracket \llcorner ok_0 \gg / \$ok' \rrbracket ;; (P2 \vdash Q2) \llbracket \llcorner ok_0 \gg / \$ok \rrbracket))$$

by (rule segr-middle, simp)

also have ...

$$= (((P1 \vdash Q1) \llbracket false/\$ok' \rrbracket ;; (P2 \vdash Q2) \llbracket false/\$ok \rrbracket) \vee ((P1 \vdash Q1) \llbracket true/\$ok' \rrbracket ;; (P2 \vdash Q2) \llbracket true/\$ok \rrbracket))$$

by (simp add: true-alt-def false-alt-def, pred-auto)

also from *assms*

$$\text{have } \dots = (((\$ok \wedge P1 \Rightarrow Q1 \llbracket true/\$ok' \rrbracket) ;; (P2 \Rightarrow \$ok' \wedge Q2 \llbracket true/\$ok \rrbracket)) \vee ((\neg (\$ok \wedge P1)) ;; true))$$

by (simp add: design-def usubst unrest, pred-auto)

$$\text{also have } \dots = ((\neg \$ok ;; true_h) \vee ((\neg P1) ;; true) \vee (Q1 \llbracket true/\$ok' \rrbracket ;; (\neg P2)) \vee (\$ok' \wedge (Q1 \llbracket true/\$ok' \rrbracket ;; Q2 \llbracket true/\$ok \rrbracket)))$$

by (rel-auto)

$$\text{also have } \dots = (((\neg (\neg P1) ;; true)) \wedge \neg (Q1 \llbracket true/\$ok' \rrbracket ;; (\neg P2))) \vdash (Q1 \llbracket true/\$ok' \rrbracket ;; Q2 \llbracket true/\$ok \rrbracket))$$

by (simp add: precond-right-unit design-def unrest, rel-auto)

finally show ?thesis .

qed

theorem *design-composition*:

assumes

$$\$ok' \# P1 \ \$ok \# P2 \ \$ok' \# Q1 \ \$ok \# Q2$$

shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg (\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$

using *assms* by (simp add: design-composition-subst usubst)

theorem *design-composition-runrest*:

assumes

$$\$ok' \# P1 \ \$ok \# P2 \ ok \# Q1 \ ok \# Q2$$

shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg (\neg P1) ;; true)) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (Q1 ;; Q2))$

proof –

$$\text{have } (\$ok \wedge \$ok' \wedge (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket)) = (\$ok \wedge \$ok' \wedge (Q1 ;; Q2))$$

proof –

$$\text{have } (\$ok \wedge \$ok' \wedge (Q1 ;; Q2)) = ((\$ok \wedge Q1) ;; (Q2 \wedge \$ok'))$$

by (metis (no-types, lifting) conj-comm segr-post-var-out segr-pre-var-out)

$$\text{also have } \dots = ((Q1 \wedge \$ok') ;; (\$ok \wedge Q2))$$

by (simp add: assms(3) assms(4) runrest-ident-var)

$$\text{also have } \dots = (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket)$$

by (metis ok-vwb-lens segr-pre-transfer segr-right-one-point true-alt-def uovar-convr upred-eq-true utp-pred-laws.inf.left-idem utp-rel.unrest-ouvar vwb-lens-mwb)

finally show ?thesis

by (metis utp-pred-laws.inf.left-commute utp-pred-laws.inf.left-idem)

qed
moreover have $(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (Q1^t ;; Q2[\![true/\$ok]\!]) =$
 $(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (\$ok \wedge \$ok' \wedge (Q1^t ;; Q2[\![true/\$ok]\!]))$
by (*metis design-export-ok design-export-ok'*)
ultimately show *?thesis* **using** *assms*
by (*simp add: design-composition-subst usubst, metis design-export-ok design-export-ok'*)
qed

theorem *rdesign-composition:*
 $((P1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = (((\neg (\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$
by (*simp add: rdesign-def design-composition unrest alpha*)

theorem *design-composition-cond:*
assumes
 $out\alpha \# p1 \ \$ok \# P2 \ \$ok' \# Q1 \ \$ok \# Q2$
shows $((p1 \vdash Q1) ;; (P2 \vdash Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$
using *assms*
by (*simp add: design-composition unrest precondition-right-unit*)

theorem *rdesign-composition-cond:*
assumes *outα # p1*
shows $((p1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$
using *assms*
by (*simp add: rdesign-def design-composition-cond unrest alpha*)

theorem *design-composition-wp:*
assumes
 $ok \# p1 \ ok \# p2$
 $\$ok \# Q1 \ \$ok' \# Q1 \ \$ok \# Q2 \ \$ok' \# Q2$
shows $((\lceil p1 \rceil_{<} \vdash Q1) ;; (\lceil p2 \rceil_{<} \vdash Q2)) = ((\lceil p1 \wedge Q1 \ wp \ p2 \rceil_{<}) \vdash (Q1 ;; Q2))$
using *assms* **by** (*rel-blast*)

theorem *rdesign-composition-wp:*
 $((\lceil p1 \rceil_{<} \vdash_r Q1) ;; (\lceil p2 \rceil_{<} \vdash_r Q2)) = ((\lceil p1 \wedge Q1 \ wp \ p2 \rceil_{<}) \vdash_r (Q1 ;; Q2))$
by (*rel-blast*)

theorem *ndesign-composition-wp [ndes-simp]:*
 $((p1 \vdash_n Q1) ;; (p2 \vdash_n Q2)) = ((p1 \wedge Q1 \ wp \ p2) \vdash_n (Q1 ;; Q2))$
by (*rel-blast*)

theorem *design-true-left-zero:* $(true ;; (P \vdash Q)) = true$
proof –
have $(true ;; (P \vdash Q)) = (\exists \ ok_0 \cdot true[\![\ll ok_0 \gg / \$ok']\!]) ;; (P \vdash Q)[\![\ll ok_0 \gg / \$ok]\!])$
by (*subst segr-middle[of ok], simp-all*)
also have $\dots = ((true[\![false/\$ok']\!]) ;; (P \vdash Q)[\![false/\$ok]\!]) \vee (true[\![true/\$ok']\!]) ;; (P \vdash Q)[\![true/\$ok]\!])$
by (*simp add: disj-comm false-alt-def true-alt-def*)
also have $\dots = ((true[\![false/\$ok']\!]) ;; true_h) \vee (true ;; ((P \vdash Q)[\![true/\$ok]\!]))$
by (*subst-tac, rel-auto*)
also have $\dots = true$
by (*subst-tac, simp add: precondition-right-unit unrest*)
finally show *?thesis* .
qed

theorem *design-left-unit-hom:*
fixes $P \ Q :: 'a \ hrel_des$

shows $(II_D ;; (P \vdash_r Q)) = (P \vdash_r Q)$
proof –
have $(II_D ;; (P \vdash_r Q)) = ((true \vdash_r II) ;; (P \vdash_r Q))$
by (*simp add: skip-d-def*)
also have $\dots = (true \wedge \neg (II ;; (\neg P))) \vdash_r (II ;; Q)$
proof –
have $out\alpha \# true$
by *unrest-tac*
thus *?thesis*
using *rdesign-composition-cond* **by** *blast*
qed
also have $\dots = (\neg (\neg P)) \vdash_r Q$
by *simp*
finally show *?thesis* **by** *simp*
qed

theorem *rdesign-left-unit* [*simp*]:
 $II_D ;; (P \vdash_r Q) = (P \vdash_r Q)$
by (*rel-auto*)

theorem *design-right-semi-unit*:
 $(P \vdash_r Q) ;; II_D = ((\neg (\neg P) ;; true) \vdash_r Q)$
by (*simp add: skip-d-def rdesign-composition*)

theorem *design-right-cond-unit* [*simp*]:
assumes $out\alpha \# p$
shows $(p \vdash_r Q) ;; II_D = (p \vdash_r Q)$
using *assms*
by (*simp add: skip-d-def rdesign-composition-cond*)

theorem *ndesign-left-unit* [*simp*]:
 $II_D ;; (p \vdash_n Q) = (p \vdash_n Q)$
by (*rel-auto*)

theorem *design-bot-left-zero*: $(\perp_D ;; (P \vdash Q)) = \perp_D$
by (*rel-auto*)

theorem *design-top-left-zero*: $(\top_D ;; (P \vdash Q)) = \top_D$
by (*rel-auto*)

1.5 Preconditions and Postconditions

theorem *design-npre*:
 $(P \vdash Q)^f = (\neg \$ok \vee \neg P^f)$
by (*rel-auto*)

theorem *design-pre*:
 $\neg (P \vdash Q)^f = (\$ok \wedge P^f)$
by (*simp add: design-def, subst-tac*)
(metis (no-types, hide-lams) not-conj-deMorgans true-not-false(2) utp-pred-laws.compl-top-eq utp-pred-laws.sup.idem utp-pred-laws.sup-compl-top)

theorem *design-post*:
 $(P \vdash Q)^t = ((\$ok \wedge P^t) \Rightarrow Q^t)$
by (*rel-auto*)

theorem *rdesign-pre* [simp]: $\text{pre}_D(P \vdash_r Q) = P$
by (*pred-auto*)

theorem *rdesign-post* [simp]: $\text{post}_D(P \vdash_r Q) = (P \Rightarrow Q)$
by (*pred-auto*)

theorem *ndesign-pre* [simp]: $\text{pre}_D(p \vdash_n Q) = \lceil p \rceil_<$
by (*pred-auto*)

theorem *ndesign-post* [simp]: $\text{post}_D(p \vdash_n Q) = (\lceil p \rceil_< \Rightarrow Q)$
by (*pred-auto*)

lemma *design-pre-choice* [simp]:
 $\text{pre}_D(P \sqcap Q) = (\text{pre}_D(P) \wedge \text{pre}_D(Q))$
by (*rel-auto*)

lemma *design-post-choice* [simp]:
 $\text{post}_D(P \sqcap Q) = (\text{post}_D(P) \vee \text{post}_D(Q))$
by (*rel-auto*)

lemma *design-pre-condr* [simp]:
 $\text{pre}_D(P \triangleleft \lceil b \rceil_D \triangleright Q) = (\text{pre}_D(P) \triangleleft b \triangleright \text{pre}_D(Q))$
by (*rel-auto*)

lemma *design-post-condr* [simp]:
 $\text{post}_D(P \triangleleft \lceil b \rceil_D \triangleright Q) = (\text{post}_D(P) \triangleleft b \triangleright \text{post}_D(Q))$
by (*rel-auto*)

lemma *preD-USUP-mem*: $\text{pre}_D(\bigsqcup_{i \in A} P \cdot i) = (\bigsqcap_{i \in A} \text{pre}_D(P \cdot i))$
by (*rel-auto*)

lemma *preD-USUP-ind*: $\text{pre}_D(\bigsqcup i \cdot P \cdot i) = (\bigsqcap i \cdot \text{pre}_D(P \cdot i))$
by (*rel-auto*)

1.6 Distribution Laws

theorem *design-choice*:
 $(P_1 \vdash P_2) \sqcap (Q_1 \vdash Q_2) = ((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2))$
by (*rel-auto*)

theorem *rdesign-choice*:
 $(P_1 \vdash_r P_2) \sqcap (Q_1 \vdash_r Q_2) = ((P_1 \wedge Q_1) \vdash_r (P_2 \vee Q_2))$
by (*rel-auto*)

theorem *ndesign-choice* [ndes-simp]:
 $(p_1 \vdash_n P_2) \sqcap (q_1 \vdash_n Q_2) = ((p_1 \wedge q_1) \vdash_n (P_2 \vee Q_2))$
by (*rel-auto*)

theorem *ndesign-choice'* [ndes-simp]:
 $((p_1 \vdash_n P_2) \vee (q_1 \vdash_n Q_2)) = ((p_1 \wedge q_1) \vdash_n (P_2 \vee Q_2))$
by (*rel-auto*)

theorem *design-inf*:
 $(P_1 \vdash P_2) \sqcup (Q_1 \vdash Q_2) = ((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow P_2) \wedge (Q_1 \Rightarrow Q_2)))$
by (*rel-auto*)

theorem *rdesign-inf*:

$(P_1 \vdash_r P_2) \sqcup (Q_1 \vdash_r Q_2) = ((P_1 \vee Q_1) \vdash_r ((P_1 \Rightarrow P_2) \wedge (Q_1 \Rightarrow Q_2)))$
by (*rel-auto*)

theorem *ndesign-inf* [*ndes-simp*]:

$(p_1 \vdash_n P_2) \sqcup (q_1 \vdash_n Q_2) = ((p_1 \vee q_1) \vdash_n (([p_1]_{<} \Rightarrow P_2) \wedge ([q_1]_{<} \Rightarrow Q_2)))$
by (*rel-auto*)

theorem *design-condr*:

$((P_1 \vdash P_2) \triangleleft b \triangleright (Q_1 \vdash Q_2)) = ((P_1 \triangleleft b \triangleright Q_1) \vdash (P_2 \triangleleft b \triangleright Q_2))$
by (*rel-auto*)

theorem *ndesign-dcond* [*ndes-simp*]:

$((p_1 \vdash_n P_2) \triangleleft b \triangleright_D (q_1 \vdash_n Q_2)) = ((p_1 \triangleleft b \triangleright q_1) \vdash_n (P_2 \triangleleft b \triangleright_r Q_2))$
by (*rel-auto*)

lemma *design-UINF-mem*:

assumes $A \neq \{\}$
shows $(\prod i \in A \cdot P(i) \vdash Q(i)) = (\bigsqcup i \in A \cdot P(i) \vdash (\prod i \in A \cdot Q(i)))$
using *assms* **by** (*rel-auto*)

lemma *ndesign-UINF-mem* [*ndes-simp*]:

assumes $A \neq \{\}$
shows $(\prod i \in A \cdot p(i) \vdash_n Q(i)) = (\bigsqcup i \in A \cdot p(i) \vdash_n (\prod i \in A \cdot Q(i)))$
using *assms* **by** (*rel-auto*)

lemma *ndesign-UINF-ind* [*ndes-simp*]:

$(\prod i \cdot p(i) \vdash_n Q(i)) = (\bigsqcup i \cdot p(i) \vdash_n (\prod i \cdot Q(i)))$
by (*rel-auto*)

lemma *design-USUP-mem*:

$(\bigsqcup i \in A \cdot P(i) \vdash Q(i)) = (\prod i \in A \cdot P(i) \vdash (\bigsqcup i \in A \cdot P(i) \Rightarrow Q(i)))$
by (*rel-auto*)

lemma *ndesign-USUP-mem* [*ndes-simp*]:

$(\bigsqcup i \in A \cdot p(i) \vdash_n Q(i)) = (\prod i \in A \cdot p(i) \vdash_n (\bigsqcup i \in A \cdot [p(i)]_{<} \Rightarrow Q(i)))$
by (*rel-auto*)

lemma *ndesign-USUP-ind* [*ndes-simp*]:

$(\bigsqcup i \cdot p(i) \vdash_n Q(i)) = (\prod i \cdot p(i) \vdash_n (\bigsqcup i \cdot [p(i)]_{<} \Rightarrow Q(i)))$
by (*rel-auto*)

1.7 Refinement Introduction

lemma *ndesign-eq-intro*:

assumes $p_1 = q_1 \ P_2 = Q_2$
shows $p_1 \vdash_n P_2 = q_1 \vdash_n Q_2$
by (*simp add: assms*)

theorem *design-refinement*:

assumes
 $\$ok \# P1 \ \$ok' \# P1 \ \$ok \# P2 \ \$ok' \# P2$
 $\$ok \# Q1 \ \$ok' \# Q1 \ \$ok \# Q2 \ \$ok' \# Q2$
shows $(P1 \vdash Q1 \sqsubseteq P2 \vdash Q2) \longleftrightarrow ('P1 \Rightarrow P2' \wedge 'P1 \wedge Q2 \Rightarrow Q1')$

proof –

have $(P1 \vdash Q1) \sqsubseteq (P2 \vdash Q2) \longleftrightarrow '(\$ok \wedge P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (\$ok \wedge P1 \Rightarrow \$ok' \wedge Q1)'$

by (pred-auto)
 also with *assms* have ... = $(P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (P1 \Rightarrow \$ok' \wedge Q1)$
 by (subst subst-bool-split[of in-var ok], simp-all, subst-tac)
 also with *assms* have ... = $(\neg P2 \Rightarrow \neg P1) \wedge ((P2 \Rightarrow Q2) \Rightarrow P1 \Rightarrow Q1)$
 by (subst subst-bool-split[of out-var ok], simp-all, subst-tac)
 also have ... $\longleftrightarrow (P1 \Rightarrow P2) \wedge (P1 \wedge Q2 \Rightarrow Q1)$
 by (pred-auto)
 finally show ?thesis .
 qed

theorem *rdesign-refinement*:

$(P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2) \longleftrightarrow (P1 \Rightarrow P2' \wedge P1 \wedge Q2 \Rightarrow Q1')$
 by (rel-auto)

lemma *design-refine-intro*:

assumes $P1 \Rightarrow P2'$ $P1 \wedge Q2 \Rightarrow Q1'$
 shows $P1 \vdash Q1 \sqsubseteq P2 \vdash Q2$
 using *assms* unfolding *upred-defs*
 by (pred-auto)

lemma *design-refine-intro'*:

assumes $P2 \sqsubseteq P1$ $Q1 \sqsubseteq (P1 \wedge Q2)$
 shows $P1 \vdash Q1 \sqsubseteq P2 \vdash Q2$
 using *assms* *design-refine-intro*[of $P1$ $P2$ $Q2$ $Q1$] by (simp add: refBy-order)

lemma *rdesign-refine-intro*:

assumes $P1 \Rightarrow P2'$ $P1 \wedge Q2 \Rightarrow Q1'$
 shows $P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2$
 using *assms* unfolding *upred-defs*
 by (pred-auto)

lemma *rdesign-refine-intro'*:

assumes $P2 \sqsubseteq P1$ $Q1 \sqsubseteq (P1 \wedge Q2)$
 shows $P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2$
 using *assms* unfolding *upred-defs*
 by (pred-auto)

lemma *ndesign-refine-intro*:

assumes $p1 \Rightarrow p2'$ $\lceil p1 \rceil_{<} \wedge Q2 \Rightarrow Q1'$
 shows $p1 \vdash_n Q1 \sqsubseteq p2 \vdash_n Q2$
 using *assms* unfolding *upred-defs*
 by (pred-auto)

lemma *design-top*:

$(P \vdash Q) \sqsubseteq \top_D$
 by (rel-auto)

lemma *design-bottom*:

$\perp_D \sqsubseteq (P \vdash Q)$
 by (rel-auto)

lemma *design-refine-thms*:

assumes $P \sqsubseteq Q$
 shows $\text{pre}_D(P) \Rightarrow \text{pre}_D(Q)$ $\text{pre}_D(P) \wedge \text{post}_D(Q) \Rightarrow \text{post}_D(P)$
 apply (metis *assms* *design-pre-choice* *disj-comm* *disj-upred-def* *order-refl* *rdesign-refinement* *utp-pred-laws.le-iff-sup*)

```

  apply (metis assms conj-comm design-post-choice disj-upred-def refBy-order semilattice-sup-class.le-iff-sup
    utp-pred-laws.inf.coboundedI1)
done

end

```

2 Design Healthiness Conditions

```

theory utp-des-healths
  imports utp-des-core
begin

```

2.1 H1: No observation is allowed before initiation

definition $H1 :: ('\alpha, '\beta) \text{rel-des} \Rightarrow ('\alpha, '\beta) \text{rel-des}$ **where**
 $[upred-defs]: H1(P) = (\$ok \Rightarrow P)$

lemma $H1\text{-idem}$:
 $H1(H1 P) = H1(P)$
by ($pred\text{-auto}$)

lemma $H1\text{-monotone}$:
 $P \sqsubseteq Q \Longrightarrow H1(P) \sqsubseteq H1(Q)$
by ($pred\text{-auto}$)

lemma $H1\text{-Continuous}$: *Continuous* $H1$
by ($rel\text{-auto}$)

lemma $H1\text{-below-top}$:
 $H1(P) \sqsubseteq \top_D$
by ($pred\text{-auto}$)

lemma $H1\text{-design-skip}$:
 $H1(\Pi) = \Pi_D$
by ($rel\text{-auto}$)

lemma $H1\text{-cond}$: $H1(P \triangleleft b \triangleright Q) = H1(P) \triangleleft b \triangleright H1(Q)$
by ($rel\text{-auto}$)

lemma $H1\text{-conj}$: $H1(P \wedge Q) = (H1(P) \wedge H1(Q))$
by ($rel\text{-auto}$)

lemma $H1\text{-disj}$: $H1(P \vee Q) = (H1(P) \vee H1(Q))$
by ($rel\text{-auto}$)

lemma $design\text{-export-}H1$: $(P \vdash Q) = (P \vdash H1(Q))$
by ($rel\text{-auto}$)

The H1 algebraic laws are valid only when $\alpha(R)$ is homogeneous. This should maybe be generalised.

theorem $H1\text{-algebraic-intro}$:
assumes
 $(true_h ;; R) = true_h$
 $(\Pi_D ;; R) = R$
shows R is $H1$

proof –
 have $R = (II_D ;; R)$ **by** (*simp add: assms(2)*)
 also have $\dots = (H1(II) ;; R)$
 by (*simp add: H1-design-skip*)
 also have $\dots = (\$ok \Rightarrow II) ;; R$
 by (*simp add: H1-def*)
 also have $\dots = (((\neg \$ok) ;; R) \vee R)$
 by (*simp add: impl-alt-def seqr-or-distl*)
 also have $\dots = (((\neg \$ok) ;; true_h) ;; R) \vee R$
 by (*simp add: precondition-right-unit unrest*)
 also have $\dots = (((\neg \$ok) ;; true_h) \vee R)$
 by (*metis assms(1) seqr-assoc*)
 also have $\dots = (\$ok \Rightarrow R)$
 by (*simp add: impl-alt-def precondition-right-unit unrest*)
 finally show *?thesis* **by** (*metis H1-def Healthy-def'*)
qed

lemma *nok-not-false*:
 $(\neg \$ok) \neq \text{false}$
by (*pred-auto*)

theorem *H1-left-zero*:
 assumes P is *H1*
 shows $(\text{true} ;; P) = \text{true}$

proof –
 from *assms* have $(\text{true} ;; P) = (\text{true} ;; (\$ok \Rightarrow P))$
 by (*simp add: H1-def Healthy-def'*)

 also from *assms* have $\dots = (\text{true} ;; (\neg \$ok \vee P))$ (*is - = (?true ;; -)*)
 by (*simp add: impl-alt-def*)
 also from *assms* have $\dots = ((?true ;; (\neg \$ok)) \vee (?true ;; P))$
 using *seqr-or-distr* **by** *blast*
 also from *assms* have $\dots = (\text{true} \vee (\text{true} ;; P))$
 by (*simp add: nok-not-false precondition-left-zero unrest*)
 finally show *?thesis*
 by (*simp add: upred-defs urel-defs*)
qed

theorem *H1-left-unit*:
 fixes $P :: 'a$ *hrel-des*
 assumes P is *H1*
 shows $(II_D ;; P) = P$

proof –
 have $(II_D ;; P) = (\$ok \Rightarrow II) ;; P$
 by (*metis H1-def H1-design-skip*)
 also have $\dots = (((\neg \$ok) ;; P) \vee P)$
 by (*simp add: impl-alt-def seqr-or-distl*)
 also from *assms* have $\dots = (((\neg \$ok) ;; true_h) ;; P) \vee P$
 by (*simp add: precondition-right-unit unrest*)
 also have $\dots = (((\neg \$ok) ;; (true_h ;; P)) \vee P)$
 by (*simp add: seqr-assoc*)
 also from *assms* have $\dots = (\$ok \Rightarrow P)$
 by (*simp add: H1-left-zero impl-alt-def precondition-right-unit unrest*)
 finally show *?thesis* **using** *assms*
 by (*simp add: H1-def Healthy-def'*)

qed

theorem *H1-algebraic*:

$P \text{ is } H1 \iff (true_h ;; P) = true_h \wedge (H_D ;; P) = P$
using *H1-algebraic-intro H1-left-unit H1-left-zero* **by** *blast*

theorem *H1-nok-left-zero*:

fixes $P :: 'a \text{ hrel-des}$
assumes $P \text{ is } H1$
shows $((\neg \$ok) ;; P) = (\neg \$ok)$

proof –

have $((\neg \$ok) ;; P) = (((\neg \$ok) ;; true_h) ;; P)$
by (*simp add: precondition-right-unit unrest*)
also have $\dots = ((\neg \$ok) ;; true_h)$
by (*metis H1-left-zero assms segr-assoc*)
also have $\dots = (\neg \$ok)$
by (*simp add: precondition-right-unit unrest*)
finally show *?thesis* .

qed

lemma *H1-design*:

$H1(P \vdash Q) = (P \vdash Q)$
by (*rel-auto*)

lemma *H1-rdesign*:

$H1(P \vdash_r Q) = (P \vdash_r Q)$
by (*rel-auto*)

lemma *H1-choice-closed* [*closure*]:

$\llbracket P \text{ is } H1; Q \text{ is } H1 \rrbracket \implies P \sqcap Q \text{ is } H1$
by (*simp add: H1-def Healthy-def' disj-upred-def impl-alt-def semilattice-sup-class.sup-left-commute*)

lemma *H1-inf-closed* [*closure*]:

$\llbracket P \text{ is } H1; Q \text{ is } H1 \rrbracket \implies P \sqcup Q \text{ is } H1$
by (*rel-blast*)

lemma *H1-UINF*:

assumes $A \neq \{\}$
shows $H1(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot H1(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *H1-Sup*:

assumes $A \neq \{\} \vee P \in A. P \text{ is } H1$
shows $(\bigsqcap A) \text{ is } H1$

proof –

from *assms*(2) **have** $H1 \text{ ' } A = A$
by (*auto simp add: Healthy-def rev-image-eqI*)
with *H1-UINF[of A id, OF assms(1)]* **show** *?thesis*
by (*simp add: UINF-as-Sup-image Healthy-def, presburger*)

qed

lemma *H1-USUP*:

shows $H1(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot H1(P(i)))$
by (*rel-auto*)

lemma *H1-Inf* [*closure*]:
assumes $\forall P \in A. P \text{ is } H1$
shows $(\bigsqcup A) \text{ is } H1$
proof –
from *assms* **have** $H1 \text{ ' } A = A$
by (*auto simp add: Healthy-def rev-image-eqI*)
with *H1-USUP*[*of A id*] **show** ?thesis
by (*simp add: USUP-as-Inf-image Healthy-def, presburger*)
qed

2.2 H2: A specification cannot require non-termination

definition *J* :: ' α *hrel-des* **where**
[*upred-defs*]: $J = ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D)$

definition *H2* **where**
[*upred-defs*]: $H2(P) \equiv P ;; J$

lemma *J-split*:
shows $(P ;; J) = (P^f \vee (P^t \wedge \$ok'))$
proof –
have $(P ;; J) = (P ;; ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D))$
by (*simp add: H2-def J-def design-def*)
also have $\dots = (P ;; ((\$ok \Rightarrow \$ok \wedge \$ok') \wedge \lceil II \rceil_D))$
by (*rel-auto*)
also have $\dots = ((P ;; (\neg \$ok \wedge \lceil II \rceil_D)) \vee (P ;; (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))))$
by (*rel-auto*)
also have $\dots = (P^f \vee (P^t \wedge \$ok'))$
proof –
have $(P ;; (\neg \$ok \wedge \lceil II \rceil_D)) = P^f$
proof –
have $(P ;; (\neg \$ok \wedge \lceil II \rceil_D)) = ((P \wedge \neg \$ok') ;; \lceil II \rceil_D)$
by (*rel-auto*)
also have $\dots = (\exists \$ok' \cdot P \wedge \$ok' =_u \text{false})$
by (*rel-auto*)
also have $\dots = P^f$
by (*metis C1 one-point out-var-uvar unrest-as-exists ok-vwb-lens vwb-lens-mwb*)
finally show ?thesis .
qed
moreover have $(P ;; (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P^t \wedge \$ok')$
proof –
have $(P ;; (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P ;; (\$ok \wedge II))$
by (*rel-auto*)
also have $\dots = (P^t \wedge \$ok')$
by (*rel-auto*)
finally show ?thesis .
qed
ultimately show ?thesis
by *simp*
qed
finally show ?thesis .
qed

lemma *H2-split*:
shows $H2(P) = (P^f \vee (P^t \wedge \$ok'))$
by (*simp add: H2-def J-split*)

theorem *H2-equivalence*:

$P \text{ is } H2 \iff 'P^f \Rightarrow P^t'$

proof –

have $'P \Leftrightarrow (P ;; J)'$ $\iff 'P \Leftrightarrow (P^f \vee (P^t \wedge \$ok'))'$

by (*simp add: J-split*)

also have $\dots \iff '(P \Leftrightarrow P^f \vee P^t \wedge \$ok')^f \wedge (P \Leftrightarrow P^f \vee P^t \wedge \$ok')^t'$

by (*simp add: subst-bool-split*)

also have $\dots = '(P^f \Leftrightarrow P^f) \wedge (P^t \Leftrightarrow P^f \vee P^t)'$

by *subst-tac*

also have $\dots = 'P^t \Leftrightarrow (P^f \vee P^t)'$

by (*pred-auto robust*)

also have $\dots = '(P^f \Rightarrow P^t)'$

by (*pred-auto*)

finally show *?thesis*

by (*metis H2-def Healthy-def' taut-iff-eq*)

qed

lemma *H2-equiv*:

$P \text{ is } H2 \iff P^t \sqsubseteq P^f$

using *H2-equivalence refBy-order* **by** *blast*

lemma *H2-design*:

assumes $\$ok' \nVdash P \ \$ok' \nVdash Q$

shows $H2(P \vdash Q) = P \vdash Q$

using *assms*

by (*simp add: H2-split design-def usubst unrest, pred-auto*)

lemma *H2-rdesign*:

$H2(P \vdash_r Q) = P \vdash_r Q$

by (*simp add: H2-design unrest rdesign-def*)

theorem *J-idem*:

$(J ;; J) = J$

by (*rel-auto*)

theorem *H2-idem*:

$H2(H2(P)) = H2(P)$

by (*metis H2-def J-idem segr-assoc*)

theorem *H2-Continuous*: *Continuous H2*

by (*rel-auto*)

theorem *H2-not-okay*: $H2 (\neg \$ok) = (\neg \$ok)$

proof –

have $H2 (\neg \$ok) = ((\neg \$ok)^f \vee ((\neg \$ok)^t \wedge \$ok'))$

by (*simp add: H2-split*)

also have $\dots = (\neg \$ok \vee (\neg \$ok) \wedge \$ok')$

by (*subst-tac*)

also have $\dots = (\neg \$ok)$

by (*pred-auto*)

finally show *?thesis* .

qed

lemma *H2-true*: $H2(true) = true$

by (*rel-auto*)

lemma *H2-choice-closed* [*closure*]:

$\llbracket P \text{ is } H2; Q \text{ is } H2 \rrbracket \implies P \sqcap Q \text{ is } H2$

by (*metis H2-def Healthy-def' disj-upred-def seqr-or-distl*)

lemma *H2-inf-closed* [*closure*]:

assumes $P \text{ is } H2$ $Q \text{ is } H2$

shows $P \sqcup Q \text{ is } H2$

proof –

have $P \sqcup Q = (P^f \vee P^t \wedge \$ok') \sqcup (Q^f \vee Q^t \wedge \$ok')$

by (*metis H2-def Healthy-def J-split assms(1) assms(2)*)

moreover have $H2(\dots) = \dots$

by (*simp add: H2-split usubst, pred-auto*)

ultimately show *?thesis*

by (*simp add: Healthy-def*)

qed

lemma *H2-USUP*:

shows $H2(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot H2(P(i)))$

by (*rel-auto*)

theorem *H1-H2-commute*:

$H1(H2 P) = H2(H1 P)$

proof –

have $H2(H1 P) = (\$ok \Rightarrow P) ;; J$

by (*simp add: H1-def H2-def*)

also have $\dots = ((\neg \$ok \vee P) ;; J)$

by (*rel-auto*)

also have $\dots = (((\neg \$ok) ;; J) \vee (P ;; J))$

using *seqr-or-distl* by *blast*

also have $\dots = ((H2(\neg \$ok)) \vee H2(P))$

by (*simp add: H2-def*)

also have $\dots = ((\neg \$ok) \vee H2(P))$

by (*simp add: H2-not-okay*)

also have $\dots = H1(H2(P))$

by (*rel-auto*)

finally show *?thesis* by *simp*

qed

2.3 Designs as *H1-H2* predicates

abbreviation $H1-H2 :: ('\alpha, '\beta) \text{ rel-des} \Rightarrow ('\alpha, '\beta) \text{ rel-des } (\mathbf{H})$ where

$H1-H2 P \equiv H1(H2 P)$

lemma *H1-H2-comp*: $\mathbf{H} = H1 \circ H2$

by (*auto*)

theorem *H1-H2-eq-design*:

$\mathbf{H}(P) = (\neg P^f) \vdash P^t$

proof –

have $\mathbf{H}(P) = (\$ok \Rightarrow H2(P))$

by (*simp add: H1-def*)

also have $\dots = (\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$

by (*metis H2-split*)

also have $\dots = (\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t)$

by (*rel-auto*)
 also have ... = $(\neg P^f) \vdash P^t$
 by (*rel-auto*)
 finally show *?thesis* .
 qed

theorem *H1-H2-is-design*:
 assumes *P is H1 P is H2*
 shows $P = (\neg P^f) \vdash P^t$
 using *assms* by (*metis H1-H2-eq-design Healthy-def*)

theorem *H1-H2-eq-rdesign*:
 $\mathbf{H}(P) = pre_D(P) \vdash_r post_D(P)$
proof –
 have $\mathbf{H}(P) = (\$ok \Rightarrow H2(P))$
 by (*simp add: H1-def Healthy-def'*)
 also have ... = $(\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$
 by (*metis H2-split*)
 also have ... = $(\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge P^t)$
 by (*pred-auto*)
 also have ... = $(\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t)$
 by (*pred-auto*)
 also have ... = $(\$ok \wedge [pre_D(P)]_D \Rightarrow \$ok' \wedge \$ok \wedge [post_D(P)]_D)$
 by (*simp add: ok-post ok-pre*)
 also have ... = $(\$ok \wedge [pre_D(P)]_D \Rightarrow \$ok' \wedge [post_D(P)]_D)$
 by (*pred-auto*)
 also have ... = $pre_D(P) \vdash_r post_D(P)$
 by (*simp add: rdesign-def design-def*)
 finally show *?thesis* .
 qed

theorem *H1-H2-is-rdesign*:
 assumes *P is H1 P is H2*
 shows $P = pre_D(P) \vdash_r post_D(P)$
 by (*metis H1-H2-eq-rdesign Healthy-def assms(1) assms(2)*)

lemma *H1-H2-refinement*:
 assumes *P is H Q is H*
 shows $P \sqsubseteq Q \longleftrightarrow ('pre_D(P) \Rightarrow pre_D(Q)' \wedge 'pre_D(P) \wedge post_D(Q) \Rightarrow post_D(P)')$
 by (*metis H1-H2-eq-rdesign Healthy-if assms rdesign-refinement*)

lemma *H1-H2-refines*:
 assumes *P is H Q is H P \sqsubseteq Q*
 shows $pre_D(Q) \sqsubseteq pre_D(P) \ post_D(P) \sqsubseteq (pre_D(P) \wedge post_D(Q))$
 using *H1-H2-refinement assms refBy-order* by *auto*

lemma *H1-H2-idempotent*: $\mathbf{H}(\mathbf{H} P) = \mathbf{H} P$
 by (*simp add: H1-H2-commute H1-idem H2-idem*)

lemma *H1-H2-Idempotent [closure]: Idempotent H*
 by (*simp add: Idempotent-def H1-H2-idempotent*)

lemma *H1-H2-monotonic [closure]: Monotonic H*
 by (*simp add: H1-monotone H2-def mono-def segr-mono*)

lemma *H1-H2-Continuous* [closure]: *Continuous* **H**
 by (simp add: Continuous-comp H1-Continuous H1-H2-comp H2-Continuous)

lemma *design-is-H1-H2* [closure]:
 $\llbracket \$ok' \# P; \$ok' \# Q \rrbracket \implies (P \vdash Q) \text{ is } \mathbf{H}$
 by (simp add: H1-design H2-design Healthy-def')

lemma *rdesign-is-H1-H2* [closure]:
 $(P \vdash_r Q) \text{ is } \mathbf{H}$
 by (simp add: Healthy-def H1-rdesign H2-rdesign)

lemma *top-d-is-H1-H2* [closure]: $\top_D \text{ is } \mathbf{H}$
 by (simp add: H1-def H2-not-okay Healthy-intro impl-alt-def)

lemma *bot-d-is-H1-H2* [closure]: $\perp_D \text{ is } \mathbf{H}$
 by (simp add: bot-d-def closure unrest)

lemma *seq-r-H1-H2-closed* [closure]:
 assumes $P \text{ is } \mathbf{H} \ Q \text{ is } \mathbf{H}$
 shows $(P ;; Q) \text{ is } \mathbf{H}$
proof –
 obtain $P_1 \ P_2$ where $P = P_1 \vdash_r P_2$
 by (metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms(1))
 moreover obtain $Q_1 \ Q_2$ where $Q = Q_1 \vdash_r Q_2$
 by (metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms(2))
 moreover have $((P_1 \vdash_r P_2) ;; (Q_1 \vdash_r Q_2)) \text{ is } \mathbf{H}$
 by (simp add: rdesign-composition rdesign-is-H1-H2)
 ultimately show ?thesis by simp
qed

lemma *UINF-H1-H2-closed* [closure]:
 assumes $A \neq \{\}$ $\forall P \in A. P \text{ is } \mathbf{H}$
 shows $(\sqcap A) \text{ is } H1-H2$
proof –
 from assms have $A: A = H1-H2 \text{ ' } A$
 by (auto simp add: Healthy-def rev-image-eqI)
 also have $(\sqcap ...) = (\sqcap P \in A \cdot H1-H2(P))$
 by (simp add: UINF-as-Sup-collect)
 also have $\dots = (\sqcap P \in A \cdot (\neg P^f) \vdash P^t)$
 by (meson H1-H2-eq-design)
 also have $\dots = (\sqcup P \in A \cdot \neg P^f) \vdash (\sqcap P \in A \cdot P^t)$
 by (simp add: design-UINF-mem assms)
 also have $\dots \text{ is } H1-H2$
 by (simp add: design-is-H1-H2 unrest)
 finally show ?thesis .
qed

definition *design-inf* :: $(' \alpha, ' \beta) \text{ rel-des set} \Rightarrow (' \alpha, ' \beta) \text{ rel-des } (\sqcap_D - [900] \ 900)$ **where**
 $\sqcap_D A = (\text{if } (A = \{\}) \text{ then } \top_D \text{ else } \sqcap A)$

abbreviation *design-sup* :: $(' \alpha, ' \beta) \text{ rel-des set} \Rightarrow (' \alpha, ' \beta) \text{ rel-des } (\sqcup_D - [900] \ 900)$ **where**
 $\sqcup_D A \equiv \sqcup A$

lemma *design-inf-H1-H2-closed*:
 assumes $\forall P \in A. P \text{ is } \mathbf{H}$

shows $(\sqcap_D A)$ *is* **H**
apply (*auto simp add: design-inf-def closure*)
apply (*simp add: H1-def H2-not-okay Healthy-def impl-alt-def*)
apply (*metis H1-def Healthy-def UINF-H1-H2-closed assms empty-iff impl-alt-def*)
done

lemma *design-sup-empty* [*simp*]: $\sqcap_D \{\} = \top_D$
by (*simp add: design-inf-def*)

lemma *design-sup-non-empty* [*simp*]: $A \neq \{\} \implies \sqcap_D A = \sqcap A$
by (*simp add: design-inf-def*)

lemma *USUP-mem-H1-H2-closed*:

assumes $\bigwedge i. i \in A \implies P\ i$ *is* **H**

shows $(\sqcup_{i \in A} P\ i)$ *is* **H**

proof –

from *assms* **have** $(\sqcup_{i \in A} P\ i) = (\sqcup_{i \in A} \mathbf{H}(P\ i))$

by (*auto intro: USUP-cong simp add: Healthy-def*)

also have $\dots = (\sqcup_{i \in A} \neg (P\ i)^f) \vdash (P\ i)^t$

by (*meson H1-H2-eq-design*)

also have $\dots = (\sqcap_{i \in A} \neg (P\ i)^f) \vdash (\sqcup_{i \in A} \neg (P\ i)^f \Rightarrow (P\ i)^t)$

by (*simp add: design-USUP-mem*)

also have \dots *is* **H**

by (*simp add: design-is-H1-H2 unrest*)

finally show *?thesis* .

qed

lemma *USUP-ind-H1-H2-closed*:

assumes $\bigwedge i. P\ i$ *is* **H**

shows $(\sqcup i \cdot P\ i)$ *is* **H**

using *assms USUP-mem-H1-H2-closed* [*of UNIV P*] **by** *simp*

lemma *Inf-H1-H2-closed*:

assumes $\forall P \in A. P$ *is* **H**

shows $(\sqcap A)$ *is* **H**

proof –

from *assms* **have** $A: A = \mathbf{H} \text{ ` } A$

by (*auto simp add: Healthy-def rev-image-eqI*)

also have $(\sqcup \dots) = (\sqcup P \in A \cdot \mathbf{H}(P))$

by (*simp add: USUP-as-Inf-collect*)

also have $\dots = (\sqcup P \in A \cdot \neg P^f) \vdash P^t$

by (*meson H1-H2-eq-design*)

also have $\dots = (\sqcap P \in A \cdot \neg P^f) \vdash (\sqcup P \in A \cdot \neg P^f \Rightarrow P^t)$

by (*simp add: design-USUP-mem*)

also have \dots *is* **H**

by (*simp add: design-is-H1-H2 unrest*)

finally show *?thesis* .

qed

lemma *rdesign-ref-monos*:

assumes P *is* **H** Q *is* **H** $P \sqsubseteq Q$

shows $\text{pre}_D(Q) \sqsubseteq \text{pre}_D(P)$ $\text{post}_D(P) \sqsubseteq (\text{pre}_D(P) \wedge \text{post}_D(Q))$

proof –

have $r: P \sqsubseteq Q \iff (\text{pre}_D(P) \Rightarrow \text{pre}_D(Q)) \wedge (\text{pre}_D(P) \wedge \text{post}_D(Q) \Rightarrow \text{post}_D(P))$

by (*metis H1-H2-eq-rdesign Healthy-if assms(1) assms(2) rdesign-refinement*)

from r *assms* **show** $\text{pre}_D(Q) \sqsubseteq \text{pre}_D(P)$
 by (*auto simp add: refBy-order*)
from r *assms* **show** $\text{post}_D(P) \sqsubseteq (\text{pre}_D(P) \wedge \text{post}_D(Q))$
 by (*auto simp add: refBy-order*)
qed

2.4 H3: The design assumption is a precondition

definition $H3 :: ('α, 'β) \text{rel-des} \Rightarrow ('α, 'β) \text{rel-des}$ **where**
 $[\text{upred-defs}]: H3(P) \equiv P ;; II_D$

theorem *H3-idem*:

$H3(H3(P)) = H3(P)$
 by (*metis H3-def design-skip-idem seqr-assoc*)

theorem *H3-mono*:

$P \sqsubseteq Q \Longrightarrow H3(P) \sqsubseteq H3(Q)$
 by (*simp add: H3-def seqr-mono*)

theorem *H3-Monotonic*:

Monotonic H3
 by (*simp add: H3-mono mono-def*)

theorem *H3-Continuous*: *Continuous H3*

by (*rel-auto*)

theorem *design-condition-is-H3*:

assumes $\text{out}\alpha \nVdash p$
shows $(p \vdash Q)$ *is* $H3$

proof –

have $((p \vdash Q) ;; II_D) = (\neg((\neg p) ;; \text{true})) \vdash (Q^t ;; II[\text{true}/\$ok])$
 by (*simp add: skip-d-alt-def design-composition-subst unrest assms*)
also have $\dots = p \vdash (Q^t ;; II[\text{true}/\$ok])$
 using *assms precondition-equiv seqr-true-lemma* **by force**
also have $\dots = p \vdash Q$
 by (*rel-auto*)
finally show *?thesis*
 by (*simp add: H3-def Healthy-def'*)

qed

theorem *rdesign-H3-iff-pre*:

$P \vdash_r Q \text{ is } H3 \iff P = (P ;; \text{true})$

proof –

have $(P \vdash_r Q) ;; II_D = (P \vdash_r Q) ;; (\text{true} \vdash_r II)$
 by (*simp add: skip-d-def*)
also have $\dots = (\neg((\neg P) ;; \text{true}) \wedge \neg(Q ;; (\neg \text{true}))) \vdash_r (Q ;; II)$
 by (*simp add: rdesign-composition*)
also have $\dots = (\neg((\neg P) ;; \text{true}) \wedge \neg(Q ;; (\neg \text{true}))) \vdash_r Q$
 by *simp*
also have $\dots = (\neg((\neg P) ;; \text{true})) \vdash_r Q$
 by (*pred-auto*)
finally have $P \vdash_r Q \text{ is } H3 \iff P \vdash_r Q = (\neg((\neg P) ;; \text{true})) \vdash_r Q$
 by (*metis H3-def Healthy-def'*)
also have $\dots \iff P = (\neg((\neg P) ;; \text{true}))$
 by (*metis rdesign-pre*)
thm *seqr-true-lemma*

also have ... $\longleftrightarrow P = (P ;; \text{true})$
 by (simp add: segr-true-lemma)
 finally show ?thesis .
 qed

theorem *design-H3-iff-pre*:

assumes $\$ok \# P \$ok' \# P \$ok \# Q \$ok' \# Q$
 shows $P \vdash Q \text{ is } H3 \longleftrightarrow P = (P ;; \text{true})$

proof –

have $P \vdash Q = \lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D$
 by (simp add: asms lift-desr-inv rdesign-def)
 moreover hence $\lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D \text{ is } H3 \longleftrightarrow \lfloor P \rfloor_D = (\lfloor P \rfloor_D ;; \text{true})$
 using *rdesign-H3-iff-pre* by blast
 ultimately show ?thesis
 by (metis asms(1,2) drop-desr-inv lift-desr-inv lift-dist-seq aext-true)

qed

theorem *H1-H3-commute*:

$H1 (H3 P) = H3 (H1 P)$
 by (rel-auto)

lemma *skip-d-absorb-J-1*:

$(II_D ;; J) = II_D$
 by (metis H2-def H2-rdesign skip-d-def)

lemma *skip-d-absorb-J-2*:

$(J ;; II_D) = II_D$

proof –

have $(J ;; II_D) = ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) ;; (true \vdash II)$
 by (simp add: J-def skip-d-alt-def)
 also have ... = $(\exists ok_0 \cdot ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) \llbracket \ll ok_0 \gg / \$ok' \rrbracket ;; (true \vdash II) \llbracket \ll ok_0 \gg / \$ok \rrbracket)$
 by (subst segr-middle[of ok], simp-all)
 also have ... = $((((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) \llbracket false / \$ok' \rrbracket ;; (true \vdash II) \llbracket false / \$ok \rrbracket) \vee (((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) \llbracket true / \$ok' \rrbracket ;; (true \vdash II) \llbracket true / \$ok \rrbracket))$
 by (simp add: disj-comm false-alt-def true-alt-def)
 also have ... = $((\neg \$ok \wedge \lceil II \rceil_D ;; true) \vee (\lceil II \rceil_D ;; \$ok' \wedge \lceil II \rceil_D))$
 by (rel-auto)
 also have ... = II_D
 by (rel-auto)
 finally show ?thesis .

qed

lemma *H2-H3-absorb*:

$H2 (H3 P) = H3 P$
 by (metis H2-def H3-def segr-assoc skip-d-absorb-J-1)

lemma *H3-H2-absorb*:

$H3 (H2 P) = H3 P$
 by (metis H2-def H3-def segr-assoc skip-d-absorb-J-2)

theorem *H2-H3-commute*:

$H2 (H3 P) = H3 (H2 P)$
 by (simp add: H2-H3-absorb H3-H2-absorb)

theorem *H3-design-pre*:

assumes $\$ok \# p \text{ out}\alpha \# p \ \$ok \# Q \ \$ok' \# Q$
shows $H3(p \vdash Q) = p \vdash Q$
using *assms*
by (*metis Healthy-def' design-H3-iff-pre precondition-right-unit unrest-out α -var ok-vwb-lens vwb-lens-mwb*)

theorem *H3-rdesign-pre*:

assumes $\text{out}\alpha \# p$
shows $H3(p \vdash_r Q) = p \vdash_r Q$
using *assms*
by (*simp add: H3-def*)

theorem *H3-ndesign*: $H3(p \vdash_n Q) = (p \vdash_n Q)$

by (*simp add: H3-def ndesign-def unrest-pre-out α*)

theorem *ndesign-is-H3 [closure]*: $p \vdash_n Q$ is *H3*

by (*simp add: H3-ndesign Healthy-def*)

2.5 Normal Designs as *H1-H2* predicates

abbreviation *H1-H3* :: $(\alpha, \beta) \text{ rel-des} \Rightarrow (\alpha, \beta) \text{ rel-des } (\mathbf{N})$ **where**
H1-H3 $p \equiv H1 (H3 p)$

lemma *H1-H3-comp*: $H1-H3 = H1 \circ H3$

by (*auto*)

theorem *H1-H3-is-design*:

assumes $P \text{ is } H1 \ P \text{ is } H3$
shows $P = (\neg P^f) \vdash P^t$
by (*metis H1-H2-eq-design H2-H3-absorb Healthy-def' assms(1) assms(2)*)

theorem *H1-H3-is-rdesign*:

assumes $P \text{ is } H1 \ P \text{ is } H3$
shows $P = \text{pre}_D(P) \vdash_r \text{post}_D(P)$
by (*metis H1-H2-is-rdesign H2-H3-absorb Healthy-def' assms*)

theorem *H1-H3-is-normal-design*:

assumes $P \text{ is } H1 \ P \text{ is } H3$
shows $P = \lfloor \text{pre}_D(P) \rfloor_{<} \vdash_n \text{post}_D(P)$
by (*metis H1-H3-is-rdesign assms drop-pre-inv ndesign-def precondition-equiv rdesign-H3-iff-pre*)

lemma *H1-H3-idempotent*: $\mathbf{N} (\mathbf{N} P) = \mathbf{N} P$

by (*simp add: H1-H3-commute H1-idem H3-idem*)

lemma *H1-H3-Idempotent [closure]*: *Idempotent* \mathbf{N}

by (*simp add: Idempotent-def H1-H3-idempotent*)

lemma *H1-H3-monotonic [closure]*: *Monotonic* \mathbf{N}

by (*simp add: H1-monotone H3-mono mono-def*)

lemma *H1-H3-Continuous [closure]*: *Continuous* \mathbf{N}

by (*simp add: Continuous-comp H1-Continuous H1-H3-comp H3-Continuous*)

lemma *H1-H3-intro*:

assumes $P \text{ is } \mathbf{H} \ \text{out}\alpha \# \text{pre}_D(P)$
shows $P \text{ is } \mathbf{N}$
by (*metis H1-H2-eq-rdesign H1-rdesign H3-rdesign-pre Healthy-def' assms*)

lemma *H1-H3-impl-H2* [closure]: P is **N** $\implies P$ is **H**

by (metis *H1-H2-commute H1-idem H2-H3-absorb Healthy-def'*)

lemma *H1-H3-eq-design-d-comp*: $\mathbf{N}(P) = ((\neg P^f) \vdash P^t) ;; \Pi_D$

by (metis *H1-H2-eq-design H1-H3-commute H3-H2-absorb H3-def*)

lemma *H1-H3-eq-design*: $\mathbf{N}(P) = (\neg (P^f ;; true)) \vdash P^t$

apply (simp add: *H1-H3-eq-design-d-comp skip-d-alt-def*)

apply (subst *design-composition-subst*)

apply (simp-all add: *usubst unrest*)

apply (rel-auto)

done

lemma *H3-unrest-out-alpha-nok* [unrest]:

assumes P is **N**

shows $\text{out}\alpha \nVdash P^f$

proof –

have $P = (\neg (P^f ;; true)) \vdash P^t$

by (metis *H1-H3-eq-design Healthy-def assms*)

also have $\text{out}\alpha \nVdash (\dots)^f$

by (simp add: *design-def usubst unrest, rel-auto*)

finally show ?thesis .

qed

lemma *H3-unrest-out-alpha* [unrest]: P is **N** $\implies \text{out}\alpha \nVdash \text{pre}_D(P)$

by (metis *H1-H3-commute H1-H3-is-rdesign H1-idem Healthy-def' precondition-equiv rdesign-H3-iff-pre*)

lemma *ndesign-H1-H3* [closure]: $p \vdash_n Q$ is **N**

by (simp add: *H1-rdesign H3-def Healthy-def' ndesign-def unrest-pre-out\alpha*)

lemma *ndesign-form*: P is **N** $\implies (\lfloor \text{pre}_D(P) \rfloor_{<} \vdash_n \text{post}_D(P)) = P$

by (metis *H1-H2-eq-rdesign H1-H3-impl-H2 H3-unrest-out-alpha Healthy-def drop-pre-inv ndesign-def*)

lemma *des-bot-H1-H3* [closure]: \perp_D is **N**

by (metis *H1-design H3-def Healthy-def' design-false-pre design-true-left-zero skip-d-alt-def bot-d-def*)

lemma *des-top-is-H1-H3* [closure]: \top_D is **N**

by (metis *ndesign-H1-H3 ndesign-miracle*)

lemma *skip-d-is-H1-H3* [closure]: Π_D is **N**

by (simp add: *ndesign-H1-H3 skip-d-ndes-def*)

lemma *seq-r-H1-H3-closed* [closure]:

assumes P is **N** Q is **N**

shows $(P ;; Q)$ is **N**

by (metis (no-types) *H1-H2-eq-design H1-H3-eq-design-d-comp H1-H3-impl-H2 Healthy-def assms(1)*)

assms(2) seq-r-H1-H2-closed seqr-assoc)

lemma *dcond-H1-H2-closed* [closure]:

assumes P is **N** Q is **N**

shows $(P \triangleleft b \triangleright_D Q)$ is **N**

by (metis *assms ndesign-H1-H3 ndesign-dcond ndesign-form*)

lemma *inf-H1-H2-closed* [closure]:

assumes P is \mathbf{N} Q is \mathbf{N}
shows $(P \sqcap Q)$ is \mathbf{N}
by (*metis assms ndesign-H1-H3 ndesign-choice ndesign-form*)

lemma *sup-H1-H2-closed* [*closure*]:
assumes P is \mathbf{N} Q is \mathbf{N}
shows $(P \sqcup Q)$ is \mathbf{N}
by (*metis assms ndesign-H1-H3 ndesign-inf ndesign-form*)

lemma *ndes-seqr-miracle*:
assumes P is \mathbf{N}
shows $P ;; \top_D = \lfloor pre_D P \rfloor_{<} \vdash_n false$
proof –
have $P ;; \top_D = (\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) ;; (true \vdash_n false)$
by (*simp add: assms ndesign-form ndesign-miracle*)
also have $\dots = \lfloor pre_D P \rfloor_{<} \vdash_n false$
by (*simp add: ndesign-composition-wp wp alpha*)
finally show ?thesis .
qed

lemma *ndes-seqr-abort*:
assumes P is \mathbf{N}
shows $P ;; \perp_D = (\lfloor pre_D P \rfloor_{<} \wedge post_D P \text{ wp } false) \vdash_n false$
proof –
have $P ;; \perp_D = (\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) ;; (false \vdash_n false)$
by (*simp add: assms bot-d-true ndesign-false-pre ndesign-form*)
also have $\dots = (\lfloor pre_D P \rfloor_{<} \wedge post_D P \text{ wp } false) \vdash_n false$
by (*simp add: ndesign-composition-wp alpha*)
finally show ?thesis .
qed

lemma *USUP-ind-H1-H3-closed* [*closure*]:
 $\llbracket \bigwedge i. P \ i \text{ is } \mathbf{N} \rrbracket \implies (\bigsqcup i \cdot P \ i) \text{ is } \mathbf{N}$
by (*rule H1-H3-intro, simp-all add: H1-H3-impl-H2 USUP-ind-H1-H2-closed preD-USUP-ind unrest*)

2.6 H4: Feasibility

definition $H_4 :: ('\alpha, '\beta) \text{ rel-des} \Rightarrow (''\alpha, ''\beta) \text{ rel-des}$ **where**
 $[upred-defs]: H_4(P) = ((P;;true) \Rightarrow P)$

theorem *H4-idem*:
 $H_4(H_4(P)) = H_4(P)$
by (*rel-auto*)

lemma *is-H4-alt-def*:
 $P \text{ is } H_4 \iff (P ;; true) = true$
by (*rel-blast*)

end

2.7 UTP theory of Designs

theory *utp-des-theory*
imports *utp-des-healths*
begin

2.8 UTP theories

typeddecl *DES*
typeddecl *NDES*

abbreviation *DES* \equiv *UTHY*(*DES*, ' α *des*)
abbreviation *NDES* \equiv *UTHY*(*NDES*, ' α *des*)

overloading

des-hcond == *utp-hcond* :: (*DES*, ' α *des*) *uthy* \Rightarrow (' α *des* \times ' α *des*) *health*
des-unit == *utp-unit* :: (*DES*, ' α *des*) *uthy* \Rightarrow ' α *hrel-des* (**unchecked**)

ndes-hcond == *utp-hcond* :: (*NDES*, ' α *des*) *uthy* \Rightarrow (' α *des* \times ' α *des*) *health*
ndes-unit == *utp-unit* :: (*NDES*, ' α *des*) *uthy* \Rightarrow ' α *hrel-des* (**unchecked**)

begin

definition *des-hcond* :: (*DES*, ' α *des*) *uthy* \Rightarrow (' α *des* \times ' α *des*) *health* **where**
[upred-defs]: *des-hcond* *t* = *H1-H2*

definition *des-unit* :: (*DES*, ' α *des*) *uthy* \Rightarrow ' α *hrel-des* **where**
[upred-defs]: *des-unit* *t* = *II_D*

definition *ndes-hcond* :: (*NDES*, ' α *des*) *uthy* \Rightarrow (' α *des* \times ' α *des*) *health* **where**
[upred-defs]: *ndes-hcond* *t* = *H1-H3*

definition *ndes-unit* :: (*NDES*, ' α *des*) *uthy* \Rightarrow ' α *hrel-des* **where**
[upred-defs]: *ndes-unit* *t* = *II_D*

end

interpretation *des-utp-theory*: *utp-theory* *DES*
by (*simp* *add*: *H1-H2-commute* *H1-idem* *H2-idem* *des-hcond-def* *utp-theory-def*)

interpretation *ndes-utp-theory*: *utp-theory* *NDES*
by (*simp* *add*: *H1-H3-commute* *H1-idem* *H3-idem* *ndes-hcond-def* *utp-theory.intro*)

interpretation *des-left-unital*: *utp-theory-left-unital* *DES*

apply (*unfold-locales*)
apply (*simp-all* *add*: *des-hcond-def* *des-unit-def*)
using *seq-r-H1-H2-closed* **apply** *blast*
apply (*simp* *add*: *rdesign-is-H1-H2* *skip-d-def*)
apply (*metis* *H1-idem* *H1-left-unit* *Healthy-def'*)

done

interpretation *ndes-unital*: *utp-theory-unital* *NDES*

apply (*unfold-locales*, *simp-all* *add*: *ndes-hcond-def* *ndes-unit-def*)
using *seq-r-H1-H3-closed* **apply** *blast*
apply (*metis* *H1-rdesign* *H3-def* *Healthy-def'* *design-skip-idem* *skip-d-def*)
apply (*metis* *H1-idem* *H1-left-unit* *Healthy-def'*)
apply (*metis* *H1-H3-commute* *H3-def* *H3-idem* *Healthy-def'*)

done

interpretation *design-theory-continuous*: *utp-theory-continuous* *DES*

rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } DES) \longleftrightarrow P \text{ is } \mathbf{H}$
and *carrier* (*uthy-order* *DES*) \rightarrow *carrier* (*uthy-order* *DES*) $\equiv \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$
and $\llbracket \mathcal{H}_{DES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{DES} \rrbracket_H \equiv \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$

and $le \text{ (uthy-order DES) } = op \sqsubseteq$
and $eq \text{ (uthy-order DES) } = op =$
by (unfold-locales, simp-all add: des-hcond-def H1-H2-Continuous utp-order-def)

interpretation *normal-design-theory-continuous: utp-theory-continuous NDES*
rewrites $\bigwedge P. P \in \text{carrier (uthy-order NDES)} \longleftrightarrow P \text{ is } \mathbf{N}$
and $\text{carrier (uthy-order NDES)} \rightarrow \text{carrier (uthy-order NDES)} \equiv \llbracket \mathbf{N} \rrbracket_H \rightarrow \llbracket \mathbf{N} \rrbracket_H$
and $\llbracket \mathcal{H}_{NDES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{NDES} \rrbracket_H \equiv \llbracket \mathbf{N} \rrbracket_H \rightarrow \llbracket \mathbf{N} \rrbracket_H$
and $le \text{ (uthy-order NDES) } = op \sqsubseteq$
and $A \subseteq \text{carrier (uthy-order NDES)} \longleftrightarrow A \subseteq \llbracket \mathbf{N} \rrbracket_H$
and $eq \text{ (uthy-order NDES) } = op =$
by (unfold-locales, simp-all add: ndes-hcond-def H1-H3-Continuous utp-order-def)

lemma *design-lat-top: $\top_{DES} = \mathbf{H}(\text{false})$*
by (simp add: design-theory-continuous.healthy-top, simp add: des-hcond-def)

lemma *design-lat-bottom: $\perp_{DES} = \mathbf{H}(\text{true})$*
by (simp add: design-theory-continuous.healthy-bottom, simp add: des-hcond-def)

lemma *ndesign-lat-top: $\top_{NDES} = \mathbf{N}(\text{false})$*
by (metis ndes-hcond-def normal-design-theory-continuous.healthy-top)

lemma *ndesign-lat-bottom: $\perp_{NDES} = \mathbf{N}(\text{true})$*
by (metis ndes-hcond-def normal-design-theory-continuous.healthy-bottom)

2.9 Galois Connection

Example Galois connection between designs and relations. Based on Jim's example in COMPASS deliverable D23.5.

definition [*upred-defs*]: $Des(R) = \mathbf{H}(\lceil R \rceil_D \wedge \$ok')$

definition [*upred-defs*]: $Rel(D) = \lfloor D \llbracket true, true / \$ok, \$ok' \rrbracket \rfloor_D$

lemma *Des-design: $Des(R) = true \vdash_r R$*
by (rel-auto)

lemma *Rel-design: $Rel(P \vdash_r Q) = (P \Rightarrow Q)$*
by (rel-auto)

interpretation *Des-Rel-coretract:*
 $\text{coretract } DES \leftarrow \langle Des, Rel \rangle \rightarrow REL$

rewrites

$\bigwedge x. x \in \text{carrier } \mathcal{X}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = (x \text{ is } \mathbf{H})$ **and**

$\bigwedge x. x \in \text{carrier } \mathcal{Y}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = \text{True}$ **and**

$\pi_{*DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = Des$ **and**

$\pi^{*DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = Rel$ **and**

$le \mathcal{X}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = op \sqsubseteq$ **and**

$le \mathcal{Y}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = op \sqsubseteq$

proof (unfold-locales, simp-all add: rel-hcond-def des-hcond-def)

show $\bigwedge x. x \text{ is } id$

by (simp add: Healthy-def)

next

show $Rel \in \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket id \rrbracket_H$

by (auto simp add: Rel-def rel-hcond-def Healthy-def)

next

```

show  $Des \in \llbracket id \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$ 
  by (auto simp add: Des-def des-hcond-def Healthy-def H1-H2-commute H1-idem H2-idem)
next
  fix  $R :: 'a \text{ hrel}$ 
  show  $R \sqsubseteq Rel (Des R)$ 
  by (simp add: Des-design Rel-design)
next
  fix  $R :: 'a \text{ hrel}$  and  $D :: 'a \text{ hrel-des}$ 
  assume  $a: D \text{ is } \mathbf{H}$ 
  then obtain  $D_1 D_2$  where  $D: D = D_1 \vdash_r D_2$ 
  by (metis H1-H2-commute H1-H2-is-rdesign H1-idem Healthy-def')
  show  $(Rel D \sqsubseteq R) = (D \sqsubseteq Des R)$ 
  proof -
    have  $(D \sqsubseteq Des R) = (D_1 \vdash_r D_2 \sqsubseteq true \vdash_r R)$ 
    by (simp add: D Des-design)
    also have  $\dots = 'D_1 \wedge R \Rightarrow D_2'$ 
    by (simp add: rdesign-refinement)
    also have  $\dots = ((D_1 \Rightarrow D_2) \sqsubseteq R)$ 
    by (rel-auto)
    also have  $\dots = (Rel D \sqsubseteq R)$ 
    by (simp add: D Rel-design)
    finally show ?thesis ..
  qed
qed

```

From this interpretation we gain many Galois theorems. Some require simplification to remove superfluous assumptions.

```

thm Des-Rel-coretract.deflation[simplified]
thm Des-Rel-coretract.inflation
thm Des-Rel-coretract.upper-comp[simplified]
thm Des-Rel-coretract.lower-comp

```

2.10 Fixed Points

abbreviation *design-lfp* :: $('a \text{ hrel-des} \Rightarrow 'a \text{ hrel-des}) \Rightarrow 'a \text{ hrel-des} (\mu_D)$ **where**
 $\mu_D F \equiv \mu_{DES} F$

abbreviation *design-gfp* :: $('a \text{ hrel-des} \Rightarrow 'a \text{ hrel-des}) \Rightarrow 'a \text{ hrel-des} (\nu_D)$ **where**
 $\nu_D F \equiv \nu_{DES} F$

syntax

```

-dmu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic  $(\mu_D \text{ - - } [0, 10] \text{ } 10)$ 
-dnu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic  $(\nu_D \text{ - - } [0, 10] \text{ } 10)$ 

```

translations

```

 $\mu_D X \cdot P == \mu_{CONST DES} (\lambda X. P)$ 
 $\nu_D X \cdot P == \nu_{CONST DES} (\lambda X. P)$ 

```

thm *design-theory-continuous.GFP-unfold*

thm *design-theory-continuous.LFP-unfold*

Specialise *mu-refine-intro* to designs.

lemma *design-mu-refine-intro*:

```

assumes  $\$ok' \# C \ \$ok' \# S \ (C \vdash S) \sqsubseteq F(C \vdash S) \ 'C \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)'$ 
shows  $(C \vdash S) \sqsubseteq \mu_D F$ 

```

proof –

from *assms* **have** $(C \vdash S) \sqsubseteq \nu_D F$
thm *design-theory-continuous.weak.GFP-upperbound*
by (*simp add: design-is-H1-H2 design-theory-continuous.weak.GFP-upperbound*)
with *assms* **show** *?thesis*
by (*rel-auto, metis (no-types, lifting)*)

qed

lemma *rdesign-mu-refine-intro*:

assumes $(C \vdash_r S) \sqsubseteq F(C \vdash_r S)$ ‘ $\lceil C \rceil_D \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)$ ’
shows $(C \vdash_r S) \sqsubseteq \mu_D F$
using *assms* **by** (*simp add: rdesign-def design-mu-refine-intro unrest*)

lemma *H1-H2-mu-refine-intro*:

assumes P is **H** $P \sqsubseteq F(P)$ ‘ $\lceil pre_D(P) \rceil_D \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)$ ’
shows $P \sqsubseteq \mu_D F$
by (*metis H1-H2-eq-rdesign Healthy-if assms rdesign-mu-refine-intro*)

Foundational theorem for recursion introduction using a well-founded relation. Contributed by Dr. Yakoub Nemouchi.

theorem *rdesign-mu-wf-refine-intro*:

assumes $WF: wf\ R$
and $M: Monotonic\ F$
and $H: F \in \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$
and *induct-step*:
 $\bigwedge st. (P \wedge \lceil e \rceil_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq F((P \wedge (\lceil e \rceil_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q)$
shows $(P \vdash_r Q) \sqsubseteq \mu_D F$

proof –

{
fix *st*
have $(P \wedge \lceil e \rceil_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq \mu_D F$
using *WF* **proof** (*induction rule: wf-induct-rule*)
case (*less st*)
hence $0: (P \wedge (\lceil e \rceil_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q \sqsubseteq \mu_D F$
by *rel-blast*
from *M H design-theory-continuous.LFP-lemma3 mono-Monotone-utp-order*
have $1: \mu_D F \sqsubseteq F(\mu_D F)$
by *blast*
from $0\ 1$ **have** $2: (P \wedge (\lceil e \rceil_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q \sqsubseteq F(\mu_D F)$
by *simp*
have $3: F((P \wedge (\lceil e \rceil_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q) \sqsubseteq F(\mu_D F)$
by (*simp add: 0 M monoD*)
have $4: (P \wedge \lceil e \rceil_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq \dots$
by (*rule induct-step*)
show *?case*
using *order-trans[OF 3 4] H M design-theory-continuous.LFP-lemma2 dual-order.trans mono-Monotone-utp-order*
by *blast*
qed
}
thus *?thesis*
by (*pred-simp*)
qed

theorem *ndesign-mu-wf-refine-intro'*:

```

assumes   WF: wf R
and       M: Monotonic F
and       H: F ∈  $\llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$ 
and   induct-step:
   $\bigwedge st. ((p \wedge e =_u \ll st \gg) \vdash_n Q) \sqsubseteq F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q)$ 
shows  $(p \vdash_n Q) \sqsubseteq \mu_D F$ 
using assms unfolding ndesign-def
by (rule-tac rdesign-mu-wf-refine-intro[of R F [p]_< e], simp-all add: alpha)

theorem ndesign-mu-wf-refine-intro:
assumes   WF: wf R
and       M: Monotonic F
and       H: F ∈  $\llbracket \mathbf{N} \rrbracket_H \rightarrow \llbracket \mathbf{N} \rrbracket_H$ 
and   induct-step:
   $\bigwedge st. ((p \wedge e =_u \ll st \gg) \vdash_n Q) \sqsubseteq F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q)$ 
shows  $(p \vdash_n Q) \sqsubseteq \mu_{NDES} F$ 
proof –
{
  fix st
  have  $(p \wedge e =_u \ll st \gg) \vdash_n Q \sqsubseteq \mu_{NDES} F$ 
  using WF proof (induction rule: wf-induct-rule)
    case (less st)
    hence 0:  $(p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q \sqsubseteq \mu_{NDES} F$ 
    by rel-blast
    from M H design-theory-continuous.LFP-lemma3 mono-Monotone-utp-order
    have 1:  $\mu_{NDES} F \sqsubseteq F (\mu_{NDES} F)$ 
    by (simp add: mono-Monotone-utp-order normal-design-theory-continuous.LFP-lemma3)
    from 0 1 have 2:  $(p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q \sqsubseteq F (\mu_{NDES} F)$ 
    by simp
    have 3:  $F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q) \sqsubseteq F (\mu_{NDES} F)$ 
    by (simp add: 0 M monoD)
    have 4:  $(p \wedge e =_u \ll st \gg) \vdash_n Q \sqsubseteq \dots$ 
    by (rule induct-step)
    show ?case
      using order-trans[OF 3 4] H M normal-design-theory-continuous.LFP-lemma2 dual-order.trans
      mono-Monotone-utp-order
      by blast
  qed
}
thus ?thesis
by (pred-simp)
qed

end

```

3 Design Proof Tactics

```

theory utp-des-tactics
imports utp-des-theory
begin

```

The tactics split apart a healthy normal design predicate into its pre-postcondition form, using elimination rules, and then attempt to prove refinement conjectures.

```

named-theorems ND-elim

```

lemma *ndes-elim*: $\llbracket P \text{ is } \mathbf{N}; Q(\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) \rrbracket \implies Q(P)$
by (*simp add: ndesign-form*)

lemma *ndes-ind-elim*: $\llbracket \bigwedge i. P \ i \text{ is } \mathbf{N}; Q(\lambda i. \lfloor pre_D(P \ i) \rfloor_{<} \vdash_n post_D(P \ i)) \rrbracket \implies Q(P)$
by (*simp add: ndesign-form*)

lemma *ndes-split* [*ND-elim*]: $\llbracket P \text{ is } \mathbf{N}; \bigwedge pre \ post. Q(pre \vdash_n post) \rrbracket \implies Q(P)$
by (*metis H1-H2-eq-rdesign H1-H3-impl-H2 H3-unrest-out-alpha Healthy-def drop-pre-inv ndesign-def*)

Use given closure laws (*cls*) to expand normal design predicates

method *ndes-expand* **uses** *cls* = (*insert cls, (erule ND-elim)+*)

Expand and simplify normal designs

method *ndes-simp* **uses** *cls* =
(*((ndes-expand cls: cls)?, (simp add: ndes-simp closure alpha usubst unrest wp prod.case-eq-if))*)

Attempt to discharge a refinement between two normal designs

method *ndes-refine* **uses** *cls* =
(*(ndes-simp cls: cls; rule-tac ndesign-refine-intro; (insert cls; rel-simp; auto?))*)

Attempt to discharge an equality between two normal designs

method *ndes-eq* **uses** *cls* =
(*(ndes-simp cls: cls; rule-tac antisym; rule-tac ndesign-refine-intro; (insert cls; rel-simp; auto?))*)

end

4 Imperative Programming in Designs

theory *utp-des-prog*
imports *utp-des-tactics*
begin

4.1 Assignment

definition *assigns-d* :: $'\alpha \text{ usubst} \Rightarrow '\alpha \text{ hrel-des } (\langle \cdot \rangle_D)$ **where**
[*upred-defs*]: *assigns-d* $\sigma = (true \vdash_r \text{assigns-r } \sigma)$

syntax

-assignmenttd :: *svids* \Rightarrow *uexprs* \Rightarrow *logic* (**infixr** $:=_D$ 72)

translations

-assignmenttd xs vs \Rightarrow *CONST assigns-d (-mk-usubst (CONST id) xs vs)*
-assignmenttd x v \leq *CONST assigns-d (CONST subst-upd (CONST id) x v)*
-assignmenttd x v \leq *-assignmenttd (-spvar x) v*
x, y :=_D u, v \leq *CONST assigns-d (CONST subst-upd (CONST subst-upd (CONST id) (CONST svar x) u) (CONST svar y) v)*

lemma *assigns-d-is-H1-H2* [*closure*]: $\langle \sigma \rangle_D$ **is** **H**
by (*simp add: assigns-d-def rdesign-is-H1-H2*)

lemma *assigns-d-H1-H3* [*closure*]: $\langle \sigma \rangle_D$ **is** **N**
by (*metis H1-rdesign H3-ndesign Healthy-def' aext-true assigns-d-def ndesign-def*)

Designs are closed under substitutions on state variables only (via lifting)

lemma *state-subst-H1-H2-closed* [closure]:

$P \text{ is } \mathbf{H} \implies \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P \text{ is } \mathbf{H}$

by (metis *H1-H2-eq-rdesign Healthy-if rdesign-is-H1-H2 state-subst-design*)

lemma *assigns-d-ndes-def* [ndes-simp]:

$\langle \sigma \rangle_D = (\text{true} \vdash_n \langle \sigma \rangle_a)$

by (rel-auto)

lemma *assigns-d-id* [simp]: $\langle id \rangle_D = II_D$

by (rel-auto)

lemma *assign-d-left-comp*:

$(\langle f \rangle_D ;; (P \vdash_r Q)) = (\lceil f \rceil_s \uparrow P \vdash_r \lceil f \rceil_s \uparrow Q)$

by (simp add: *assigns-d-def rdesign-composition assigns-r-comp subst-not*)

lemma *assign-d-right-comp*:

$((P \vdash_r Q) ;; \langle f \rangle_D) = ((\neg ((\neg P) ;; \text{true})) \vdash_r (Q ;; \langle f \rangle_a))$

by (simp add: *assigns-d-def rdesign-composition*)

lemma *assigns-d-comp*:

$(\langle f \rangle_D ;; \langle g \rangle_D) = \langle g \circ f \rangle_D$

by (simp add: *assigns-d-def rdesign-composition assigns-comp*)

lemma *assigns-d-comp-ext*:

fixes $P :: 'a \text{ hrel-des}$

assumes $P \text{ is } \mathbf{H}$

shows $(\langle \sigma \rangle_D ;; P) = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$

proof –

have $\langle \sigma \rangle_D ;; P = \langle \sigma \rangle_D ;; (\text{pre}_D(P) \vdash_r \text{post}_D(P))$

by (metis *H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms*)

also have $\dots = \lceil \sigma \rceil_s \uparrow \text{pre}_D(P) \vdash_r \lceil \sigma \rceil_s \uparrow \text{post}_D(P)$

by (simp add: *assign-d-left-comp*)

also have $\dots = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow (\text{pre}_D(P) \vdash_r \text{post}_D(P))$

by (rel-auto)

also have $\dots = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$

by (metis *H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms*)

finally show ?thesis .

qed

Normal designs are closed under substitutions on state variables only

lemma *state-subst-H1-H3-closed* [closure]:

$P \text{ is } \mathbf{N} \implies \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P \text{ is } \mathbf{N}$

by (metis *H1-H2-eq-rdesign H1-H3-impl-H2 Healthy-if assign-d-left-comp assigns-d-H1-H3 seq-r-H1-H3-closed state-subst-design*)

lemma *H4-assigs-d*: $\langle \sigma \rangle_D \text{ is } H4$

proof –

have $(\langle \sigma \rangle_D ;; (\text{false} \vdash_r \text{true}_h)) = (\text{false} \vdash_r \text{true})$

by (simp add: *assigns-d-def rdesign-composition assigns-r-feasible*)

moreover have $\dots = \text{true}$

by (rel-auto)

ultimately show ?thesis

using *is-H4-alt-def* by auto

qed

4.2 Guarded Commands

definition $GrdCommD :: 'a \text{ upred} \Rightarrow ('a, 'b) \text{ rel-des} \Rightarrow ('a, 'b) \text{ rel-des} \ (- \rightarrow_D - [85, 86] \ 85) \text{ where}$
 $[upred-defs]: b \rightarrow_D P = P \triangleleft b \triangleright_D \top_D$

lemma $GrdCommD-ndes-simp \ [ndes-simp]:$
 $b \rightarrow_D (p_1 \vdash_n P_2) = ((b \Rightarrow p_1) \vdash_n (\lceil b \rceil_{<} \wedge P_2))$
by $(rel-auto)$

lemma $GrdCommD-H1-H3-closed \ [closure]: P \text{ is } \mathbf{N} \Longrightarrow b \rightarrow_D P \text{ is } \mathbf{N}$
by $(simp \ add: GrdCommD-def \ closure)$

lemma $GrdCommD-true \ [simp]: true \rightarrow_D P = P$
by $(rel-auto)$

lemma $GrdCommD-false \ [simp]: false \rightarrow_D P = \top_D$
by $(rel-auto)$

lemma $GrdCommD-abort \ [simp]: b \rightarrow_D true = ((\neg b) \vdash_n false)$
by $(rel-auto)$

4.3 Alternation

consts

$ualtern \quad :: 'a \text{ set} \Rightarrow ('a \Rightarrow 'p) \Rightarrow ('a \Rightarrow 'r) \Rightarrow 'r \Rightarrow 'r$
 $ualtern-list \quad :: ('a \times 'r) \text{ list} \Rightarrow 'r \Rightarrow 'r$

definition $AlternateD :: 'a \text{ set} \Rightarrow ('a \Rightarrow 'a \text{ upred}) \Rightarrow ('a \Rightarrow ('a, 'b) \text{ rel-des}) \Rightarrow ('a, 'b) \text{ rel-des} \Rightarrow ('a, 'b) \text{ rel-des} \text{ where}$
 $[upred-defs, ndes-simp]:$
 $AlternateD \ A \ g \ P \ Q = (\bigwedge i \in A \cdot g(i) \rightarrow_D P(i)) \sqcap (\bigwedge i \in A \cdot \neg g(i) \rightarrow_D Q)$

This lemma shows that our generalised alternation is the same operator as Marcel Oliveira's definition of alternation when the else branch is abort.

lemma $AlternateD-abort-alternate:$

assumes $\bigwedge i. P(i) \text{ is } \mathbf{N}$

shows

$AlternateD \ A \ g \ P \perp_D =$
 $((\bigvee i \in A \cdot g(i)) \wedge (\bigwedge i \in A \cdot g(i) \Rightarrow \lfloor pre_D(P \ i) \rfloor_{<})) \vdash_n (\bigvee i \in A \cdot \lceil g(i) \rceil_{<} \wedge post_D(P \ i))$

proof $(cases \ A = \{\})$

case $False$

have $AlternateD \ A \ g \ P \perp_D =$

$(\bigwedge i \in A \cdot g(i) \rightarrow_D (\lfloor pre_D(P \ i) \rfloor_{<} \vdash_n post_D(P \ i))) \sqcap (\bigwedge i \in A \cdot \neg g(i) \rightarrow_D (false \vdash_n true))$

by $(simp \ add: AlternateD-def \ ndesign-form \ bot-d-ndes-def \ assms)$

also have $\dots = ((\bigvee i \in A \cdot g(i)) \wedge (\bigwedge i \in A \cdot g(i) \Rightarrow \lfloor pre_D(P \ i) \rfloor_{<})) \vdash_n (\bigvee i \in A \cdot \lceil g(i) \rceil_{<} \wedge post_D(P \ i))$

by $(simp \ add: ndes-simp \ False, \ rel-auto)$

finally show $?thesis$ **by** $simp$

next

case $True$

thus $?thesis$

by $(simp \ add: AlternateD-def, \ rel-auto)$

qed

definition $AlternateD-list :: ('a \text{ upred} \times ('a, 'b) \text{ rel-des}) \text{ list} \Rightarrow ('a, 'b) \text{ rel-des} \Rightarrow ('a, 'b) \text{ rel-des}$
where

[upred-defs, ndes-simp]:

AlternateD-list xs P =

AlternateD {0.. $\text{length } xs$ } ($\lambda i. \text{map fst } xs ! i$) ($\lambda i. \text{map snd } xs ! i$) P

adhoc-overloading

ualtern AlternateD and

ualtern-list AlternateD-list

nonterminal gcomm and gcomms

syntax

-altind-els :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \Rightarrow logic \Rightarrow logic (if - \in - \cdot - \rightarrow - else - fi)

-altind :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \Rightarrow logic (if - \in - \cdot - \rightarrow - fi)

-gcomm :: logic \Rightarrow logic \Rightarrow gcomm (- \rightarrow - [65, 66] 65)

-gcomm-nil :: gcomm \Rightarrow gcomms (-)

-gcomm-cons :: gcomm \Rightarrow gcomms \Rightarrow gcomms (- | - [60, 61] 61)

-gcomm-show :: logic \Rightarrow logic

-altgcomm-els :: gcomms \Rightarrow logic \Rightarrow logic (if - else - fi)

-altgcomm :: gcomms \Rightarrow logic (if - fi)

translations

-altind-els x A g P Q \Rightarrow CONST ualtern A ($\lambda x. g$) ($\lambda x. P$) Q

-altind-els x A g P Q \Leftarrow CONST ualtern A ($\lambda x. g$) ($\lambda x'. P$) Q

-altind x A g P \Rightarrow CONST ualtern A ($\lambda x. g$) ($\lambda x. P$) (CONST Orderings.top)

-altind x A g P \Leftarrow CONST ualtern A ($\lambda x. g$) ($\lambda x'. P$) (CONST Orderings.top)

-altgcomm cs \Rightarrow CONST ualtern-list cs (CONST Orderings.top)

-altgcomm (-gcomm-show cs) \Leftarrow CONST ualtern-list cs (CONST Orderings.top)

-altgcomm-els cs P \Rightarrow CONST ualtern-list cs P

-altgcomm-els (-gcomm-show cs) P \Leftarrow CONST ualtern-list cs P

-gcomm g P \Rightarrow (g, P)

-gcomm g P \Leftarrow -gcomm-show (g, P)

-gcomm-cons c cs \Rightarrow c # cs

-gcomm-cons (-gcomm-show c) (-gcomm-show (d # cs)) \Leftarrow -gcomm-show (c # d # cs)

-gcomm-nil c \Rightarrow [c]

-gcomm-nil (-gcomm-show c) \Leftarrow -gcomm-show [c]

lemma AlternateD-H1-H3-closed [closure]:

assumes $\bigwedge i. i \in A \Rightarrow P \ i \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$

shows if $i \in A \cdot g(i) \rightarrow P(i)$ else $Q \text{ fi is } \mathbf{N}$

proof (cases A = {})

case True

then show ?thesis

by (simp add: AlternateD-def closure false-upred-def assms)

next

case False

then show ?thesis

by (simp add: AlternateD-def closure assms)

qed

lemma AltD-ndes-simp [ndes-simp]:

if $i \in A \cdot g(i) \rightarrow (P_1(i) \vdash_n P_2(i))$ else $Q_1 \vdash_n Q_2 \text{ fi}$

= ($(\bigwedge i \in A \cdot g \ i \Rightarrow P_1 \ i) \wedge ((\bigwedge i \in A \cdot \neg g \ i) \Rightarrow Q_1)) \vdash_n$

$((\bigvee i \in A \cdot [g \ i]_{<} \wedge P_2 \ i) \vee (\bigwedge i \in A \cdot \neg [g \ i]_{<}) \wedge Q_2)$

proof (cases A = {})

```

case True
then show ?thesis by (simp add: AlternateD-def)
next
case False
then show ?thesis
  by (simp add: ndes-simp, rel-auto)
qed

declare UINF-upto-expand-first [ndes-simp]
declare UINF-Suc-shift [ndes-simp]
declare USUP-upto-expand-first [ndes-simp]
declare USUP-Suc-shift [ndes-simp]
declare true-upred-def [THEN sym, ndes-simp]

lemma AlternateD-mono-refine:
  assumes  $\bigwedge i. P\ i \sqsubseteq Q\ i \ R \sqsubseteq S$ 
  shows  $(\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } R\ \text{fi}) \sqsubseteq (\text{if } i \in A \cdot g(i) \rightarrow Q(i) \text{ else } S\ \text{fi})$ 
  using assms by (rel-auto, meson)

lemma Monotonic-AlternateD [closure]:
   $\llbracket \bigwedge i. \text{Monotonic } (F\ i); \text{Monotonic } G \rrbracket \implies \text{Monotonic } (\lambda X. \text{if } i \in A \cdot g(i) \rightarrow F\ i\ X \text{ else } G(X)\ \text{fi})$ 
  by (rel-auto, meson)

lemma AlternateD-eq:
  assumes  $A = B \ \bigwedge i. i \in A \implies g(i) = h(i) \ \bigwedge i. i \in A \implies P(i) = Q(i) \ R = S$ 
  shows  $\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } R\ \text{fi} = \text{if } i \in B \cdot h(i) \rightarrow Q(i) \text{ else } S\ \text{fi}$ 
  by (insert assms, rel-blast)

lemma AlternateD-empty:
   $\text{if } i \in \{\} \cdot g(i) \rightarrow P(i) \text{ else } Q\ \text{fi} = Q$ 
  by (rel-auto)

lemma AlternateD-true-singleton:
  assumes  $P\ \text{is } \mathbf{N}$ 
  shows  $\text{if } \text{true} \rightarrow P\ \text{fi} = P$ 
  by (ndes-eq cls: assms)

lemma AlternateD-no-ind:
  assumes  $A \neq \{\} \ P\ \text{is } \mathbf{N} \ Q\ \text{is } \mathbf{N}$ 
  shows  $\text{if } i \in A \cdot b \rightarrow P \text{ else } Q\ \text{fi} = \text{if } b \rightarrow P \text{ else } Q\ \text{fi}$ 
  by (ndes-eq cls: assms)

lemma AlternateD-singleton:
  assumes  $P\ k\ \text{is } \mathbf{N} \ Q\ \text{is } \mathbf{N}$ 
  shows  $\text{if } i \in \{k\} \cdot b(i) \rightarrow P(i) \text{ else } Q\ \text{fi} = \text{if } b(k) \rightarrow P(k) \text{ else } Q\ \text{fi} \ (\text{is } ?lhs = ?rhs)$ 
proof -
  have  $?lhs = \text{if } i \in \{k\} \cdot b(k) \rightarrow P(k) \text{ else } Q\ \text{fi}$ 
  by (auto intro: AlternateD-eq simp add: assms ndesign-form)
  also have  $\dots = ?rhs$ 
  by (simp add: AlternateD-no-ind assms closure)
  finally show ?thesis .
qed

lemma AlternateD-commute:
  assumes  $P\ \text{is } \mathbf{N} \ Q\ \text{is } \mathbf{N}$ 

```

shows if $g_1 \rightarrow P \mid g_2 \rightarrow Q$ fi = if $g_2 \rightarrow Q \mid g_1 \rightarrow P$ fi
by (*ndes-eq cls:assms*)

lemma *AlternateD-dcond*:

assumes P is \mathbf{N} Q is \mathbf{N}
shows if $g \rightarrow P$ else Q fi = $P \triangleleft g \triangleright_D Q$
by (*ndes-eq cls:assms*)

lemma *AlternateD-cover*:

assumes P is \mathbf{N} Q is \mathbf{N}
shows if $g \rightarrow P$ else Q fi = if $g \rightarrow P \mid (\neg g) \rightarrow Q$ fi
by (*ndes-eq cls:assms*)

lemma *UINF-ndes-expand*:

assumes $\bigwedge i. i \in A \implies P(i)$ is \mathbf{N}
shows $(\bigcap i \in A \cdot \lfloor pre_D(P(i)) \rfloor_{<} \vdash_n post_D(P(i))) = (\bigcap i \in A \cdot P(i))$
by (*rule UINF-cong, simp add: assms ndesign-form*)

lemma *USUP-ndes-expand*:

assumes $\bigwedge i. i \in A \implies P(i)$ is \mathbf{N}
shows $(\bigcup i \in A \cdot \lfloor pre_D(P(i)) \rfloor_{<} \vdash_n post_D(P(i))) = (\bigcup i \in A \cdot P(i))$
by (*rule USUP-cong, simp add: assms ndesign-form*)

lemma *AlternateD-ndes-expand*:

assumes $\bigwedge i. i \in A \implies P(i)$ is \mathbf{N} Q is \mathbf{N}
shows if $i \in A \cdot g(i) \rightarrow P(i)$ else Q fi =
 if $i \in A \cdot g(i) \rightarrow (\lfloor pre_D(P(i)) \rfloor_{<} \vdash_n post_D(P(i)))$ else $\lfloor pre_D(Q) \rfloor_{<} \vdash_n post_D(Q)$ fi
apply (*simp add: AlternateD-def*)
apply (*subst UINF-ndes-expand[THEN sym]*)
apply (*simp add: assms closure*)
apply (*ndes-simp cls: assms*)
apply (*rel-auto*)
done

lemma *AlternateD-ndes-expand'*:

assumes $\bigwedge i. i \in A \implies P(i)$ is \mathbf{N}
shows if $i \in A \cdot g(i) \rightarrow P(i)$ fi = if $i \in A \cdot g(i) \rightarrow (\lfloor pre_D(P(i)) \rfloor_{<} \vdash_n post_D(P(i)))$ fi
apply (*simp add: AlternateD-def*)
apply (*subst UINF-ndes-expand[THEN sym]*)
apply (*simp add: assms closure*)
apply (*ndes-simp cls: assms*)
apply (*rel-auto*)
done

lemma *ndesign-ind-form*:

assumes $\bigwedge i. P(i)$ is \mathbf{N}
shows $(\lambda i. \lfloor pre_D(P(i)) \rfloor_{<} \vdash_n post_D(P(i))) = P$
by (*simp add: assms ndesign-form*)

lemma *AlternateD-insert*:

assumes $\bigwedge i. i \in (\text{insert } x \ A) \implies P(i)$ is \mathbf{N} Q is \mathbf{N}
shows if $i \in (\text{insert } x \ A) \cdot g(i) \rightarrow P(i)$ else Q fi =
 if $g(x) \rightarrow P(x) \mid$
 $(\bigvee i \in A \cdot g(i)) \rightarrow$ if $i \in A \cdot g(i) \rightarrow P(i)$ fi
 else Q

```

      fi (is ?lhs = ?rhs)
proof -
  have ?lhs = if i ∈ (insert x A) · g(i) → ([preD(P(i))] < ⊢n postD(P(i))) else ([preD(Q)] < ⊢n
postD(Q)) fi
  using AlternateD-ndes-expand assms(1) assms(2) by blast
also
have ... =
  if g(x) → ([preD(P(x))] < ⊢n postD(P(x))) |
  (⋁ i ∈ A · g(i) → if i ∈ A · g(i) → [preD(P(i))] < ⊢n postD(P(i)) fi
  else [preD(Q)] < ⊢n postD(Q)
  fi
  by (ndes-simp cls:assms, rel-auto)
also have ... = ?rhs
  by (simp add: AlternateD-ndes-expand' ndesign-form assms)
finally show ?thesis .
qed

```

4.4 Iteration

```

theorem ndesign-iteration-wp [ndes-simp]:
  (p ⊢n Q) ;; (p ⊢n Q) ^ n = ((⋀ i ∈ {0..n} · (Q ^ i) wp p) ⊢n Q ^ Suc n)
proof (induct n)
  case 0
  then show ?case by (rel-auto)
next
  case (Suc n) note hyp = this
  have (p ⊢n Q) ;; (p ⊢n Q) ^ Suc n = (p ⊢n Q) ;; (p ⊢n Q) ;; (p ⊢n Q) ^ n
  by (simp)
  also have ... = (p ⊢n Q) ;; ((⋂ i ∈ {0..n} · Q ^ i wp p) ⊢n Q ^ Suc n)
  by (simp add: hyp)
  also have ... = (p ∧ Q wp (⋂ i ∈ {0..n} · Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: ndesign-composition-wp segr-assoc)
  also have ... = (p ∧ (⋂ i ∈ {0..n} · Q ^ Suc i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: wp)
  also have ... = (p ∧ (⋂ i ∈ {0..n}. Q ^ Suc i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: USUP-as-Inf-image)
  also have ... = (p ∧ (⋂ i ∈ {1..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (metis (no-types, lifting) One-nat-def image-Suc-atLeastAtMost image-cong image-image)
  also have ... = (Q ^ 0 wp p ∧ (⋂ i ∈ {1..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: wp)
  also have ... = ((⋂ i ∈ {0..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: lic-Suc-eq-insert-0 atLeast0AtMost conj-upred-def image-Suc-atMost)
  also have ... = (⋂ i ∈ {0..Suc n} · Q ^ i wp p) ⊢n Q ^ Suc (Suc n)
  by (simp add: USUP-as-Inf-image upred-semiring.mult-assoc)
  finally show ?case .
qed

```

Overloadable Syntax

consts

```

uiterate      :: 'a set ⇒ ('a ⇒ 'p) ⇒ ('a ⇒ 'r) ⇒ 'r
uiterate-list :: ('a × 'r) list ⇒ 'r

```

syntax

```

-iterind      :: ptnr ⇒ logic ⇒ logic ⇒ logic ⇒ logic (do -∈- · - → - od)
-itergcomm    :: gcomms ⇒ logic (do - od)

```

translations

$-iterind\ x\ A\ g\ P \Rightarrow CONST\ uiterate\ A\ (\lambda\ x.\ g)\ (\lambda\ x.\ P)$
 $-iterind\ x\ A\ g\ P \Leftarrow CONST\ uiterate\ A\ (\lambda\ x.\ g)\ (\lambda\ x'.\ P)$
 $-itergcomm\ cs \Rightarrow CONST\ uiterate-list\ cs$
 $-itergcomm\ (-gcomm-show\ cs) \Leftarrow CONST\ uiterate-list\ cs$

definition $IterateD :: 'a\ set \Rightarrow ('a \Rightarrow 'a\ upred) \Rightarrow ('a \Rightarrow 'a\ hrel-des) \Rightarrow 'a\ hrel-des$ **where**
 $[upred-defs, ndes-simp]:$

$IterateD\ A\ g\ P = (\mu_{NDES}\ X \cdot \text{if } i \in A \cdot g(i) \rightarrow P(i) ;; X\ \text{else } II_D\ fi)$

definition $IterateD-list :: ('a\ upred \times 'a\ hrel-des)\ list \Rightarrow 'a\ hrel-des$ **where**
 $[upred-defs, ndes-simp]:$

$IterateD-list\ xs = IterateD\ \{0..<length\ xs\}\ (\lambda\ i.\ fst\ (nth\ xs\ i))\ (\lambda\ i.\ snd\ (nth\ xs\ i))$

ad hoc-overloading

$uiterate\ IterateD$ **and**
 $uiterate-list\ IterateD-list$

lemma $IterateD-H1-H3-closed\ [closure]:$

$\text{assumes } \bigwedge i. i \in A \implies P\ i\ \text{is } \mathbf{N}$
 $\text{shows } do\ i \in A \cdot g(i) \rightarrow P(i)\ \text{od is } \mathbf{N}$

proof $(cases\ A = \{\})$

case $True$

then show $?thesis$

by $(simp\ add: IterateD-def\ closure\ assms)$

next

case $False$

then show $?thesis$

by $(simp\ add: IterateD-def\ closure\ assms)$

qed

lemma $IterateD-empty:$

$do\ i \in \{\} \cdot g(i) \rightarrow P(i)\ \text{od} = II_D$

by $(simp\ add: IterateD-def\ AlternateD-empty\ normal-design-theory-continuous.LFP-const\ skip-d-is-H1-H3)$

lemma $IterateD-list-single-expand:$

$do\ b \rightarrow P\ \text{od} = (\mu_{NDES}\ X \cdot \text{if } b \rightarrow P ;; X\ \text{else } II_D\ fi)$

oops

lemma $IterateD-singleton:$

$\text{assumes } P\ \text{is } \mathbf{N}$

$\text{shows } do\ b \rightarrow P\ \text{od} = do\ i \in \{0\} \cdot b \rightarrow P\ \text{od}$

apply $(simp\ add: IterateD-list-def\ IterateD-def\ AlternateD-singleton\ assms)$

apply $(subst\ AlternateD-singleton)$

apply $(simp)$

apply $(rel-auto)$

oops

lemma $IterateD-mono-refine:$

assumes

$\bigwedge i. P\ i\ \text{is } \mathbf{N} \wedge i. Q\ i\ \text{is } \mathbf{N}$

$\bigwedge i. P\ i \sqsubseteq Q\ i$

$\text{shows } (do\ i \in A \cdot g(i) \rightarrow P(i)\ \text{od}) \sqsubseteq (do\ i \in A \cdot g(i) \rightarrow Q(i)\ \text{od})$

apply $(simp\ add: IterateD-def\ normal-design-theory-continuous.utp-lfp-def)$

apply $(subst\ normal-design-theory-continuous.utp-lfp-def)$

```

apply (simp-all add: closure assms)
apply (subst normal-design-theory-continuous.utp-lfp-def)
apply (simp-all add: closure assms)
apply (simp add: ndes-hcond-def)
apply (rule gfp-mono)
apply (rule AlternateD-mono-refine)
apply (simp-all add: closure segr-mono assms)
done

```

lemma *IterateD-single-refine*:

```

assumes
   $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N} \ P \sqsubseteq Q$ 
shows  $(do\ g \rightarrow P\ od) \sqsubseteq (do\ g \rightarrow Q\ od)$ 
oops

```

lemma *IterateD-refine-intro*:

```

fixes  $V :: (nat, 'a) \text{ uexpr}$ 
assumes vwb-lens  $w$ 
shows
 $I \vdash_n (w: [I \wedge \neg (\bigvee_{i \in A} \bullet g(i))]_{>}) \sqsubseteq$ 
 $do\ i \in A \bullet g(i) \rightarrow (I \wedge g(i)) \vdash_n (w: [I]_{>} \wedge [V]_{>} <_u [V]_{<})\ od$ 
proof (cases  $A = \{\}$ )
case True
with assms show ?thesis
by (simp add: IterateD-empty, rel-auto)
next
case False
then show ?thesis
using assms
apply (simp add: IterateD-def)
apply (rule ndesign-mu-wf-refine-intro[where  $e = V$  and  $R = \{(x, y). x < y\}$ ])
apply (simp-all add: wf closure)
apply (simp add: ndes-simp unrest)
apply (rule ndesign-refine-intro)
apply (rel-auto)
apply (rel-auto)
apply (metis mwb-lens.put-put vwb-lens-mwb)
done
qed

```

lemma *IterateD-single-refine-intro*:

```

fixes  $V :: (nat, 'a) \text{ uexpr}$ 
assumes vwb-lens  $w$ 
shows
 $I \vdash_n (w: [I \wedge \neg g]_{>}) \sqsubseteq$ 
 $do\ g \rightarrow ((I \wedge g) \vdash_n (w: [I]_{>} \wedge [V]_{>} <_u [V]_{<}))\ od$ 
apply (rule order-trans)
defer
apply (rule IterateD-refine-intro[of  $w\ \{0\}\ \lambda i. g\ I\ V$ , simplified, OF assms(1)])
oops

```

4.5 Let and Local Variables

definition $LetD :: ('a, 'a) \text{ uexpr} \Rightarrow ('a \Rightarrow 'a \text{ hrel-des}) \Rightarrow 'a \text{ hrel-des}$ **where**
 $[upred-defs]: LetD\ v\ P = (P\ x) \llbracket x \rightarrow [v]_{D<} \rrbracket$

syntax

$-LetD \quad :: [letbinds, 'a] \Rightarrow 'a \quad ((let_D (-) / in (-)) [0, 10] 10)$

translations

$-LetD (-binds b bs) e \Rightarrow -LetD b (-LetD bs e)$
 $let_D x = a \text{ in } e \Rightarrow CONST LetD a (\lambda x. e)$

lemma *LetD-ndes-simp* [ndes-simp]:

$LetD v (\lambda x. p(x) \vdash_n Q(x)) = (p(x)[x \rightarrow v]) \vdash_n (Q(x)[x \rightarrow [v]_{<}])$
by (*rel-auto*)

lemma *LetD-H1-H3-closed* [closure]:

$\llbracket \bigwedge x. P(x) \text{ is } \mathbf{N} \rrbracket \Longrightarrow LetD v P \text{ is } \mathbf{N}$
by (*rel-auto*)

4.6 Deep Local Variables

definition *des-local-state* ::

$'a::countable \text{ itself} \Rightarrow ((nat, 's) \text{ local-scheme } des, 's, nat, 'a::countable) \text{ local-prim}$ **where**
 $des\text{-local-state } t = (\mid sstate = \Sigma_D, sassigns = assigns\text{-d}, inj\text{-local} = nat\text{-inj-univ} \mid)$

syntax

$-des\text{-local-state-type} :: type \Rightarrow logic (\mathcal{L}_D[-])$
 $-des\text{-var-scope-type} :: id \Rightarrow type \Rightarrow logic \Rightarrow logic (var_D - :: - \cdot - [0, 0, 10] 10)$

translations

$\mathcal{L}_D['a] == CONST des\text{-local-state } TYPE('a)$
 $-des\text{-var-scope-type } x \ t \ P \Rightarrow -var\text{-scope-type } (-des\text{-local-state-type } t) \ x \ t \ P$
 $var_D x :: 'a \cdot P \leq var[\mathcal{L}_D['a]] \ x \cdot P$

lemma *get-rel-local* [lens-defs]:

$gets_{\mathcal{L}_D['a::countable]} = get_{\Sigma_D}$
by (*simp add: des-local-state-def*)

lemma *des-local-state* [simp]: *utp-local-state* $\mathcal{L}_D['a::countable]$

by (*unfold-locales, simp-all add: upred-defs assigns-comp des-local-state-def, rel-auto*)
(metis local.cases-scheme)

lemma *sassigns-des-state* [simp]: $\langle \sigma \rangle_{\mathcal{L}_D['a::countable]} = \langle \sigma \rangle_D$

by (*simp add: des-local-state-def*)

lemma *des-var-open-H1-H3-closed* [closure]:

$open[\mathcal{L}_D['a::countable]] \text{ is } \mathbf{N}$
by (*simp add: utp-local-state.var-open-def closure*)

lemma *des-var-close-H1-H3-closed* [closure]:

$close[\mathcal{L}_D['a::countable]] \text{ is } \mathbf{N}$
by (*simp add: utp-local-state.var-close-def closure*)

lemma *unrest-ok-vtop-des* [unrest]: $ok \ \# \ top[\mathcal{L}_D['a::countable]]$

by (*simp add: utp-local-state.top-var-def, simp add: des-local-state-def unrest*)

lemma *msubst-H1-H3-closed* [closure]:

$\llbracket \$ok \ \# \ v; out\alpha \ \# \ v; (\bigwedge x. P \ x \text{ is } \mathbf{N}) \rrbracket \Longrightarrow (P(x)[x \rightarrow v]) \text{ is } \mathbf{N}$
by (*rel-auto, metis+*)

lemma *var-block-H1-H3-closed* [closure]:
 $(\bigwedge x. P \ x \text{ is } \mathbf{N}) \implies \mathcal{V}[\mathcal{L}_D['a::\text{countable}], P] \text{ is } \mathbf{N}$
by (*simp add: utp-local-state.var-scope-def closure unrest*)

lemma *inj-local-rel* [simp]: $\text{inj-local } R_l = \mathcal{U}_{\mathbf{N}}$
by (*simp add: rel-local-state-def*)

lemma *sstate-rel* [simp]: $\mathbf{s}_{R_l} = 1_L$
by (*simp add: rel-local-state-def*)

lemma *inj-local-des* [simp]:
 $\text{inj-local } \mathcal{L}_D['a::\text{countable}] = \mathcal{U}_{\mathbf{N}}$
by (*simp add: des-local-state-def*)

lemma *sstate-des* [simp]: $\mathbf{s}_{\mathcal{L}_D['a::\text{countable}]} = \Sigma_D$
by (*simp add: des-local-state-def*)

lemma *ndesign-msubst-top* [usubst]:
 $(p \ x \vdash_n Q \ x) \llbracket x \rightarrow \text{top}[\mathcal{L}_D['a::\text{countable}]] \rrbracket_{<} = ((p \ x) \llbracket x \rightarrow \text{top}[R_l['a]] \rrbracket \vdash_n (Q \ x) \llbracket x \rightarrow \text{top}[R_l['a]] \rrbracket_{<})$
by (*rel-auto'*)

First attempt at a law for expanding design variable blocks. Far from adequate at the moment though.

lemma *ndesign-local-expand-1* [ndes-simp]:
 $(\text{var}_D \ x :: 'a :: \text{countable} \cdot p(x) \vdash_n Q(x)) =$
 $(\bigsqcup v \cdot (p \ x) \llbracket x \rightarrow \text{top}[R_l] \rrbracket \llbracket \&\text{store} \hat{^}_u \langle \llbracket v \rrbracket \rrbracket / \text{store} \rrbracket \vdash_n$
 $(\prod v \cdot \text{store} := \&\text{store} \hat{^}_u \langle \llbracket v \rrbracket \rrbracket ;; (Q \ x) \llbracket x \rightarrow \text{top}[R_l] \rrbracket_{<} ;; \text{store} := (\text{front}_u(\&\text{store}) \triangleleft 0 <_u$
 $\#_u(\&\text{store}) \triangleright \&\text{store}))$
apply (*simp add: utp-local-state.var-scope-def utp-local-state.var-open-def utp-local-state.var-close-def*
seq-UINF-distr' usubst)
apply (*simp add: ndes-simp wp unrest*)
apply (*rel-auto'*)
done

end

5 Design Weakest Preconditions

theory *utp-des-wp*
imports *utp-des-prog*
begin

definition *wp-design* :: $(' \alpha, ' \beta) \text{ rel-des} \Rightarrow ' \beta \text{ cond} \Rightarrow ' \alpha \text{ cond}$ (**infix** *wp_D* 60) **where**
 $[\text{upred-defs}]: Q \text{ wp}_D r = (\lfloor \text{pre}_D(Q) \rfloor ;; \text{true} :: (' \alpha, ' \beta) \text{ urel} \rfloor_{<} \wedge (\text{post}_D(Q) \text{ wp } r))$

If two normal designs have the same weakest precondition for any given postcondition, then the two designs are equivalent.

theorem *wpd-eq-intro*: $\llbracket \bigwedge r. (p_1 \vdash_n Q_1) \text{ wp}_D r = (p_2 \vdash_n Q_2) \text{ wp}_D r \rrbracket \implies (p_1 \vdash_n Q_1) = (p_2 \vdash_n Q_2)$
apply (*rel-simp robust; metis curry-conv*)
done

theorem *wpd-H3-eq-intro*: $\llbracket P \text{ is } H1-H3; Q \text{ is } H1-H3; \bigwedge r. P \text{ wp}_D r = Q \text{ wp}_D r \rrbracket \implies P = Q$
by (*metis H1-H3-commute H1-H3-is-normal-design H3-idem Healthy-def' wpd-eq-intro*)

lemma *wp-assigns-d* [wp]: $\langle \sigma \rangle_D \text{ wp}_D r = \sigma \dagger r$
by (*rel-auto*)

theorem *rdesign-wp* [wp]:
 $([p]_{<} \vdash_r Q) \text{ wp}_D r = (p \wedge Q \text{ wp } r)$
by (*rel-auto*)

theorem *ndesign-wp* [wp]:
 $(p \vdash_n Q) \text{ wp}_D r = (p \wedge Q \text{ wp } r)$
by (*simp add: ndesign-def rdesign-wp*)

theorem *wpd-seq-r*:
fixes *Q1 Q2* :: $'\alpha \text{ hrel}$
shows $(([p1]_{<} \vdash_r Q1) ;; ([p2]_{<} \vdash_r Q2)) \text{ wp}_D r = ([p1]_{<} \vdash_r Q1) \text{ wp}_D (([p2]_{<} \vdash_r Q2) \text{ wp}_D r)$
apply (*simp add: wp*)
apply (*subst rdesign-composition-wp*)
apply (*simp only: wp*)
apply (*rel-auto*)
done

theorem *wpnd-seq-r* [wp]:
fixes *Q1 Q2* :: $'\alpha \text{ hrel}$
shows $((p1 \vdash_n Q1) ;; (p2 \vdash_n Q2)) \text{ wp}_D r = (p1 \vdash_n Q1) \text{ wp}_D ((p2 \vdash_n Q2) \text{ wp}_D r)$
by (*simp add: ndesign-def wpd-seq-r*)

theorem *wpd-seq-r-H1-H3* [wp]:
fixes *P Q* :: $'\alpha \text{ hrel-des}$
assumes *P is N Q is N*
shows $(P ;; Q) \text{ wp}_D r = P \text{ wp}_D (Q \text{ wp}_D r)$
by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def' assms(1) assms(2) wpnd-seq-r*)

end

6 Refinement Calculus

theory *utp-des-refcalc*
imports *utp-des-prog*
begin

definition *des-spec* :: $('a \Rightarrow '\alpha) \Rightarrow '\alpha \text{ upred} \Rightarrow ('a \Rightarrow '\alpha \text{ upred}) \Rightarrow '\alpha \text{ hrel-des}$ **where**
 $[upred-defs]: \text{des-spec } x \ p \ q = (\bigsqcup \ v \cdot ((p \wedge \&\mathbf{v} =_u \ll v \gg) \vdash_n x: [[q(v)]_{>}]))$

syntax

-init-var :: *logic*
-des-spec :: $\text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (-: [-, / -]_D [99, 0, 0] \ 100)$
-des-log-const :: $\text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\text{con}_D \ - \cdot - \ [0, 10] \ 10)$

translations

-des-spec $x \ p \ q \Rightarrow \text{CONST des-spec } x \ p \ (\lambda \text{ -init-var. } q)$
-des-spec $(\text{-salphaset } (\text{-salphamk } x)) \ p \ q \Leftarrow \text{CONST des-spec } x \ p \ (\lambda \text{ iv. } q)$
-des-log-const $x \ P \Rightarrow \bigsqcup \ x \cdot P$

parse-translation \ll
let

```

  fun init-var-tr [] = Syntax.free iv
    | init-var-tr - = raise Match;
in
  [(@{syntax-const -init-var}, K init-var-tr)]
end
>>

```

abbreviation $choose_D x \equiv \{\&x\}:[true,true]_D$

lemma *des-spec-simple-def*:

```

  x:[pre,post]D = (pre ⊢n x:[post]>])
  by (rel-auto)

```

lemma *des-spec-abort*:

```

  x:[false,post]D = ⊥D
  by (rel-auto)

```

lemma *des-spec-skip*: $\emptyset:[true,true]_D = II_D$

```

  by (rel-auto)

```

lemma *des-spec-strengthen-post*:

```

  assumes 'post' ⇒ post'
  shows w:[pre, post]D ⊆ w:[pre, post']D
  using assms by (rel-auto)

```

lemma *des-spec-weaken-pre*:

```

  assumes 'pre ⇒ pre'
  shows w:[pre, post]D ⊆ w:[pre', post]D
  using assms by (rel-auto)

```

lemma *des-spec-refine-skip*:

```

  assumes vwb-lens w 'pre ⇒ post'
  shows w:[pre, post]D ⊆ IID
  using assms by (rel-auto)

```

lemma *rc-iter*:

```

  fixes V :: (nat, 'a) uexpr

```

```

  assumes vwb-lens w

```

```

  shows w:[ivr, ivr ∧ ¬ (⋁ i∈A · g(i))]D

```

```

    ⊆ (do i∈A · g(i) → ⋒ iv · w:[ivr ∧ g(i) ∧ «iv» =u &v, ivr ∧ (V <u V[«iv»/v])]D od) (is

```

```

  ?lhs ⊆ ?rhs)

```

```

  apply (rule order-trans)

```

```

  defer

```

```

  apply (simp add: des-spec-simple-def)

```

```

  apply (rule IterateD-refine-intro[of - - - V])

```

```

  apply (simp add: assms)

```

```

  apply (rule IterateD-mono-refine)

```

```

    apply (simp-all add: ndes-simp closure)

```

```

  apply (rel-auto)

```

```

  using assms

```

```

  apply (rel-auto)

```

```

  done

```

end

7 Theory of Invariants

```
theory utp-des-invariants
  imports utp-des-theory
begin
```

The theory of invariants formalises operation and state invariants based on the theory of designs. For more information, please see the associated paper [1, Section 4].

7.1 Operation Invariants

definition $OIH(\psi)(D) = (D \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

declare $OIH-def$ [*upred-defs*]

lemma $OIH-design$:

assumes D is $H1-H2$

shows $OIH(\psi)(D) = ((\neg D^f) \vdash (D^t \wedge \psi))$

proof –

have $OIH(\psi)(D) = (((\neg D^f) \vdash D^t) \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

by (*metis H1-H2-commute H1-H2-is-design H1-idem Healthy-def' OIH-def assms*)

also have $\dots = ((\$ok \wedge \neg D^f \Rightarrow \$ok' \wedge D^t) \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

by (*simp add: design-def*)

also have $\dots = ((\neg D^f) \vdash (D^t \wedge \psi))$

by (*pred-auto*)

finally show *?thesis* .

qed

lemma $OIH-idem$:

assumes D is $H1-H2$ $\$ok' \# \psi$

shows $OIH(\psi)(OIH(\psi)(D)) = OIH(\psi)(D)$

using *assms*

by (*simp add: OIH-design design-is-H1-H2 unrest (simp add: design-def usubst, rel-auto)*)

lemma $OIH-of-design$:

$\$ok' \# P \Longrightarrow OIH(\psi)(P \vdash Q) = (P \vdash (Q \wedge \psi))$

by (*simp add: OIH-def design-def usubst, rel-auto*)

7.2 State Invariants

definition $ISH(\psi)(D) = (D \vee (\$ok \wedge \neg D^f \wedge [\psi]_{<} \Rightarrow \$ok' \wedge D^t))$

declare $ISH-def$ [*upred-defs*]

lemma $ISH-design$: $ISH(\psi)(D) = (\neg D^f \wedge [\psi]_{<}) \vdash D^t$

by (*rel-auto, metis+*)

lemma $ISH-idem$: $ISH(\psi)(ISH(\psi)(D)) = ISH(\psi)(D)$

by (*simp add: ISH-design usubst design-def, pred-auto*)

lemma $ISH-of-design$:

$\llbracket \$ok' \# P; \$ok' \# Q \rrbracket \Longrightarrow ISH(\psi)(P \vdash Q) = ((P \wedge [\psi]_{<}) \vdash Q)$

by (*simp add: ISH-design design-def usubst, pred-auto*)

definition $OSH(\psi)(D) = (D \wedge (\$ok \wedge \neg D^f \wedge [\psi]_{<} \Rightarrow [\psi]_{>}))$

declare *OSH-def* [*upred-defs*]

lemma *OSH-as-OIH*:

$OSH(\psi)(D) = OIH(\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})(D)$
by (*simp add: OSH-def OIH-def, pred-auto*)

lemma *OSH-design*:

assumes *D is H1-H2*
shows $OSH(\psi)(D) = ((\neg D^f) \vdash (D^t \wedge (\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})))$
by (*simp add: OSH-as-OIH OIH-design assms*)

lemma *OSH-of-design*:

$\llbracket \$ok' \# P; \$ok' \# Q \rrbracket \Longrightarrow OSH(\psi)(P \vdash Q) = (P \vdash (Q \wedge (\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})))$
by (*simp add: OSH-design design-is-H1-H2 unrest, simp add: design-def usubst, pred-auto*)

definition $SIH(\psi) = ISH(\psi) \circ OSH(\psi)$

declare *SIH-def* [*upred-defs*]

lemma *SIH-of-design*:

$\llbracket \$ok' \# P; \$ok' \# Q; ok \# \psi \rrbracket \Longrightarrow SIH(\psi)(P \vdash Q) = ((P \wedge \lceil\psi\rceil_{<}) \vdash (Q \wedge \lceil\psi\rceil_{>}))$
by (*simp add: SIH-def OSH-of-design ISH-of-design unrest, pred-auto*)

end

8 Meta Theory for UTP Designs

theory *utp-designs*

imports

utp-des-core
utp-des-healths
utp-des-theory
utp-des-tactics
utp-des-prog
utp-des-wp
utp-des-refcalc
utp-des-invariants

begin end

References

- [1] A. Cavalcanti, A. Wellings, and J. Woodcock. The Safety-Critical Java memory model formalised. *Formal Aspects of Computing*, 25(1):37–57, 2012.
- [2] A. Cavalcanti and J. Woodcock. A tutorial introduction to designs in unifying theories of programming. In *Proc. 4th Intl. Conf. on Integrated Formal Methods (IFM)*, volume 2999 of *LNCIS*, pages 40–66. Springer, 2004.
- [3] T. Hoare and J. He. *Unifying Theories of Programming*. Prentice-Hall, 1998.