Mathematical Toolkit for Isabelle/UTP

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Abstract

This document describes our mathematical toolkit for Isabelle/UTP, which provides a foundational collection of definition, theorems, and proof facilities. This includes extensions to existing HOL libraries, such as for list and partial functions, and also new type definitions, theorems, and Isabelle/HOL commands.

Contents

1 Introduction

This document contains the description of our mathematical toolkit for Isabelle/UTP [2, 3, 4, 7], a mechanisation of Hoare and He's *Unifying Theories of Programming* [5, 1]. The toolkit provides a foundational collection of additional HOL theorems, new abstract types, and proof facilities, upon which Isabelle/UTP depends. In brief, the toolkit contains the following principal items:

- additional laws and functions for the list, map (partial functions), countable set, and finite set types;
- type definitions for partial and finite functions, together with additional functions and laws derived from the Z mathematical toolkit [6];
- positive subtypes of existing types;
- infinite sequences;
- the "total recall" package, which allows us to precisely control overriding of existing syntax annotations.

A few other theories exist that add smaller utilities and additional laws.

2 Lists: extra functions and properties

theory *List-Extra*
  imports
  HOL−Library.Sublist
  HOL−Library.Monad-Syntax
  HOL−Library.Prefix-Order
  Optics.Lens-Instances
begin

2.1 Useful Abbreviations

abbreviation *list-sum xs* $\equiv$ *foldr* $(+)$ *xs 0*

2.2 Folds

context *abel-semigroup*
begin

lemma *foldr-snoc*: *foldr* $(*)$ $(xs\ @\ [x])$ k $=$ $(foldr\ (*)\ xs\ k) * x$
by (*induct xs*, *simp-all add*: *commute left-commute*)

end

2.3 List Lookup

The following variant of the standard *nth* function returns $\perp$ if the index is out of range.

primrec
nth-el :: $'a\ list \Rightarrow nat \Rightarrow 'a\ option$ $(\text{-}\langle\text{-}\rangle_l\ [90,\ 0]\ 91)$
where
$[]\langle i\rangle_l = None$
$|\ (x\ \#\ xs)\langle i\rangle_l = (case\ i\ of\ 0 \Rightarrow Some\ x\ |\ Suc\ j \Rightarrow xs\ \langle j\rangle_l)$

lemma *nth-el-appendl*[*simp*]: $i < length\ xs \Longrightarrow (xs\ @\ ys)\langle i\rangle_l = xs\langle i\rangle_l$
apply (*induct xs arbitrary*: *i*)
apply *simp*
apply (*case-tac i*)
apply *simp-all*
done

lemma *nth-el-appendr*[*simp*]: $length\ xs \leq i \Longrightarrow (xs\ @\ ys)\langle i\rangle_l = ys\langle i - length\ xs\rangle_l$
apply (*induct xs arbitrary*: *i*)
apply *simp*
apply (*case-tac i*)
apply *simp-all*
done

2.4 Extra List Theorems

2.4.1 Map

lemma *map-nth-Cons-atLeastLessThan*:
$map\ (nth\ (x\ \#\ xs))\ [Suc\ m..<n] = map\ (nth\ xs)\ [m..<n - 1]$
proof $-$
have $nth\ xs = nth\ (x\ \#\ xs) \circ Suc$
by *auto*
hence $map\ (nth\ xs)\ [m..<n - 1] = map\ (nth\ (x\ \#\ xs) \circ Suc)\ [m..<n - 1]$
by *simp*
also have $... = map\ (nth\ (x\ \#\ xs))\ (map\ Suc\ [m..<n - 1])$
by *simp*
also have $... = map\ (nth\ (x\ \#\ xs))\ [Suc\ m..<n]$
by (*metis Suc-diff-1 le-0-eq length-upt list.simps*(*8*) *list.size*(*3*) *map-Suc-upt not-less upt-0*)
finally show *?thesis* **..**
qed

2.4.2 Sorted Lists

lemma *sorted-last* [*simp*]: $[\![\ x \in set\ xs;\ sorted\ xs\ ]\!] \Longrightarrow x \leq last\ xs$
  by (*induct xs*, *auto*)

lemma *sorted-prefix*:
  assumes $xs \leq ys$ *sorted ys*
  shows *sorted xs*
proof −
  obtain *zs* **where** $ys = xs\ @\ zs$
    using *Prefix-Order.prefixE assms*(*1*) **by** *auto*
  thus *?thesis*
    using *assms*(*2*) *sorted-append* **by** *blast*
qed

lemma *sorted-map*: $[\![\ sorted\ xs;\ mono\ f\ ]\!] \Longrightarrow sorted\ (map\ f\ xs)$
  by (*simp add*: *monoD sorted-iff-nth-mono*)

lemma *sorted-distinct* [*intro*]: $[\![\ sorted\ (xs);\ distinct(xs)\ ]\!] \Longrightarrow (\forall\ i{<}length\ xs - 1.\ xs!i < xs!(i + 1))$
  apply (*induct xs*)
   apply (*auto*)
 apply (*metis* (*no-types*, *hide-lams*) *Suc-leI Suc-less-eq Suc-pred gr0-conv-Suc not-le not-less-iff-gr-or-eq nth-Cons-Suc nth-mem nth-non-equal-first-eq*)
  done

Is the given list a permutation of the given set?

definition *is-sorted-list-of-set* :: $('a{::}ord)\ set \Rightarrow 'a\ list \Rightarrow bool$ **where**
$is\text{-}sorted\text{-}list\text{-}of\text{-}set\ A\ xs = ((\forall\ i{<}length(xs) - 1.\ xs!i < xs!(i + 1)) \wedge set(xs) = A)$

lemma *sorted-is-sorted-list-of-set*:
  assumes *is-sorted-list-of-set A xs*
  shows *sorted*(*xs*) **and** *distinct*(*xs*)
using *assms* **proof** (*induct xs arbitrary*: *A*)
  show *sorted* []
    by *auto*
next
  show *distinct* []
    by *auto*
next
  fix $A :: 'a\ set$
  case (*Cons x xs*) **note** *hyps* = *this*
  assume *isl*: *is-sorted-list-of-set A* (x # xs)
  hence *srt*: $(\forall\, i{<}length\ xs - Suc\ 0.\ xs\ !\ i < xs\ !\ Suc\ i)$
    using *less-diff-conv* **by** (*auto simp add*: *is-sorted-list-of-set-def*)
  with *hyps*(*1*) **have** *srtd*: *sorted xs*
    by (*simp add*: *is-sorted-list-of-set-def*)
  with *isl* **show** *sorted* (x # xs)
    apply (*auto simp add*: *is-sorted-list-of-set-def*)
   apply (*metis* (*mono-tags*, *lifting*) *all-nth-imp-all-set less-le-trans linorder-not-less not-less-iff-gr-or-eq nth-Cons-0 sorted-iff-nth-mono zero-order*(*3*))
    done
  from *srt hyps*(*2*) **have** *distinct xs*
    by (*simp add*: *is-sorted-list-of-set-def*)
  with *isl* **show** *distinct* (x # xs)
  proof −
    have $(\forall\, n.\ \neg\ n < length\ (x \# xs) - 1 \vee (x \# xs)\ !\ n < (x \# xs)\ !\ (n + 1)) \wedge set\ (x \# xs) = A$

by (*meson* ‹*is-sorted-list-of-set A* (*x* # *xs*)› *is-sorted-list-of-set-def*)
then show *?thesis*
by (*metis* ‹*distinct xs*› *add.commute add-diff-cancel-left′ distinct.simps*(*2*) *leD length-Cons length-greater-0-conv length-pos-if-in-set less-le nth-Cons-0 nth-Cons-Suc plus-1-eq-Suc set-ConsD sorted.elims*(*2*) *srtd*)
qed
qed

lemma *is-sorted-list-of-set-alt-def*:
is-sorted-list-of-set A xs ⟷ *sorted* (*xs*) ∧ *distinct* (*xs*) ∧ *set*(*xs*) = *A*
apply (*auto intro*: *sorted-is-sorted-list-of-set*)
apply (*auto simp add*: *is-sorted-list-of-set-def*)
apply (*metis Nat.add-0-right One-nat-def add-Suc-right sorted-distinct*)
done

definition *sorted-list-of-set-alt* :: (*′a*::*ord*) *set* ⇒ *′a list* **where**
sorted-list-of-set-alt A =
(*if* (*A* = {}) *then* [] *else* (*THE xs. is-sorted-list-of-set A xs*))

lemma *is-sorted-list-of-set*:
finite A ⟹ *is-sorted-list-of-set A* (*sorted-list-of-set A*)
apply (*simp add*: *is-sorted-list-of-set-def*)
apply (*metis One-nat-def add.right-neutral add-Suc-right sorted-distinct sorted-list-of-set*)
done

lemma *sorted-list-of-set-other-def*:
finite A ⟹ *sorted-list-of-set*(*A*) = (*THE xs. sorted*(*xs*) ∧ *distinct*(*xs*) ∧ *set xs* = *A*)
apply (*rule sym*)
apply (*rule the-equality*)
apply (*auto*)
apply (*simp add*: *sorted-distinct-set-unique*)
done

lemma *sorted-list-of-set-alt* [*simp*]:
finite A ⟹ *sorted-list-of-set-alt*(*A*) = *sorted-list-of-set*(*A*)
apply (*rule sym*)
apply (*auto simp add*: *sorted-list-of-set-alt-def is-sorted-list-of-set-alt-def sorted-list-of-set-other-def*)
done

Sorting lists according to a relation

definition *is-sorted-list-of-set-by* :: *′a rel* ⇒ *′a set* ⇒ *′a list* ⇒ *bool* **where**
is-sorted-list-of-set-by R A xs = ((∀ *i*<*length*(*xs*) − *1*. (*xs*!*i*, *xs*!(*i* + *1*)) ∈ *R*) ∧ *set*(*xs*) = *A*)

definition *sorted-list-of-set-by* :: *′a rel* ⇒ *′a set* ⇒ *′a list* **where**
sorted-list-of-set-by R A = (*THE xs. is-sorted-list-of-set-by R A xs*)

definition *fin-set-lexord* :: *′a rel* ⇒ *′a set rel* **where**
fin-set-lexord R = {(*A*, *B*). *finite A* ∧ *finite B* ∧
(∃ *xs ys. is-sorted-list-of-set-by R A xs* ∧ *is-sorted-list-of-set-by R B ys*
∧ (*xs*, *ys*) ∈ *lexord R*)}

lemma *is-sorted-list-of-set-by-mono*:
⟦ *R* ⊆ *S*; *is-sorted-list-of-set-by R A xs* ⟧ ⟹ *is-sorted-list-of-set-by S A xs*
by (*auto simp add*: *is-sorted-list-of-set-by-def*)

lemma *lexord-mono′*:

⟦ ($\bigwedge$ *x y*. *f x y* ⟶ *g x y*); (*xs*, *ys*) ∈ *lexord* {(*x*, *y*). *f x y*} ⟧ ⟹ (*xs*, *ys*) ∈ *lexord* {(*x*, *y*). *g x y*}
by (*metis case-prodD case-prodI lexord-take-index-conv mem-Collect-eq*)

lemma *fin-set-lexord-mono* [*mono*]:
($\bigwedge$ *x y*. *f x y* ⟶ *g x y*) ⟹ (*xs*, *ys*) ∈ *fin-set-lexord* {(*x*, *y*). *f x y*} ⟶ (*xs*, *ys*) ∈ *fin-set-lexord* {(*x*, *y*). *g x y*}
proof
assume
fin: (*xs*, *ys*) ∈ *fin-set-lexord* {(*x*, *y*). *f x y*} **and**
hyp: ($\bigwedge$ *x y*. *f x y* ⟶ *g x y*)

from *fin* **have** *finite xs finite ys*
using *fin-set-lexord-def* **by** *fastforce+*

with *fin hyp* **show** (*xs*, *ys*) ∈ *fin-set-lexord* {(*x*, *y*). *g x y*}
apply (*auto simp add*: *fin-set-lexord-def*)
apply (*rename-tac xs′ ys′*)
apply (*rule-tac x=xs′* **in** *exI*)
apply (*auto*)
apply (*metis case-prodD case-prodI is-sorted-list-of-set-by-def mem-Collect-eq*)
apply (*metis case-prodD case-prodI is-sorted-list-of-set-by-def lexord-mono′ mem-Collect-eq*)
done
qed

definition *distincts* :: ′*a set* ⇒ ′*a list set* **where**
distincts A = {*xs* ∈ *lists A*. *distinct*(*xs*)}

lemma *tl-element*:
⟦ *x* ∈ *set xs*; *x* ≠ *hd*(*xs*) ⟧ ⟹ *x* ∈ *set*(*tl*(*xs*))
by (*metis in-set-insert insert-Nil list.collapse list.distinct*(*2*) *set-ConsD*)

2.4.3 List Update

lemma *listsum-update*:
fixes *xs* :: ′*a*::*ring list*
assumes *i* < *length xs*
shows *list-sum* (*xs*[*i* := *v*]) = *list-sum xs* − *xs* ! *i* + *v*
using *assms* **proof** (*induct xs arbitrary*: *i*)
case *Nil*
then show *?case* **by** (*simp*)
next
case (*Cons a xs*)
then show *?case*
proof (*cases i*)
case *0*
thus *?thesis*
by (*simp add*: *add.commute*)
next
case (*Suc i′*)
with *Cons* **show** *?thesis*
by (*auto*)
qed
qed

2.4.4 Drop While and Take While

lemma *dropWhile-sorted-le-above*:
$[\![$ *sorted xs*; $x \in$ *set* (*dropWhile* (λ x. $x \leq n$) *xs*) $]\!] \Longrightarrow x > n$
apply (*induct xs*)
apply (*auto*)
apply (*rename-tac a xs*)
apply (*case-tac* $a \leq n$)
apply (*auto*)
done

lemma *set-dropWhile-le*:
sorted xs $\Longrightarrow$ *set* (*dropWhile* (λ x. $x \leq n$) *xs*) = {x∈*set xs*. $x > n$}
apply (*induct xs*)
apply (*simp*)
apply (*rename-tac x xs*)
apply (*subgoal-tac sorted xs*)
apply (*simp*)
apply (*safe*)
apply (*auto*)
done

lemma *set-takeWhile-less-sorted*:
$[\![$ *sorted I*; $x \in$ *set I*; $x < n$ $]\!] \Longrightarrow x \in$ *set* (*takeWhile* (λx. $x < n$) *I*)
proof (*induct I arbitrary*: *x*)
case *Nil* **thus** *?case*
by (*simp*)
next
case (*Cons a I*) **thus** *?case*
by *auto*
qed

lemma *nth-le-takeWhile-ord*: $[\![$ *sorted xs*; $i \geq$ *length* (*takeWhile* (λ x. $x \leq n$) *xs*); $i <$ *length xs* $]\!] \Longrightarrow$
$n \leq$ *xs* ! i
apply (*induct xs arbitrary*: *i*, *auto*)
apply (*rename-tac x xs i*)
apply (*case-tac* $x \leq n$)
apply (*auto*)
apply (*metis One-nat-def Suc-eq-plus1 le-less-linear le-less-trans less-imp-le list.size*(*4*) *nth-mem set-ConsD*)
done

lemma *length-takeWhile-less*:
$[\![$ $a \in$ *set xs*; $\neg$ *P a* $]\!] \Longrightarrow$ *length* (*takeWhile P xs*) $<$ *length xs*
by (*metis in-set-conv-nth length-takeWhile-le nat-neq-iff not-less set-takeWhileD takeWhile-nth*)

lemma *nth-length-takeWhile-less*:
$[\![$ *sorted xs*; *distinct xs*; ($\exists$ $a \in$ *set xs*. $a \geq n$) $]\!] \Longrightarrow$ *xs* ! *length* (*takeWhile* (λx. $x < n$) *xs*) $\geq n$
by (*induct xs*, *auto*)

2.4.5 Last and But Last

lemma *length-gt-zero-butlast-concat*:
assumes *length ys* > 0
shows *butlast* (*xs* @ *ys*) = *xs* @ (*butlast ys*)
using *assms* **by** (*metis butlast-append length-greater-0-conv*)

lemma *length-eq-zero-butlast-concat*:
  assumes *length ys* = *0*
  shows *butlast* (*xs* @ *ys*) = *butlast xs*
  using *assms* **by** (*metis append-Nil2 length-0-conv*)

lemma *butlast-single-element*:
  shows *butlast* [*e*] = []
  by (*metis butlast.simps*(*2*))

lemma *last-single-element*:
  shows *last* [*e*] = *e*
  by (*metis last.simps*)

lemma *length-zero-last-concat*:
  assumes *length t* = *0*
  shows *last* (*s* @ *t*) = *last s*
  by (*metis append-Nil2 assms length-0-conv*)

lemma *length-gt-zero-last-concat*:
  assumes *length t* > *0*
  shows *last* (*s* @ *t*) = *last t*
  by (*metis assms last-append length-greater-0-conv*)

2.4.6 Prefixes and Strict Prefixes

lemma *prefix-length-eq*:
  ⟦ *length xs* = *length ys*; *prefix xs ys* ⟧ ⟹ *xs* = *ys*
  by (*metis not-equal-is-parallel parallel-def*)

lemma *prefix-Cons-elim* [*elim*]:
  assumes *prefix* (*x* # *xs*) *ys*
  obtains *ys'* **where** *ys* = *x* # *ys'* *prefix xs ys'*
  using *assms*
  by (*metis append-Cons prefix-def*)

lemma *prefix-map-inj*:
  ⟦ *inj-on f* (*set xs* ∪ *set ys*); *prefix* (*map f xs*) (*map f ys*) ⟧ ⟹
   prefix xs ys
  apply (*induct xs arbitrary*:*ys*)
   apply (*simp-all*)
  apply (*erule prefix-Cons-elim*)
  apply (*auto*)
  apply (*metis image-insert insertI1 insert-Diff-if singletonE*)
  done

lemma *prefix-map-inj-eq* [*simp*]:
  inj-on f (*set xs* ∪ *set ys*) ⟹
   prefix (*map f xs*) (*map f ys*) ⟷ *prefix xs ys*
  using *map-mono-prefix prefix-map-inj* **by** *blast*

lemma *strict-prefix-Cons-elim* [*elim*]:
  assumes *strict-prefix* (*x* # *xs*) *ys*
  obtains *ys'* **where** *ys* = *x* # *ys'* *strict-prefix xs ys'*
  using *assms*
  by (*metis Sublist.strict-prefixE' Sublist.strict-prefixI' append-Cons*)

lemma *strict-prefix-map-inj*:
$[\![$ *inj-on f* (*set xs* $\cup$ *set ys*); *strict-prefix* (*map f xs*) (*map f ys*) $]\!] \Longrightarrow$
strict-prefix xs ys
apply (*induct xs arbitrary*:*ys*)
apply (*auto*)
using *prefix-bot.bot.not-eq-extremum* **apply** *fastforce*
apply (*erule strict-prefix-Cons-elim*)
apply (*auto*)
apply (*metis* (*hide-lams*, *full-types*) *image-insert insertI1 insert-Diff-if singletonE*)
done

lemma *strict-prefix-map-inj-eq* [*simp*]:
inj-on f (*set xs* $\cup$ *set ys*) $\Longrightarrow$
strict-prefix (*map f xs*) (*map f ys*) $\longleftrightarrow$ *strict-prefix xs ys*
by (*simp add*: *inj-on-map-eq-map strict-prefix-def*)

lemma *prefix-drop*:
$[\![$ *drop* (*length xs*) *ys* = *zs*; *prefix xs ys* $]\!]$
$\Longrightarrow$ *ys* = *xs* @ *zs*
by (*metis append-eq-conv-conj prefix-def*)

lemma *list-append-prefixD* [*dest*]: x @ $y \leq z \Longrightarrow x \leq z$
using *append-prefixD less-eq-list-def* **by** *blast*

lemma *prefix-not-empty*:
assumes *strict-prefix xs ys* **and** *xs* $\neq$ []
shows *ys* $\neq$ []
using *Sublist.strict-prefix-simps*(*1*) *assms*(*1*) **by** *blast*

lemma *prefix-not-empty-length-gt-zero*:
assumes *strict-prefix xs ys* **and** *xs* $\neq$ []
shows *length ys* > *0*
using *assms prefix-not-empty* **by** *auto*

lemma *butlast-prefix-suffix-not-empty*:
assumes *strict-prefix* (*butlast xs*) *ys*
shows *ys* $\neq$ []
using *assms prefix-not-empty-length-gt-zero* **by** *fastforce*

lemma *prefix-and-concat-prefix-is-concat-prefix*:
assumes *prefix s t prefix* (*e* @ *t*) *u*
shows *prefix* (*e* @ *s*) *u*
using *Sublist.same-prefix-prefix assms*(*1*) *assms*(*2*) *prefix-order.dual-order.trans* **by** *blast*

lemma *prefix-eq-exists*:
prefix s t $\longleftrightarrow$ ($\exists$ *xs* . *s* @ *xs* = *t*)
using *prefix-def* **by** *auto*

lemma *strict-prefix-eq-exists*:
strict-prefix s t $\longleftrightarrow$ ($\exists$ *xs* . *s* @ *xs* = *t* $\wedge$ (*length xs*) > *0*)
using *prefix-def strict-prefix-def* **by** *auto*

lemma *butlast-strict-prefix-eq-butlast*:
assumes *length s* = *length t* **and** *strict-prefix* (*butlast s*) *t*

shows *strict-prefix* (*butlast s*) *t* $\longleftrightarrow$ (*butlast s*) = (*butlast t*)
by (*metis append-butlast-last-id append-eq-append-conv assms*(*1*) *assms*(*2*) *length-0-conv length-butlast strict-prefix-eq-exists*)

lemma *butlast-eq-if-eq-length-and-prefix*:
assumes *length s* > *0 length z* > *0*
length s = *length z strict-prefix* (*butlast s*) *t strict-prefix* (*butlast z*) *t*
shows (*butlast s*) = (*butlast z*)
using *assms* **by** (*auto simp add*:*strict-prefix-eq-exists*)

lemma *prefix-imp-length-lteq*:
assumes *prefix s t*
shows *length s* ≤ *length t*
using *assms* **by** (*simp add*: *Sublist.prefix-length-le*)

lemma *prefix-imp-length-not-gt*:
assumes *prefix s t*
shows ¬ *length t* < *length s*
using *assms* **by** (*simp add*: *Sublist.prefix-length-le leD*)

lemma *prefix-and-eq-length-imp-eq-list*:
assumes *prefix s t* **and** *length t* = *length s*
shows *s*=*t*
using *assms* **by** (*simp add*: *prefix-length-eq*)

lemma *butlast-prefix-imp-length-not-gt*:
assumes *length s* > *0 strict-prefix* (*butlast s*) *t*
shows ¬ (*length t* < *length s*)
using *assms prefix-length-less* **by** *fastforce*

lemma *length-not-gt-iff-eq-length*:
assumes *length s* > *0* **and** *strict-prefix* (*butlast s*) *t*
shows (¬ (*length s* < *length t*)) = (*length s* = *length t*)
proof −
have (¬ (*length s* < *length t*)) = ((*length t* < *length s*) ∨ (*length s* = *length t*))
by (*metis not-less-iff-gr-or-eq*)
also have ... = (*length s* = *length t*)
using *assms*
by (*simp add*:*butlast-prefix-imp-length-not-gt*)

finally show *?thesis* **.**
qed

Greatest common prefix

fun *gcp* :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
gcp [] *ys* = [] |
gcp (*x* # *xs*) (*y* # *ys*) = (*if* (*x* = *y*) *then x* # *gcp xs ys else* []) |
gcp - - = []

lemma *gcp-right* [*simp*]: *gcp xs* [] = []
by (*induct xs*, *auto*)

lemma *gcp-append* [*simp*]: *gcp* (*xs* @ *ys*) (*xs* @ *zs*) = *xs* @ *gcp ys zs*
by (*induct xs*, *auto*)

lemma *gcp-lb1*: *prefix* (*gcp xs ys*) *xs*
  apply (*induct xs arbitrary*: *ys*, *auto*)
  apply (*case-tac ys*, *auto*)
  done

lemma *gcp-lb2*: *prefix* (*gcp xs ys*) *ys*
  apply (*induct ys arbitrary*: *xs*, *auto*)
  apply (*case-tac xs*, *auto*)
  done

interpretation *prefix-semilattice*: *semilattice-inf gcp prefix strict-prefix*
proof
  fix *xs ys* :: *'a list*
  show *prefix* (*gcp xs ys*) *xs*
    by (*induct xs arbitrary*: *ys*, *auto*, *case-tac ys*, *auto*)
  show *prefix* (*gcp xs ys*) *ys*
    by (*induct ys arbitrary*: *xs*, *auto*, *case-tac xs*, *auto*)
next
  fix *xs ys zs* :: *'a list*
  assume *prefix xs ys prefix xs zs*
  thus *prefix xs* (*gcp ys zs*)
    by (*simp add*: *prefix-def*, *auto*)
qed

2.4.7 Lexicographic Order

lemma *lexord-append*:
  assumes $(xs_1 @ ys_1, xs_2 @ ys_2) \in$ *lexord R length*(xs_1) = *length*(xs_2)
  shows $(xs_1, xs_2) \in$ *lexord R* $\vee$ $(xs_1 = xs_2 \wedge (ys_1, ys_2) \in$ *lexord R*$)$
using *assms*
proof (*induct* xs_2 *arbitrary*: xs_1)
  case (*Cons* x_2 xs_2') **note** *hyps* = *this*
  from *hyps*(*3*) **obtain** x_1 xs_1' **where** xs_1: $xs_1 = x_1 \# xs_1'$ *length*(xs_1') = *length*(xs_2')
    by (*auto*, *metis Suc-length-conv*)
  with *hyps*(*2*) **have** *xcases*: $(x_1, x_2) \in R \vee (xs_1' @ ys_1, xs_2' @ ys_2) \in$ *lexord R*
    by (*auto*)
  show *?case*
  proof (*cases* $(x_1, x_2) \in R$)
    case *True* **with** xs_1 **show** *?thesis*
      by (*auto*)
  next
    case *False*
    with *xcases* **have** $(xs_1' @ ys_1, xs_2' @ ys_2) \in$ *lexord R*
      by (*auto*)
    with *hyps*(*1*) xs_1 **have** *dichot*: $(xs_1', xs_2') \in$ *lexord R* $\vee$ $(xs_1' = xs_2' \wedge (ys_1, ys_2) \in$ *lexord R*$)$
      by (*auto*)
    have $x_1 = x_2$
      using *False hyps*(*2*) $xs_1(1)$ **by** *auto*
    with *dichot* xs_1 **show** *?thesis*
      by (*simp*)
  qed
next
  case *Nil* **thus** *?case*
    by *auto*
qed

```
lemma strict-prefix-lexord-rel:
  strict-prefix xs ys ⟹ (xs, ys) ∈ lexord R
  by (metis Sublist.strict-prefixE' lexord-append-rightI)

lemma strict-prefix-lexord-left:
  assumes trans R (xs, ys) ∈ lexord R strict-prefix xs' xs
  shows (xs', ys) ∈ lexord R
  by (metis assms lexord-trans strict-prefix-lexord-rel)

lemma prefix-lexord-right:
  assumes trans R (xs, ys) ∈ lexord R strict-prefix ys ys'
  shows (xs, ys') ∈ lexord R
  by (metis assms lexord-trans strict-prefix-lexord-rel)

lemma lexord-eq-length:
  assumes (xs, ys) ∈ lexord R length xs = length ys
  shows ∃ i. (xs!i, ys!i) ∈ R ∧ i < length xs ∧ (∀ j<i. xs!j = ys!j)
using assms proof (induct xs arbitrary: ys)
  case (Cons x xs) note hyps = this
  then obtain y ys' where ys: ys = y # ys' length ys' = length xs
    by (metis Suc-length-conv)
  show ?case
  proof (cases (x, y) ∈ R)
    case True with ys show ?thesis
      by (rule-tac x=0 in exI, simp)
  next
    case False
    with ys hyps(2) have xy: x = y (xs, ys') ∈ lexord R
      by auto
    with hyps(1,3) ys obtain i where (xs!i, ys'!i) ∈ R i < length xs (∀ j<i. xs!j = ys'!j)
      by force
    with xy ys show ?thesis
      apply (rule-tac x=Suc i in exI)
      apply (auto simp add: less-Suc-eq-0-disj)
    done
  qed
next
  case Nil thus ?case by (auto)
qed

lemma lexord-intro-elems:
  assumes length xs > i length ys > i (xs!i, ys!i) ∈ R ∀ j<i. xs!j = ys!j
  shows (xs, ys) ∈ lexord R
using assms proof (induct i arbitrary: xs ys)
  case 0 thus ?case
    by (auto, metis lexord-cons-cons list.exhaust nth-Cons-0)
next
  case (Suc i) note hyps = this
  then obtain x' y' xs' ys' where xs = x' # xs' ys = y' # ys'
    by (metis Suc-length-conv Suc-lessE)
  moreover with hyps(5) have ∀j<i. xs' ! j = ys' ! j
    by (auto)
  ultimately show ?case using hyps
    by (auto)
qed
```

2.5 Distributed Concatenation

definition *uncurry* :: $('a \Rightarrow 'b \Rightarrow\ 'c) \Rightarrow ('a \times 'b \Rightarrow 'c)$ **where**
[*simp*]: *uncurry f* = $(\lambda(x, y).\ f\ x\ y)$

definition *dist-concat* ::
  $'a$ *list set* $\Rightarrow$ $'a$ *list set* $\Rightarrow$ $'a$ *list set* (**infixr** $\frown$ *100*) **where**
dist-concat ls1 ls2 = (*uncurry* (@) ‘ (*ls1* × *ls2*))

lemma *dist-concat-left-empty* [*simp*]:
  {} $\frown$ *ys* = {}
  by (*simp add*: *dist-concat-def*)

lemma *dist-concat-right-empty* [*simp*]:
  xs $\frown$ {} = {}
  by (*simp add*: *dist-concat-def*)

lemma *dist-concat-insert* [*simp*]:
insert l ls1 $\frown$ *ls2* = ((@) *l* ‘ (*ls2*)) ∪ (*ls1* $\frown$ *ls2*)
  by (*auto simp add*: *dist-concat-def*)

2.6 List Domain and Range

abbreviation *seq-dom* :: $'a$ *list* $\Rightarrow$ *nat set* (dom_l) **where**
seq-dom xs ≡ {*0*..<*length xs*}

abbreviation *seq-ran* :: $'a$ *list* $\Rightarrow$ $'a$ *set* (ran_l) **where**
seq-ran xs ≡ *set xs*

2.7 Extracting List Elements

definition *seq-extract* :: *nat set* $\Rightarrow$ $'a$ *list* $\Rightarrow$ $'a$ *list* (**infix** $\upharpoonright_l$ *80*) **where**
seq-extract A xs = *nths xs A*

lemma *seq-extract-Nil* [*simp*]: *A* $\upharpoonright_l$ [] = []
  by (*simp add*: *seq-extract-def*)

lemma *seq-extract-Cons*:
  A $\upharpoonright_l$ (*x* # *xs*) = (*if 0* ∈ *A then* [*x*] *else* []) @ {*j*. *Suc j* ∈ *A*} $\upharpoonright_l$ *xs*
  by (*simp add*: *seq-extract-def nths-Cons*)

lemma *seq-extract-empty* [*simp*]: {} $\upharpoonright_l$ *xs* = []
  by (*simp add*: *seq-extract-def*)

lemma *seq-extract-ident* [*simp*]: {*0*..<*length xs*} $\upharpoonright_l$ *xs* = *xs*
  unfolding *list-eq-iff-nth-eq*
  by (*auto simp add*: *seq-extract-def length-nths atLeast0LessThan*)

lemma *seq-extract-split*:
  assumes *i* ≤ *length xs*
  shows {*0*..<*i*} $\upharpoonright_l$ *xs* @ {*i*..<*length xs*} $\upharpoonright_l$ *xs* = *xs*
using *assms*
proof (*induct xs arbitrary*: *i*)
  case *Nil* **thus** *?case* **by** (*simp add*: *seq-extract-def*)
next
  case (*Cons x xs*) **note** *hyp* = *this*

have $\{j.\ Suc\ j < i\} = \{0..<i - 1\}$
  by (*auto*)
moreover have $\{j.\ i \leq Suc\ j \wedge j < length\ xs\} = \{i - 1..<length\ xs\}$
  by (*auto*)
ultimately show *?case*
  using *hyp* **by** (*force simp add*: *seq-extract-def nths-Cons*)
qed

lemma *seq-extract-append*:
$A \upharpoonright_l (xs\ @\ ys) = (A \upharpoonright_l xs)\ @\ (\{j.\ j + length\ xs \in A\} \upharpoonright_l ys)$
by (*simp add*: *seq-extract-def nths-append*)

lemma *seq-extract-range*: $A \upharpoonright_l xs = (A \cap dom_l(xs)) \upharpoonright_l xs$
apply (*auto simp add*: *seq-extract-def nths-def*)
apply (*metis* (*no-types*, *lifting*) *atLeastLessThan-iff filter-cong in-set-zip nth-mem set-upt*)
done

lemma *seq-extract-out-of-range*:
$A \cap dom_l(xs) = \{\} \Longrightarrow A \upharpoonright_l xs = []$
by (*metis seq-extract-def seq-extract-range nths-empty*)

lemma *seq-extract-length* [*simp*]:
$length\ (A \upharpoonright_l xs) = card\ (A \cap dom_l(xs))$
proof −
have $\{i.\ i < length(xs) \wedge i \in A\} = (A \cap \{0..<length(xs)\})$
  by (*auto*)
thus *?thesis*
  by (*simp add*: *seq-extract-def length-nths*)
qed

lemma *seq-extract-Cons-atLeastLessThan*:
assumes $m < n$
shows $\{m..<n\} \upharpoonright_l (x \# xs) = (if\ (m = 0)\ then\ x \# (\{0..<n-1\} \upharpoonright_l xs)\ else\ \{m-1..<n-1\} \upharpoonright_l xs)$
proof −
have $\{j.\ Suc\ j < n\} = \{0..<n - Suc\ 0\}$
  by (*auto*)
moreover have $\{j.\ m \leq Suc\ j \wedge Suc\ j < n\} = \{m - Suc\ 0..<n - Suc\ 0\}$
  by (*auto*)

ultimately show *?thesis* **using** *assms*
  by (*auto simp add*: *seq-extract-Cons*)
qed

lemma *seq-extract-singleton*:
assumes $i < length\ xs$
shows $\{i\} \upharpoonright_l xs = [xs\ !\ i]$
using *assms*
apply (*induct xs arbitrary*: *i*)
apply (*auto simp add*: *seq-extract-Cons*)
apply (*rename-tac xs i*)
apply (*subgoal-tac* $\{j.\ Suc\ j = i\} = \{i - 1\}$)
apply (*auto*)
done

lemma *seq-extract-as-map*:

assumes $m < n$ $n \leq$ *length xs*
shows $\{m..<n\} \upharpoonright_l xs = map\ (nth\ xs)\ [m..<n]$
using *assms* **proof** (*induct xs arbitrary*: *m n*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
have $[m..<n] = m \# [m+1..<n]$
using *Cons.prems(1) upt-eq-Cons-conv* **by** *blast*
moreover have $map\ (nth\ (x \# xs))\ [Suc\ m..<n] = map\ (nth\ xs)\ [m..<n-1]$
by (*simp add*: *map-nth-Cons-atLeastLessThan*)
ultimately show *?case*
using *Cons upt-rec*
by (*auto simp add*: *seq-extract-Cons-atLeastLessThan*)
qed

lemma *seq-append-as-extract*:
$xs = ys\ @\ zs \longleftrightarrow (\exists\ i \leq length(xs).\ ys = \{0..<i\} \upharpoonright_l xs \wedge zs = \{i..<length(xs)\} \upharpoonright_l xs)$
proof
assume *xs*: $xs = ys\ @\ zs$
moreover have $ys = \{0..<length\ ys\} \upharpoonright_l (ys\ @\ zs)$
by (*simp add*: *seq-extract-append*)
moreover have $zs = \{length\ ys..<length\ ys + length\ zs\} \upharpoonright_l (ys\ @\ zs)$
proof −
have $\{length\ ys..<length\ ys + length\ zs\} \cap \{0..<length\ ys\} = \{\}$
by *auto*
moreover have *s1*: $\{j.\ j < length\ zs\} = \{0..<length\ zs\}$
by *auto*
ultimately show *?thesis*
by (*simp add*: *seq-extract-append seq-extract-out-of-range*)
qed
ultimately show $(\exists\ i \leq length(xs).\ ys = \{0..<i\} \upharpoonright_l xs \wedge zs = \{i..<length(xs)\} \upharpoonright_l xs)$
by (*rule-tac x=length ys* **in** *exI*, *auto*)
next
assume $\exists\, i \leq length\ xs.\ ys = \{0..<i\} \upharpoonright_l xs \wedge zs = \{i..<length\ xs\} \upharpoonright_l xs$
thus $xs = ys\ @\ zs$
by (*auto simp add*: *seq-extract-split*)
qed

2.8 Filtering a list according to a set

definition *seq-filter* :: $'a\ list \Rightarrow\ 'a\ set \Rightarrow\ 'a\ list$ (**infix** $\restriction_l$ *80*) **where**
$seq\text{-}filter\ xs\ A = filter\ (\lambda\ x.\ x \in A)\ xs$

lemma *seq-filter-Cons-in* [*simp*]:
$x \in cs \Longrightarrow (x \# xs) \restriction_l cs = x \# (xs \restriction_l cs)$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-Cons-out* [*simp*]:
$x \notin cs \Longrightarrow (x \# xs) \restriction_l cs = (xs \restriction_l cs)$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-Nil* [*simp*]: $[] \restriction_l A = []$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-empty* [*simp*]: $xs \restriction_l \{\} = []$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-append*: $(xs\ @\ ys) \upharpoonright_l A = (xs \upharpoonright_l A)\ @\ (ys \upharpoonright_l A)$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-UNIV* [*simp*]: $xs \upharpoonright_l UNIV = xs$
by (*simp add*: *seq-filter-def*)

lemma *seq-filter-twice* [*simp*]: $(xs \upharpoonright_l A) \upharpoonright_l B = xs \upharpoonright_l (A \cap B)$
by (*simp add*: *seq-filter-def*)

2.9 Minus on lists

instantiation *list* :: (*type*) *minus*
begin

We define list minus so that if the second list is not a prefix of the first, then an arbitrary list longer than the combined length is produced. Thus we can always determined from the output whether the minus is defined or not.

definition $xs - ys = (if\ (prefix\ ys\ xs)\ then\ drop\ (length\ ys)\ xs\ else\ [])$

instance ..
end

lemma *minus-cancel* [*simp*]: $xs - xs = []$
by (*simp add*: *minus-list-def*)

lemma *append-minus* [*simp*]: $(xs\ @\ ys) - xs = ys$
by (*simp add*: *minus-list-def*)

lemma *minus-right-nil* [*simp*]: $xs - [] = xs$
by (*simp add*: *minus-list-def*)

lemma *list-concat-minus-list-concat*: $(s\ @\ t) - (s\ @\ z) = t - z$
by (*simp add*: *minus-list-def*)

lemma *length-minus-list*: $y \leq x \Longrightarrow length(x - y) = length(x) - length(y)$
by (*simp add*: *less-eq-list-def minus-list-def*)

lemma *map-list-minus*:
$xs \leq ys \Longrightarrow map\ f\ (ys - xs) = map\ f\ ys - map\ f\ xs$
by (*simp add*: *drop-map less-eq-list-def map-mono-prefix minus-list-def*)

lemma *list-minus-first-tl* [*simp*]:
$[x] \leq xs \Longrightarrow (xs - [x]) = tl\ xs$
by (*metis Prefix-Order.prefixE append.left-neutral append-minus list.sel*(*3*) *not-Cons-self2 tl-append2*)

Extra lemmas about *prefix* and *strict-prefix*

lemma *prefix-concat-minus*:
assumes *prefix xs ys*
shows $xs\ @\ (ys - xs) = ys$
using *assms* **by** (*metis minus-list-def prefix-drop*)

lemma *prefix-minus-concat*:
assumes *prefix s t*
shows $(t - s)\ @\ z = (t\ @\ z) - s$

using *assms* **by** (*simp add*: *Sublist.prefix-length-le minus-list-def*)

lemma *strict-prefix-minus-not-empty*:
assumes *strict-prefix xs ys*
shows $ys - xs \neq []$
using *assms* **by** (*metis append-Nil2 prefix-concat-minus strict-prefix-def*)

lemma *strict-prefix-diff-minus*:
assumes *prefix xs ys* **and** $xs \neq ys$
shows $(ys - xs) \neq []$
using *assms* **by** (*simp add*: *strict-prefix-minus-not-empty*)

lemma *length-tl-list-minus-butlast-gt-zero*:
assumes *length s* < *length t* **and** *strict-prefix* (*butlast s*) *t* **and** *length s* > *0*
shows *length* (*tl* (*t* − (*butlast s*))) > *0*
using *assms*
by (*metis Nitpick.size-list-simp*(*2*) *butlast-snoc hd-Cons-tl length-butlast length-greater-0-conv length-tl less-trans nat-neq-iff strict-prefix-minus-not-empty prefix-order.dual-order.strict-implies-order prefix-concat-minus*)

lemma *list-minus-butlast-eq-butlast-list*:
assumes *length t* = *length s* **and** *strict-prefix* (*butlast s*) *t*
shows *t* − (*butlast s*) = [*last t*]
using *assms*
by (*metis append-butlast-last-id append-eq-append-conv butlast.simps*(*1*) *length-butlast less-numeral-extra*(*3*) *list.size*(*3*) *prefix-order.dual-order.strict-implies-order prefix-concat-minus prefix-length-less*)

lemma *butlast-strict-prefix-length-lt-imp-last-tl-minus-butlast-eq-last*:
assumes *length s* > *0 strict-prefix* (*butlast s*) *t length s* < *length t*
shows *last* (*tl* (*t* − (*butlast s*))) = (*last t*)
using *assms* **by** (*metis last-append last-tl length-tl-list-minus-butlast-gt-zero less-numeral-extra*(*3*) *list.size*(*3*) *append-minus strict-prefix-eq-exists*)

lemma *tl-list-minus-butlast-not-empty*:
assumes *strict-prefix* (*butlast s*) *t* **and** *length s* > *0* **and** *length t* > *length s*
shows *tl* (*t* − (*butlast s*)) ≠ []
using *assms length-tl-list-minus-butlast-gt-zero* **by** *fastforce*

lemma *tl-list-minus-butlast-empty*:
assumes *strict-prefix* (*butlast s*) *t* **and** *length s* > *0* **and** *length t* = *length s*
shows *tl* (*t* − (*butlast s*)) = []
using *assms* **by** (*simp add*: *list-minus-butlast-eq-butlast-list*)

lemma *concat-minus-list-concat-butlast-eq-list-minus-butlast*:
assumes *prefix* (*butlast u*) *s*
shows (*t* @ *s*) − (*t* @ (*butlast u*)) = *s* − (*butlast u*)
using *assms* **by** (*metis append-assoc prefix-concat-minus append-minus*)

lemma *tl-list-minus-butlast-eq-empty*:
assumes *strict-prefix* (*butlast s*) *t* **and** *length s* = *length t*
shows *tl* (*t* − (*butlast s*)) = []
using *assms* **by** (*metis list.sel*(*3*) *list-minus-butlast-eq-butlast-list*)

lemma *prefix-length-tl-minus*:
assumes *strict-prefix s t*

shows *length* (*tl* (*t*−*s*)) = (*length* (*t*−*s*)) − *1*
by (*auto*)

lemma *length-list-minus*:
assumes *strict-prefix s t*
shows *length*(*t* − *s*) = *length*(*t*) − *length*(*s*)
using *assms* **by** (*simp add*: *minus-list-def prefix-order.dual-order.strict-implies-order*)

2.10 Laws on *take*, *drop*, and *nths*

lemma *take-prefix*: $m \leq n \Longrightarrow$ *take m xs* ≤ *take n xs*
by (*metis Prefix-Order.prefixI append-take-drop-id min-absorb2 take-append take-take*)

lemma *nths-atLeastAtMost-0-take*: *nths xs* $\{0..m\}$ = *take* (*Suc m*) *xs*
by (*metis atLeast0AtMost lessThan-Suc-atMost nths-upt-eq-take*)

lemma *nths-atLeastLessThan-0-take*: *nths xs* $\{0..<m\}$ = *take m xs*
by (*simp add*: *atLeast0LessThan*)

lemma *nths-atLeastAtMost-prefix*: $m \leq n \Longrightarrow$ *nths xs* $\{0..m\} \leq$ *nths xs* $\{0..n\}$
by (*simp add*: *nths-atLeastAtMost-0-take take-prefix*)

lemma *sorted-nths-atLeastAtMost-0*: ⟦ $m \leq n$; *sorted* (*nths xs* $\{0..n\}$) ⟧ $\Longrightarrow$ *sorted* (*nths xs* $\{0..m\}$)
using *nths-atLeastAtMost-prefix sorted-prefix* **by** *blast*

lemma *sorted-nths-atLeastLessThan-0*: ⟦ $m \leq n$; *sorted* (*nths xs* $\{0..<n\}$) ⟧ $\Longrightarrow$ *sorted* (*nths xs* $\{0..<m\}$)
by (*metis atLeast0LessThan nths-upt-eq-take sorted-prefix take-prefix*)

lemma *list-augment-as-update*:
$k <$ *length xs* $\Longrightarrow$ *list-augment xs k x* = *list-update xs k x*
by (*metis list-augment-def list-augment-idem list-update-overwrite*)

lemma *nths-list-update-out*: $k \notin A \Longrightarrow$ *nths* (*list-update xs k x*) *A* = *nths xs A*
apply (*induct xs arbitrary*: *k x A*)
apply (*auto*)
apply (*rename-tac a xs k x A*)
apply (*case-tac k*)
apply (*auto simp add*: *nths-Cons*)
done

lemma *nths-list-augment-out*: ⟦ $k <$ *length xs*; $k \notin A$ ⟧ $\Longrightarrow$ *nths* (*list-augment xs k x*) *A* = *nths xs A*
by (*simp add*: *list-augment-as-update nths-list-update-out*)

2.11 List power

overloading
listpow ≡ *compow* :: *nat* ⇒ *'a list* ⇒ *'a list*
begin

fun *listpow* :: *nat* ⇒ *'a list* ⇒ *'a list* **where**
listpow 0 xs = []
| *listpow* (*Suc n*) *xs* = *xs* @ *listpow n xs*

end

lemma *listpow-Nil* [*simp*]: [] ^^ *n* = []

by (*induct n*) *simp-all*

lemma *listpow-Suc-right*: *xs ^^ Suc n = xs ^^ n @ xs*
by (*induct n*) *simp-all*

lemma *listpow-add*: *xs ^^ (m + n) = xs ^^ m @ xs ^^ n*
by (*induct m*) *simp-all*

end

3 Infinite Sequences

theory *Sequence*
imports
HOL.Real
List-Extra
HOL−Library.Sublist
HOL−Library.Nat-Bijection
begin

typedef *'a seq = UNIV :: (nat ⇒ 'a) set*
by (*auto*)

setup-lifting *type-definition-seq*

definition *ssubstr :: nat ⇒ nat ⇒ 'a seq ⇒ 'a list* **where**
ssubstr i j xs = map (Rep-seq xs) [i ..< j]

lift-definition *nth-seq :: 'a seq ⇒ nat ⇒ 'a* (**infixl** $!_s$ *100*)
is *λ f i. f i* **.**

abbreviation *sinit :: nat ⇒ 'a seq ⇒ 'a list* **where**
sinit i xs ≡ ssubstr 0 i xs

lemma *sinit-len* [*simp*]:
length (sinit i xs) = i
by (*simp add: ssubstr-def*)

lemma *sinit-0* [*simp*]: *sinit 0 xs = []*
by (*simp add: ssubstr-def*)

lemma *prefix-upt-0* [*intro*]:
i ≤ j ⟹ prefix [0..<i] [0..<j]
by (*induct i, auto, metis append-prefixD le0 prefix-order.lift-Suc-mono-le prefix-order.order-refl upt-Suc*)

lemma *sinit-prefix*:
i ≤ j ⟹ prefix (sinit i xs) (sinit j xs)
by (*simp add: map-mono-prefix prefix-upt-0 ssubstr-def*)

lemma *sinit-strict-prefix*:
i < j ⟹ strict-prefix (sinit i xs) (sinit j xs)
by (*metis sinit-len sinit-prefix le-less nat-neq-iff prefix-order.dual-order.strict-iff-order*)

lemma *nth-sinit*:
i < n ⟹ sinit n xs ! i = xs $!_s$ *i*

apply (*auto simp add*: *ssubstr-def*)
apply (*transfer*, *auto*)
done

lemma *sinit-append-split*:
assumes $i < j$
shows *sinit j xs* = *sinit i xs* @ *ssubstr i j xs*
proof −
have $[0..<i]$ @ $[i..<j] = [0..<j]$
by (*metis assms le0 le-add-diff-inverse le-less upt-add-eq-append*)
thus *?thesis*
by (*auto simp add*: *ssubstr-def*, *transfer*, *simp add*: *map-append*[*THEN sym*])
qed

lemma *sinit-linear-asym-lemma1*:
assumes *asym R* $i < j$ (*sinit i xs*, *sinit i ys*) ∈ *lexord R* (*sinit j ys*, *sinit j xs*) ∈ *lexord R*
shows *False*
proof −
have *sinit-xs*: *sinit j xs* = *sinit i xs* @ *ssubstr i j xs*
by (*metis assms*(*2*) *sinit-append-split*)
have *sinit-ys*: *sinit j ys* = *sinit i ys* @ *ssubstr i j ys*
by (*metis assms*(*2*) *sinit-append-split*)
from *sinit-xs sinit-ys assms*(*4*)
have (*sinit i ys*, *sinit i xs*) ∈ *lexord R* ∨ (*sinit i ys* = *sinit i xs* ∧ (*ssubstr i j ys*, *ssubstr i j xs*) ∈ *lexord R*)
by (*auto dest*: *lexord-append*)
with *assms lexord-asymmetric* **show** *False*
by (*force*)
qed

lemma *sinit-linear-asym-lemma2*:
assumes *asym R* (*sinit i xs*, *sinit i ys*) ∈ *lexord R* (*sinit j ys*, *sinit j xs*) ∈ *lexord R*
shows *False*
proof (*cases i j rule*: *linorder-cases*)
case *less* **with** *assms* **show** *?thesis*
by (*auto dest*: *sinit-linear-asym-lemma1*)
next
case *equal* **with** *assms* **show** *?thesis*
by (*simp add*: *lexord-asymmetric*)
next
case *greater* **with** *assms* **show** *?thesis*
by (*auto dest*: *sinit-linear-asym-lemma1*)
qed

lemma *range-ext*:
assumes $\forall\, i :: nat.\ \forall\, x{\in}\{0..<i\}.\ f(x) = g(x)$
shows $f = g$
proof (*rule ext*)
fix $x :: nat$
obtain $i :: nat$ **where** $i > x$
by (*metis lessI*)
with *assms* **show** $f(x) = g(x)$
by (*auto*)
qed

lemma *sinit-ext*:
  $(\forall\ i.\ sinit\ i\ xs = sinit\ i\ ys) \Longrightarrow xs = ys$
  by (*simp add*: *ssubstr-def*, *transfer*, *auto intro*: *range-ext*)

definition *seq-lexord* :: $'a\ rel \Rightarrow ('a\ seq)\ rel$ **where**
$seq\text{-}lexord\ R = \{(xs, ys).\ (\exists\ i.\ (sinit\ i\ xs, sinit\ i\ ys) \in lexord\ R)\}$

lemma *seq-lexord-irreflexive*:
  $\forall x.\ (x, x) \notin R \Longrightarrow (xs, xs) \notin seq\text{-}lexord\ R$
  by (*auto dest*: *lexord-irreflexive simp add*: *irrefl-def seq-lexord-def*)

lemma *seq-lexord-irrefl*:
  $irrefl\ R \Longrightarrow irrefl\ (seq\text{-}lexord\ R)$
  by (*simp add*: *irrefl-def seq-lexord-irreflexive*)

lemma *seq-lexord-transitive*:
  assumes *trans R*
  shows *trans* (*seq-lexord R*)
unfolding *seq-lexord-def*
proof (*rule transI*, *clarify*)
  fix *xs ys zs* :: $'a\ seq$ **and** *m n* :: *nat*
  assume *las*: $(sinit\ m\ xs, sinit\ m\ ys) \in lexord\ R\ (sinit\ n\ ys, sinit\ n\ zs) \in lexord\ R$
  hence *inz*: $m > 0$
    using *gr0I* **by** *force*
  from *las*(*1*) **obtain** *i* **where** *sinitm*: $(sinit\ m\ xs!i, sinit\ m\ ys!i) \in R\ i < m\ \forall\ j{<}i.\ sinit\ m\ xs!j = sinit\ m\ ys!j$
    using *lexord-eq-length* **by** *force*
  from *las*(*2*) **obtain** *j* **where** *sinitn*: $(sinit\ n\ ys!j, sinit\ n\ zs!j) \in R\ j < n\ \forall\ k{<}j.\ sinit\ n\ ys!k = sinit\ n\ zs!k$
    using *lexord-eq-length* **by** *force*
  show $\exists i.\ (sinit\ i\ xs, sinit\ i\ zs) \in lexord\ R$
  proof (*cases* $i \le j$)
    case *True* **note** *lt* = *this*
    with *sinitm sinitn* **have** $(sinit\ n\ xs!i, sinit\ n\ zs!i) \in R$
      by (*metis assms le-eq-less-or-eq le-less-trans nth-sinit transD*)
    moreover from *lt sinitm sinitn* **have** $\forall\ j{<}i.\ sinit\ m\ xs!j = sinit\ m\ zs!j$
      by (*metis less-le-trans less-trans nth-sinit*)
    ultimately have $(sinit\ n\ xs, sinit\ n\ zs) \in lexord\ R$ **using** *sinitm*(*2*) *sinitn*(*2*) *lt*
      apply (*rule-tac lexord-intro-elems*)
        apply (*auto*)
      apply (*metis less-le-trans less-trans nth-sinit*)
      done
    thus *?thesis* **by** *auto*
  next
    case *False*
    then have *ge*: $i > j$ **by** *auto*
    with *assms sinitm sinitn* **have** $(sinit\ n\ xs!j, sinit\ n\ zs!j) \in R$
      by (*metis less-trans nth-sinit*)
    moreover from *ge sinitm sinitn* **have** $\forall\ k{<}j.\ sinit\ m\ xs!k = sinit\ m\ zs!k$
      by (*metis dual-order.strict-trans nth-sinit*)
    ultimately have $(sinit\ n\ xs, sinit\ n\ zs) \in lexord\ R$ **using** *sinitm*(*2*) *sinitn*(*2*) *ge*
      apply (*rule-tac lexord-intro-elems*)
        apply (*auto*)
      apply (*metis less-trans nth-sinit*)
      done

thus *?thesis* **by** *auto*
qed
qed

lemma *seq-lexord-trans*:
$[\![$ $(xs, ys) \in$ *seq-lexord* R; $(ys, zs) \in$ *seq-lexord* R; *trans* R $]\!] \Longrightarrow (xs, zs) \in$ *seq-lexord* R
by (*meson seq-lexord-transitive transE*)

lemma *seq-lexord-antisym*:
$[\![$ *asym* R; $(a, b) \in$ *seq-lexord* R $]\!] \Longrightarrow (b, a) \notin$ *seq-lexord* R
by (*auto dest*: *sinit-linear-asym-lemma2 simp add*: *seq-lexord-def*)

lemma *seq-lexord-asym*:
assumes *asym* R
shows *asym* (*seq-lexord* R)
by (*meson assms asym.simps seq-lexord-antisym seq-lexord-irrefl*)

lemma *seq-lexord-total*:
assumes *total* R
shows *total* (*seq-lexord* R)
using *assms* **by** (*auto simp add*: *total-on-def seq-lexord-def*, *meson lexord-linear sinit-ext*)

lemma *seq-lexord-strict-linear-order*:
assumes *strict-linear-order* R
shows *strict-linear-order* (*seq-lexord* R)
using *assms*
by (*auto simp add*: *strict-linear-order-on-def partial-order-on-def preorder-on-def*
intro: *seq-lexord-transitive seq-lexord-irrefl seq-lexord-total*)

lemma *seq-lexord-linear*:
assumes $(\forall\ a\ b.\ (a,b) \in R \lor a = b \lor (b,a) \in R)$
shows $(x,y) \in$ *seq-lexord* $R \lor x = y \lor (y,x) \in$ *seq-lexord* R
proof −
have *total* R
using *assms total-on-def* **by** *blast*
hence *total* (*seq-lexord* R)
using *seq-lexord-total* **by** *blast*
thus *?thesis*
by (*auto simp add*: *total-on-def*)
qed

instantiation *seq* :: (*ord*) *ord*
begin

definition *less-seq* :: $'a$ *seq* $\Rightarrow$ $'a$ *seq* $\Rightarrow$ *bool* **where**
less-seq xs ys $\longleftrightarrow$ $(xs, ys) \in$ *seq-lexord* $\{(xs, ys).\ xs < ys\}$

definition *less-eq-seq* :: $'a$ *seq* $\Rightarrow$ $'a$ *seq* $\Rightarrow$ *bool* **where**
less-eq-seq xs ys $= (xs = ys \lor xs < ys)$

instance ..

end

instance *seq* :: (*order*) *order*

proof
  fix *xs* :: *'a seq*
  show $xs \leq xs$ **by** (*simp add*: *less-eq-seq-def*)
next
  fix *xs ys zs* :: *'a seq*
  assume $xs \leq ys$ **and** $ys \leq zs$
  then show $xs \leq zs$
    by (*force dest*: *seq-lexord-trans simp add*: *less-eq-seq-def less-seq-def trans-def*)
next
  fix *xs ys* :: *'a seq*
  assume $xs \leq ys$ **and** $ys \leq xs$
  then show $xs = ys$
    apply (*auto simp add*: *less-eq-seq-def less-seq-def*)
    apply (*rule seq-lexord-irreflexive* [*THEN notE*])
     defer
     apply (*rule seq-lexord-trans*)
       apply (*auto intro*: *transI*)
    done
next
  fix *xs ys* :: *'a seq*
  show $xs < ys \longleftrightarrow xs \leq ys \land \neg\, ys \leq xs$
    apply (*auto simp add*: *less-seq-def less-eq-seq-def*)
     defer
     apply (*rule seq-lexord-irreflexive* [*THEN notE*])
      apply *auto*
     apply (*rule seq-lexord-irreflexive* [*THEN notE*])
      defer
      apply (*rule seq-lexord-trans*)
        apply (*auto intro*: *transI*)
    apply (*simp add*: *seq-lexord-irreflexive*)
    done
qed

instance *seq* :: (*linorder*) *linorder*
proof
  fix *xs ys* :: *'a seq*
  have $(xs,\ ys) \in$ *seq-lexord* $\{(u,\ v).\ u < v\} \lor xs = ys \lor (ys,\ xs) \in$ *seq-lexord* $\{(u,\ v).\ u < v\}$
    by (*rule seq-lexord-linear*) *auto*
  then show $xs \leq ys \lor ys \leq xs$
    by (*auto simp add*: *less-eq-seq-def less-seq-def*)
qed

lemma *seq-lexord-mono* [*mono*]:
  $(\bigwedge x\ y.\ f\ x\ y \longrightarrow g\ x\ y) \Longrightarrow (xs,\ ys) \in$ *seq-lexord* $\{(x,\ y).\ f\ x\ y\} \longrightarrow (xs,\ ys) \in$ *seq-lexord* $\{(x,\ y).\ g\ x\ y\}$
  apply (*auto simp add*: *seq-lexord-def*)
  apply (*metis case-prodD case-prodI lexord-take-index-conv mem-Collect-eq*)
done

fun *insort-rel* :: *'a rel* ⇒ *'a* ⇒ *'a list* ⇒ *'a list* **where**
insort-rel $R\ x\ [] = [x]$ |
insort-rel $R\ x\ (y\ \#\ ys) = ($*if* $(x = y \lor (x,y) \in R)$ *then* $x\ \#\ y\ \#\ ys$ *else* $y\ \#$ *insort-rel* $R\ x\ ys)$

inductive *sorted-rel* :: *'a rel* ⇒ *'a list* ⇒ *bool* **where**
Nil-rel [*iff*]: *sorted-rel* R [] |

Cons-rel: ∀ y ∈ set xs. (x = y ∨ (x, y) ∈ R) ⟹ sorted-rel R xs ⟹ sorted-rel R (x # xs)

definition *list-of-set* :: 'a rel ⇒ 'a set ⇒ 'a list **where**
list-of-set R = folding.F (insort-rel R) []

lift-definition *seq-inj* :: 'a seq seq ⇒ 'a seq **is**
λ f i. f (fst (prod-decode i)) (snd (prod-decode i)) **.**

lift-definition *seq-proj* :: 'a seq ⇒ 'a seq seq **is**
λ f i j. f (prod-encode (i, j)) **.**

lemma *seq-inj-inverse*: *seq-proj (seq-inj x) = x*
  by (*transfer*, *simp*)

lemma *seq-proj-inverse*: *seq-inj (seq-proj x) = x*
  by (*transfer*, *simp*)

lemma *seq-inj*: *inj seq-inj*
  by (*metis injI seq-inj-inverse*)

lemma *seq-inj-surj*: *bij seq-inj*
  apply (*rule bijI*)
   apply (*auto simp add*: *seq-inj*)
  apply (*metis rangeI seq-proj-inverse*)
  done
end

4 Finite Sets: extra functions and properties

theory *FSet-Extra*
imports
  HOL−Library.FSet
  HOL−Library.Countable-Set-Type
begin

setup-lifting *type-definition-fset*

notation *fempty* ({||})
notation *fset* (⟨-⟩$_f$)
notation *fminus* (**infixl** $-_f$ *65*)

syntax
  -FinFset :: *args => 'a fset*   ({|(-)|})

translations
  {|x, xs|} == *CONST finsert x* {|xs|}
  {|x|} == *CONST finsert x* {||}

term *fBall*

syntax
  -fBall :: *pttrn => 'a fset => bool => bool* ((*3*∀ -|∈|-./ -) [*0*, *0*, *10*] *10*)
  -fBex :: *pttrn => 'a fset => bool => bool* ((*3*∃ -|∈|-./ -) [*0*, *0*, *10*] *10*)

translations

$\forall\ x|\in|A.\ P == CONST\ fBall\ A\ (\%x.\ P)$
$\exists\ x|\in|A.\ P == CONST\ fBex\ A\ (\%x.\ P)$

definition $FUnion :: 'a\ fset\ fset \Rightarrow 'a\ fset\ (\bigcup_f\text{-}\ [90]\ 90)$ **where**
$FUnion\ xs = Abs\text{-}fset\ (\bigcup x \in \langle xs\rangle_f.\ \langle x\rangle_f)$

definition $FInter :: 'a\ fset\ fset \Rightarrow 'a\ fset\ (\bigcap_f\text{-}\ [90]\ 90)$ **where**
$FInter\ xs = Abs\text{-}fset\ (\bigcap x \in \langle xs\rangle_f.\ \langle x\rangle_f)$

Finite power set

definition $FinPow :: 'a\ fset \Rightarrow 'a\ fset\ fset$ **where**
$FinPow\ xs = Abs\text{-}fset\ (Abs\text{-}fset\ `\ Pow\ \langle xs\rangle_f)$

Set of all finite subsets of a set

definition $Fow :: 'a\ set \Rightarrow 'a\ fset\ set$ **where**
$Fow\ A = \{x.\ \langle x\rangle_f \subseteq A\}$

declare *Abs-fset-inverse* [*simp*]

lemma *fset-intro*:
$fset\ x = fset\ y \Longrightarrow x = y$
by (*simp add:fset-inject*)

lemma *fset-elim*:
$[\![\ x = y;\ fset\ x = fset\ y \Longrightarrow P\ ]\!] \Longrightarrow P$
by (*auto*)

lemma *fmember-intro*:
$[\![\ x \in fset(xs)\ ]\!] \Longrightarrow x\ |\in|\ xs$
by (*metis fmember.rep-eq*)

lemma *fmember-elim*:
$[\![\ x\ |\in|\ xs;\ x \in fset(xs) \Longrightarrow P\ ]\!] \Longrightarrow P$
by (*metis fmember.rep-eq*)

lemma *fnmember-intro* [*intro*]:
$[\![\ x \notin fset(xs)\ ]\!] \Longrightarrow x\ |\notin|\ xs$
by (*metis fmember.rep-eq*)

lemma *fnmember-elim* [*elim*]:
$[\![\ x\ |\notin|\ xs;\ x \notin fset(xs) \Longrightarrow P\ ]\!] \Longrightarrow P$
by (*metis fmember.rep-eq*)

lemma *fsubset-intro* [*intro*]:
$\langle xs\rangle_f \subseteq \langle ys\rangle_f \Longrightarrow xs\ |\subseteq|\ ys$
by (*metis less-eq-fset.rep-eq*)

lemma *fsubset-elim* [*elim*]:
$[\![\ xs\ |\subseteq|\ ys;\ \langle xs\rangle_f \subseteq \langle ys\rangle_f \Longrightarrow P\ ]\!] \Longrightarrow P$
by (*metis less-eq-fset.rep-eq*)

lemma *fBall-intro* [*intro*]:
$Ball\ \langle A\rangle_f\ P \Longrightarrow fBall\ A\ P$
by (*metis* (*poly-guards-query*) *fBallI fmember.rep-eq*)

lemma *fBall-elim* [*elim*]:
  ⟦ *fBall A P*; *Ball* $\langle A \rangle_f$ *P* ⟹ *Q* ⟧ ⟹ *Q*
  by (*metis fBallE fmember.rep-eq*)

lift-definition *finset* :: ′*a list* ⇒ ′*a fset* **is** *set* **..**

context *linorder*
begin

lemma *sorted-list-of-set-inj*:
  ⟦ *finite xs*; *finite ys*; *sorted-list-of-set xs* = *sorted-list-of-set ys* ⟧
  ⟹ *xs* = *ys*
  apply (*simp add*:*sorted-list-of-set-def*)
  apply (*induct xs rule*:*finite-induct*)
   apply (*induct ys rule*:*finite-induct*)
    apply (*simp-all*)
  apply (*metis finite.insertI insert-not-empty sorted-list-of-set-def sorted-list-of-set-empty sorted-list-of-set-eq-Nil-iff*)
 apply (*metis finite.insertI finite-list set-remdups set-sort sorted-list-of-set-def sorted-list-of-set-sort-remdups*)
  done

definition *flist* :: ′*a fset* ⇒ ′*a list* **where**
flist xs = *sorted-list-of-set* (*fset xs*)

lemma *flist-inj*: *inj flist*
  apply (*simp add*:*flist-def inj-on-def*)
  apply (*clarify*)
  apply (*rename-tac x y*)
  apply (*subgoal-tac fset x* = *fset y*)
   apply (*simp add*:*fset-inject*)
  apply (*rule sorted-list-of-set-inj*, *simp-all*)
  done

lemma *flist-props* [*simp*]:
  sorted (*flist xs*)
  distinct (*flist xs*)
  by (*simp-all add*:*flist-def*)

lemma *flist-empty* [*simp*]:
  flist {||} = []
  by (*simp add*:*flist-def*)

lemma *flist-inv* [*simp*]: *finset* (*flist xs*) = *xs*
  by (*simp add*:*finset-def flist-def fset-inverse*)

lemma *flist-set* [*simp*]: *set* (*flist xs*) = *fset xs*
  by (*simp add*:*finset-def flist-def fset-inverse*)

lemma *fset-inv* [*simp*]: ⟦ *sorted xs*; *distinct xs* ⟧ ⟹ *flist* (*finset xs*) = *xs*
  apply (*simp add*:*finset-def flist-def fset-inverse*)
  apply (*metis local.sorted-list-of-set-sort-remdups local.sorted-sort-id remdups-id-iff-distinct*)
  done

lemma *fcard-flist*:
  fcard xs = *length* (*flist xs*)
  apply (*simp add*:*fcard-def*)

apply (*fold flist-set*)
apply (*unfold distinct-card*[*OF flist-props*(*2*)])
apply (*rule refl*)
done

lemma *flist-nth*:
$i < fcard\ vs \Longrightarrow flist\ vs\ !\ i\ |{\in}|\ vs$
apply (*simp add*: *fmember-def flist-def fcard-def*)
apply (*metis fcard.rep-eq fcard-flist finset.rep-eq flist-def flist-inv nth-mem*)
done

definition *fmax* :: $'a\ fset \Rightarrow 'a$ **where**
$fmax\ xs = (if\ (xs = \{||\})\ then\ undefined\ else\ last\ (flist\ xs))$

end

definition *flists* :: $'a\ fset \Rightarrow 'a\ list\ set$ **where**
$flists\ A = \{xs.\ distinct\ xs \wedge finset\ xs = A\}$

lemma *flists-nonempty*: $\exists\ xs.\ xs \in flists\ A$
apply (*simp add*: *flists-def*)
apply (*metis Abs-fset-cases Abs-fset-inverse finite-distinct-list finite-fset finset.rep-eq*)
done

lemma *flists-elem-uniq*: $[\![\ x \in flists\ A;\ x \in flists\ B\ ]\!] \Longrightarrow A = B$
by (*simp add*: *flists-def*)

definition *flist-arb* :: $'a\ fset \Rightarrow 'a\ list$ **where**
$flist\text{-}arb\ A = (SOME\ xs.\ xs \in flists\ A)$

lemma *flist-arb-distinct* [*simp*]: *distinct* (*flist-arb A*)
by (*metis* (*mono-tags*) *flist-arb-def flists-def flists-nonempty mem-Collect-eq someI-ex*)

lemma *flist-arb-inv* [*simp*]: *finset* (*flist-arb A*) = *A*
by (*metis* (*mono-tags*) *flist-arb-def flists-def flists-nonempty mem-Collect-eq someI-ex*)

lemma *flist-arb-inj*:
inj flist-arb
by (*metis flist-arb-inv injI*)

lemma *flist-arb-lists*: $flist\text{-}arb\ `\ Fow\ A \subseteq lists\ A$
apply (*auto*)
using *Fow-def finset.rep-eq* **apply** *fastforce*
done

lemma *countable-Fow*:
fixes *A* :: *'a set*
assumes *countable A*
shows *countable* (*Fow A*)
proof –
from *assms* **obtain** *to-nat-list* :: $'a\ list \Rightarrow nat$ **where** *inj-on to-nat-list* (*lists A*)
by *blast*
thus *?thesis*
apply (*simp add*: *countable-def*)
apply (*rule-tac x=to-nat-list* $\circ$ *flist-arb* **in** *exI*)

apply (*rule comp-inj-on*)
 apply (*metis flist-arb-inv inj-on-def*)
apply (*simp add*: *flist-arb-lists subset-inj-on*)
done
qed

definition *flub* :: $'a$ *fset set* $\Rightarrow$ $'a$ *fset* $\Rightarrow$ $'a$ *fset* **where**
flub A t = (*if* ($\forall$ $a{\in}A$. a $|\subseteq|$ t) *then Abs-fset* ($\bigcup x{\in}A$. $\langle x\rangle_f$) *else t*)

lemma *finite-Union-subsets*:
 $[\![$ $\forall$ $a \in A$. $a \subseteq b$; *finite b* $]\!]$ $\Longrightarrow$ *finite* ($\bigcup A$)
 by (*metis Sup-le-iff finite-subset*)

lemma *finite-UN-subsets*:
 $[\![$ $\forall$ $a \in A$. $B\ a \subseteq b$; *finite b* $]\!]$ $\Longrightarrow$ *finite* ($\bigcup a{\in}A$. $B\ a$)
 by (*metis UN-subset-iff finite-subset*)

lemma *flub-rep-eq*:
 $\langle$*flub A t*$\rangle_f$ = (*if* ($\forall$ $a{\in}A$. a $|\subseteq|$ t) *then* ($\bigcup x{\in}A$. $\langle x\rangle_f$) *else* $\langle t\rangle_f$)
 apply (*subgoal-tac* (*if* ($\forall$ $a{\in}A$. a $|\subseteq|$ t) *then* ($\bigcup x{\in}A$. $\langle x\rangle_f$) *else* $\langle t\rangle_f$) $\in$ {*x*. *finite x*})
  apply (*auto simp add*:*flub-def*)
 apply (*rule finite-UN-subsets*[*of* - - $\langle t\rangle_f$])
  apply (*auto*)
 done

definition *fglb* :: $'a$ *fset set* $\Rightarrow$ $'a$ *fset* $\Rightarrow$ $'a$ *fset* **where**
fglb A t = (*if* (A = {}) *then t else Abs-fset* ($\bigcap x{\in}A$. $\langle x\rangle_f$))

lemma *fglb-rep-eq*:
 $\langle$*fglb A t*$\rangle_f$ = (*if* (A = {}) *then* $\langle t\rangle_f$ *else* ($\bigcap x{\in}A$. $\langle x\rangle_f$))
 apply (*subgoal-tac* (*if* (A = {}) *then* $\langle t\rangle_f$ *else* ($\bigcap x{\in}A$. $\langle x\rangle_f$)) $\in$ {*x*. *finite x*})
  apply (*metis Abs-fset-inverse fglb-def*)
 apply (*auto*)
 apply (*metis finite-INT finite-fset*)
 done

lemma *FinPow-rep-eq* [*simp*]:
 fset (*FinPow xs*) = {*ys*. *ys* $|\subseteq|$ *xs*}
 apply (*subgoal-tac finite* (*Abs-fset* ‘ *Pow* $\langle xs\rangle_f$))
  apply (*auto simp add*: *fmember-def FinPow-def*)
  apply (*rename-tac* x' y')
  apply (*subgoal-tac finite* x')
   apply (*auto*)
  apply (*metis finite-fset finite-subset*)
 apply (*metis* (*full-types*) *Pow-iff fset-inverse imageI less-eq-fset.rep-eq*)
 done

lemma *FUnion-rep-eq* [*simp*]:
 $\langle\bigcup_f\ xs\rangle_f$ = ($\bigcup x{\in}\langle xs\rangle_f$. $\langle x\rangle_f$)
 by (*simp add*:*FUnion-def*)

lemma *FInter-rep-eq* [*simp*]:
 $xs \neq \{\!|\!|\!\}$ $\Longrightarrow$ $\langle\bigcap_f\ xs\rangle_f$ = ($\bigcap x{\in}\langle xs\rangle_f$. $\langle x\rangle_f$)
 apply (*simp add*:*FInter-def*)
 apply (*subgoal-tac finite* ($\bigcap x{\in}\langle xs\rangle_f$. $\langle x\rangle_f$))

apply (*simp*)
apply (*metis* (*poly-guards-query*) *bot-fset.rep-eq fglb-rep-eq finite-fset fset-inverse*)
done

lemma *FUnion-empty* [*simp*]:
$\bigcup_f$ {||} = {||}
by (*auto simp add*:*FUnion-def fmember-def*)

lemma *FinPow-member* [*simp*]:
xs |∈| *FinPow xs*
by (*auto simp add*:*fmember-def*)

lemma *FUnion-FinPow* [*simp*]:
$\bigcup_f$ (*FinPow x*) = *x*
by (*auto simp add*:*fmember-def less-eq-fset-def*)

lemma *Fow-mem* [*iff*]: $x \in Fow\ A \longleftrightarrow \langle x\rangle_f \subseteq A$
by (*auto simp add*:*Fow-def*)

lemma *Fow-UNIV* [*simp*]: *Fow UNIV* = *UNIV*
by (*simp add*:*Fow-def*)

lift-definition *FMax* :: (*'a*::*linorder*) *fset* ⇒ *'a* **is** *Max* **.**

end

5 Countable Sets: Extra functions and properties

theory *Countable-Set-Extra*
imports
HOL−Library.Countable-Set-Type
Sequence
FSet-Extra
HOL−Library.Bit
begin

5.1 Extra syntax

notation *cempty* ($\{\}_c$)
notation *cin* (**infix** $\in_c$ *50*)
notation *cUn* (**infixl** $\cup_c$ *65*)
notation *cInt* (**infixl** $\cap_c$ *70*)
notation *cDiff* (**infixl** $-_c$ *65*)
notation *cUnion* ($\bigcup_c$- [*900*] *900*)
notation *cimage* (**infixr** $`_c$ *90*)

abbreviation *csubseteq* :: *'a cset* ⇒ *'a cset* ⇒ *bool* ((-/ $\subseteq_c$ -) [*51*, *51*] *50*)
where $A \subseteq_c B \equiv A \leq B$

abbreviation *csubset* :: *'a cset* ⇒ *'a cset* ⇒ *bool* ((-/ $\subset_c$ -) [*51*, *51*] *50*)
where $A \subset_c B \equiv A < B$

5.2 Countable set functions

setup-lifting *type-definition-cset*

lift-definition *cnin* :: $'a \Rightarrow 'a\ cset \Rightarrow bool$ (**infix** $\notin_c$ *50*) **is** ($\notin$) **.**

definition *cBall* :: $'a\ cset \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$ **where**
$cBall\ A\ P = (\forall x.\ x \in_c A \longrightarrow P\ x)$

definition *cBex* :: $'a\ cset \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$ **where**
$cBex\ A\ P = (\exists x.\ x \in_c A \longrightarrow P\ x)$

declare *cBall-def* [*mono*,*simp*]
declare *cBex-def* [*mono*,*simp*]

syntax
  -cBall :: *pttrn* => *'a cset* => *bool* => *bool* ((*3*∀ -$\in_c$-./ -) [*0*, *0*, *10*] *10*)
  -cBex :: *pttrn* => *'a cset* => *bool* => *bool* ((*3*∃ -$\in_c$-./ -) [*0*, *0*, *10*] *10*)

translations
  ∀ $x \in_c A$. *P* == *CONST cBall A* (%*x*. *P*)
  ∃ $x \in_c A$. *P* == *CONST cBex A* (%*x*. *P*)

definition *cset-Collect* :: $('a \Rightarrow bool) \Rightarrow 'a\ cset$ **where**
cset-Collect = (*acset o Collect*)

lift-definition *cset-Coll* :: $'a\ cset \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a\ cset$ **is** $\lambda\ A\ P.\ \{x \in A.\ P\ x\}$
  by (*auto*)

lemma *cset-Coll-equiv*: *cset-Coll A P* = *cset-Collect* ($\lambda\ x.\ x \in_c A \wedge P\ x$)
  by (*simp add*:*cset-Collect-def cset-Coll-def cin-def*)

declare *cset-Collect-def* [*simp*]

syntax
  -cColl :: *pttrn* => *bool* => *'a cset* ((*1*{-./ -}$_c$))

translations
  $\{x\ .\ P\}_c \rightleftharpoons$ (*CONST cset-Collect*) ($\lambda\ x\ .\ P$)

syntax (*xsymbols*)
  -cCollect :: *pttrn* => *'a cset* => *bool* => *'a cset* ((*1*{- $\in_c$/ -./ -}$_c$))
translations
  $\{x \in_c A.\ P\}_c$ => *CONST cset-Coll A* ($\lambda\ x.\ P$)

lemma *cset-CollectI*: *P* (*a* :: *'a*::*countable*) $\Longrightarrow a \in_c \{x.\ P\ x\}_c$
  by (*simp add*: *cin-def*)

lemma *cset-CollI*: $[\![\ a \in_c A;\ P\ a\ ]\!] \Longrightarrow a \in_c \{x \in_c A.\ P\ x\}_c$
  by (*simp add*: *cin.rep-eq cset-Coll.rep-eq*)

lemma *cset-CollectD*: (*a* :: *'a*::*countable*) $\in_c \{x.\ P\ x\}_c \Longrightarrow P\ a$
  by (*simp add*: *cin-def*)

lemma *cset-Collect-cong*: $(\bigwedge x.\ P\ x = Q\ x)$ ==> $\{x.\ P\ x\}_c = \{x.\ Q\ x\}_c$
  by *simp*

Avoid eta-contraction for robust pretty-printing.

print-translation ‹
 [*Syntax-Trans.preserve-binder-abs-tr′*
 @{*const-syntax cset-Collect*} @{*syntax-const -cColl*}]
›

lift-definition *cset-set* :: *′a list* ⇒ *′a cset* **is** *set*
 using *countable-finite* **by** *blast*

lemma *countable-finite-power*:
 $countable(A) \Longrightarrow countable\ \{B.\ B \subseteq A \wedge finite(B)\}$
 by (*metis Collect-conj-eq Int-commute countable-Collect-finite-subset*)

lift-definition *cINTER* :: *′a cset* ⇒ (*′a* ⇒ *′b cset*) ⇒ *′b cset* **is**
$\lambda\ A\ f.\ if\ (A = \{\})\ then\ \{\}\ else\ INTER\ A\ f$
 by (*auto*)

definition *cInter* :: *′a cset cset* ⇒ *′a cset* ($\bigcap_c$- [*900*] *900*) **where**
$\bigcap_c\ A = cINTER\ A\ id$

lift-definition *cfinite* :: *′a cset* ⇒ *bool* **is** *finite* **.**
lift-definition *cInfinite* :: *′a cset* ⇒ *bool* **is** *infinite* **.**
lift-definition *clist* :: *′a*::*linorder cset* ⇒ *′a list* **is** *sorted-list-of-set* **.**
lift-definition *ccard* :: *′a cset* ⇒ *nat* **is** *card* **.**
lift-definition *cPow* :: *′a cset* ⇒ *′a cset cset* **is** $\lambda\ A.\ \{B.\ B \subseteq_c A \wedge cfinite(B)\}$
proof −
 fix *A*
 have $\{B :: 'a\ cset.\ B \subseteq_c A \wedge cfinite\ B\} = acset\ `\ \{B :: 'a\ set.\ B \subseteq rcset\ A \wedge finite\ B\}$
  apply (*auto simp add*: *cfinite.rep-eq cin-def less-eq-cset-def countable-finite*)
  using *image-iff* **apply** *fastforce*
  done

 moreover have *countable* $\{B :: 'a\ set.\ B \subseteq rcset\ A \wedge finite\ B\}$
  by (*auto intro*: *countable-finite-power*)

 ultimately show *countable* $\{B.\ B \subseteq_c A \wedge cfinite\ B\}$
  by *simp*
qed

definition *CCollect* :: (*′a* ⇒ *bool option*) ⇒ *′a cset option* **where**
$CCollect\ p = (if\ (None \notin range\ p)\ then\ Some\ (cset\text{-}Collect\ (the \circ p))\ else\ None)$

definition *cset-mapM* :: *′a option cset* ⇒ *′a cset option* **where**
$cset\text{-}mapM\ A = (if\ (None \in_c A)\ then\ None\ else\ Some\ (the\ `_c\ A))$

lemma *cset-mapM-Some-image* [*simp*]:
 $cset\text{-}mapM\ (cimage\ Some\ A) = Some\ A$
 apply (*auto simp add*: *cset-mapM-def*)
 apply (*metis cimage-cinsert cinsertI1 option.sel set-cinsert*)
 done

definition *CCollect-ext* :: (*′a* ⇒ *′b option*) ⇒ (*′a* ⇒ *bool option*) ⇒ *′b cset option* **where**
$CCollect\text{-}ext\ f\ p = do\ \{\ xs \leftarrow CCollect\ p;\ cset\text{-}mapM\ (f\ `_c\ xs)\ \}$

lemma *the-Some-image* [*simp*]:
 $the\ `\ Some\ `\ xs = xs$

by (*auto simp add*:*image-iff*)

lemma *CCollect-ext-Some* [*simp*]:
CCollect-ext Some xs = *CCollect xs*
apply (*case-tac CCollect xs*)
apply (*auto simp add*:*CCollect-ext-def*)
done

lift-definition *list-of-cset* :: *'a* :: *linorder cset* ⇒ *'a list* **is** *sorted-list-of-set* **.**

lift-definition *fset-cset* :: *'a fset* ⇒ *'a cset* **is** *id*
using *uncountable-infinite* **by** *auto*

definition *cset-count* :: *'a cset* ⇒ *'a* ⇒ *nat* **where**
cset-count A =
(*if* (*finite* (*rcset A*))
then (*SOME f*::*'a*⇒*nat*. *inj-on f* (*rcset A*))
else (*SOME f*::*'a*⇒*nat*. *bij-betw f* (*rcset A*) *UNIV*))

lemma *cset-count-inj-seq*:
inj-on (*cset-count A*) (*rcset A*)
proof (*cases finite* (*rcset A*))
case *True* **note** *fin* = *this*
obtain *count* :: *'a* ⇒ *nat* **where** *count-inj*: *inj-on count* (*rcset A*)
by (*metis countable-def mem-Collect-eq rcset*)
with *fin* **show** *?thesis*
by (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
next
case *False* **note** *inf* = *this*
obtain *count* :: *'a* ⇒ *nat* **where** *count-bij*: *bij-betw count* (*rcset A*) *UNIV*
by (*metis countableE-infinite inf mem-Collect-eq rcset*)
with *inf* **have** *bij-betw* (*cset-count A*) (*rcset A*) *UNIV*
by (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
thus *?thesis*
by (*metis bij-betw-imp-inj-on*)
qed

lemma *cset-count-infinite-bij*:
assumes *infinite* (*rcset A*)
shows *bij-betw* (*cset-count A*) (*rcset A*) *UNIV*
proof −
from *assms* **obtain** *count* :: *'a* ⇒ *nat* **where** *count-bij*: *bij-betw count* (*rcset A*) *UNIV*
by (*metis countableE-infinite mem-Collect-eq rcset*)
with *assms* **show** *?thesis*
by (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
qed

definition *cset-seq* :: *'a cset* ⇒ (*nat* ⇀ *'a*) **where**
cset-seq A i = (*if* (*i* ∈ *range* (*cset-count A*) ∧ *inv-into* (*rcset A*) (*cset-count A*) *i* $\in_c$ *A*)
then Some (*inv-into* (*rcset A*) (*cset-count A*) *i*)
else None)

lemma *cset-seq-ran*: *ran* (*cset-seq A*) = *rcset*(*A*)
apply (*auto simp add*: *ran-def cset-seq-def cin.rep-eq*)
apply (*metis cset-count-inj-seq inv-into-f-f rangeI*)

done

lemma *cset-seq-inj*: *inj cset-seq*
proof (*rule injI*)
fix A B :: $'a$ *cset*
assume *cset-seq* A = *cset-seq* B
thus $A = B$
by (*metis cset-seq-ran rcset-inverse*)
qed

lift-definition *cset2seq* :: $'a$ *cset* $\Rightarrow$ $'a$ *seq*
is (λ A i. *if* ($i \in$ *cset-count* A ‘ *rcset* A) *then inv-into* (*rcset* A) (*cset-count* A) i *else* (*SOME* x. $x \in_c$ A)) **.**

lemma *range-cset2seq*:
$A \neq \{\}_c \Longrightarrow$ *range* (*Rep-seq* (*cset2seq* A)) = *rcset* A
by (*force intro*: *someI2 simp add*: *cset2seq.rep-eq cset-count-inj-seq bot-cset.rep-eq cin.rep-eq*)

lemma *infinite-cset-count-surj*: *infinite* (*rcset* A) $\Longrightarrow$ *surj* (*cset-count* A)
using *bij-betw-imp-surj cset-count-infinite-bij* **by** *auto*

lemma *cset2seq-inj*:
inj-on cset2seq $\{A.\ A \neq \{\}_c\}$
apply (*rule inj-onI*)
apply (*simp*)
apply (*metis range-cset2seq rcset-inject*)
done

lift-definition *nat-seq2set* :: *nat seq* $\Rightarrow$ *nat set* **is**
λ f. *prod-encode* ‘ $\{(x, f\,x) \mid x.\ True\}$ **.**

lemma *inj-nat-seq2set*: *inj nat-seq2set*
proof (*rule injI*, *transfer*)
fix f g
assume *prod-encode* ‘ $\{(x, f\,x) \mid x.\ True\}$ = *prod-encode* ‘ $\{(x, g\,x) \mid x.\ True\}$
hence $\{(x, f\,x) \mid x.\ True\} = \{(x, g\,x) \mid x.\ True\}$
by (*simp add*: *inj-image-eq-iff*[*OF inj-prod-encode*])
thus $f = g$
by (*auto simp add*: *set-eq-iff*)
qed

lift-definition *bit-seq-of-nat-set* :: *nat set* $\Rightarrow$ *bit seq*
is λ A i. *if* ($i \in A$) *then 1 else 0* **.**

lemma *bit-seq-of-nat-set-inj*: *inj bit-seq-of-nat-set*
apply (*rule injI*)
apply (*transfer*, *auto*)
apply (*metis bit.distinct*(*1*))
apply (*meson zero-neq-one*)
done

lemma *bit-seq-of-nat-cset-bij*: *bij bit-seq-of-nat-set*
apply (*rule bijI*)
apply (*fact bit-seq-of-nat-set-inj*)
apply (*auto simp add*: *image-def*)

apply (*transfer*)
apply (*rename-tac x*)
apply (*rule-tac x*=$\{$*i. x i* = *1*$\}$ **in** *exI*)
apply (*auto*)
done

This function is a partial injection from countable sets of natural sets to natural sets. When used with the Schroeder-Bernstein theorem, it can be used to conjure a total bijection between these two types.

definition *nat-set-cset-collapse* :: *nat set cset* $\Rightarrow$ *nat set* **where**
nat-set-cset-collapse = *inv bit-seq-of-nat-set* $\circ$ *seq-inj* $\circ$ *cset2seq* $\circ$ (λ *A*. (*bit-seq-of-nat-set* ‘$_c$ *A*))

lemma *nat-set-cset-collapse-inj*: *inj-on nat-set-cset-collapse* $\{A.\ A \neq \{\}_c\}$
proof –
have (‘$_c$) *bit-seq-of-nat-set* ‘ $\{A.\ A \neq \{\}_c\} \subseteq \{A.\ A \neq \{\}_c\}$
by (*auto simp add*:*cimage.rep-eq*)
thus *?thesis*
apply (*simp add*: *nat-set-cset-collapse-def*)
apply (*rule comp-inj-on*)
apply (*meson bit-seq-of-nat-set-inj cset.inj-map injD inj-onI*)
apply (*rule comp-inj-on*)
apply (*metis cset2seq-inj subset-inj-on*)
apply (*rule comp-inj-on*)
apply (*rule subset-inj-on*)
apply (*rule seq-inj*)
apply (*simp*)
apply (*meson UNIV-I bij-imp-bij-inv bij-is-inj bit-seq-of-nat-cset-bij subsetI subset-inj-on*)
done
qed

lemma *inj-csingle*:
inj csingle
by (*auto intro*: *injI simp add*: *cinsert-def bot-cset.rep-eq*)

lemma *range-csingle*:
range csingle $\subseteq \{A.\ A \neq \{\}_c\}$
by (*auto*)

lift-definition *csets* :: *'a set* $\Rightarrow$ *'a cset set* **is**
λ *A*. $\{B.\ B \subseteq A \wedge$ *countable B*$\}$ **by** *auto*

lemma *csets-finite*: *finite A* $\Longrightarrow$ *finite* (*csets A*)
by (*auto simp add*: *csets-def*)

lemma *csets-infinite*: *infinite A* $\Longrightarrow$ *infinite* (*csets A*)
by (*auto simp add*: *csets-def*, *metis csets.abs-eq csets.rep-eq finite-countable-subset finite-imageI*)

lemma *csets-UNIV*:
csets (*UNIV* :: *'a set*) = (*UNIV* :: *'a cset set*)
by (*auto simp add*: *csets-def*, *metis image-iff rcset rcset-inverse*)

lemma *infinite-nempty-cset*:
assumes *infinite* (*UNIV* :: *'a set*)
shows *infinite* ($\{A.\ A \neq \{\}_c\}$ :: *'a cset set*)
proof –

have *infinite* (*UNIV* :: *'a cset set*)
by (*metis assms csets-UNIV csets-infinite*)
hence *infinite* ((*UNIV* :: *'a cset set*) $-$ $\{\{\}_c\}$)
by (*rule infinite-remove*)
thus *?thesis*
by (*auto*)
qed

lemma *nat-set-cset-partial-bij*:
obtains *f* :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij-betw f* $\{A.\ A \neq \{\}_c\}$ *UNIV*
using *Schroeder-Bernstein*[*OF nat-set-cset-collapse-inj*, *of UNIV csingle*, *simplified*, *OF inj-csingle range-csingle*]
by (*auto*)

lemma *nat-set-cset-bij*:
obtains *f* :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij f*
proof $-$
obtain *g* :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij-betw g* $\{A.\ A \neq \{\}_c\}$ *UNIV*
using *nat-set-cset-partial-bij* **by** *blast*
moreover obtain *h* :: *nat set cset* $\Rightarrow$ *nat set cset* **where** *bij-betw h UNIV* $\{A.\ A \neq \{\}_c\}$
proof $-$
have *infinite* (*UNIV* :: *nat set cset set*)
by (*metis Finite-Set.finite-set csets-UNIV csets-infinite infinite-UNIV-char-0*)
then obtain h' :: *nat set cset* $\Rightarrow$ *nat set cset* **where** *bij-betw* h' *UNIV* (*UNIV* $-$ $\{\{\}_c\}$)
using *infinite-imp-bij-betw*[*of UNIV* :: *nat set cset set* $\{\}_c$] **by** *auto*
moreover have (*UNIV* :: *nat set cset set*) $-$ $\{\{\}_c\} = \{A.\ A \neq \{\}_c\}$
by (*auto*)
ultimately show *?thesis*
using *that* **by** (*auto*)
qed
ultimately have *bij* ($g \circ h$)
using *bij-betw-trans* **by** *blast*
with *that* **show** *?thesis*
by (*auto*)
qed

definition *nat-set-cset-bij* = (*SOME f* :: *nat set cset* $\Rightarrow$ *nat set*. *bij f*)

lemma *bij-nat-set-cset-bij*:
bij nat-set-cset-bij
by (*metis nat-set-cset-bij nat-set-cset-bij-def someI-ex*)

lemma *inj-on-image-csets*:
inj-on f A $\Longrightarrow$ *inj-on* ((`$_c$) *f*) (*csets A*)
by (*fastforce simp add*: *inj-on-def cimage-def cin-def csets-def*)

lemma *image-csets-surj*:
$[\![$ *inj-on f A*; *f* ` *A* = *B* $]\!]$ $\Longrightarrow$ (`$_c$) *f* ` *csets A* = *csets B*
apply (*auto simp add*: *cimage-def csets-def image-mono map-fun-def*)
apply (*simp add*: *image-comp*)
apply (*auto simp add*: *image-Collect*)
apply (*erule subset-imageE*)
apply (*auto*)
apply (*metis countable-image rcset-inverse rcset-to-rcset subset-inj-on the-inv-into-onto*)
done

lemma *bij-betw-image-csets*:
bij-betw f A B $\Longrightarrow$ *bij-betw* $((`_c)\ f)$ (*csets A*) (*csets B*)
by (*simp add*: *bij-betw-def inj-on-image-csets image-csets-surj*)
end

6 Extra Relational Definitions and Theorems

theory *Relation-Extra*
imports *HOL−Library.FuncSet*
begin

We set up some nice syntax for heterogeneous relations at the type level

syntax
-rel-type :: *type* $\Rightarrow$ *type* $\Rightarrow$ *type* (**infixr** $\leftrightarrow$ *0*)

translations
(*type*) $'a \leftrightarrow 'b$ == (*type*) $('a \times 'b)$ *set*

6.1 Relational Function Operations

definition *rel-apply* :: $('a \leftrightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ $(\text{-}'(\text{-}')_r$ [*999*,*0*] *999*) **where**
rel-apply R x = (*if x* $\in$ *Domain*(*R*) *then THE y*. $(x, y) \in R$ *else undefined*)

definition *rel-domres* :: $'a$ *set* $\Rightarrow ('a \leftrightarrow 'b) \Rightarrow 'a \leftrightarrow 'b$ (**infixr** $\lhd_r$ *85*) **where**
rel-domres A R = $\{(k, v) \in R.\ k \in A\}$

definition *rel-override* :: $('a \leftrightarrow 'b) \Rightarrow ('a \leftrightarrow 'b) \Rightarrow 'a \leftrightarrow 'b$ (**infixl** $+_r$ *65*) **where**
rel-override R S = $(-$ *Domain S*$)\ \lhd_r R \cup S$

definition *rel-update* :: $('a \leftrightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow 'a \leftrightarrow 'b$ **where**
rel-update R k v = *rel-override R* $\{(k, v)\}$

6.2 Domain Restriction

lemma *Domain-rel-domres* [*simp*]: *Domain* $(A \lhd_r R) = A \cap$ *Domain*(*R*)
by (*auto simp add*: *rel-domres-def*)

lemma *rel-domres-empty* [*simp*]: $\{\} \lhd_r R = \{\}$
by (*simp add*: *rel-domres-def*)

lemma *rel-domres-UNIV* [*simp*]: *UNIV* $\lhd_r R = R$
by (*simp add*: *rel-domres-def*)

lemma *rel-domres-nil* [*simp*]: $A \lhd_r \{\} = \{\}$
by (*simp add*: *rel-domres-def*)

lemma *rel-domres-inter* [*simp*]: $A \lhd_r B \lhd_r R = (A \cap B) \lhd_r R$
by (*auto simp add*: *rel-domres-def*)

6.3 Relational Override

interpretation *rel-override-monoid*: *monoid-add* $(+_r)$ $\{\}$
by (*unfold-locales*, *simp-all add*: *rel-override-def*, *auto simp add*: *rel-domres-def*)

lemma *Domain-rel-override* [*simp*]: $Domain\ (R +_r S) = Domain(R) \cup Domain(S)$
  by (*auto simp add*: *rel-override-def Domain-Un-eq*)

lemma *Range-rel-override*: $Range(R +_r S) \subseteq Range(R) \cup Range(S)$
  by (*auto simp add*: *rel-override-def rel-domres-def*)

6.4 Functional Relations

definition *functional* :: $('a \leftrightarrow 'b) \Rightarrow bool$ **where**
functional g = *inj-on fst g*

lemma *functional-algebraic*: *functional R* $\longleftrightarrow R^{-1}\ O\ R \subseteq Id$
  apply (*auto simp add*: *functional-def subset-iff relcomp-unfold*)
  using *inj-on-eq-iff* **apply** *fastforce*
  apply (*metis inj-onI surjective-pairing*)
  done

lemma *functional-determine*: $\llbracket$ *functional R*; $(x, y) \in R$; $(x, z) \in R$ $\rrbracket \Longrightarrow y = z$
  by (*auto simp add*: *functional-algebraic subset-iff relcomp-unfold*)

lemma *functional-apply*:
  assumes *functional R* $(x, y) \in R$
  shows $R(x)_r = y$
  by (*metis* (*no-types*, *lifting*) *Domain.intros DomainE assms*(*1*) *assms*(*2*) *functional-determine rel-apply-def theI-unique*)

lemma *functional-elem*:
  assumes *functional R* $x \in Domain(R)$
  shows $(x, R(x)_r) \in R$
  using *assms*(*1*) *assms*(*2*) *functional-apply* **by** *fastforce*

lemma *functional-empty* [*simp*]: *functional* {}
  by (*simp add*: *functional-def*)

lemma *functional-override* [*intro*]: $\llbracket$ *functional R*; *functional S* $\rrbracket \Longrightarrow$ *functional* $(R +_r S)$
  by (*auto simp add*: *functional-algebraic rel-override-def rel-domres-def*)

definition *functional-list* :: $('a \times 'b)$ *list* $\Rightarrow bool$ **where**
functional-list xs = $(\forall\ x\ y\ z.\ ListMem\ (x,y)\ xs \wedge ListMem\ (x,z)\ xs \longrightarrow y = z)$

lemma *functional-insert* [*simp*]: *functional* (*insert* (x,y) *g*) $\longleftrightarrow$ ($g``\{x\} \subseteq \{y\} \wedge$ *functional g*)
  by (*auto simp add*:*functional-def inj-on-def image-def*)

lemma *functional-list-nil*[*simp*]: *functional-list* []
  by (*simp add*:*functional-list-def ListMem-iff*)

lemma *functional-list*: *functional-list xs* $\longleftrightarrow$ *functional* (*set xs*)
  apply (*induct xs*)
   apply (*simp add*:*functional-def*)
  apply (*simp add*:*functional-def functional-list-def ListMem-iff*)
  apply (*safe*)
        apply (*force*)
       apply (*force*)
      apply (*force*)
     apply (*force*)
    apply (*force*)

apply (*force*)
apply (*force*)
apply (*force*)
done

definition *fun-rel* :: $('a \Rightarrow 'b) \Rightarrow ('a \leftrightarrow 'b)$ **where**
$\mathit{fun\text{-}rel}\ f = \{(x, y).\ y = f\ x\}$

lemma *functional-fun-rel*: *functional* (*fun-rel f*)
by (*simp add*: *fun-rel-def functional-def*)
(*metis* (*mono-tags*, *lifting*) *Product-Type.Collect-case-prodD inj-onI prod.expand*)

lemma *rel-apply-fun* [*simp*]: $(\mathit{fun\text{-}rel}\ f)(x)_r = f\ x$
by (*simp add*: *fun-rel-def rel-apply-def*)

6.5 Left-Total Relations

definition *left-totalr-on* :: $'a\ set \Rightarrow ('a \leftrightarrow 'b) \Rightarrow bool$ **where**
$\mathit{left\text{-}totalr\text{-}on}\ A\ R \longleftrightarrow (\forall x{\in}A.\ \exists y.\ (x, y) \in R)$

abbreviation $\mathit{left\text{-}totalr}\ R \equiv \mathit{left\text{-}totalr\text{-}on}\ \mathit{UNIV}\ R$

lemma *left-totalr-algebraic*: $\mathit{left\text{-}totalr}\ R \longleftrightarrow \mathit{Id} \subseteq R\ O\ R^{-1}$
by (*auto simp add*: *left-totalr-on-def*)

lemma *left-totalr-fun-rel*: *left-totalr* (*fun-rel f*)
by (*simp add*: *left-totalr-on-def fun-rel-def*)

6.6 Relation Sets

definition *rel-typed* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\leftrightarrow_r$ *55*) **where**
$\mathit{rel\text{-}typed}\ A\ B = \{R.\ \mathit{Domain}(R) \subseteq A \wedge \mathit{Range}(R) \subseteq B\}$

lemma *rel-typed-intro*: $[\![\ \mathit{Domain}(R) \subseteq A;\ \mathit{Range}(R) \subseteq B\ ]\!] \Longrightarrow R \in A \leftrightarrow_r B$
by (*simp add*: *rel-typed-def*)

definition *rel-pfun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\rightharpoonup_r$ *55*) **where**
$\mathit{rel\text{-}pfun}\ A\ B = \{R.\ R \in A \leftrightarrow_r B \wedge \mathit{functional}\ R\}$

lemma *rel-pfun-intro*: $[\![\ R \in A \leftrightarrow_r B;\ \mathit{functional}\ R\ ]\!] \Longrightarrow R \in A \rightharpoonup_r B$
by (*simp add*: *rel-pfun-def*)

definition *rel-tfun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\rightarrow_r$ *55*) **where**
$\mathit{rel\text{-}tfun}\ A\ B = \{R.\ R \in A \rightharpoonup_r B \wedge \mathit{left\text{-}totalr}\ R\}$

definition *rel-ffun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ **where**
$\mathit{rel\text{-}ffun}\ A\ B = \{R.\ R \in A \rightharpoonup_r B \wedge \mathit{finite}(\mathit{Domain}\ R)\}$

6.7 Closure Properties

These can be seen as typing rules for relational functions

named-theorems *rclos*

lemma *rel-ffun-is-pfun* [*rclos*]: $R \in \mathit{rel\text{-}ffun}\ A\ B \Longrightarrow R \in A \rightharpoonup_r B$
by (*simp add*: *rel-ffun-def*)

lemma *rel-tfun-is-pfun* [*rclos*]: $R \in A \rightarrow_r B \Longrightarrow R \in A \rightharpoonup_r B$
by (*simp add*: *rel-tfun-def*)

lemma *rel-pfun-is-typed* [*rclos*]: $R \in A \rightharpoonup_r B \Longrightarrow R \in A \leftrightarrow_r B$
by (*simp add*: *rel-pfun-def*)

lemma *rel-ffun-empty* [*rclos*]: $\{\} \in$ *rel-ffun A B*
by (*simp add*: *rel-ffun-def rel-pfun-def rel-typed-def*)

lemma *rel-pfun-apply* [*rclos*]: $[\![x \in Domain(R); R \in A \rightharpoonup_r B]\!] \Longrightarrow R(x)_r \in B$
using *functional-apply* **by** (*fastforce simp add*: *rel-pfun-def rel-typed-def*)

lemma *rel-tfun-apply* [*rclos*]: $[\![x \in A; R \in A \rightarrow_r B]\!] \Longrightarrow R(x)_r \in B$
by (*metis* (*no-types*, *lifting*) *Domain-iff iso-tuple-UNIV-I left-totalr-on-def mem-Collect-eq rel-pfun-apply rel-tfun-def*)

lemma *rel-typed-insert* [*rclos*]: $[\![R \in A \leftrightarrow_r B; x \in A; y \in B]\!] \Longrightarrow$ *insert* (x, y) $R \in A \leftrightarrow_r B$
by (*simp add*: *rel-typed-def*)

lemma *rel-pfun-insert* [*rclos*]: $[\![R \in A \rightharpoonup_r B; x \in A; y \in B; x \notin Domain(R)]\!] \Longrightarrow$ *insert* (x, y) $R \in A \rightharpoonup_r B$
by (*auto intro*: *rclos simp add*: *rel-pfun-def*)

lemma *rel-pfun-override* [*rclos*]: $[\![R \in A \rightharpoonup_r B; S \in A \rightharpoonup_r B]\!] \Longrightarrow (R +_r S) \in A \rightharpoonup_r B$
apply (*rule rel-pfun-intro*)
apply (*rule rel-typed-intro*)
apply (*auto simp add*: *rel-pfun-def rel-typed-def*)
apply (*metis* (*no-types*, *hide-lams*) *Range.simps Range-Un-eq Range-rel-override Un-iff rev-subsetD*)
done

end

7 Map Type: extra functions and properties

theory *Map-Extra*
imports
Relation-Extra
HOL−Library.Countable-Set
HOL−Library.Monad-Syntax
begin

7.1 Graphing Maps

definition *map-graph* :: $('a \rightharpoonup 'b) \Rightarrow ('a \leftrightarrow 'b)$ **where**
map-graph $f = \{(x,y) \mid x\ y.\ f\ x = Some\ y\}$

definition *graph-map* :: $('a \leftrightarrow 'b) \Rightarrow ('a \rightharpoonup 'b)$ **where**
graph-map $g = (\lambda\ x.\ if\ (x \in fst\ `\ g)\ then\ Some\ (SOME\ y.\ (x,y) \in g)\ else\ None)$

definition *graph-map′* :: $('a \leftrightarrow 'b) \rightharpoonup ('a \rightharpoonup 'b)$ **where**
graph-map′ $R = (if\ (functional\ R)\ then\ Some\ (graph\text{-}map\ R)\ else\ None)$

lemma *map-graph-mem-equiv*: $(x, y) \in$ *map-graph* $f \longleftrightarrow f(x) = Some\ y$
by (*simp add*: *map-graph-def*)

lemma *map-graph-functional*[*simp*]: *functional* (*map-graph f*)
  by (*simp add*:*functional-def map-graph-def inj-on-def*)

lemma *map-graph-countable* [*simp*]: *countable* (*dom f*) $\Longrightarrow$ *countable* (*map-graph f*)
  apply (*auto simp add*:*map-graph-def countable-def*)
  apply (*rename-tac f'*)
  apply (*rule-tac x*=*f'* $\circ$ *fst* **in** *exI*)
  apply (*auto simp add*:*inj-on-def dom-def*)
  apply *fastforce*
  done

lemma *map-graph-inv* [*simp*]: *graph-map* (*map-graph f*) = *f*
  by (*auto intro*!:*ext simp add*:*map-graph-def graph-map-def image-def*)

lemma *graph-map-empty*[*simp*]: *graph-map* {} = *Map.empty*
  by (*simp add*:*graph-map-def*)

lemma *graph-map-insert* [*simp*]: ⟦*functional g*; *g*‘‘{*x*} ⊆ {*y*}⟧ $\Longrightarrow$ *graph-map* (*insert* (*x*,*y*) *g*) = (*graph-map g*)($x \mapsto y$)
  by (*rule ext*, *auto simp add*:*graph-map-def*)

lemma *dom-map-graph*: *dom f* = *Domain*(*map-graph f*)
  by (*simp add*: *map-graph-def dom-def image-def*)

lemma *ran-map-graph*: *ran f* = *Range*(*map-graph f*)
  by (*simp add*: *map-graph-def ran-def image-def*)

lemma *rel-apply-map-graph*:
  $x \in dom(f) \Longrightarrow (map\text{-}graph\ f)(x)_r = the\ (f\ x)$
  by (*auto simp add*: *rel-apply-def map-graph-def*)

lemma *ran-map-add-subset*:
  ran (*x* ++ *y*) ⊆ (*ran x*) ∪ (*ran y*)
  by (*auto simp add*:*ran-def*)

lemma *finite-dom-graph*: *finite* (*dom f*) $\Longrightarrow$ *finite* (*map-graph f*)
  by (*metis dom-map-graph finite-imageD fst-eq-Domain functional-def map-graph-functional*)

lemma *finite-dom-ran* [*simp*]: *finite* (*dom f*) $\Longrightarrow$ *finite* (*ran f*)
  by (*metis finite-Range finite-dom-graph ran-map-graph*)

lemma *graph-map-inv* [*simp*]: *functional g* $\Longrightarrow$ *map-graph* (*graph-map g*) = *g*
  apply (*auto simp add*:*map-graph-def graph-map-def functional-def*)
    apply (*metis* (*lifting*, *no-types*) *image-iff option.distinct*(*1*) *option.inject someI surjective-pairing*)
   apply (*simp add*:*inj-on-def*)
   apply (*metis fst-conv snd-conv some-equality*)
  apply (*metis* (*lifting*) *fst-conv image-iff*)
  done

lemma *graph-map-dom*: *dom* (*graph-map R*) = *fst* ‘ *R*
  by (*simp add*: *graph-map-def dom-def*)

lemma *graph-map-countable-dom*: *countable R* $\Longrightarrow$ *countable* (*dom* (*graph-map R*))
  by (*simp add*: *graph-map-dom*)

lemma *countable-ran*:
assumes *countable* (*dom f*)
shows *countable* (*ran f*)
proof −
have *countable* (*map-graph f*)
by (*simp add*: *assms*)
then have *countable* (*Range*(*map-graph f*))
by (*simp add*: *Range-snd*)
thus *?thesis*
by (*simp add*: *ran-map-graph*)
qed

lemma *map-graph-inv′* [*simp*]:
graph-map′ (*map-graph f*) = *Some f*
by (*simp add*: *graph-map′-def*)

lemma *map-graph-inj*:
inj map-graph
by (*metis injI map-graph-inv*)

lemma *map-eq-graph*: $f = g \longleftrightarrow$ *map-graph f* = *map-graph g*
by (*auto simp add*: *inj-eq map-graph-inj*)

lemma *map-le-graph*: $f \subseteq_m g \longleftrightarrow$ *map-graph f* $\subseteq$ *map-graph g*
by (*force simp add*: *map-le-def map-graph-def*)

lemma *map-graph-comp*: *map-graph* ($g \circ_m f$) = (*map-graph f*) *O* (*map-graph g*)
apply (*auto simp add*: *map-comp-def map-graph-def relcomp-unfold*)
apply (*rename-tac a b*)
apply (*case-tac f a*, *auto*)
done

7.2 Map Application

definition *map-apply* :: ($'a \rightharpoonup 'b$) $\Rightarrow$ $'a \Rightarrow 'b$ (-′(-′)$_m$ [*999*,*0*] *999*) **where**
map-apply = (λ *f x*. *the* (*f x*))

7.3 Map Membership

fun *map-member* :: $'a \times 'b \Rightarrow ('a \rightharpoonup 'b) \Rightarrow$ *bool* (**infix** $\in_m$ *50*) **where**
$(k,\ v) \in_m m \longleftrightarrow m(k) = Some(v)$

lemma *map-ext*:
$[\![\bigwedge x\ y.\ (x,\ y) \in_m A \longleftrightarrow (x,\ y) \in_m B]\!] \Longrightarrow A = B$
by (*rule ext*, *auto*, *metis not-Some-eq*)

lemma *map-member-alt-def*:
$(x,\ y) \in_m A \longleftrightarrow (x \in dom\ A \wedge A(x)_m = y)$
by (*auto simp add*: *map-apply-def*)

lemma *map-le-member*:
$f \subseteq_m g \longleftrightarrow (\forall\ x\ y.\ (x,y) \in_m f \longrightarrow (x,y) \in_m g)$
by (*force simp add*: *map-le-def*)

7.4 Preimage

definition *preimage* :: $('a \rightharpoonup 'b) \Rightarrow 'b\ set \Rightarrow 'a\ set$ **where**
preimage f B $= \{x \in dom(f).\ the(f(x)) \in B\}$

lemma *preimage-range*: *preimage f* $(ran\ f) = dom\ f$
  by (*auto simp add*: *preimage-def ran-def*)

lemma *dom-preimage*: $dom\ (m \circ_m f) = preimage\ f\ (dom\ m)$
  apply (*auto simp add*: *dom-def preimage-def*)
   apply (*meson map-comp-Some-iff*)
  apply (*metis map-comp-def option.case-eq-if option.distinct*(*1*))
  done

lemma *countable-preimage*:
  $\llbracket$ *countable A*; *inj-on f* (*preimage f A*) $\rrbracket \Longrightarrow$ *countable* (*preimage f A*)
  apply (*auto simp add*: *countable-def*)
  apply (*rename-tac g*)
  apply (*rule-tac* $x = g \circ the \circ f$ **in** *exI*)
  apply (*rule inj-onI*)
  apply (*drule inj-onD*)
    apply (*auto simp add*: *preimage-def inj-onD*)
  done

7.5 Minus operation for maps

definition *map-minus* :: $('a \rightharpoonup 'b) \Rightarrow ('a \rightharpoonup 'b) \Rightarrow ('a \rightharpoonup 'b)$ (**infixl** $--$ *100*)
where *map-minus f g* $= (\lambda\ x.\ if\ (f\ x = g\ x)\ then\ None\ else\ f\ x)$

lemma *map-minus-apply* [*simp*]: $y \in dom(f\ --\ g) \Longrightarrow (f\ --\ g)(y)_m = f(y)_m$
  by (*auto simp add*: *map-minus-def dom-def map-apply-def*)

lemma *map-member-plus*:
  $(x,\ y) \in_m f\ {+}{+}\ g \longleftrightarrow ((x \notin dom(g) \wedge (x,\ y) \in_m f) \vee (x,\ y) \in_m g)$
  by (*auto simp add*: *map-add-Some-iff*)

lemma *map-member-minus*:
  $(x,\ y) \in_m f\ --\ g \longleftrightarrow (x,\ y) \in_m f \wedge (\neg\ (x,\ y) \in_m g)$
  by (*auto simp add*: *map-minus-def*)

lemma *map-minus-plus-commute*:
  $dom(g) \cap dom(h) = \{\} \Longrightarrow (f\ --\ g)\ {+}{+}\ h = (f\ {+}{+}\ h)\ --\ g$
  apply (*rule map-ext*)
  apply (*auto simp add*: *map-member-plus map-member-minus simp del*: *map-member.simps*)
  apply (*auto simp add*: *map-member-alt-def*)
  done

lemma *map-graph-minus*: *map-graph* $(f\ --\ g) =$ *map-graph f* $-$ *map-graph g*
  by (*auto simp add*: *map-minus-def map-graph-def*, (*meson option.distinct*(*1*))+)

lemma *map-minus-common-subset*:
  $\llbracket h \subseteq_m f;\ h \subseteq_m g \rrbracket \Longrightarrow (f\ --\ h = g\ --\ h) = (f = g)$
  by (*auto simp add*: *map-eq-graph map-graph-minus map-le-graph*)

7.6 Map Bind

Create some extra intro/elim rules to help dealing with proof about option bind.

lemma *option-bindSomeE* [*elim*!]:
  $[\![\ X >>= F = Some(v);\ \bigwedge x.\ [\![\ X = Some(x);\ F(x) = Some(v)\ ]\!] \Longrightarrow P\ ]\!] \Longrightarrow P$
  by (*case-tac* X, *auto*)

lemma *option-bindSomeI* [*intro*]:
  $[\![\ X = Some(x);\ F(x) = Some(y)\ ]\!] \Longrightarrow X >>= F = Some(y)$
  by (*simp*)

lemma *ifSomeE* [*elim*]: $[\![\ (if\ c\ then\ Some(x)\ else\ None) = Some(y);\ [\![\ c;\ x = y\ ]\!] \Longrightarrow P\ ]\!] \Longrightarrow P$
  by (*case-tac* c, *auto*)

7.7 Range Restriction

A range restriction operator; only domain restriction is provided in HOL.

definition *ran-restrict-map* :: $('a \rightharpoonup 'b) \Rightarrow 'b\ set \Rightarrow 'a \rightharpoonup 'b$ (-⌉- [*111*,*110*] *110*) **where**
ran-restrict-map $f\ B = (\lambda x.\ do\ \{\ v <- f(x);\ if\ (v \in B)\ then\ Some(v)\ else\ None\ \})$

lemma *ran-restrict-empty* [*simp*]: $f\rceil_{\{\}} = Map.empty$
  by (*simp add*:*ran-restrict-map-def*)

lemma *ran-restrict-ran* [*simp*]: $f\rceil_{ran(f)} = f$
  apply (*auto simp add*:*ran-restrict-map-def ran-def*)
  apply (*rule ext*)
  apply (*case-tac* $f(x)$, *auto*)
  done

lemma *ran-ran-restrict* [*simp*]: $ran(f\rceil_B) = ran(f) \cap B$
  by (*auto intro*!:*option-bindSomeI simp add*:*ran-restrict-map-def ran-def*)

lemma *dom-ran-restrict*: $dom(f\rceil_B) \subseteq dom(f)$
  by (*auto simp add*:*ran-restrict-map-def dom-def*)

lemma *ran-restrict-finite-dom* [*intro*]:
  $finite(dom(f)) \Longrightarrow finite(dom(f\rceil_B))$
  by (*metis finite-subset dom-ran-restrict*)

lemma *dom-Some* [*simp*]: $dom\ (Some \circ f) = UNIV$
  by (*auto*)

lemma *dom-left-map-add* [*simp*]: $x \in dom\ g \Longrightarrow (f\ {+}{+}\ g)\ x = g\ x$
  by (*auto simp add*:*map-add-def dom-def*)

lemma *dom-right-map-add* [*simp*]: $[\![\ x \notin dom\ g;\ x \in dom\ f\ ]\!] \Longrightarrow (f\ {+}{+}\ g)\ x = f\ x$
  by (*auto simp add*:*map-add-def dom-def*)

lemma *map-add-restrict*:
  $f\ {+}{+}\ g = (f\ |`\ (-\ dom\ g))\ {+}{+}\ g$
  by (*rule ext*, *auto simp add*: *map-add-def restrict-map-def*)

7.8 Map Inverse and Identity

definition *map-inv* :: $('a \rightharpoonup 'b) \Rightarrow ('b \rightharpoonup 'a)$ **where**

map-inv f ≡ λ *y*. *if* (*y* ∈ *ran f*) *then Some* (*SOME x*. *f x* = *Some y*) *else None*

definition *map-id-on* :: *'a set* ⇒ (*'a* ⇀ *'a*) **where**
map-id-on xs ≡ λ *x*. *if* (*x* ∈ *xs*) *then Some x else None*

lemma *map-id-on-in* [*simp*]:
  x ∈ *xs* ⟹ *map-id-on xs x* = *Some x*
  by (*simp add*:*map-id-on-def*)

lemma *map-id-on-out* [*simp*]:
  x ∉ *xs* ⟹ *map-id-on xs x* = *None*
  by (*simp add*:*map-id-on-def*)

lemma *map-id-dom* [*simp*]: *dom* (*map-id-on xs*) = *xs*
  by (*simp add*:*dom-def map-id-on-def*)

lemma *map-id-ran* [*simp*]: *ran* (*map-id-on xs*) = *xs*
  by (*force simp add*:*ran-def map-id-on-def*)

lemma *map-id-on-UNIV*[*simp*]: *map-id-on UNIV* = *Some*
  by (*simp add*:*map-id-on-def*)

lemma *map-id-on-inj* [*simp*]:
  inj-on (*map-id-on xs*) *xs*
  by (*simp add*:*inj-on-def*)

lemma *map-inv-empty* [*simp*]: *map-inv Map.empty* = *Map.empty*
  by (*simp add*:*map-inv-def*)

lemma *map-inv-id* [*simp*]:
  map-inv (*map-id-on xs*) = *map-id-on xs*
  by (*force simp add*:*map-inv-def map-id-on-def ran-def*)

lemma *map-inv-Some* [*simp*]: *map-inv Some* = *Some*
  by (*simp add*:*map-inv-def ran-def*)

lemma *map-inv-f-f* [*simp*]:
  ⟦ *inj-on f* (*dom f*); *f x* = *Some y* ⟧ ⟹ *map-inv f y* = *Some x*
  apply (*auto simp add*: *map-inv-def*)
   apply (*rule some-equality*)
    apply (*auto simp add*:*inj-on-def dom-def ran-def*)
  done

lemma *dom-map-inv* [*simp*]:
  dom (*map-inv f*) = *ran f*
  by (*auto simp add*:*map-inv-def*)

lemma *ran-map-inv* [*simp*]:
  inj-on f (*dom f*) ⟹ *ran* (*map-inv f*) = *dom f*
  apply (*auto simp add*:*map-inv-def ran-def*)
   apply (*rename-tac a b*)
   apply (*rule-tac x*=*a* **in** *exI*)
   apply (*force intro*:*someI*)
  apply (*rename-tac x y*)
  apply (*rule-tac x*=*y* **in** *exI*)

apply (*auto*)
apply (*rule some-equality*, *simp-all*)
apply (*auto simp add*:*inj-on-def dom-def*)
done

lemma *dom-image-ran*: *f* ‘ *dom f* = *Some* ‘ *ran f*
by (*auto simp add*:*dom-def ran-def image-def*)

lemma *inj-map-inv* [*intro*]:
inj-on f (*dom f*) $\Longrightarrow$ *inj-on* (*map-inv f*) (*ran f*)
apply (*auto simp add*:*map-inv-def inj-on-def dom-def ran-def*)
apply (*rename-tac x y u v*)
apply (*frule-tac P*=λ *xa*. *f xa* = *Some x* **in** *some-equality*)
apply (*auto*)
apply (*metis* (*mono-tags*) *option.sel someI*)
done

lemma *inj-map-bij*: *inj-on f* (*dom f*) $\Longrightarrow$ *bij-betw f* (*dom f*) (*Some* ‘ *ran f*)
by (*auto simp add*:*inj-on-def dom-def ran-def image-def bij-betw-def*)

lemma *map-inv-map-inv* [*simp*]:
assumes *inj-on f* (*dom f*)
shows *map-inv* (*map-inv f*) = *f*
proof −

from *assms* **have** *inj-on* (*map-inv f*) (*ran f*)
by *auto*

thus *?thesis*
apply (*rule-tac ext*)
apply (*rename-tac x*)
apply (*case-tac* ∃ *y*. *map-inv f y* = *Some x*)
apply (*auto*)[*1*]
apply (*simp add*:*map-inv-def*)
apply (*rename-tac x y*)
apply (*case-tac y* ∈ *ran f*, *simp-all*)
apply (*auto*)
apply (*rule someI2-ex*)
apply (*simp add*:*ran-def*)
apply (*simp*)
apply (*metis assms dom-image-ran dom-map-inv image-iff map-add-dom-app-simps*(*2*) *map-add-dom-app-simps*(*3*) *ran-map-inv*)
done
qed

lemma *map-self-adjoin-complete* [*intro*]:
assumes *dom f* ∩ *ran f* = {} *inj-on f* (*dom f*)
shows *inj-on* (*map-inv f* ++ *f*) (*dom f* ∪ *ran f*)
apply (*rule inj-onI*)
apply (*insert assms*)
apply (*rename-tac x y*)
apply (*case-tac x* ∈ *dom f*)
apply (*simp*)
apply (*case-tac y* ∈ *dom f*)
apply (*simp add*:*inj-on-def*)

apply (*case-tac* $y \in$ *ran* f)
apply (*subgoal-tac* $y \in$ *dom* (*map-inv* f))
apply (*simp*)
apply (*metis Int-iff domD empty-iff ranI ran-map-inv*)
apply (*simp*)
apply (*simp*)
apply (*simp*)
apply (*case-tac* $y \in$ *dom* f)
apply (*simp*)
apply (*case-tac* $y \in$ *ran* f)
apply (*subgoal-tac* $y \in$ *dom* (*map-inv* f))
apply (*simp*)
apply (*metis Int-iff empty-iff*)
apply (*simp*)
apply (*metis Int-iff domD empty-iff ranI ran-map-inv*)
apply (*simp*)
apply (*metis* (*lifting*) *inj-map-inv inj-on-contraD*)
done

lemma *inj-completed-map* [*intro*]:
$\llbracket$ *dom* f = *ran* f; *inj-on* f (*dom* f) $\rrbracket \Longrightarrow$ *inj* (*Some* ++ f)
apply (*drule inj-map-bij*)
apply (*auto simp add*:*bij-betw-def*)
apply (*auto simp add*:*inj-on-def*)[*1*]
apply (*rename-tac* x y)
apply (*case-tac* $x \in$ *dom* f)
apply (*simp*)
apply (*case-tac* $y \in$ *dom* f)
apply (*simp*)
apply (*simp add*:*ran-def*)
apply (*case-tac* $y \in$ *dom* f)
apply (*auto intro*:*ranI*)
done

lemma *bij-completed-map* [*intro*]:
$\llbracket$ *dom* f = *ran* f; *inj-on* f (*dom* f) $\rrbracket \Longrightarrow$
bij-betw (*Some* ++ f) *UNIV* (*range Some*)
apply (*auto simp add*:*bij-betw-def*)
apply (*rename-tac* x)
apply (*case-tac* $x \in$ *dom* f)
apply (*simp*)
apply (*metis domD rangeI*)
apply (*simp*)
apply (*simp add*:*image-def*)
apply (*metis* (*full-types*) *dom-image-ran dom-left-map-add image-iff map-add-dom-app-simps*(*3*))
done

lemma *bij-map-Some*:
bij-betw f a (*Some* ‘ b) $\Longrightarrow$ *bij-betw* (*the* $\circ$ f) a b
apply (*simp add*:*bij-betw-def*)
apply (*safe*)
apply (*metis* (*hide-lams*, *no-types*) *comp-inj-on-iff f-the-inv-into-f inj-on-inverseI option.sel*)
apply (*metis* (*hide-lams*, *no-types*) *comp-apply image-iff option.sel*)
apply (*metis imageI image-comp option.sel*)
done

lemma *ran-map-add* [*simp*]:
  $m`(dom\ m \cap dom\ n) = n`(dom\ m \cap dom\ n) \Longrightarrow$
   $ran(m{+}{+}n) = ran\ n \cup ran\ m$
  apply (*auto simp add*:*ran-def*)
   apply (*metis map-add-find-right*)
  apply (*rename-tac x a*)
  apply (*case-tac* $a \in dom\ n$)
   apply (*subgoal-tac* $\exists\ b.\ n\ b = Some\ x$)
    apply (*auto*)
    apply (*rename-tac x a b y*)
    apply (*rule-tac x=b* **in** *exI*)
    apply (*simp*)
   apply (*metis* (*hide-lams*, *no-types*) *IntI domI image-iff*)
 apply (*metis* (*full-types*) *map-add-None map-add-dom-app-simps*(*1*) *map-add-dom-app-simps*(*3*) *not-None-eq*)
  done

lemma *ran-maplets* [*simp*]:
  $[\![$ *length xs = length ys*; *distinct xs* $]\!] \Longrightarrow$ *ran* [*xs* [$\mapsto$] *ys*] = *set ys*
    by (*induct rule*:*list-induct2*, *simp-all*)

lemma *inj-map-add*:
  $[\![$ *inj-on f* (*dom f*); *inj-on g* (*dom g*); $ran\ f \cap ran\ g = \{\}$ $]\!] \Longrightarrow$
   inj-on (*f* ++ *g*) ($dom\ f \cup dom\ g$)
  apply (*auto simp add*:*inj-on-def*)
      apply (*metis* (*full-types*) *disjoint-iff-not-equal domI dom-left-map-add map-add-dom-app-simps*(*3*)
ranI)
     apply (*metis domI*)
    apply (*metis disjoint-iff-not-equal ranI*)
   apply (*metis disjoint-iff-not-equal domIff map-add-Some-iff ranI*)
  apply (*metis domI*)
  done

lemma *map-inv-add* [*simp*]:
  assumes *inj-on f* (*dom f*) *inj-on g* (*dom g*)
          $dom\ f \cap dom\ g = \{\}$ $ran\ f \cap ran\ g = \{\}$
  shows *map-inv* (*f* ++ *g*) = *map-inv f* ++ *map-inv g*
proof (*rule ext*)

  from *assms* **have** *minj*: *inj-on* (*f* ++ *g*) (*dom* (*f* ++ *g*))
    by (*simp*, *metis inj-map-add sup-commute*)

  fix *x*
  have $x \in ran\ g \Longrightarrow$ *map-inv* (*f* ++ *g*) *x* = (*map-inv f* ++ *map-inv g*) *x*
  proof −
    assume *ran*:$x \in ran\ g$
    then obtain *y* **where** *dom*:*g y = Some x* $y \in dom\ g$
      by (*auto simp add*:*ran-def*)

    hence (*f* ++ *g*) *y = Some x*
      by *simp*

    with *assms minj ran dom* **show** *map-inv* (*f* ++ *g*) *x* = (*map-inv f* ++ *map-inv g*) *x*
      by *simp*
  qed

moreover have $\llbracket$ $x \notin$ *ran* g; $x \in$ *ran* f $\rrbracket \Longrightarrow$ *map-inv* $(f$ ++ $g)$ x = (*map-inv* f ++ *map-inv* g) x
proof −
assume *ran*:$x \notin$ *ran* g $x \in$ *ran* f
with *assms* **obtain** y **where** *dom*:f y = *Some* x $y \in$ *dom* f $y \notin$ *dom* g
by (*auto simp add*:*ran-def*)

with *ran* **have** $(f$ ++ $g)$ y = *Some* x
by (*simp*)

with *assms minj ran dom* **show** *map-inv* $(f$ ++ $g)$ x = (*map-inv* f ++ *map-inv* g) x
by *simp*
qed

moreover from *assms minj* **have** $\llbracket$ $x \notin$ *ran* g; $x \notin$ *ran* f $\rrbracket \Longrightarrow$ *map-inv* $(f$ ++ $g)$ x = (*map-inv* f ++ *map-inv* g) x
apply (*auto simp add*:*map-inv-def ran-def map-add-def*)
apply (*metis dom-left-map-add map-add-def map-add-dom-app-simps*(*3*))
done

ultimately show *map-inv* $(f$ ++ $g)$ x = (*map-inv* f ++ *map-inv* g) x
apply (*case-tac* $x \in$ *ran* g)
apply (*simp*)
apply (*case-tac* $x \in$ *ran* f)
apply (*simp-all*)
done
qed

lemma *map-add-lookup* [*simp*]:
$x \notin$ *dom* $f \Longrightarrow$ ([$x \mapsto y$] ++ f) x = *Some* y
by (*simp add*:*map-add-def dom-def*)

lemma *map-add-Some*: *Some* ++ f = *map-id-on* (− *dom* f) ++ f
apply (*rule ext*)
apply (*rename-tac* x)
apply (*case-tac* $x \in$ *dom* f)
apply (*simp-all*)
done

lemma *distinct-map-dom*:
$x \notin$ *set xs* $\Longrightarrow x \notin$ *dom* [*xs* [$\mapsto$] *ys*]
by (*simp add*:*dom-def*)

lemma *distinct-map-ran*:
$\llbracket$ *distinct xs*; $y \notin$ *set ys*; *length xs* = *length ys* $\rrbracket \Longrightarrow$
$y \notin$ *ran* ([*xs* [$\mapsto$] *ys*])
apply (*simp add*:*map-upds-def*)
apply (*subgoal-tac distinct* (*map fst* (*rev* (*zip xs ys*))))
apply (*simp add*:*ran-distinct*)
apply (*metis* (*hide-lams*, *no-types*) *image-iff set-zip-rightD surjective-pairing*)
apply (*simp add*:*zip-rev*[*THEN sym*])
done

lemma *maplets-lookup*[*rule-format*,*dest*]:
$\llbracket$ *length xs* = *length ys*; *distinct xs* $\rrbracket \Longrightarrow$

$\forall\ y.\ [xs\ [\mapsto]\ ys]\ x = Some\ y \longrightarrow y \in set\ ys$
by (*induct rule*:*list-induct2*, *auto*)

lemma *maplets-distinct-inj* [*intro*]:
$[\![$ *length xs* = *length ys*; *distinct xs*; *distinct ys*; *set xs* $\cap$ *set ys* = {} $]\!] \Longrightarrow$
inj-on [*xs* [$\mapsto$] *ys*] (*set xs*)
apply (*induct rule*:*list-induct2*)
apply (*simp-all*)
apply (*rule conjI*)
apply (*rule inj-onI*)
apply (*rename-tac x xs y ys xa ya*)
apply (*case-tac xa* = *x*)
apply (*simp*)
apply (*case-tac xa* = *y*)
apply (*simp*)
apply (*simp*)
apply (*case-tac ya* = *x*)
apply (*simp*)
apply (*simp add*:*inj-on-def*)
apply (*auto*)
apply (*rename-tac x xs y ys xa*)
apply (*case-tac xa* = *y*)
apply (*simp*)
apply (*metis maplets-lookup*)
done

lemma *map-inv-maplet*[*simp*]: *map-inv* [$x \mapsto y$] = [$y \mapsto x$]
by (*auto simp add*:*map-inv-def*)

lemma *map-inv-maplets* [*simp*]:
$[\![$ *length xs* = *length ys*; *distinct xs*; *distinct ys*; *set xs* $\cap$ *set ys* = {} $]\!] \Longrightarrow$
map-inv [*xs* [$\mapsto$] *ys*] = [*ys* [$\mapsto$] *xs*]
apply (*induct rule*:*list-induct2*)
apply (*simp-all*)
apply (*rename-tac x xs y ys*)
apply (*subgoal-tac map-inv* ([*xs* [$\mapsto$] *ys*] ++ [$x \mapsto y$]) = *map-inv* [*xs* [$\mapsto$] *ys*] ++ *map-inv* [$x \mapsto y$])
apply (*simp*)
apply (*rule map-inv-add*)
apply (*auto*)
done

lemma *maplets-lookup-nth* [*rule-format*,*simp*]:
$[\![$ *length xs* = *length ys*; *distinct xs* $]\!] \Longrightarrow$
$\forall\ i <$ *length ys*. [*xs* [$\mapsto$] *ys*] (*xs* ! *i*) = *Some* (*ys* ! *i*)
apply (*induct rule*:*list-induct2*)
apply (*auto*)
apply (*rename-tac x xs y ys i*)
apply (*case-tac i*)
apply (*simp-all*)
apply (*metis nth-mem*)
apply (*rename-tac x xs y ys i*)
apply (*case-tac i*)
apply (*auto*)
done

theorem *inv-map-inv*:
  ⟦ *inj-on f* (*dom f*); *ran f* = *dom f* ⟧
  $\Longrightarrow$ *inv* (*the* ∘ (*Some* ++ *f*)) = *the* ∘ *map-inv* (*Some* ++ *f*)
  apply (*rule ext*)
  apply (*simp add*:*map-add-Some*)
  apply (*simp add*:*inv-def*)
  apply (*rename-tac x*)
  apply (*case-tac* ∃ *y*. *f y* = *Some x*)
   apply (*erule exE*)
   apply (*rename-tac x y*)
   apply (*subgoal-tac x* ∈ *ran f*)
    apply (*subgoal-tac y* ∈ *dom f*)
     apply (*simp*)
     apply (*rule some-equality*)
      apply (*simp*)
     apply (*metis* (*hide-lams*, *mono-tags*) *domD domI dom-left-map-add inj-on-contraD map-add-Some map-add-dom-app-simps*(*3*) *option.sel*)
    apply (*simp add*:*dom-def*)
   apply (*metis ranI*)
  apply (*simp*)
  apply (*rename-tac x*)
  apply (*subgoal-tac x* ∉ *ran f*)
   apply (*simp*)
   apply (*rule some-equality*)
    apply (*simp*)
   apply (*metis domD dom-left-map-add map-add-Some map-add-dom-app-simps*(*3*) *option.sel*)
  apply (*metis dom-image-ran image-iff*)
  done

lemma *map-comp-dom*: *dom* ($g \circ_m f$) ⊆ *dom f*
  by (*metis* (*lifting*, *full-types*) *Collect-mono dom-def map-comp-simps*(*1*))

lemma *map-comp-assoc*: $f \circ_m (g \circ_m h) = f \circ_m g \circ_m h$
proof
  fix *x*
  show $(f \circ_m (g \circ_m h))\ x = (f \circ_m g \circ_m h)\ x$
  proof (*cases h x*)
    case *None* **thus** *?thesis*
      by (*auto simp add*: *map-comp-def*)
  next
    case (*Some y*) **thus** *?thesis*
      by (*auto simp add*: *map-comp-def*)
  qed
qed

lemma *map-comp-runit* [*simp*]: $f \circ_m$ *Some* = *f*
  by (*simp add*: *map-comp-def*)

lemma *map-comp-lunit* [*simp*]: *Some* $\circ_m f = f$
proof
  fix *x*
  show (*Some* $\circ_m f$) *x* = *f x*
  proof (*cases f x*)
    case *None* **thus** *?thesis*
      by (*simp add*: *map-comp-def*)

next
  case (*Some y*) **thus** *?thesis*
    by (*simp add*: *map-comp-def*)
  qed
qed

lemma *map-comp-apply* [*simp*]: $(f \circ_m g)\ x = g(x) >>= f$
  by (*auto simp add*: *map-comp-def option.case-eq-if*)

7.9 Merging of compatible maps

definition *comp-map* :: $('a \rightharpoonup 'b) \Rightarrow ('a \rightharpoonup 'b) \Rightarrow bool$ (**infixl** $\|_m$ *60*) **where**
$comp\text{-}map\ f\ g = (\forall\ x \in dom(f) \cap dom(g).\ the(f(x)) = the(g(x)))$

lemma *comp-map-unit*: $Map.empty \|_m f$
  by (*simp add*: *comp-map-def*)

lemma *comp-map-refl*: $f \|_m f$
  by (*simp add*: *comp-map-def*)

lemma *comp-map-sym*: $f \|_m g \Longrightarrow g \|_m f$
  by (*simp add*: *comp-map-def*)

definition *merge* :: $('a \rightharpoonup 'b)\ set \Rightarrow 'a \rightharpoonup 'b$ **where**
merge fs =
  $(\lambda\ x.\ if\ (\exists\ f \in fs.\ x \in dom(f))\ then\ (THE\ y.\ \forall\ f \in fs.\ x \in dom(f) \longrightarrow f(x) = y)\ else\ None)$

lemma *merge-empty*: $merge\ \{\} = Map.empty$
  by (*simp add*: *merge-def*)

lemma *merge-singleton*: $merge\ \{f\} = f$
  apply (*auto intro*!: *ext simp add*: *merge-def*)
  using *option.collapse* **apply** *fastforce*
  done

7.10 Conversion between lists and maps

definition *map-of-list* :: $'a\ list \Rightarrow (nat \rightharpoonup 'a)$ **where**
$map\text{-}of\text{-}list\ xs = (\lambda\ i.\ if\ (i < length\ xs)\ then\ Some\ (xs!i)\ else\ None)$

lemma *map-of-list-nil* [*simp*]: $map\text{-}of\text{-}list\ [] = Map.empty$
  by (*simp add*: *map-of-list-def*)

lemma *dom-map-of-list* [*simp*]: $dom\ (map\text{-}of\text{-}list\ xs) = \{0..<length\ xs\}$
  by (*auto simp add*: *map-of-list-def dom-def*)

lemma *ran-map-of-list* [*simp*]: $ran\ (map\text{-}of\text{-}list\ xs) = set\ xs$
  apply (*simp add*: *ran-def map-of-list-def*)
  apply (*safe*)
   apply (*force*)
  apply (*meson in-set-conv-nth*)
  done

definition *list-of-map* :: $(nat \rightharpoonup 'a) \Rightarrow 'a\ list$ **where**
$list\text{-}of\text{-}map\ f = (if\ (f = Map.empty)\ then\ []\ else\ map\ (the \circ f)\ [0\ ..<\ Suc(GREATEST\ x.\ x \in dom\ f)])$

lemma *list-of-map-empty* [*simp*]: *list-of-map Map.empty* = []
  by (*simp add*: *list-of-map-def*)

definition *list-of-map'* :: ($nat \rightharpoonup 'a$) $\rightharpoonup$ *'a list* **where**
list-of-map' f = (*if* ($\exists$ *n*. *dom f* = {*0*..<*n*}) *then Some* (*list-of-map f*) *else None*)

lemma *map-of-list-inv* [*simp*]: *list-of-map* (*map-of-list xs*) = *xs*
proof (*cases xs* = [])
  case *True* **thus** *?thesis* **by** (*simp*)
next
  case *False*
  moreover hence (*GREATEST x*. $x \in$ *dom* (*map-of-list xs*)) = *length xs* − *1*
    by (*auto intro*: *Greatest-equality*)
  moreover from *False* **have** *map-of-list xs* $\neq$ *Map.empty*
    by (*metis ran-empty ran-map-of-list set-empty*)
  ultimately show *?thesis*
    by (*auto simp add*: *list-of-map-def map-of-list-def nth-equalityI*)
qed

7.11 Map Comprehension

Map comprehension simply converts a relation built through set comprehension into a map.

syntax
  -Mapcompr :: $'a \Rightarrow 'b \Rightarrow idts \Rightarrow bool \Rightarrow 'a \rightharpoonup 'b$ ((*1*[- $\mapsto$ - |/-./ -]))

translations
  -Mapcompr F G xs P == *CONST graph-map* {(*F*, *G*) | *xs*. *P*}

lemma *map-compr-eta*:
  [$x \mapsto y$ | *x y*. (*x*, *y*) $\in_m$ *f*] = *f*
  apply (*rule ext*)
  apply (*auto simp add*: *graph-map-def*)
  apply (*metis* (*mono-tags*, *lifting*) *Domain.DomainI fst-eq-Domain mem-Collect-eq old.prod.case option.distinct*(*1*) *option.expand option.sel*)
  done

lemma *map-compr-simple*:
  [$x \mapsto$ *F x y* | *x y*. (*x*, *y*) $\in_m$ *f*] = (λ *x*. *do* { *y* $\leftarrow$ *f*(*x*); *Some*(*F x y*) })
  apply (*rule ext*)
  apply (*auto simp add*: *graph-map-def image-Collect*)
  done

lemma *map-compr-dom-simple* [*simp*]:
  dom [$x \mapsto$ *f x* | *x*. *P x*] = {*x*. *P x*}
  by (*force simp add*: *graph-map-dom image-Collect*)

lemma *map-compr-ran-simple* [*simp*]:
  ran [$x \mapsto$ *f x* | *x*. *P x*] = {*f x* | *x*. *P x*}
  apply (*auto simp add*: *graph-map-def ran-def*)
  apply (*metis* (*mono-tags*, *lifting*) *fst-eqD image-eqI mem-Collect-eq someI*)
  done

lemma *map-compr-eval-simple* [*simp*]:
  [$x \mapsto$ *f x* | *x*. *P x*] *x* = (*if* (*P x*) *then Some* (*f x*) *else None*)
  by (*auto simp add*: *graph-map-def image-Collect*)

7.12 Sorted lists from maps

definition *sorted-list-of-map* :: ('*a*::*linorder* ⇀ '*b*) ⇒ ('*a* × '*b*) *list* **where**
sorted-list-of-map f = *map* (λ *k*. (*k*, *the* (*f k*))) (*sorted-list-of-set*(*dom*(*f*)))

lemma *sorted-list-of-map-empty* [*simp*]:
  sorted-list-of-map Map.empty = []
  by (*simp add*: *sorted-list-of-map-def*)

lemma *sorted-list-of-map-inv*:
  assumes *finite*(*dom*(*f*))
  shows *map-of* (*sorted-list-of-map f*) = *f*
proof −
  obtain *A* **where** *finite A A* = *dom*(*f*)
    by (*simp add*: *assms*)
  thus *?thesis*
  proof (*induct A rule*: *finite-induct*)
    case *empty* **thus** *?thesis*
      by (*simp add*: *sorted-list-of-map-def*, *metis dom-empty empty-iff map-le-antisym map-le-def*)
  next
    case (*insert x A*) **thus** *?thesis*
      by (*simp add*: *assms sorted-list-of-map-def map-of-map-keys*)
  qed
qed

declare *map-member.simps* [*simp del*]

7.13 Extra map lemmas

lemma *map-eqI*:
  ⟦ *dom f* = *dom g*; ∀ *x*∈*dom*(*f*). *the*(*f x*) = *the*(*g x*) ⟧ ⟹ *f* = *g*
  by (*metis domIff map-le-antisym map-le-def option.expand*)

lemma *map-restrict-dom* [*simp*]: *f* |‘ *dom f* = *f*
  by (*simp add*: *map-eqI*)

lemma *map-restrict-dom-compl*: *f* |‘ (− *dom f*) = *Map.empty*
  by (*metis dom-eq-empty-conv dom-restrict inf-compl-bot*)

lemma *restrict-map-neg-disj*:
  dom(*f*) ∩ *A* = {} ⟹ *f* |‘ (− *A*) = *f*
  by (*auto simp add*: *restrict-map-def*, *rule ext*, *auto*, *metis disjoint-iff-not-equal domIff*)

lemma *map-plus-restrict-dist*: (*f* ++ *g*) |‘ *A* = (*f* |‘ *A*) ++ (*g* |‘ *A*)
  by (*auto simp add*: *restrict-map-def map-add-def*)

lemma *map-plus-eq-left*:
  assumes *f* ++ *h* = *g* ++ *h*
  shows (*f* |‘ (− *dom h*)) = (*g* |‘ (− *dom h*))
proof −
  have *h* |‘ (− *dom h*) = *Map.empty*
    by (*metis Compl-disjoint dom-eq-empty-conv dom-restrict*)
  then have *f2*: *f* |‘ (− *dom h*) = (*f* ++ *h*) |‘ (− *dom h*)
    by (*simp add*: *map-plus-restrict-dist*)
  have *h* |‘ (− *dom h*) = *Map.empty*
    by (*metis* (*no-types*) *Compl-disjoint dom-eq-empty-conv dom-restrict*)

then show *?thesis*
using *f2 assms* **by** (*simp add*: *map-plus-restrict-dist*)
qed

lemma *map-add-split*:
$dom(f) = A \cup B \Longrightarrow (f \mid` A) ++ (f \mid` B) = f$
by (*rule ext*, *auto simp add*: *map-add-def restrict-map-def option.case-eq-if*)

lemma *map-le-via-restrict*:
$f \subseteq_m g \longleftrightarrow g \mid` dom(f) = f$
by (*auto simp add*: *map-le-def restrict-map-def dom-def fun-eq-iff*)

lemma *map-add-cancel*:
$f \subseteq_m g \Longrightarrow f ++ (g -- f) = g$
by (*auto simp add*: *map-le-def map-add-def map-minus-def fun-eq-iff option.case-eq-if*)
(*metis domIff*)

lemma *map-le-iff-add*: $f \subseteq_m g \longleftrightarrow (\exists\ h.\ dom(f) \cap dom(h) = \{\} \wedge f ++ h = g)$
apply (*auto*)
apply (*rule-tac x=g -- f* **in** *exI*)
apply (*metis* (*no-types*, *lifting*) *Int-emptyI domIff map-add-cancel map-le-def map-minus-def*)
apply (*simp add*: *map-add-comm*)
done

lemma *map-add-comm-weak*: $(\forall\ k \in dom\ m1 \cap dom\ m2.\ m1(k) = m2(k)) \Longrightarrow m1 ++ m2 = m2 ++ m1$
by (*auto simp add*: *map-add-def option.case-eq-if fun-eq-iff*)
(*metis IntI domI option.inject*)

end

8 Alternative List Lexicographic Order

theory *List-Lexord-Alt*
imports *Main*
begin

Since we can't instantiate the order class twice for lists, and we want prefix as the default order for the UTP we here add syntax for the lexicographic order relation.

definition *list-lex-less* :: *'a::linorder list* ⇒ *'a list* ⇒ *bool* (**infix** $<_l$ *50*)
where $xs <_l ys \longleftrightarrow (xs, ys) \in lexord\ \{(u, v).\ u < v\}$

lemma *list-lex-less-neq* [*simp*]: $x <_l y \Longrightarrow x \neq y$
apply (*simp add*: *list-lex-less-def*)
apply (*meson case-prodD less-irrefl lexord-irreflexive mem-Collect-eq*)
done

lemma *not-less-Nil* [*simp*]: $\neg\ x <_l []$
by (*simp add*: *list-lex-less-def*)

lemma *Nil-less-Cons* [*simp*]: $[] <_l a \# x$
by (*simp add*: *list-lex-less-def*)

lemma *Cons-less-Cons* [*simp*]: $a \# x <_l b \# y \longleftrightarrow a < b \vee a = b \wedge x <_l y$
by (*simp add*: *list-lex-less-def*)

end

9 Partial Functions

theory *Partial-Fun*
imports *Optics.Lenses Map-Extra*
begin

I'm not completely satisfied with partial functions as provided by Map.thy, since they don't have a unique type and so we can't instantiate classes, make use of adhoc-overloading etc. Consequently I've created a new type and derived the laws.

9.1 Partial function type and operations

typedef $('a, 'b)$ *pfun* $=$ *UNIV* $:: ('a \rightharpoonup 'b)$ *set* **..**

setup-lifting *type-definition-pfun*

lift-definition *pfun-app* $:: ('a, 'b)$ *pfun* $\Rightarrow 'a \Rightarrow 'b$ $(\text{-}'(\text{-}')_p\ [999,0]\ 999)$ **is**
$\lambda\, f\, x.$ *if* $(x \in dom\ f)$ *then the* $(f\, x)$ *else undefined* **.**

lift-definition *pfun-upd* $:: ('a, 'b)$ *pfun* $\Rightarrow 'a \Rightarrow 'b \Rightarrow ('a, 'b)$ *pfun*
is $\lambda\, f\, k\, v.\ f(k := Some\ v)$ **.**

lift-definition *pdom* $:: ('a, 'b)$ *pfun* $\Rightarrow 'a$ *set* **is** *dom* **.**

lift-definition *pran* $:: ('a, 'b)$ *pfun* $\Rightarrow 'b$ *set* **is** *ran* **.**

lift-definition *pfun-comp* $:: ('b, 'c)$ *pfun* $\Rightarrow ('a, 'b)$ *pfun* $\Rightarrow ('a, 'c)$ *pfun* (**infixl** $\circ_p$ *55*) **is** *map-comp* **.**

lift-definition *pfun-member* $:: 'a \times 'b \Rightarrow ('a, 'b)$ *pfun* $\Rightarrow$ *bool* (**infix** $\in_p$ *50*) **is** $(\in_m)$ **.**

lift-definition *pId-on* $:: 'a$ *set* $\Rightarrow ('a, 'a)$ *pfun* **is** $\lambda\ A\ x.$ *if* $(x \in A)$ *then Some x else None* **.**

abbreviation *pId* $:: ('a, 'a)$ *pfun* **where**
pId $\equiv$ *pId-on UNIV*

lift-definition *plambda* $:: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a, 'b)$ *pfun*
is $\lambda\ P\, f\, x.$ *if* $(P\ x)$ *then Some* $(f\, x)$ *else None* **.**

lift-definition *pdom-res* $:: 'a$ *set* $\Rightarrow ('a, 'b)$ *pfun* $\Rightarrow ('a, 'b)$ *pfun* (**infixr** $\lhd_p$ *85*)
is $\lambda\ A\ f.$ *restrict-map f A* **.**

lift-definition *pran-res* $:: ('a, 'b)$ *pfun* $\Rightarrow 'b$ *set* $\Rightarrow ('a, 'b)$ *pfun* (**infixl** $\rhd_p$ *85*)
is *ran-restrict-map* **.**

lift-definition *pfun-graph* $:: ('a, 'b)$ *pfun* $\Rightarrow ('a \times 'b)$ *set* **is** *map-graph* **.**

lift-definition *graph-pfun* $:: ('a \times 'b)$ *set* $\Rightarrow ('a, 'b)$ *pfun* **is** *graph-map* **.**

lift-definition *pfun-entries* $:: 'k$ *set* $\Rightarrow ('k \Rightarrow 'v) \Rightarrow ('k, 'v)$ *pfun* **is**
$\lambda\ d\, f\, x.$ *if* $(x \in d)$ *then Some* $(f\, x)$ *else None* **.**

definition *pcard* $:: ('a, 'b)$ *pfun* $\Rightarrow$ *nat*
where *pcard f* $=$ *card* $(pdom\ f)$

instantiation *pfun* :: (*type*, *type*) *zero*
begin
lift-definition *zero-pfun* :: ('*a*, '*b*) *pfun* **is** *Map.empty* **.**
instance ..
end

abbreviation *pempty* :: ('*a*, '*b*) *pfun* ($\{\}_p$)
where *pempty* ≡ *0*

instantiation *pfun* :: (*type*, *type*) *plus*
begin
lift-definition *plus-pfun* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* **is** (++) **.**
instance ..
end

instantiation *pfun* :: (*type*, *type*) *minus*
begin
lift-definition *minus-pfun* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* **is** (−−) **.**
instance ..
end

instance *pfun* :: (*type*, *type*) *monoid-add*
  by (*intro-classes*, (*transfer*, *auto*)+)

instantiation *pfun* :: (*type*, *type*) *inf*
begin
lift-definition *inf-pfun* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* **is**
λ *f g x*. *if* (*x* ∈ *dom*(*f*) ∩ *dom*(*g*) ∧ *f*(*x*) = *g*(*x*)) *then f*(*x*) *else None* **.**
instance ..
end

abbreviation *pfun-inter* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* (**infixl** $\cap_p$ *80*)
where *pfun-inter* ≡ *inf*

instantiation *pfun* :: (*type*, *type*) *order*
begin
  lift-definition *less-eq-pfun* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ *bool* **is**
  λ *f g*. *f* $\subseteq_m$ *g* **.**
  lift-definition *less-pfun* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ *bool* **is**
  λ *f g*. *f* $\subseteq_m$ *g* ∧ *f* ≠ *g* **.**
instance
  by (*intro-classes*, (*transfer*, *auto intro*: *map-le-trans simp add*: *map-le-antisym*)+)
end

abbreviation *pfun-subset* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ *bool* (**infix** $\subset_p$ *50*)
where *pfun-subset* ≡ *less*

abbreviation *pfun-subset-eq* :: ('*a*, '*b*) *pfun* ⇒ ('*a*, '*b*) *pfun* ⇒ *bool* (**infix** $\subseteq_p$ *50*)
where *pfun-subset-eq* ≡ *less-eq*

instance *pfun* :: (*type*, *type*) *semilattice-inf*
  by (*intro-classes*, (*transfer*, *auto simp add*: *map-le-def dom-def*)+)

lemma *pfun-subset-eq-least* [*simp*]:

$\{\}_p \subseteq_p f$
by (*transfer*, *auto*)

syntax
-PfunUpd :: [('a, 'b) *pfun*, *maplets*] => ('a, 'b) *pfun* (-'(-')$_p$ [*900*,*0*]*900*)
-Pfun :: *maplets* => ('a, 'b) *pfun* ((*1*{-}$_p$))
-plam :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (λ - | - . - [*0*,*0*,*10*] *10*)

translations
-PfunUpd m (*-Maplets xy ms*) == *-PfunUpd* (*-PfunUpd m xy*) *ms*
-PfunUpd m (*-maplet x y*) == *CONST pfun-upd m x y*
-Pfun ms => *-PfunUpd* (*CONST pempty*) *ms*
-Pfun (*-Maplets ms1 ms2*) <= *-PfunUpd* (*-Pfun ms1*) *ms2*
-Pfun ms <= *-PfunUpd* (*CONST pempty*) *ms*
λ *x* | *P* . *e* => *CONST plambda* (λ *x*. *P*) (λ *x*. *e*)
λ *x* | *P* . *e* <= *CONST plambda* (λ *x*. *P*) (λ *y*. *e*)
λ *y* | *P* . *e* <= *CONST plambda* (λ *x*. *P*) (λ *y*. *e*)
λ *y* | *f v y* . *e* <= *CONST plambda* (*f v*) (λ *y*. *e*)

9.2 Algebraic laws

lemma *pfun-comp-assoc*: $f \circ_p (g \circ_p h) = (f \circ_p g) \circ_p h$
by (*transfer*, *simp add*: *map-comp-assoc*)

lemma *pfun-comp-left-id* [*simp*]: $pId \circ_p f = f$
by (*transfer*, *auto*)

lemma *pfun-comp-right-id* [*simp*]: $f \circ_p pId = f$
by (*transfer*, *auto*)

lemma *pfun-override-dist-comp*:
$(f + g) \circ_p h = (f \circ_p h) + (g \circ_p h)$
apply (*transfer*)
apply (*rule ext*)
apply (*auto simp add*: *map-add-def*)
apply (*rename-tac f g h x*)
apply (*case-tac h x*)
apply (*auto*)
apply (*rename-tac f g h x y*)
apply (*case-tac g y*)
apply (*auto*)
done

lemma *pfun-minus-unit* [*simp*]:
fixes *f* :: ('a, 'b) *pfun*
shows $f - 0 = f$
by (*transfer*, *simp add*: *map-minus-def*)

lemma *pfun-minus-zero* [*simp*]:
fixes *f* :: ('a, 'b) *pfun*
shows $0 - f = 0$
by (*transfer*, *simp add*: *map-minus-def*)

lemma *pfun-minus-self* [*simp*]:
fixes *f* :: ('a, 'b) *pfun*
shows $f - f = 0$

by (*transfer*, *simp add*: *map-minus-def*)

lemma *pfun-plus-commute*:
$pdom(f) \cap pdom(g) = \{\} \Longrightarrow f + g = g + f$
by (*transfer*, *metis map-add-comm*)

lemma *pfun-plus-commute-weak*:
$(\forall\ k \in pdom(f) \cap pdom(g).\ f(k)_p = g(k)_p) \Longrightarrow f + g = g + f$
by (*transfer*, *simp*, *metis IntD1 IntD2 domD map-add-comm-weak option.sel*)

lemma *pfun-minus-plus-commute*:
$pdom(g) \cap pdom(h) = \{\} \Longrightarrow (f - g) + h = (f + h) - g$
by (*transfer*, *simp add*: *map-minus-plus-commute*)

lemma *pfun-plus-minus*:
$f \subseteq_p g \Longrightarrow (g - f) + f = g$
by (*transfer*, *rule ext*, *auto simp add*: *map-le-def map-minus-def map-add-def option.case-eq-if*)

lemma *pfun-minus-common-subset*:
$[\![\ h \subseteq_p f;\ h \subseteq_p g\ ]\!] \Longrightarrow (f - h = g - h) = (f = g)$
by (*transfer*, *simp add*: *map-minus-common-subset*)

lemma *pfun-minus-plus*:
$pdom(f) \cap pdom(g) = \{\} \Longrightarrow (f + g) - g = f$
by (*transfer*, *simp add*: *map-add-def map-minus-def option.case-eq-if*, *rule ext*, *auto*)
(*metis Int-commute domIff insert-disjoint*(*1*) *insert-dom*)

lemma *pfun-plus-pos*: $x + y = \{\}_p \Longrightarrow x = \{\}_p$
by (*transfer*, *simp*)

lemma *pfun-le-plus*: $pdom\ x \cap pdom\ y = \{\} \Longrightarrow x \leq x + y$
by (*transfer*, *auto simp add*: *map-le-iff-add*)

9.3 Lambda abstraction

lemma *plambda-app* [*simp*]: $(\lambda\ x\ |\ P\ x\ .\ f\ x)(v)_p = (\mathit{if}\ (P\ v)\ \mathit{then}\ (f\ v)\ \mathit{else}\ \mathit{undefined})$
by (*transfer*, *auto*)

lemma *plambda-eta* [*simp*]: $(\lambda\ x\ |\ x \in pdom(f).\ f(x)_p) = f$
by (*transfer*; *auto simp add*: *domIff*)

lemma *plambda-id* [*simp*]: $(\lambda\ x\ |\ P\ x\ .\ x) = pId\text{-}on\ \{x.\ P\ x\}$
by (*transfer*, *simp*)

9.4 Membership, application, and update

lemma *pfun-ext*: $[\![\ \bigwedge x\ y.\ (x, y) \in_p f \longleftrightarrow (x, y) \in_p g\ ]\!] \Longrightarrow f = g$
by (*transfer*, *simp add*: *map-ext*)

lemma *pfun-member-alt-def*:
$(x, y) \in_p f \longleftrightarrow (x \in pdom\ f \wedge f(x)_p = y)$
by (*transfer*, *auto simp add*: *map-member-alt-def map-apply-def*)

lemma *pfun-member-plus*:
$(x, y) \in_p f + g \longleftrightarrow ((x \notin pdom(g) \wedge (x, y) \in_p f) \vee (x, y) \in_p g)$
by (*transfer*, *simp add*: *map-member-plus*)

lemma *pfun-member-minus*:
  $(x, y) \in_p f - g \longleftrightarrow (x, y) \in_p f \wedge (\neg (x, y) \in_p g)$
  by (*transfer*, *simp add*: *map-member-minus*)

lemma *pfun-app-upd-1* [*simp*]: $x = y \Longrightarrow (f(x \mapsto v)_p)(y)_p = v$
  by (*transfer*, *simp*)

lemma *pfun-app-upd-2* [*simp*]: $x \neq y \Longrightarrow (f(x \mapsto v)_p)(y)_p = f(y)_p$
  by (*transfer*, *simp*)

lemma *pfun-graph-apply* [*simp*]: *rel-apply* (*pfun-graph* f) $x = f(x)_p$
  by (*transfer*, *auto simp add*: *rel-apply-def map-graph-def*)

lemma *pfun-upd-ext* [*simp*]: $x \in pdom(f) \Longrightarrow f(x \mapsto f(x)_p)_p = f$
  by (*transfer*, *simp add*: *domIff*)

lemma *pfun-app-add* [*simp*]: $x \in pdom(g) \Longrightarrow (f + g)(x)_p = g(x)_p$
  by (*transfer*, *auto*)

lemma *pfun-upd-add* [*simp*]: $f + g(x \mapsto v)_p = (f + g)(x \mapsto v)_p$
  by (*transfer*, *simp*)

lemma *pfun-upd-twice* [*simp*]: $f(x \mapsto u, x \mapsto v)_p = f(x \mapsto v)_p$
  by (*transfer*, *simp*)

lemma *pfun-upd-comm*:
  assumes $x \neq y$
  shows $f(y \mapsto u, x \mapsto v)_p = f(x \mapsto v, y \mapsto u)_p$
  using *assms* **by** (*transfer*, *auto*)

lemma *pfun-upd-comm-linorder* [*simp*]:
  fixes $x\ y$ :: $'a$ :: *linorder*
  assumes $x < y$
  shows $f(y \mapsto u, x \mapsto v)_p = f(x \mapsto v, y \mapsto u)_p$
  using *assms* **by** (*transfer*, *auto*)

lemma *pfun-app-minus* [*simp*]: $x \notin pdom\ g \Longrightarrow (f - g)(x)_p = f(x)_p$
  by (*transfer*, *auto simp add*: *map-minus-def*)

lemma *pfun-app-empty* [*simp*]: $\{\}_p(x)_p = undefined$
  by (*transfer*, *simp*)

lemma *pfun-app-not-in-dom*:
  $x \notin pdom(f) \Longrightarrow f(x)_p = undefined$
  by (*transfer*, *simp*)

lemma *pfun-upd-minus* [*simp*]:
  $x \notin pdom\ g \Longrightarrow (f - g)(x \mapsto v)_p = (f(x \mapsto v)_p - g)$
  by (*transfer*, *auto simp add*: *map-minus-def*)

lemma *pdom-member-minus-iff* [*simp*]:
  $x \notin pdom\ g \Longrightarrow x \in pdom(f - g) \longleftrightarrow x \in pdom(f)$
  by (*transfer*, *simp add*: *domIff map-minus-def*)

lemma *psubseteq-pfun-upd1* [*intro*]:
$\llbracket f \subseteq_p g;\ x \notin pdom(g) \rrbracket \Longrightarrow f \subseteq_p g(x \mapsto v)_p$
by (*transfer*, *auto simp add*: *map-le-def dom-def*)

lemma *psubseteq-pfun-upd2* [*intro*]:
$\llbracket f \subseteq_p g;\ x \notin pdom(f) \rrbracket \Longrightarrow f \subseteq_p g(x \mapsto v)_p$
by (*transfer*, *auto simp add*: *map-le-def dom-def*)

lemma *psubseteq-pfun-upd3* [*intro*]:
$\llbracket f \subseteq_p g;\ g(x)_p = v \rrbracket \Longrightarrow f \subseteq_p g(x \mapsto v)_p$
by (*transfer*, *auto simp add*: *map-le-def dom-def*)

lemma *psubseteq-dom-subset*:
$f \subseteq_p g \Longrightarrow pdom(f) \subseteq pdom(g)$
by (*transfer*, *auto simp add*: *map-le-def dom-def*)

lemma *psubseteq-ran-subset*:
$f \subseteq_p g \Longrightarrow pran(f) \subseteq pran(g)$
by (*transfer*, *auto simp add*: *map-le-def dom-def ran-def*, *fastforce*)

9.5 Domain laws

lemma *pdom-zero* [*simp*]: $pdom\ 0 = \{\}$
by (*transfer*, *simp*)

lemma *pdom-pId-on* [*simp*]: $pdom\ (pId\text{-}on\ A) = A$
by (*transfer*, *auto*)

lemma *pdom-plus* [*simp*]: $pdom\ (f + g) = pdom\ f \cup pdom\ g$
by (*transfer*, *auto*)

lemma *pdom-minus* [*simp*]: $g \leq f \Longrightarrow pdom\ (f - g) = pdom\ f - pdom\ g$
apply (*transfer*, *auto simp add*: *map-minus-def*)
apply (*meson option.distinct*(*1*))
apply (*metis domIff map-le-def option.simps*(*3*))
done

lemma *pdom-inter*: $pdom\ (f \cap_p g) \subseteq pdom\ f \cap pdom\ g$
by (*transfer*, *auto simp add*: *dom-def*)

lemma *pdom-comp* [*simp*]: $pdom\ (g \circ_p f) = pdom\ (f \rhd_p pdom\ g)$
by (*transfer*, *auto simp add*: *ran-restrict-map-def*)

lemma *pdom-upd* [*simp*]: $pdom\ (f(k \mapsto v)_p) = insert\ k\ (pdom\ f)$
by (*transfer*, *simp*)

lemma *pdom-plamda* [*simp*]: $pdom\ (\lambda\ x \mid P\ x\ .\ f\ x) = \{x.\ P\ x\}$
by (*transfer*, *auto*)

lemma *pdom-pdom-res* [*simp*]: $pdom\ (A \lhd_p f) = A \cap pdom(f)$
by (*transfer*, *auto*)

lemma *pdom-graph-pfun* [*simp*]: $pdom\ (graph\text{-}pfun\ R) = Domain\ R$
by (*transfer*, *simp add*: *Domain-fst graph-map-dom*)

lemma *pdom-pran-res-finite* [*simp*]:

$finite\ (pdom\ f) \Longrightarrow finite\ (pdom\ (f \rhd_p A))$
by (*transfer*, *auto*)

lemma *pdom-pfun-graph-finite* [*simp*]:
$finite\ (pdom\ f) \Longrightarrow finite\ (pfun\text{-}graph\ f)$
by (*transfer*, *simp add*: *finite-dom-graph*)

9.6 Range laws

lemma *pran-zero* [*simp*]: $pran\ 0 = \{\}$
by (*transfer*, *simp*)

lemma *pran-pId-on* [*simp*]: $pran\ (pId\text{-}on\ A) = A$
by (*transfer*, *auto simp add*: *ran-def*)

lemma *pran-upd* [*simp*]: $pran\ (f(k \mapsto v)_p) = insert\ v\ (pran\ ((- \{k\}) \lhd_p f))$
by (*transfer*, *auto simp add*: *ran-def restrict-map-def*)

lemma *pran-plamda* [*simp*]: $pran\ (\lambda\ x \mid P\ x\ .\ f\ x) = \{f\ x \mid x.\ P\ x\}$
by (*transfer*, *auto simp add*: *ran-def*)

lemma *pran-pran-res* [*simp*]: $pran\ (f \rhd_p A) = pran(f) \cap A$
by (*transfer*, *auto*)

lemma *pran-comp* [*simp*]: $pran\ (g \circ_p f) = pran\ (pran\ f \lhd_p g)$
by (*transfer*, *auto simp add*: *ran-def restrict-map-def*)

lemma *pran-finite* [*simp*]: $finite\ (pdom\ f) \Longrightarrow finite\ (pran\ f)$
by (*transfer*, *auto*)

9.7 Domain restriction laws

lemma *pdom-res-zero* [*simp*]: $A \lhd_p \{\}_p = \{\}_p$
by (*transfer*, *auto*)

lemma *pdom-res-empty* [*simp*]:
$(\{\} \lhd_p f) = \{\}_p$
by (*transfer*, *auto*)

lemma *pdom-res-pdom* [*simp*]:
$pdom(f) \lhd_p f = f$
by (*transfer*, *auto*)

lemma *pdom-res-UNIV* [*simp*]: $UNIV \lhd_p f = f$
by (*transfer*, *auto*)

lemma *pdom-res-alt-def*: $A \lhd_p f = f \circ_p pId\text{-}on\ A$
by (*transfer*, *rule ext*, *auto simp add*: *restrict-map-def*)

lemma *pdom-res-upd-in* [*simp*]:
$k \in A \Longrightarrow A \lhd_p f(k \mapsto v)_p = (A \lhd_p f)(k \mapsto v)_p$
by (*transfer*, *auto*)

lemma *pdom-res-upd-out* [*simp*]:
$k \notin A \Longrightarrow A \lhd_p f(k \mapsto v)_p = A \lhd_p f$
by (*transfer*, *auto*)

lemma *pfun-pdom-antires-upd* [*simp*]:
$k \in A \Longrightarrow ((- A) \lhd_p m)(k \mapsto v)_p = ((- (A - \{k\})) \lhd_p m)(k \mapsto v)_p$
by (*transfer*, *simp*)

lemma *pdom-antires-insert-notin* [*simp*]:
$k \notin pdom(f) \Longrightarrow (- \textit{insert } k\ A) \lhd_p f = (- A) \lhd_p f$
by (*transfer*, *auto simp add*: *restrict-map-def*)

lemma *pdom-res-override* [*simp*]: $A \lhd_p (f + g) = (A \lhd_p f) + (A \lhd_p g)$
by (*simp add*: *pdom-res-alt-def pfun-override-dist-comp*)

lemma *pdom-res-minus* [*simp*]: $A \lhd_p (f - g) = (A \lhd_p f) - g$
by (*transfer*, *auto simp add*: *map-minus-def restrict-map-def*)

lemma *pdom-res-swap*: $(A \lhd_p f) \rhd_p B = A \lhd_p (f \rhd_p B)$
by (*transfer*, *auto simp add*: *restrict-map-def ran-restrict-map-def*)

lemma *pdom-res-twice* [*simp*]: $A \lhd_p (B \lhd_p f) = (A \cap B) \lhd_p f$
by (*transfer*, *auto simp add*: *Int-commute*)

lemma *pdom-res-comp* [*simp*]: $A \lhd_p (g \circ_p f) = g \circ_p (A \lhd_p f)$
by (*simp add*: *pdom-res-alt-def pfun-comp-assoc*)

lemma *pdom-res-apply* [*simp*]:
$x \in A \Longrightarrow (A \lhd_p f)(x)_p = f(x)_p$
by (*transfer*, *auto*)

9.8 Range restriction laws

lemma *pran-res-zero* [*simp*]: $\{\}_p \rhd_p A = \{\}_p$
by (*transfer*, *auto simp add*: *ran-restrict-map-def*)

lemma *pran-res-upd-1* [*simp*]: $v \in A \Longrightarrow f(x \mapsto v)_p \rhd_p A = (f \rhd_p A)(x \mapsto v)_p$
by (*transfer*, *auto simp add*: *ran-restrict-map-def*)

lemma *pran-res-upd-2* [*simp*]: $v \notin A \Longrightarrow f(x \mapsto v)_p \rhd_p A = ((- \{x\}) \lhd_p f) \rhd_p A$
by (*transfer*, *auto simp add*: *ran-restrict-map-def*)

lemma *pran-res-alt-def*: $f \rhd_p A = \textit{pId-on } A \circ_p f$
by (*transfer*, *rule ext*, *auto simp add*: *ran-restrict-map-def*)

lemma *pran-res-override*: $(f + g) \rhd_p A \subseteq_p (f \rhd_p A) + (g \rhd_p A)$
apply (*transfer*, *auto simp add*: *map-add-def ran-restrict-map-def map-le-def*)
apply (*rename-tac f g A a y x*)
apply (*case-tac g a*)
 apply (*auto*)
done

9.9 Graph laws

lemma *pfun-graph-inv*: *graph-pfun* (*pfun-graph f*) = *f*
by (*transfer*, *simp*)

lemma *pfun-graph-zero*: *pfun-graph* $0 = \{\}$
by (*transfer*, *simp add*: *map-graph-def*)

lemma *pfun-graph-pId-on*: *pfun-graph* (*pId-on A*) = *Id-on A*
  by (*transfer*, *auto simp add*: *map-graph-def*)

lemma *pfun-graph-minus*: *pfun-graph* ($f - g$) = *pfun-graph f* − *pfun-graph g*
  by (*transfer*, *simp add*: *map-graph-minus*)

lemma *pfun-graph-inter*: *pfun-graph* ($f \cap_p g$) = *pfun-graph f* ∩ *pfun-graph g*
  apply (*transfer*, *auto simp add*: *map-graph-def*)
   apply (*metis option.discI*)+
  done

9.10 Entries

lemma *pfun-entries-empty* [*simp*]: *pfun-entries* {} f = $\{\}_p$
  by (*transfer*, *simp*)

lemma *pfun-entries-apply-1* [*simp*]:
  $x \in d \Longrightarrow$ (*pfun-entries d f*)$(x)_p = f\ x$
  by (*transfer*, *auto*)

lemma *pfun-entries-apply-2* [*simp*]:
  $x \notin d \Longrightarrow$ (*pfun-entries d f*)$(x)_p$ = *undefined*
  by (*transfer*, *auto*)

9.11 Summation

definition *pfun-sum* :: (*'k*, *'v*::*comm-monoid-add*) *pfun* ⇒ *'v* **where**
pfun-sum f = *sum* (*pfun-app f*) (*pdom f*)

lemma *pfun-sum-empty* [*simp*]: *pfun-sum* $\{\}_p$ = *0*
  by (*simp add*: *pfun-sum-def*)

lemma *pfun-sum-upd-1*:
  assumes *finite*(*pdom*(*m*)) $k \notin$ *pdom*(*m*)
  shows *pfun-sum* $(m(k \mapsto v)_p)$ = *pfun-sum m* + *v*
  by (*simp-all add*: *pfun-sum-def assms*, *metis add.commute assms*(*2*) *pfun-app-upd-2 sum.cong*)

lemma *pfun-sums-upd-2*:
  assumes *finite*(*pdom*(*m*))
  shows *pfun-sum* $(m(k \mapsto v)_p)$ = *pfun-sum* ((− {*k*}) $\lhd_p$ *m*) + *v*
proof (*cases* $k \notin$ *pdom*(*m*))
  case *True*
  then show *?thesis*
    by (*simp add*: *pfun-sum-upd-1 assms*)
next
  case *False*
  then show *?thesis*
    using *assms pfun-sum-upd-1*[*of* ((− {*k*}) $\lhd_p$ *m*) *k v*]
    by (*simp add*: *pfun-sum-upd-1*)
qed

lemma *pfun-sum-dom-res-insert* [*simp*]:
  assumes $x \in$ *pdom f* $x \notin A$ *finite A*
  shows *pfun-sum* ((*insert x A*) $\lhd_p$ *f*) = $f(x)_p$ + *pfun-sum* ($A \lhd_p f$)
  using *assms* **by** (*simp add*: *pfun-sum-def*)

lemma *pfun-sum-pdom-res*:
fixes f :: ('a,'b::ab-group-add) pfun
assumes *finite*(*pdom* f)
shows *pfun-sum* $(A \lhd_p f) = $ *pfun-sum* f $-$ (*pfun-sum* $((- A) \lhd_p f)$)
proof $-$
have *1*:$A \cap$ *pdom*(f) = *pdom*(f) $-$ (*pdom*(f) $- A$)
by (*auto*)
show *?thesis*
apply (*simp add*: *pfun-sum-def*)
apply (*subst 1*)
apply (*subst sum-diff*)
apply (*auto simp add*: *sum-diff Diff-subset Int-commute boolean-algebra-class.diff-eq assms*)
done
qed

lemma *pfun-sum-pdom-antires* [*simp*]:
fixes f :: ('a,'b::ab-group-add) pfun
assumes *finite*(*pdom* f)
shows *pfun-sum* $((- A) \lhd_p f) = $ *pfun-sum* f $-$ *pfun-sum* $(A \lhd_p f)$
by (*subst pfun-sum-pdom-res*, *simp-all add*: *assms*)

9.12 Partial Function Lens

definition *pfun-lens* :: $'a \Rightarrow ('b \Longrightarrow ('a, 'b)$ *pfun*) **where**
[*lens-defs*]: *pfun-lens* i = (| *lens-get* = λ s. $s(i)_p$, *lens-put* = λ s v. $s(i \mapsto v)_p$ |)

lemma *pfun-lens-mwb* [*simp*]: *mwb-lens* (*pfun-lens* i)
by (*unfold-locales*, *simp-all add*: *pfun-lens-def*)

lemma *pfun-lens-src*: $\mathcal{S}_{pfun\text{-}lens\ i} = \{f.\ i \in pdom(f)\}$
by (*auto simp add*: *lens-defs lens-source-def*, *transfer*, *force*)

Hide implementation details for partial functions

lifting-update *pfun.lifting*
lifting-forget *pfun.lifting*

end

10 Finite Functions

theory *Finite-Fun*
imports *Map-Extra Partial-Fun FSet-Extra*
begin

10.1 Finite function type and operations

typedef ('a, 'b) *ffun* = {f :: ('a, 'b) *pfun*. *finite*(*pdom*(f))}
morphisms *pfun-of Abs-pfun*
by (*rule-tac* x={}$_p$ **in** *exI*, *auto*)

setup-lifting *type-definition-ffun*

lift-definition *ffun-app* :: ('a, 'b) *ffun* $\Rightarrow$ 'a $\Rightarrow$ 'b (-'(-')$_f$ [*999*,*0*] *999*) **is** *pfun-app* **.**

lift-definition *ffun-upd* :: $('a, 'b)$ *ffun* $\Rightarrow$ $'a \Rightarrow 'b \Rightarrow$ $('a, 'b)$ *ffun* **is** *pfun-upd* **by** *simp*

lift-definition *fdom* :: $('a, 'b)$ *ffun* $\Rightarrow$ $'a$ *set* **is** *pdom* **.**

lift-definition *fran* :: $('a, 'b)$ *ffun* $\Rightarrow$ $'b$ *set* **is** *pran* **.**

lift-definition *ffun-comp* :: $('b, 'c)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'c)$ *ffun* (**infixl** $\circ_f$ *55*) **is** *pfun-comp* **by** *auto*

lift-definition *ffun-member* :: $'a \times 'b \Rightarrow ('a, 'b)$ *ffun* $\Rightarrow$ *bool* (**infix** $\in_f$ *50*) **is** $(\in_p)$ **.**

lift-definition *fdom-res* :: $'a$ *set* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* (**infixl** $\lhd_f$ *85*)
is *pdom-res* **by** *simp*

lift-definition *fran-res* :: $('a, 'b)$ *ffun* $\Rightarrow$ $'b$ *set* $\Rightarrow$ $('a, 'b)$ *ffun* (**infixl** $\rhd_f$ *85*)
is *pran-res* **by** *simp*

lift-definition *ffun-graph* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a \times 'b)$ *set* **is** *pfun-graph* **.**

lift-definition *graph-ffun* :: $('a \times 'b)$ *set* $\Rightarrow$ $('a, 'b)$ *ffun* **is**
λ *R. if* (*finite* (*Domain R*)) *then graph-pfun R else pempty*
by (*simp add*: *finite-Domain*)

instantiation *ffun* :: (*type*, *type*) *zero*
begin
lift-definition *zero-ffun* :: $('a, 'b)$ *ffun* **is** *0* **by** *simp*
instance **..**
end

abbreviation *fempty* :: $('a, 'b)$ *ffun* $(\{\}_f)$
where *fempty* $\equiv$ *0*

instantiation *ffun* :: (*type*, *type*) *plus*
begin
lift-definition *plus-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** $(+)$ **by** *simp*
instance **..**
end

instantiation *ffun* :: (*type*, *type*) *minus*
begin
lift-definition *minus-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** $(-)$
by (*metis finite-Diff finite-Domain pdom-graph-pfun pdom-pfun-graph-finite pfun-graph-inv pfun-graph-minus*)
instance **..**
end

instance *ffun* :: (*type*, *type*) *monoid-add*
by (*intro-classes*, (*transfer*, *simp add*: *add.assoc*)+)

instantiation *ffun* :: (*type*, *type*) *inf*
begin
lift-definition *inf-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** *inf*
by (*meson finite-Int infinite-super pdom-inter*)
instance **..**
end

abbreviation *ffun-inter* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* (**infixl** $\cap_f$ *80*)
where *ffun-inter* $\equiv$ *inf*

instantiation *ffun* :: (*type*, *type*) *order*
begin
  lift-definition *less-eq-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* **is**
  $\lambda\, f\, g.\ f \subseteq_p g$ **.**
  lift-definition *less-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* **is**
  $\lambda\, f\, g.\ f < g$ **.**
instance
  by (*intro-classes*, (*transfer*, *auto*)+)
end

abbreviation *ffun-subset* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* (**infix** $\subset_f$ *50*)
where *ffun-subset* $\equiv$ *less*

abbreviation *ffun-subset-eq* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* (**infix** $\subseteq_f$ *50*)
where *ffun-subset-eq* $\equiv$ *less-eq*

instance *ffun* :: (*type*, *type*) *semilattice-inf*
  by (*intro-classes*, (*transfer*, *auto*)+)

lemma *ffun-subset-eq-least* [*simp*]:
  $\{\}_f \subseteq_f f$
  by (*transfer*, *auto*)

syntax
  -FfunUpd :: [$('a, 'b)$ *ffun*, *maplets*] => $('a, 'b)$ *ffun* (-'(-')$_f$ [*900*,*0*]*900*)
  -Ffun :: *maplets* => $('a, 'b)$ *ffun* ((*1*{-}$_f$))

translations
  -FfunUpd m (*-Maplets xy ms*) == *-FfunUpd* (*-FfunUpd m xy*) *ms*
  -FfunUpd m (*-maplet x y*) == *CONST ffun-upd m x y*
  -Ffun ms => *-FfunUpd* (*CONST fempty*) *ms*
  -Ffun (*-Maplets ms1 ms2*) <= *-FfunUpd* (*-Ffun ms1*) *ms2*
  -Ffun ms <= *-FfunUpd* (*CONST fempty*) *ms*

10.2 Algebraic laws

lemma *ffun-comp-assoc*: $f \circ_f (g \circ_f h) = (f \circ_f g) \circ_f h$
  by (*transfer*, *simp add*: *pfun-comp-assoc*)

lemma *pfun-override-dist-comp*:
  $(f + g) \circ_f h = (f \circ_f h) + (g \circ_f h)$
  by (*transfer*, *simp add*: *pfun-override-dist-comp*)

lemma *ffun-minus-unit* [*simp*]:
  fixes f :: $('a, 'b)$ *ffun*
  shows $f - 0 = f$
  by (*transfer*, *simp*)

lemma *ffun-minus-zero* [*simp*]:
  fixes f :: $('a, 'b)$ *ffun*
  shows $0 - f = 0$
  by (*transfer*, *simp*)

lemma *ffun-minus-self* [*simp*]:
  fixes f :: $('a, 'b)$ *ffun*
  shows $f - f = 0$
  by (*transfer*, *simp*)

lemma *ffun-plus-commute*:
  $fdom(f) \cap fdom(g) = \{\} \Longrightarrow f + g = g + f$
  by (*transfer*, *metis pfun-plus-commute*)

lemma *ffun-minus-plus-commute*:
  $fdom(g) \cap fdom(h) = \{\} \Longrightarrow (f - g) + h = (f + h) - g$
  by (*transfer*, *simp add*: *pfun-minus-plus-commute*)

lemma *ffun-plus-minus*:
  $f \subseteq_f g \Longrightarrow (g - f) + f = g$
  by (*transfer*, *simp add*: *pfun-plus-minus*)

lemma *ffun-minus-common-subset*:
  $[\![h \subseteq_f f; h \subseteq_f g]\!] \Longrightarrow (f - h = g - h) = (f = g)$
  by (*transfer*, *simp add*: *pfun-minus-common-subset*)

lemma *ffun-minus-plus*:
  $fdom(f) \cap fdom(g) = \{\} \Longrightarrow (f + g) - g = f$
  by (*transfer*, *simp add*: *pfun-minus-plus*)

lemma *ffun-plus-pos*: $x + y = \{\}_f \Longrightarrow x = \{\}_f$
  by (*transfer*, *simp add*: *pfun-plus-pos*)

lemma *ffun-le-plus*: $fdom\ x \cap fdom\ y = \{\} \Longrightarrow x \leq x + y$
  by (*transfer*, *simp add*: *pfun-le-plus*)

10.3 Membership, application, and update

lemma *ffun-ext*: $[\![\bigwedge x\ y.\ (x, y) \in_f f \longleftrightarrow (x, y) \in_f g]\!] \Longrightarrow f = g$
  by (*transfer*, *simp add*: *pfun-ext*)

lemma *ffun-member-alt-def*:
  $(x, y) \in_f f \longleftrightarrow (x \in fdom\ f \wedge f(x)_f = y)$
  by (*transfer*, *simp add*: *pfun-member-alt-def*)

lemma *ffun-member-plus*:
  $(x, y) \in_f f + g \longleftrightarrow ((x \notin fdom(g) \wedge (x, y) \in_f f) \vee (x, y) \in_f g)$
  by (*transfer*, *simp add*: *pfun-member-plus*)

lemma *ffun-member-minus*:
  $(x, y) \in_f f - g \longleftrightarrow (x, y) \in_f f \wedge (\neg (x, y) \in_f g)$
  by (*transfer*, *simp add*: *pfun-member-minus*)

lemma *ffun-app-upd-1* [*simp*]: $x = y \Longrightarrow (f(x \mapsto v)_f)(y)_f = v$
  by (*transfer*, *simp*)

lemma *ffun-app-upd-2* [*simp*]: $x \neq y \Longrightarrow (f(x \mapsto v)_f)(y)_f = f(y)_f$
  by (*transfer*, *simp*)

lemma *ffun-upd-ext* [*simp*]: $x \in fdom(f) \Longrightarrow f(x \mapsto f(x)_f)_f = f$
  by (*transfer*, *simp*)

lemma *ffun-app-add* [*simp*]: $x \in fdom(g) \Longrightarrow (f + g)(x)_f = g(x)_f$
by (*transfer*, *simp*)

lemma *ffun-upd-add* [*simp*]: $f + g(x \mapsto v)_f = (f + g)(x \mapsto v)_f$
by (*transfer*, *simp*)

lemma *ffun-upd-twice* [*simp*]: $f(x \mapsto u, x \mapsto v)_f = f(x \mapsto v)_f$
by (*transfer*, *simp*)

lemma *ffun-upd-comm*:
assumes $x \neq y$
shows $f(y \mapsto u, x \mapsto v)_f = f(x \mapsto v, y \mapsto u)_f$
using *assms* **by** (*transfer*, *simp add*: *pfun-upd-comm*)

lemma *ffun-upd-comm-linorder* [*simp*]:
fixes $x\ y$ :: $'a$ :: *linorder*
assumes $x < y$
shows $f(y \mapsto u, x \mapsto v)_f = f(x \mapsto v, y \mapsto u)_f$
using *assms* **by** (*transfer*, *auto*)

lemma *ffun-app-minus* [*simp*]: $x \notin fdom\ g \Longrightarrow (f - g)(x)_f = f(x)_f$
by (*transfer*, *auto*)

lemma *ffun-upd-minus* [*simp*]:
$x \notin fdom\ g \Longrightarrow (f - g)(x \mapsto v)_f = (f(x \mapsto v)_f - g)$
by (*transfer*, *auto*)

lemma *fdom-member-minus-iff* [*simp*]:
$x \notin fdom\ g \Longrightarrow x \in fdom(f - g) \longleftrightarrow x \in fdom(f)$
by (*transfer*, *simp*)

lemma *fsubseteq-ffun-upd1* [*intro*]:
$[\![f \subseteq_f g; x \notin fdom(g)]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
by (*transfer*, *auto*)

lemma *fsubseteq-ffun-upd2* [*intro*]:
$[\![f \subseteq_f g; x \notin fdom(f)]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
by (*transfer*, *auto*)

lemma *psubseteq-pfun-upd3* [*intro*]:
$[\![f \subseteq_f g; g(x)_f = v]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
by (*transfer*, *auto*)

lemma *fsubseteq-dom-subset*:
$f \subseteq_f g \Longrightarrow fdom(f) \subseteq fdom(g)$
by (*transfer*, *auto simp add*: *psubseteq-dom-subset*)

lemma *fsubseteq-ran-subset*:
$f \subseteq_f g \Longrightarrow fran(f) \subseteq fran(g)$
by (*transfer*, *simp add*: *psubseteq-ran-subset*)

10.4 Domain laws

lemma *fdom-finite* [*simp*]: $finite(fdom(f))$
by (*transfer*, *simp*)

lemma *fdom-zero* [*simp*]: *fdom 0* = {}
  by (*transfer*, *simp*)

lemma *fdom-plus* [*simp*]: *fdom* (f + g) = *fdom* $f \cup$ *fdom* g
  by (*transfer*, *auto*)

lemma *fdom-inter*: *fdom* ($f \cap_f g$) $\subseteq$ *fdom* $f \cap$ *fdom* g
  by (*transfer*, *meson pdom-inter*)

lemma *fdom-comp* [*simp*]: *fdom* ($g \circ_f f$) = *fdom* ($f \rhd_f$ *fdom* g)
  by (*transfer*, *auto*)

lemma *fdom-upd* [*simp*]: *fdom* ($f(k \mapsto v)_f$) = *insert* k (*fdom* f)
  by (*transfer*, *simp*)

lemma *fdom-fdom-res* [*simp*]: *fdom* ($A \lhd_f f$) = $A \cap$ *fdom*(f)
  by (*transfer*, *auto*)

lemma *fdom-graph-ffun* [*simp*]:
  finite (*Domain R*) $\Longrightarrow$ *fdom* (*graph-ffun R*) = *Domain R*
  by (*transfer*, *simp add*: *Domain-fst graph-map-dom*)

10.5 Range laws

lemma *fran-zero* [*simp*]: *fran 0* = {}
  by (*transfer*, *simp*)

lemma *fran-upd* [*simp*]: *fran* ($f(k \mapsto v)_f$) = *insert* v (*fran* (($-$ $\{k\}$) $\lhd_f f$))
  by (*transfer*, *auto*)

lemma *fran-fran-res* [*simp*]: *fran* ($f \rhd_f A$) = *fran*(f) $\cap A$
  by (*transfer*, *auto*)

lemma *fran-comp* [*simp*]: *fran* ($g \circ_f f$) = *fran* (*fran* $f \lhd_f g$)
  by (*transfer*, *auto*)

10.6 Domain restriction laws

lemma *fdom-res-zero* [*simp*]: $A \lhd_f \{\}_f = \{\}_f$
  by (*transfer*, *auto*)

lemma *fdom-res-empty* [*simp*]:
  $(\{\} \lhd_f f) = \{\}_f$
  by (*transfer*, *auto*)

lemma *fdom-res-fdom* [*simp*]:
  fdom(f) $\lhd_f f = f$
  by (*transfer*, *auto*)

lemma *pdom-res-upd-in* [*simp*]:
  $k \in A \Longrightarrow A \lhd_f f(k \mapsto v)_f = (A \lhd_f f)(k \mapsto v)_f$
  by (*transfer*, *auto*)

lemma *pdom-res-upd-out* [*simp*]:
  $k \notin A \Longrightarrow A \lhd_f f(k \mapsto v)_f = A \lhd_f f$

by (*transfer*, *auto*)

lemma *fdom-res-override* [*simp*]: $A \lhd_f (f + g) = (A \lhd_f f) + (A \lhd_f g)$
by (*metis fdom-res.rep-eq pdom-res-override pfun-of-inject plus-ffun.rep-eq*)

lemma *fdom-res-minus* [*simp*]: $A \lhd_f (f - g) = (A \lhd_f f) - g$
by (*transfer*, *auto*)

lemma *fdom-res-swap*: $(A \lhd_f f) \rhd_f B = A \lhd_f (f \rhd_f B)$
by (*transfer*, *simp add*: *pdom-res-swap*)

lemma *fdom-res-twice* [*simp*]: $A \lhd_f (B \lhd_f f) = (A \cap B) \lhd_f f$
by (*transfer*, *auto*)

lemma *fdom-res-comp* [*simp*]: $A \lhd_f (g \circ_f f) = g \circ_f (A \lhd_f f)$
by (*transfer*, *simp*)

10.7 Range restriction laws

lemma *fran-res-zero* [*simp*]: $\{\}_f \rhd_f A = \{\}_f$
by (*transfer*, *auto*)

lemma *fran-res-upd-1* [*simp*]: $v \in A \Longrightarrow f(x \mapsto v)_f \rhd_f A = (f \rhd_f A)(x \mapsto v)_f$
by (*transfer*, *auto*)

lemma *fran-res-upd-2* [*simp*]: $v \notin A \Longrightarrow f(x \mapsto v)_f \rhd_f A = ((- \{x\}) \lhd_f f) \rhd_f A$
by (*transfer*, *auto*)

lemma *fran-res-override*: $(f + g) \rhd_f A \subseteq_f (f \rhd_f A) + (g \rhd_f A)$
by (*transfer*, *simp add*: *pran-res-override*)

10.8 Graph laws

lemma *ffun-graph-inv*: *graph-ffun* (*ffun-graph* f) = f
by (*transfer*, *auto simp add*: *pfun-graph-inv finite-Domain*)

lemma *ffun-graph-zero*: *ffun-graph* $0 = \{\}$
by (*transfer*, *simp add*: *pfun-graph-zero*)

lemma *ffun-graph-minus*: *ffun-graph* $(f - g)$ = *ffun-graph* f − *ffun-graph* g
by (*transfer*, *simp add*: *pfun-graph-minus*)

lemma *ffun-graph-inter*: *ffun-graph* $(f \cap_f g)$ = *ffun-graph* f ∩ *ffun-graph* g
by (*transfer*, *simp add*: *pfun-graph-inter*)

10.9 Partial Function Lens

definition *ffun-lens* :: $'a \Rightarrow ('b \Longrightarrow ('a, 'b)$ *ffun*) **where**
[*lens-defs*]: *ffun-lens* i = (| *lens-get* = $\lambda\ s.\ s(i)_f$, *lens-put* = $\lambda\ s\ v.\ s(i \mapsto v)_f$ |)

lemma *ffun-lens-mwb* [*simp*]: *mwb-lens* (*ffun-lens* i)
by (*unfold-locales*, *simp-all add*: *ffun-lens-def*)

lemma *ffun-lens-src*: $\mathcal{S}_{\textit{ffun-lens}\ i} = \{f.\ i \in \textit{fdom}(f)\}$
by (*auto simp add*: *lens-defs lens-source-def*, *metis ffun-upd-ext*)

Hide implementation details for finite functions

lifting-update *ffun.lifting*
lifting-forget *ffun.lifting*

end

11 Infinity Supplement

theory *Infinity*
imports *HOL.Real*
HOL−Library.Infinite-Set
Optics.Two
begin

This theory introduces a type class *infinite* that guarantees that the underlying universe of the type is infinite. It also provides useful theorems to prove infinity of the universes for various HOL types.

11.1 Type class *infinite*

The type class postulates that the universe (carrier) of a type is infinite.

class *infinite* =
assumes *infinite-UNIV* [*simp*]: *infinite* (*UNIV* :: *'a set*)

11.2 Infinity Theorems

Useful theorems to prove that a type's *UNIV* is infinite.

Note that *infinite-UNIV-nat* is already a simplification rule by default.

lemmas *infinite-UNIV-int* [*simp*]

theorem *infinite-UNIV-real* [*simp*]:
infinite (*UNIV* :: *real set*)
by (*rule infinite-UNIV-char-0*)

theorem *infinite-UNIV-fun1* [*simp*]:
infinite (*UNIV* :: *'a set*) $\Longrightarrow$
card (*UNIV* :: *'b set*) $\neq$ *Suc 0* $\Longrightarrow$
infinite (*UNIV* :: (*'a* $\Rightarrow$ *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*erule finite-fun-UNIVD1*)
apply (*assumption*)
done

theorem *infinite-UNIV-fun2* [*simp*]:
infinite (*UNIV* :: *'b set*) $\Longrightarrow$
infinite (*UNIV* :: (*'a* $\Rightarrow$ *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*erule finite-fun-UNIVD2*)
done

theorem *infinite-UNIV-set* [*simp*]:
infinite (*UNIV* :: *'a set*) $\Longrightarrow$

infinite (*UNIV* :: *'a set set*)
apply (*erule contrapos-nn*)
apply (*simp add*: *Finite-Set.finite-set*)
done

theorem *infinite-UNIV-prod1* [*simp*]:
infinite (*UNIV* :: *'a set*) $\Longrightarrow$
infinite (*UNIV* :: (*'a* $\times$ *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*simp add*: *finite-prod*)
done

theorem *infinite-UNIV-prod2* [*simp*]:
infinite (*UNIV* :: *'b set*) $\Longrightarrow$
infinite (*UNIV* :: (*'a* $\times$ *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*simp add*: *finite-prod*)
done

theorem *infinite-UNIV-sum1* [*simp*]:
infinite (*UNIV* :: *'a set*) $\Longrightarrow$
infinite (*UNIV* :: (*'a* + *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*simp*)
done

theorem *infinite-UNIV-sum2* [*simp*]:
infinite (*UNIV* :: *'b set*) $\Longrightarrow$
infinite (*UNIV* :: (*'a* + *'b*) *set*)
apply (*erule contrapos-nn*)
apply (*simp*)
done

theorem *infinite-UNIV-list* [*simp*]:
infinite (*UNIV* :: *'a list set*)
apply (*rule infinite-UNIV-listI*)
done

theorem *infinite-UNIV-option* [*simp*]:
infinite (*UNIV* :: *'a set*) $\Longrightarrow$
infinite (*UNIV* :: *'a option set*)
apply (*erule contrapos-nn*)
apply (*simp*)
done

theorem *infinite-image* [*intro*]:
infinite A $\Longrightarrow$ *inj-on f A* $\Longrightarrow$ *infinite* (*f* ` *A*)
apply (*metis finite-imageD*)
done

theorem *infinite-transfer* :
infinite B $\Longrightarrow$ *B* $\subseteq$ *f* ` *A* $\Longrightarrow$ *infinite A*
using *infinite-super*
apply (*blast*)
done

11.3 Instantiations

The instantiations for product and sum types have stronger caveats than in principle needed. Namely, it would be sufficient for one type of a product or sum to be infinite. A corresponding rule, however, cannot be formulated using type classes. Generally, classes are not entirely adequate for the purpose of deriving the infinity of HOL types, which is perhaps why a class such as *infinite* was omitted from the Isabelle/HOL library.

instance *nat* :: *infinite* **by** (*intro-classes*, *simp*)
instance *int* :: *infinite* **by** (*intro-classes*, *simp*)
instance *real* :: *infinite* **by** (*intro-classes*, *simp*)
instance *fun* :: (*type*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
instance *set* :: (*infinite*) *infinite* **by** (*intro-classes*, *simp*)
instance *prod* :: (*infinite*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
instance *sum* :: (*infinite*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
instance *list* :: (*type*) *infinite* **by** (*intro-classes*, *simp*)
instance *option* :: (*infinite*) *infinite* **by** (*intro-classes*, *simp*)

subclass (**in** *infinite*) *two* **by** (*intro-classes*, *auto*)

end

12 Positive Subtypes

theory *Positive*
imports
  Infinity
  HOL−Library.Countable
begin

12.1 Type Definition

typedef (**overloaded**) *'a*::{*zero*, *linorder*} *pos* = {*x*::*'a*. $x \geq 0$}
  apply (*rule-tac* $x = 0$ **in** *exI*)
  apply (*clarsimp*)
  done

syntax
  -type-pos :: *type* ⇒ *type* (-$^+$ [*999*] *999*)

translations
  (*type*) $'a^+$ == (*type*) *'a pos*

setup-lifting *type-definition-pos*

type-synonym *preal* = *real pos*

12.2 Operators

lift-definition *mk-pos* :: *'a*::{*zero*, *linorder*} ⇒ *'a pos* **is**
λ *n*. *if* ($n \geq 0$) *then n else 0* **by** *auto*

lift-definition *real-of-pos* :: *real pos* ⇒ *real* **is** *id* **.**

declare [[*coercion real-of-pos*]]

12.3 Instantiations

```
instantiation pos :: ({zero, linorder}) zero
begin
  lift-definition zero-pos :: 'a pos
    is 0 :: 'a ..
  instance ..
end

instantiation pos :: ({zero, linorder}) linorder
begin
  lift-definition less-eq-pos :: 'a pos ⇒ 'a pos ⇒ bool
    is (≤) :: 'a ⇒ 'a ⇒ bool .
  lift-definition less-pos :: 'a pos ⇒ 'a pos ⇒ bool
    is (<) :: 'a ⇒ 'a ⇒ bool .
  instance
    apply (intro-classes; transfer)
        apply (auto)
  done
end

instance pos :: ({zero, linorder, no-top}) no-top
  apply (intro-classes)
  apply (transfer)
  apply (clarsimp)
  apply (meson gt-ex less-imp-le order.strict-trans1)
  done

instance pos :: ({zero, linorder, no-top}) infinite
  apply (intro-classes)
  apply (rule notI)
  apply (subgoal-tac ∀ x::'a pos. x ≤ Max UNIV)
  using gt-ex leD apply (blast)
  apply (simp)
  done

instantiation pos :: (linordered-semidom) linordered-semidom
begin
  lift-definition one-pos :: 'a pos
    is 1 :: 'a by (simp)
  lift-definition plus-pos :: 'a pos ⇒ 'a pos ⇒ 'a pos
    is (+) by (simp)
  lift-definition minus-pos :: 'a pos ⇒ 'a pos ⇒ 'a pos
    is λx y. if y ≤ x then x − y else 0
    by (simp add: add-le-imp-le-diff)
  lift-definition times-pos :: 'a pos ⇒ 'a pos ⇒ 'a pos
    is times by (simp)
  instance
    apply (intro-classes; transfer; simp?)
            apply (simp add: add.assoc)
           apply (simp add: add.commute)
          apply (safe; clarsimp?) [1]
             apply (simp add: diff-diff-add)
            apply (metis add-le-cancel-left le-add-diff-inverse)
           apply (simp add: add.commute add-le-imp-le-diff)
          apply (metis add-increasing2 antisym linear)
```

apply (*simp add*: *mult.assoc*)
apply (*simp add*: *mult.commute*)
apply (*simp add*: *comm-semiring-class.distrib*)
apply (*simp add*: *mult-strict-left-mono*)
apply (*safe*; *clarsimp?*) [*1*]
apply (*simp add*: *right-diff-distrib'*)
apply (*simp add*: *mult-left-mono*)
using *mult-left-le-imp-le* **apply** (*fastforce*)
apply (*simp add*: *distrib-left*)
done
end

instantiation *pos* :: (*linordered-field*) *semidom-divide*
begin
lift-definition *divide-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ $'a$ *pos*
is *divide* **by** (*simp*)
instance
apply (*intro-classes*; *transfer*)
apply (*simp-all*)
done
end

instantiation *pos* :: (*linordered-field*) *inverse*
begin
lift-definition *inverse-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos*
is *inverse* **by** (*simp*)
instance ..
end

lemma *pos-positive* [*simp*]: $0 \leq (x::'a::\{zero,linorder\}\ pos)$
by (*transfer*, *simp*)

12.4 Theorems

lemma *mk-pos-zero* [*simp*]: *mk-pos* $0 = 0$
by (*transfer*, *simp*)

lemma *mk-pos-one* [*simp*]: *mk-pos* $1 = 1$
by (*transfer*, *simp*)

lemma *mk-pos-leq*:
$[\![\ 0 \leq x;\ x \leq y\ ]\!] \Longrightarrow$ *mk-pos* $x \leq$ *mk-pos* y
by (*transfer*, *auto*)

lemma *mk-pos-less*:
$[\![\ 0 \leq x;\ x < y\ ]\!] \Longrightarrow$ *mk-pos* $x <$ *mk-pos* y
by (*transfer*, *auto*)

lemma *real-of-pos* [*simp*]: $x \geq 0 \Longrightarrow$ *real-of-pos* (*mk-pos* x) $= x$
by (*transfer*, *simp*)

lemma *mk-pos-real-of-pos* [*simp*]: *mk-pos* (*real-of-pos* x) $= x$
by (*transfer*, *simp*)

12.5 Transfer to Reals

named-theorems *pos-transfer*

lemma *real-of-pos-0* [*pos-transfer*]:
real-of-pos 0 = *0*
by (*transfer*, *auto*)

lemma *real-of-pos-1* [*pos-transfer*]:
real-of-pos 1 = *1*
by (*transfer*, *auto*)

lemma *real-op-pos-plus* [*pos-transfer*]:
real-of-pos $(x + y)$ = *real-of-pos* x + *real-of-pos* y
by (*transfer*, *simp*)

lemma *real-op-pos-minus* [*pos-transfer*]:
$x \geq y \Longrightarrow$ *real-of-pos* $(x - y)$ = *real-of-pos* x − *real-of-pos* y
by (*transfer*, *simp*)

lemma *real-op-pos-mult* [*pos-transfer*]:
real-of-pos $(x * y)$ = *real-of-pos* x ∗ *real-of-pos* y
by (*transfer*, *simp*)

lemma *real-op-pos-div* [*pos-transfer*]:
real-of-pos (x / y) = *real-of-pos* x / *real-of-pos* y
by (*transfer*, *simp*)

lemma *real-of-pos-numeral* [*pos-transfer*]:
real-of-pos (*numeral n*) = *numeral n*
by (*induct n*, *simp-all only*: *numeral.simps pos-transfer*)

lemma *real-of-pos-eq-transfer* [*pos-transfer*]:
$x = y \longleftrightarrow$ *real-of-pos* x = *real-of-pos* y
by (*transfer*, *auto*)

lemma *real-of-pos-less-eq-transfer* [*pos-transfer*]:
$x \leq y \longleftrightarrow$ *real-of-pos* $x \leq$ *real-of-pos* y
by (*transfer*, *auto*)

lemma *real-of-pos-less-transfer* [*pos-transfer*]:
$x < y \longleftrightarrow$ *real-of-pos* $x <$ *real-of-pos* y
by (*transfer*, *auto*)

end

13 Recall Undeclarations

theory *Total-Recall*
imports *Main*
keywords
purge-syntax :: *thy-decl* **and**
purge-notation :: *thy-decl* **and**
recall-syntax :: *thy-decl*
begin

13.1 ML File Import

ML-file *Total-Recall.ML*

13.2 Outer Commands

ML ‹
```
  val - =
    Outer-Syntax.command @{command-keyword purge-syntax}
      purge raw syntax clauses
      ((Parse.syntax-mode -- Scan.repeat1 Parse.const-decl) >>
        (Toplevel.theory o (fn (mode, args) =>
          (TotalRecall.record-no-syntax mode args) o
          (Sign.del-syntax-cmd mode args))));

  val - =
    Outer-Syntax.local-theory @{command-keyword purge-notation}
      purge concrete syntax for constants / fixed variables
      ((Parse.syntax-mode -- Parse.and-list1 (Parse.const -- Parse.mixfix)) >>
        (fn (mode, args) =>
          (Local-Theory.background-theory
            (TotalRecall.record-no-notation mode args)) o
          (Specification.notation-cmd false mode args)));

  val - =
    Outer-Syntax.command @{command-keyword recall-syntax}
      recall undecarations of all purged items
      (Scan.succeed (Toplevel.theory TotalRecall.execute-all))
```
›
end

14 Meta-theory for UTP Toolkit

theory *utp-toolkit*
 imports
 HOL.Deriv
 HOL−Library.Adhoc-Overloading
 HOL−Library.Char-ord
 HOL−Library.Countable-Set
 HOL−Library.FSet
 HOL−Library.Monad-Syntax
 HOL−Library.Countable
 HOL−Library.Order-Continuity
 HOL−Library.Prefix-Order
 HOL−Library.Product-Order
 HOL−Library.Sublist
 HOL−Algebra.Complete-Lattice
 HOL−Algebra.Galois-Connection
 HOL−Eisbach.Eisbach
 Optics.Lenses
 Countable-Set-Extra
 FSet-Extra
 Relation-Extra
 Map-Extra
 List-Extra
 List-Lexord-Alt

Partial-Fun
Finite-Fun
Infinity
Positive
Total-Recall
begin end

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