

Theory of Designs in Isabelle/UTP

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January 21, 2019

Abstract

This document describes a mechanisation of the UTP theory of designs in Isabelle/UTP. Designs enrich UTP relations with explicit precondition/postcondition pairs, as present in formal notations like VDM, B, and the refinement calculus. If a program's precondition holds, then it is guaranteed to terminate and establish its postcondition, which is an approach known as total correctness. If the precondition does not hold, the behaviour is maximally nondeterministic, which represents unspecified behaviour. In this mechanisation, we create the theory of designs, including its alphabet, signature, and healthiness conditions. We then use these to prove the key algebraic laws of programming. This development can be used to support program verification based on total correctness.

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1 Design Signature and Core Laws

```
theory utp-des-core
imports UTP-KAT.utp-kleene
begin
```

UTP designs [2, 4] are a subset of the alphabetised relations that use a boolean observational variable *ok* to record the start and termination of a program. For more information on designs please see Chapter 3 of the UTP book [4], or the more accessible designs tutorial [2].

1.1 Definitions

Two named theorem sets exist are created to group theorems that, respectively, provide pre-postcondition definitions, and simplify operators to their normal design form.

```
named-theorems ndes and ndes-simp
```

```
alphabet des-vars =
  ok :: bool
```

The two locale interpretations below are a technicality to improve automatic proof support via the predicate and relational tactics. This is to enable the (re-)interpretation of state spaces to remove any occurrences of lens types after the proof tactics *pred-simp* and *rel-simp*, or any of their derivatives have been applied. Eventually, it would be desirable to automate both interpretations as part of a custom outer command for defining alphabets.

```
type-synonym 'α des = 'α des-vars-scheme
type-synonym ('α, 'β) rel-des = ('α des, 'β des) urel
type-synonym 'α hrel-des = ('α des) hrel
```

```
translations
```

```
(type) 'α des <= (type) 'α des-vars-scheme
(type) 'α des <= (type) 'α des-vars-ext
(type) ('α, 'β) rel-des <= (type) ('α des, 'β des) urel
```

$(type) \ ' \alpha \ hrel\text{-}des \leq (type) \ ' \alpha \ des \ hrel$

notation $des\text{-}vars.more_L \ (\Sigma_D)$

syntax

$\text{-}svid\text{-}des\text{-}\alpha \ :: \ svid \ (\mathbf{v}_D)$

translations

$\text{-}svid\text{-}des\text{-}\alpha \Rightarrow CONST \ des\text{-}vars.more_L$

lemma $ok\text{-}des\text{-}bij\text{-}lens: \ bij\text{-}lens \ (ok \ +_L \ \Sigma_D) \ (\text{is } bij\text{-}lens \ ?P)$

proof –

have $?P \approx_L 1_L$

by $(meson \ des\text{-}vars.equivs(1) \ des\text{-}vars.equivs(2) \ des\text{-}vars.indeps(1) \ lens\text{-}equiv\text{-}sym \ lens\text{-}equiv\text{-}trans \ lens\text{-}plus\text{-}eq\text{-}left)$

thus $?thesis$

by $(simp \ add: \ bij\text{-}lens\text{-}equiv\text{-}id)$

qed

Define the lens functor for designs

definition $lmap\text{-}des\text{-}vars \ :: \ (' \alpha \Rightarrow ' \beta) \Rightarrow (' \alpha \ des\text{-}vars\text{-}scheme \Rightarrow ' \beta \ des\text{-}vars\text{-}scheme) \ (lmap_D)$

where $[lens\text{-}defs]: \ lmap\text{-}des\text{-}vars = lmap[des\text{-}vars]$

lemma $lmap\text{-}des\text{-}vars: \ vwb\text{-}lens \ f \Rightarrow vwb\text{-}lens \ (lmap\text{-}des\text{-}vars \ f)$

by $(unfold\text{-}locales, \ auto \ simp \ add: \ lens\text{-}defs)$

lemma $lmap\text{-}id: \ lmap_D \ 1_L = 1_L$

by $(simp \ add: \ lens\text{-}defs \ fun\text{-}eq\text{-}iff)$

lemma $lmap\text{-}comp: \ lmap_D \ (f \ ;_L \ g) = lmap_D \ f \ ;_L \ lmap_D \ g$

by $(simp \ add: \ lens\text{-}defs \ fun\text{-}eq\text{-}iff)$

The following notations define liftings from non-design predicates into design predicates using alphabet extensions.

abbreviation $lift\text{-}desr \ (\lceil \cdot \rceil_D)$

where $\lceil P \rceil_D \equiv P \oplus_p (\Sigma_D \times_L \Sigma_D)$

abbreviation $lift\text{-}pre\text{-}desr \ (\lceil \cdot \rceil_{D<})$

where $\lceil p \rceil_{D<} \equiv \lceil \lceil p \rceil_{<} \rceil_D$

abbreviation $lift\text{-}post\text{-}desr \ (\lceil \cdot \rceil_{D>})$

where $\lceil p \rceil_{D>} \equiv \lceil \lceil p \rceil_{>} \rceil_D$

abbreviation $drop\text{-}desr \ (\lfloor \cdot \rfloor_D)$

where $\lfloor P \rfloor_D \equiv P \upharpoonright_e (\Sigma_D \times_L \Sigma_D)$

abbreviation $dcond \ :: \ (' \alpha, ' \beta) \ rel\text{-}des \Rightarrow ' \alpha \ upred \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des$

where $dcond \ P \ b \ Q \equiv P \triangleleft \lceil b \rceil_{D<} \triangleright Q$

syntax $\text{-}dcond \ :: \ logic \Rightarrow uexp \Rightarrow logic \Rightarrow logic \ ((\beta \triangleleft \text{-} \triangleright_D / \text{-}) \ [52,0,53] \ 52)$

translations $\text{-}dcond \ P \ b \ Q == CONST \ dcond \ P \ b \ Q$

definition $design::(' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \Rightarrow (' \alpha, ' \beta) \ rel\text{-}des \ (\text{infixl} \vdash 59) \ \text{where}$

$[upred\text{-}defs]: \ P \vdash Q = (\$ok \wedge P \Rightarrow \$ok' \wedge Q)$

An rdesign is a design that uses the Isabelle type system to prevent reference to ok in the

assumption and commitment.

definition $rdesign::('α, 'β) urel \Rightarrow ('α, 'β) urel \Rightarrow ('α, 'β) rel-des$ (**infixl** \vdash_r 59) **where**
 $[upred-defs]: (P \vdash_r Q) = \lceil P \rceil_D \vdash \lceil Q \rceil_D$

An ndesign is a normal design, i.e. where the assumption is a condition

definition $ndesign::'α cond \Rightarrow ('α, 'β) urel \Rightarrow ('α, 'β) rel-des$ (**infixl** \vdash_n 59) **where**
 $[upred-defs]: (p \vdash_n Q) = (\lceil p \rceil_{<} \vdash_r Q)$

definition $skip-d :: 'α hrel-des (II_D)$ **where**
 $[upred-defs]: II_D \equiv (true \vdash_r II)$

definition $bot-d :: ('α, 'β) rel-des (\perp_D)$ **where**
 $[upred-defs]: \perp_D = (false \vdash false)$

definition $pre-design :: ('α, 'β) rel-des \Rightarrow ('α, 'β) urel (pre_D)$ **where**
 $[upred-defs]: pre_D(P) = \lfloor \neg P \llbracket true, false / \$ok, \$ok' \rrbracket \rfloor_D$

definition $post-design :: ('α, 'β) rel-des \Rightarrow ('α, 'β) urel (post_D)$ **where**
 $[upred-defs]: post_D(P) = \lfloor P \llbracket true, true / \$ok, \$ok' \rrbracket \rfloor_D$

syntax

$-ok-f :: logic \Rightarrow logic \ (-^f \ [1000] \ 1000)$
 $-ok-t :: logic \Rightarrow logic \ (-^t \ [1000] \ 1000)$
 $-top-d :: logic \ (\top_D)$

translations

$P^f \Rightarrow CONST \ usubst \ (CONST \ subst-upd \ CONST \ id \ (CONST \ ovar \ CONST \ ok) \ false) \ P$
 $P^t \Rightarrow CONST \ usubst \ (CONST \ subst-upd \ CONST \ id \ (CONST \ ovar \ CONST \ ok) \ true) \ P$
 $\top_D \Rightarrow CONST \ not-upred \ (CONST \ utp-expr.var \ (CONST \ ivar \ CONST \ ok))$

1.2 Lifting, Unrestriction, and Substitution

lemma $drop-desr-inv \ [simp]: \lfloor \lceil P \rceil_D \rfloor_D = P$
by ($simp \ add: \ prod-mwb-lens$)

lemma $lift-desr-inv:$

fixes $P :: ('α, 'β) rel-des$
assumes $\$ok \# P \ \$ok' \# P$
shows $\lceil \lfloor P \rfloor_D \rceil_D = P$

proof –

have $bij-lens \ (\Sigma_D \times_L \Sigma_D +_L (in-var \ ok +_L out-var \ ok)) :: (-, 'α \ des-vars-scheme \times 'β \ des-vars-scheme)$
 $lens$)

(is $bij-lens \ (?P)$)

proof –

have $?P \approx_L (ok +_L \Sigma_D) \times_L (ok +_L \Sigma_D)$ **(is** $?P \approx_L ?Q$)

apply ($simp \ add: in-var-def \ out-var-def \ prod-as-plus$)

apply ($simp \ add: prod-as-plus[THEN \ sym]$)

apply ($meson \ lens-equiv-sym \ lens-equiv-trans \ lens-indep-prod \ lens-plus-comm \ lens-plus-prod-exchange$
 $des-vars.indeps(1)$)

done

moreover **have** $bij-lens \ ?Q$

by ($simp \ add: ok-des-bij-lens \ prod-bij-lens$)

ultimately **show** $?thesis$

by ($metis \ bij-lens-equiv \ lens-equiv-sym$)

qed

```

with assms show ?thesis
  apply (rule-tac aext-arestr[of - in-var ok +L out-var ok])
  apply (simp add: prod-mwb-lens)
  apply (simp)
  apply (metis alpha-in-var lens-indep-prod lens-indep-sym des-vars.indeps(1) out-var-def prod-as-plus)
  using unrest-var-comp apply blast
done
qed

lemma unrest-out-des-lift [unrest]:  $out\alpha \# p \implies out\alpha \# [p]_D$ 
  by (pred-simp)

lemma lift-dist-seq [simp]:
 $[P ;; Q]_D = ([P]_D ;; [Q]_D)$ 
  by (rel-auto)

lemma lift-des-skip-dr-unit [simp]:
 $([P]_D ;; [II]_D) = [P]_D$ 
 $([II]_D ;; [P]_D) = [P]_D$ 
  by (rel-auto)+

lemma lift-des-skip-dr-unit-unrest:  $\$ok' \# P \implies (P ;; [II]_D) = P$ 
  by (rel-auto)

lemma state-subst-design [usubst]:
 $[\sigma \oplus_s \Sigma_D]_s \dagger (P \vdash_r Q) = ([\sigma]_s \dagger P) \vdash_r ([\sigma]_s \dagger Q)$ 
  by (rel-auto)

lemma design-subst [usubst]:
 $\llbracket \$ok \# \sigma; \$ok' \# \sigma \rrbracket \implies \sigma \dagger (P \vdash Q) = (\sigma \dagger P) \vdash (\sigma \dagger Q)$ 
  by (simp add: design-def usubst)

lemma design-msubst [usubst]:
 $(P(x) \vdash Q(x)) \llbracket x \rightarrow v \rrbracket = (P(x) \llbracket x \rightarrow v \rrbracket \vdash Q(x) \llbracket x \rightarrow v \rrbracket)$ 
  by (rel-auto)

lemma design-ok-false [usubst]:  $(P \vdash Q) \llbracket false / \$ok \rrbracket = true$ 
  by (simp add: design-def usubst)

lemma ok-pre:  $(\$ok \wedge [pre_D(P)]_D) = (\$ok \wedge (\neg P^f))$ 
  apply (simp add: pre-design-def alpha unrest usubst)
  apply (subst aext-arestr')
  apply (rel-simp)
  apply (rel-auto)
done

lemma ok-post:  $(\$ok \wedge [post_D(P)]_D) = (\$ok \wedge (P^t))$ 
  apply (simp add: post-design-def alpha unrest usubst)
  apply (subst aext-arestr')
  apply (rel-simp)
  apply (rel-auto)
done

```

1.3 Basic Design Laws

lemma *design-export-ok*: $P \vdash Q = (P \vdash (\$ok \wedge Q))$
by (*rel-auto*)

lemma *design-export-ok'*: $P \vdash Q = (P \vdash (\$ok' \wedge Q))$
by (*rel-auto*)

lemma *design-export-pre*: $P \vdash (P \wedge Q) = P \vdash Q$
by (*rel-auto*)

lemma *design-export-spec*: $P \vdash (P \Rightarrow Q) = P \vdash Q$
by (*rel-auto*)

lemma *design-ok-pre-conj*: $(\$ok \wedge P) \vdash Q = P \vdash Q$
by (*rel-auto*)

lemma *true-is-design*: $(false \vdash true) = true$
by (*rel-auto*)

lemma *true-is-rdesign*: $(false \vdash_r true) = true$
by (*rel-auto*)

lemma *bot-d-true*: $\perp_D = true$
by (*rel-auto*)

lemma *bot-d-ndes-def* [*ndes-simp*]: $\perp_D = (false \vdash_n true)$
by (*rel-auto*)

lemma *design-false-pre*: $(false \vdash P) = true$
by (*rel-auto*)

lemma *rdesign-false-pre*: $(false \vdash_r P) = true$
by (*rel-auto*)

lemma *ndesign-false-pre*: $(false \vdash_n P) = true$
by (*rel-auto*)

lemma *ndesign-miracle*: $(true \vdash_n false) = \top_D$
by (*rel-auto*)

lemma *top-d-ndes-def* [*ndes-simp*]: $\top_D = (true \vdash_n false)$
by (*rel-auto*)

lemma *skip-d-alt-def*: $II_D = true \vdash II$
by (*rel-auto*)

lemma *skip-d-ndes-def* [*ndes-simp*]: $II_D = true \vdash_n II$
by (*rel-auto*)

lemma *design-subst-ok*:
 $(P \llbracket true/\$ok \rrbracket \vdash Q \llbracket true/\$ok \rrbracket) = (P \vdash Q)$
by (*rel-auto*)

lemma *design-subst-ok-ok'*:
 $(P \llbracket true/\$ok \rrbracket \vdash Q \llbracket true, true/\$ok, \$ok' \rrbracket) = (P \vdash Q)$

proof –

have $(P \vdash Q) = ((\$ok \wedge P) \vdash (\$ok \wedge \$ok' \wedge Q))$

by (*pred-auto*)

also have $\dots = ((\$ok \wedge P[\![true/\$ok]\!]) \vdash (\$ok \wedge (\$ok' \wedge Q[\![true/\$ok']\!])[\![true/\$ok]\!]))$

by (*metis conj-eq-out-var-subst conj-pos-var-subst upred-eq-true utp-pred-laws.inf-commute ok-vwb-lens*)

also have $\dots = ((\$ok \wedge P[\![true/\$ok]\!]) \vdash (\$ok \wedge \$ok' \wedge Q[\![true,true/\$ok,\$ok']\!]))$

by (*simp add: usubst*)

also have $\dots = (P[\![true/\$ok]\!] \vdash Q[\![true,true/\$ok,\$ok']\!])$

by (*pred-auto*)

finally show *?thesis* ..

qed

lemma *design-subst-ok'*:

$(P \vdash Q[\![true/\$ok']\!]) = (P \vdash Q)$

proof –

have $(P \vdash Q) = (P \vdash (\$ok' \wedge Q))$

by (*pred-auto*)

also have $\dots = (P \vdash (\$ok' \wedge Q[\![true/\$ok']\!]))$

by (*metis conj-eq-out-var-subst upred-eq-true utp-pred-laws.inf-commute ok-vwb-lens*)

also have $\dots = (P \vdash Q[\![true/\$ok']\!])$

by (*pred-auto*)

finally show *?thesis* ..

qed

1.4 Sequential Composition Laws

theorem *design-skip-idem* [*simp*]:

$(II_D ;; II_D) = II_D$

by (*rel-auto*)

theorem *design-composition-subst*:

assumes

$\$ok' \# P1 \ \$ok \# P2$

shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) =$

$((\neg (\neg P1) ;; true) \wedge \neg (Q1[\![true/\$ok']\!] ;; (\neg P2))) \vdash (Q1[\![true/\$ok']\!] ;; Q2[\![true/\$ok]\!]))$

proof –

have $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (\exists \ ok_0 \cdot ((P1 \vdash Q1)[\![\ll ok_0 \gg / \$ok']\!] ;; (P2 \vdash Q2)[\![\ll ok_0 \gg / \$ok]\!]))$

by (*rule seqr-middle, simp*)

also have \dots

$= (((P1 \vdash Q1)[\![false/\$ok']\!] ;; (P2 \vdash Q2)[\![false/\$ok]\!])$

$\vee ((P1 \vdash Q1)[\![true/\$ok']\!] ;; (P2 \vdash Q2)[\![true/\$ok]\!]))$

by (*metis (no-types, lifting) calculation disj-comm ok-vwb-lens seqr-bool-split*)

also from *assms*

have $\dots = (((\$ok \wedge P1 \Rightarrow Q1[\![true/\$ok']\!]) ;; (P2 \Rightarrow \$ok' \wedge Q2[\![true/\$ok]\!])) \vee ((\neg (\$ok \wedge P1)) ;;$

$true))$

by (*simp add: design-def usubst unrest, pred-auto*)

also have $\dots = ((\neg \$ok ;; true_h) \vee ((\neg P1) ;; true) \vee (Q1[\![true/\$ok']\!] ;; (\neg P2)) \vee (\$ok' \wedge (Q1[\![true/\$ok']\!]$

$;; Q2[\![true/\$ok]\!]))$

by (*rel-auto*)

also have $\dots = (((\neg (\neg P1) ;; true) \wedge \neg (Q1[\![true/\$ok']\!] ;; (\neg P2))) \vdash (Q1[\![true/\$ok']\!] ;; Q2[\![true/\$ok]\!]))$

by (*simp add: precondition-right-unit design-def unrest, rel-auto*)

finally show *?thesis* .

qed

theorem *design-composition*:

assumes

$\$ok' \# P1 \ \$ok \# P2 \ \$ok' \# Q1 \ \$ok \# Q2$
shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$
using *assms* **by** (*simp add: design-composition-subst usubst*)

theorem *design-composition-runrest:*

assumes

$\$ok' \# P1 \ \$ok \# P2 \ ok \#\# Q1 \ ok \#\# Q2$

shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (Q1 ;; Q2))$

proof –

have $(\$ok \wedge \$ok' \wedge (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket)) = (\$ok \wedge \$ok' \wedge (Q1 ;; Q2))$

proof –

have $(\$ok \wedge \$ok' \wedge (Q1 ;; Q2)) = ((\$ok \wedge Q1) ;; (Q2 \wedge \$ok'))$

by (*metis (no-types, lifting) conj-comm segr-post-var-out segr-pre-var-out*)

also have $\dots = ((Q1 \wedge \$ok') ;; (\$ok \wedge Q2))$

by (*simp add: assms(3) assms(4) runrest-ident-var*)

also have $\dots = (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket)$

by (*metis ok-vwb-lens segr-pre-transfer segr-right-one-point true-alt-def uovar-convr upred-eq-true utp-pred-laws.inf.left-idem utp-rel.unrest-ouvar vwb-lens-mwb*)

finally show *?thesis*

by (*metis utp-pred-laws.inf.left-commute utp-pred-laws.inf-left-idem*)

qed

moreover have $(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket) =$

$(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (\$ok \wedge \$ok' \wedge (Q1^t ;; Q2 \llbracket true/\$ok \rrbracket))$

by (*metis design-export-ok design-export-ok'*)

ultimately show *?thesis* **using** *assms*

by (*simp add: design-composition-subst usubst, metis design-export-ok design-export-ok'*)

qed

theorem *rdesign-composition:*

$((P1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$

by (*simp add: rdesign-def design-composition unrest alpha*)

theorem *design-composition-cond:*

assumes

$out\alpha \# p1 \ \$ok \# P2 \ \$ok' \# Q1 \ \$ok \# Q2$

shows $((p1 \vdash Q1) ;; (P2 \vdash Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$

using *assms*

by (*simp add: design-composition unrest precondition-right-unit*)

theorem *rdesign-composition-cond:*

assumes $out\alpha \# p1$

shows $((p1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$

using *assms*

by (*simp add: rdesign-def design-composition-cond unrest alpha*)

theorem *design-composition-wp:*

assumes

$ok \# p1 \ ok \# p2$

$\$ok \# Q1 \ \$ok' \# Q1 \ \$ok \# Q2 \ \$ok' \# Q2$

shows $((\llbracket p1 \rrbracket_{<} \vdash Q1) ;; (\llbracket p2 \rrbracket_{<} \vdash Q2)) = ((\llbracket p1 \wedge Q1 \ wp \ p2 \rrbracket_{<} \vdash (Q1 ;; Q2))$

using *assms* **by** (*rel-blast*)

theorem *rdesign-composition-wp:*

$((\llbracket p1 \rrbracket_{<} \vdash_r Q1) ;; (\llbracket p2 \rrbracket_{<} \vdash_r Q2)) = ((\llbracket p1 \wedge Q1 \ wp \ p2 \rrbracket_{<} \vdash_r (Q1 ;; Q2))$

by (*rel-blast*)

theorem *ndesign-composition-wp* [*ndes-simp*]:

$((p1 \vdash_n Q1) ;; (p2 \vdash_n Q2)) = ((p1 \wedge Q1 \text{ wp } p2) \vdash_n (Q1 ;; Q2))$
by (*rel-blast*)

theorem *design-true-left-zero*: $(\text{true} ;; (P \vdash Q)) = \text{true}$

proof –

have $(\text{true} ;; (P \vdash Q)) = ((\text{true} \llbracket \text{false}/\$ok' \rrbracket ;; (P \vdash Q) \llbracket \text{false}/\$ok \rrbracket) \vee (\text{true} \llbracket \text{true}/\$ok' \rrbracket ;; (P \vdash Q) \llbracket \text{true}/\$ok \rrbracket))$

by (*rel-auto*)

also have $\dots = ((\text{true} \llbracket \text{false}/\$ok' \rrbracket ;; \text{true}_h) \vee (\text{true} ;; ((P \vdash Q) \llbracket \text{true}/\$ok \rrbracket)))$

by (*subst-tac, rel-auto*)

also have $\dots = \text{true}$

by (*subst-tac, simp add: precondition-right-unit unrest*)

finally show *?thesis* .

qed

theorem *design-left-unit-hom*:

fixes $P Q :: 'a \text{ hrel-des}$

shows $(II_D ;; (P \vdash_r Q)) = (P \vdash_r Q)$

proof –

have $(II_D ;; (P \vdash_r Q)) = ((\text{true} \vdash_r II) ;; (P \vdash_r Q))$

by (*simp add: skip-d-def*)

also have $\dots = (\text{true} \wedge \neg (II ;; (\neg P))) \vdash_r (II ;; Q)$

proof –

have $\text{out}\alpha \nVdash \text{true}$

by *unrest-tac*

thus *?thesis*

using *rdesign-composition-cond* **by** *blast*

qed

also have $\dots = (\neg (\neg P)) \vdash_r Q$

by *simp*

finally show *?thesis* **by** *simp*

qed

theorem *rdesign-left-unit* [*simp*]:

$II_D ;; (P \vdash_r Q) = (P \vdash_r Q)$

by (*rel-auto*)

theorem *design-right-semi-unit*:

$(P \vdash_r Q) ;; II_D = ((\neg (\neg P) ;; \text{true}) \vdash_r Q)$

by (*simp add: skip-d-def rdesign-composition*)

theorem *design-right-cond-unit* [*simp*]:

assumes $\text{out}\alpha \nVdash p$

shows $(p \vdash_r Q) ;; II_D = (p \vdash_r Q)$

using *assms*

by (*simp add: skip-d-def rdesign-composition-cond*)

theorem *ndesign-left-unit* [*simp*]:

$II_D ;; (p \vdash_n Q) = (p \vdash_n Q)$

by (*rel-auto*)

theorem *design-bot-left-zero*: $(\perp_D ;; (P \vdash Q)) = \perp_D$

by (*rel-auto*)

theorem *design-top-left-zero*: $(\top_D ;; (P \vdash Q)) = \top_D$
by (*rel-auto*)

1.5 Preconditions and Postconditions

theorem *design-npre*:
 $(P \vdash Q)^f = (\neg \$ok \vee \neg P^f)$
by (*rel-auto*)

theorem *design-pre*:
 $\neg (P \vdash Q)^f = (\$ok \wedge P^f)$
by (*simp add: design-def, subst-tac*)
 $(metis (no-types, hide-lams) not-conj-deMorgans true-not-false(2) utp-pred-laws.compl-top-eq$
 $utp-pred-laws.sup.idem utp-pred-laws.sup-compl-top)$

theorem *design-post*:
 $(P \vdash Q)^t = ((\$ok \wedge P^t) \Rightarrow Q^t)$
by (*rel-auto*)

theorem *rdesign-pre [simp]*: $pre_D(P \vdash_r Q) = P$
by (*pred-auto*)

theorem *rdesign-post [simp]*: $post_D(P \vdash_r Q) = (P \Rightarrow Q)$
by (*pred-auto*)

theorem *ndesign-pre [simp]*: $pre_D(p \vdash_n Q) = [p]_<$
by (*pred-auto*)

theorem *ndesign-post [simp]*: $post_D(p \vdash_n Q) = ([p]_< \Rightarrow Q)$
by (*pred-auto*)

lemma *design-pre-choice [simp]*:
 $pre_D(P \sqcap Q) = (pre_D(P) \wedge pre_D(Q))$
by (*rel-auto*)

lemma *design-post-choice [simp]*:
 $post_D(P \sqcap Q) = (post_D(P) \vee post_D(Q))$
by (*rel-auto*)

lemma *design-pre-condr [simp]*:
 $pre_D(P \triangleleft [b]_D \triangleright Q) = (pre_D(P) \triangleleft b \triangleright pre_D(Q))$
by (*rel-auto*)

lemma *design-post-condr [simp]*:
 $post_D(P \triangleleft [b]_D \triangleright Q) = (post_D(P) \triangleleft b \triangleright post_D(Q))$
by (*rel-auto*)

lemma *preD-USUP-mem*: $pre_D(\bigsqcup_{i \in A} P \cdot i) = (\bigcap_{i \in A} pre_D(P \cdot i))$
by (*rel-auto*)

lemma *preD-USUP-ind*: $pre_D(\bigsqcup i \cdot P \cdot i) = (\bigcap i \cdot pre_D(P \cdot i))$
by (*rel-auto*)

1.6 Distribution Laws

theorem *design-choice*:

$$(P_1 \vdash P_2) \sqcap (Q_1 \vdash Q_2) = ((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2))$$

by (*rel-auto*)

theorem *rdesign-choice*:

$$(P_1 \vdash_r P_2) \sqcap (Q_1 \vdash_r Q_2) = ((P_1 \wedge Q_1) \vdash_r (P_2 \vee Q_2))$$

by (*rel-auto*)

theorem *ndesign-choice* [*ndes-simp*]:

$$(p_1 \vdash_n P_2) \sqcap (q_1 \vdash_n Q_2) = ((p_1 \wedge q_1) \vdash_n (P_2 \vee Q_2))$$

by (*rel-auto*)

theorem *ndesign-choice'* [*ndes-simp*]:

$$((p_1 \vdash_n P_2) \vee (q_1 \vdash_n Q_2)) = ((p_1 \wedge q_1) \vdash_n (P_2 \vee Q_2))$$

by (*rel-auto*)

theorem *design-inf*:

$$(P_1 \vdash P_2) \sqcup (Q_1 \vdash Q_2) = ((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow P_2) \wedge (Q_1 \Rightarrow Q_2)))$$

by (*rel-auto*)

theorem *rdesign-inf*:

$$(P_1 \vdash_r P_2) \sqcup (Q_1 \vdash_r Q_2) = ((P_1 \vee Q_1) \vdash_r ((P_1 \Rightarrow P_2) \wedge (Q_1 \Rightarrow Q_2)))$$

by (*rel-auto*)

theorem *ndesign-inf* [*ndes-simp*]:

$$(p_1 \vdash_n P_2) \sqcup (q_1 \vdash_n Q_2) = ((p_1 \vee q_1) \vdash_n (([p_1]_{<} \Rightarrow P_2) \wedge ([q_1]_{<} \Rightarrow Q_2)))$$

by (*rel-auto*)

theorem *design-condr*:

$$((P_1 \vdash P_2) \triangleleft b \triangleright (Q_1 \vdash Q_2)) = ((P_1 \triangleleft b \triangleright Q_1) \vdash (P_2 \triangleleft b \triangleright Q_2))$$

by (*rel-auto*)

theorem *ndesign-dcond* [*ndes-simp*]:

$$((p_1 \vdash_n P_2) \triangleleft b \triangleright_D (q_1 \vdash_n Q_2)) = ((p_1 \triangleleft b \triangleright q_1) \vdash_n (P_2 \triangleleft b \triangleright_r Q_2))$$

by (*rel-auto*)

lemma *design-UNIF-mem*:

$$\begin{aligned} &\text{assumes } A \neq \{\} \\ &\text{shows } (\prod i \in A \cdot P(i) \vdash Q(i)) = (\prod i \in A \cdot P(i)) \vdash (\prod i \in A \cdot Q(i)) \\ &\text{using } \textit{assms} \text{ by } (\textit{rel-auto}) \end{aligned}$$

lemma *ndesign-UNIF-mem* [*ndes-simp*]:

$$\begin{aligned} &\text{assumes } A \neq \{\} \\ &\text{shows } (\prod i \in A \cdot p(i) \vdash_n Q(i)) = (\prod i \in A \cdot p(i)) \vdash_n (\prod i \in A \cdot Q(i)) \\ &\text{using } \textit{assms} \text{ by } (\textit{rel-auto}) \end{aligned}$$

lemma *ndesign-UNIF-ind* [*ndes-simp*]:

$$(\prod i \cdot p(i) \vdash_n Q(i)) = (\prod i \cdot p(i)) \vdash_n (\prod i \cdot Q(i))$$

by (*rel-auto*)

lemma *design-USUP-mem*:

$$(\prod i \in A \cdot P(i) \vdash Q(i)) = (\prod i \in A \cdot P(i)) \vdash (\prod i \in A \cdot P(i) \Rightarrow Q(i))$$

by (*rel-auto*)

lemma *ndesign-USUP-mem* [*ndes-simp*]:

$(\sqcup i \in A \cdot p(i) \vdash_n Q(i)) = (\prod i \in A \cdot p(i)) \vdash_n (\sqcup i \in A \cdot \lceil p(i) \rceil_{<} \Rightarrow Q(i))$
by (*rel-auto*)

lemma *ndesign-USUP-ind* [*ndes-simp*]:

$(\sqcup i \cdot p(i) \vdash_n Q(i)) = (\prod i \cdot p(i)) \vdash_n (\sqcup i \cdot \lceil p(i) \rceil_{<} \Rightarrow Q(i))$
by (*rel-auto*)

1.7 Refinement Introduction

lemma *ndesign-eq-intro*:

assumes $p_1 = q_1 \ P_2 = Q_2$
shows $p_1 \vdash_n P_2 = q_1 \vdash_n Q_2$
by (*simp add: assms*)

theorem *design-refinement*:

assumes
 $\$ok \# P1 \ \$ok' \# P1 \ \$ok \# P2 \ \$ok' \# P2$
 $\$ok \# Q1 \ \$ok' \# Q1 \ \$ok \# Q2 \ \$ok' \# Q2$
shows $(P1 \vdash Q1 \sqsubseteq P2 \vdash Q2) \longleftrightarrow ('P1 \Rightarrow P2' \wedge 'P1 \wedge Q2 \Rightarrow Q1')$

proof –

have $(P1 \vdash Q1) \sqsubseteq (P2 \vdash Q2) \longleftrightarrow '(\$ok \wedge P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (\$ok \wedge P1 \Rightarrow \$ok' \wedge Q1)'$
by (*pred-auto*)

also with *assms* **have** $\dots = '(P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (P1 \Rightarrow \$ok' \wedge Q1)'$
by (*subst subst-bool-split[of in-var ok], simp-all, subst-tac*)

also with *assms* **have** $\dots = '(\neg P2 \Rightarrow \neg P1) \wedge ((P2 \Rightarrow Q2) \Rightarrow P1 \Rightarrow Q1)'$
by (*subst subst-bool-split[of out-var ok], simp-all, subst-tac*)

also have $\dots \longleftrightarrow '(P1 \Rightarrow P2)' \wedge 'P1 \wedge Q2 \Rightarrow Q1'$
by (*pred-auto*)

finally show *?thesis* .

qed

theorem *rdesign-refinement*:

$(P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2) \longleftrightarrow ('P1 \Rightarrow P2' \wedge 'P1 \wedge Q2 \Rightarrow Q1')$
by (*rel-auto*)

lemma *design-refine-intro*:

assumes $'P1 \Rightarrow P2' \ 'P1 \wedge Q2 \Rightarrow Q1'$
shows $P1 \vdash Q1 \sqsubseteq P2 \vdash Q2$
using *assms* **unfolding** *upred-defs*
by (*pred-auto*)

lemma *design-refine-intro'*:

assumes $P2 \sqsubseteq P1 \ Q1 \sqsubseteq (P1 \wedge Q2)$
shows $P1 \vdash Q1 \sqsubseteq P2 \vdash Q2$
using *assms* *design-refine-intro*[*of P1 P2 Q2 Q1*] **by** (*simp add: refBy-order*)

lemma *rdesign-refine-intro*:

assumes $'P1 \Rightarrow P2' \ 'P1 \wedge Q2 \Rightarrow Q1'$
shows $P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2$
using *assms* **unfolding** *upred-defs*
by (*pred-auto*)

lemma *rdesign-refine-intro'*:

assumes $P2 \sqsubseteq P1 \ Q1 \sqsubseteq (P1 \wedge Q2)$
shows $P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2$

using *assms* **unfolding** *upred-defs*
by (*pred-auto*)

lemma *ndesign-refinement*:

$p1 \vdash_n Q1 \sqsubseteq p2 \vdash_n Q2 \iff (p1 \Rightarrow p2' \wedge '[p1]_< \wedge Q2 \Rightarrow Q1')$
by (*simp add: ndesign-def rdesign-def design-refinement unrest, rel-auto*)

lemma *ndesign-refine-intro*:

assumes ' $p1 \Rightarrow p2'$ ' ' $[p1]_< \wedge Q2 \Rightarrow Q1'$ '
shows $p1 \vdash_n Q1 \sqsubseteq p2 \vdash_n Q2$
using *assms* **unfolding** *upred-defs*
by (*pred-auto*)

lemma *design-top*:

$(P \vdash Q) \sqsubseteq \top_D$
by (*rel-auto*)

lemma *design-bottom*:

$\perp_D \sqsubseteq (P \vdash Q)$
by (*rel-auto*)

lemma *design-refine-thms*:

assumes $P \sqsubseteq Q$
shows ' $\text{pre}_D(P) \Rightarrow \text{pre}_D(Q)$ ' ' $\text{pre}_D(P) \wedge \text{post}_D(Q) \Rightarrow \text{post}_D(P)$ '
apply (*metis assms design-pre-choice disj-comm disj-upred-def order-refl rdesign-refinement utp-pred-laws.le-iff-sup*)
apply (*metis assms conj-comm design-post-choice disj-upred-def refBy-order semilattice-sup-class.le-iff-sup*
utp-pred-laws.inf.coboundedI1)
done

end

2 Design Healthiness Conditions

theory *utp-des-healths*

imports *utp-des-core*

begin

2.1 H1: No observation is allowed before initiation

definition *H1* :: $('α, 'β) \text{rel-des} \Rightarrow ('α, 'β) \text{rel-des}$ **where**

$[upred-defs]: H1(P) = (\$ok \Rightarrow P)$

lemma *H1-idem*:

$H1(H1 P) = H1(P)$
by (*pred-auto*)

lemma *H1-monotone*:

$P \sqsubseteq Q \implies H1(P) \sqsubseteq H1(Q)$
by (*pred-auto*)

lemma *H1-Continuous*: *Continuous H1*

by (*rel-auto*)

lemma *H1-below-top*:

$H1(P) \sqsubseteq \top_D$

by (pred-auto)

lemma *H1-design-skip*:

$H1(II) = II_D$

by (rel-auto)

lemma *H1-cond*: $H1(P \triangleleft b \triangleright Q) = H1(P) \triangleleft b \triangleright H1(Q)$

by (rel-auto)

lemma *H1-conj*: $H1(P \wedge Q) = (H1(P) \wedge H1(Q))$

by (rel-auto)

lemma *H1-disj*: $H1(P \vee Q) = (H1(P) \vee H1(Q))$

by (rel-auto)

lemma *design-export-H1*: $(P \vdash Q) = (P \vdash H1(Q))$

by (rel-auto)

The H1 algebraic laws are valid only when $\alpha(R)$ is homogeneous. This should maybe be generalised.

theorem *H1-algebraic-intro*:

assumes

$(true_h ;; R) = true_h$

$(II_D ;; R) = R$

shows *R is H1*

proof –

have $R = (II_D ;; R)$ by (simp add: assms(2))

also have $\dots = (H1(II) ;; R)$

by (simp add: H1-design-skip)

also have $\dots = (\$ok \Rightarrow II) ;; R$

by (simp add: H1-def)

also have $\dots = (((\neg \$ok) ;; R) \vee R)$

by (simp add: impl-alt-def seqr-or-distl)

also have $\dots = (((\neg \$ok) ;; true_h) ;; R) \vee R$

by (simp add: precondition-right-unit unrest)

also have $\dots = (((\neg \$ok) ;; true_h) \vee R)$

by (metis assms(1) seqr-assoc)

also have $\dots = (\$ok \Rightarrow R)$

by (simp add: impl-alt-def precondition-right-unit unrest)

finally show ?thesis by (metis H1-def Healthy-def')

qed

lemma *nok-not-false*:

$(\neg \$ok) \neq false$

by (pred-auto)

theorem *H1-left-zero*:

assumes *P is H1*

shows $(true ;; P) = true$

proof –

from assms have $(true ;; P) = (true ;; (\$ok \Rightarrow P))$

by (simp add: H1-def Healthy-def')

also from assms have $\dots = (true ;; (\neg \$ok \vee P))$ (is - = (?true ;; -))

by (simp add: impl-alt-def)

also from *assms* have ... = $((?true ;; (\neg \$ok)) \vee (?true ;; P))$
 using *seqr-or-distr* by *blast*
 also from *assms* have ... = $(true \vee (true ;; P))$
 by (*simp* add: *nok-not-false precondition-left-zero unrest*)
 finally show *?thesis*
 by (*simp* add: *upred-defs urel-defs*)
 qed

theorem *H1-left-unit*:

fixes $P :: 'a \text{ hrel-des}$

assumes *P is H1*

shows $(II_D ;; P) = P$

proof –

have $(II_D ;; P) = ((\$ok \Rightarrow II) ;; P)$
 by (*metis H1-def H1-design-skip*)
 also have ... = $((\neg \$ok) ;; P) \vee P$
 by (*simp* add: *impl-alt-def seqr-or-distl*)
 also from *assms* have ... = $((\neg \$ok) ;; true_h) ;; P) \vee P$
 by (*simp* add: *precond-right-unit unrest*)
 also have ... = $((\neg \$ok) ;; (true_h ;; P)) \vee P$
 by (*simp* add: *seqr-assoc*)
 also from *assms* have ... = $(\$ok \Rightarrow P)$
 by (*simp* add: *H1-left-zero impl-alt-def precondition-right-unit unrest*)
 finally show *?thesis* using *assms*
 by (*simp* add: *H1-def Healthy-def'*)
 qed

theorem *H1-algebraic*:

$P \text{ is } H1 \longleftrightarrow (true_h ;; P) = true_h \wedge (II_D ;; P) = P$

using *H1-algebraic-intro H1-left-unit H1-left-zero* by *blast*

theorem *H1-nok-left-zero*:

fixes $P :: 'a \text{ hrel-des}$

assumes *P is H1*

shows $((\neg \$ok) ;; P) = (\neg \$ok)$

proof –

have $((\neg \$ok) ;; P) = ((\neg \$ok) ;; true_h) ;; P$
 by (*simp* add: *precond-right-unit unrest*)
 also have ... = $((\neg \$ok) ;; true_h)$
 by (*metis H1-left-zero assms seqr-assoc*)
 also have ... = $(\neg \$ok)$
 by (*simp* add: *precond-right-unit unrest*)
 finally show *?thesis* .
 qed

lemma *H1-design*:

$H1(P \vdash Q) = (P \vdash Q)$

by (*rel-auto*)

lemma *H1-rdesign*:

$H1(P \vdash_r Q) = (P \vdash_r Q)$

by (*rel-auto*)

lemma *H1-choice-closed [closure]*:

$\llbracket P \text{ is } H1; Q \text{ is } H1 \rrbracket \Longrightarrow P \sqcap Q \text{ is } H1$

by (simp add: H1-def Healthy-def' disj-upred-def impl-alt-def semilattice-sup-class.sup-left-commute)

lemma *H1-inf-closed* [closure]:

[[*P* is *H1*; *Q* is *H1*]] $\implies P \sqcup Q$ is *H1*
by (rel-blast)

lemma *H1-UINF*:

assumes $A \neq \{\}$
shows $H1(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot H1(P(i)))$
using *assms* by (rel-auto)

lemma *H1-Sup*:

assumes $A \neq \{\} \ \forall P \in A. P$ is *H1*
shows $(\bigsqcap A)$ is *H1*

proof –

from *assms*(2) have $H1 \text{ ‘ } A = A$
by (auto simp add: Healthy-def rev-image-eqI)
with *H1-UINF*[of *A* *id*, OF *assms*(1)] **show** ?thesis
by (simp add: UINF-as-Sup-image Healthy-def, presburger)

qed

lemma *H1-USUP*:

shows $H1(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot H1(P(i)))$
by (rel-auto)

lemma *H1-Inf* [closure]:

assumes $\forall P \in A. P$ is *H1*
shows $(\bigsqcup A)$ is *H1*

proof –

from *assms* have $H1 \text{ ‘ } A = A$
by (auto simp add: Healthy-def rev-image-eqI)
with *H1-USUP*[of *A* *id*] **show** ?thesis
by (simp add: USUP-as-Inf-image Healthy-def, presburger)

qed

lemma *msubst-H1*: $(\bigwedge x. P \ x \text{ is } H1) \implies P \ x \llbracket x \rightarrow v \rrbracket$ is *H1*

by (rel-auto)

2.2 H2: A specification cannot require non-termination

definition *J* :: ' α hrel-des **where**

[upred-defs]: $J = ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D)$

definition *H2* **where**

[upred-defs]: $H2 \ (P) \equiv P ;; J$

lemma *J-split*:

shows $(P ;; J) = (P^f \vee (P^t \wedge \$ok'))$

proof –

have $(P ;; J) = (P ;; ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D))$
by (simp add: H2-def J-def design-def)
also have $\dots = (P ;; ((\$ok \Rightarrow \$ok' \wedge \$ok') \wedge \lceil II \rceil_D))$
by (rel-auto)
also have $\dots = ((P ;; (\neg \$ok' \wedge \lceil II \rceil_D)) \vee (P ;; (\$ok' \wedge (\lceil II \rceil_D \wedge \$ok'))))$
by (rel-auto)
also have $\dots = (P^f \vee (P^t \wedge \$ok'))$


```

proof –
  have  $(P ;; (\neg \$ok \wedge \lceil II \rceil_D)) = P^f$ 
  proof –
    have  $(P ;; (\neg \$ok \wedge \lceil II \rceil_D)) = ((P \wedge \neg \$ok') ;; \lceil II \rceil_D)$ 
      by (rel-auto)
    also have  $\dots = (\exists \$ok' \cdot P \wedge \$ok' =_u \text{false})$ 
      by (rel-auto)
    also have  $\dots = P^f$ 
      by (metis C1 one-point out-var-uvar unrest-as-exists ok-vwb-lens vwb-lens-mwb)
    finally show ?thesis .
  qed
moreover have  $(P ;; (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P^t \wedge \$ok')$ 
proof –
  have  $(P ;; (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P ;; (\$ok \wedge II))$ 
    by (rel-auto)
  also have  $\dots = (P^t \wedge \$ok')$ 
    by (rel-auto)
  finally show ?thesis .
qed
ultimately show ?thesis
  by simp
qed
finally show ?thesis .
qed

```

lemma *H2-split*:

```

shows  $H2(P) = (P^f \vee (P^t \wedge \$ok'))$ 
by (simp add: H2-def J-split)

```

theorem *H2-equivalence*:

$P \text{ is } H2 \iff 'P^f \Rightarrow P^t'$

```

proof –
  have  $'P \Leftrightarrow (P ;; J)' \iff 'P \Leftrightarrow (P^f \vee (P^t \wedge \$ok'))'$ 
    by (simp add: J-split)
  also have  $\dots \iff '(P \Leftrightarrow P^f \vee P^t \wedge \$ok')^f \wedge (P \Leftrightarrow P^f \vee P^t \wedge \$ok')^t'$ 
    by (simp add: subst-bool-split)
  also have  $\dots = '(P^f \Leftrightarrow P^f) \wedge (P^t \Leftrightarrow P^f \vee P^t)'$ 
    by subst-tac
  also have  $\dots = 'P^t \Leftrightarrow (P^f \vee P^t)'$ 
    by (pred-auto robust)
  also have  $\dots = '(P^f \Rightarrow P^t)'$ 
    by (pred-auto)
  finally show ?thesis
    by (metis H2-def Healthy-def' taut-iff-eq)
qed

```

lemma *H2-equiv*:

$P \text{ is } H2 \iff P^t \sqsubseteq P^f$

using *H2-equivalence refBy-order* **by** *blast*

lemma *H2-design*:

assumes $\$ok' \nVdash P \ \$ok' \nVdash Q$

shows $H2(P \vdash Q) = P \vdash Q$

using *assms*

by *(simp add: H2-split design-def usubst unrest, pred-auto)*

lemma *H2-rdesign*:

$H2(P \vdash_r Q) = P \vdash_r Q$

by (*simp add: H2-design unrest rdesign-def*)

theorem *J-idem*:

$(J ;; J) = J$

by (*rel-auto*)

theorem *H2-idem*:

$H2(H2(P)) = H2(P)$

by (*metis H2-def J-idem segr-assoc*)

theorem *H2-Continuous: Continuous H2*

by (*rel-auto*)

theorem *H2-not-okay*: $H2(\neg \$ok) = (\neg \$ok)$

proof –

have $H2(\neg \$ok) = ((\neg \$ok)^f \vee ((\neg \$ok)^t \wedge \$ok')$

by (*simp add: H2-split*)

also have $\dots = (\neg \$ok \vee (\neg \$ok) \wedge \$ok')$

by (*subst-tac*)

also have $\dots = (\neg \$ok)$

by (*pred-auto*)

finally show *?thesis* .

qed

lemma *H2-true*: $H2(true) = true$

by (*rel-auto*)

lemma *H2-choice-closed [closure]*:

$\llbracket P \text{ is } H2; Q \text{ is } H2 \rrbracket \implies P \sqcap Q \text{ is } H2$

by (*metis H2-def Healthy-def' disj-upred-def segr-or-distl*)

lemma *H2-inf-closed [closure]*:

assumes $P \text{ is } H2 \ Q \text{ is } H2$

shows $P \sqcup Q \text{ is } H2$

proof –

have $P \sqcup Q = (P^f \vee P^t \wedge \$ok') \sqcup (Q^f \vee Q^t \wedge \$ok')$

by (*metis H2-def Healthy-def J-split assms(1) assms(2)*)

moreover have $H2(\dots) = \dots$

by (*simp add: H2-split usubst, pred-auto*)

ultimately show *?thesis*

by (*simp add: Healthy-def*)

qed

lemma *H2-USUP*:

shows $H2(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot H2(P(i)))$

by (*rel-auto*)

theorem *H1-H2-commute*:

$H1(H2 P) = H2(H1 P)$

proof –

have $H2(H1 P) = ((\$ok \Rightarrow P) ;; J)$

by (*simp add: H1-def H2-def*)

```

also have ... = ((¬ $ok ∨ P) ;; J)
  by (rel-auto)
also have ... = (((¬ $ok) ;; J) ∨ (P ;; J))
  using segr-or-distl by blast
also have ... = ((H2 (¬ $ok)) ∨ H2(P))
  by (simp add: H2-def)
also have ... = ((¬ $ok) ∨ H2(P))
  by (simp add: H2-not-okay)
also have ... = H1(H2(P))
  by (rel-auto)
finally show ?thesis by simp
qed

```

2.3 Designs as $H1$ - $H2$ predicates

abbreviation $H1\text{-}H2 :: ('\alpha, '\beta) \text{rel-des} \Rightarrow ('\alpha, '\beta) \text{rel-des}$ (**H**) **where**
 $H1\text{-}H2\ P \equiv H1\ (H2\ P)$

lemma $H1\text{-}H2\text{-comp}$: $\mathbf{H} = H1 \circ H2$
 by (auto)

theorem $H1\text{-}H2\text{-eq-design}$:

$\mathbf{H}(P) = (\neg P^f) \vdash P^t$

proof –

```

have  $\mathbf{H}(P) = (\$ok \Rightarrow H2(P))$ 
  by (simp add: H1-def)
also have ... =  $(\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$ 
  by (metis H2-split)
also have ... =  $(\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t)$ 
  by (rel-auto)
also have ... =  $(\neg P^f) \vdash P^t$ 
  by (rel-auto)
finally show ?thesis .

```

qed

theorem $H1\text{-}H2\text{-is-design}$:

assumes P is $H1\ P$ is $H2$

shows $P = (\neg P^f) \vdash P^t$

using *assms* **by** (metis $H1\text{-}H2\text{-eq-design}$ *Healthy-def*)

theorem $H1\text{-}H2\text{-eq-rdesign}$:

$\mathbf{H}(P) = pre_D(P) \vdash_r post_D(P)$

proof –

```

have  $\mathbf{H}(P) = (\$ok \Rightarrow H2(P))$ 
  by (simp add: H1-def Healthy-def)
also have ... =  $(\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$ 
  by (metis H2-split)
also have ... =  $(\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge P^t)$ 
  by (pred-auto)
also have ... =  $(\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t)$ 
  by (pred-auto)
also have ... =  $(\$ok \wedge [pre_D(P)]_D \Rightarrow \$ok' \wedge \$ok \wedge [post_D(P)]_D)$ 
  by (simp add: ok-post ok-pre)
also have ... =  $(\$ok \wedge [pre_D(P)]_D \Rightarrow \$ok' \wedge [post_D(P)]_D)$ 
  by (pred-auto)
also have ... =  $pre_D(P) \vdash_r post_D(P)$ 

```

by (simp add: rdesign-def design-def)
 finally show ?thesis .
 qed

theorem *H1-H2-is-rdesign*:
 assumes *P is H1 P is H2*
 shows $P = pre_D(P) \vdash_r post_D(P)$
 by (metis *H1-H2-eq-rdesign Healthy-def assms(1) assms(2)*)

lemma *H1-H2-refinement*:
 assumes *P is H Q is H*
 shows $P \sqsubseteq Q \longleftrightarrow ('pre_D(P) \Rightarrow pre_D(Q)' \wedge 'pre_D(P) \wedge post_D(Q) \Rightarrow post_D(P)')$
 by (metis *H1-H2-eq-rdesign Healthy-if assms rdesign-refinement*)

lemma *H1-H2-refines*:
 assumes *P is H Q is H P \sqsubseteq Q*
 shows $pre_D(Q) \sqsubseteq pre_D(P) \text{ } post_D(P) \sqsubseteq (pre_D(P) \wedge post_D(Q))$
 using *H1-H2-refinement assms refBy-order* by auto

lemma *H1-H2-idempotent*: $\mathbf{H} (\mathbf{H} P) = \mathbf{H} P$
 by (simp add: *H1-H2-commute H1-idem H2-idem*)

lemma *H1-H2-Idempotent [closure]*: *Idempotent H*
 by (simp add: *Idempotent-def H1-H2-idempotent*)

lemma *H1-H2-monotonic [closure]*: *Monotonic H*
 by (simp add: *H1-monotone H2-def mono-def seqr-mono*)

lemma *H1-H2-Continuous [closure]*: *Continuous H*
 by (simp add: *Continuous-comp H1-Continuous H1-H2-comp H2-Continuous*)

lemma *design-is-H1-H2 [closure]*:
 $\llbracket \$ok' \# P; \$ok' \# Q \rrbracket \Longrightarrow (P \vdash Q) \text{ is } \mathbf{H}$
 by (simp add: *H1-design H2-design Healthy-def'*)

lemma *rdesign-is-H1-H2 [closure]*:
 $(P \vdash_r Q) \text{ is } \mathbf{H}$
 by (simp add: *Healthy-def H1-rdesign H2-rdesign*)

lemma *top-d-is-H1-H2 [closure]*: $\top_D \text{ is } \mathbf{H}$
 by (simp add: *H1-def H2-not-okay Healthy-intro impl-alt-def*)

lemma *bot-d-is-H1-H2 [closure]*: $\perp_D \text{ is } \mathbf{H}$
 by (simp add: *bot-d-def closure unrest*)

lemma *seq-r-H1-H2-closed [closure]*:

assumes *P is H Q is H*

shows $(P ;; Q) \text{ is } \mathbf{H}$

proof –

obtain $P_1 P_2$ where $P = P_1 \vdash_r P_2$

by (metis *H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms(1)*)

moreover obtain $Q_1 Q_2$ where $Q = Q_1 \vdash_r Q_2$

by (metis *H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms(2)*)

moreover have $((P_1 \vdash_r P_2) ;; (Q_1 \vdash_r Q_2)) \text{ is } \mathbf{H}$

by (simp add: *rdesign-composition rdesign-is-H1-H2*)

ultimately show *?thesis* by *simp*
qed

lemma *H1-H2-left-unit*: P is **H** $\implies \Pi_D \;; P = P$
by (*metis H1-H2-eq-rdesign Healthy-def' rdesign-left-unit*)

lemma *UINF-H1-H2-closed* [*closure*]:
assumes $A \neq \{\}$ $\forall P \in A. P$ is **H**
shows $(\sqcap A)$ is *H1-H2*

proof –

from *assms* have $A: A = H1-H2 \text{ ' } A$
by (*auto simp add: Healthy-def rev-image-eqI*)
also have $(\sqcap \dots) = (\sqcap P \in A \cdot H1-H2(P))$
by (*simp add: UINF-as-Sup-collect*)
also have $\dots = (\sqcap P \in A \cdot (\neg P^f) \vdash P^t)$
by (*meson H1-H2-eq-design*)
also have $\dots = (\sqcup P \in A \cdot \neg P^f) \vdash (\sqcap P \in A \cdot P^t)$
by (*simp add: design-UINF-mem assms*)
also have \dots is *H1-H2*
by (*simp add: design-is-H1-H2 unrest*)
finally show *?thesis* .

qed

definition *design-inf* :: $(' \alpha, ' \beta)$ *rel-des set* $\Rightarrow (' \alpha, ' \beta)$ *rel-des* $(\sqcap_D - [900] 900)$ **where**
 $\sqcap_D A = (\text{if } (A = \{\}) \text{ then } \top_D \text{ else } \sqcap A)$

abbreviation *design-sup* :: $(' \alpha, ' \beta)$ *rel-des set* $\Rightarrow (' \alpha, ' \beta)$ *rel-des* $(\sqcup_D - [900] 900)$ **where**
 $\sqcup_D A \equiv \sqcup A$

lemma *design-inf-H1-H2-closed*:
assumes $\forall P \in A. P$ is **H**
shows $(\sqcap_D A)$ is **H**
apply (*auto simp add: design-inf-def closure*)
apply (*simp add: H1-def H2-not-okay Healthy-def impl-alt-def*)
apply (*metis H1-def Healthy-def UINF-H1-H2-closed assms empty-iff impl-alt-def*)
done

lemma *design-sup-empty* [*simp*]: $\sqcap_D \{\} = \top_D$
by (*simp add: design-inf-def*)

lemma *design-sup-non-empty* [*simp*]: $A \neq \{\} \implies \sqcap_D A = \sqcap A$
by (*simp add: design-inf-def*)

lemma *USUP-mem-H1-H2-closed*:
assumes $\bigwedge i. i \in A \implies P \text{ } i$ is **H**
shows $(\sqcup i \in A \cdot P \text{ } i)$ is **H**

proof –

from *assms* have $(\sqcup i \in A \cdot P \text{ } i) = (\sqcup i \in A \cdot \mathbf{H}(P \text{ } i))$
by (*auto intro: USUP-cong simp add: Healthy-def*)
also have $\dots = (\sqcup i \in A \cdot (\neg (P \text{ } i)^f) \vdash (P \text{ } i)^t)$
by (*meson H1-H2-eq-design*)
also have $\dots = (\sqcap i \in A \cdot \neg (P \text{ } i)^f) \vdash (\sqcup i \in A \cdot \neg (P \text{ } i)^f \Rightarrow (P \text{ } i)^t)$
by (*simp add: design-USUP-mem*)
also have \dots is **H**
by (*simp add: design-is-H1-H2 unrest*)

finally show ?thesis .
qed

lemma *USUP-ind-H1-H2-closed*:
assumes $\bigwedge i. P\ i$ is **H**
shows $(\bigsqcup i \cdot P\ i)$ is **H**
using *assms USUP-mem-H1-H2-closed*[of *UNIV P*] by *simp*

lemma *Inf-H1-H2-closed*:
assumes $\forall P \in A. P$ is **H**
shows $(\bigsqcup A)$ is **H**

proof –
from *assms* have $A: A = \mathbf{H} \cdot A$
by (*auto simp add: Healthy-def rev-image-eqI*)
also have $(\bigsqcup \dots) = (\bigsqcup P \in A \cdot \mathbf{H}(P))$
by (*simp add: USUP-as-Inf-collect*)
also have $\dots = (\bigsqcup P \in A \cdot (\neg P^f) \vdash P^t)$
by (*meson H1-H2-eq-design*)
also have $\dots = (\bigcap P \in A \cdot \neg P^f) \vdash (\bigsqcup P \in A \cdot \neg P^f \Rightarrow P^t)$
by (*simp add: design-USUP-mem*)
also have \dots is **H**
by (*simp add: design-is-H1-H2 unrest*)
finally show ?thesis .
qed

lemma *rdesign-ref-monos*:
assumes P is **H** Q is **H** $P \sqsubseteq Q$
shows $\text{pre}_D(Q) \sqsubseteq \text{pre}_D(P)$ $\text{post}_D(P) \sqsubseteq (\text{pre}_D(P) \wedge \text{post}_D(Q))$
proof –
have $r: P \sqsubseteq Q \iff (' \text{pre}_D(P) \Rightarrow \text{pre}_D(Q)' \wedge ' \text{pre}_D(P) \wedge \text{post}_D(Q) \Rightarrow \text{post}_D(P)')$
by (*metis H1-H2-eq-rdesign Healthy-if assms(1) assms(2) rdesign-refinement*)
from r *assms* show $\text{pre}_D(Q) \sqsubseteq \text{pre}_D(P)$
by (*auto simp add: refBy-order*)
from r *assms* show $\text{post}_D(P) \sqsubseteq (\text{pre}_D(P) \wedge \text{post}_D(Q))$
by (*auto simp add: refBy-order*)
qed

2.4 H3: The design assumption is a precondition

definition $H3 :: ('\alpha, '\beta) \text{rel-des} \Rightarrow ('\alpha, '\beta) \text{rel-des}$ where
[*upred-defs*]: $H3\ (P) \equiv P \;; \Pi_D$

theorem *H3-idem*:
 $H3(H3(P)) = H3(P)$
by (*metis H3-def design-skip-idem seqr-assoc*)

theorem *H3-mono*:
 $P \sqsubseteq Q \implies H3(P) \sqsubseteq H3(Q)$
by (*simp add: H3-def seqr-mono*)

theorem *H3-Monotonic*:
Monotonic H3
by (*simp add: H3-mono mono-def*)

theorem *H3-Continuous*: *Continuous H3*
by (*rel-auto*)

theorem *design-condition-is-H3*:

assumes $out\alpha \nmid p$

shows $(p \vdash Q)$ is *H3*

proof –

have $((p \vdash Q) ;; II_D) = (\neg ((\neg p) ;; true)) \vdash (Q^t ;; II\llbracket true/\$ok \rrbracket)$

by (*simp add: skip-d-alt-def design-composition-subst unrest assms*)

also have $\dots = p \vdash (Q^t ;; II\llbracket true/\$ok \rrbracket)$

using *assms precondition-equiv segr-true-lemma* by *force*

also have $\dots = p \vdash Q$

by (*rel-auto*)

finally show *?thesis*

by (*simp add: H3-def Healthy-def'*)

qed

theorem *rdesign-H3-iff-pre*:

$P \vdash_r Q$ is *H3* $\longleftrightarrow P = (P ;; true)$

proof –

have $(P \vdash_r Q) ;; II_D = (P \vdash_r Q) ;; (true \vdash_r II)$

by (*simp add: skip-d-def*)

also have $\dots = (\neg ((\neg P) ;; true) \wedge \neg (Q ;; (\neg true))) \vdash_r (Q ;; II)$

by (*simp add: rdesign-composition*)

also have $\dots = (\neg ((\neg P) ;; true) \wedge \neg (Q ;; (\neg true))) \vdash_r Q$

by *simp*

also have $\dots = (\neg ((\neg P) ;; true)) \vdash_r Q$

by (*pred-auto*)

finally have $P \vdash_r Q$ is *H3* $\longleftrightarrow P \vdash_r Q = (\neg ((\neg P) ;; true)) \vdash_r Q$

by (*metis H3-def Healthy-def'*)

also have $\dots \longleftrightarrow P = (\neg ((\neg P) ;; true))$

by (*metis rdesign-pre*)

thm *segr-true-lemma*

also have $\dots \longleftrightarrow P = (P ;; true)$

by (*simp add: segr-true-lemma*)

finally show *?thesis* .

qed

theorem *design-H3-iff-pre*:

assumes $\$ok \nmid P \$ok' \nmid P \$ok \nmid Q \$ok' \nmid Q$

shows $P \vdash Q$ is *H3* $\longleftrightarrow P = (P ;; true)$

proof –

have $P \vdash Q = \lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D$

by (*simp add: assms lift-desr-inv rdesign-def*)

moreover hence $\lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D$ is *H3* $\longleftrightarrow \lfloor P \rfloor_D = (\lfloor P \rfloor_D ;; true)$

using *rdesign-H3-iff-pre* by *blast*

ultimately show *?thesis*

by (*metis assms(1,2) drop-desr-inv lift-desr-inv lift-dist-seq aext-true*)

qed

theorem *H1-H3-commute*:

$H1 (H3 P) = H3 (H1 P)$

by (*rel-auto*)

lemma *skip-d-absorb-J-1*:

$(II_D ;; J) = II_D$

by (*metis H2-def H2-rdesign skip-d-def*)

lemma *skip-d-absorb-J-2*:

$$(J ;; II_D) = II_D$$

proof –

have $(J ;; II_D) = ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) ;; (true \vdash II)$

by (*simp add: J-def skip-d-alt-def*)

also have $\dots = (((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) \llbracket false / \$ok' \rrbracket ;; (true \vdash II) \llbracket false / \$ok \rrbracket)$
 $\vee (((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D) \llbracket true / \$ok' \rrbracket ;; (true \vdash II) \llbracket true / \$ok \rrbracket)$

by (*rel-auto*)

also have $\dots = ((\neg \$ok \wedge \lceil II \rceil_D ;; true) \vee (\lceil II \rceil_D ;; \$ok' \wedge \lceil II \rceil_D))$

by (*rel-auto*)

also have $\dots = II_D$

by (*rel-auto*)

finally show *?thesis* .

qed

lemma *H2-H3-absorb*:

$$H2 (H3 P) = H3 P$$

by (*metis H2-def H3-def segr-assoc skip-d-absorb-J-1*)

lemma *H3-H2-absorb*:

$$H3 (H2 P) = H3 P$$

by (*metis H2-def H3-def segr-assoc skip-d-absorb-J-2*)

theorem *H2-H3-commute*:

$$H2 (H3 P) = H3 (H2 P)$$

by (*simp add: H2-H3-absorb H3-H2-absorb*)

theorem *H3-design-pre*:

assumes $\$ok \# p \text{ out}\alpha \# p \ \$ok \# Q \ \$ok' \# Q$

shows $H3(p \vdash Q) = p \vdash Q$

using *assms*

by (*metis Healthy-def' design-H3-iff-pre precondition-right-unit unrest-out α -var ok-vwb-lens vwb-lens-mwb*)

theorem *H3-rdesign-pre*:

assumes $\text{out}\alpha \# p$

shows $H3(p \vdash_r Q) = p \vdash_r Q$

using *assms*

by (*simp add: H3-def*)

theorem *H3-ndesign*: $H3(p \vdash_n Q) = (p \vdash_n Q)$

by (*simp add: H3-def ndesign-def unrest-pre-out α*)

theorem *ndesign-is-H3 [closure]*: $p \vdash_n Q$ is *H3*

by (*simp add: H3-ndesign Healthy-def*)

lemma *msubst-pre-H3*: $(\bigwedge x. P \ x \text{ is } H3) \implies P \ x \llbracket x \rightarrow \lceil v \rceil_{<} \rrbracket \text{ is } H3$

by (*rel-auto*)

2.5 Normal Designs as *H1-H3* predicates

A normal design [3] refers only to initial state variables in the precondition.

abbreviation *H1-H3* :: $(\alpha, \beta) \text{ rel-des} \implies (\alpha, \beta) \text{ rel-des } (\mathbf{N})$ **where**

H1-H3 $p \equiv H1 (H3 \ p)$

lemma *H1-H3-comp*: $H1-H3 = H1 \circ H3$
by (*auto*)

theorem *H1-H3-is-design*:
assumes P is $H1$ P is $H3$
shows $P = (\neg P^f) \vdash P^t$
by (*metis H1-H2-eq-design H2-H3-absorb Healthy-def' assms(1) assms(2)*)

theorem *H1-H3-is-rdesign*:
assumes P is $H1$ P is $H3$
shows $P = pre_D(P) \vdash_r post_D(P)$
by (*metis H1-H2-is-rdesign H2-H3-absorb Healthy-def' assms*)

theorem *H1-H3-is-normal-design*:
assumes P is $H1$ P is $H3$
shows $P = \lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)$
by (*metis H1-H3-is-rdesign assms drop-pre-inv ndesign-def precondition-equiv rdesign-H3-iff-pre*)

lemma *H1-H3-idempotent*: $\mathbf{N} (\mathbf{N} P) = \mathbf{N} P$
by (*simp add: H1-H3-commute H1-idem H3-idem*)

lemma *H1-H3-Idempotent [closure]*: *Idempotent* \mathbf{N}
by (*simp add: Idempotent-def H1-H3-idempotent*)

lemma *H1-H3-monotonic [closure]*: *Monotonic* \mathbf{N}
by (*simp add: H1-monotone H3-mono mono-def*)

lemma *H1-H3-Continuous [closure]*: *Continuous* \mathbf{N}
by (*simp add: Continuous-comp H1-Continuous H1-H3-comp H3-Continuous*)

lemma *H1-H3-intro*:
assumes P is \mathbf{H} $out\alpha \nVdash pre_D(P)$
shows P is \mathbf{N}
by (*metis H1-H2-eq-rdesign H1-rdesign H3-rdesign-pre Healthy-def' assms*)

lemma *H1-H3-left-unit*: P is $\mathbf{N} \implies \Pi_D ;; P = P$
by (*metis H1-H2-left-unit H1-H3-commute H2-H3-absorb H3-idem Healthy-def*)

lemma *H1-H3-right-unit*: P is $\mathbf{N} \implies P ;; \Pi_D = P$
by (*metis H1-H3-commute H3-def H3-idem Healthy-def*)

lemma *H1-H3-top-left*: P is $\mathbf{N} \implies \top_D ;; P = \top_D$
by (*metis H1-H2-eq-design H2-H3-absorb Healthy-if design-top-left-zero*)

lemma *H1-H3-bot-left*: P is $\mathbf{N} \implies \perp_D ;; P = \perp_D$
by (*metis H1-idem H1-left-zero Healthy-def bot-d-true*)

lemma *H1-H3-impl-H2 [closure]*: P is $\mathbf{N} \implies P$ is \mathbf{H}
by (*metis H1-H2-commute H1-idem H2-H3-absorb Healthy-def'*)

lemma *H1-H3-eq-design-d-comp*: $\mathbf{N}(P) = ((\neg P^f) \vdash P^t) ;; \Pi_D$
by (*metis H1-H2-eq-design H1-H3-commute H3-H2-absorb H3-def*)

lemma *H1-H3-eq-design*: $\mathbf{N}(P) = (\neg (P^f ;; true)) \vdash P^t$
apply (*simp add: H1-H3-eq-design-d-comp skip-d-alt-def*)

apply (*subst design-composition-subst*)
apply (*simp-all add: usubst unrest*)
apply (*rel-auto*)
done

lemma *H3-unrest-out-alpha-nok* [*unrest*]:
assumes *P is N*
shows $\text{out}\alpha \nVdash P^f$
proof –
have $P = (\neg (P^f \;; \text{true})) \vdash P^t$
by (*metis H1-H3-eq-design Healthy-def assms*)
also have $\text{out}\alpha \nVdash (\dots)^f$
by (*simp add: design-def usubst unrest, rel-auto*)
finally show *?thesis* .
qed

lemma *H3-unrest-out-alpha* [*unrest*]: $P \text{ is } \mathbf{N} \implies \text{out}\alpha \nVdash \text{pre}_D(P)$
by (*metis H1-H3-commute H1-H3-is-rdesign H1-idem Healthy-def' precond-equiv rdesign-H3-iff-pre*)

lemma *ndesign-H1-H3* [*closure*]: $p \vdash_n Q \text{ is } \mathbf{N}$
by (*simp add: H1-rdesign H3-def Healthy-def' ndesign-def unrest-pre-out α*)

lemma *ndesign-form*: $P \text{ is } \mathbf{N} \implies (\lfloor \text{pre}_D(P) \rfloor < \vdash_n \text{post}_D(P)) = P$
by (*metis H1-H2-eq-rdesign H1-H3-impl-H2 H3-unrest-out-alpha Healthy-def drop-pre-inv ndesign-def*)

lemma *des-bot-H1-H3* [*closure*]: $\perp_D \text{ is } \mathbf{N}$
by (*metis H1-design H3-def Healthy-def' design-false-pre design-true-left-zero skip-d-alt-def bot-d-def*)

lemma *des-top-is-H1-H3* [*closure*]: $\top_D \text{ is } \mathbf{N}$
by (*metis ndesign-H1-H3 ndesign-miracle*)

lemma *skip-d-is-H1-H3* [*closure*]: $\text{II}_D \text{ is } \mathbf{N}$
by (*simp add: ndesign-H1-H3 skip-d-ndes-def*)

lemma *seq-r-H1-H3-closed* [*closure*]:
assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $(P \;; \ Q) \text{ is } \mathbf{N}$
by (*metis (no-types) H1-H2-eq-design H1-H3-eq-design-d-comp H1-H3-impl-H2 Healthy-def assms(1) assms(2) seq-r-H1-H2-closed seqr-assoc*)

lemma *dcond-H1-H2-closed* [*closure*]:
assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $(P \triangleleft b \triangleright_D Q) \text{ is } \mathbf{N}$
by (*metis assms ndesign-H1-H3 ndesign-dcond ndesign-form*)

lemma *inf-H1-H2-closed* [*closure*]:
assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $(P \sqcap Q) \text{ is } \mathbf{N}$
by (*metis assms ndesign-H1-H3 ndesign-choice ndesign-form*)

lemma *sup-H1-H2-closed* [*closure*]:
assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $(P \sqcup Q) \text{ is } \mathbf{N}$
by (*metis assms ndesign-H1-H3 ndesign-inf ndesign-form*)

lemma *ndes-segr-miracle*:

assumes P is \mathbf{N}

shows $P \;; \top_D = \lfloor pre_D P \rfloor_{<} \vdash_n false$

proof –

have $P \;; \top_D = (\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) \;; (true \vdash_n false)$

by (*simp add: assms ndesign-form ndesign-miracle*)

also have $\dots = \lfloor pre_D P \rfloor_{<} \vdash_n false$

by (*simp add: ndesign-composition-wp wp alpha*)

finally show *?thesis* .

qed

lemma *ndes-segr-abort*:

assumes P is \mathbf{N}

shows $P \;; \perp_D = (\lfloor pre_D P \rfloor_{<} \wedge post_D P \text{ wp } false) \vdash_n false$

proof –

have $P \;; \perp_D = (\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) \;; (false \vdash_n false)$

by (*simp add: assms bot-d-true ndesign-false-pre ndesign-form*)

also have $\dots = (\lfloor pre_D P \rfloor_{<} \wedge post_D P \text{ wp } false) \vdash_n false$

by (*simp add: ndesign-composition-wp alpha*)

finally show *?thesis* .

qed

lemma *USUP-ind-H1-H3-closed [closure]*:

$\llbracket \bigwedge i. P i \text{ is } \mathbf{N} \rrbracket \implies (\bigsqcup i. P i) \text{ is } \mathbf{N}$

by (*rule H1-H3-intro, simp-all add: H1-H3-impl-H2 USUP-ind-H1-H2-closed preD-USUP-ind unrest*)

lemma *msubst-pre-H1-H3 [closure]*: $(\bigwedge x. P x \text{ is } \mathbf{N}) \implies P x \llbracket x \mapsto [v]_{<} \rrbracket \text{ is } \mathbf{N}$

by (*metis H1-H3-right-unit H3-def Healthy-if Healthy-intro msubst-H1 msubst-pre-H3*)

2.6 H4: Feasibility

definition $H4 \:: (' \alpha, ' \beta) \text{ rel-des} \Rightarrow (' \alpha, ' \beta) \text{ rel-des}$ **where**

[upred-defs]: $H4(P) = ((P;;true) \Rightarrow P)$

theorem *H4-idem*:

$H4(H4(P)) = H4(P)$

by (*rel-auto*)

lemma *is-H4-alt-def*:

$P \text{ is } H4 \iff (P;;true) = true$

by (*rel-blast*)

end

2.7 UTP theory of Designs

theory *utp-des-theory*

imports *utp-des-healths*

begin

2.8 UTP theories

typedecl *DES*

typedecl *NDES*

abbreviation $DES \equiv UTHY(DES, ' \alpha \text{ des})$

abbreviation $NDES \equiv UTHY(NDES, ' \alpha \text{ des})$

overloading

$des\text{-}hcond == utp\text{-}hcond :: (DES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow (' \alpha \text{ des} \times ' \alpha \text{ des}) \text{ health}$
 $des\text{-}unit == utp\text{-}unit :: (DES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow ' \alpha \text{ hrel-des (unchecked)}$

$ndes\text{-}hcond == utp\text{-}hcond :: (NDES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow (' \alpha \text{ des} \times ' \alpha \text{ des}) \text{ health}$
 $ndes\text{-}unit == utp\text{-}unit :: (NDES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow ' \alpha \text{ hrel-des (unchecked)}$

begin

definition $des\text{-}hcond :: (DES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow (' \alpha \text{ des} \times ' \alpha \text{ des}) \text{ health}$ **where**
 $[upred\text{-}defs]: des\text{-}hcond \text{ } t = H1\text{-}H2$

definition $des\text{-}unit :: (DES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow ' \alpha \text{ hrel-des}$ **where**
 $[upred\text{-}defs]: des\text{-}unit \text{ } t = II_D$

definition $ndes\text{-}hcond :: (NDES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow (' \alpha \text{ des} \times ' \alpha \text{ des}) \text{ health}$ **where**
 $[upred\text{-}defs]: ndes\text{-}hcond \text{ } t = H1\text{-}H3$

definition $ndes\text{-}unit :: (NDES, ' \alpha \text{ des}) \text{ uthy} \Rightarrow ' \alpha \text{ hrel-des}$ **where**
 $[upred\text{-}defs]: ndes\text{-}unit \text{ } t = II_D$

end

interpretation $des\text{-}utp\text{-}theory: utp\text{-}theory \text{ } DES$

by ($simp \text{ add: } H1\text{-}H2\text{-}commute \text{ } H1\text{-}idem \text{ } H2\text{-}idem \text{ } des\text{-}hcond\text{-}def \text{ } utp\text{-}theory\text{-}def$)

interpretation $ndes\text{-}utp\text{-}theory: utp\text{-}theory \text{ } NDES$

by ($simp \text{ add: } H1\text{-}H3\text{-}commute \text{ } H1\text{-}idem \text{ } H3\text{-}idem \text{ } ndes\text{-}hcond\text{-}def \text{ } utp\text{-}theory.intro$)

interpretation $des\text{-}left\text{-}unital: utp\text{-}theory\text{-}left\text{-}unital \text{ } DES$

apply ($unfold\text{-}locales$)
apply ($simp\text{-}all \text{ add: } des\text{-}hcond\text{-}def \text{ } des\text{-}unit\text{-}def$)
using $seq\text{-}r\text{-}H1\text{-}H2\text{-}closed$ **apply** $blast$
apply ($simp \text{ add: } rdesign\text{-}is\text{-}H1\text{-}H2 \text{ } skip\text{-}d\text{-}def$)
apply ($metis \text{ } H1\text{-}idem \text{ } H1\text{-}left\text{-}unit \text{ } Healthy\text{-}def'$)

done

interpretation $ndes\text{-}unital: utp\text{-}theory\text{-}unital \text{ } NDES$

apply ($unfold\text{-}locales, simp\text{-}all \text{ add: } ndes\text{-}hcond\text{-}def \text{ } ndes\text{-}unit\text{-}def$)
using $seq\text{-}r\text{-}H1\text{-}H3\text{-}closed$ **apply** $blast$
apply ($metis \text{ } H1\text{-}rdesign \text{ } H3\text{-}def \text{ } Healthy\text{-}def' \text{ } design\text{-}skip\text{-}idem \text{ } skip\text{-}d\text{-}def$)
apply ($metis \text{ } H1\text{-}idem \text{ } H1\text{-}left\text{-}unit \text{ } Healthy\text{-}def'$)
apply ($metis \text{ } H1\text{-}H3\text{-}commute \text{ } H3\text{-}def \text{ } H3\text{-}idem \text{ } Healthy\text{-}def'$)

done

interpretation $design\text{-}theory\text{-}continuous: utp\text{-}theory\text{-}continuous \text{ } DES$

rewrites $\bigwedge P. P \in carrier \text{ } (uthy\text{-}order \text{ } DES) \longleftrightarrow P \text{ is } \mathbf{H}$
and $carrier \text{ } (uthy\text{-}order \text{ } DES) \rightarrow carrier \text{ } (uthy\text{-}order \text{ } DES) \equiv \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$
and $\llbracket \mathcal{H}_{DES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{DES} \rrbracket_H \equiv \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$
and $le \text{ } (uthy\text{-}order \text{ } DES) = (\sqsubseteq)$
and $eq \text{ } (uthy\text{-}order \text{ } DES) = (=)$
by ($unfold\text{-}locales, simp\text{-}all \text{ add: } des\text{-}hcond\text{-}def \text{ } H1\text{-}H2\text{-}Continuous \text{ } utp\text{-}order\text{-}def$)

interpretation $normal\text{-}design\text{-}theory\text{-}continuous: utp\text{-}theory\text{-}continuous \text{ } NDES$

rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } NDES) \longleftrightarrow P \text{ is } \mathbf{N}$
and $\text{carrier } (\text{uthy-order } NDES) \rightarrow \text{carrier } (\text{uthy-order } NDES) \equiv [\mathbf{N}]_H \rightarrow [\mathbf{N}]_H$
and $[\mathcal{H}_{NDES}]_H \rightarrow [\mathcal{H}_{NDES}]_H \equiv [\mathbf{N}]_H \rightarrow [\mathbf{N}]_H$
and $\text{le } (\text{uthy-order } NDES) = (\sqsubseteq)$
and $A \subseteq \text{carrier } (\text{uthy-order } NDES) \longleftrightarrow A \subseteq [\mathbf{N}]_H$
and $\text{eq } (\text{uthy-order } NDES) = (=)$
by (*unfold-locales, simp-all add: ndes-hcond-def H1-H3-Continuous utp-order-def*)

lemma *design-lat-top*: $\top_{DES} = \mathbf{H}(\text{false})$
by (*simp add: design-theory-continuous.healthy-top, simp add: des-hcond-def*)

lemma *design-lat-bottom*: $\perp_{DES} = \mathbf{H}(\text{true})$
by (*simp add: design-theory-continuous.healthy-bottom, simp add: des-hcond-def*)

lemma *ndesign-lat-top*: $\top_{NDES} = \top_D$
by (*metis (mono-tags, lifting) H1-below-top antisym-conv des-top-is-H1-H3 ndes-hcond-def normal-design-theory-continuous.healthy-top normal-design-theory-continuous.top-higher*)

lemma *ndesign-lat-bottom*: $\perp_{NDES} = \mathbf{N}(\text{true})$
by (*metis ndes-hcond-def normal-design-theory-continuous.healthy-bottom*)

interpretation *ndes-kleene*: *utp-theory-kleene* $UTHY(NDES, ' \alpha \text{ des})$
rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } NDES) \longleftrightarrow P \text{ is } \mathbf{N}$
and $P \text{ is } \mathcal{H}_{NDES} \longleftrightarrow P \text{ is } \mathbf{N}$
and $\text{carrier } (\text{uthy-order } NDES) \rightarrow \text{carrier } (\text{uthy-order } NDES) \equiv [\mathbf{N}]_H \rightarrow [\mathbf{N}]_H$
and $[\mathcal{H}_{NDES}]_H \rightarrow [\mathcal{H}_{NDES}]_H \equiv [\mathbf{N}]_H \rightarrow [\mathbf{N}]_H$
and $\top_{NDES} = \top_D$
and $\mathcal{II}_{NDES} = \mathcal{II}_D$
and $\text{le } (\text{uthy-order } NDES) = (\sqsubseteq)$
by (*unfold-locales, simp-all add: ndesign-lat-top, simp-all add: H1-H3-top-left ndes-hcond-def ndes-unit-def closure*)

abbreviation *ndes-star* :: $- \Rightarrow -$ (*D [999] 999) **where**
 $P^{*D} \equiv P \star_{NDES}$

2.9 Galois Connection

Example Galois connection between designs and relations. Based on Jim's example in COM-PASS deliverable D23.5.

definition [*upred-defs*]: $\text{Des}(R) = \mathbf{H}([R]_D \wedge \$ok')$

definition [*upred-defs*]: $\text{Rel}(D) = \lfloor D[\text{true}, \text{true}/\$ok, \$ok'] \rfloor_D$

lemma *Des-design*: $\text{Des}(R) = \text{true} \vdash_r R$
by (*rel-auto*)

lemma *Rel-design*: $\text{Rel}(P \vdash_r Q) = (P \Rightarrow Q)$
by (*rel-auto*)

interpretation *Des-Rel-coretract*:
 $\text{coretract } DES \leftarrow \langle \text{Des}, \text{Rel} \rangle \rightarrow REL$
rewrites

$\bigwedge x. x \in \text{carrier } \mathcal{X}_{DES} \leftarrow \langle \text{Des}, \text{Rel} \rangle \rightarrow REL = (x \text{ is } \mathbf{H})$ **and**
 $\bigwedge x. x \in \text{carrier } \mathcal{Y}_{DES} \leftarrow \langle \text{Des}, \text{Rel} \rangle \rightarrow REL = \text{True}$ **and**
 $\pi_{*} DES \leftarrow \langle \text{Des}, \text{Rel} \rangle \rightarrow REL = \text{Des}$ **and**

```

     $\pi^*_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = Rel$  and
     $le \mathcal{X}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = (\sqsubseteq)$  and
     $le \mathcal{Y}_{DES} \leftarrow \langle Des, Rel \rangle \rightarrow REL = (\sqsubseteq)$ 
proof (unfold-locales, simp-all add: rel-hcond-def des-hcond-def)
  show  $\bigwedge x. x \text{ is } id$ 
    by (simp add: Healthy-def)
next
  show  $Rel \in \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket id \rrbracket_H$ 
    by (auto simp add: Rel-def rel-hcond-def Healthy-def)
next
  show  $Des \in \llbracket id \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$ 
    by (auto simp add: Des-def des-hcond-def Healthy-def H1-H2-commute H1-idem H2-idem)
next
  fix  $R :: 'a \text{ hrel}$ 
  show  $R \sqsubseteq Rel (Des R)$ 
    by (simp add: Des-design Rel-design)
next
  fix  $R :: 'a \text{ hrel}$  and  $D :: 'a \text{ hrel-des}$ 
  assume  $a: D \text{ is } \mathbf{H}$ 
  then obtain  $D_1 D_2$  where  $D: D = D_1 \vdash_r D_2$ 
    by (metis H1-H2-commute H1-H2-is-rdesign H1-idem Healthy-def')
  show  $(Rel D \sqsubseteq R) = (D \sqsubseteq Des R)$ 
proof –
  have  $(D \sqsubseteq Des R) = (D_1 \vdash_r D_2 \sqsubseteq true \vdash_r R)$ 
    by (simp add: D Des-design)
  also have  $\dots = 'D_1 \wedge R \Rightarrow D_2'$ 
    by (simp add: rdesign-refinement)
  also have  $\dots = ((D_1 \Rightarrow D_2) \sqsubseteq R)$ 
    by (rel-auto)
  also have  $\dots = (Rel D \sqsubseteq R)$ 
    by (simp add: D Rel-design)
  finally show ?thesis ..
qed
qed

```

From this interpretation we gain many Galois theorems. Some require simplification to remove superfluous assumptions.

```

thm Des-Rel-coretract.deflation[simplified]
thm Des-Rel-coretract.inflation
thm Des-Rel-coretract.upper-comp[simplified]
thm Des-Rel-coretract.lower-comp

```

2.10 Fixed Points

abbreviation $design_lfp :: ('\alpha \text{ hrel-des} \Rightarrow '\alpha \text{ hrel-des}) \Rightarrow '\alpha \text{ hrel-des } (\mu_D)$ **where**
 $\mu_D F \equiv \mu_{DES} F$

abbreviation $design_gfp :: ('\alpha \text{ hrel-des} \Rightarrow '\alpha \text{ hrel-des}) \Rightarrow '\alpha \text{ hrel-des } (\nu_D)$ **where**
 $\nu_D F \equiv \nu_{DES} F$

syntax

```

-dmu :: p  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\mu_D \cdot \cdot [0, 10] 10$ )
-dnu :: p  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\nu_D \cdot \cdot [0, 10] 10$ )

```

translations

$\mu_D X \cdot P == \mu_{CONST\ DES} (\lambda X. P)$
 $\nu_D X \cdot P == \nu_{CONST\ DES} (\lambda X. P)$

thm *design-theory-continuous.GFP-unfold*

thm *design-theory-continuous.LFP-unfold*

Specialise *mu-refine-intro* to designs.

lemma *design-mu-refine-intro*:

assumes $\$ok' \# C \$ok' \# S \ (C \vdash S) \sqsubseteq F(C \vdash S) \ 'C \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)'$

shows $(C \vdash S) \sqsubseteq \mu_D F$

proof –

from *assms* **have** $(C \vdash S) \sqsubseteq \nu_D F$

thm *design-theory-continuous.weak.GFP-upperbound*

by (*simp add: design-is-H1-H2 design-theory-continuous.weak.GFP-upperbound*)

with *assms* **show** *?thesis*

by (*rel-auto, metis (no-types, lifting)*)

qed

lemma *rdesign-mu-refine-intro*:

assumes $(C \vdash_r S) \sqsubseteq F(C \vdash_r S) \ '[C]_D \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)'$

shows $(C \vdash_r S) \sqsubseteq \mu_D F$

using *assms* **by** (*simp add: rdesign-def design-mu-refine-intro unrest*)

lemma *H1-H2-mu-refine-intro*:

assumes *P is H* $P \sqsubseteq F(P) \ '[pre_D(P)]_D \Rightarrow (\mu_D F \Leftrightarrow \nu_D F)'$

shows $P \sqsubseteq \mu_D F$

by (*metis H1-H2-eq-rdesign Healthy-if assms rdesign-mu-refine-intro*)

Foundational theorem for recursion introduction using a well-founded relation. Contributed by Dr. Yakoub Nemouchi.

theorem *rdesign-mu-wf-refine-intro*:

assumes *WF*: *wf R*

and *M*: *Monotonic F*

and *H*: $F \in [\mathbf{H}]_H \rightarrow [\mathbf{H}]_H$

and *induct-step*:

$\bigwedge st. (P \wedge [e]_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq F ((P \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q)$

shows $(P \vdash_r Q) \sqsubseteq \mu_D F$

proof –

{

fix *st*

have $(P \wedge [e]_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq \mu_D F$

using *WF* **proof** (*induction rule: wf-induct-rule*)

case (*less st*)

hence *0*: $(P \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q \sqsubseteq \mu_D F$

by *rel-blast*

from *M H design-theory-continuous.LFP-lemma3 mono-Monotone-utp-order*

have *1*: $\mu_D F \sqsubseteq F (\mu_D F)$

by *blast*

from *0 1* **have** *2*: $(P \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q \sqsubseteq F (\mu_D F)$

by *simp*

have *3*: $F ((P \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg) \vdash_r Q) \sqsubseteq F (\mu_D F)$

by (*simp add: 0 M monoD*)

have *4*: $(P \wedge [e]_{<} =_u \ll st \gg) \vdash_r Q \sqsubseteq \dots$

by (*rule induct-step*)

show *?case*

```

using order-trans[OF 3 4] H M design-theory-continuous.LFP-lemma2 dual-order.trans mono-Monotone-utp-order

  by blast
qed
}
thus ?thesis
  by (pred-simp)
qed

theorem ndesign-mu-wf-refine-intro':
  assumes WF: wf R
  and M: Monotonic F
  and H:  $F \in \llbracket \mathbf{H} \rrbracket_H \rightarrow \llbracket \mathbf{H} \rrbracket_H$ 
  and induct-step:
     $\bigwedge st. ((p \wedge e =_u \ll st \gg) \vdash_n Q) \sqsubseteq F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q)$ 
  shows  $(p \vdash_n Q) \sqsubseteq \mu_D F$ 
  using assms unfolding ndesign-def
  by (rule-tac rdesign-mu-wf-refine-intro[of R F [p]_< e], simp-all add: alpha)

theorem ndesign-mu-wf-refine-intro:
  assumes WF: wf R
  and M: Monotonic F
  and H:  $F \in \llbracket \mathbf{N} \rrbracket_H \rightarrow \llbracket \mathbf{N} \rrbracket_H$ 
  and induct-step:
     $\bigwedge st. ((p \wedge e =_u \ll st \gg) \vdash_n Q) \sqsubseteq F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q)$ 
  shows  $(p \vdash_n Q) \sqsubseteq \mu_{NDES} F$ 
proof -
  {
  fix st
  have  $(p \wedge e =_u \ll st \gg) \vdash_n Q \sqsubseteq \mu_{NDES} F$ 
  using WF proof (induction rule: wf-induct-rule)
    case (less st)
    hence 0:  $(p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q \sqsubseteq \mu_{NDES} F$ 
    by rel-blast
    from M H design-theory-continuous.LFP-lemma3 mono-Monotone-utp-order
    have 1:  $\mu_{NDES} F \sqsubseteq F (\mu_{NDES} F)$ 
    by (simp add: mono-Monotone-utp-order normal-design-theory-continuous.LFP-lemma3)
    from 0 1 have 2:  $(p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q \sqsubseteq F (\mu_{NDES} F)$ 
    by simp
    have 3:  $F ((p \wedge (e, \ll st \gg)_u \in_u \ll R \gg) \vdash_n Q) \sqsubseteq F (\mu_{NDES} F)$ 
    by (simp add: 0 M monoD)
    have 4:  $(p \wedge e =_u \ll st \gg) \vdash_n Q \sqsubseteq \dots$ 
    by (rule induct-step)
    show ?case
    using order-trans[OF 3 4] H M normal-design-theory-continuous.LFP-lemma2 dual-order.trans
    mono-Monotone-utp-order
    by blast
  }
qed
thus ?thesis
  by (pred-simp)
qed

end

```


3 Design Proof Tactics

```
theory utp-des-tactics
  imports utp-des-theory
begin
```

The tactics split apart a healthy normal design predicate into its pre-postcondition form, using elimination rules, and then attempt to prove refinement conjectures.

```
named-theorems ND-elim
```

```
lemma ndes-elim:  $\llbracket P \text{ is } \mathbf{N}; Q(\lfloor \text{pre}_D(P) \rfloor_{<} \vdash_n \text{post}_D(P)) \rrbracket \implies Q(P)$ 
  by (simp add: ndesign-form)
```

```
lemma ndes-ind-elim:  $\llbracket \bigwedge i. P \text{ is } \mathbf{N}; Q(\lambda i. \lfloor \text{pre}_D(P \ i) \rfloor_{<} \vdash_n \text{post}_D(P \ i)) \rrbracket \implies Q(P)$ 
  by (simp add: ndesign-form)
```

```
lemma ndes-split [ND-elim]:  $\llbracket P \text{ is } \mathbf{N}; \bigwedge \text{pre post}. Q(\text{pre} \vdash_n \text{post}) \rrbracket \implies Q(P)$ 
  by (metis H1-H2-eq-rdesign H1-H3-impl-H2 H3-unrest-out-alpha Healthy-def drop-pre-inv ndesign-def)
```

Use given closure laws (cls) to expand normal design predicates

```
method ndes-expand uses cls = (insert cls, (erule ND-elim)+)
```

Expand and simplify normal designs

```
method ndes-simp uses cls =
  ((ndes-expand cls: cls)?, (simp add: ndes-simp closure alpha usubst unrest wp prod.case-eq-if))
```

Attempt to discharge a refinement between two normal designs

```
method ndes-refine uses cls =
  (ndes-simp cls: cls; rule-tac ndesign-refine-intro; (insert cls; rel-simp; auto?))
```

Attempt to discharge an equality between two normal designs

```
method ndes-eq uses cls =
  (ndes-simp cls: cls; rule-tac antisym; rule-tac ndesign-refine-intro; (insert cls; rel-simp; auto?))
```

```
end
```

4 Imperative Programming in Designs

```
theory utp-des-prog
  imports utp-des-tactics
begin
```

4.1 Assignment

```
definition assigns-d :: ' $\alpha$  usubst  $\Rightarrow$  ' $\alpha$  hrel-des ( $\langle \cdot \rangle_D$ ) where
  [upred-defs]: assigns-d  $\sigma = (\text{true} \vdash_r \text{assigns-r } \sigma)$ 
```

```
syntax
```

```
-assignmenttd :: svids  $\Rightarrow$  uexprs  $\Rightarrow$  logic (infixr :=D 62)
```

```
translations
```

```
-assignmenttd xs vs => CONST assigns-d (-mk-usubst (CONST id) xs vs)
-assignmenttd x v <= CONST assigns-d (CONST subst-upd (CONST id) x v)
-assignmenttd x v <= -assignmenttd (-spvar x) v
```

$x, y :=_D u, v \leq \text{CONST assigns-d } (\text{CONST subst-upd } (\text{CONST subst-upd } (\text{CONST id}) (\text{CONST svar } x) u) (\text{CONST svar } y) v)$

lemma *assigns-d-is-H1-H2* [closure]: $\langle \sigma \rangle_D$ is **H**
 by (simp add: assigns-d-def rdesign-is-H1-H2)

lemma *assigns-d-H1-H3* [closure]: $\langle \sigma \rangle_D$ is **N**
 by (metis H1-rdesign H3-ndesign Healthy-def' aext-true assigns-d-def ndesign-def)

Designs are closed under substitutions on state variables only (via lifting)

lemma *state-subst-H1-H2-closed* [closure]:
 P is **H** $\implies \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$ is **H**
 by (metis H1-H2-eq-rdesign Healthy-if rdesign-is-H1-H2 state-subst-design)

lemma *assigns-d-ndes-def* [ndes-simp]:
 $\langle \sigma \rangle_D = (\text{true} \vdash_n \langle \sigma \rangle_a)$
 by (rel-auto)

lemma *assigns-d-id* [simp]: $\langle \text{id} \rangle_D = \Pi_D$
 by (rel-auto)

lemma *assign-d-left-comp*:
 $(\langle f \rangle_D ;; (P \vdash_r Q)) = (\lceil f \rceil_s \uparrow P \vdash_r \lceil f \rceil_s \uparrow Q)$
 by (simp add: assigns-d-def rdesign-composition assigns-r-comp subst-not)

lemma *assign-d-right-comp*:
 $((P \vdash_r Q) ;; \langle f \rangle_D) = ((\neg ((\neg P) ;; \text{true})) \vdash_r (Q ;; \langle f \rangle_a))$
 by (simp add: assigns-d-def rdesign-composition)

lemma *assigns-d-comp*:
 $(\langle f \rangle_D ;; \langle g \rangle_D) = \langle g \circ f \rangle_D$
 by (simp add: assigns-d-def rdesign-composition assigns-comp)

lemma *assigns-d-comp-ext*:
 fixes $P :: 'a \text{ hrel-des}$
 assumes P is **H**
 shows $(\langle \sigma \rangle_D ;; P) = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$

proof –

have $\langle \sigma \rangle_D ;; P = \langle \sigma \rangle_D ;; (\text{pre}_D(P) \vdash_r \text{post}_D(P))$
 by (metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms)
 also have $\dots = \lceil \sigma \rceil_s \uparrow \text{pre}_D(P) \vdash_r \lceil \sigma \rceil_s \uparrow \text{post}_D(P)$
 by (simp add: assign-d-left-comp)
 also have $\dots = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow (\text{pre}_D(P) \vdash_r \text{post}_D(P))$
 by (rel-auto)
 also have $\dots = \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$
 by (metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms)
 finally show ?thesis .

qed

Normal designs are closed under substitutions on state variables only

lemma *state-subst-H1-H3-closed* [closure]:
 P is **N** $\implies \lceil \sigma \oplus_s \Sigma_D \rceil_s \uparrow P$ is **N**
 by (metis H1-H2-eq-rdesign H1-H3-impl-H2 Healthy-if assign-d-left-comp assigns-d-H1-H3 seq-r-H1-H3-closed state-subst-design)

lemma $H4$ -assigns-d: $\langle \sigma \rangle_D$ is $H4$
proof –
 have $(\langle \sigma \rangle_D ;; (false \vdash_r true_h)) = (false \vdash_r true)$
 by (simp add: assigns-d-def rdesign-composition assigns-r-feasible)
 moreover have ... = true
 by (rel-auto)
 ultimately show ?thesis
 using is- $H4$ -alt-def by auto
qed

4.2 Guarded Commands

definition $GrdCommD :: 'a \text{ upred} \Rightarrow ('a, 'b) \text{ rel-des} \Rightarrow ('a, 'b) \text{ rel-des}$ **where**
 $[upred-defs]: GrdCommD \ b \ P = P \triangleleft b \triangleright_D \top_D$

syntax $-GrdCommD :: uexp \Rightarrow logic \Rightarrow logic \ (- \rightarrow_D - [60, 61] \ 61)$
translations $-GrdCommD \ b \ P == CONST \ GrdCommD \ b \ P$

lemma $GrdCommD$ -ndes-simp $[ndes-simp]$:
 $b \rightarrow_D (p_1 \vdash_n P_2) = ((b \Rightarrow p_1) \vdash_n (\lceil b \rceil_{<} \wedge P_2))$
 by (rel-auto)

lemma $GrdCommD$ -H1-H3-closed $[closure]$: $P \text{ is } \mathbf{N} \Longrightarrow b \rightarrow_D P \text{ is } \mathbf{N}$
 by (simp add: $GrdCommD$ -def closure)

lemma $GrdCommD$ -true $[simp]$: $true \rightarrow_D P = P$
 by (rel-auto)

lemma $GrdCommD$ -false $[simp]$: $false \rightarrow_D P = \top_D$
 by (rel-auto)

lemma $GrdCommD$ -abort $[simp]$: $b \rightarrow_D true = ((\neg b) \vdash_n false)$
 by (rel-auto)

4.3 Frames and Extensions

definition $des-frame :: ('a \Longrightarrow 'b) \Rightarrow 'b \text{ hrel-des} \Rightarrow 'b \text{ hrel-des}$ **where**
 $[upred-defs]: des-frame \ x \ P = frame \ (ok \ +_L \ x \ ;_L \ \Sigma_D) \ P$

definition $des-frame-ext :: ('a \Longrightarrow 'b) \Rightarrow 'a \text{ hrel-des} \Rightarrow 'b \text{ hrel-des}$ **where**
 $[upred-defs]: des-frame-ext \ a \ P = des-frame \ a \ (rel-aext \ P \ (lmap_D \ a))$

syntax
 $-des-frame \quad :: salpha \Rightarrow logic \Rightarrow logic \ (-:[-]_D [99,0] \ 100)$
 $-des-frame-ext \quad :: salpha \Rightarrow logic \Rightarrow logic \ (-:[-]_D^+ [99,0] \ 100)$

translations
 $-des-frame \ x \ P \Rightarrow CONST \ des-frame \ x \ P$
 $-des-frame \ (-salphaset \ (-salphamk \ x)) \ P \leq CONST \ des-frame \ x \ P$
 $-des-frame-ext \ x \ P \Rightarrow CONST \ des-frame-ext \ x \ P$
 $-des-frame-ext \ (-salphaset \ (-salphamk \ x)) \ P \leq CONST \ des-frame-ext \ x \ P$

4.4 Alternation

consts
 $ualtern \quad :: 'a \text{ set} \Rightarrow ('a \Rightarrow 'p) \Rightarrow ('a \Rightarrow 'r) \Rightarrow 'r \Rightarrow 'r$

ualtern-list :: ('a × 'r) list ⇒ 'r ⇒ 'r

definition *AlternateD* :: 'a set ⇒ ('a ⇒ 'α upred) ⇒ ('a ⇒ ('α, 'β) rel-des) ⇒ ('α, 'β) rel-des ⇒ ('α, 'β) rel-des **where**

[upred-defs, ndes-simp]:

AlternateD A g P Q = (⋀ i ∈ A • g(i) →_D P(i)) □ ((⋀ i ∈ A • ¬ g(i)) →_D Q)

This lemma shows that our generalised alternation is the same operator as Marcel Oliveira's definition of alternation when the else branch is abort.

lemma *AlternateD-abort-alternate*:

assumes ⋀ i. P(i) is **N**

shows

AlternateD A g P ⊥_D =

((⋀ i ∈ A • g(i)) ∧ (⋀ i ∈ A • g(i) ⇒ [pre_D(P i)]_<)) ⊢_n (⋀ i ∈ A • [g(i)]_< ∧ post_D(P i))

proof (cases A = {})

case *False*

have *AlternateD* A g P ⊥_D =

(⋀ i ∈ A • g(i) →_D ([pre_D(P i)]_< ⊢_n post_D(P i))) □ ((⋀ i ∈ A • ¬ g(i)) →_D (false ⊢_n true))

by (simp add: *AlternateD-def* ndesign-form bot-d-ndes-def assms)

also have ... = ((⋀ i ∈ A • g(i)) ∧ (⋀ i ∈ A • g(i) ⇒ [pre_D(P i)]_<)) ⊢_n (⋀ i ∈ A • [g(i)]_< ∧ post_D(P i))

by (simp add: ndes-simp *False*, rel-auto)

finally show ?thesis **by** simp

next

case *True*

thus ?thesis

by (simp add: *AlternateD-def*, rel-auto)

qed

definition *AlternateD-list* :: ('α upred × ('α, 'β) rel-des) list ⇒ ('α, 'β) rel-des ⇒ ('α, 'β) rel-des **where**

[upred-defs, ndes-simp]:

AlternateD-list xs P =

AlternateD {0..*length* xs} (λ i. map fst xs ! i) (λ i. map snd xs ! i) P

adhoc-overloading

ualtern *AlternateD* **and**

ualtern-list *AlternateD-list*

nonterminal *gcomm* **and** *gcomms*

syntax

-altind-els :: pttrn ⇒ uexp ⇒ uexp ⇒ logic ⇒ logic ⇒ logic (if -∈- • - → - else - fi)

-altind :: pttrn ⇒ uexp ⇒ uexp ⇒ logic ⇒ logic (if -∈- • - → - fi)

-gcomm :: uexp ⇒ logic ⇒ gcomm (- → - [60, 60] 61)

-gcomm-nil :: gcomm ⇒ gcomms (-)

-gcomm-cons :: gcomm ⇒ gcomms ⇒ gcomms (- | - [60, 61] 61)

-gcomm-show :: logic ⇒ logic

-altgcomm-els :: gcomms ⇒ logic ⇒ logic (if / - /else - /fi)

-altgcomm :: gcomms ⇒ logic (if / - /fi)

translations

-altind-els x A g P Q => *CONST* *ualtern* A (λ x. g) (λ x. P) Q

-altind-els x A g P Q <= *CONST* *ualtern* A (λ x. g) (λ x'. P) Q

-altind x A g P => *CONST* *ualtern* A (λ x. g) (λ x. P) (*CONST* *Orderings.top*)

$\text{-altind } x \ A \ g \ P \leq \text{CONST } \text{ualtern } A \ (\lambda x. g) \ (\lambda x'. P) \ (\text{CONST } \text{Orderings.top})$
 $\text{-altgcomm } cs \Rightarrow \text{CONST } \text{ualtern-list } cs \ (\text{CONST } \text{Orderings.top})$
 $\text{-altgcomm } (-\text{gcomm-show } cs) \leq \text{CONST } \text{ualtern-list } cs \ (\text{CONST } \text{Orderings.top})$
 $\text{-altgcomm-els } cs \ P \Rightarrow \text{CONST } \text{ualtern-list } cs \ P$
 $\text{-altgcomm-els } (-\text{gcomm-show } cs) \ P \leq \text{CONST } \text{ualtern-list } cs \ P$

$\text{-gcomm } g \ P \Rightarrow (g, P)$
 $\text{-gcomm } g \ P \leq \text{-gcomm-show } (g, P)$
 $\text{-gcomm-cons } c \ cs \Rightarrow c \# cs$
 $\text{-gcomm-cons } (-\text{gcomm-show } c) \ (-\text{gcomm-show } (d \# cs)) \leq \text{-gcomm-show } (c \# d \# cs)$
 $\text{-gcomm-nil } c \Rightarrow [c]$
 $\text{-gcomm-nil } (-\text{gcomm-show } c) \leq \text{-gcomm-show } [c]$

lemma *AlternateD-H1-H3-closed* [closure]:

assumes $\bigwedge i. i \in A \Rightarrow P \ i \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi is } \mathbf{N}$

proof (cases $A = \{\}$)

case *True*

then show *?thesis*

by (simp add: *AlternateD-def closure false-upred-def assms*)

next

case *False*

then show *?thesis*

by (simp add: *AlternateD-def closure assms*)

qed

lemma *AltD-ndes-simp* [ndes-simp]:

$\text{if } i \in A \cdot g(i) \rightarrow (P_1(i) \vdash_n P_2(i)) \text{ else } Q_1 \vdash_n Q_2 \text{ fi}$
 $= ((\bigwedge i \in A \cdot g \ i \Rightarrow P_1 \ i) \wedge ((\bigwedge i \in A \cdot \neg g \ i) \Rightarrow Q_1)) \vdash_n$
 $((\bigvee i \in A \cdot [g \ i]_{<} \wedge P_2 \ i) \vee (\bigwedge i \in A \cdot \neg [g \ i]_{<} \wedge Q_2))$

proof (cases $A = \{\}$)

case *True*

then show *?thesis* **by** (simp add: *AlternateD-def*)

next

case *False*

then show *?thesis*

by (simp add: *ndes-simp, rel-auto*)

qed

declare *UINF-upto-expand-first* [ndes-simp]

declare *UINF-Suc-shift* [ndes-simp]

declare *USUP-upto-expand-first* [ndes-simp]

declare *USUP-Suc-shift* [ndes-simp]

declare *true-upred-def* [*THEN sym, ndes-simp*]

lemma *AlternateD-mono-refine*:

assumes $\bigwedge i. P \ i \sqsubseteq Q \ i \ R \sqsubseteq S$

shows $(\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } R \text{ fi}) \sqsubseteq (\text{if } i \in A \cdot g(i) \rightarrow Q(i) \text{ else } S \text{ fi})$

using *assms* **by** (*rel-auto, meson*)

lemma *Monotonic-AlternateD* [closure]:

$\llbracket \bigwedge i. \text{Monotonic } (F \ i); \text{Monotonic } G \rrbracket \Rightarrow \text{Monotonic } (\lambda X. \text{if } i \in A \cdot g(i) \rightarrow F \ i \ X \text{ else } G(X) \text{ fi})$

by (*rel-auto, meson*)

lemma *AlternateD-eq*:

assumes $A = B \wedge i. i \in A \implies g(i) = h(i) \wedge i. i \in A \implies P(i) = Q(i) \ R = S$
shows $\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } R \text{ fi} = \text{if } i \in B \cdot h(i) \rightarrow Q(i) \text{ else } S \text{ fi}$
by (*insert assms, rel-blast*)

lemma *AlternateD-empty:*

if $i \in \{\}$ **·** $g(i) \rightarrow P(i) \text{ else } Q \text{ fi} = Q$
by (*rel-auto*)

lemma *AlternateD-true-singleton:*

assumes $P \text{ is } \mathbf{N}$
shows $\text{if } \text{true} \rightarrow P \text{ fi} = P$
by (*ndes-eq cls: assms*)

lemma *AlternateD-no-ind:*

assumes $A \neq \{\} \ P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } i \in A \cdot b \rightarrow P \text{ else } Q \text{ fi} = \text{if } b \rightarrow P \text{ else } Q \text{ fi}$
by (*ndes-eq cls: assms*)

lemma *AlternateD-singleton:*

assumes $P \ k \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } i \in \{k\} \cdot b(i) \rightarrow P(i) \text{ else } Q \text{ fi} = \text{if } b(k) \rightarrow P(k) \text{ else } Q \text{ fi}$ (**is** $?lhs = ?rhs$)

proof –

have $?lhs = \text{if } i \in \{k\} \cdot b(k) \rightarrow P(k) \text{ else } Q \text{ fi}$
by (*auto intro: AlternateD-eq simp add: assms ndesign-form*)
also have $\dots = ?rhs$
by (*simp add: AlternateD-no-ind assms closure*)
finally show $?thesis$.

qed

lemma *AlternateD-commute:*

assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } g_1 \rightarrow P \mid g_2 \rightarrow Q \text{ fi} = \text{if } g_2 \rightarrow Q \mid g_1 \rightarrow P \text{ fi}$
by (*ndes-eq cls: assms*)

lemma *AlternateD-dcond:*

assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } g \rightarrow P \text{ else } Q \text{ fi} = P \triangleleft g \triangleright_D Q$
by (*ndes-eq cls: assms*)

lemma *AlternateD-cover:*

assumes $P \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } g \rightarrow P \text{ else } Q \text{ fi} = \text{if } g \rightarrow P \mid (\neg g) \rightarrow Q \text{ fi}$
by (*ndes-eq cls: assms*)

lemma *UINF-ndes-expand:*

assumes $\bigwedge i. i \in A \implies P(i) \text{ is } \mathbf{N}$
shows $(\bigcap i \in A \cdot \lfloor \text{pre}_D(P(i)) \rfloor < \vdash_n \text{post}_D(P(i))) = (\bigcap i \in A \cdot P(i))$
by (*rule UINF-cong, simp add: assms ndesign-form*)

lemma *USUP-ndes-expand:*

assumes $\bigwedge i. i \in A \implies P(i) \text{ is } \mathbf{N}$
shows $(\bigsqcup i \in A \cdot \lfloor \text{pre}_D(P(i)) \rfloor < \vdash_n \text{post}_D(P(i))) = (\bigsqcup i \in A \cdot P(i))$
by (*rule USUP-cong, simp add: assms ndesign-form*)

lemma *AlternateD-ndes-expand:*

assumes $\bigwedge i. i \in A \implies P(i) \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi} =$
 $\text{if } i \in A \cdot g(i) \rightarrow (\lfloor \text{pre}_D(P(i)) \rfloor_{<} \vdash_n \text{post}_D(P(i))) \text{ else } \lfloor \text{pre}_D(Q) \rfloor_{<} \vdash_n \text{post}_D(Q) \text{ fi}$
apply (*simp add: AlternateD-def*)
apply (*subst UINF-ndes-expand[THEN sym]*)
apply (*simp add: assms closure*)
apply (*ndes-simp cls: assms*)
apply (*rel-auto*)
done

lemma *AlternateD-ndes-expand'*:

assumes $\bigwedge i. i \in A \implies P(i) \text{ is } \mathbf{N}$
shows $\text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ fi} = \text{if } i \in A \cdot g(i) \rightarrow (\lfloor \text{pre}_D(P(i)) \rfloor_{<} \vdash_n \text{post}_D(P(i))) \text{ fi}$
apply (*simp add: AlternateD-def*)
apply (*subst UINF-ndes-expand[THEN sym]*)
apply (*simp add: assms closure*)
apply (*ndes-simp cls: assms*)
apply (*rel-auto*)
done

lemma *ndesign-ind-form*:

assumes $\bigwedge i. P(i) \text{ is } \mathbf{N}$
shows $(\lambda i. \lfloor \text{pre}_D(P(i)) \rfloor_{<} \vdash_n \text{post}_D(P(i))) = P$
by (*simp add: assms ndesign-form*)

lemma *AlternateD-insert*:

assumes $\bigwedge i. i \in (\text{insert } x \ A) \implies P(i) \text{ is } \mathbf{N} \ Q \text{ is } \mathbf{N}$
shows $\text{if } i \in (\text{insert } x \ A) \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi} =$
 $\text{if } g(x) \rightarrow P(x) \mid$
 $(\bigvee i \in A \cdot g(i)) \rightarrow \text{if } i \in A \cdot g(i) \rightarrow P(i) \text{ fi}$
 $\text{else } Q$
 $\text{fi} \text{ (is } ?lhs = ?rhs)$

proof –

have $?lhs = \text{if } i \in (\text{insert } x \ A) \cdot g(i) \rightarrow (\lfloor \text{pre}_D(P(i)) \rfloor_{<} \vdash_n \text{post}_D(P(i))) \text{ else } (\lfloor \text{pre}_D(Q) \rfloor_{<} \vdash_n \text{post}_D(Q)) \text{ fi}$

using *AlternateD-ndes-expand assms(1) assms(2)* **by** *blast*

also

have ... =

$\text{if } g(x) \rightarrow (\lfloor \text{pre}_D(P(x)) \rfloor_{<} \vdash_n \text{post}_D(P(x))) \mid$
 $(\bigvee i \in A \cdot g(i)) \rightarrow \text{if } i \in A \cdot g(i) \rightarrow \lfloor \text{pre}_D(P(i)) \rfloor_{<} \vdash_n \text{post}_D(P(i)) \text{ fi}$
 $\text{else } \lfloor \text{pre}_D(Q) \rfloor_{<} \vdash_n \text{post}_D(Q)$
 fi

by (*ndes-simp cls:assms, rel-auto*)

also have ... = *?rhs*

by (*simp add: AlternateD-ndes-expand' ndesign-form assms*)

finally show *?thesis* .

qed

4.5 Iteration

theorem *ndesign-iteration-wp* [*ndes-simp*]:

$(p \vdash_n Q) ;; (p \vdash_n Q) \wedge^n = ((\bigwedge i \in \{0..n\} \cdot (Q \wedge i) \text{ wp } p) \vdash_n Q \wedge \text{Suc } n)$

proof (*induct n*)

case 0

then show *?case* **by** (*rel-auto*)

next

```

case (Suc n) note hyp = this
have (p ⊢n Q) ;; (p ⊢n Q) ^ Suc n = (p ⊢n Q) ;; (p ⊢n Q) ;; (p ⊢n Q) ^ n
  by (simp add: upred-semiring.power-Suc)
also have ... = (p ⊢n Q) ;; (⋀ i ∈ {0..n}. Q ^ i wp p) ⊢n Q ^ Suc n
  by (simp add: hyp)
also have ... = (p ∧ Q wp (⋀ i ∈ {0..n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: upred-semiring.power-Suc ndesign-composition-wp segr-assoc)
also have ... = (p ∧ (⋀ i ∈ {0..n}. Q ^ Suc i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: upred-semiring.power-Suc wp)
also have ... = (p ∧ (⋀ i ∈ {0..n}. Q ^ Suc i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: USUP-as-Inf-image)
also have ... = (p ∧ (⋀ i ∈ {1..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (metis (no-types, lifting) One-nat-def image-Suc-atLeastAtMost image-cong image-image)
also have ... = (Q ^ 0 wp p ∧ (⋀ i ∈ {1..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: wp)
also have ... = ((⋀ i ∈ {0..Suc n}. Q ^ i wp p)) ⊢n (Q ;; Q) ;; Q ^ n
  by (simp add: atMost-Suc-eq-insert-0 atLeast0AtMost conj-upred-def image-Suc-atMost)
also have ... = (⋀ i ∈ {0..Suc n}. Q ^ i wp p) ⊢n Q ^ Suc (Suc n)
  by (simp add: upred-semiring.power-Suc USUP-as-Inf-image upred-semiring.mult-assoc)
finally show ?case .
qed

```

Overloadable Syntax

consts

```

uiterate      :: 'a set ⇒ ('a ⇒ 'p) ⇒ ('a ⇒ 'r) ⇒ 'r
uiterate-list :: ('a × 'r) list ⇒ 'r

```

syntax

```

-iterind      :: pttm ⇒ uexp ⇒ uexp ⇒ logic ⇒ logic (do -∈- · - → - od)
-itergcomm    :: gcomms ⇒ logic (do - od)

```

translations

```

-iterind x A g P => CONST uiterate A (λ x. g) (λ x. P)
-iterind x A g P <= CONST uiterate A (λ x. g) (λ x'. P)
-itergcomm cs => CONST uiterate-list cs
-itergcomm (-gcomm-show cs) <= CONST uiterate-list cs

```

definition *IterateD* :: 'a set ⇒ ('a ⇒ 'α upred) ⇒ ('a ⇒ 'α hrel-des) ⇒ 'α hrel-des **where**
[upred-defs, ndes-simp]:

IterateD A g P = (μ_{NDES} X · if i ∈ A · g(i) → P(i) ;; X else II_D fi)

definition *IterateD-list* :: ('α upred × 'α hrel-des) list ⇒ 'α hrel-des **where**
[upred-defs, ndes-simp]:

IterateD-list xs = *IterateD* {0..*length* xs} (λ i. fst (nth xs i)) (λ i. snd (nth xs i))

adhoc-overloading

```

uiterate IterateD and
uiterate-list IterateD-list

```

lemma *IterateD-H1-H3-closed* [closure]:

```

assumes ⋀ i. i ∈ A ⇒ P i is N
shows do i ∈ A · g(i) → P(i) od is N

```

proof (cases A = {}) **then show** ?thesis


```

    by (simp add: IterateD-def closure assms)
next
case False
then show ?thesis
    by (simp add: IterateD-def closure assms)
qed

lemma IterateD-empty:
  do i ∈ {} · g(i) → P(i) od = II_D
  by (simp add: IterateD-def AlternateD-empty normal-design-theory-continuous.LFP-const skip-d-is-H1-H3)

lemma IterateD-list-single-expand:
  do b → P od = (μNDES X · if b → P ;; X else II_D fi)
oops

lemma IterateD-singleton:
  assumes P is N
  shows do b → P od = do i ∈ {0} · b → P od
  apply (simp add: IterateD-list-def IterateD-def AlternateD-singleton assms)
  apply (subst AlternateD-singleton)
  apply (simp)
  apply (rel-auto)
oops

lemma IterateD-mono-refine:
  assumes
    ∧ i. P i is N ∧ i. Q i is N
    ∧ i. P i ⊆ Q i
  shows (do i ∈ A · g(i) → P(i) od) ⊆ (do i ∈ A · g(i) → Q(i) od)
  apply (simp add: IterateD-def normal-design-theory-continuous.utp-lfp-def)
  apply (subst normal-design-theory-continuous.utp-lfp-def)
  apply (simp-all add: closure assms)
  apply (subst normal-design-theory-continuous.utp-lfp-def)
  apply (simp-all add: closure assms)
  apply (simp add: ndes-hcond-def)
  apply (rule gfp-mono)
  apply (rule AlternateD-mono-refine)
  apply (simp-all add: closure seqr-mono assms)
done

lemma IterateD-single-refine:
  assumes
    P is N Q is N P ⊆ Q
  shows (do g → P od) ⊆ (do g → Q od)
oops

lemma IterateD-refine-intro:
  fixes V :: (nat, 'a) uexpr
  assumes vwb-lens w
  shows
    I ⊢n (w: [I ∧ ¬ (∨ i ∈ A · g(i))] >]) ⊆
    do i ∈ A · g(i) → (I ∧ g(i)) ⊢n (w: [I] > ∧ [V] > <u [V] <]) od
proof (cases A = {})
case True
with assms show ?thesis

```

```

    by (simp add: IterateD-empty, rel-auto)
next
case False
then show ?thesis
using assms
  apply (simp add: IterateD-def)
  apply (rule ndesign-mu-wf-refine-intro[where e=V and R={ (x, y). x < y }])
  apply (simp-all add: wf_closure)
  apply (simp add: ndes-simp unrest)
  apply (rule ndesign-refine-intro)
  apply (rel-auto)
  apply (rel-auto)
  apply (metis mwb-lens.put-put vwb-lens-mwb)
done
qed

lemma IterateD-single-refine-intro:
  fixes V :: (nat, 'a) uexpr
  assumes vwb-lens w
  shows
    I ⊢n (w: [I ∧ ¬ g]>) ⊆
    do g → ((I ∧ g) ⊢n (w: [I]> ∧ [V]> <u [V]<)) od
  apply (rule order-trans)
  defer
  apply (rule IterateD-refine-intro[of w {0} λ i. g I V, simplified, OF assms(1)])
oops

```

4.6 Let and Local Variables

definition $LetD :: ('a, 'α) uexpr ⇒ ('a ⇒ 'α hrel-des) ⇒ 'α hrel-des$ **where**
 $[upred-defs]: LetD\ v\ P = (P\ x) \llbracket x \rightarrow [v]_{D<} \rrbracket$

syntax
 $-LetD \quad :: [letbinds, 'a] ⇒ 'a \quad ((let_D\ (-) / in\ (-))\ [0, 10]\ 10)$

translations
 $-LetD\ (-binds\ b\ bs)\ e \rightleftharpoons -LetD\ b\ (-LetD\ bs\ e)$
 $let_D\ x = a\ in\ e \rightleftharpoons CONST\ LetD\ a\ (λx. e)$

lemma $LetD-ndes-simp\ [ndes-simp]:$
 $LetD\ v\ (λ\ x. p(x) ⊢_n Q(x)) = (p(x) \llbracket x \rightarrow v \rrbracket) ⊢_n (Q(x) \llbracket x \rightarrow [v]_{D<} \rrbracket)$
by (rel-auto)

lemma $LetD-H1-H3-closed\ [closure]:$
 $\llbracket \bigwedge x. P(x)\ is\ \mathbf{N} \rrbracket \implies LetD\ v\ P\ is\ \mathbf{N}$
by (rel-auto)

end

4.7 Design Hoare Logic

theory *utp-des-hoare*
imports *utp-des-prog*
begin

definition $HoareD :: 's\ upred ⇒ 's\ hrel-des ⇒ 's\ upred ⇒ bool\ (\{-\}\{-\}_D)$ **where**

$[upred-defs, ndes-simp]: HoareD\ p\ S\ q = ((p \vdash_n [q]_>) \sqsubseteq S)$

lemma *assigns-hoare-d* $[hoare-safe]: 'p \Rightarrow \sigma \dagger q' \Longrightarrow \{p\}\langle\sigma\rangle_D\{q\}_D$
by *rel-auto*

lemma *skip-hoare-d*: $\{p\}II_D\{p\}_D$
by (*rel-auto*)

lemma *assigns-backward-hoare-d*:
 $\{\sigma \dagger p\}\langle\sigma\rangle_D\{p\}_D$
by *rel-auto*

lemma *seq-hoare-d*:
assumes $C\ is\ N\ D\ is\ N\ \{p\}C\{q\}_D\ \{q\}D\{r\}_D$
shows $\{p\}C\ ;\ ;\ D\{r\}_D$

proof –

obtain $c_1\ C_2$ **where** $C: C = c_1 \vdash_n C_2$
by (*metis assms(1) ndesign-form*)

obtain $d_1\ D_2$ **where** $D: D = d_1 \vdash_n D_2$
by (*metis assms(2) ndesign-form*)

from *assms(3-4)* **show** *?thesis*

apply (*simp add: C D*)

apply (*ndes-simp*)

apply (*simp add: ndesign-refinement*)

apply (*rel-blast*)

done

qed

end

5 Design Weakest Preconditions

theory *utp-des-wp*

imports *utp-des-prog utp-des-hoare*

begin

definition *wp-design* :: $('α, 'β)\ rel-des \Rightarrow 'β\ cond \Rightarrow 'α\ cond$ (**infix** *wp_D* 60) **where**
 $[upred-defs]: Q\ wp_D\ r = (\lfloor pre_D(Q) \rfloor ;\ ;\ true :: ('α, 'β)\ urel \rfloor_{<} \wedge (post_D(Q)\ wp\ r))$

If two normal designs have the same weakest precondition for any given postcondition, then the two designs are equivalent.

theorem *wpd-eq-intro*: $\llbracket \bigwedge r. (p_1 \vdash_n Q_1)\ wp_D\ r = (p_2 \vdash_n Q_2)\ wp_D\ r \rrbracket \Longrightarrow (p_1 \vdash_n Q_1) = (p_2 \vdash_n Q_2)$
apply (*rel-simp robust; metis curry-conv*)
done

theorem *wpd-H3-eq-intro*: $\llbracket P\ is\ H1-H3; Q\ is\ H1-H3; \bigwedge r. P\ wp_D\ r = Q\ wp_D\ r \rrbracket \Longrightarrow P = Q$
by (*metis H1-H3-commute H1-H3-is-normal-design H3-idem Healthy-def' wpd-eq-intro*)

lemma *wp-d-abort* $[wp]: true\ wp_D\ p = false$
by (*rel-auto*)

lemma *wp-assigns-d* $[wp]: \langle\sigma\rangle_D\ wp_D\ r = \sigma \dagger r$
by (*rel-auto*)

theorem *rdesign-wp* $[wp]:$

$([p]_{<} \vdash_r Q) \text{ wp}_D r = (p \wedge Q \text{ wp } r)$
by (*rel-auto*)

theorem *ndesign-wp [wp]*:
 $(p \vdash_n Q) \text{ wp}_D r = (p \wedge Q \text{ wp } r)$
by (*simp add: ndesign-def rdesign-wp*)

theorem *wpd-seq-r*:
fixes $Q1\ Q2 :: 'a\ hrel$
shows $(([p1]_{<} \vdash_r Q1) ;; ([p2]_{<} \vdash_r Q2)) \text{ wp}_D r = ([p1]_{<} \vdash_r Q1) \text{ wp}_D (([p2]_{<} \vdash_r Q2) \text{ wp}_D r)$
apply (*simp add: wp*)
apply (*subst rdesign-composition-wp*)
apply (*simp only: wp*)
apply (*rel-auto*)
done

theorem *wpnd-seq-r [wp]*:
fixes $Q1\ Q2 :: 'a\ hrel$
shows $((p1 \vdash_n Q1) ;; (p2 \vdash_n Q2)) \text{ wp}_D r = (p1 \vdash_n Q1) \text{ wp}_D ((p2 \vdash_n Q2) \text{ wp}_D r)$
by (*simp add: ndesign-def wpd-seq-r*)

theorem *wpd-seq-r-H1-H3 [wp]*:
fixes $P\ Q :: 'a\ hrel\text{-des}$
assumes $P\ \text{is}\ \mathbf{N}\ Q\ \text{is}\ \mathbf{N}$
shows $(P ;; Q) \text{ wp}_D r = P \text{ wp}_D (Q \text{ wp}_D r)$
by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def' assms(1) assms(2) wpnd-seq-r*)

theorem *wp-hoare-d-link*:
assumes $Q\ \text{is}\ \mathbf{N}$
shows $\{p\}Q\{r\}_D \longleftrightarrow (Q \text{ wp}_D r \sqsubseteq p)$
by (*ndes-simp cls: assms, rel-auto*)

end

6 Refinement Calculus

theory *utp-des-refcalc*
imports *utp-des-prog*
begin

definition *des-spec* :: $('a \Rightarrow 'a) \Rightarrow 'a\ \text{upred} \Rightarrow ('a \Rightarrow 'a\ \text{upred}) \Rightarrow 'a\ hrel\text{-des}$ **where**
 $[upred\text{-defs}, ndes\text{-simp}]: \text{des-spec } x\ p\ q = (\bigsqcup v \cdot ((p \wedge \&\mathbf{v} =_u \ll v \gg) \vdash_n x: [q(v)]_{>})))$

syntax
 $\text{-init-var} \quad :: \text{logic}$
 $\text{-des-spec} \quad :: \text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (-: [-, /]_D [99, 0, 0] 100)$
 $\text{-des-log-const} :: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (-: [-, /]_D [0, 10] 10)$

translations
 $\text{-des-spec } x\ p\ q \Rightarrow \text{CONST des-spec } x\ p\ (\lambda \text{-init-var. } q)$
 $\text{-des-spec } (-\text{salphaset } (-\text{salphamk } x))\ p\ q \Leftarrow \text{CONST des-spec } x\ p\ (\lambda \text{iv. } q)$
 $\text{-des-log-const } x\ P \Rightarrow \bigsqcup x \cdot P$

parse-translation \ll
 let

```

  fun init-var-tr [] = Syntax.free iv
    | init-var-tr - = raise Match;
in
[(@{syntax-const -init-var}, K init-var-tr)]
end
>>

```

abbreviation $choose_D x \equiv \{\&x\}:[true,true]_D$

lemma *des-spec-simple-def*:

```

  x:[pre,post]D = (pre ⊢n x:[post]>])
  by (rel-auto)

```

lemma *des-spec-abort*:

```

  x:[false,post]D = ⊥D
  by (rel-auto)

```

lemma *des-spec-skip*: $\emptyset:[true,true]_D = II_D$

```

  by (rel-auto)

```

lemma *des-spec-strengthen-post*:

```

  assumes 'post' ⇒ post'
  shows w:[pre, post]D ⊆ w:[pre, post']D
  using assms by (rel-auto)

```

lemma *des-spec-weaken-pre*:

```

  assumes 'pre ⇒ pre'
  shows w:[pre, post]D ⊆ w:[pre', post]D
  using assms by (rel-auto)

```

lemma *des-spec-refine-skip*:

```

  assumes vwb-lens w 'pre ⇒ post'
  shows w:[pre, post]D ⊆ IID
  using assms by (rel-auto)

```

lemma *rc-iter*:

```

  fixes V :: (nat, 'a) uexpr
  assumes vwb-lens w
  shows w:[ivr, ivr ∧ ¬ (⋁ i∈A • g(i))]D
    ⊆ (do i∈A • g(i) → ⋒ iv • w:[ivr ∧ g(i) ∧ «iv» =u &v, ivr ∧ (V <u V[«iv»/v])]D od) (is
?lhs ⊆ ?rhs)
  apply (rule order-trans)
  defer
  apply (simp add: des-spec-simple-def)
  apply (rule IterateD-refine-intro[of - - - V])
  apply (simp add: assms)
  apply (rule IterateD-mono-refine)
  apply (simp-all add: ndes-simp closure)
  apply (rel-auto)
done

```

end

7 Theory of Invariants

```
theory utp-des-invariants
  imports utp-des-theory
begin
```

The theory of invariants formalises operation and state invariants based on the theory of designs. For more information, please see the associated paper [1, Section 4].

7.1 Operation Invariants

definition $OIH(\psi)(D) = (D \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

declare $OIH\text{-}def$ [*upred-defs*]

lemma $OIH\text{-}design$:

assumes D is $H1\text{-}H2$

shows $OIH(\psi)(D) = ((\neg D^f) \vdash (D^t \wedge \psi))$

proof –

have $OIH(\psi)(D) = (((\neg D^f) \vdash D^t) \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

by (*metis H1-H2-commute H1-H2-is-design H1-idem Healthy-def' OIH-def assms*)

also have $\dots = ((\$ok \wedge \neg D^f \Rightarrow \$ok' \wedge D^t) \wedge (\$ok \wedge \neg D^f \Rightarrow \psi))$

by (*simp add: design-def*)

also have $\dots = ((\neg D^f) \vdash (D^t \wedge \psi))$

by (*pred-auto*)

finally show *?thesis* .

qed

lemma $OIH\text{-}idem$:

assumes D is $H1\text{-}H2$ $\$ok' \nVdash \psi$

shows $OIH(\psi)(OIH(\psi)(D)) = OIH(\psi)(D)$

using *assms*

by (*simp add: OIH-design design-is-H1-H2 unrest (simp add: design-def usubst, rel-auto)*)

lemma $OIH\text{-}of\text{-}design$:

$\$ok' \nVdash P \Longrightarrow OIH(\psi)(P \vdash Q) = (P \vdash (Q \wedge \psi))$

by (*simp add: OIH-def design-def usubst, rel-auto*)

7.2 State Invariants

definition $ISH(\psi)(D) = (D \vee (\$ok \wedge \neg D^f \wedge [\psi]_{<} \Rightarrow \$ok' \wedge D^t))$

declare $ISH\text{-}def$ [*upred-defs*]

lemma $ISH\text{-}design$: $ISH(\psi)(D) = (\neg D^f \wedge [\psi]_{<}) \vdash D^t$

by (*rel-auto, metis+*)

lemma $ISH\text{-}idem$: $ISH(\psi)(ISH(\psi)(D)) = ISH(\psi)(D)$

by (*simp add: ISH-design usubst design-def, pred-auto*)

lemma $ISH\text{-}of\text{-}design$:

$\llbracket \$ok' \nVdash P; \$ok' \nVdash Q \rrbracket \Longrightarrow ISH(\psi)(P \vdash Q) = ((P \wedge [\psi]_{<}) \vdash Q)$

by (*simp add: ISH-design design-def usubst, pred-auto*)

definition $OSH(\psi)(D) = (D \wedge (\$ok \wedge \neg D^f \wedge [\psi]_{<} \Rightarrow [\psi]_{>}))$

declare *OSH-def* [*upred-defs*]

lemma *OSH-as-OIH*:

$OSH(\psi)(D) = OIH(\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})(D)$

by (*simp add: OSH-def OIH-def, pred-auto*)

lemma *OSH-design*:

assumes *D is H1-H2*

shows $OSH(\psi)(D) = ((\neg D^f) \vdash (D^t \wedge (\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})))$

by (*simp add: OSH-as-OIH OIH-design assms*)

lemma *OSH-of-design*:

$\llbracket \$ok' \# P; \$ok' \# Q \rrbracket \Longrightarrow OSH(\psi)(P \vdash Q) = (P \vdash (Q \wedge (\lceil\psi\rceil_{<} \Rightarrow \lceil\psi\rceil_{>})))$

by (*simp add: OSH-design design-is-H1-H2 unrest, simp add: design-def usubst, pred-auto*)

definition $SIH(\psi) = ISH(\psi) \circ OSH(\psi)$

declare *SIH-def* [*upred-defs*]

lemma *SIH-of-design*:

$\llbracket \$ok' \# P; \$ok' \# Q; ok \# \psi \rrbracket \Longrightarrow SIH(\psi)(P \vdash Q) = ((P \wedge \lceil\psi\rceil_{<}) \vdash (Q \wedge \lceil\psi\rceil_{>}))$

by (*simp add: SIH-def OSH-of-design ISH-of-design unrest, pred-auto*)

end

8 Meta Theory for UTP Designs

theory *utp-designs*

imports

utp-des-core

utp-des-healths

utp-des-theory

utp-des-tactics

utp-des-hoare

utp-des-prog

utp-des-wp

utp-des-refcalc

utp-des-invariants

begin end

References

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