

# Generalised Reactive Processes in Isabelle/UTP

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## Abstract

Hoare and He’s UTP theory of reactive processes provides a unifying foundation for the semantics of process calculi and reactive programming. A reactive process is a form of UTP relation which can refer to both state variables and also a trace history of events. In their original presentation, a trace was modelled solely by a discrete sequence of events. Here, we generalise the trace model using “trace algebra”, which characterises traces abstractly using cancellative monoids, and thus enables application of the theory to a wider family of computational models, including hybrid computation. We recast the reactive healthiness conditions in this setting, and prove all the associated distributivity laws. We tackle parallel composition of reactive processes using the “parallel-by-merge” scheme from UTP. We also identify the associated theory of “reactive relations”, and use it to define generic reactive laws, a Hoare logic, and a weakest precondition calculus.

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# 1 Trace Algebras

```

theory Trace-Algebra
  imports
    UTP-Toolkit.List-Extra
    UTP-Toolkit.Positive
begin

```

Trace algebras provide a useful way in the UTP of characterising different notions of trace history. They can characterise notions as diverse as discrete event sequences and piecewise continuous functions, as employed by hybrid systems. For more information, please see our journal publication [4].

## 1.1 Ordered Semigroups

```

class ordered-semigroup = semigroup-add + order +
  assumes add-left-mono:  $a \leq b \implies c + a \leq c + b$ 
  and add-right-mono:  $a \leq b \implies a + c \leq b + c$ 
begin

```

```

lemma add-mono:
   $a \leq b \implies c \leq d \implies a + c \leq b + d$ 
  using local.add-left-mono local.add-right-mono local.order.trans by blast

end

```

## 1.2 Monoid Subclasses

```

class left-cancel-monoid = monoid-add +
  assumes add-left-imp-eq:  $a + b = a + c \implies b = c$ 

class right-cancel-monoid = monoid-add +
  assumes add-right-imp-eq:  $b + a = c + a \implies b = c$ 

```

Positive Monoids

```

class monoid-pos = monoid-add +
  assumes zero-sum-left:  $a + b = 0 \implies a = 0$ 
begin

lemma zero-sum-right:  $a + b = 0 \implies b = 0$ 
  by (metis local.add-0-left local.zero-sum-left)

lemma zero-sum:  $a + b = 0 \iff a = 0 \wedge b = 0$ 
  by (metis local.add-0-right zero-sum-right)

```

**end**

```

context monoid-add
begin

```

An additive monoid gives rise to natural notions of order, which we here define.

```

definition monoid-le (infix  $\leq_m$  50)
where  $a \leq_m b \iff (\exists c. b = a + c)$ 

```

We can also define a subtraction operator that remove a prefix from a monoid, if possible.

```

definition monoid-subtract (infixl  $-_m$  65)
where  $a -_m b = (\text{if } (b \leq_m a) \text{ then } \text{THE } c. a = b + c \text{ else } 0)$ 

```

We derive some basic properties of the preorder

```

lemma monoid-le-least-zero:  $0 \leq_m a$ 
  by (simp add: monoid-le-def)

lemma monoid-le-add:  $a \leq_m a + b$ 
  by (auto simp add: monoid-le-def)

lemma monoid-le-refl:  $a \leq_m a$ 
  by (simp add: monoid-le-def, metis add.right-neutral)

lemma monoid-le-trans:  $\llbracket a \leq_m b; b \leq_m c \rrbracket \implies a \leq_m c$ 
  by (metis add.assoc monoid-le-def)

lemma monoid-le-add-left-mono:  $a \leq_m b \implies c + a \leq_m c + b$ 
  using add-assoc by (auto simp add: monoid-le-def)

```

end

```

class ordered-monoid-pos = monoid-pos + ord +
  assumes le-is-monoid-le:  $a \leq b \longleftrightarrow (a \leq_m b)$ 
  and less-iff:  $a < b \longleftrightarrow a \leq b \wedge \neg (b \leq a)$ 
begin

  subclass preorder
  proof
    fix x y z :: 'a
    show  $(x < y) = (x \leq y \wedge \neg y \leq x)$ 
      by (simp add: local.less-iff)
    show  $x \leq x$ 
      by (simp add: local.le-is-monoid-le local.monoid-le-reft)
    show  $x \leq y \implies y \leq z \implies x \leq z$ 
      using local.le-is-monoid-le local.monoid-le-trans by blast
  qed
end

```

end

### 1.3 Trace Algebras

A pre-trace algebra is based on a left-cancellative monoid with the additional property that plus has no additive inverse. The latter is required to ensure that there are no “negative traces”. A pre-trace algebra has all the trace algebra axioms, but does not export the definitions of  $(\leq)$  and  $(-)$ .

```

class pre-trace = left-cancel-monoid + monoid-pos
begin

```

From our axiom set, we can derive a variety of properties of the monoid order

```

lemma monoid-le-antisym:
  assumes  $a \leq_m b$   $b \leq_m a$ 
  shows  $a = b$ 
proof -
  obtain a' where a':  $b = a + a'$ 
    using assms(1) monoid-le-def by auto

  obtain b' where b':  $a = b + b'$ 
    using assms(2) monoid-le-def by auto

  have  $b' = (b' + a' + b')$ 
    by (metis a' add-assoc b' local.add-left-imp-eq)

  hence  $a' + b' = 0$ 
    by (metis add-assoc local.add-0-right local.add-left-imp-eq)

  hence  $a' = 0$   $b' = 0$ 
    by (simp add: zero-sum)+

  with a' b' show ?thesis
    by simp
qed

```

The monoid minus operator is also the inverse of plus in this context, as expected.

```

lemma add-monoid-diff-cancel-left [simp]:  $(a + b) -_m a = b$ 
  apply (simp add: monoid-subtract-def monoid-le-add)
  apply (rule the-equality)
  apply (simp)
  using local.add-left-imp-eq apply blast
done

```

Iterating a trace

```

fun tr-iter :: nat  $\Rightarrow$  'a  $\Rightarrow$  'a where
  tr-iter-0: tr-iter 0 t = 0 |
  tr-iter-Suc: tr-iter (Suc n) t = tr-iter n t + t

```

```

lemma tr-iter-empty [simp]: tr-iter m 0 = 0
  by (induct m, simp-all)

```

**end**

We now construct the trace algebra by also exporting the order and minus operators.

```

class trace = pre-trace + ord + minus +
  assumes le-is-monoid-le:  $a \leq b \iff (a \leq_m b)$ 
  and less-iff:  $a < b \iff a \leq b \wedge \neg (b \leq a)$ 
  and minus-def:  $a - b = a -_m b$ 
begin

```

Next we prove all the trace algebra lemmas.

```

lemma le-iff-add:  $a \leq b \iff (\exists c. b = a + c)$ 
  by (simp add: local.le-is-monoid-le local.monoid-le-def)

```

```

lemma least-zero [simp]:  $0 \leq a$ 
  by (simp add: local.le-is-monoid-le local.monoid-le-least-zero)

```

```

lemma le-add [simp]:  $a \leq a + b$ 
  by (simp add: le-is-monoid-le local.monoid-le-add)

```

```

lemma not-le-minus [simp]:  $\neg (a \leq b) \implies b - a = 0$ 
  by (simp add: le-is-monoid-le local.minus-def local.monoid-subtract-def)

```

```

lemma add-diff-cancel-left [simp]:  $(a + b) - a = b$ 
  by (simp add: minus-def)

```

```

lemma diff-zero [simp]:  $a - 0 = a$ 
  by (metis local.add-0-left local.add-diff-cancel-left)

```

```

lemma diff-cancel [simp]:  $a - a = 0$ 
  by (metis local.add-0-right local.add-diff-cancel-left)

```

```

lemma add-left-mono:  $a \leq b \implies c + a \leq c + b$ 
  by (simp add: local.le-is-monoid-le local.monoid-le-add-left-mono)

```

```

lemma add-le-imp-le-left:  $c + a \leq c + b \implies a \leq b$ 
  by (auto simp add: le-iff-add, metis add-assoc local.add-diff-cancel-left)

```

```

lemma add-diff-cancel-left' [simp]:  $(c + a) - (c + b) = a - b$ 
proof (cases  $b \leq a$ )
  case True thus ?thesis

```

```

    by (metis add-assoc local.add-diff-cancel-left local.le-iff-add)
next
case False thus ?thesis
  using local.add-le-imp-le-left not-le-minus by blast
qed

lemma minus-zero-eq:  $\llbracket b \leq a; a - b = 0 \rrbracket \implies a = b$ 
  using local.le-iff-add local.monoid-le-def by auto

lemma diff-add-cancel-left':  $a \leq b \implies a + (b - a) = b$ 
  using local.le-iff-add local.monoid-le-def by auto

lemma add-left-strict-mono:  $\llbracket a + b < a + c \rrbracket \implies b < c$ 
  using local.add-le-imp-le-left local.add-left-mono local.less-iff by blast

lemma sum-minus-left:  $c \leq a \implies (a + b) - c = (a - c) + b$ 
  by (metis add-assoc diff-add-cancel-left' local.add-monoid-diff-cancel-left local.minus-def)

lemma neq-zero-impl-greater:
   $x \neq 0 \implies 0 < x$ 
  using le-is-monoid-le less-iff monoid-le-antisym monoid-le-least-zero by auto

lemma minus-cancel-le:
   $\llbracket x \leq y; y \leq z \rrbracket \implies y - x \leq z - x$ 
  using add-assoc le-iff-add by auto

lemma sum-minus-right:  $c \geq a \implies a + b - c = b - (c - a)$ 
  by (metis diff-add-cancel-left' local.add-diff-cancel-left')

lemma minus-gr-zero-iff [simp]:
   $0 < x - y \iff y < x$ 
  by (metis diff-cancel le-is-monoid-le least-zero less-iff minus-zero-eq monoid-le-antisym not-le-minus)

lemma le-zero-iff [simp]:  $x \leq 0 \iff x = 0$ 
  using local.le-iff-add local.zero-sum by auto

lemma minus-assoc [simp]:  $x - y - z = x - (y + z)$ 
  by (metis diff-add-cancel-left' le-add local.add-0-right local.add-diff-cancel-left' local.zero-sum minus-cancel-le
    not-le-minus)

```

end

class trace-split = trace +

assumes

sum-eq-sum-conv:  $(a + b) = (c + d) \implies \exists e. a = c + e \wedge e + b = d \vee a + e = c \wedge b = e + d$   
 $\text{--- } ?a + ?b = ?c + ?d \implies \exists e. ?a = ?c + e \wedge e + ?b = ?d \vee ?a + e = ?c \wedge ?b = e + ?d$  shows

how two equal traces that are each composed of two subtraces, can be expressed in terms of each other.

begin

The set subtraces of a common trace  $c$  is totally ordered.

lemma le-common-total:  $\llbracket a \leq c; b \leq c \rrbracket \implies a \leq b \vee b \leq a$   
 by (metis diff-add-cancel-left' le-add local.sum-eq-sum-conv)

lemma le-sum-cases:  $a \leq b + c \implies a \leq b \vee b \leq a$   
 by (simp add: le-common-total)

```

lemma le-sum-cases':
   $a \leq b + c \implies a \leq b \vee b \leq a \wedge a - b \leq c$ 
  by (auto, metis le-sum-cases, metis minus-def le-is-monoid-le add-monoid-diff-cancel-left monoid-le-def sum-eq-sum-conv)

lemma le-sum-iff:  $a \leq b + c \iff a \leq b \vee b \leq a \wedge a - b \leq c$ 
  by (metis le-sum-cases' add-monoid-diff-cancel-left le-is-monoid-le minus-def monoid-le-add-left-monoid-le-def monoid-le-trans)

```

**end**

Trace algebra give rise to a partial order on traces.

```

instance trace  $\subseteq$  order
  apply (intro-classes)
  apply (simp-all add: less-iff le-is-monoid-le monoid-le-refl)
  using monoid-le-trans apply blast
  apply (simp add: monoid-le-antisym)
  done

```

## 1.4 Models

Lists form a trace algebra.

```

instantiation list :: (type) monoid-add
begin

```

```

  definition zero-list :: 'a list where zero-list = []
  definition plus-list :: 'a list  $\Rightarrow$  'a list  $\Rightarrow$  'a list where plus-list = (@)

```

```

instance
  by (intro-classes, simp-all add: zero-list-def plus-list-def)

```

**end**

```

lemma monoid-le-list:
   $(xs :: 'a \text{ list}) \leq_m ys \iff xs \leq ys$ 
  apply (simp add: monoid-le-def plus-list-def)
  apply (meson Prefix-Order.prefixE Prefix-Order.prefixI)
  done

```

```

lemma monoid-subtract-list:
   $(xs :: 'a \text{ list}) -_m ys = xs - ys$ 
  apply (auto simp add: monoid-subtract-def monoid-le-list minus-list-def less-eq-list-def)
  apply (rule the-equality)
  apply (simp-all add: zero-list-def plus-list-def prefix-drop)
  done

```

```

instance list :: (type) trace-split
  apply (intro-classes, simp-all add: zero-list-def plus-list-def monoid-le-def monoid-subtract-list)
  using Prefix-Order.prefixE apply blast
  apply (simp add: less-list-def)
  apply (simp add: append-eq-append-conv2)
  done

```

```

lemma monoid-le-nat:

```

```

(x :: nat) ≤m y ↔ x ≤ y
by (simp add: monoid-le-def nat-le-iff-add)

```

**lemma** *monoid-subtract-nat*:

```

(x :: nat) −m y = x − y
by (auto simp add: monoid-subtract-def monoid-le-nat)

```

**instance** *nat :: trace-split*

```

  apply (intro-classes, simp-all add: monoid-subtract-nat)
  apply (simp add: nat-le-iff-add monoid-le-def)
  apply linarith+
  apply (metis Nat.diff-add-assoc Nat.diff-add-assoc2 add-diff-cancel-right' add-le-cancel-left add-le-cancel-right
    add-less-mono cancel-ab-semigroup-add-class.add-diff-cancel-left' less-irrefl not-le)
  done

```

Positives form a trace algebra.

**instance** *pos :: (linordered-semidom) trace-split*

**proof** (*intro-classes, simp-all*)

```

  fix a b c d :: 'a pos

```

```

  show a + b = 0 ⇒ a = 0

```

```

    by (transfer, simp add: add-nonneg-eq-0-iff)

```

```

  show a + b = c + d ⇒ ∃ e. a = c + e ∧ e + b = d ∨ a + e = c ∧ b = e + d

```

```

    apply (cases c ≤ a)

```

```

    apply (metis (no-types, lifting) cancel-semigroup-add-class.add-left-imp-eq le-add-diff-inverse semiring-normalization-

```

```

    semiring-normalization-rules(21))
  done

```

```

  show (a < b) = (a ≤ b ∧ ¬ b ≤ a)

```

```

    by auto

```

```

  show le-def: ∧ a b :: 'a pos. (a ≤ b) = (a ≤m b)

```

```

    by (auto simp add: monoid-le-def, metis le-add-diff-inverse)

```

```

  show a − b = a −m b

```

```

    apply (auto simp add: monoid-subtract-def le-def[THEN sym])

```

```

    apply (rule sym)

```

```

    apply (rule the-equality)

```

```

    apply (simp-all)

```

```

    apply (transfer, simp)

```

```

  done

```

**qed**

**end**

## 2 Reactive Processes Core Definitions

**theory** *utp-rea-core*

**imports**

*Trace-Algebra*

*UTP.utp-concurrency*

*UTP-Designs.utp-designs*

**begin** *recall-syntax*

### 2.1 Alphabet and Signature

The alphabet of reactive processes contains a boolean variable *wait*, which denotes whether a process is exhibiting an intermediate observation. It also has the variable *tr* which denotes



the trace history of a process. The type parameter  $'t$  represents the trace model being used, which must form a trace algebra [4], and thus provides the theory of “generalised reactive processes” [4]. The reactive process alphabet also extends the design alphabet, and thus includes the  $ok$  variable. For more information on these, see the UTP book [5], or the associated tutorial [2].

**alphabet**  $'t::trace$   $rp\text{-}vars = des\text{-}vars +$   
 $wait :: bool$   
 $tr :: 't$

**type-synonym**  $( 't, ' \alpha ) rp = ( 't, ' \alpha ) rp\text{-}vars\text{-}scheme$

**type-synonym**  $( 't, ' \alpha, ' \beta ) rel\text{-}rp = ( ( 't, ' \alpha ) rp, ( 't, ' \beta ) rp ) urel$

**type-synonym**  $( 't, ' \alpha ) hrel\text{-}rp = ( 't, ' \alpha ) rp hrel$

#### translations

$(type) ( 't, ' \alpha ) rp \leq (type) ( 't, ' \alpha ) rp\text{-}vars\text{-}scheme$   
 $(type) ( 't, ' \alpha ) rp \leq (type) ( 't, ' \alpha ) rp\text{-}vars\text{-}ext$   
 $(type) ( 't, ' \alpha, ' \beta ) rel\text{-}rp \leq (type) ( ( 't, ' \alpha ) rp, ( ' \gamma, ' \beta ) rp ) urel$   
 $(type) ( 't, ' \alpha ) hrel\text{-}rp \leq (type) ( 't, ' \alpha ) rp hrel$

As for designs, we set up various types to represent reactive processes. The main types to be used are  $( 't, ' \alpha, ' \beta ) rel\text{-}rp$  and  $( 't, ' \alpha ) hrel\text{-}rp$ , which correspond to heterogeneous/homogeneous reactive processes whose trace model is  $'t$  and alphabet types are  $' \alpha$  and  $' \beta$ . We also set up some useful syntax translations for these.

**notation**  $rp\text{-}vars.more_L (\Sigma_R)$

#### syntax

$-svid\text{-}rea\text{-}\alpha :: svid (\Sigma_R)$

#### translations

$-svid\text{-}rea\text{-}\alpha \Rightarrow CONST rp\text{-}vars.more_L$

Lens  $\Sigma_R$  exists because reactive alphabets are extensible.  $\Sigma_R$  points to the portion of the alphabet / state space that is neither  $ok$ ,  $wait$ , or  $tr$ .

**declare**  $des\text{-}vars.splits [alpha\text{-}splits del]$   
**declare**  $des\text{-}vars.splits [alpha\text{-}splits]$   
**declare**  $zero\text{-}list\text{-}def [upred\text{-}defs]$   
**declare**  $plus\text{-}list\text{-}def [upred\text{-}defs]$   
**declare**  $prefixE [elim]$

**abbreviation**  $wait\text{-}f::( 't::trace, ' \alpha, ' \beta ) rel\text{-}rp \Rightarrow ( 't, ' \alpha, ' \beta ) rel\text{-}rp$   
**where**  $wait\text{-}f R \equiv R \llbracket false / \$wait \rrbracket$

**abbreviation**  $wait\text{-}t::( 't::trace, ' \alpha, ' \beta ) rel\text{-}rp \Rightarrow ( 't, ' \alpha, ' \beta ) rel\text{-}rp$   
**where**  $wait\text{-}t R \equiv R \llbracket true / \$wait \rrbracket$

#### syntax

$-wait\text{-}f :: logic \Rightarrow logic (-_f [1000] 1000)$   
 $-wait\text{-}t :: logic \Rightarrow logic (-_t [1000] 1000)$

#### translations

$P_f \Leftarrow CONST usubst (CONST subst\text{-}upd CONST id (CONST ivar CONST wait) false) P$

$P_t \Rightarrow \text{CONST usubst } (\text{CONST subst-upd } \text{CONST id } (\text{CONST ivar } \text{CONST wait}) \text{ true}) P$

**abbreviation** *lift-rea* ::  $- \Rightarrow - ([\cdot]_R)$  **where**  
 $[P]_R \equiv P \oplus_p (\Sigma_R \times_L \Sigma_R)$

**abbreviation** *drop-rea* ::  $(t::\text{trace}, ' \alpha, ' \beta) \text{ rel-rp} \Rightarrow (' \alpha, ' \beta) \text{ urel } ([\cdot]_R)$  **where**  
 $[P]_R \equiv P \upharpoonright_e (\Sigma_R \times_L \Sigma_R)$

**abbreviation** *rea-pre-lift* ::  $- \Rightarrow - ([\cdot]_{R<})$  **where**  $[n]_{R<} \equiv [[n]_{<}]_R$

## 2.2 Reactive Lemmas

**lemma** *unrest-ok-lift-rea* [*unrest*]:

$\$ok \# [P]_R \$ok' \# [P]_R$

**by** (*pred-auto*) $+$

**lemma** *unrest-wait-lift-rea* [*unrest*]:

$\$wait \# [P]_R \$wait' \# [P]_R$

**by** (*pred-auto*) $+$

**lemma** *unrest-tr-lift-rea* [*unrest*]:

$\$tr \# [P]_R \$tr' \# [P]_R$

**by** (*pred-auto*) $+$

**lemma** *des-lens-equiv-wait-tr-rest*:  $\Sigma_D \approx_L \text{wait} +_L \text{tr} +_L \Sigma_R$

**by** *simp*

**lemma** *rea-lens-bij*: *bij-lens* ( $ok +_L \text{wait} +_L \text{tr} +_L \Sigma_R$ )

**proof**  $-$

**have**  $ok +_L \text{wait} +_L \text{tr} +_L \Sigma_R \approx_L ok +_L \Sigma_D$

**using** *des-lens-equiv-wait-tr-rest des-vars.indeps lens-equiv-sym lens-plus-eq-right* **by** *blast*

**also have**  $\dots \approx_L 1_L$

**using** *bij-lens-equiv-id*[*of*  $ok +_L \Sigma_D$ ] **by** (*simp add: ok-des-bij-lens*)

**finally show** *?thesis*

**by** (*simp add: bij-lens-equiv-id*)

**qed**

**lemma** *segr-wait-true* [*usubst*]:  $(P ;; Q)_t = (P_t ;; Q)$

**by** (*rel-auto*)

**lemma** *segr-wait-false* [*usubst*]:  $(P ;; Q)_f = (P_f ;; Q)$

**by** (*rel-auto*)

## 2.3 Trace contribution lens

The following lens represents the portion of the state-space that is the difference between *tr'* and *tr*, that is the contribution that a process is making to the trace history.

**definition** *tcontr* ::  $t::\text{trace} \Rightarrow ('t, ' \alpha) \text{ rp} \times ('t, ' \alpha) \text{ rp } (tt)$  **where**

[*lens-defs*]:

$tcontr = (\text{ lens-get } = (\lambda s. \text{get}(\$tr')_v s - \text{get}(\$tr)_v s) ,$

$\text{ lens-put } = (\lambda s v. \text{put}(\$tr')_v s (\text{get}(\$tr)_v s + v)) )$

**definition** *itrace* ::  $t::\text{trace} \Rightarrow ('t, ' \alpha) \text{ rp} \times ('t, ' \alpha) \text{ rp } (it)$  **where**

[*lens-defs*]:

$itrace = (\text{ lens-get } = \text{get}(\$tr)_v ,$

$$\text{lens-put} = (\lambda s v. \text{put}(\$tr')_v (\text{put}(\$tr)_v s v) v) \text{ ()}$$

**lemma** *tcontr-mwb-lens* [simp]: *mwb-lens* *tt*  
 by (unfold-locales, simp-all add: lens-defs prod.case-eq-if)

**lemma** *itrace-mwb-lens* [simp]: *mwb-lens* **it**  
 by (unfold-locales, simp-all add: lens-defs prod.case-eq-if)

**syntax**

-svid-tcontr :: svid (*tt*)  
 -svid-itrace :: svid (**it**)  
 -utr-iter :: logic  $\Rightarrow$  logic  $\Rightarrow$  logic (*iter*[-]'(-))

**translations**

-svid-tcontr == CONST *tcontr*  
 -svid-itrace == CONST *itrace*  
*iter*[*n*](*P*) == CONST *uop* (CONST *tr-iter* *n*) *P*

**lemma** *tcontr-alt-def*:  $\&tt = (\$tr' - \$tr)$   
 by (rel-auto)

**lemma** *tcontr-alt-def'*: *utp-expr.var* *tt* =  $(\$tr' - \$tr)$   
 by (rel-auto)

**lemma** *tt-indeps* [simp]:  
 assumes  $x \bowtie (\$tr)_v$   $x \bowtie (\$tr')_v$   
 shows  $x \bowtie tt$   $tt \bowtie x$   
 using *assms*  
 by (unfold lens-indep-def, safe, simp-all add: tcontr-def, (metis lens-indep-get var-update-out)+)

We lift a few trace properties from the trace class using *transfer*.

**lemma** *ueexpr-diff-zero* [simp]:  
 fixes *a* :: ( $\alpha::\text{trace}$ , 'a) *ueexpr*  
 shows  $a - 0 = a$   
 by (simp add: minus-ueexpr-def zero-ueexpr-def, transfer, auto)

**lemma** *ueexpr-add-diff-cancel-left* [simp]:  
 fixes *a b* :: ( $\alpha::\text{trace}$ , 'a) *ueexpr*  
 shows  $(a + b) - a = b$   
 by (simp add: minus-ueexpr-def plus-ueexpr-def, transfer, auto)

**lemma** *iter-0* [simp]: *iter*[0](*t*) =  $\langle \rangle$   
 by (transfer, simp add: zero-list-def)

end

### 3 Events for Reactive Processes

theory *utp-rea-event*  
 imports *UTP.utp*  
 begin

#### 3.1 Events

Events of some type '*v*' are just the elements of that type.

**type-synonym**  $'\vartheta \text{ event} = '\vartheta$

### 3.2 Channels

Typed channels are modelled as functions. Below,  $'a$  determines the channel type and  $'\vartheta$  the underlying event type. As with values, it is difficult to introduce channels as monomorphic types due to the fact that they can have arbitrary parametrisations in term of  $'a$ . Applying a channel to an element of its type yields an event, as we may expect. Though this is not formalised here, we may also sensibly assume that all channel- representing functions are injective. Note: is there benefit in formalising this here?

**type-synonym**  $('a, '\vartheta) \text{ chan} = 'a \Rightarrow '\vartheta \text{ event}$

A downside of the approach is that the event type  $'\vartheta$  must be able to encode *all* events of a process model, and hence cannot be fixed upfront for a single channel or channel set. To do so, we actually require a notion of ‘extensible’ datatypes, in analogy to extensible record types. Another solution is to encode a notion of channel scoping that namely uses *sum* types to lift channel types into extensible ones, that is using channel-set specific scoping operators. This is a current work in progress.

#### 3.2.1 Operators

The Z type of a channel corresponds to the entire carrier of the underlying HOL type of that channel.

**definition**  $\text{chan-type} :: ('a, '\vartheta) \text{ chan} \Rightarrow 'a \text{ set } (\delta_u)$  **where**  
 $[\text{upred-defs}]: \delta_u \ c = \text{UNIV}$

The next lifted function creates an expression that yields a channel event, from an expression on the channel type  $'a$ .

**definition**  $\text{chan-apply} ::$   
 $('a, '\vartheta) \text{ chan} \Rightarrow ('a, '\alpha) \text{ uexpr} \Rightarrow (' \vartheta \text{ event}, '\alpha) \text{ uexpr } (('(-/-)_u)$  **where**  
 $[\text{upred-defs}]: (c \cdot e)_u = \text{uop } c \ e$

**lemma**  $\text{unrest-chan-apply } [\text{unrest}]: x \# e \Longrightarrow x \# (c \cdot e)_u$   
**by**  $(\text{rel-auto})$

**lemma**  $\text{usubst-chan-apply } [\text{usubst}]: \sigma \dagger (c \cdot v)_u = (c \cdot \sigma \dagger v)_u$   
**by**  $(\text{rel-auto})$

**lemma**  $\text{msubst-event } [\text{usubst}]:$   
 $(c \cdot v \ x)_u \llbracket x \rightarrow u \rrbracket = (c \cdot (v \ x) \llbracket x \rightarrow u \rrbracket)_u$   
**by**  $(\text{pred-simp})$

**lemma**  $\text{msubst-event-2 } [\text{usubst}]:$   
 $(c \cdot v \ x \ y)_u \llbracket (x, y) \rightarrow u \rrbracket = (c \cdot (v \ x \ y) \llbracket (x, y) \rightarrow u \rrbracket)_u$   
**by**  $(\text{pred-simp})+$

**lemma**  $\text{aext-event } [\text{alpha}]: (c \cdot v)_u \oplus_p a = (c \cdot v \oplus_p a)_u$   
**by**  $(\text{pred-auto})$

**end**

## 4 Reactive Healthiness Conditions

**theory** *utp-rea-healths*  
**imports** *utp-rea-core*  
**begin**

### 4.1 R1: Events cannot be undone

**definition**  $R1 :: ('t::trace, 'α, 'β) \text{rel-rp} \Rightarrow ('t, 'α, 'β) \text{rel-rp}$  **where**  
 $R1\text{-def}$  [*upred-defs*]:  $R1(P) = (P \wedge (\$tr \leq_u \$tr'))$

**lemma**  $R1\text{-idem}$ :  $R1(R1(P)) = R1(P)$   
**by** *pred-auto*

**lemma**  $R1\text{-Idempotent}$  [*closure*]: *Idempotent*  $R1$   
**by** (*simp add: Idempotent-def R1-idem*)

**lemma**  $R1\text{-mono}$ :  $P \sqsubseteq Q \Longrightarrow R1(P) \sqsubseteq R1(Q)$   
**by** *pred-auto*

**lemma**  $R1\text{-Monotonic}$ : *Monotonic*  $R1$   
**by** (*simp add: mono-def R1-mono*)

**lemma**  $R1\text{-Continuous}$ : *Continuous*  $R1$   
**by** (*auto simp add: Continuous-def, rel-auto*)

**lemma**  $R1\text{-unrest}$  [*unrest*]:  $\llbracket x \bowtie \text{in-var } tr; x \bowtie \text{out-var } tr; x \# P \rrbracket \Longrightarrow x \# R1(P)$   
**by** (*simp add: R1-def unrest lens-indep-sym*)

**lemma**  $R1\text{-false}$ :  $R1(\text{false}) = \text{false}$   
**by** *pred-auto*

**lemma**  $R1\text{-conj}$ :  $R1(P \wedge Q) = (R1(P) \wedge R1(Q))$   
**by** *pred-auto*

**lemma**  $\text{conj-}R1\text{-closed-1}$  [*closure*]:  $P \text{ is } R1 \Longrightarrow (P \wedge Q) \text{ is } R1$   
**by** (*rel-blast*)

**lemma**  $\text{conj-}R1\text{-closed-2}$  [*closure*]:  $Q \text{ is } R1 \Longrightarrow (P \wedge Q) \text{ is } R1$   
**by** (*rel-blast*)

**lemma**  $R1\text{-disj}$ :  $R1(P \vee Q) = (R1(P) \vee R1(Q))$   
**by** *pred-auto*

**lemma**  $\text{disj-}R1\text{-closed}$  [*closure*]:  $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \Longrightarrow (P \vee Q) \text{ is } R1$   
**by** (*simp add: Healthy-def R1-def utp-pred-laws.inf-sup-distrib2*)

**lemma**  $R1\text{-impl}$ :  $R1(P \Rightarrow Q) = ((\neg R1(\neg P)) \Rightarrow R1(Q))$   
**by** (*rel-auto*)

**lemma**  $R1\text{-inf}$ :  $R1(P \sqcap Q) = (R1(P) \sqcap R1(Q))$   
**by** *pred-auto*

**lemma**  $R1\text{-USUP}$ :  
 $R1(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot R1(P(i)))$   
**by** (*rel-auto*)

**lemma** *R1-Sup [closure]*:  $\llbracket \bigwedge P. P \in A \implies P \text{ is } R1; A \neq \{\} \rrbracket \implies \bigcap A \text{ is } R1$   
**using** *R1-Continuous* **by** (*auto simp add: Continuous-def Healthy-def*)

**lemma** *R1-UINF*:  
**assumes**  $A \neq \{\}$   
**shows**  $R1(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot R1(P(i)))$   
**using** *assms* **by** (*rel-auto*)

**lemma** *R1-UINF-ind*:  
 $R1(\bigsqcup i \cdot P(i)) = (\bigsqcup i \cdot R1(P(i)))$   
**by** (*rel-auto*)

**lemma** *UINF-ind-R1-closed [closure]*:  
 $\llbracket \bigwedge i. P(i) \text{ is } R1 \rrbracket \implies (\bigcap i \cdot P(i)) \text{ is } R1$   
**by** (*rel-blast*)

**lemma** *UINF-R1-closed [closure]*:  
 $\llbracket \bigwedge i. P \text{ is } R1 \rrbracket \implies (\bigcap i \in A \cdot P \ i) \text{ is } R1$   
**by** (*rel-blast*)

**lemma** *tr-ext-conj-R1 [closure]*:  
 $\$tr' =_u \$tr \hat{\ }_u e \wedge P \text{ is } R1$   
**by** (*rel-auto, simp add: Prefix-Order.prefixI*)

**lemma** *tr-id-conj-R1 [closure]*:  
 $\$tr' =_u \$tr \wedge P \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *R1-extend-conj*:  $R1(P \wedge Q) = (R1(P) \wedge Q)$   
**by** *pred-auto*

**lemma** *R1-extend-conj'*:  $R1(P \wedge Q) = (P \wedge R1(Q))$   
**by** *pred-auto*

**lemma** *R1-cond*:  $R1(P \triangleleft b \triangleright Q) = (R1(P) \triangleleft b \triangleright R1(Q))$   
**by** (*rel-auto*)

**lemma** *R1-cond'*:  $R1(P \triangleleft b \triangleright Q) = (R1(P) \triangleleft R1(b) \triangleright R1(Q))$   
**by** (*rel-auto*)

**lemma** *R1-negate-R1*:  $R1(\neg R1(P)) = R1(\neg P)$   
**by** *pred-auto*

**lemma** *R1-wait-true [usubst]*:  $(R1 \ P)_t = R1(P)_t$   
**by** *pred-auto*

**lemma** *R1-wait-false [usubst]*:  $(R1 \ P)_f = R1(P)_f$   
**by** *pred-auto*

**lemma** *R1-wait'-true [usubst]*:  $(R1 \ P) \llbracket true / \$wait' \rrbracket = R1(P \llbracket true / \$wait' \rrbracket)$   
**by** (*rel-auto*)

**lemma** *R1-wait'-false [usubst]*:  $(R1 \ P) \llbracket false / \$wait' \rrbracket = R1(P \llbracket false / \$wait' \rrbracket)$   
**by** (*rel-auto*)

**lemma** *R1-wait-false-closed* [closure]:  $P \text{ is } R1 \implies P[\text{false}/\$wait] \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *R1-wait'-false-closed* [closure]:  $P \text{ is } R1 \implies P[\text{false}/\$wait'] \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *R1-skip*:  $R1(II) = II$   
**by** (*rel-auto*)

**lemma** *skip-is-R1* [closure]:  $II \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *subst-R1*:  $\llbracket \$tr \# \sigma; \$tr' \# \sigma \rrbracket \implies \sigma \dagger (R1 P) = R1(\sigma \dagger P)$   
**by** (*simp add: R1-def usubst*)

**lemma** *subst-R1-closed* [closure]:  $\llbracket \$tr \# \sigma; \$tr' \# \sigma; P \text{ is } R1 \rrbracket \implies \sigma \dagger P \text{ is } R1$   
**by** (*metis Healthy-def subst-R1*)

**lemma** *R1-by-refinement*:  
 $P \text{ is } R1 \iff ((\$tr \leq_u \$tr') \sqsubseteq P)$   
**by** (*rel-blast*)

**lemma** *R1-trace-extension* [closure]:  
 $\$tr' \geq_u \$tr \hat{^}_u e \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *tr-le-trans*:  
 $((\$tr \leq_u \$tr') ;; (\$tr \leq_u \$tr')) = (\$tr \leq_u \$tr')$   
**by** (*rel-auto*)

**lemma** *R1-seqr*:  
 $R1(R1(P) ;; R1(Q)) = (R1(P) ;; R1(Q))$   
**by** (*rel-auto*)

**lemma** *R1-seqr-closure* [closure]:  
**assumes**  $P \text{ is } R1 \ Q \text{ is } R1$   
**shows**  $(P ;; Q) \text{ is } R1$   
**using** *assms unfolding R1-by-refinement*  
**by** (*metis seqr-mono tr-le-trans*)

**lemma** *R1-power* [closure]:  $P \text{ is } R1 \implies P^n \text{ is } R1$   
**by** (*induct n, simp-all add: upred-semiring.power-Suc closure*)

**lemma** *R1-true-comp* [simp]:  $(R1(\text{true}) ;; R1(\text{true})) = R1(\text{true})$   
**by** (*rel-auto*)

**lemma** *R1-ok'-true*:  $(R1(P))^t = R1(P^t)$   
**by** *pred-auto*

**lemma** *R1-ok'-false*:  $(R1(P))^f = R1(P^f)$   
**by** *pred-auto*

**lemma** *R1-ok-true*:  $(R1(P))\llbracket \text{true}/\$ok \rrbracket = R1(P\llbracket \text{true}/\$ok \rrbracket)$   
**by** *pred-auto*

**lemma** *R1-ok-false*:  $(R1(P))\llbracket false/\$ok \rrbracket = R1(P\llbracket false/\$ok \rrbracket)$   
**by** *pred-auto*

**lemma** *seqr-R1-true-right*:  $((P ;; R1(true)) \vee P) = (P ;; (\$tr \leq_u \$tr'))$   
**by** (*rel-auto*)

**lemma** *conj-R1-true-right*:  $(P;;R1(true) \wedge Q;;R1(true)) ;; R1(true) = (P;;R1(true) \wedge Q;;R1(true))$   
**apply** (*rel-auto*) **using** *dual-order.trans* **by** *blast+*

**lemma** *R1-extend-conj-unrest*:  $\llbracket \$tr \# Q; \$tr' \# Q \rrbracket \implies R1(P \wedge Q) = (R1(P) \wedge Q)$   
**by** *pred-auto*

**lemma** *R1-extend-conj-unrest'*:  $\llbracket \$tr \# P; \$tr' \# P \rrbracket \implies R1(P \wedge Q) = (P \wedge R1(Q))$   
**by** *pred-auto*

**lemma** *R1-tr'-eq-tr*:  $R1(\$tr' =_u \$tr) = (\$tr' =_u \$tr)$   
**by** (*rel-auto*)

**lemma** *R1-tr-less-tr'*:  $R1(\$tr <_u \$tr') = (\$tr <_u \$tr')$   
**by** (*rel-auto*)

**lemma** *tr-strict-prefix-R1-closed* [*closure*]:  $\$tr <_u \$tr'$  is *R1*  
**by** (*rel-auto*)

**lemma** *R1-H2-commute*:  $R1(H2(P)) = H2(R1(P))$   
**by** (*simp add: H2-split R1-def usubst, rel-auto*)

## 4.2 R2: No dependence upon trace history

There are various ways of expressing *R2*, which are enumerated below.

**definition** *R2a* ::  $(t::trace, '\alpha, '\beta) \text{ rel-rp} \Rightarrow (t, '\alpha, '\beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $R2a(P) = (\bigcap s \cdot P\llbracket \langle s \rangle, (\langle s \rangle + (\$tr' - \$tr)) / \$tr, \$tr' \rrbracket)$

**definition** *R2a'* ::  $(t::trace, '\alpha, '\beta) \text{ rel-rp} \Rightarrow (t, '\alpha, '\beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $R2a' P = (R2a(P) \triangleleft R1(true) \triangleright P)$

**definition** *R2s* ::  $(t::trace, '\alpha, '\beta) \text{ rel-rp} \Rightarrow (t, '\alpha, '\beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $R2s(P) = (P\llbracket 0/\$tr \rrbracket\llbracket (\$tr' - \$tr) / \$tr' \rrbracket)$

**definition** *R2* ::  $(t::trace, '\alpha, '\beta) \text{ rel-rp} \Rightarrow (t, '\alpha, '\beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $R2(P) = R1(R2s(P))$

**definition** *R2c* ::  $(t::trace, '\alpha, '\beta) \text{ rel-rp} \Rightarrow (t, '\alpha, '\beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $R2c(P) = (R2s(P) \triangleleft R1(true) \triangleright P)$

*R2a* and *R2s* are the standard definitions from the UTP book [5]. An issue with these forms is that their definition depends upon *R1* also being satisfied [4], since otherwise the trace minus operator is not well defined. We overcome this with our own version, *R2c*, which applies *R2s* if *R1* holds, and otherwise has no effect. This latter healthiness condition can therefore be reasoned about independently of *R1*, which is useful in some circumstances.

**lemma** *unrest-ok-R2s* [*unrest*]:  $\$ok \# P \implies \$ok \# R2s(P)$   
**by** (*simp add: R2s-def unrest*)



**lemma** *unrest-ok'-R2s* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R2s(P)$   
**by** (*simp add: R2s-def unrest*)

**lemma** *unrest-ok-R2c* [*unrest*]:  $\$ok \# P \implies \$ok \# R2c(P)$   
**by** (*simp add: R2c-def unrest*)

**lemma** *unrest-ok'-R2c* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R2c(P)$   
**by** (*simp add: R2c-def unrest*)

**lemma** *R2s-unrest* [*unrest*]:  $\llbracket vwb\text{-}lens\ x; x \bowtie in\text{-}var\ tr; x \bowtie out\text{-}var\ tr; x \# P \rrbracket \implies x \# R2s(P)$   
**by** (*simp add: R2s-def unrest usubst lens-indep-sym*)

**lemma** *R2s-subst-wait-true* [*usubst*]:  
 $(R2s(P))\llbracket true/\$wait \rrbracket = R2s(P\llbracket true/\$wait \rrbracket)$   
**by** (*simp add: R2s-def usubst unrest*)

**lemma** *R2s-subst-wait'-true* [*usubst*]:  
 $(R2s(P))\llbracket true/\$wait' \rrbracket = R2s(P\llbracket true/\$wait' \rrbracket)$   
**by** (*simp add: R2s-def usubst unrest*)

**lemma** *R2-subst-wait-true* [*usubst*]:  
 $(R2(P))\llbracket true/\$wait \rrbracket = R2(P\llbracket true/\$wait \rrbracket)$   
**by** (*simp add: R2-def R1-def R2s-def usubst unrest*)

**lemma** *R2-subst-wait'-true* [*usubst*]:  
 $(R2(P))\llbracket true/\$wait' \rrbracket = R2(P\llbracket true/\$wait' \rrbracket)$   
**by** (*simp add: R2-def R1-def R2s-def usubst unrest*)

**lemma** *R2-subst-wait-false* [*usubst*]:  
 $(R2(P))\llbracket false/\$wait \rrbracket = R2(P\llbracket false/\$wait \rrbracket)$   
**by** (*simp add: R2-def R1-def R2s-def usubst unrest*)

**lemma** *R2-subst-wait'-false* [*usubst*]:  
 $(R2(P))\llbracket false/\$wait' \rrbracket = R2(P\llbracket false/\$wait' \rrbracket)$   
**by** (*simp add: R2-def R1-def R2s-def usubst unrest*)

**lemma** *R2c-R2s-absorb*:  $R2c(R2s(P)) = R2s(P)$   
**by** (*rel-auto*)

**lemma** *R2a-R2s*:  $R2a(R2s(P)) = R2s(P)$   
**by** (*rel-auto*)

**lemma** *R2s-R2a*:  $R2s(R2a(P)) = R2a(P)$   
**by** (*rel-auto*)

**lemma** *R2a-equiv-R2s*:  $P \text{ is } R2a \longleftrightarrow P \text{ is } R2s$   
**by** (*metis Healthy-def' R2a-R2s R2s-R2a*)

**lemma** *R2a-idem*:  $R2a(R2a(P)) = R2a(P)$   
**by** (*rel-auto*)

**lemma** *R2a'-idem*:  $R2a'(R2a'(P)) = R2a'(P)$   
**by** (*rel-auto*)

**lemma** *R2a-mono*:  $P \sqsubseteq Q \implies R2a(P) \sqsubseteq R2a(Q)$

by (rel-blast)

**lemma** *R2a'-mono*:  $P \sqsubseteq Q \implies R2a'(P) \sqsubseteq R2a'(Q)$   
by (rel-blast)

**lemma** *R2a'-weakening*:  $R2a'(P) \sqsubseteq P$   
apply (rel-simp)  
apply (rename-tac ok wait tr more ok' wait' tr' more')  
apply (rule-tac x=tr in exI)  
apply (simp add: diff-add-cancel-left)  
done

**lemma** *R2s-idem*:  $R2s(R2s(P)) = R2s(P)$   
by (pred-auto)

**lemma** *R2-idem*:  $R2(R2(P)) = R2(P)$   
by (pred-auto)

**lemma** *R2-mono*:  $P \sqsubseteq Q \implies R2(P) \sqsubseteq R2(Q)$   
by (pred-auto)

**lemma** *R2-implies-R1 [closure]*:  $P \text{ is } R2 \implies P \text{ is } R1$   
by (rel-blast)

**lemma** *R2c-Continuous*: *Continuous*  $R2c$   
by (rel-simp)

**lemma** *R2c-lit*:  $R2c(\ll x \gg) = \ll x \gg$   
by (rel-auto)

**lemma** *tr-strict-prefix-R2c-closed [closure]*:  $\$tr <_u \$tr' \text{ is } R2c$   
by (rel-auto)

**lemma** *R2s-conj*:  $R2s(P \wedge Q) = (R2s(P) \wedge R2s(Q))$   
by (pred-auto)

**lemma** *R2-conj*:  $R2(P \wedge Q) = (R2(P) \wedge R2(Q))$   
by (pred-auto)

**lemma** *R2s-disj*:  $R2s(P \vee Q) = (R2s(P) \vee R2s(Q))$   
by pred-auto

**lemma** *R2s-USUP*:  
 $R2s(\bigcap i \in A \cdot P(i)) = (\bigcap i \in A \cdot R2s(P(i)))$   
by (simp add: R2s-def usubst)

**lemma** *R2c-USUP*:  
 $R2c(\bigcap i \in A \cdot P(i)) = (\bigcap i \in A \cdot R2c(P(i)))$   
by (rel-auto)

**lemma** *R2s-UINF*:  
 $R2s(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot R2s(P(i)))$   
by (simp add: R2s-def usubst)

**lemma** *R2c-UINF*:

$R2c(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot R2c(P(i)))$   
**by** (*rel-auto*)

**lemma** *R2-disj*:  $R2(P \vee Q) = (R2(P) \vee R2(Q))$   
**by** (*pred-auto*)

**lemma** *R2s-not*:  $R2s(\neg P) = (\neg R2s(P))$   
**by** *pred-auto*

**lemma** *R2s-condr*:  $R2s(P \triangleleft b \triangleright Q) = (R2s(P) \triangleleft R2s(b) \triangleright R2s(Q))$   
**by** (*rel-auto*)

**lemma** *R2-condr*:  $R2(P \triangleleft b \triangleright Q) = (R2(P) \triangleleft R2(b) \triangleright R2(Q))$   
**by** (*rel-auto*)

**lemma** *R2-condr'*:  $R2(P \triangleleft b \triangleright Q) = (R2(P) \triangleleft R2s(b) \triangleright R2(Q))$   
**by** (*rel-auto*)

**lemma** *R2s-ok*:  $R2s(\$ok) = \$ok$   
**by** (*rel-auto*)

**lemma** *R2s-ok'*:  $R2s(\$ok') = \$ok'$   
**by** (*rel-auto*)

**lemma** *R2s-wait*:  $R2s(\$wait) = \$wait$   
**by** (*rel-auto*)

**lemma** *R2s-wait'*:  $R2s(\$wait') = \$wait'$   
**by** (*rel-auto*)

**lemma** *R2s-true*:  $R2s(true) = true$   
**by** *pred-auto*

**lemma** *R2s-false*:  $R2s(false) = false$   
**by** *pred-auto*

**lemma** *true-is-R2s*:  
*true is R2s*  
**by** (*simp add: Healthy-def R2s-true*)

**lemma** *R2s-lift-rea*:  $R2s(\lceil P \rceil_R) = \lceil P \rceil_R$   
**by** (*simp add: R2s-def usubst unrest*)

**lemma** *R2c-lift-rea*:  $R2c(\lceil P \rceil_R) = \lceil P \rceil_R$   
**by** (*simp add: R2c-def R2s-lift-rea cond-idem usubst unrest*)

**lemma** *R2c-true*:  $R2c(true) = true$   
**by** (*rel-auto*)

**lemma** *R2c-false*:  $R2c(false) = false$   
**by** (*rel-auto*)

**lemma** *R2c-and*:  $R2c(P \wedge Q) = (R2c(P) \wedge R2c(Q))$   
**by** (*rel-auto*)

**lemma** *conj-R2c-closed* [*closure*]:  $\llbracket P \text{ is } R2c; Q \text{ is } R2c \rrbracket \implies (P \wedge Q) \text{ is } R2c$   
**by** (*simp add: Healthy-def R2c-and*)

**lemma** *R2c-disj*:  $R2c(P \vee Q) = (R2c(P) \vee R2c(Q))$   
**by** (*rel-auto*)

**lemma** *R2c-inf*:  $R2c(P \sqcap Q) = (R2c(P) \sqcap R2c(Q))$   
**by** (*rel-auto*)

**lemma** *R2c-not*:  $R2c(\neg P) = (\neg R2c(P))$   
**by** (*rel-auto*)

**lemma** *R2c-ok*:  $R2c(\$ok) = (\$ok)$   
**by** (*rel-auto*)

**lemma** *R2c-ok'*:  $R2c(\$ok') = (\$ok')$   
**by** (*rel-auto*)

**lemma** *R2c-wait*:  $R2c(\$wait) = \$wait$   
**by** (*rel-auto*)

**lemma** *R2c-wait'*:  $R2c(\$wait') = \$wait'$   
**by** (*rel-auto*)

**lemma** *R2c-wait'-true* [*usubst*]:  $(R2c\ P) \llbracket true/\$wait' \rrbracket = R2c(P \llbracket true/\$wait' \rrbracket)$   
**by** (*rel-auto*)

**lemma** *R2c-wait'-false* [*usubst*]:  $(R2c\ P) \llbracket false/\$wait' \rrbracket = R2c(P \llbracket false/\$wait' \rrbracket)$   
**by** (*rel-auto*)

**lemma** *R2c-tr'-minus-tr*:  $R2c(\$tr' =_u \$tr) = (\$tr' =_u \$tr)$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

**lemma** *R2c-tr'-ge-tr*:  $R2c(\$tr' \geq_u \$tr) = (\$tr' \geq_u \$tr)$   
**by** (*rel-auto*)

**lemma** *R2c-tr'-less-tr'*:  $R2c(\$tr <_u \$tr') = (\$tr <_u \$tr')$   
**by** (*rel-auto*)

**lemma** *R2c-condr*:  $R2c(P \triangleleft b \triangleright Q) = (R2c(P) \triangleleft R2c(b) \triangleright R2c(Q))$   
**by** (*rel-auto*)

**lemma** *R2c-shAll*:  $R2c(\forall x \cdot P\ x) = (\forall x \cdot R2c(P\ x))$   
**by** (*rel-auto*)

**lemma** *R2c-impl*:  $R2c(P \implies Q) = (R2c(P) \implies R2c(Q))$   
**by** (*metis (no-types, lifting) R2c-and R2c-not double-negation impl-alt-def not-conj-deMorgans*)

**lemma** *R2c-skip-r*:  $R2c(II) = II$   
**proof** –  
**have**  $R2c(II) = R2c(\$tr' =_u \$tr \wedge II \upharpoonright_{\alpha} tr)$   
**by** (*subst skip-r-unfold[of tr], simp-all*)  
**also have**  $\dots = (R2c(\$tr' =_u \$tr) \wedge II \upharpoonright_{\alpha} tr)$   
**by** (*simp add: R2c-and, simp add: R2c-def R2s-def usubst unrest cond-idem*)  
**also have**  $\dots = (\$tr' =_u \$tr \wedge II \upharpoonright_{\alpha} tr)$

by (simp add: R2c-tr'-minus-tr)  
 finally show ?thesis  
 by (subst skip-r-unfold[of tr], simp-all)  
 qed

lemma R1-R2c-commute:  $R1(R2c(P)) = R2c(R1(P))$   
 by (rel-auto)

lemma R1-R2c-is-R2:  $R1(R2c(P)) = R2(P)$   
 by (rel-auto)

lemma R1-R2s-R2c:  $R1(R2s(P)) = R1(R2c(P))$   
 by (rel-auto)

lemma R1-R2s-tr-wait:  
 $R1(R2s(\$tr' =_u \$tr \wedge \$wait')) = (\$tr' =_u \$tr \wedge \$wait')$   
 apply rel-auto using minus-zero-eq by blast

lemma R1-R2s-tr'-eq-tr:  
 $R1(R2s(\$tr' =_u \$tr)) = (\$tr' =_u \$tr)$   
 apply (rel-auto) using minus-zero-eq by blast

lemma R1-R2s-tr'-extend-tr:  
 $\llbracket \$tr \# v; \$tr' \# v \rrbracket \implies R1(R2s(\$tr' =_u \$tr \hat{^}_u v)) = (\$tr' =_u \$tr \hat{^}_u v)$   
 apply (rel-auto)  
 apply (metis append-minus)  
 apply (simp add: Prefix-Order.prefixI)  
 done

lemma R2-tr-prefix:  $R2(\$tr \leq_u \$tr') = (\$tr \leq_u \$tr')$   
 by (pred-auto)

lemma R2-form:  
 $R2(P) = (\exists tt_0 \cdot P[\llbracket 0/\$tr \rrbracket \llbracket \llbracket tt_0 \rrbracket / \$tr' \rrbracket] \wedge \$tr' =_u \$tr + \llbracket tt_0 \rrbracket)$   
 by (rel-auto, metis trace-class.add-diff-cancel-left trace-class.le-iff-add)

lemma R2-subst-tr:  
 assumes  $P$  is  $R2$   
 shows  $[\$tr \mapsto_s tr_0, \$tr' \mapsto_s tr_0 + t] \uparrow P = [\$tr \mapsto_s 0, \$tr' \mapsto_s t] \uparrow P$   
 proof –  
 have  $[\$tr \mapsto_s tr_0, \$tr' \mapsto_s tr_0 + t] \uparrow R2 P = [\$tr \mapsto_s 0, \$tr' \mapsto_s t] \uparrow R2 P$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

lemma R2-seqr-form:  
 shows  $(R2(P) ;; R2(Q)) =$   
 $(\exists tt_1 \cdot \exists tt_2 \cdot ((P[\llbracket 0/\$tr \rrbracket \llbracket \llbracket tt_1 \rrbracket / \$tr' \rrbracket] ;; (Q[\llbracket 0/\$tr \rrbracket \llbracket \llbracket tt_2 \rrbracket / \$tr' \rrbracket]))$   
 $\wedge (\$tr' =_u \$tr + \llbracket tt_1 \rrbracket + \llbracket tt_2 \rrbracket))$   
 proof –  
 have  $(R2(P) ;; R2(Q)) = (\exists tr_0 \cdot (R2(P))[\llbracket \llbracket tr_0 \rrbracket / \$tr' \rrbracket] ;; (R2(Q))[\llbracket \llbracket tr_0 \rrbracket / \$tr \rrbracket])$   
 by (subst seqr-middle[of tr], simp-all)  
 also have ... =  
 $(\exists tr_0 \cdot \exists tt_1 \cdot \exists tt_2 \cdot ((P[\llbracket 0/\$tr \rrbracket \llbracket \llbracket tt_1 \rrbracket / \$tr' \rrbracket] \wedge \llbracket tr_0 \rrbracket =_u \$tr + \llbracket tt_1 \rrbracket) ;;$

$(Q[0/\$tr][\langle tt_2 \rangle / \$tr'] \wedge \$tr' =_u \langle tr_0 \rangle + \langle tt_2 \rangle))$   
 by (simp add: R2-form usubst unrest uquant-lift, rel-blast)  
 also have ... =  
 $(\exists tr_0 \cdot \exists tt_1 \cdot \exists tt_2 \cdot ((\langle tr_0 \rangle =_u \$tr + \langle tt_1 \rangle \wedge P[0/\$tr][\langle tt_1 \rangle / \$tr']) ;;$   
 $(Q[0/\$tr][\langle tt_2 \rangle / \$tr'] \wedge \$tr' =_u \langle tr_0 \rangle + \langle tt_2 \rangle)))$   
 by (simp add: conj-comm)  
 also have ... =  
 $(\exists tt_1 \cdot \exists tt_2 \cdot \exists tr_0 \cdot ((P[0/\$tr][\langle tt_1 \rangle / \$tr']) ; (Q[0/\$tr][\langle tt_2 \rangle / \$tr'])))$   
 $\wedge \langle tr_0 \rangle =_u \$tr + \langle tt_1 \rangle \wedge \$tr' =_u \langle tr_0 \rangle + \langle tt_2 \rangle)$   
 by (rel-blast)  
 also have ... =  
 $(\exists tt_1 \cdot \exists tt_2 \cdot ((P[0/\$tr][\langle tt_1 \rangle / \$tr']) ; (Q[0/\$tr][\langle tt_2 \rangle / \$tr'])))$   
 $\wedge (\exists tr_0 \cdot \langle tr_0 \rangle =_u \$tr + \langle tt_1 \rangle \wedge \$tr' =_u \langle tr_0 \rangle + \langle tt_2 \rangle))$   
 by (rel-auto)  
 also have ... =  
 $(\exists tt_1 \cdot \exists tt_2 \cdot ((P[0/\$tr][\langle tt_1 \rangle / \$tr']) ; (Q[0/\$tr][\langle tt_2 \rangle / \$tr'])))$   
 $\wedge (\$tr' =_u \$tr + \langle tt_1 \rangle + \langle tt_2 \rangle))$   
 by (rel-auto)  
 finally show ?thesis .  
 qed

lemma R2-seqr-form':  
 assumes  $P$  is R2  $Q$  is R2  
 shows  $P ; Q =$   
 $(\exists tt_1 \cdot \exists tt_2 \cdot ((P[0/\$tr][\langle tt_1 \rangle / \$tr']) ; (Q[0/\$tr][\langle tt_2 \rangle / \$tr'])))$   
 $\wedge (\$tr' =_u \$tr + \langle tt_1 \rangle + \langle tt_2 \rangle))$   
 using R2-seqr-form[of  $P$   $Q$ ] by (simp add: Healthy-if assms)

lemma R2-seqr-form'':  
 assumes  $P$  is R2  $Q$  is R2  
 shows  $P ; Q =$   
 $(\exists (tt_1, tt_2) \cdot ((P[0, \langle tt_1 \rangle / \$tr, \$tr']) ; (Q[0, \langle tt_2 \rangle / \$tr, \$tr'])))$   
 $\wedge (\$tr' =_u \$tr + \langle tt_1 \rangle + \langle tt_2 \rangle))$   
 by (subst R2-seqr-form', simp-all add: assms, rel-auto)

lemma R2-tr-middle:  
 assumes  $P$  is R2  $Q$  is R2  
 shows  $(\exists tr_0 \cdot (P[\langle tr_0 \rangle / \$tr'] ; Q[\langle tr_0 \rangle / \$tr]) \wedge \langle tr_0 \rangle \leq_u \$tr') = (P ; Q)$   
 proof –  
 have  $(P ; Q) = (\exists tr_0 \cdot (P[\langle tr_0 \rangle / \$tr'] ; Q[\langle tr_0 \rangle / \$tr]))$   
 by (simp add: seqr-middle)  
 also have ... =  $(\exists tr_0 \cdot ((R2\ P)[\langle tr_0 \rangle / \$tr'] ; (R2\ Q)[\langle tr_0 \rangle / \$tr]))$   
 by (simp add: assms Healthy-if)  
 also have ... =  $(\exists tr_0 \cdot ((R2\ P)[\langle tr_0 \rangle / \$tr'] ; (R2\ Q)[\langle tr_0 \rangle / \$tr]) \wedge \langle tr_0 \rangle \leq_u \$tr')$   
 by (rel-auto)  
 also have ... =  $(\exists tr_0 \cdot (P[\langle tr_0 \rangle / \$tr'] ; Q[\langle tr_0 \rangle / \$tr]) \wedge \langle tr_0 \rangle \leq_u \$tr')$   
 by (simp add: assms Healthy-if)  
 finally show ?thesis ..  
 qed

lemma R2-seqr-distribute:  
 fixes  $P :: ('t::trace, 'α, 'β)$  rel-rp and  $Q :: ('t, 'β, 'γ)$  rel-rp  
 shows  $R2(R2(P) ; R2(Q)) = (R2(P) ; R2(Q))$   
 proof –  
 have  $R2(R2(P) ; R2(Q)) =$

$((\exists tt_1 \cdot \exists tt_2 \cdot (P[0/\$tr][\langle\langle tt_1 \rangle\rangle/\$tr'] ;; Q[0/\$tr][\langle\langle tt_2 \rangle\rangle/\$tr'])(\$tr' - \$tr)/\$tr') \wedge \$tr' - \$tr =_u \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle) \wedge \$tr' \geq_u \$tr)$   
**by** (*simp add: R2-seqr-form, simp add: R2s-def usubst unrest, rel-auto*)  
**also have** ... =  
 $((\exists tt_1 \cdot \exists tt_2 \cdot (P[0/\$tr][\langle\langle tt_1 \rangle\rangle/\$tr'] ;; Q[0/\$tr][\langle\langle tt_2 \rangle\rangle/\$tr'])(\langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle)/\$tr') \wedge \$tr' - \$tr =_u \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle) \wedge \$tr' \geq_u \$tr)$   
**by** (*subst subst-eq-replace, simp*)  
**also have** ... =  
 $((\exists tt_1 \cdot \exists tt_2 \cdot (P[0/\$tr][\langle\langle tt_1 \rangle\rangle/\$tr'] ;; Q[0/\$tr][\langle\langle tt_2 \rangle\rangle/\$tr'])(\$tr' - \$tr =_u \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle) \wedge \$tr' \geq_u \$tr)$   
**by** (*rel-auto*)  
**also have** ... =  
 $(\exists tt_1 \cdot \exists tt_2 \cdot (P[0/\$tr][\langle\langle tt_1 \rangle\rangle/\$tr'] ;; Q[0/\$tr][\langle\langle tt_2 \rangle\rangle/\$tr'] \wedge (\$tr' - \$tr =_u \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle) \wedge \$tr' \geq_u \$tr))$   
**by** *pred-auto*  
**also have** ... =  
 $((\exists tt_1 \cdot \exists tt_2 \cdot (P[0/\$tr][\langle\langle tt_1 \rangle\rangle/\$tr'] ;; Q[0/\$tr][\langle\langle tt_2 \rangle\rangle/\$tr'] \wedge \$tr' =_u \$tr + \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle))$   
**proof** –  
**have**  $\bigwedge tt_1 tt_2. (((\$tr' - \$tr =_u \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle) \wedge \$tr' \geq_u \$tr) :: ('t, 'α, 'γ) rel-rp)$   
 $= (\$tr' =_u \$tr + \langle\langle tt_1 \rangle\rangle + \langle\langle tt_2 \rangle\rangle)$   
**apply** (*rel-auto*)  
**apply** (*metis add.assoc diff-add-cancel-left'*)  
**apply** (*simp add: add.assoc*)  
**apply** (*meson le-add order-trans*)  
**done**  
**thus** *?thesis* **by** *simp*  
**qed**  
**also have** ... = (*R2(P) ;; R2(Q)*)  
**by** (*simp add: R2-seqr-form*)  
**finally show** *?thesis* .  
**qed**

**lemma** *R2-seqr-closure* [*closure*]:  
**assumes** *P is R2 Q is R2*  
**shows** (*P ;; Q*) *is R2*  
**by** (*metis Healthy-def' R2-seqr-distribute assms(1) assms(2)*)

**lemma** *false-R2* [*closure*]: *false is R2*  
**by** (*rel-auto*)

**lemma** *R1-R2-commute*:  
 $R1(R2(P)) = R2(R1(P))$   
**by** *pred-auto*

**lemma** *R2-R1-form*:  $R2(R1(P)) = R1(R2s(P))$   
**by** (*rel-auto*)

**lemma** *R2s-H1-commute*:  
 $R2s(H1(P)) = H1(R2s(P))$   
**by** (*rel-auto*)

**lemma** *R2s-H2-commute*:  
 $R2s(H2(P)) = H2(R2s(P))$   
**by** (*simp add: H2-split R2s-def usubst*)

**lemma** *R2-R1-seq-drop-left*:

$R2(R1(P) ;; R1(Q)) = R2(P ;; R1(Q))$   
**by** (*rel-auto*)

**lemma** *R2c-idem*:  $R2c(R2c(P)) = R2c(P)$

**by** (*rel-auto*)

**lemma** *R2c-Idempotent [closure]*: *Idempotent R2c*

**by** (*simp add: Idempotent-def R2c-idem*)

**lemma** *R2c-Monotonic [closure]*: *Monotonic R2c*

**by** (*rel-auto*)

**lemma** *R2c-H2-commute*:  $R2c(H2(P)) = H2(R2c(P))$

**by** (*simp add: H2-split R2c-disj R2c-def R2s-def usubst, rel-auto*)

**lemma** *R2c-seq*:  $R2c(R2(P) ;; R2(Q)) = (R2(P) ;; R2(Q))$

**by** (*metis (no-types, lifting) R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute R2c-idem*)

**lemma** *R2-R2c-def*:  $R2(P) = R1(R2c(P))$

**by** (*rel-auto*)

**lemma** *R2-comp-def*:  $R2 = R1 \circ R2c$

**by** (*auto simp add: R2-R2c-def*)

**lemma** *R2c-R1-seq*:  $R2c(R1(R2c(P)) ;; R1(R2c(Q))) = (R1(R2c(P)) ;; R1(R2c(Q)))$

**using** *R2c-seq[of P Q]* **by** (*simp add: R2-R2c-def*)

**lemma** *R1-R2c-seqr-distribute*:

**fixes**  $P :: ('t::trace, 'α, 'β) \text{ rel-rp}$  **and**  $Q :: ('t, 'β, 'γ) \text{ rel-rp}$

**assumes**  $P \text{ is } R1 \ P \text{ is } R2c \ Q \text{ is } R1 \ Q \text{ is } R2c$

**shows**  $R1(R2c(P ;; Q)) = P ;; Q$

**by** (*metis Healthy-if R1-seqr R2c-R1-seq assms*)

**lemma** *R2-R1-true*:

$R2(R1(true)) = R1(true)$

**by** (*simp add: R2-R1-form R2s-true*)

**lemma** *R1-true-R2 [closure]*:  $R1(true) \text{ is } R2$

**by** (*rel-auto*)

**lemma** *R1-R2s-R1-true-lemma*:

$R1(R2s(R1(\neg R2s P) ;; R1 true)) = R1(R2s((\neg P) ;; R1 true))$

**by** (*rel-auto*)

**lemma** *R2c-healthy-R2s*:  $P \text{ is } R2c \implies R1(R2s(P)) = R1(P)$

**by** (*simp add: Healthy-def R1-R2s-R2c*)

### 4.3 R3: No activity while predecessor is waiting

**definition**  $R3 :: ('t::trace, 'α) \text{ hrel-rp} \Rightarrow ('t, 'α) \text{ hrel-rp}$  **where**

*[upred-defs]*:  $R3(P) = (II \triangleleft \$wait \triangleright P)$

**lemma** *R3-idem*:  $R3(R3(P)) = R3(P)$

**by** (*rel-auto*)



**lemma** *R3-Idempotent [closure]: Idempotent R3*  
**by** (*simp add: Idempotent-def R3-idem*)

**lemma** *R3-mono:  $P \sqsubseteq Q \implies R3(P) \sqsubseteq R3(Q)$*   
**by** (*rel-auto*)

**lemma** *R3-Monotonic: Monotonic R3*  
**by** (*simp add: mono-def R3-mono*)

**lemma** *R3-Continuous: Continuous R3*  
**by** (*rel-auto*)

**lemma** *R3-conj:  $R3(P \wedge Q) = (R3(P) \wedge R3(Q))$*   
**by** (*rel-auto*)

**lemma** *R3-disj:  $R3(P \vee Q) = (R3(P) \vee R3(Q))$*   
**by** (*rel-auto*)

**lemma** *R3-USUP:*  
**assumes**  $A \neq \{\}$   
**shows**  $R3(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot R3(P(i)))$   
**using** *assms* **by** (*rel-auto*)

**lemma** *R3-UINF:*  
**assumes**  $A \neq \{\}$   
**shows**  $R3(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot R3(P(i)))$   
**using** *assms* **by** (*rel-auto*)

**lemma** *R3-condr:  $R3(P \triangleleft b \triangleright Q) = (R3(P) \triangleleft b \triangleright R3(Q))$*   
**by** (*rel-auto*)

**lemma** *R3-skipr:  $R3(II) = II$*   
**by** (*rel-auto*)

**lemma** *R3-form:  $R3(P) = ((\$wait \wedge II) \vee (\neg \$wait \wedge P))$*   
**by** (*rel-auto*)

**lemma** *wait-R3:*  
 $(\$wait \wedge R3(P)) = (II \wedge \$wait')$   
**by** (*rel-auto*)

**lemma** *nwait-R3:*  
 $(\neg \$wait \wedge R3(P)) = (\neg \$wait \wedge P)$   
**by** (*rel-auto*)

**lemma** *R3-semir-form:*  
 $(R3(P) ;; R3(Q)) = R3(P ;; R3(Q))$   
**by** (*rel-auto*)

**lemma** *R3-semir-closure:*  
**assumes**  $P$  is *R3*  $Q$  is *R3*  
**shows**  $(P ;; Q)$  is *R3*  
**using** *assms*  
**by** (*metis Healthy-def' R3-semir-form*)

**lemma** *R1-R3-commute*:  $R1(R3(P)) = R3(R1(P))$   
**by** (*rel-auto*)

**lemma** *R2-R3-commute*:  $R2(R3(P)) = R3(R2(P))$   
**apply** (*rel-auto*)  
**using** *minus-zero-eq* **apply** *blast+*  
**done**

#### 4.4 R4: The trace strictly increases

**definition**  $R4 :: ('t::trace, 'α, 'β) rel-rp \Rightarrow ('t, 'α, 'β) rel-rp$  **where**  
 $[upred-defs]: R4(P) = (P \wedge \$tr <_u \$tr')$

**lemma** *R4-implies-R1* [*closure*]:  $P \text{ is } R4 \implies P \text{ is } R1$   
**using** *less-iff* **by** *rel-blast*

**lemma** *R4-iff-refine*:  
 $P \text{ is } R4 \longleftrightarrow (\$tr <_u \$tr') \sqsubseteq P$   
**by** (*rel-blast*)

**lemma** *R4-idem*:  $R4(R4 P) = R4 P$   
**by** (*rel-auto*)

**lemma** *R4-false*:  $R4(false) = false$   
**by** (*rel-auto*)

**lemma** *R4-conj*:  $R4(P \wedge Q) = (R4(P) \wedge R4(Q))$   
**by** (*rel-auto*)

**lemma** *R4-disj*:  $R4(P \vee Q) = (R4(P) \vee R4(Q))$   
**by** (*rel-auto*)

**lemma** *R4-is-R4* [*closure*]:  $R4(P) \text{ is } R4$   
**by** (*rel-auto*)

**lemma** *false-R4* [*closure*]:  $false \text{ is } R4$   
**by** (*rel-auto*)

**lemma** *UINF-R4-closed* [*closure*]:  
 $\llbracket \bigwedge i. P \text{ is } R4 \rrbracket \implies (\bigcap i. P \text{ is } R4)$   
**by** (*rel-blast*)

**lemma** *conj-R4-closed* [*closure*]:  
 $\llbracket P \text{ is } R4; Q \text{ is } R4 \rrbracket \implies (P \wedge Q) \text{ is } R4$   
**by** (*simp add: Healthy-def R4-conj*)

**lemma** *disj-R4-closed* [*closure*]:  
 $\llbracket P \text{ is } R4; Q \text{ is } R4 \rrbracket \implies (P \vee Q) \text{ is } R4$   
**by** (*simp add: Healthy-def R4-disj*)

**lemma** *seq-R4-closed-1* [*closure*]:  
 $\llbracket P \text{ is } R4; Q \text{ is } R1 \rrbracket \implies (P ;; Q) \text{ is } R4$   
**using** *less-le-trans* **by** *rel-blast*

**lemma** *seq-R4-closed-2* [*closure*]:

$\llbracket P \text{ is } R1; Q \text{ is } R4 \rrbracket \implies (P ;; Q) \text{ is } R4$   
**using** *le-less-trans* **by** *rel-blast*

#### 4.5 R5: The trace does not increase

**definition** *R5* :: (*t*::*trace*, ' $\alpha$ ', ' $\beta$ ') *rel-rp*  $\Rightarrow$  (*t*, ' $\alpha$ ', ' $\beta$ ') *rel-rp* **where**  
*[upred-defs]*:  $R5(P) = (P \wedge \$tr =_u \$tr')$

**lemma** *R5-implies-R1* [*closure*]:  $P \text{ is } R5 \implies P \text{ is } R1$   
**using** *eq-iff* **by** *rel-blast*

**lemma** *R5-iff-refine*:  
 $P \text{ is } R5 \longleftrightarrow (\$tr =_u \$tr') \sqsubseteq P$   
**by** (*rel-blast*)

**lemma** *R5-conj*:  $R5(P \wedge Q) = (R5(P) \wedge R5(Q))$   
**by** (*rel-auto*)

**lemma** *R5-disj*:  $R5(P \vee Q) = (R5(P) \vee R5(Q))$   
**by** (*rel-auto*)

**lemma** *R4-R5*:  $R4(R5 P) = false$   
**by** (*rel-auto*)

**lemma** *R5-R4*:  $R5(R4 P) = false$   
**by** (*rel-auto*)

**lemma** *UINF-R5-closed* [*closure*]:  
 $\llbracket \bigwedge i. P \text{ is } R5 \rrbracket \implies (\bigcap i. P \text{ is } R5)$   
**by** (*rel-blast*)

**lemma** *conj-R5-closed* [*closure*]:  
 $\llbracket P \text{ is } R5; Q \text{ is } R5 \rrbracket \implies (P \wedge Q) \text{ is } R5$   
**by** (*simp add: Healthy-def R5-conj*)

**lemma** *disj-R5-closed* [*closure*]:  
 $\llbracket P \text{ is } R5; Q \text{ is } R5 \rrbracket \implies (P \vee Q) \text{ is } R5$   
**by** (*simp add: Healthy-def R5-disj*)

**lemma** *seq-R5-closed* [*closure*]:  
 $\llbracket P \text{ is } R5; Q \text{ is } R5 \rrbracket \implies (P ;; Q) \text{ is } R5$   
**by** (*rel-auto, metis*)

#### 4.6 RP laws

**definition** *RP-def* [*upred-defs*]:  $RP(P) = R1(R2c(R3(P)))$

**lemma** *RP-comp-def*:  $RP = R1 \circ R2c \circ R3$   
**by** (*auto simp add: RP-def*)

**lemma** *RP-alt-def*:  $RP(P) = R1(R2(R3(P)))$   
**by** (*metis R1-R2c-is-R2 R1-idem RP-def*)

**lemma** *RP-intro*:  $\llbracket P \text{ is } R1; P \text{ is } R2; P \text{ is } R3 \rrbracket \implies P \text{ is } RP$   
**by** (*simp add: Healthy-def' RP-alt-def*)

**lemma** *RP-idem*:  $RP(RP(P)) = RP(P)$   
 by (simp add: R1-R2c-is-R2 R2-R3-commute R2-idem R3-idem RP-def)

**lemma** *RP-Idempotent [closure]*: *Idempotent RP*  
 by (simp add: Idempotent-def RP-idem)

**lemma** *RP-mono*:  $P \sqsubseteq Q \implies RP(P) \sqsubseteq RP(Q)$   
 by (simp add: R1-R2c-is-R2 R2-mono R3-mono RP-def)

**lemma** *RP-Monotonic*: *Monotonic RP*  
 by (simp add: mono-def RP-mono)

**lemma** *RP-Continuous*: *Continuous RP*  
 by (simp add: Continuous-comp R1-Continuous R2c-Continuous R3-Continuous RP-comp-def)

**lemma** *RP-skip*:  
 $RP(II) = II$   
 by (simp add: R1-skip R2c-skip-r R3-skipr RP-def)

**lemma** *RP-skip-closure [closure]*:  
 $II$  is *RP*  
 by (simp add: Healthy-def' RP-skip)

**lemma** *RP-seq-closure [closure]*:  
 assumes  $P$  is *RP*  $Q$  is *RP*  
 shows  $(P ;; Q)$  is *RP*  
**proof** (rule *RP-intro*)  
 show  $(P ;; Q)$  is *R1*  
 by (metis Healthy-def R1-seqr RP-def assms)  
 thus  $(P ;; Q)$  is *R2*  
 by (metis Healthy-def' R2-R2c-def R2c-R1-seq RP-def assms)  
 show  $(P ;; Q)$  is *R3*  
 by (metis (no-types, lifting) Healthy-def' R1-R2c-is-R2 R2-R3-commute R3-idem R3-semir-form RP-def assms)  
**qed**

## 4.7 UTP theories

**interpretation** *rea-theory*: *utp-theory-continuous RP*  
 rewrites  $P \in \text{carrier } \text{rea-theory.thy-order} \longleftrightarrow P \text{ is } RP$   
 and  $\text{le } \text{des-theory.thy-order} = (\sqsubseteq)$   
 and  $\text{eq } \text{des-theory.thy-order} = (=)$   
**proof** –  
 show *utp-theory-continuous RP*  
 by (unfold-locales, simp-all add: RP-idem RP-Continuous)  
**qed** (simp-all)

**notation** *rea-theory.utp-top* ( $\top_r$ )  
**notation** *rea-theory.utp-bottom* ( $\perp_r$ )

**interpretation** *rea-theory-rel*: *utp-theory-unital RP skip-r*  
 by (unfold-locales, simp-all add: closure)

**lemma** *rea-top*:  $\top_r = (\$wait \wedge II)$   
**proof** –  
 have  $\top_r = RP(\text{false})$

by (simp add: rea-theory.healthy-top)  
 also have ... = (\$wait  $\wedge$  II)  
 by (rel-auto, metis minus-zero-eq)  
 finally show ?thesis .  
 qed

**lemma** *rea-top-left-zero*:

assumes *P is RP*  
 shows  $(\top_r ;; P) = \top_r$   
**proof** –  
 have  $(\top_r ;; P) = ((\$wait \wedge II) ;; R3(P))$   
 by (metis (no-types, lifting) Healthy-def R1-R2c-is-R2 R2-R3-commute R3-idem RP-def assms  
 rea-top)  
 also have ... = (\$wait  $\wedge$  R3(P))  
 by (rel-auto)  
 also have ... = (\$wait  $\wedge$  II)  
 by (metis R3-skipr wait-R3)  
 also have ... =  $\top_r$   
 by (simp add: rea-top)  
 finally show ?thesis .  
 qed

**lemma** *rea-bottom*:  $\perp_r = R1(\$wait \Rightarrow II)$

**proof** –  
 have  $\perp_r = RP(true)$   
 by (simp add: rea-theory.healthy-bottom)  
 also have ... =  $R1(\$wait \Rightarrow II)$   
 by (rel-auto, metis minus-zero-eq)  
 finally show ?thesis .  
 qed

end

## 5 Reactive Parallel-by-Merge

**theory** *utp-rea-parallel*  
**imports** *utp-rea-healths*  
**begin**

We show closure of parallel by merge under the reactive healthiness conditions by means of suitable restrictions on the merge predicate [4]. We first define healthiness conditions for *R1* and *R2* merge predicates.

**definition** *R1m* ::  $('t :: trace, 'a) \text{ rp merge} \Rightarrow ('t, 'a) \text{ rp merge}$   
**where** [upred-defs]:  $R1m(M) = (M \wedge \$tr_{<} \leq_u \$tr')$

**definition** *R1m'* ::  $('t :: trace, 'a) \text{ rp merge} \Rightarrow ('t, 'a) \text{ rp merge}$   
**where** [upred-defs]:  $R1m'(M) = (M \wedge \$tr_{<} \leq_u \$tr' \wedge \$tr_{<} \leq_u \$0-tr \wedge \$tr_{<} \leq_u \$1-tr)$

A merge predicate can access the history through *tr*, as usual, but also through *0.tr* and *1.tr*. Thus we have to remove the latter two histories as well to satisfy *R2* for the overall construction.

**definition** *R2m* ::  $('t :: trace, 'a) \text{ rp merge} \Rightarrow ('t, 'a) \text{ rp merge}$   
**where** [upred-defs]:  $R2m(M) = R1m(M \llbracket 0, (\$tr' - \$tr_{<}), (\$0-tr - \$tr_{<}), (\$1-tr - \$tr_{<}) / \$tr_{<}, \$tr', \$0-tr, \$1-tr \rrbracket)$

**definition** *R2m'* ::  $('t :: trace, 'a) \text{ rp merge} \Rightarrow ('t, 'a) \text{ rp merge}$

**where**  $[upred-defs]: R2m'(M) = R1m'(M[\![0, (\$tr' - \$tr_<), (\$0 - tr - \$tr_<), (\$1 - tr - \$tr_<)]/\$tr_<, \$tr', \$0 - tr, \$1 - tr\])$

**definition**  $R2cm :: ('t :: trace, 'α) rp\ merge \Rightarrow ('t, 'α) rp\ merge$

**where**  $[upred-defs]: R2cm(M) = M[\![0, (\$tr' - \$tr_<), (\$0 - tr - \$tr_<), (\$1 - tr - \$tr_<)]/\$tr_<, \$tr', \$0 - tr, \$1 - tr\]$   
 $\triangleleft \$tr_< \leq_u \$tr' \triangleright M$

**lemma**  $R2m'$ -form:

$R2m'(M) =$   
 $(\exists (tt_p, tt_0, tt_1) \cdot M[\![0, \ll tt_p \gg, \ll tt_0 \gg, \ll tt_1 \gg]/\$tr_<, \$tr', \$0 - tr, \$1 - tr\]$   
 $\quad \wedge \$tr' =_u \$tr_< + \ll tt_p \gg$   
 $\quad \wedge \$0 - tr =_u \$tr_< + \ll tt_0 \gg$   
 $\quad \wedge \$1 - tr =_u \$tr_< + \ll tt_1 \gg)$   
**by**  $(rel-auto, metis\ diff-add-cancel-left')$

**lemma**  $R1m-idem$ :  $R1m(R1m(P)) = R1m(P)$

**by**  $(rel-auto)$

**lemma**  $R1m-seq-lemma$ :  $R1m(R1m(M) ;; R1(P)) = R1m(M) ;; R1(P)$

**by**  $(rel-auto)$

**lemma**  $R1m-seq$  [closure]:

**assumes**  $M$  is  $R1m$   $P$  is  $R1$

**shows**  $M ;; P$  is  $R1m$

**proof** –

**from**  $assms$  **have**  $R1m(M ;; P) = R1m(R1m(M) ;; R1(P))$

**by**  $(simp\ add: Healthy-if)$

**also have**  $\dots = R1m(M) ;; R1(P)$

**by**  $(simp\ add: R1m-seq-lemma)$

**also have**  $\dots = M ;; P$

**by**  $(simp\ add: Healthy-if\ assms)$

**finally show**  $?thesis$

**by**  $(simp\ add: Healthy-def)$

**qed**

**lemma**  $R2m-idem$ :  $R2m(R2m(P)) = R2m(P)$

**by**  $(rel-auto)$

**lemma**  $R2m-seq-lemma$ :  $R2m'(R2m'(M) ;; R2(P)) = R2m'(M) ;; R2(P)$

**apply**  $(simp\ add: R2m'$ -form  $R2$ -form)

**apply**  $(rel-auto)$

**apply**  $(metis\ (no-types, lifting)\ add.assoc)+$

**done**

**lemma**  $R2m'$ -seq [closure]:

**assumes**  $M$  is  $R2m'$   $P$  is  $R2$

**shows**  $M ;; P$  is  $R2m'$

**by**  $(metis\ Healthy-def'\ R2m-seq-lemma\ assms(1)\ assms(2))$

**lemma**  $R1-par-by-merge$  [closure]:

$M$  is  $R1m \implies (P \parallel_M Q)$  is  $R1$

**by**  $(rel-blast)$

**lemma**  $R2-R2m'$ -pbm:  $R2(P \parallel_M Q) = (R2(P) \parallel_{R2m'(M)} R2(Q))$

**proof** –

**have**  $(R2(P) \parallel_{R2m'(M)} R2(Q)) = ((R2(P) \parallel_s R2(Q)) ;;$

$$\begin{aligned}
& (\exists (tt_p, tt_0, tt_1) \cdot M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr] \\
& \quad \wedge \$tr' =_u \$tr_{<} + \langle\langle tt_p \rangle\rangle \\
& \quad \wedge \$0-tr =_u \$tr_{<} + \langle\langle tt_0 \rangle\rangle \\
& \quad \wedge \$1-tr =_u \$tr_{<} + \langle\langle tt_1 \rangle\rangle)) \\
& \text{by (simp add: par-by-merge-def R2m'-form)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (((R2(P) \parallel_s R2(Q)) \;; (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr] \\
& \quad \wedge \$tr' =_u \$tr_{<} + \langle\langle tt_p \rangle\rangle \\
& \quad \wedge \$0-tr =_u \$tr_{<} + \langle\langle tt_0 \rangle\rangle \\
& \quad \wedge \$1-tr =_u \$tr_{<} + \langle\langle tt_1 \rangle\rangle))) \\
& \text{by (rel-blast)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (((R2(P) \parallel_s R2(Q)) \wedge \$0-tr' =_u \$tr_{<}' + \langle\langle tt_0 \rangle\rangle \wedge \$1-tr' =_u \\
& \$tr_{<}' + \langle\langle tt_1 \rangle\rangle) \;; \\
& \quad (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr] \wedge \$tr' =_u \$tr_{<} + \\
& \langle\langle tt_p \rangle\rangle))) \\
& \text{by (rel-blast)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (((R2(P) \parallel_s R2(Q)) \wedge \$0-tr' =_u \$tr_{<}' + \langle\langle tt_0 \rangle\rangle \wedge \$1-tr' =_u \\
& \$tr_{<}' + \langle\langle tt_1 \rangle\rangle) \;; \\
& \quad (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr])) \wedge \$tr' =_u \$tr_{<} + \\
& \langle\langle tt_p \rangle\rangle) \\
& \text{by (rel-blast)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (((R2(P) \wedge \$tr' =_u \$tr + \langle\langle tt_0 \rangle\rangle) \parallel_s (R2(Q) \wedge \$tr' =_u \$tr + \\
& \langle\langle tt_1 \rangle\rangle)) \;; \\
& \quad (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr])) \wedge \$tr' =_u \$tr_{<} + \\
& \langle\langle tt_p \rangle\rangle) \\
& \text{by (rel-auto, blast, metis le-add trace-class.add-diff-cancel-left)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (( (\exists tt_0' \cdot P[0, \langle\langle tt_0' \rangle\rangle / \$tr, \$tr'] \wedge \$tr' =_u \$tr + \langle\langle tt_0' \rangle\rangle) \wedge \\
& \$tr' =_u \$tr + \langle\langle tt_0 \rangle\rangle) \\
& \quad \parallel_s ((\exists tt_1' \cdot Q[0, \langle\langle tt_1' \rangle\rangle / \$tr, \$tr'] \wedge \$tr' =_u \$tr + \langle\langle tt_1' \rangle\rangle) \wedge \$tr' =_u \\
& \$tr + \langle\langle tt_1 \rangle\rangle) \;; \\
& \quad (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr])) \wedge \$tr' =_u \$tr_{<} + \\
& \langle\langle tt_p \rangle\rangle) \\
& \text{by (simp add: R2-form usubst)} \\
& \text{also have } \dots = (\exists (tt_p, tt_0, tt_1) \cdot (( (P[0, \langle\langle tt_0 \rangle\rangle / \$tr, \$tr'] \wedge \$tr' =_u \$tr + \langle\langle tt_0 \rangle\rangle) \\
& \quad \parallel_s (Q[0, \langle\langle tt_1 \rangle\rangle / \$tr, \$tr'] \wedge \$tr' =_u \$tr + \langle\langle tt_1 \rangle\rangle) \;; \\
& \quad (M[0, \langle\langle tt_p \rangle\rangle, \langle\langle tt_0 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$tr_{<}, \$tr', \$0-tr, \$1-tr])) \wedge \$tr' =_u \$tr_{<} + \\
& \langle\langle tt_p \rangle\rangle) \\
& \text{by (rel-auto, metis left-cancel-monoid-class.add-left-imp-eq, blast)} \\
& \text{also have } \dots = R2(P \parallel_M Q) \\
& \text{by (rel-auto, blast, metis diff-add-cancel-left')} \\
& \text{finally show ?thesis ..} \\
& \text{qed}
\end{aligned}$$

**lemma** *R2m-R2m'-pbm*:  $(R2(P) \parallel_{R2m(M)} R2(Q)) = (R2(P) \parallel_{R2m'(M)} R2(Q))$   
**by** (rel-blast)

**lemma** *R2-par-by-merge [closure]*:  
**assumes** *P is R2 Q is R2 M is R2m*  
**shows**  $(P \parallel_M Q) \text{ is } R2$   
**by** (metis *Healthy-def' R2-R2m'-pbm R2m-R2m'-pbm assms(1) assms(2) assms(3)*)

**lemma** *R2-par-by-merge' [closure]*:  
**assumes** *P is R2 Q is R2 M is R2m'*  
**shows**  $(P \parallel_M Q) \text{ is } R2$   
**by** (metis *Healthy-def' R2-R2m'-pbm assms(1) assms(2) assms(3)*)

**lemma** *R1m-skip-merge*:  $R1m(skip_m) = skip_m$   
**by** (*rel-auto*)

**lemma** *R1m-disj*:  $R1m(P \vee Q) = (R1m(P) \vee R1m(Q))$   
**by** (*rel-auto*)

**lemma** *R1m-conj*:  $R1m(P \wedge Q) = (R1m(P) \wedge R1m(Q))$   
**by** (*rel-auto*)

**lemma** *R2m-skip-merge*:  $R2m(skip_m) = skip_m$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

**lemma** *R2m-disj*:  $R2m(P \vee Q) = (R2m(P) \vee R2m(Q))$   
**by** (*rel-auto*)

**lemma** *R2m-conj*:  $R2m(P \wedge Q) = (R2m(P) \wedge R2m(Q))$   
**by** (*rel-auto*)

**definition** *R3m* ::  $(t :: trace, 'a) \text{ rp merge} \Rightarrow (t, 'a) \text{ rp merge}$  **where**  
 $[upred-defs]: R3m(M) = skip_m \triangleleft \$wait_{<} \triangleright M$

**lemma** *R3-par-by-merge*:

**assumes**

*P is R3 Q is R3 M is R3m*

**shows**  $(P \parallel_M Q) \text{ is } R3$

**proof** –

**have**  $(P \parallel_M Q) = ((P \parallel_M Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*metis cond-L6 cond-var-split in-var-uvar wait-vwb-lens*)

**also have**  $\dots = (((R3\ P)[\![true/\$wait]\!] \parallel_{(R3m\ M)[\![true/\$wait_{<}]}\ (R3\ Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*subst-tac, simp add: Healthy-if assms*)

**also have**  $\dots = ((II[\![true/\$wait]\!] \parallel_{skip_m[\![true/\$wait_{<}]}\ II[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*simp add: R3-def R3m-def usubst*)

**also have**  $\dots = ((II \parallel_{skip_m} II)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*subst-tac*)

**also have**  $\dots = (II \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*simp add: cond-var-subst-left par-by-merge-skip*)

**also have**  $\dots = R3(P \parallel_M Q)$

**by** (*simp add: R3-def*)

**finally show** *?thesis*

**by** (*simp add: Healthy-def*)

**qed**

**lemma** *SymMerge-R1-true* [*closure*]:

*M is SymMerge*  $\implies M \;; R1(true)$  *is SymMerge*

**by** (*rel-auto*)

**end**

## 6 Reactive Relations

**theory** *utp-rea-rel*

**imports**

*utp-rea-healths*

*UTP-KAT.utp-kleene*



**begin**

This theory defines a reactive relational calculus for  $R1$ - $R2$  predicates as an extension of the standard alphabetised predicate calculus. This enables us to formally characterise relational programs that refer to both state variables and a trace history. For more details on reactive relations, please see the associated journal paper [3].

## 6.1 Healthiness Conditions

**definition**  $RR :: ('t::trace, 'α, 'β) rel-rp \Rightarrow ('t, 'α, 'β) rel-rp$  **where**  
 $[upred-defs]: RR(P) = (\exists \{ \$ok, \$ok', \$wait, \$wait' \} \cdot R2(P))$

**lemma**  $RR-idem: RR(RR(P)) = RR(P)$   
**by** (*rel-auto*)

**lemma**  $RR-Idempotent [closure]: Idempotent\ RR$   
**by** (*simp add: Idempotent-def RR-idem*)

**lemma**  $RR-Continuous [closure]: Continuous\ RR$   
**by** (*rel-blast*)

**lemma**  $R1-RR: R1(RR(P)) = RR(P)$   
**by** (*rel-auto*)

**lemma**  $R2c-RR: R2c(RR(P)) = RR(P)$   
**by** (*rel-auto*)

**lemma**  $RR-implies-R1 [closure]: P\ is\ RR \Longrightarrow P\ is\ R1$   
**by** (*metis Healthy-def R1-RR*)

**lemma**  $RR-implies-R2c: P\ is\ RR \Longrightarrow P\ is\ R2c$   
**by** (*metis Healthy-def R2c-RR*)

**lemma**  $RR-implies-R2 [closure]: P\ is\ RR \Longrightarrow P\ is\ R2$   
**by** (*metis Healthy-def R1-RR R2-R2c-def R2c-RR*)

**lemma**  $RR-intro:$   
**assumes**  $\$ok \# P\ \$ok' \# P\ \$wait \# P\ \$wait' \# P\ P\ is\ R1\ P\ is\ R2c$   
**shows**  $P\ is\ RR$   
**by** (*simp add: RR-def Healthy-def ex-plus R2-R2c-def, simp add: Healthy-if assms ex-unrest*)

**lemma**  $RR-R2-intro:$   
**assumes**  $\$ok \# P\ \$ok' \# P\ \$wait \# P\ \$wait' \# P\ P\ is\ R2$   
**shows**  $P\ is\ RR$   
**by** (*simp add: RR-def Healthy-def ex-plus, simp add: Healthy-if assms ex-unrest*)

**lemma**  $RR-unrests [unrest]:$   
**assumes**  $P\ is\ RR$   
**shows**  $\$ok \# P\ \$ok' \# P\ \$wait \# P\ \$wait' \# P$   
**proof** –  
**have**  $\$ok \# RR(P)\ \$ok' \# RR(P)\ \$wait \# RR(P)\ \$wait' \# RR(P)$   
**by** (*simp-all add: RR-def ex-plus unrest*)  
**thus**  $\$ok \# P\ \$ok' \# P\ \$wait \# P\ \$wait' \# P$   
**by** (*simp-all add: assms Healthy-if*)  
**qed**

**lemma** *RR-refine-intro*:  
 assumes  $P$  is *RR*  $Q$  is *RR*  $\wedge t. P \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket \sqsubseteq Q \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket$   
 shows  $P \sqsubseteq Q$   
**proof** –  
 have  $\wedge t. (RR\ P) \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket \sqsubseteq (RR\ Q) \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket$   
   by (*simp add: Healthy-if assms*)  
 hence  $RR(P) \sqsubseteq RR(Q)$   
   by (*rel-auto*)  
 thus ?thesis  
   by (*simp add: Healthy-if assms*)  
**qed**

**lemma** *R4-RR-closed* [*closure*]:  
 assumes  $P$  is *RR*  
 shows  $R_4(P)$  is *RR*  
**proof** –  
 have  $R_4(RR(P))$  is *RR*  
   by (*rel-blast*)  
 thus ?thesis  
   by (*simp add: Healthy-if assms*)  
**qed**

**lemma** *R5-RR-closed* [*closure*]:  
 assumes  $P$  is *RR*  
 shows  $R_5(P)$  is *RR*  
**proof** –  
 have  $R_5(RR(P))$  is *RR*  
   using *minus-zero-eq* by *rel-auto*  
 thus ?thesis  
   by (*simp add: Healthy-if assms*)  
**qed**

## 6.2 Reactive relational operators

**named-theorems** *rpred*

**abbreviation** *rea-true* ::  $(t::trace, 'α, 'β)$  *rel-rp* ( $true_r$ ) **where**  
 $true_r \equiv R1(true)$

**definition** *rea-not* ::  $(t::trace, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp*  $(\neg_r - [40] 40)$   
**where** [*upred-defs*]:  $(\neg_r P) = R1(\neg P)$

**definition** *rea-diff* ::  $(t::trace, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp* (**infixl**  $\neg_r$  65)  
**where** [*upred-defs*]:  $rea\_diff\ P\ Q = (P \wedge \neg_r Q)$

**definition** *rea-impl* ::  
 $(t::trace, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp* (**infixr**  $\Rightarrow_r$  25)  
**where** [*upred-defs*]:  $(P \Rightarrow_r Q) = (\neg_r P \vee Q)$

**definition** *rea-lift* ::  $(t::trace, 'α, 'β)$  *rel-rp*  $\Rightarrow (t, 'α, 'β)$  *rel-rp*  $([ ]_r)$   
**where** [*upred-defs*]:  $[P]_r = R1(P)$

**definition** *rea-skip* ::  $(t::trace, 'α)$  *hrel-rp* ( $II_r$ )  
**where** [*upred-defs*]:  $II_r = (\$tr' =_u \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R)$

**definition** *rea-assert* :: ( $t::\text{trace}, \alpha$ ) *hrel-rp*  $\Rightarrow$  ( $t, \alpha$ ) *hrel-rp* ( $\{-\}_r$ )  
**where** [*upred-defs*]:  $\{b\}_r = (II_r \vee \neg_r b)$

Convert from one trace algebra to another using renamer functions, which are a kind of monoid homomorphism.

**locale** *renamer* =  
**fixes**  $f :: 'a::\text{trace} \Rightarrow 'b::\text{trace}$   
**assumes**  
*injective*: *inj*  $f$  **and**  
*add*:  $f (x + y) = f x + f y$   
**begin**  
**lemma** *zero*:  $f 0 = 0$   
**by** (*metis add add.right-neutral add-monoid-diff-cancel-left*)  
  
**lemma** *monotonic*: *mono*  $f$   
**by** (*metis add monoI trace-class.le-iff-add*)  
  
**lemma** *minus*:  $x \leq y \Longrightarrow f (y - x) = f(y) - f(x)$   
**by** (*metis add diff-add-cancel-left' trace-class.add-diff-cancel-left*)  
**end**  
  
**declare** *renamer.add* [*simp*]  
**declare** *renamer.zero* [*simp*]  
**declare** *renamer.minus* [*simp*]

**lemma** *renamer-id*: *renamer id*  
**by** (*unfold-locales, simp-all*)

**lemma** *renamer-comp*:  $\llbracket \text{renamer } f; \text{renamer } g \rrbracket \Longrightarrow \text{renamer } (f \circ g)$   
**by** (*unfold-locales, simp-all add: inj-compose renamer.injective*)

**lemma** *renamer-map*: *inj*  $f \Longrightarrow \text{renamer } (\text{map } f)$   
**by** (*unfold-locales, simp-all add: plus-list-def*)

**definition** *rea-rename* :: ( $t_1::\text{trace}, \alpha$ ) *hrel-rp*  $\Rightarrow$  ( $t_1 \Rightarrow t_2$ )  $\Rightarrow$  ( $t_2::\text{trace}, \alpha$ ) *hrel-rp* ( $(\cdot) \llbracket \cdot \rrbracket_r$  [999, 0]  
999) **where**  
[*upred-defs*]: *rea-rename*  $P f = R2((\$tr' =_u 0 \wedge \$\Sigma_R' =_u \$\Sigma_R) ;; P ;; (\$tr' =_u \text{uop } f \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R))$

Trace contribution substitution: make a substitution for the trace contribution lens *tt*, and apply *R1* to make the resulting predicate healthy again.

**definition** *rea-subst* :: ( $t::\text{trace}, \alpha$ ) *hrel-rp*  $\Rightarrow$  ( $t, (t, \alpha) \text{rp}$ ) *hexpr*  $\Rightarrow$  ( $t, \alpha$ ) *hrel-rp* ( $(\cdot) \llbracket \cdot \rrbracket_r$  [999, 0]  
999)  
**where** [*upred-defs*]:  $P \llbracket v \rrbracket_r = R1(P \llbracket v / \&tt \rrbracket)$

### 6.3 Unrestriction and substitution laws

**lemma** *rea-true-unrest* [*unrest*]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v \rrbracket \Longrightarrow x \# \text{true}_r$   
**by** (*simp add: R1-def unrest lens-indep-sym*)

**lemma** *rea-not-unrest* [*unrest*]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \# P \rrbracket \Longrightarrow x \# \neg_r P$   
**by** (*simp add: rea-not-def R1-def unrest lens-indep-sym*)

**lemma** *rea-impl-unrest* [*unrest*]:

$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \# P; x \# Q \rrbracket \Longrightarrow x \# (P \Rightarrow_r Q)$   
**by** (*simp add: rea-impl-def unrest*)

**lemma** *rea-true-usubst* [*usubst*]:

$\llbracket \$tr \# \sigma; \$tr' \# \sigma \rrbracket \Longrightarrow \sigma \dagger true_r = true_r$   
**by** (*simp add: R1-def usubst*)

**lemma** *rea-not-usubst* [*usubst*]:

$\llbracket \$tr \# \sigma; \$tr' \# \sigma \rrbracket \Longrightarrow \sigma \dagger (\neg_r P) = (\neg_r \sigma \dagger P)$   
**by** (*simp add: rea-not-def R1-def usubst*)

**lemma** *rea-impl-usubst* [*usubst*]:

$\llbracket \$tr \# \sigma; \$tr' \# \sigma \rrbracket \Longrightarrow \sigma \dagger (P \Rightarrow_r Q) = (\sigma \dagger P \Rightarrow_r \sigma \dagger Q)$   
**by** (*simp add: rea-impl-def usubst R1-def*)

**lemma** *rea-true-usubst-tt* [*usubst*]:

$R1(true) \llbracket e/\&tt \rrbracket = true$   
**by** (*rel-simp*)

**lemma** *unrests-rea-rename* [*unrest*]:

$\$ok \# P \Longrightarrow \$ok \# P \llbracket f \rrbracket_r$   
 $\$ok' \# P \Longrightarrow \$ok' \# P \llbracket f \rrbracket_r$   
 $\$wait \# P \Longrightarrow \$wait \# P \llbracket f \rrbracket_r$   
 $\$wait' \# P \Longrightarrow \$wait' \# P \llbracket f \rrbracket_r$   
**by** (*simp-all add: rea-rename-def R2-def unrest*)

**lemma** *unrest-rea-subst* [*unrest*]:

$\llbracket mwb\text{-}lens\ x; x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \# v; x \# P \rrbracket \Longrightarrow x \# P \llbracket v \rrbracket_r$   
**by** (*simp add: rea-subst-def R1-def unrest lens-indep-sym*)

**lemma** *rea-substs* [*usubst*]:

$true_r \llbracket v \rrbracket_r = true_r$   $true \llbracket v \rrbracket_r = true_r$   $false \llbracket v \rrbracket_r = false$   
 $(\neg_r P) \llbracket v \rrbracket_r = (\neg_r P \llbracket v \rrbracket_r)$   $(P \wedge Q) \llbracket v \rrbracket_r = (P \llbracket v \rrbracket_r \wedge Q \llbracket v \rrbracket_r)$   $(P \vee Q) \llbracket v \rrbracket_r = (P \llbracket v \rrbracket_r \vee Q \llbracket v \rrbracket_r)$   
 $(P \Rightarrow_r Q) \llbracket v \rrbracket_r = (P \llbracket v \rrbracket_r \Rightarrow_r Q \llbracket v \rrbracket_r)$   
**by** *rel-auto+*

**lemma** *rea-substs-lattice* [*usubst*]:

$(\bigcap i \cdot P(i)) \llbracket v \rrbracket_r = (\bigcap i \cdot (P(i)) \llbracket v \rrbracket_r)$   
 $(\bigcap i \in A \cdot P(i)) \llbracket v \rrbracket_r = (\bigcap i \in A \cdot (P(i)) \llbracket v \rrbracket_r)$   
 $(\bigcup i \cdot P(i)) \llbracket v \rrbracket_r = (\bigcup i \cdot (P(i)) \llbracket v \rrbracket_r)$   
**by** (*rel-auto*) $+$

**lemma** *rea-subst-USUP-set* [*usubst*]:

$A \neq \{\} \Longrightarrow (\bigcup i \in A \cdot P(i)) \llbracket v \rrbracket_r = (\bigcup i \in A \cdot (P(i)) \llbracket v \rrbracket_r)$   
**by** (*rel-auto*) $+$

## 6.4 Closure laws

**lemma** *rea-lift-R1* [*closure*]:  $[P]_r$  is *R1*

**by** (*rel-simp*)

**lemma** *R1-rea-not*:  $R1(\neg_r P) = (\neg_r P)$

**by** *rel-auto*

**lemma** *R1-rea-not'*:  $R1(\neg_r P) = (\neg_r R1(P))$

by *rel-auto*

**lemma** *R2c-rea-not*:  $R2c(\neg_r P) = (\neg_r R2c(P))$   
by *rel-auto*

**lemma** *RR-rea-not*:  $RR(\neg_r RR(P)) = (\neg_r RR(P))$   
by *(rel-auto)*

**lemma** *R1-rea-impl*:  $R1(P \Rightarrow_r Q) = (P \Rightarrow_r R1(Q))$   
by *(rel-auto)*

**lemma** *R1-rea-impl'*:  $R1(P \Rightarrow_r Q) = (R1(P) \Rightarrow_r R1(Q))$   
by *(rel-auto)*

**lemma** *R2c-rea-impl*:  $R2c(P \Rightarrow_r Q) = (R2c(P) \Rightarrow_r R2c(Q))$   
by *(rel-auto)*

**lemma** *RR-rea-impl*:  $RR(RR(P) \Rightarrow_r RR(Q)) = (RR(P) \Rightarrow_r RR(Q))$   
by *(rel-auto)*

**lemma** *rea-true-R1* [*closure*]: *true<sub>r</sub> is R1*  
by *(rel-auto)*

**lemma** *rea-true-R2c* [*closure*]: *true<sub>r</sub> is R2c*  
by *(rel-auto)*

**lemma** *rea-true-RR* [*closure*]: *true<sub>r</sub> is RR*  
by *(rel-auto)*

**lemma** *rea-not-R1* [*closure*]:  $\neg_r P$  *is R1*  
by *(rel-auto)*

**lemma** *rea-not-R2c* [*closure*]:  $P$  *is R2c*  $\implies \neg_r P$  *is R2c*  
by (*simp add: Healthy-def rea-not-def R1-R2c-commute*[*THEN sym*] *R2c-not*)

**lemma** *rea-not-R2-closed* [*closure*]:  
 $P$  *is R2*  $\implies (\neg_r P)$  *is R2*  
by (*simp add: Healthy-def' R1-rea-not' R2-R2c-def R2c-rea-not*)

**lemma** *rea-no-RR* [*closure*]:  
 $\llbracket P \text{ is } RR \rrbracket \implies (\neg_r P)$  *is RR*  
by (*metis Healthy-def' RR-rea-not*)

**lemma** *rea-impl-R1* [*closure*]:  
 $Q$  *is R1*  $\implies (P \Rightarrow_r Q)$  *is R1*  
by *(rel-blast)*

**lemma** *rea-impl-R2c* [*closure*]:  
 $\llbracket P \text{ is } R2c; Q \text{ is } R2c \rrbracket \implies (P \Rightarrow_r Q)$  *is R2c*  
by (*simp add: rea-impl-def Healthy-def rea-not-def R1-R2c-commute*[*THEN sym*] *R2c-not R2c-disj*)

**lemma** *rea-impl-R2* [*closure*]:  
 $\llbracket P \text{ is } R2; Q \text{ is } R2 \rrbracket \implies (P \Rightarrow_r Q)$  *is R2*  
by *(rel-blast)*

**lemma** *rea-impl-RR* [closure]:

$\llbracket P \text{ is } RR; Q \text{ is } RR \rrbracket \implies (P \Rightarrow_r Q) \text{ is } RR$   
**by** (*metis Healthy-def' RR-rea-impl*)

**lemma** *conj-RR* [closure]:

$\llbracket P \text{ is } RR; Q \text{ is } RR \rrbracket \implies (P \wedge Q) \text{ is } RR$

**by** (*meson RR-implies-R1 RR-implies-R2c RR-intro RR-unrests(1-4) conj-R1-closed-1 conj-R2c-closed unrest-conj*)

**lemma** *disj-RR* [closure]:

$\llbracket P \text{ is } RR; Q \text{ is } RR \rrbracket \implies (P \vee Q) \text{ is } RR$

**by** (*metis Healthy-def' R1-RR R1-idem R1-rea-not' RR-rea-impl RR-rea-not disj-comm double-negation rea-impl-def rea-not-def*)

**lemma** *USUP-mem-RR-closed* [closure]:

**assumes**  $\bigwedge i. i \in A \implies P \ i \text{ is } RR \ A \neq \{\}$

**shows**  $(\bigsqcup_{i \in A} P(i)) \text{ is } RR$

**proof** –

**have** 1:  $(\bigsqcup_{i \in A} P(i)) \text{ is } R1$

**by** (*unfold Healthy-def, subst R1-UNIF, simp-all add: Healthy-if assms closure cong: USUP-cong*)

**have** 2:  $(\bigsqcup_{i \in A} P(i)) \text{ is } R2c$

**by** (*unfold Healthy-def, subst R2c-UNIF, simp-all add: Healthy-if assms RR-implies-R2c closure cong: USUP-cong*)

**show** ?thesis

**using** 1 2 **by** (*rule-tac RR-intro, simp-all add: unrest assms*)

**qed**

**lemma** *USUP-ind-RR-closed* [closure]:

**assumes**  $\bigwedge i. P \ i \text{ is } RR$

**shows**  $(\bigsqcup i \cdot P(i)) \text{ is } RR$

**using** *USUP-mem-RR-closed*[of *UNIV P*] **by** (*simp add: assms*)

**lemma** *UNIF-mem-RR-closed* [closure]:

**assumes**  $\bigwedge i. i \in A \implies P \ i \text{ is } RR$

**shows**  $(\bigsqcap_{i \in A} P(i)) \text{ is } RR$

**proof** –

**have** 1:  $(\bigsqcap_{i \in A} P(i)) \text{ is } R1$

**by** (*unfold Healthy-def, subst R1-USUP, simp add: Healthy-if RR-implies-R1 assms cong: UNIF-cong*)

**have** 2:  $(\bigsqcap_{i \in A} P(i)) \text{ is } R2c$

**by** (*unfold Healthy-def, subst R2c-USUP, simp add: Healthy-if RR-implies-R2c assms cong: UNIF-cong*)

**show** ?thesis

**using** 1 2 **by** (*rule-tac RR-intro, simp-all add: unrest assms*)

**qed**

**lemma** *UNIF-ind-RR-closed* [closure]:

**assumes**  $\bigwedge i. P \ i \text{ is } RR$

**shows**  $(\bigsqcap i \cdot P(i)) \text{ is } RR$

**by** (*simp add: assms closure*)

**lemma** *USUP-elem-RR* [closure]:

**assumes**  $\bigwedge i. P \ i \text{ is } RR \ A \neq \{\}$

**shows**  $(\bigsqcup i \in A \cdot P \ i) \text{ is } RR$

**proof** –

**have** 1:  $(\bigsqcup_{i \in A} P(i)) \text{ is } R1$

**by** (*unfold Healthy-def, subst R1-UNIF, simp-all add: Healthy-if assms closure*)

```

have 2:( $\bigwedge i \in A \cdot P(i)$ ) is R2c
  by (unfold Healthy-def, subst R2c-UINF, simp-all add: Healthy-if assms RR-implies-R2c closure)
show ?thesis
  using 1 2 by (rule-tac RR-intro, simp-all add: unrest assms)
qed

lemma seq-RR-closed [closure]:
  assumes  $P$  is RR  $Q$  is RR
  shows  $P ;; Q$  is RR
  unfolding Healthy-def
  by (simp add: RR-def Healthy-if assms closure RR-implies-R2 ex-unrest unrest)

lemma power-Suc-RR-closed [closure]:
   $P$  is RR  $\implies P ;; P \wedge i$  is RR
  by (induct i, simp-all add: closure upred-semiring.power-Suc)

lemma seqr-iter-RR-closed [closure]:
   $\llbracket I \neq []; \bigwedge i. i \in \text{set}(I) \implies P(i) \text{ is RR} \rrbracket \implies (i : I \cdot P(i)) \text{ is RR}$ 
  apply (induct I, simp-all)
  apply (rename-tac i I)
  apply (case-tac I)
  apply (simp-all add: seq-RR-closed)
done

lemma cond-tt-RR-closed [closure]:
  assumes  $P$  is RR  $Q$  is RR
  shows  $P \triangleleft \$tr' =_u \$tr \triangleright Q$  is RR
  apply (rule RR-intro)
  apply (simp-all add: unrest assms)
  apply (simp-all add: Healthy-def)
  apply (simp-all add: R1-cond R2c-condr Healthy-if assms RR-implies-R2c closure R2c-tr'-minus-tr)
done

lemma rea-skip-RR [closure]:
   $II_r$  is RR
  apply (rel-auto) using minus-zero-eq by blast

lemma tr'-eq-tr-RR-closed [closure]:  $\$tr' =_u \$tr$  is RR
  apply (rel-auto) using minus-zero-eq by auto

lemma inf-RR-closed [closure]:
   $\llbracket P \text{ is RR}; Q \text{ is RR} \rrbracket \implies P \sqcap Q \text{ is RR}$ 
  by (simp add: disj-RR uinf-or)

lemma conj-tr-strict-RR-closed [closure]:
  assumes  $P$  is RR
  shows  $(P \wedge \$tr <_u \$tr')$  is RR
proof -
  have  $RR(RR(P) \wedge \$tr <_u \$tr') = (RR(P) \wedge \$tr <_u \$tr')$ 
  by (rel-auto)
  thus ?thesis
  by (metis Healthy-def assms)
qed

lemma rea-assert-RR-closed [closure]:

```

**assumes**  $b$  is  $RR$   
**shows**  $\{b\}_r$  is  $RR$   
**by** (*simp add: closure assms rea-assert-def*)

**lemma** *upower-RR-closed* [*closure*]:  
 $\llbracket i > 0; P \text{ is } RR \rrbracket \implies P \wedge i \text{ is } RR$   
**apply** (*induct i, simp-all*)  
**apply** (*rename-tac i*)  
**apply** (*case-tac i = 0*)  
**apply** (*simp-all add: closure upred-semiring.power-Suc*)  
**done**

**lemma** *seq-power-RR-closed* [*closure*]:  
**assumes**  $P$  is  $RR$   $Q$  is  $RR$   
**shows**  $(P \wedge i) ;; Q$  is  $RR$   
**by** (*metis assms neq0-conv seq-RR-closed seqr-left-unit upower-RR-closed upred-semiring.power-0*)

**lemma** *ustar-right-RR-closed* [*closure*]:  
**assumes**  $P$  is  $RR$   $Q$  is  $RR$   
**shows**  $P ;; Q^*$  is  $RR$   
**proof** –  
**have**  $P ;; Q^* = P ;; (\bigsqcap i \in \{0..\} \cdot Q \wedge i)$   
**by** (*simp add: ustar-def*)  
**also have**  $\dots = P ;; (II \sqcap (\bigsqcap i \in \{1..\} \cdot Q \wedge i))$   
**by** (*metis One-nat-def UINF-atLeast-first upred-semiring.power-0*)  
**also have**  $\dots = (P \vee P ;; (\bigsqcap i \in \{1..\} \cdot Q \wedge i))$   
**by** (*simp add: disj-upred-def[THEN sym] seqr-or-distr*)  
**also have**  $\dots$  is  $RR$   
**proof** –  
**have**  $(\bigsqcap i \in \{1..\} \cdot Q \wedge i)$  is  $RR$   
**by** (*rule UINF-mem-Continuous-closed, simp-all add: assms closure*)  
**thus** ?thesis  
**by** (*simp add: assms closure*)  
**qed**  
**finally show** ?thesis .  
**qed**

**lemma** *ustar-left-RR-closed* [*closure*]:  
**assumes**  $P$  is  $RR$   $Q$  is  $RR$   
**shows**  $P^* ;; Q$  is  $RR$   
**proof** –  
**have**  $P^* ;; Q = (\bigsqcap i \in \{0..\} \cdot P \wedge i) ;; Q$   
**by** (*simp add: ustar-def*)  
**also have**  $\dots = (II \sqcap (\bigsqcap i \in \{1..\} \cdot P \wedge i)) ;; Q$   
**by** (*metis One-nat-def UINF-atLeast-first upred-semiring.power-0*)  
**also have**  $\dots = (Q \vee (\bigsqcap i \in \{1..\} \cdot P \wedge i) ;; Q)$   
**by** (*simp add: disj-upred-def[THEN sym] seqr-or-distl*)  
**also have**  $\dots$  is  $RR$   
**proof** –  
**have**  $(\bigsqcap i \in \{1..\} \cdot P \wedge i)$  is  $RR$   
**by** (*rule UINF-mem-Continuous-closed, simp-all add: assms closure*)  
**thus** ?thesis  
**by** (*simp add: assms closure*)  
**qed**  
**finally show** ?thesis .



qed

**lemma** *uplus-RR-closed* [closure]:  $P$  is  $RR \implies P^+$  is  $RR$   
 by (simp add: uplus-def ustar-right-RR-closed)

**lemma** *trace-ext-prefix-RR* [closure]:  
 $\llbracket \$tr \# e; \$ok \# e; \$wait \# e; out\alpha \# e \rrbracket \implies \$tr \hat{\ }_u e \leq_u \$tr' \text{ is } RR$   
 apply (rel-auto)  
 apply (metis (no-types, lifting) Prefix-Order.same-prefix-prefix less-eq-list-def prefix-concat-minus zero-list-def)  
 apply (metis append-minus list-append-prefixD minus-cancel-le order-refl)  
 done

**lemma** *rea-subst-R1-closed* [closure]:  $P\llbracket v \rrbracket_r$  is  $R1$   
 by (rel-auto)

**lemma** *R5-comp* [rpred]:  
 assumes  $P$  is  $RR$   $Q$  is  $RR$   
 shows  $R5(P ;; Q) = R5(P) ;; R5(Q)$   
**proof** –  
 have  $R5(RR(P) ;; RR(Q)) = R5(RR(P)) ;; R5(RR(Q))$   
 by (rel-auto; force)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

**lemma** *R4-comp* [rpred]:  
 assumes  $P$  is  $R4$   $Q$  is  $RR$   
 shows  $R4(P ;; Q) = P ;; Q$   
**proof** –  
 have  $R4(R4(P) ;; RR(Q)) = R4(P) ;; RR(Q)$   
 by (rel-auto, blast)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

**lemma** *rea-rename-RR-closed* [closure]:  
 assumes  $P$  is  $RR$   
 shows  $P\llbracket f \rrbracket_r$  is  $RR$   
**proof** –  
 have  $(RR P)\llbracket f \rrbracket_r$  is  $RR$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

## 6.5 Reactive relational calculus

**lemma** *rea-skip-unit* [rpred]:  
 assumes  $P$  is  $RR$   
 shows  $P ;; II_r = P II_r ;; P = P$   
**proof** –  
 have 1:  $RR(P) ;; II_r = RR(P)$   
 by (rel-auto)  
 have 2:  $II_r ;; RR(P) = RR(P)$   
 by (rel-auto)  
 from 1 2 show  $P ;; II_r = P II_r ;; P = P$

by (*simp-all add: Healthy-if assms*)  
qed

**lemma** *rea-true-conj* [*rpred*]:  
**assumes** *P is R1*  
**shows**  $(\text{true}_r \wedge P) = P \ (P \wedge \text{true}_r) = P$   
**using** *assms*  
**by** (*simp-all add: Healthy-def R1-def utp-pred-laws.inf-commute*)

**lemma** *rea-true-disj* [*rpred*]:  
**assumes** *P is R1*  
**shows**  $(\text{true}_r \vee P) = \text{true}_r \ (P \vee \text{true}_r) = \text{true}_r$   
**using** *assms* **by** (*metis Healthy-def R1-disj disj-comm true-disj-zero*)<sup>+</sup>

**lemma** *rea-not-not* [*rpred*]: *P is R1*  $\implies (\neg_r \neg_r P) = P$   
**by** (*simp add: rea-not-def R1-negate-R1 Healthy-if*)

**lemma** *rea-not-rea-true* [*simp*]:  $(\neg_r \text{true}_r) = \text{false}$   
**by** (*simp add: rea-not-def R1-negate-R1 R1-false*)

**lemma** *rea-not-false* [*simp*]:  $(\neg_r \text{false}) = \text{true}_r$   
**by** (*simp add: rea-not-def*)

**lemma** *rea-true-impl* [*rpred*]:  
*P is R1*  $\implies (\text{true}_r \Rightarrow_r P) = P$   
**by** (*simp add: rea-not-def rea-impl-def R1-negate-R1 R1-false Healthy-if*)

**lemma** *rea-true-impl'* [*rpred*]:  
*P is R1*  $\implies (\text{true} \Rightarrow_r P) = P$   
**by** (*simp add: rea-not-def rea-impl-def R1-negate-R1 R1-false Healthy-if*)

**lemma** *rea-false-impl* [*rpred*]:  
*P is R1*  $\implies (\text{false} \Rightarrow_r P) = \text{true}_r$   
**by** (*simp add: rea-impl-def rpred Healthy-if*)

**lemma** *rea-impl-true* [*simp*]:  $(P \Rightarrow_r \text{true}_r) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *rea-impl-false* [*simp*]:  $(P \Rightarrow_r \text{false}) = (\neg_r P)$   
**by** (*rel-simp*)

**lemma** *rea-imp-refl* [*rpred*]: *P is R1*  $\implies (P \Rightarrow_r P) = \text{true}_r$   
**by** (*rel-blast*)

**lemma** *rea-impl-conj* [*rpred*]:  
 $(P \Rightarrow_r Q \Rightarrow_r R) = ((P \wedge Q) \Rightarrow_r R)$   
**by** (*rel-auto*)

**lemma** *rea-impl-mp* [*rpred*]:  
 $(P \wedge (P \Rightarrow_r Q)) = (P \wedge Q)$   
**by** (*rel-auto*)

**lemma** *rea-impl-conj-combine* [*rpred*]:  
 $((P \Rightarrow_r Q) \wedge (P \Rightarrow_r R)) = (P \Rightarrow_r Q \wedge R)$   
**by** (*rel-auto*)

**lemma** *rea-impl-alt-def*:  
 assumes  $Q$  is  $R1$   
 shows  $(P \Rightarrow_r Q) = R1(P \Rightarrow Q)$   
**proof** –  
 have  $(P \Rightarrow_r R1(Q)) = R1(P \Rightarrow Q)$   
 by (*rel-auto*)  
 thus ?thesis  
 by (*simp add: assms Healthy-if*)  
**qed**

**lemma** *rea-impl-disj*:  
 $(P \Rightarrow_r Q \vee R) = (Q \vee (P \Rightarrow_r R))$   
 by (*rel-auto*)

**lemma** *rea-not-true* [*simp*]:  $(\neg_r \text{true}) = \text{false}$   
 by (*rel-auto*)

**lemma** *rea-not-demorgan1* [*simp*]:  
 $(\neg_r (P \wedge Q)) = (\neg_r P \vee \neg_r Q)$   
 by (*rel-auto*)

**lemma** *rea-not-demorgan2* [*simp*]:  
 $(\neg_r (P \vee Q)) = (\neg_r P \wedge \neg_r Q)$   
 by (*rel-auto*)

**lemma** *rea-not-or* [*rpred*]:  
 $P$  is  $R1 \implies (P \vee \neg_r P) = \text{true}_r$   
 by (*rel-blast*)

**lemma** *rea-not-and* [*simp*]:  
 $(P \wedge \neg_r P) = \text{false}$   
 by (*rel-auto*)

**lemma** *truer-bottom-rpred* [*rpred*]:  $P$  is  $RR \implies R1(\text{true}) \sqsubseteq P$   
 by (*metis Healthy-def R1-RR R1-mono utp-pred-laws.top-greatest*)

**lemma** *ext-close-weakening*:  $P ;; \text{true}_r \sqsubseteq P$   
 by (*rel-auto*)

**lemma** *rea-not-INFIMUM* [*simp*]:  
 $(\neg_r (\bigsqcup_{i \in A} Q(i))) = (\bigcap_{i \in A} \neg_r Q(i))$   
 by (*rel-auto*)

**lemma** *rea-not-USUP* [*simp*]:  
 $(\neg_r (\bigsqcup_{i \in A} Q(i))) = (\bigcap_{i \in A} \neg_r Q(i))$   
 by (*rel-auto*)

**lemma** *rea-not-SUPREMUM* [*simp*]:  
 $A \neq \{\} \implies (\neg_r (\bigcap_{i \in A} Q(i))) = (\bigsqcup_{i \in A} \neg_r Q(i))$   
 by (*rel-auto*)

**lemma** *rea-not-UNIF* [*simp*]:  
 $A \neq \{\} \implies (\neg_r (\bigcap_{i \in A} Q(i))) = (\bigsqcup_{i \in A} \neg_r Q(i))$   
 by (*rel-auto*)

**lemma** *USUP-mem-rea-true* [simp]:  $A \neq \{\}$   $\implies (\bigsqcup i \in A \cdot \text{true}_r) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *USUP-ind-rea-true* [simp]:  $(\bigsqcup i \cdot \text{true}_r) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *UINF-ind-rea-true* [rpred]:  $A \neq \{\}$   $\implies (\prod i \in A \cdot \text{true}_r) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *UINF-rea-impl*:  $(\prod P \in A \cdot F(P) \Rightarrow_r G(P)) = ((\bigsqcup P \in A \cdot F(P)) \Rightarrow_r (\prod P \in A \cdot G(P)))$   
**by** (*rel-auto*)

**lemma** *rea-not-shEx* [rpred]:  $(\neg_r \text{shEx } P) = (\text{shAll } (\lambda x. \neg_r P x))$   
**by** (*rel-auto*)

**lemma** *rea-assert-true*:  
 $\{\text{true}_r\}_r = \text{II}_r$   
**by** (*rel-auto*)

**lemma** *rea-false-true*:  
 $\{\text{false}\}_r = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *rea-rename-id* [rpred]:  
**assumes**  $P$  is *RR*  
**shows**  $P(\downarrow \text{id})_r = P$   
**proof** –  
**have**  $(RR \ P)(\downarrow \text{id})_r = RR \ P$   
**by** (*rel-auto*)  
**thus** ?thesis **by** (*simp add: Healthy-if assms*)  
**qed**

**lemma** *rea-rename-comp* [rpred]:  
**assumes** *renamer*  $f$  *renamer*  $g$   $P$  is *RR*  
**shows**  $P(\downarrow g \circ f)_r = P(\downarrow g)_r(\downarrow f)_r$   
**oops**

**lemma** *rea-rename-false* [rpred]:  $\text{false}(\downarrow f)_r = \text{false}$   
**by** (*rel-auto*)

**lemma** *rea-rename-disj* [rpred]:  
 $(P \vee Q)(\downarrow f)_r = (P(\downarrow f)_r \vee Q(\downarrow f)_r)$   
**by** (*rel-blast*)

**lemma** *rea-rename-UINF-ind* [rpred]:  
 $(\prod i \cdot P \ i)(\downarrow f)_r = (\prod i \cdot (P \ i)(\downarrow f)_r)$   
**by** (*rel-blast*)

**lemma** *rea-rename-UINF-mem* [rpred]:  
 $(\prod i \in A \cdot P \ i)(\downarrow f)_r = (\prod i \in A \cdot (P \ i)(\downarrow f)_r)$   
**by** (*rel-blast*)

**lemma** *rea-rename-conj* [rpred]:  
**assumes** *renamer*  $f$   $P$  is *RR*  $Q$  is *RR*

**shows**  $(P \wedge Q)\langle f \rangle_r = (P\langle f \rangle_r \wedge Q\langle f \rangle_r)$   
**proof** –  
**interpret** *ren: renamer f* **by** (*simp add: assms*)  
**have**  $(RR\ P \wedge RR\ Q)\langle f \rangle_r = ((RR\ P)\langle f \rangle_r \wedge (RR\ Q)\langle f \rangle_r)$   
**using** *injD[OF ren.injective]*  
**by** (*rel-auto; blast*)  
**thus** *?thesis* **by** (*simp add: Healthy-if assms*)  
**qed**

**lemma** *rea-rename-USUP-ind* [*rpred*]:  
**assumes** *renamer f*  $\bigwedge i. P\ i\ is\ RR$   
**shows**  $(\bigsqcup i \cdot P\ i)\langle f \rangle_r = (\bigsqcup i \cdot (P\ i)\langle f \rangle_r)$   
**proof** –  
**interpret** *ren: renamer f* **by** (*simp add: assms*)  
**have**  $(\bigsqcup i \cdot RR(P\ i))\langle f \rangle_r = (\bigsqcup i \cdot (RR\ (P\ i))\langle f \rangle_r)$   
**using** *injD[OF ren.injective]*  
**by** (*rel-auto, blast, metis (mono-tags, hide-lams)*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms cong: USUP-all-cong*)  
**qed**

**lemma** *rea-rename-USUP-mem* [*rpred*]:  
**assumes** *renamer f*  $A \neq \{\}$   $\bigwedge i. i \in A \implies P\ i\ is\ RR$   
**shows**  $(\bigsqcup i \in A \cdot P\ i)\langle f \rangle_r = (\bigsqcup i \in A \cdot (P\ i)\langle f \rangle_r)$   
**proof** –  
**interpret** *ren: renamer f* **by** (*simp add: assms*)  
**have**  $(\bigsqcup i \in A \cdot RR(P\ i))\langle f \rangle_r = (\bigsqcup i \in A \cdot (RR\ (P\ i))\langle f \rangle_r)$   
**using** *injD[OF ren.injective] assms(2)*  
**by** (*rel-auto, blast, metis (no-types, hide-lams)*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms cong: USUP-cong*)  
**qed**

**lemma** *rea-rename-skip-rea* [*rpred*]: *renamer f*  $\implies II_r\langle f \rangle_r = II_r$   
**using** *minus-zero-eq* **by** (*rel-auto*)

**lemma** *rea-rename-seq* [*rpred*]:  
**assumes** *renamer f*  $P\ is\ RR\ Q\ is\ RR$   
**shows**  $(P\ ;;\ Q)\langle f \rangle_r = P\langle f \rangle_r\ ;;\ Q\langle f \rangle_r$   
**proof** –  
**interpret** *ren: renamer f* **by** (*simp add: assms*)  
**from** *assms(1)* **have**  $(RR(P)\ ;;\ RR(Q))\langle f \rangle_r = (RR\ P)\langle f \rangle_r\ ;;\ (RR\ Q)\langle f \rangle_r$   
**by** (*rel-auto*)  
*(metis (no-types, lifting) diff-add-cancel-left' le-add minus-assoc mono-def ren.minus ren.monotonic trace-class.add-diff-cancel-left trace-class.add-left-mono)+*  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

**declare** *R4-idem* [*rpred*]  
**declare** *R4-false* [*rpred*]  
**declare** *R4-conj* [*rpred*]  
**declare** *R4-disj* [*rpred*]

**declare** *R4-R5* [*rpred*]

**declare** *R5-R4* [*rpred*]

**declare** *R5-conj* [*rpred*]

**declare** *R5-disj* [*rpred*]

**lemma** *R4-USUP* [*rpred*]:  $I \neq \{\}$   $\implies R4(\bigsqcup_{i \in I} P(i)) = (\bigsqcup_{i \in I} R4(P(i)))$   
**by** (*rel-auto*)

**lemma** *R5-USUP* [*rpred*]:  $I \neq \{\}$   $\implies R5(\bigsqcup_{i \in I} P(i)) = (\bigsqcup_{i \in I} R5(P(i)))$   
**by** (*rel-auto*)

**lemma** *R4-UINF* [*rpred*]:  $R4(\bigsqcap_{i \in I} P(i)) = (\bigsqcap_{i \in I} R4(P(i)))$   
**by** (*rel-auto*)

**lemma** *R5-UINF* [*rpred*]:  $R5(\bigsqcap_{i \in I} P(i)) = (\bigsqcap_{i \in I} R5(P(i)))$   
**by** (*rel-auto*)

## 6.6 UTP theory

We create a UTP theory of reactive relations which in particular provides Kleene star theorems

**interpretation** *rrel-theory*: *utp-theory-kleene* *RR* *II<sub>r</sub>*  
**rewrites**  $P \in \text{carrier } rrel\text{-theory.thy-order} \longleftrightarrow P \text{ is } RR$   
**and**  $le \text{ } rrel\text{-theory.thy-order} = (\sqsubseteq)$   
**and**  $eq \text{ } rrel\text{-theory.thy-order} = (=)$   
**and** *rrel-top*: *rrel-theory.utp-top* = *false*  
**and** *rrel-bottom*: *rrel-theory.utp-bottom* = *true<sub>r</sub>*

**proof** –

**interpret** *utp-theory-continuous* *RR*  
**by** (*unfold-locales*, *simp-all add*: *add*: *RR-idem* *RR-Continuous*)  
**show** *top:utp-top* = *false*  
**by** (*simp add*: *healthy-top*, *rel-auto*)  
**show** *bot:utp-bottom* = *true<sub>r</sub>*  
**by** (*simp add*: *healthy-bottom*, *rel-auto*)  
**show** *utp-theory-kleene* *RR* *II<sub>r</sub>*  
**by** (*unfold-locales*, *simp-all add*: *closure* *rpred* *top*)  
**qed** (*simp-all*)

**abbreviation** *rea-star* ::  $- \Rightarrow -$  ( $-^{*r}$  [999] 999) **where**  
 $P^{*r} \equiv rrel\text{-theory.utp-star } P$

The supernova tactic explodes conjectures using the Kleene star laws and relational calculus

**method** *supernova* = ((*safe intro!*: *rrel-theory.Star-inductr* *rrel-theory.Star-inductl*, *simp-all add*: *closure*) ; *rel-auto*)[1]

## 6.7 Instantaneous Reactive Relations

Instantaneous Reactive Relations, where the trace stays the same.

**abbreviation** *Instant* :: ( $t::\text{trace}$ ,  $\alpha$ ) *hrel-rp*  $\Rightarrow$  ( $t$ ,  $\alpha$ ) *hrel-rp* **where**  
 $Instant(P) \equiv RID(tr)(P)$

**lemma** *skip-rea-Instant* [*closure*]: *II<sub>r</sub>* is *Instant*  
**by** (*rel-auto*)

**end**

## 7 Reactive Conditions

**theory** *utp-rea-cond*  
**imports** *utp-rea-rel*  
**begin**

### 7.1 Healthiness Conditions

**definition**  $RC1 :: ('t::trace, 'α, 'β) \text{ rel-rp} \Rightarrow ('t, 'α, 'β) \text{ rel-rp}$  **where**  
 $[upred-defs]: RC1(P) = (\neg_r (\neg_r P) ;; true_r)$

**definition**  $RC :: ('t::trace, 'α, 'β) \text{ rel-rp} \Rightarrow ('t, 'α, 'β) \text{ rel-rp}$  **where**  
 $[upred-defs]: RC = RC1 \circ RR$

**lemma**  $RC\text{-intro}$ :  $\llbracket P \text{ is } RR; ((\neg_r (\neg_r P) ;; true_r) = P) \rrbracket \Longrightarrow P \text{ is } RC$   
**by** (*simp add: Healthy-def RC1-def RC-def*)

**lemma**  $RC\text{-intro}'$ :  $\llbracket P \text{ is } RR; P \text{ is } RC1 \rrbracket \Longrightarrow P \text{ is } RC$   
**by** (*simp add: Healthy-def RC1-def RC-def*)

**lemma**  $RC1\text{-idem}$ :  $RC1(RC1(P)) = RC1(P)$   
**by** (*rel-auto, (blast intro: dual-order.trans)+*)

**lemma**  $RC1\text{-mono}$ :  $P \sqsubseteq Q \Longrightarrow RC1(P) \sqsubseteq RC1(Q)$   
**by** (*rel-blast*)

**lemma**  $RC1\text{-prop}$ :  
**assumes**  $P \text{ is } RC1$   
**shows**  $(\neg_r P) ;; R1 \text{ true} = (\neg_r P)$   
**proof** –  
**have**  $(\neg_r P) = (\neg_r (RC1 P))$   
**by** (*simp add: Healthy-if assms*)  
**also have**  $\dots = (\neg_r P) ;; R1 \text{ true}$   
**by** (*simp add: RC1-def rpred closure*)  
**finally show** *?thesis* ..  
**qed**

**lemma**  $R2\text{-RC}$ :  $R2 (RC P) = RC P$   
**proof** –  
**have**  $\neg_r RR P \text{ is } RR$   
**by** (*metis (no-types) Healthy-Idempotent RR-Idempotent RR-rea-not*)  
**then show** *?thesis*  
**by** (*metis (no-types) Healthy-def' R1-R2c-seqr-distribute R2-R2c-def RC1-def RC-def RR-implies-R1 RR-implies-R2c comp-apply rea-not-R2-closed rea-true-R1 rea-true-R2c*)  
**qed**

**lemma**  $RC\text{-R2-def}$ :  $RC = RC1 \circ RR$   
**by** (*auto simp add: RC-def fun-eq-iff R1-R2c-commute[THEN sym] R1-R2c-is-R2*)

**lemma**  $RC\text{-implies-R2}$ :  $P \text{ is } RC \Longrightarrow P \text{ is } R2$   
**by** (*metis Healthy-def' R2-RC*)

**lemma**  $RC\text{-ex-ok-wait}$ :  $(\exists \{ \$ok, \$ok', \$wait, \$wait' \} \cdot RC P) = RC P$   
**by** (*rel-auto*)

An important property of reactive conditions is they are monotonic with respect to the trace. That is,  $P$  with a shorter trace is refined by  $P$  with a longer trace.

**lemma** *RC-prefix-refine*:  
**assumes**  $P$  is *RC*  $s \leq t$   
**shows**  $P \llbracket 0, \langle s \rangle / \$tr, \$tr' \rrbracket \sqsubseteq P \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket$   
**proof** –  
**from** *assms*(2) **have**  $(RC\ P) \llbracket 0, \langle s \rangle / \$tr, \$tr' \rrbracket \sqsubseteq (RC\ P) \llbracket 0, \langle t \rangle / \$tr, \$tr' \rrbracket$   
**apply** (*rel-auto*)  
**using** *dual-order.trans* **apply** *blast*  
**done**  
**thus** *?thesis*  
**by** (*simp only: assms*(1) *Healthy-if*)  
**qed**

The *RC* healthy relations can also be defined in terms of prefix closure, which is characterised by the healthiness condition below.

**definition** *RC2* ::  $(t :: trace, 'a, 'b) \text{ rel-rp} \Rightarrow (t, 'a, 'b) \text{ rel-rp}$  **where**  
 $[upred-defs]: RC2(P) = R1(P ;; (\$tr' \leq_u \$tr))$

**lemma** *RC2-RR-commute*:  
 $RC2(RR(P)) = RR(RC2(P))$   
**apply** (*rel-auto*)  
**using** *minus-cancel-le* **apply** *blast*  
**apply** (*metis diff-add-cancel-left' le-add trace-class.add-diff-cancel-left trace-class.add-left-mono*)  
**done**

Intuitive meaning of *RC2*

**lemma** *RC2-form-1*:  
**assumes**  $P$  is *RR*  
**shows**  $RC2(P) = (\exists\ tr_0 \cdot (\exists\ \$\Sigma_R' \cdot P) \llbracket \langle tr_0 \rangle / \$tr' \rrbracket \wedge \$tr' \leq_u \langle tr_0 \rangle \wedge \$tr \leq_u \$tr')$   
**proof** –  
**have**  $RC2(RR(P)) = (\exists\ tr_0 \cdot (\exists\ \$\Sigma_R' \cdot RR\ P) \llbracket \langle tr_0 \rangle / \$tr' \rrbracket \wedge \$tr' \leq_u \langle tr_0 \rangle \wedge \$tr \leq_u \$tr')$   
**by** (*rel-blast*)  
**thus** *?thesis*  
**by** (*metis (mono-tags, lifting) Healthy-if assms shEx-cong*)  
**qed**

**lemma** *RC2-form-2*:  
**assumes**  $P$  is *RR*  
**shows**  $RC2(P) = (\exists\ (t_0, t_1) \cdot (\exists\ \$\Sigma_R' \cdot P) \llbracket 0, \langle t_1 \rangle / \$tr, \$tr' \rrbracket \wedge \langle t_0 \rangle \leq_u \langle t_1 \rangle \wedge \$tr' =_u \$tr + \langle t_0 \rangle)$   
**proof** –  
**have**  $RC2(RR(P)) = (\exists\ (t_0, t_1) \cdot (\exists\ \$\Sigma_R' \cdot RR(P)) \llbracket 0, \langle t_1 \rangle / \$tr, \$tr' \rrbracket \wedge \langle t_0 \rangle \leq_u \langle t_1 \rangle \wedge \$tr' =_u \$tr + \langle t_0 \rangle)$   
**apply** (*rel-auto*)  
**apply** (*metis diff-add-cancel-left' trace-class.add-le-imp-le-left*)  
**apply** (*metis le-add trace-class.add-diff-cancel-left trace-class.add-left-mono*)  
**done**  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

Every reactive condition is prefix closed

**lemma** *RC-prefix-closed*:  
**assumes**  $P$  is *RC*  
**shows**  $P$  is *RC2*  
**proof** –



```

have  $RC2(RC(P)) = RC(P)$ 
  apply (rel-auto) using dual-order.trans by blast
thus ?thesis
  by (metis Healthy-def assms)
qed

```

```

lemma RC2-RR-is-RC1:
  assumes  $P$  is  $RR$   $P$  is  $RC2$ 
  shows  $P$  is  $RC1$ 
proof -
  have  $RC1(RC2(RR(P))) = RC2(RR(P))$ 
  apply (rel-auto) using dual-order.trans by blast
  thus ?thesis
    by (metis Healthy-def assms(1) assms(2))
qed

```

$RC$  closure can be demonstrated in terms of prefix closure.

```

lemma RC-intro-prefix-closed:
  assumes  $P$  is  $RR$   $P$  is  $RC2$ 
  shows  $P$  is  $RC$ 
  by (simp add: RC2-RR-is-RC1 RC-intro' assms)

```

## 7.2 Closure laws

```

lemma RC-implies-RR [closure]:
  assumes  $P$  is  $RC$ 
  shows  $P$  is  $RR$ 
  by (metis Healthy-def RC-ex-ok-wait RC-implies-R2 RR-def assms)

```

```

lemma RC-implies-RC1:  $P$  is  $RC \implies P$  is  $RC1$ 
  by (metis Healthy-def RC-R2-def RC-implies-RR comp-eq-dest-lhs)

```

```

lemma RC1-trace-ext-prefix:
   $out\alpha \nmid e \implies RC1(\neg_r \$tr \hat{\_}_u e \leq_u \$tr') = (\neg_r \$tr \hat{\_}_u e \leq_u \$tr')$ 
  by (rel-auto, blast, metis (no-types, lifting) dual-order.trans)

```

```

lemma RC1-conj [rpred]:  $RC1(P \wedge Q) = (RC1(P) \wedge RC1(Q))$ 
  by (rel-blast)

```

```

lemma conj-RC1-closed [closure]:
   $\llbracket P$  is  $RC1$ ;  $Q$  is  $RC1 \rrbracket \implies P \wedge Q$  is  $RC1$ 
  by (simp add: Healthy-def RC1-conj)

```

```

lemma disj-RC1-closed [closure]:
  assumes  $P$  is  $RC1$   $Q$  is  $RC1$ 
  shows  $(P \vee Q)$  is  $RC1$ 
proof -
  have  $1: RC1(RC1(P) \vee RC1(Q)) = (RC1(P) \vee RC1(Q))$ 
  apply (rel-auto) using dual-order.trans by blast+
  show ?thesis
    by (metis (no-types) Healthy-def 1 assms)
qed

```

```

lemma conj-RC-closed [closure]:
  assumes  $P$  is  $RC$   $Q$  is  $RC$ 
  shows  $(P \wedge Q)$  is  $RC$ 

```

by (metis Healthy-def RC-R2-def RC-implies-RR assms comp-apply conj-RC1-closed conj-RR)

**lemma** *rea-true-RC* [closure]: *true<sub>r</sub> is RC*  
 by (rel-auto)

**lemma** *false-RC* [closure]: *false is RC*  
 by (rel-auto)

**lemma** *disj-RC-closed* [closure]:  $\llbracket P \text{ is } RC; Q \text{ is } RC \rrbracket \implies (P \vee Q) \text{ is } RC$   
 by (metis Healthy-def RC-R2-def RC-implies-RR comp-apply disj-RC1-closed disj-RR)

**lemma** *UINF-mem-RC1-closed* [closure]:  
 assumes  $\bigwedge i. P \ i \text{ is } RC1$   
 shows  $(\bigcap i \in A. P \ i) \text{ is } RC1$   
**proof** –  
 have  $1: RC1(\bigcap i \in A. RC1(P \ i)) = (\bigcap i \in A. RC1(P \ i))$   
 by (rel-auto, meson order.trans)  
 show ?thesis  
 by (metis (mono-tags, lifting) 1 Healthy-def' UINF-all-cong UINF-alt-def assms)  
**qed**

**lemma** *UINF-mem-RC-closed* [closure]:  
 assumes  $\bigwedge i. P \ i \text{ is } RC$   
 shows  $(\bigcap i \in A. P \ i) \text{ is } RC$   
**proof** –  
 have  $RC(\bigcap i \in A. P \ i) = (RC1 \circ RR)(\bigcap i \in A. P \ i)$   
 by (simp add: RC-def)  
 also have  $\dots = RC1(\bigcap i \in A. RR(P \ i))$   
 by (rel-blast)  
 also have  $\dots = RC1(\bigcap i \in A. RC1(P \ i))$   
 by (simp add: Healthy-if RC-implies-RR RC-implies-RC1 assms)  
 also have  $\dots = (\bigcap i \in A. RC1(P \ i))$   
 by (rel-auto, meson order.trans)  
 also have  $\dots = (\bigcap i \in A. P \ i)$   
 by (simp add: Healthy-if RC-implies-RC1 assms)  
 finally show ?thesis  
 by (simp add: Healthy-def)  
**qed**

**lemma** *UINF-ind-RC-closed* [closure]:  
 assumes  $\bigwedge i. P \ i \text{ is } RC$   
 shows  $(\bigcap i. P \ i) \text{ is } RC$   
 by (metis (no-types) UINF-as-Sup-collect' UINF-as-Sup-image UINF-mem-RC-closed assms)

**lemma** *USUP-mem-RC1-closed* [closure]:  
 assumes  $\bigwedge i. i \in A \implies P \ i \text{ is } RC1 \ A \neq \{\}$   
 shows  $(\bigcup i \in A. P \ i) \text{ is } RC1$   
**proof** –  
 have  $RC1(\bigcup i \in A. P \ i) = RC1(\bigcup i \in A. RC1(P \ i))$   
 by (simp add: Healthy-if assms(1) cong: USUP-cong)  
 also from assms(2) have  $\dots = (\bigcup i \in A. RC1(P \ i))$   
 using dual-order.trans by (rel-blast)  
 also have  $\dots = (\bigcup i \in A. P \ i)$   
 by (simp add: Healthy-if assms(1) cong: USUP-cong)  
 finally show ?thesis

using *Healthy-def* by *blast*  
qed

**lemma** *USUP-mem-RC-closed* [closure]:  
assumes  $\bigwedge i. i \in A \implies P\ i\ \text{is}\ RC\ A \neq \{\}$   
shows  $(\bigsqcup_{i \in A} P\ i)\ \text{is}\ RC$   
by (rule *RC-intro'*, simp-all add: closure assms *RC-implies-RC1*)

**lemma** *USUP-ind-RC-closed* [closure]:  
 $\llbracket \bigwedge i. P\ i\ \text{is}\ RC \rrbracket \implies (\bigsqcup i. P\ i)\ \text{is}\ RC$   
by (metis *UNIV-not-empty USUP-mem-RC-closed USUP-mem-UNIV*)

**lemma** *neg-trace-ext-prefix-RC* [closure]:  
 $\llbracket \$tr \# e; \$ok \# e; \$wait \# e; out\alpha \# e \rrbracket \implies \neg_r \$tr \hat{\ }_u e \leq_u \$tr' \ \text{is}\ RC$   
by (rule *RC-intro*, simp add: closure, metis *RC1-def RC1-trace-ext-prefix*)

**lemma** *RC1-unrest*:  
 $\llbracket mwb\text{-}lens\ x; x \bowtie tr \rrbracket \implies \$x' \# RC1(P)$   
by (simp add: *RC1-def unrest*)

**lemma** *RC-unrest-dashed* [unrest]:  
 $\llbracket P\ \text{is}\ RC; mwb\text{-}lens\ x; x \bowtie tr \rrbracket \implies \$x' \# P$   
by (metis *Healthy-if RC1-unrest RC-implies-RC1*)

**lemma** *RC1-RR-closed* [closure]:  $P\ \text{is}\ RR \implies RC1(P)\ \text{is}\ RR$   
by (simp add: *RC1-def closure*)

end

## 8 Reactive Programs

**theory** *utp-rea-prog*  
imports *utp-rea-cond*  
begin

### 8.1 Stateful reactive alphabet

*R3* as presented in the UTP book and related publications is not sensitive to state, although reactive programs often need this property. Thus it is necessary to use a modification of *R3* from Butterfield et al. [1] that explicitly states that intermediate waiting states do not propagate final state variables. In order to do this we need an additional observational variable that captures the program state that we call *st*. Upon this foundation, we can define operators for reactive programs [3].

**alphabet**  $(t, s)\ rsp\text{-}vars = t::trace\ rp\text{-}vars +$   
 $st :: s$

**print-theorems**

**type-synonym**  $(s, t, \alpha)\ rsp = (t, s, \alpha)\ rsp\text{-}vars\text{-}scheme$   
**type-synonym**  $(s, t, \alpha, \beta)\ rel\text{-}rsp = ((s, t, \alpha)\ rsp, (s, t, \beta)\ rsp)\ urel$   
**type-synonym**  $(s, t, \alpha)\ hrel\text{-}rsp = (s, t, \alpha)\ rsp\ hrel$   
**type-synonym**  $(s, t)\ rdes = (s, t, unit)\ hrel\text{-}rsp$

**translations**

$(type) ('s, 't, 'α) rsp \leq (type) ('t, ('s, 'α) rsp-vars-ext) rp$   
 $(type) ('s, 't, 'α) rsp \leq (type) ('t, ('s, 'α) rsp-vars-scheme) rp$   
 $(type) ('s, 't, unit) rsp \leq (type) ('t, 's rsp-vars) rp$   
 $(type) ('s, 't, 'α, 'β) rel-rsp \leq (type) (('s, 't, 'α) rsp, ('s1, 't1, 'β) rsp) urel$   
 $(type) ('s, 't, 'α) hrel-rsp \leq (type) ('s, 't, 'α) rsp hrel$   
 $(type) ('s, 't) rdes \leq (type) ('s, 't, unit) hrel-rsp$

**notation**  $rsp-vars.more_L (\Sigma_S)$

**syntax**

$-svid-st-alpha :: svid (\Sigma_S)$

**translations**

$-svid-st-alpha \Rightarrow CONST\ rsp-vars.more_L$

**lemma**  $rea-lens-equiv-st-rest: \Sigma_R \approx_L st +_L \Sigma_S$

**by**  $simp$

**lemma**  $srea-lens-bij: bij-lens (ok +_L wait +_L tr +_L st +_L \Sigma_S)$

**proof** –

**have**  $ok +_L wait +_L tr +_L st +_L \Sigma_S \approx_L ok +_L wait +_L tr +_L \Sigma_R$

**by**  $(auto\ intro!\ lens-plus-cong, rule\ lens-equiv-sym, simp\ add: rea-lens-equiv-st-rest)$

**also have**  $\dots \approx_L 1_L$

**using**  $bij-lens-equiv-id[of\ ok +_L wait +_L tr +_L \Sigma_R]$  **by**  $(simp\ add: rea-lens-bij)$

**finally show**  $?thesis$

**by**  $(simp\ add: bij-lens-equiv-id)$

**qed**

**lemma**  $st-qual-alpha\ [alpha]: x ;_L fst_L ;_L st \times_L st = (\$st:x)_v$

**by**  $(metis\ (no-types, hide-lams)\ in-var-def\ in-var-prod-lens\ lens-comp-assoc\ st-vwb-lens\ vwb-lens-wb)$

**declare**  $des-vars.splits\ [alpha-splits\ del]$

**declare**  $rp-vars.splits\ [alpha-splits\ del]$

**declare**  $rp-vars.splits\ [alpha-splits]$

**declare**  $des-vars.splits\ [alpha-splits]$

**lemma**  $unrest-st'-neg-RC\ [unrest]:$

**assumes**  $P\ is\ RR\ P\ is\ RC$

**shows**  $\$st' \# P$

**proof** –

**have**  $P = (\neg_r \neg_r P)$

**by**  $(simp\ add: closure\ rpred\ assms)$

**also have**  $\dots = (\neg_r (\neg_r P) ;; true_r)$

**by**  $(metis\ Healthy-if\ RC1-def\ RC-implies-RC1\ assms(2)\ calculation)$

**also have**  $\$st' \# \dots$

**by**  $(rel-auto)$

**finally show**  $?thesis$  .

**qed**

**lemma**  $ex-st'-RR-closed\ [closure]:$

**assumes**  $P\ is\ RR$

shows  $(\exists \$st' \cdot P)$  is  $RR$   
**proof** –  
 have  $RR (\exists \$st' \cdot RR(P)) = (\exists \$st' \cdot RR(P))$   
 by  $(rel-auto)$   
 thus  $?thesis$   
 by  $(metis Healthy-def assms)$   
**qed**

**lemma** *unrest-st'-R4* [*unrest*]:  
 $\$st' \# P \implies \$st' \# R4(P)$   
 by  $(rel-auto)$

**lemma** *unrest-st'-R5* [*unrest*]:  
 $\$st' \# P \implies \$st' \# R5(P)$   
 by  $(rel-auto)$

## 8.2 State Lifting

**abbreviation** *lift-state-rel*  $(\lceil \cdot \rceil_S)$   
**where**  $\lceil P \rceil_S \equiv P \oplus_p (st \times_L st)$

**abbreviation** *drop-state-rel*  $(\lfloor \cdot \rfloor_S)$   
**where**  $\lfloor P \rfloor_S \equiv P \upharpoonright_e (st \times_L st)$

**abbreviation** *lift-state-pre*  $(\lceil \cdot \rceil_{S<})$   
**where**  $\lceil p \rceil_{S<} \equiv \lceil \lceil p \rceil \rfloor_S$

**abbreviation** *drop-state-pre*  $(\lfloor \cdot \rfloor_{S<})$   
**where**  $\lfloor p \rfloor_{S<} \equiv \lfloor \lfloor p \rfloor_S \rfloor_{S<}$

**abbreviation** *lift-state-post*  $(\lceil \cdot \rceil_{S>})$   
**where**  $\lceil p \rceil_{S>} \equiv \lceil \lceil p \rceil \rfloor_S$

**abbreviation** *drop-state-post*  $(\lfloor \cdot \rfloor_{S>})$   
**where**  $\lfloor p \rfloor_{S>} \equiv \lfloor \lfloor p \rfloor_S \rfloor_{S>}$

**lemma** *st-unrest-state-pre* [*unrest*]:  $\&\mathbf{v} \# s \implies \$st \# \lceil s \rceil_{S<}$   
 by  $(rel-auto)$

**lemma** *st'-unrest-st-lift-pred* [*unrest*]:  
 $\$st' \# \lceil a \rceil_{S<}$   
 by  $(pred-auto)$

**lemma** *out-alpha-unrest-st-lift-pre* [*unrest*]:  
 $out\alpha \# \lceil a \rceil_{S<}$   
 by  $(rel-auto)$

**lemma** *R1-st'-unrest* [*unrest*]:  $\$st' \# P \implies \$st' \# R1(P)$   
 by  $(simp add: R1-def unrest)$

**lemma** *R2c-st'-unrest* [*unrest*]:  $\$st' \# P \implies \$st' \# R2c(P)$   
 by  $(simp add: R2c-def unrest)$

**lemma** *unrest-st-rea-rename* [*unrest*]:  
 $\$st \# P \implies \$st \# P(\lfloor f \rfloor_r)$   
 $\$st' \# P \implies \$st' \# P(\lfloor f \rfloor_r)$

by (rel-blast)+

**lemma** *st-lift-R1-true-right*:  $\lceil b \rceil_{S<} ;; R1(true) = \lceil b \rceil_{S<}$   
by (rel-auto)

**lemma** *R2c-lift-state-pre*:  $R2c(\lceil b \rceil_{S<}) = \lceil b \rceil_{S<}$   
by (rel-auto)

## 8.3 Reactive Program Operators

### 8.3.1 State Substitution

Lifting substitutions on the reactive state

**definition** *usubst-st-lift* ::

$'s \text{ usubst} \Rightarrow (('s, 't::\text{trace}, ' \alpha) \text{ rsp} \times ('s, 't, ' \beta) \text{ rsp}) \text{ usubst } (\lceil - \rceil_{S\sigma})$  **where**  
[upred-defs]:  $\lceil \sigma \rceil_{S\sigma} = \lceil \sigma \oplus_s st \rceil_s$

**abbreviation** *st-subst* ::  $'s \text{ usubst} \Rightarrow ('s, 't::\text{trace}, ' \alpha, ' \beta) \text{ rel-rsp} \Rightarrow ('s, 't, ' \alpha, ' \beta) \text{ rel-rsp}$  (**infixr**  $\dagger_S$  80)  
**where**

$\sigma \dagger_S P \equiv \lceil \sigma \rceil_{S\sigma} \dagger P$

**translations**

$\sigma \dagger_S P \leq \lceil \sigma \oplus_s st \rceil_s \dagger P$   
 $\sigma \dagger_S P \leq \lceil \sigma \rceil_{S\sigma} \dagger P$

**lemma** *st-lift-lemma*:

$\lceil \sigma \rceil_{S\sigma} = \sigma \oplus_s (fst_L ;_L (st \times_L st))$   
**by** (auto simp add: upred-defs lens-defs prod.case-eq-if)

**lemma** *unrest-st-lift* [unrest]:

**fixes**  $x :: 'a \Longrightarrow ('s, 't::\text{trace}, ' \alpha) \text{ rsp} \times ('s, 't, ' \alpha) \text{ rsp}$

**assumes**  $x \bowtie (\$st)_v$

**shows**  $x \# \lceil \sigma \rceil_{S\sigma}$  (**is** ?P)

**by** (simp add: st-lift-lemma)

(metis assms in-var-def in-var-prod-lens lens-comp-left-id st-vwb-lens unrest-subst-alpha-ext vwb-lens-wb)

**lemma** *id-st-subst* [usubst]:

$\lceil id \rceil_{S\sigma} = id$   
**by** (pred-auto)

**lemma** *st-subst-comp* [usubst]:

$\lceil \sigma \rceil_{S\sigma} \circ \lceil \varrho \rceil_{S\sigma} = \lceil \sigma \circ \varrho \rceil_{S\sigma}$   
**by** (rel-auto)

**definition** *lift-cond-srea* ( $\lceil - \rceil_{S\leftarrow}$ ) **where**

[upred-defs]:  $\lceil b \rceil_{S\leftarrow} = \lceil b \rceil_{S<}$

**lemma** *unrest-lift-cond-srea* [unrest]:

$x \# \lceil b \rceil_{S<} \Longrightarrow x \# \lceil b \rceil_{S\leftarrow}$   
**by** (simp add: lift-cond-srea-def)

**lemma** *st-subst-RR-closed* [closure]:

**assumes**  $P$  is RR

**shows**  $\lceil \sigma \rceil_{S\sigma} \dagger P$  is RR

**proof** —

**have**  $RR(\lceil \sigma \rceil_{S\sigma} \dagger RR(P)) = \lceil \sigma \rceil_{S\sigma} \dagger RR(P)$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*metis Healthy-def assms*)  
**qed**

**lemma** *subst-lift-cond-srea* [*usubst*]:  $\sigma \dagger_S \lceil P \rceil_{S\leftarrow} = \lceil \sigma \dagger P \rceil_{S\leftarrow}$   
**by** (*rel-auto*)

**lemma** *st-subst-rea-not* [*usubst*]:  $\sigma \dagger_S (\neg_r P) = (\neg_r \sigma \dagger_S P)$   
**by** (*rel-auto*)

**lemma** *st-subst-seq* [*usubst*]:  $\sigma \dagger_S (P ;; Q) = \sigma \dagger_S P ;; Q$   
**by** (*rel-auto*)

**lemma** *st-subst-RC-closed* [*closure*]:  
**assumes** *P is RC*  
**shows**  $\sigma \dagger_S P$  *is RC*  
**apply** (*rule RC-intro, simp add: closure assms*)  
**apply** (*simp add: st-subst-rea-not[THEN sym] st-subst-seq[THEN sym]*)  
**apply** (*metis Healthy-if RC1-def RC-implies-RC1 assms*)  
**done**

### 8.3.2 Assignment

**definition** *rea-assigns* ::  $(\iota s \Rightarrow \iota s) \Rightarrow (\iota s, \iota t :: \text{trace}, \iota \alpha) \text{ hrel-rsp } (\langle \cdot \rangle_r)$  **where**  
*[upred-defs]*:  $\langle \sigma \rangle_r = (\$tr' =_u \$tr \wedge \lceil \langle \sigma \rangle_a \rceil_S \wedge \$\Sigma_S' =_u \$\Sigma_S)$

**syntax**

*-assign-rea* ::  $svids \Rightarrow uexprs \Rightarrow logic$  ( $\iota(-) :=_r \iota(-)$ )  
*-assign-rea* ::  $svids \Rightarrow uexprs \Rightarrow logic$  (**infixr**  $:=_r$  62)

**translations**

*-assign-rea*  $xs\ vs \Rightarrow CONST\ rea-assigns\ (-mk-usubst\ (CONST\ id)\ xs\ vs)$   
*-assign-rea*  $x\ v \leq CONST\ rea-assigns\ (CONST\ subst-upd\ (CONST\ id)\ x\ v)$   
*-assign-rea*  $x\ v \leq -assign-rea\ (-spvar\ x)\ v$   
 $x, y :=_r u, v \leq CONST\ rea-assigns\ (CONST\ subst-upd\ (CONST\ subst-upd\ (CONST\ id)\ (CONST\ svar\ x)\ u)\ (CONST\ svar\ y)\ v)$

**lemma** *rea-assigns-RR-closed* [*closure*]:  
 $\langle \sigma \rangle_r$  *is RR*  
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *auto*

**lemma** *st-subst-assigns-rea* [*usubst*]:  
 $\sigma \dagger_S \langle \varrho \rangle_r = \langle \varrho \circ \sigma \rangle_r$   
**by** (*rel-auto*)

**lemma** *st-subst-rea-skip* [*usubst*]:  
 $\sigma \dagger_S II_r = \langle \sigma \rangle_r$   
**by** (*rel-auto*)

**lemma** *rea-assigns-comp* [*rpred*]:  
**assumes** *P is RR*  
**shows**  $\langle \sigma \rangle_r ;; P = \sigma \dagger_S P$   
**proof** –  
**have**  $\langle \sigma \rangle_r ;; (RR\ P) = \sigma \dagger_S (RR\ P)$

by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def assms)  
 qed

**lemma** *rea-assigns-rename* [rpred]:  
 $\text{renamer } f \implies \langle \sigma \rangle_r \llbracket f \rrbracket_r = \langle \sigma \rangle_r$   
 using minus-zero-eq by rel-auto

**lemma** *st-subst-RR* [closure]:  
 assumes  $P$  is  $RR$   
 shows  $(\sigma \uparrow_S P)$  is  $RR$   
**proof** –  
 have  $(\sigma \uparrow_S RR(P))$  is  $RR$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

**lemma** *rea-assigns-st-subst* [usubst]:  
 $\llbracket \sigma \oplus_s st \rrbracket_s \uparrow \langle \varrho \rangle_r = \langle \varrho \circ \sigma \rangle_r$   
 by (rel-auto)

### 8.3.3 Conditional

We guard the reactive conditional condition so that it can't be simplified by alphabet laws unless explicitly simplified.

**abbreviation** *cond-srea* ::  
 $(s, t :: \text{trace}, \alpha, \beta) \text{ rel-rsp} \Rightarrow$   
 $s \text{ upred} \Rightarrow$   
 $(s, t, \alpha, \beta) \text{ rel-rsp} \Rightarrow$   
 $(s, t, \alpha, \beta) \text{ rel-rsp}$  **where**  
 $\text{cond-srea } P \text{ b } Q \equiv P \triangleleft \llbracket b \rrbracket_{S \leftarrow} \triangleright Q$

**syntax**  
 $\text{-cond-srea} :: \text{logic} \Rightarrow \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic} ((\exists - \triangleleft - \triangleright_R / -) [52, 0, 53] 52)$

**translations**  
 $\text{-cond-srea } P \text{ b } Q == \text{CONST cond-srea } P \text{ b } Q$

**lemma** *st-cond-assigns* [rpred]:  
 $\langle \sigma \rangle_r \triangleleft b \triangleright_R \langle \varrho \rangle_r = \langle \sigma \triangleleft b \triangleright_s \varrho \rangle_r$   
 by (rel-auto)

**lemma** *cond-srea-RR-closed* [closure]:  
 assumes  $P$  is  $RR$   $Q$  is  $RR$   
 shows  $P \triangleleft b \triangleright_R Q$  is  $RR$   
**proof** –  
 have  $RR(RR(P) \triangleleft b \triangleright_R RR(Q)) = RR(P) \triangleleft b \triangleright_R RR(Q)$   
 by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def' assms(1) assms(2))  
 qed

**lemma** *cond-srea-RC1-closed*:



**assumes**  $P$  is RC1  $Q$  is RC1  
**shows**  $P \triangleleft b \triangleright_R Q$  is RC1  
**proof** –  
**have**  $RC1(RC1(P) \triangleleft b \triangleright_R RC1(Q)) = RC1(P) \triangleleft b \triangleright_R RC1(Q)$   
**using** *dual-order.trans* **by** (*rel-blast*)  
**thus** *?thesis*  
**by** (*metis Healthy-def' assms*)  
**qed**

**lemma** *cond-srea-RC-closed* [*closure*]:  
**assumes**  $P$  is RC  $Q$  is RC  
**shows**  $P \triangleleft b \triangleright_R Q$  is RC  
**by** (*rule RC-intro', simp-all add: closure cond-srea-RC1-closed RC-implies-RC1 assms*)

**lemma** *R4-cond* [*rpred*]:  $R4(P \triangleleft b \triangleright_R Q) = (R4(P) \triangleleft b \triangleright_R R4(Q))$   
**by** (*rel-auto*)

**lemma** *R5-cond* [*rpred*]:  $R5(P \triangleleft b \triangleright_R Q) = (R5(P) \triangleleft b \triangleright_R R5(Q))$   
**by** (*rel-auto*)

**lemma** *rea-rename-cond* [*rpred*]:  $(P \triangleleft b \triangleright_R Q)(f)_r = P(f)_r \triangleleft b \triangleright_R Q(f)_r$   
**by** (*rel-auto*)

### 8.3.4 Assumptions

**definition** *rea-assume* ::  $'s \text{ upred} \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp } ([\cdot]^\top_r)$  **where**  
*[upred-defs]*:  $[b]^\top_r = (II_r \triangleleft b \triangleright_R \text{false})$

**lemma** *rea-assume-RR* [*closure*]:  $[b]^\top_r$  is RR  
**by** (*simp add: rea-assume-def closure*)

**lemma** *rea-assume-false* [*rpred*]:  $[\text{false}]^\top_r = \text{false}$   
**by** (*rel-auto*)

**lemma** *rea-assume-true* [*rpred*]:  $[\text{true}]^\top_r = II_r$   
**by** (*rel-auto*)

**lemma** *rea-assume-comp* [*rpred*]:  $[b]^\top_r ;; [c]^\top_r = [b \wedge c]^\top_r$   
**by** (*rel-auto*)

### 8.3.5 State Abstraction

We introduce state abstraction by creating some lens functors that allow us to lift a lens on the state-space to one on the whole stateful reactive alphabet.

**definition** *lmap<sub>R</sub>* ::  $('a \Rightarrow 'b) \Rightarrow ('t::\text{trace}, 'a) \text{ rp} \Rightarrow ('t, 'b) \text{ rp}$  **where**  
*[lens-defs]*: *lmap<sub>R</sub>* = *lmap*[*rp-vars*]

This construction lens is useful for conversion between a record and its product representation; it would be helpful if this could be automatically generated.

**definition** *rsp-make-lens* ::  $('s, 't::\text{trace}, 'a) \text{ rsp} \Rightarrow \text{bool} \times \text{bool} \times 't \times 's \times 'a$  **where**  
*[lens-defs]*: *rsp-make-lens* =  $(\lambda \text{ lens-get} = \lambda (ok, wait, tr, st, more). (\lambda ok_v = ok, wait_v = wait, tr_v = tr, st_v = st, \dots = more) \lambda \text{ lens-put} = (\lambda s v. (ok_v v, wait_v v, tr_v v, st_v v, more v)))$

**lemma** *rsp-make-lens-alt*: *rsp-make-lens* = *inv<sub>L</sub>* (*ok* +<sub>L</sub> *wait* +<sub>L</sub> *tr* +<sub>L</sub> *st* +<sub>L</sub> *rsp-vars.more<sub>L</sub>*)

by (auto simp add: lens-defs)

**lemma** make-lens-bij [simp]: bij-lens rsp-make-lens

by (unfold-locales, simp-all add: lens-defs prod.case-eq-if)

The following is an intuitive definition of the  $st$  functorial lens, which frames all the state space excluding  $st$ , to which another lens  $l$  is applied. We do this by splitting the state space into a product, including the application of  $l$  to  $st$ , and then invert the product creation lens to reconstruct the reactive state space.

**definition** map-st-lens ::

$(\sigma \Rightarrow \psi) \Rightarrow$   
 $((\sigma, \tau::trace, \alpha) \text{ rsp} \Rightarrow (\psi, \tau::trace, \alpha) \text{ rsp}) \text{ (map'-st}_L\text{) where}$   
 $\text{map-st-lens } l = \text{inv}_L (ok +_L \text{wait} +_L \text{tr} +_L \text{st} +_L \text{rsp-vars.more}_L) ;_L$   
 $(ok +_L \text{wait} +_L \text{tr} +_L (l ;_L \text{st}) +_L \text{rsp-vars.more}_L)$

The above definition is intuitive, but helpful in proof automaton. Consequently, we the following optimised definition below.

**lemma** map-st-lens-alt-def [lens-defs]:

$\text{map-st-lens } l = \langle \text{lens-get} = \lambda s. \langle ok_v = ok_v s, \text{wait}_v = \text{wait}_v s, \text{tr}_v = \text{tr}_v s, \text{st}_v = \text{get}_l(\text{st}_v s), \dots$   
 $= \text{more } s \rangle$   
 $, \text{lens-put} = \lambda s v. \langle ok_v = ok_v v, \text{wait}_v = \text{wait}_v v, \text{tr}_v = \text{tr}_v v, \text{st}_v = \text{put}_l(\text{st}_v s)(\text{st}_v$   
 $v), \dots = \text{more } v \rangle \rangle$   
 by (auto simp add: map-st-lens-def lens-defs fun-eq-iff)

**lemma** map-set-vwb [simp]: vwb-lens  $X \Rightarrow \text{vwb-lens (map-st}_L X)$

by (simp add: map-st-lens-def rsp-make-lens-alt[THEN sym])

**syntax**

-map-st-lens :: logic  $\Rightarrow$  salpha (map'-st<sub>L</sub>[-])

**translations**

-map-st-lens  $a \Rightarrow \text{CONST map-st-lens } a$

**abbreviation** abs-st<sub>L</sub>  $\equiv (\text{map-st}_L 0_L) \times_L (\text{map-st}_L 0_L)$

**abbreviation** abs-st ( $\langle \cdot \rangle_S$ ) **where**

abs-st  $P \equiv P \upharpoonright_e \text{abs-st}_L$

**lemma** rea-impl-aext-st [alpha]:

$(P \Rightarrow_r Q) \oplus_r \text{map-st}_L[a] = (P \oplus_r \text{map-st}_L[a] \Rightarrow_r Q \oplus_r \text{map-st}_L[a])$   
 by (rel-auto)

**lemma** rea-true-ext-st [alpha]:

$\text{true}_r \oplus_p \text{abs-st}_L = \text{true}_r$   
 by (rel-auto)

### 8.3.6 Reactive Frames and Extensions

**definition** rea-frame ::  $(\alpha \Rightarrow \beta) \Rightarrow (\beta, t::trace, r) \text{ hrel-rsp} \Rightarrow (\beta, t, r) \text{ hrel-rsp}$  **where**  
 [upred-defs]: rea-frame  $x P = \text{frame } (ok +_L \text{wait} +_L \text{tr} +_L (x ;_L \text{st}) +_L \Sigma_S) P$

**definition** rea-frame-ext ::  $(\alpha \Rightarrow \beta) \Rightarrow (\alpha, t::trace, r) \text{ hrel-rsp} \Rightarrow (\beta, t, r) \text{ hrel-rsp}$  **where**  
 [upred-defs]: rea-frame-ext  $a P = \text{rea-frame } a (P \oplus_r \text{map-st}_L[a])$

**syntax**

$\text{-rea-frame} \quad :: \text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\cdot:[]_r \ [99,0] \ 100)$   
 $\text{-rea-frame-ext} \quad :: \text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\cdot:[]_r^+ \ [99,0] \ 100)$

#### translations

$\text{-rea-frame } x \ P \Rightarrow \text{CONST } \text{rea-frame } x \ P$   
 $\text{-rea-frame } (\text{-salphaset } (\text{-salphamk } x)) \ P \leq \text{CONST } \text{rea-frame } x \ P$   
 $\text{-rea-frame-ext } x \ P \Rightarrow \text{CONST } \text{rea-frame-ext } x \ P$   
 $\text{-rea-frame-ext } (\text{-salphaset } (\text{-salphamk } x)) \ P \leq \text{CONST } \text{rea-frame-ext } x \ P$

**lemma** *rea-frame-R1-closed* [closure]:

**assumes**  $P$  is  $R1$   
**shows**  $x:[P]_r$  is  $R1$

**proof** –

**have**  $R1(x:[R1 \ P]_r) = x:[R1 \ P]_r$   
**by** (*rel-auto*)  
**thus** ?thesis  
**by** (*metis Healthy-if Healthy-intro assms*)

**qed**

**lemma** *rea-frame-R2-closed* [closure]:

**assumes**  $P$  is  $R2$   
**shows**  $x:[P]_r$  is  $R2$

**proof** –

**have**  $R2(x:[R2 \ P]_r) = x:[R2 \ P]_r$   
**by** (*rel-auto*)  
**thus** ?thesis  
**by** (*metis Healthy-if Healthy-intro assms*)

**qed**

**lemma** *rea-frame-RR-closed* [closure]:

**assumes**  $P$  is  $RR$   
**shows**  $x:[P]_r$  is  $RR$

**proof** –

**have**  $RR(x:[RR \ P]_r) = x:[RR \ P]_r$   
**by** (*rel-auto*)  
**thus** ?thesis  
**by** (*metis Healthy-if Healthy-intro assms*)

**qed**

**lemma** *rea-aext-R1* [closure]:

**assumes**  $P$  is  $R1$   
**shows**  $\text{rel-aext } P \ (\text{map-st}_L \ x)$  is  $R1$

**proof** –

**have**  $\text{rel-aext } (R1 \ P) \ (\text{map-st}_L \ x)$  is  $R1$   
**by** (*rel-auto*)  
**thus** ?thesis  
**by** (*simp add: Healthy-if assms*)

**qed**

**lemma** *rea-aext-R2* [closure]:

**assumes**  $P$  is  $R2$   
**shows**  $\text{rel-aext } P \ (\text{map-st}_L \ x)$  is  $R2$

**proof** –

**have**  $\text{rel-aext } (R2 \ P) \ (\text{map-st}_L \ x)$  is  $R2$   
**by** (*rel-auto*)

**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

**lemma** *rea-aext-RR* [*closure*]:  
**assumes** *P is RR*  
**shows** *rel-aext P (map-st<sub>L</sub> x) is RR*

**proof** –  
**have** *rel-aext (RR P) (map-st<sub>L</sub> x) is RR*  
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

**lemma** *true-rea-map-st* [*alpha*]: (*R1 true*  $\oplus_r$  *map-st<sub>L</sub> [a]*) = *R1 true*  
**by** (*rel-auto*)

**lemma** *rea-frame-ext-R1-closed* [*closure*]:  
*P is R1*  $\implies$  *x:[P]<sub>r</sub><sup>+</sup> is R1*  
**by** (*simp add: rea-frame-ext-def closure*)

**lemma** *rea-frame-ext-R2-closed* [*closure*]:  
*P is R2*  $\implies$  *x:[P]<sub>r</sub><sup>+</sup> is R2*  
**by** (*simp add: rea-frame-ext-def closure*)

**lemma** *rea-frame-ext-RR-closed* [*closure*]:  
*P is RR*  $\implies$  *x:[P]<sub>r</sub><sup>+</sup> is RR*  
**by** (*simp add: rea-frame-ext-def closure*)

**lemma** *rel-aext-st-Instant-closed* [*closure*]:  
*P is Instant*  $\implies$  *rel-aext P (map-st<sub>L</sub> x) is Instant*  
**by** (*rel-auto*)

**lemma** *rea-frame-ext-false* [*frame*]:  
*x:[false]<sub>r</sub><sup>+</sup> = false*  
**by** (*rel-auto*)

**lemma** *rea-frame-ext-skip* [*frame*]:  
*vwb-lens x*  $\implies$  *x:[II]<sub>r</sub><sup>+</sup> = II<sub>r</sub>*  
**by** (*rel-auto*)

**lemma** *rea-frame-ext-assigns* [*frame*]:  
*vwb-lens x*  $\implies$  *x:[⟨σ⟩<sub>r</sub>]<sub>r</sub><sup>+</sup> = ⟨σ  $\oplus_s$  x⟩<sub>r</sub>*  
**by** (*rel-auto*)

**lemma** *rea-frame-ext-cond* [*frame*]:  
*x:[P  $\triangleleft$  b  $\triangleright_R$  Q]<sub>r</sub><sup>+</sup> = *x:[P]<sub>r</sub><sup>+</sup>  $\triangleleft$  (b  $\oplus_p$  x)  $\triangleright_R$  x:[Q]<sub>r</sub><sup>+</sup>*  
**by** (*rel-auto*)*

**lemma** *rea-frame-ext-seq* [*frame*]:  
*vwb-lens x*  $\implies$  *x:[P ;; Q]<sub>r</sub><sup>+</sup> = x:[P]<sub>r</sub><sup>+</sup> ;; x:[Q]<sub>r</sub><sup>+</sup>*  
**apply** (*simp add: rea-frame-ext-def rea-frame-def alpha frame*)  
**apply** (*subst frame-seq*)  
**apply** (*simp-all add: plus-vwb-lens closure*)  
**apply** (*rel-auto*)<sup>+</sup>

done

**lemma** *rea-frame-ext-subst-indep* [*usubst*]:

**assumes**  $x \bowtie y \ \Sigma \ \# \ v \ P \text{ is } RR$

**shows**  $\sigma(y \mapsto_s v) \uparrow_S x:[P]_r^+ = (\sigma \uparrow_S x:[P]_r^+) ;; y :=_r v$

**proof** –

**from** *assms(1-2)* **have**  $\sigma(y \mapsto_s v) \uparrow_S x:[RR \ P]_r^+ = (\sigma \uparrow_S x:[RR \ P]_r^+) ;; y :=_r v$

**by** (*rel-auto*, (*metis* (*no-types*, *lifting*) *lens-indep.lens-put-comm lens-indep-get*)+)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms*)

**qed**

**lemma** *rea-frame-ext-subst-within* [*usubst*]:

**assumes** *vwb-lens*  $x \ vwb\text{-lens} \ y \ \Sigma \ \# \ v \ P \text{ is } RR$

**shows**  $\sigma(x:y \mapsto_s v) \uparrow_S x:[P]_r^+ = (\sigma \uparrow_S x:[y :=_r (v \downarrow_e x) ;; P]_r^+)$

**proof** –

**from** *assms(1,3)* **have**  $\sigma(x:y \mapsto_s v) \uparrow_S x:[RR \ P]_r^+ = (\sigma \uparrow_S x:[y :=_r (v \downarrow_e x) ;; RR(P)]_r^+)$

**by** (*rel-auto*, *metis*+) )

**thus** *?thesis*

**by** (*simp add: assms Healthy-if*)

**qed**

**lemma** *rea-frame-ext-UINF-ind* [*frame*]:

$a:[\bigwedge x \cdot P \ x]_r^+ = (\bigwedge x \cdot a:[P \ x]_r^+)$

**by** (*rel-auto*)

**lemma** *rea-frame-ext-UINF-mem* [*frame*]:

$a:[\bigwedge x \in A \cdot P \ x]_r^+ = (\bigwedge x \in A \cdot a:[P \ x]_r^+)$

**by** (*rel-auto*)

## 8.4 Stateful Reactive specifications

**definition** *rea-st-rel* ::  $'s \ hrel \Rightarrow ('s, 't::trace, 'a, 'b) \ rel\text{-rsp} \ ([\cdot]_S)$  **where**

[*upred-defs*]: *rea-st-rel*  $b = ([b]_S \wedge \$tr' =_u \$tr)$

**definition** *rea-st-rel'* ::  $'s \ hrel \Rightarrow ('s, 't::trace, 'a, 'b) \ rel\text{-rsp} \ ([\cdot]_S')$  **where**

[*upred-defs*]: *rea-st-rel'*  $b = R1([\![b]\!]_S)$

**definition** *rea-st-cond* ::  $'s \ upred \Rightarrow ('s, 't::trace, 'a, 'b) \ rel\text{-rsp} \ ([\cdot]_{S<})$  **where**

[*upred-defs*]: *rea-st-cond*  $b = R1([\![b]\!]_{S<})$

**definition** *rea-st-post* ::  $'s \ upred \Rightarrow ('s, 't::trace, 'a, 'b) \ rel\text{-rsp} \ ([\cdot]_{S>})$  **where**

[*upred-defs*]: *rea-st-post*  $b = R1([\![b]\!]_{S>})$

**lemma** *lift-state-pre-unrest* [*unrest*]:  $x \bowtie (\$st)_v \Longrightarrow x \ \# \ [P]_{S<}$

**by** (*rel-simp*, *simp add: lens-indep-def*)

**lemma** *rea-st-rel-unrest* [*unrest*]:

$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$st')_v \rrbracket \Longrightarrow x \ \# \ [P]_{S<}$

**by** (*simp add: add: rea-st-cond-def R1-def unrest lens-indep-sym*)

**lemma** *rea-st-cond-unrest* [*unrest*]:

$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v \rrbracket \Longrightarrow x \ \# \ [P]_{S<}$

**by** (*simp add: add: rea-st-cond-def R1-def unrest lens-indep-sym*)

**lemma** *subst-st-cond* [*usubst*]:  $[\sigma]_{S\sigma} \uparrow [P]_{S<} = [\sigma \uparrow P]_{S<}$

**by** (*rel-auto*)

**lemma** *rea-st-cond-R1* [*closure*]:  $[b]_{S<}$  is *R1*  
**by** (*rel-auto*)

**lemma** *rea-st-cond-R2c* [*closure*]:  $[b]_{S<}$  is *R2c*  
**by** (*rel-auto*)

**lemma** *rea-st-rel-RR* [*closure*]:  $[P]_S$  is *RR*  
**using** *minus-zero-eq* **by** (*rel-auto*)

**lemma** *rea-st-rel'-RR* [*closure*]:  $[P]_{S'}$  is *RR*  
**by** (*rel-auto*)

**lemma** *rea-st-post-RR* [*closure*]:  $[b]_{S>}$  is *RR*  
**by** (*rel-auto*)

**lemma** *st-subst-rel* [*usubst*]:  
 $\sigma \upharpoonright_S [P]_S = [[\sigma]_s \upharpoonright P]_S$   
**by** (*rel-auto*)

**lemma** *st-rel-cond* [*rpred*]:  
 $[P \triangleleft b \triangleright_r Q]_S = [P]_S \triangleleft b \triangleright_R [Q]_S$   
**by** (*rel-auto*)

**lemma** *st-rel-false* [*rpred*]:  $[false]_S = false$   
**by** (*rel-auto*)

**lemma** *st-rel-skip* [*rpred*]:  
 $[II]_S = (II_r :: ('s, 't::trace) rdes)$   
**by** (*rel-auto*)

**lemma** *st-rel-seq* [*rpred*]:  
 $[P ;; Q]_S = [P]_S ;; [Q]_S$   
**by** (*rel-auto*)

**lemma** *st-rel-conj* [*rpred*]:  
 $([P]_S \wedge [Q]_S) = [P \wedge Q]_S$   
**by** (*rel-auto*)

**lemma** *st-cond-disj* [*rpred*]:  
 $([P]_{S<} \vee [Q]_{S<}) = [P \vee Q]_{S<}$   
**by** (*rel-auto*)

**lemma** *rea-st-cond-RR* [*closure*]:  $[b]_{S<}$  is *RR*  
**by** (*rule RR-intro, simp-all add: unrest closure*)

**lemma** *rea-st-cond-RC* [*closure*]:  $[b]_{S<}$  is *RC*  
**by** (*rule RC-intro, simp add: closure, rel-auto*)

**lemma** *rea-st-cond-true* [*rpred*]:  $[true]_{S<} = true_r$   
**by** (*rel-auto*)

**lemma** *rea-st-cond-false* [*rpred*]:  $[false]_{S<} = false$   
**by** (*rel-auto*)

**lemma** *st-cond-not* [*rpred*]:  $(\neg_r [P]_{S<}) = [\neg P]_{S<}$   
**by** (*rel-auto*)

**lemma** *st-cond-conj* [*rpred*]:  $([P]_{S<} \wedge [Q]_{S<}) = [P \wedge Q]_{S<}$   
**by** (*rel-auto*)

**lemma** *st-rel-assigns* [*rpred*]:  
 $[\langle \sigma \rangle_a]_S = (\langle \sigma \rangle_r :: (' \alpha, 't::trace) rdes)$   
**by** (*rel-auto*)

**lemma** *cond-st-distr*:  $(P \triangleleft b \triangleright_R Q) ;; R = (P ;; R \triangleleft b \triangleright_R Q ;; R)$   
**by** (*rel-auto*)

**lemma** *cond-st-miracle* [*rpred*]:  $P \text{ is } R1 \implies P \triangleleft b \triangleright_R false = ([b]_{S<} \wedge P)$   
**by** (*rel-blast*)

**lemma** *cond-st-true* [*rpred*]:  $P \triangleleft true \triangleright_R Q = P$   
**by** (*rel-blast*)

**lemma** *cond-st-false* [*rpred*]:  $P \triangleleft false \triangleright_R Q = Q$   
**by** (*rel-blast*)

**lemma** *st-cond-true-or* [*rpred*]:  $P \text{ is } R1 \implies (R1 \text{ true } \triangleleft b \triangleright_R P) = ([b]_{S<} \vee P)$   
**by** (*rel-blast*)

**lemma** *st-cond-left-impl-RC-closed* [*closure*]:  
 $P \text{ is } RC \implies ([b]_{S<} \Rightarrow_r P) \text{ is } RC$   
**by** (*simp add: rea-impl-def rpred closure*)

**end**

## 9 Reactive Weakest Preconditions

**theory** *utp-rea-wp*  
**imports** *utp-rea-prog*  
**begin**

Here, we create a weakest precondition calculus for reactive relations, using the recast boolean algebra and relational operators. Please see our journal paper [3] for more information.

**definition** *wp-rea* ::  
 $('t::trace, ' \alpha) hrel\text{-}rp \Rightarrow$   
 $('t, ' \alpha) hrel\text{-}rp \Rightarrow$   
 $('t, ' \alpha) hrel\text{-}rp \text{ (infix } wp_r \text{ 60)}$   
**where** [*upred-defs*]:  $P wp_r Q = (\neg_r P ;; (\neg_r Q))$

**lemma** *in-var-unrest-wp-rea* [*unrest*]:  $\llbracket \$x \# P; tr \bowtie x \rrbracket \implies \$x \# (P wp_r Q)$   
**by** (*simp add: wp-rea-def unrest R1-def rea-not-def*)

**lemma** *out-var-unrest-wp-rea* [*unrest*]:  $\llbracket \$x' \# Q; tr \bowtie x \rrbracket \implies \$x' \# (P wp_r Q)$   
**by** (*simp add: wp-rea-def unrest R1-def rea-not-def*)

**lemma** *wp-rea-R1* [*closure*]:  $P wp_r Q \text{ is } R1$   
**by** (*rel-auto*)

**lemma** *wp-rea-RR-closed* [closure]:  $\llbracket P \text{ is } RR; Q \text{ is } RR \rrbracket \implies P \text{ wp}_r Q \text{ is } RR$   
**by** (*simp add: wp-rea-def closure*)

**lemma** *wp-rea-impl-lemma*:  
 $((P \text{ wp}_r Q) \Rightarrow_r (R1(P) ;; R1(Q \Rightarrow_r R))) = ((P \text{ wp}_r Q) \Rightarrow_r (R1(P) ;; R1(R)))$   
**by** (*rel-auto, blast*)

**lemma** *wpR-impl-post-spec*:  
**assumes**  $P \text{ is } RR$   
**shows**  $(P \text{ wp}_r Q_1 \Rightarrow_r (P ;; (Q_1 \Rightarrow_r Q_2))) = (P ;; (Q_1 \Rightarrow_r Q_2))$   
**by** (*simp add: R1-seqr-closure RR-implies-R1 assms rea-impl-def rea-not-R1 rea-not-not seqr-or-distr wp-rea-def*)

**lemma** *wpR-R1-right* [wp]:  
 $P \text{ wp}_r R1(Q) = P \text{ wp}_r Q$   
**by** (*rel-auto*)

**lemma** *wp-rea-true* [wp]:  $P \text{ wp}_r \text{ true} = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *wp-rea-conj* [wp]:  $P \text{ wp}_r (Q \wedge R) = (P \text{ wp}_r Q \wedge P \text{ wp}_r R)$   
**by** (*simp add: wp-rea-def seqr-or-distr*)

**lemma** *wp-rea-USUP-mem* [wp]:  
 $A \neq \{\} \implies P \text{ wp}_r (\bigsqcup_{i \in A} Q(i)) = (\bigsqcup_{i \in A} P \text{ wp}_r Q(i))$   
**by** (*simp add: wp-rea-def seq-UINF-distl*)

**lemma** *wp-rea-Inf-pre* [wp]:  
 $P \text{ wp}_r (\bigsqcup_{i \in \{0..n::nat\}} Q(i)) = (\bigsqcup_{i \in \{0..n\}} P \text{ wp}_r Q(i))$   
**by** (*simp add: wp-rea-def seq-SUP-distl*)

**lemma** *wp-rea-div* [wp]:  
 $(\neg_r P ;; \text{true}_r) = \text{true}_r \implies \text{true}_r \text{ wp}_r P = \text{false}$   
**by** (*simp add: wp-rea-def rpred, rel-blast*)

**lemma** *wp-rea-st-cond-div* [wp]:  
 $P \neq \text{true} \implies \text{true}_r \text{ wp}_r [P]_{S<} = \text{false}$   
**by** (*rel-auto*)

**lemma** *wp-rea-cond* [wp]:  
 $\text{out}\alpha \nmid b \implies (P \triangleleft b \triangleright Q) \text{ wp}_r R = (P \text{ wp}_r R) \triangleleft b \triangleright (Q \text{ wp}_r R)$   
**by** (*simp add: wp-rea-def cond-seq-left-distr, rel-auto*)

**lemma** *wp-rea-RC-false* [wp]:  
 $P \text{ is } RC \implies (\neg_r P) \text{ wp}_r \text{ false} = P$   
**by** (*metis Healthy-if RC1-def RC-implies-RC1 rea-not-false wp-rea-def*)

**lemma** *wp-rea-seq* [wp]:  
**assumes**  $Q \text{ is } R1$   
**shows**  $(P ;; Q) \text{ wp}_r R = P \text{ wp}_r (Q \text{ wp}_r R)$  (*is ?lhs = ?rhs*)  
**proof** –  
**have**  $?rhs = R1 (\neg P ;; R1 (Q ;; R1 (\neg R)))$   
**by** (*simp add: wp-rea-def rea-not-def R1-negate-R1 assms*)  
**also have**  $\dots = R1 (\neg P ;; (Q ;; R1 (\neg R)))$   
**by** (*metis Healthy-if R1-seqr assms*)



also have ... =  $R1 \ (\neg (P ;; Q) ;; R1 \ (\neg R))$   
 by (simp add: segr-assoc)  
 finally show ?thesis  
 by (simp add: wp-rea-def rea-not-def)  
 qed

**lemma** wp-rea-skip [wp]:  
 assumes  $Q$  is  $R1$   
 shows  $II \ wp_r \ Q = Q$   
 by (simp add: wp-rea-def rpred assms Healthy-if)

**lemma** wp-rea-rea-skip [wp]:  
 assumes  $Q$  is  $RR$   
 shows  $II_r \ wp_r \ Q = Q$   
 by (simp add: wp-rea-def rpred closure assms Healthy-if)

**lemma** power-wp-rea-RR-closed [closure]:  
 $\llbracket R \text{ is } RR; P \text{ is } RR \rrbracket \implies R \wedge i \ wp_r \ P \text{ is } RR$   
 by (induct i, simp-all add: wp closure)

**lemma** wp-rea-rea-assigns [wp]:  
 assumes  $P$  is  $RR$   
 shows  $\langle \sigma \rangle_r \ wp_r \ P = \lceil \sigma \rceil_{S\sigma} \dagger P$   
**proof** –  
 have  $\langle \sigma \rangle_r \ wp_r \ (RR \ P) = \lceil \sigma \rceil_{S\sigma} \dagger (RR \ P)$   
 by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def assms)  
 qed

**lemma** wp-rea-miracle [wp]:  $false \ wp_r \ Q = true_r$   
 by (simp add: wp-rea-def)

**lemma** wp-rea-disj [wp]:  $(P \vee Q) \ wp_r \ R = (P \ wp_r \ R \wedge Q \ wp_r \ R)$   
 by (rel-blast)

**lemma** wp-rea-UINF [wp]:  
 assumes  $A \neq \{\}$   
 shows  $(\bigcap x \in A \cdot P(x)) \ wp_r \ Q = (\bigcap x \in A \cdot P(x) \ wp_r \ Q)$   
 by (simp add: wp-rea-def rea-not-def seq-UINF-distr not-UINF R1-UINF assms)

**lemma** wp-rea-choice [wp]:  
 $(P \sqcap Q) \ wp_r \ R = (P \ wp_r \ R \wedge Q \ wp_r \ R)$   
 by (rel-blast)

**lemma** wp-rea-UINF-ind [wp]:  
 $(\bigcap i \cdot P(i)) \ wp_r \ Q = (\bigcap i \cdot P(i) \ wp_r \ Q)$   
 by (simp add: wp-rea-def rea-not-def seq-UINF-distr' not-UINF-ind R1-UINF-ind)

**lemma** rea-assume-wp [wp]:  
 assumes  $P$  is  $RC$   
 shows  $(\lceil b \rceil_r^\top \ wp_r \ P) = (\lceil b \rceil_{S<} \Rightarrow_r P)$   
**proof** –  
 have  $(\lceil b \rceil_r^\top \ wp_r \ RC \ P) = (\lceil b \rceil_{S<} \Rightarrow_r RC \ P)$   
 by (rel-auto)

thus ?thesis  
 by (simp add: Healthy-if assms)  
 qed

lemma rea-star-wp [wp]:  
 assumes  $P$  is RR  $Q$  is RR  
 shows  $P^{*r} \text{ wp}_r Q = (\bigsqcup i \cdot P \hat{\ } i \text{ wp}_r Q)$   
 proof –  
 have  $P^{*r} \text{ wp}_r Q = (Q \wedge P^+ \text{ wp}_r Q)$   
 by (simp add: assms rrel-theory.Star-alt-def wp-rea-choice wp-rea-rea-skip)  
 also have  $\dots = (II \text{ wp}_r Q \wedge (\bigsqcup i \cdot P \hat{\ } \text{Suc } i \text{ wp}_r Q))$   
 by (simp add: uplus-power-def wp closure assms)  
 also have  $\dots = (\bigsqcup i \cdot P \hat{\ } i \text{ wp}_r Q)$   
 proof –  
 have  $P^* \text{ wp}_r Q = P^{*r} \text{ wp}_r Q$   
 by (metis (no-types) RA1 assms(2) rea-no-RR rea-skip-unit(2) rrel-theory.Star-def wp-rea-def)  
 then show ?thesis  
 by (simp add: calculation ustar-def wp-rea-UINF-ind)  
 qed  
 finally show ?thesis .  
 qed

lemma wp-rea-R2-closed [closure]:  
 $\llbracket P \text{ is R2}; Q \text{ is R2} \rrbracket \implies P \text{ wp}_r Q \text{ is R2}$   
 by (simp add: wp-rea-def closure)

lemma wp-rea-false-RC1':  $P \text{ is R2} \implies \text{RC1}(P \text{ wp}_r \text{false}) = P \text{ wp}_r \text{false}$   
 by (simp add: wp-rea-def RC1-def closure rpred seqr-assoc)

lemma wp-rea-false-RC1:  $P \text{ is R2} \implies P \text{ wp}_r \text{false} \text{ is RC1}$   
 by (simp add: Healthy-def wp-rea-false-RC1')

lemma wp-rea-false-RR:  
 $\llbracket \$ok \ \# \ P; \$wait \ \# \ P; P \text{ is R2} \rrbracket \implies P \text{ wp}_r \text{false} \text{ is RR}$   
 by (rule RR-R2-intro, simp-all add: unrest closure)

lemma wp-rea-false-RC:  
 $\llbracket \$ok \ \# \ P; \$wait \ \# \ P; P \text{ is R2} \rrbracket \implies P \text{ wp}_r \text{false} \text{ is RC}$   
 by (rule RC-intro', simp-all add: wp-rea-false-RC1 wp-rea-false-RR)

lemma wp-rea-RC1:  $\llbracket P \text{ is RR}; Q \text{ is RC} \rrbracket \implies P \text{ wp}_r Q \text{ is RC1}$   
 by (rule Healthy-intro, simp add: wp-rea-def RC1-def rpred closure seqr-assoc RC1-prop RC-implies-RC1)

lemma wp-rea-RC [closure]:  $\llbracket P \text{ is RR}; Q \text{ is RC} \rrbracket \implies P \text{ wp}_r Q \text{ is RC}$   
 by (rule RC-intro', simp-all add: wp-rea-RC1 closure)

lemma wpR-power-RC-closed [closure]:  
 assumes  $P$  is RR  $Q$  is RC  
 shows  $P \hat{\ } i \text{ wp}_r Q \text{ is RC}$   
 by (metis RC-implies-RR RR-implies-R1 assms power.power-eq-if power-Suc-RR-closed wp-rea-RC wp-rea-skip)

end

## 10 Reactive Hoare Logic

**theory** *utp-rea-hoare*  
**imports** *utp-rea-prog*  
**begin**

**definition** *hoare-rp* ::  $'\alpha \text{ upred} \Rightarrow (' \alpha, \text{real pos}) \text{ rdes} \Rightarrow ' \alpha \text{ upred} \Rightarrow \text{bool}$  ( $\{\!\{-\}\!\} / - / \{\!\{-\}\!\}_r$ ) **where**  
 $[\text{upred-defs}]: \text{hoare-rp } p \ Q \ r = (([p]_{S<} \Rightarrow [r]_{S>}) \sqsubseteq Q)$

**lemma** *hoare-rp-conseq*:  
 $\llbracket 'p \Rightarrow p'; 'q' \Rightarrow q'; \{\!\{p'\}\!\} S \{\!\{q'\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} S \{\!\{q\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *hoare-rp-weaken*:  
 $\llbracket 'p \Rightarrow p'; \{\!\{p'\}\!\} S \{\!\{q\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} S \{\!\{q\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *hoare-rp-strengthen*:  
 $\llbracket 'q' \Rightarrow q'; \{\!\{p\}\!\} S \{\!\{q'\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} S \{\!\{q\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *false-pre-hoare-rp* [*hoare-safe*]:  $\{\!\{false\}\!\} P \{\!\{q\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *true-post-hoare-rp* [*hoare-safe*]:  $\{\!\{p\}\!\} Q \{\!\{true\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *miracle-hoare-rp* [*hoare-safe*]:  $\{\!\{p\}\!\} false \{\!\{q\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *assigns-hoare-rp* [*hoare-safe*]:  $'p \Rightarrow \sigma \dagger q' \Longrightarrow \{\!\{p\}\!\} \langle \sigma \rangle_r \{\!\{q\}\!\}_r$   
**by** *rel-auto*

**lemma** *skip-hoare-rp* [*hoare-safe*]:  $\{\!\{p\}\!\} II_r \{\!\{p\}\!\}_r$   
**by** *rel-auto*

**lemma** *seq-hoare-rp*:  $\llbracket \{\!\{p\}\!\} Q_1 \{\!\{s\}\!\}_r ; \{\!\{s\}\!\} Q_2 \{\!\{r\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} Q_1 ;; Q_2 \{\!\{r\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *seq-est-hoare-rp* [*hoare-safe*]:  
 $\llbracket \{\!\{true\}\!\} Q_1 \{\!\{p\}\!\}_r ; \{\!\{p\}\!\} Q_2 \{\!\{p\}\!\}_r \rrbracket \Longrightarrow \{\!\{true\}\!\} Q_1 ;; Q_2 \{\!\{p\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *seq-inv-hoare-rp* [*hoare-safe*]:  
 $\llbracket \{\!\{p\}\!\} Q_1 \{\!\{p\}\!\}_r ; \{\!\{p\}\!\} Q_2 \{\!\{p\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} Q_1 ;; Q_2 \{\!\{p\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *cond-hoare-rp* [*hoare-safe*]:  
 $\llbracket \{\!\{b \wedge p\}\!\} P \{\!\{r\}\!\}_r ; \{\!\{\neg b \wedge p\}\!\} Q \{\!\{r\}\!\}_r \rrbracket \Longrightarrow \{\!\{p\}\!\} P \triangleleft b \triangleright_R Q \{\!\{r\}\!\}_r$   
**by** (*rel-auto*)

**lemma** *repeat-hoare-rp* [*hoare-safe*]:  
 $\{\!\{p\}\!\} Q \{\!\{p\}\!\}_r \Longrightarrow \{\!\{p\}\!\} Q \hat{\ } n \{\!\{p\}\!\}_r$   
**by** (*induct n, rel-auto+*)

**lemma** *UINF-ind-hoare-rp* [*hoare-safe*]:

$\llbracket \bigwedge i. \{p\} Q(i) \{r\}_r \rrbracket \Longrightarrow \{p\} \sqcap i \cdot Q(i) \{r\}_r$   
**by** (*rel-auto*)

**lemma** *star-hoare-rp* [*hoare-safe*]:  
 $\{p\} Q \{p\}_r \Longrightarrow \{p\} Q^* \{p\}_r$   
**by** (*simp add: ustar-def hoare-safe*)

**lemma** *conj-hoare-rp* [*hoare-safe*]:  
 $\llbracket \{p_1\} Q_1 \{r_1\}_r; \{p_2\} Q_2 \{r_2\}_r \rrbracket \Longrightarrow \{p_1 \wedge p_2\} Q_1 \wedge Q_2 \{r_1 \wedge r_2\}_r$   
**by** (*rel-auto*)

**lemma** *iter-hoare-rp* [*hoare-safe*]:  
 $\{I\} P \{I\}_r \Longrightarrow \{I\} P^{*r} \{I\}_r$   
**by** (*metis rrel-theory.utp-star-def seq-hoare-rp skip-hoare-rp star-hoare-rp*)

**end**

## 11 Meta-theory for Generalised Reactive Processes

**theory** *utp-reactive*  
**imports**  
*utp-rea-core*  
*utp-rea-event*  
*utp-rea-healths*  
*utp-rea-parallel*  
*utp-rea-rel*  
*utp-rea-cond*  
*utp-rea-prog*  
*utp-rea-wp*  
*utp-rea-hoare*  
**begin end**

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