

Circus Modelling Language in Isabelle/UTP

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1 Introduction

This document contains a mechanisation in Isabelle/UTP [1] of Circus [2].

2 Circus Core Types

```
theory utp-circus-core
  imports UTP-Reactive-Designs.utp-rea-designs
begin
```

2.1 Circus Alphabet

alphabet $'\varphi$ *csp-vars* = $'\sigma$ *rsp-vars* +
 $ref :: '\varphi$ *set*

declare *csp-vars.defs* [*lens-defs*]
declare *csp-vars.splits* [*alpha-splits*]

The following two locale interpretations are a technicality to improve the behaviour of the automatic tactics. They enable (re)interpretation of state spaces in order to remove any occurrences of lens types, replacing them by tuple types after the tactics *pred-simp* and *rel-simp* are applied. Eventually, it would be desirable to automate preform these interpretations automatically as part of the **alphabet** command.

interpretation *alphabet-csp-prd*:
 $lens_interp \lambda(ok, wait, tr, m). (ok, wait, tr, ref_v m, more m)$
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

interpretation *alphabet-csp-rel*:
 $lens_interp \lambda(ok, ok', wait, wait', tr, tr', m, m').$
 $(ok, ok', wait, wait', tr, tr', ref_v m, ref_v m', more m, more m')$
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

lemma *circus-var-ords* [*usubst*]:
 $\$ref \prec_v \ref'
 $\$ok \prec_v \$ref \ \$ok' \prec_v \$ref' \ \$ok \prec_v \$ref' \ \$ok' \prec_v \ref
 $\$ref \prec_v \$wait \ \$ref' \prec_v \$wait' \ \$ref \prec_v \$wait' \ \$ref' \prec_v \$wait$
 $\$ref \prec_v \$st \ \$ref' \prec_v \$st' \ \$ref \prec_v \$st' \ \$ref' \prec_v \st
 $\$ref \prec_v \$tr \ \$ref' \prec_v \$tr' \ \$ref \prec_v \$tr' \ \$ref' \prec_v \tr
by (*simp-all add: var-name-ord-def*)

type-synonym $(' \sigma, '\varphi)$ *st-csp* = $(' \sigma, '\varphi$ *list*, $(' \varphi, unit)$ *csp-vars-scheme*) *rsp*
type-synonym $(' \sigma, '\varphi)$ *action* = $(' \sigma, '\varphi)$ *st-csp* *hrel*
type-synonym $'\varphi$ *csp* = $(unit, '\varphi)$ *st-csp*
type-synonym $'\varphi$ *rel-csp* = $'\varphi$ *csp* *hrel*

There is some slight imprecision with the translations, in that we don't bother to check if the trace event type and refusal set event types are the same. Essentially this is because its very difficult to construct processes where this would be the case. However, it may be better to add a proper ML print translation in the future.

translations
 $(type) (' \sigma, '\varphi)$ *st-csp* $\leq (type) (' \sigma, '\varphi$ *list*, $'\varphi 1$ *csp-vars*) *rsp*
 $(type) (' \sigma, '\varphi)$ *action* $\leq (type) (' \sigma, '\varphi)$ *st-csp* *hrel*

notation *csp-vars-child-lens_a* (Σ_c)
notation *csp-vars-child-lens* (Σ_C)

2.2 Basic laws

lemma *R2c-tr-ext*: $R2c (\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle) = (\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle)$

by (rel-auto)

lemma *circus-alpha-bij-lens*:

bij-lens ($\{\$ok, \$ok', \$wait, \$wait', \$tr, \$tr', \$st, \$st', \$ref, \$ref'\}_\alpha :: - \implies ('s, 'e) \text{ st-csp} \times ('s, 'e) \text{ st-csp}$)
by (*unfold-locales*, *lens-simp+*)

2.3 Unrestriction laws

lemma *pre-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# pre_R(P)$
by (*simp add: pre_R-def unrest*)

lemma *peri-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# peri_R(P)$
by (*simp add: peri_R-def unrest*)

lemma *post-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# post_R(P)$
by (*simp add: post_R-def unrest*)

lemma *cmt-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# cmt_R(P)$
by (*simp add: cmt_R-def unrest*)

lemma *st-lift-unrest-ref'* [*unrest*]: $\$ref' \# \lceil b \rceil_{S<} \implies$
by (*rel-auto*)

lemma *RHS-design-ref-unrest* [*unrest*]:

$\llbracket \$ref \# P; \$ref \# Q \rrbracket \implies \$ref \# (\mathbf{R}_s(P \vdash Q)) \llbracket false / \$wait \rrbracket$
by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

lemma *R1-ref-unrest* [*unrest*]: $\$ref \# P \implies \$ref \# R1(P)$
by (*simp add: R1-def unrest*)

lemma *R2c-ref-unrest* [*unrest*]: $\$ref \# P \implies \$ref \# R2c(P)$
by (*simp add: R2c-def unrest*)

lemma *R1-ref'-unrest* [*unrest*]: $\$ref' \# P \implies \$ref' \# R1(P)$
by (*simp add: R1-def unrest*)

lemma *R2c-ref'-unrest* [*unrest*]: $\$ref' \# P \implies \$ref' \# R2c(P)$
by (*simp add: R2c-def unrest*)

lemma *R2s-notin-ref'*: $R2s(\lceil \ll x \gg \rceil_{S<} \notin_u \$ref') = (\lceil \ll x \gg \rceil_{S<} \notin_u \$ref')$
by (*pred-auto*)

lemma *unrest-circus-alpha*:

fixes $P :: ('e, 't) \text{ action}$

assumes

$\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$tr \# P$
 $\$tr' \# P \ \$st \# P \ \$st' \# P \ \$ref \# P \ \$ref' \# P$

shows $\Sigma \# P$

by (*rule bij-lens-unrest-all*[*OF circus-alpha-bij-lens*], *simp add: unrest assms*)

lemma *unrest-all-circus-vars*:

fixes $P :: ('s, 'e) \text{ action}$

assumes $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \Sigma \# r' \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$
shows $\Sigma \# [\$ref' \mapsto_s r', \$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$

using *assms*

by (*simp add: bij-lens-unrest-all-eq*[*OF circus-alpha-bij-lens*] *unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

lemma *unrest-all-circus-vars-st-st'*:

fixes $P :: ('s, 'e) \text{ action}$

assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \Sigma \# s \Sigma \# s' \Sigma \# t \Sigma \# t'$

shows $\Sigma \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

using *assms*

by (simp add: bij-lens-unrest-all-eq[*OF circus-alpha-bij-lens*] *unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

lemma *unrest-all-circus-vars-st*:

fixes $P :: ('s, 'e) \text{ action}$

assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$

shows $\Sigma \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

using *assms*

by (simp add: bij-lens-unrest-all-eq[*OF circus-alpha-bij-lens*] *unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

lemma *unrest-any-circus-var*:

fixes $P :: ('s, 'e) \text{ action}$

assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \Sigma \# s \Sigma \# s' \Sigma \# t \Sigma \# t'$

shows $x \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

by (simp add: unrest-all-var unrest-all-circus-vars-st-st' *assms*)

lemma *unrest-any-circus-var-st*:

fixes $P :: ('s, 'e) \text{ action}$

assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$

shows $x \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

by (simp add: unrest-all-var unrest-all-circus-vars-st *assms*)

end

3 Circus Reactive Relations

theory *utp-circus-rel*

imports *utp-circus-core*

begin

3.1 Healthiness Conditions

CSP Reactive Relations

definition $CRR :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ **where**

[*upred-defs*]: $CRR(P) = (\exists \$ref \cdot RR(P))$

lemma *CRR-idem*: $CRR(CRR(P)) = CRR(P)$

by (*rel-auto*)

lemma *Idempotent-CRR* [*closure*]: *Idempotent CRR*

by (simp add: *CRR-idem Idempotent-def*)

lemma *CRR-intro*:

assumes $\$ref \# P$ *P is RR*

shows *P is CRR*

by (simp add: *CRR-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

CSP Reactive Conditions

definition $CRC :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ **where**
 $[upred-defs]: CRC(P) = (\exists \$ref \cdot RC(P))$

lemma $CRC\text{-}intro$:
assumes $\$ref \# P$ P *is* RC
shows P *is* CRC
by (*simp add: CRC-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

lemma $ref\text{-}unrest\text{-}RR$ [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RR\ P$
by (*rel-auto, blast+*)

lemma $ref\text{-}unrest\text{-}RC1$ [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RC1\ P$
by (*rel-auto, blast+*)

lemma $ref\text{-}unrest\text{-}RC$ [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RC\ P$
by (*simp add: RC-R2-def ref-unrest-RC1 ref-unrest-RR*)

lemma $RR\text{-}ex\text{-}ref$: $RR (\exists \$ref \cdot RR\ P) = (\exists \$ref \cdot RR\ P)$
by (*rel-auto*)

lemma $RC1\text{-}ex\text{-}ref$: $RC1 (\exists \$ref \cdot RC1\ P) = (\exists \$ref \cdot RC1\ P)$
by (*rel-auto, meson dual-order.trans*)

lemma $ex\text{-}ref'\text{-}RR\text{-}closed$ [*closure*]:
assumes P *is* RR
shows $(\exists \$ref' \cdot P)$ *is* RR
proof –
have $RR (\exists \$ref' \cdot RR(P)) = (\exists \$ref' \cdot RR(P))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma $CRC\text{-}idem$: $CRC(CRC(P)) = CRC(P)$
apply (*simp add: CRC-def ex-unrest unrest*)
apply (*simp add: RC-def RR-ex-ref*)
apply (*metis (no-types, hide-lams) Healthy-def RC1-RR-closed RC1-ex-ref RR-ex-ref RR-idem*)
done

lemma $Idempotent\text{-}CRC$ [*closure*]: *Idempotent* CRC
by (*simp add: CRC-idem Idempotent-def*)

3.2 Closure Properties

lemma $CRR\text{-}implies\text{-}RR$ [*closure*]:
assumes P *is* CRR
shows P *is* RR
proof –
have $RR(CRR(P)) = CRR(P)$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def' assms*)
qed

lemma *CRC-implies-RR* [closure]:
 assumes *P* is CRC
 shows *P* is RR
proof –
 have $RR(CRC(P)) = CRC(P)$
 by (*rel-auto*)
 (*metis* (*no-types*, *lifting*) *Prefix-Order.prefixE Prefix-Order.prefixI append.assoc append-minus*) +
 thus ?thesis
 by (*metis Healthy-def assms*)
qed

lemma *CRC-implies-RC* [closure]:
 assumes *P* is CRC
 shows *P* is RC
proof –
 have $RC1(CRC(P)) = CRC(P)$
 by (*rel-auto*, *meson dual-order.trans*)
 thus ?thesis
 by (*simp add: CRC-implies-RR Healthy-if RC1-def RC-intro assms*)
qed

lemma *CRR-unrest-ref* [*unrest*]: P is CRR \implies $\$ref \# P$
 by (*metis CRR-def CRR-implies-RR Healthy-def in-var-uvar ref-vwb-lens unrest-as-exists*)

lemma *CRC-implies-CRR* [closure]:
 assumes *P* is CRC
 shows *P* is CRR
 apply (*rule CRR-intro*)
 apply (*simp-all add: unrest assms closure*)
 apply (*metis CRC-def CRC-implies-RC Healthy-def assms in-var-uvar ref-vwb-lens unrest-as-exists*)
 done

lemma *unrest-ref'-neg-RC* [*unrest*]:
 assumes *P* is RR *P* is RC
 shows $\$ref' \# P$
proof –
 have $P = (\neg_r \neg_r P)$
 by (*simp add: closure rpred assms*)
 also have $\dots = (\neg_r (\neg_r P) ;; true_r)$
 by (*metis Healthy-if RC1-def RC-implies-RC1 assms(2) calculation*)
 also have $\$ref' \# \dots$
 by (*rel-auto*)
 finally show ?thesis .
qed

lemma *rea-true-CRR* [closure]: $true_r$ is CRR
 by (*rel-auto*)

lemma *rea-true-CRC* [closure]: $true_r$ is CRC
 by (*rel-auto*)

lemma *false-CRR* [closure]: *false* is CRR
 by (*rel-auto*)

lemma *false-CRC* [closure]: *false* is CRC

by (rel-auto)

lemma *st-pred-CRR* [closure]: $[P]_{S<} \text{ is CRR}$
by (rel-auto)

lemma *st-cond-CRC* [closure]: $[P]_{S<} \text{ is CRC}$
by (rel-auto)

lemma *conj-CRC-closed* [closure]:
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \wedge Q) \text{ is CRC}$
by (rule CRC-intro, simp-all add: unrest closure)

lemma *disj-CRC-closed* [closure]:
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \vee Q) \text{ is CRC}$
by (rule CRC-intro, simp-all add: unrest closure)

lemma *shEx-CRR-closed* [closure]:
assumes $\bigwedge x. P\ x \text{ is CRR}$
shows $(\exists x \cdot P(x)) \text{ is CRR}$
proof –
have $\text{CRR}(\exists x \cdot \text{CRR}(P(x))) = (\exists x \cdot \text{CRR}(P(x)))$
by (rel-auto)
thus ?thesis
by (metis Healthy-def assms shEx-cong)
qed

lemma *USUP-ind-CRR-closed* [closure]:
assumes $\bigwedge i. P\ i \text{ is CRR}$
shows $(\bigsqcup i \cdot P(i)) \text{ is CRR}$
by (rule CRR-intro, simp-all add: assms unrest closure)

lemma *UINF-ind-CRR-closed* [closure]:
assumes $\bigwedge i. P\ i \text{ is CRR}$
shows $(\bigsqcap i \cdot P(i)) \text{ is CRR}$
by (rule CRR-intro, simp-all add: assms unrest closure)

lemma *cond-tt-CRR-closed* [closure]:
assumes $P \text{ is CRR } Q \text{ is CRR}$
shows $P \triangleleft \$tr' =_u \$tr \triangleright Q \text{ is CRR}$
by (rule CRR-intro, simp-all add: unrest assms closure)

lemma *rea-implies-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \Rightarrow_r Q) \text{ is CRR}$
by (simp-all add: CRR-intro closure unrest)

lemma *conj-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \wedge Q) \text{ is CRR}$
by (simp-all add: CRR-intro closure unrest)

lemma *disj-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \vee Q) \text{ is CRR}$
by (rule CRR-intro, simp-all add: unrest closure)

lemma *rea-not-CRR-closed* [closure]:
 $P \text{ is CRR} \implies (\neg_r P) \text{ is CRR}$

using *false-CRR rea-implies-CRR-closed* **by** *fastforce*

lemma *disj-R1-closed* [closure]: $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \vee Q) \text{ is } R1$
by (*rel-blast*)

lemma *st-cond-R1-closed* [closure]: $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } R1$
by (*rel-blast*)

lemma *cond-st-RR-closed* [closure]:
assumes $P \text{ is } RR \ Q \text{ is } RR$
shows $(P \triangleleft b \triangleright_R Q) \text{ is } RR$
apply (*rule RR-intro, simp-all add: unrest closure assms, simp add: Healthy-def R2c-condr*)
apply (*simp add: Healthy-if assms RR-implies-R2c*)
apply (*rel-auto*)
done

lemma *cond-st-CRR-closed* [closure]:
 $\llbracket P \text{ is } CRR; Q \text{ is } CRR \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } CRR$
by (*simp-all add: CRR-intro closure unrest*)

lemma *seq-CRR-closed* [closure]:
assumes $P \text{ is } CRR \ Q \text{ is } RR$
shows $(P ;; Q) \text{ is } CRR$
by (*rule CRR-intro, simp-all add: unrest assms closure*)

lemma *tr-extend-seqr-lit* [rdes]:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \ \$wait \# P \ \$ref \# P$
shows $(\$tr' =_u \$tr \hat{\ }_u \langle \ll a \gg \rangle \wedge \$st' =_u \$st) ;; P = P[\$tr \hat{\ }_u \langle \ll a \gg \rangle / \$tr]$
using *assms* **by** (*rel-auto, meson*)

lemma *tr-assign-comp* [rdes]:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \ \$wait \# P \ \$ref \# P$
shows $(\$tr' =_u \$tr \wedge \lceil \langle \sigma \rangle_a \rceil_s) ;; P = \lceil \sigma \rceil_{s\sigma} \dagger P$
using *assms* **by** (*rel-auto, meson*)

lemma *RR-msubst-tt*: $RR((P \ t) \llbracket t \rightarrow \&tt \rrbracket) = (RR \ (P \ t)) \llbracket t \rightarrow \&tt \rrbracket$
by (*rel-auto*)

lemma *RR-msubst-ref'*: $RR((P \ r) \llbracket r \rightarrow \$ref' \rrbracket) = (RR \ (P \ r)) \llbracket r \rightarrow \$ref' \rrbracket$
by (*rel-auto*)

lemma *msubst-tt-RR* [closure]: $\llbracket \bigwedge t. P \ t \text{ is } RR \rrbracket \implies (P \ t) \llbracket t \rightarrow \&tt \rrbracket \text{ is } RR$
by (*simp add: Healthy-def RR-msubst-tt*)

lemma *msubst-ref'-RR* [closure]: $\llbracket \bigwedge r. P \ r \text{ is } RR \rrbracket \implies (P \ r) \llbracket r \rightarrow \$ref' \rrbracket \text{ is } RR$
by (*simp add: Healthy-def RR-msubst-ref'*)

lemma *conj-less-tr-RR-closed* [closure]:
assumes $P \text{ is } CRR$
shows $(P \wedge \$tr <_u \$tr') \text{ is } CRR$
proof –
have $CRR(CRR(P) \wedge \$tr <_u \$tr') = (CRR(P) \wedge \$tr <_u \$tr')$
apply (*rel-auto, blast+*)

using *less-le* apply *fastforce+*
 done
 thus ?thesis
 by (metis *Healthy-def* *assms*)
 qed

lemma *conj-eq-tr-RR-closed* [closure]:

assumes *P* is *CRR*
 shows $(P \wedge \$tr' =_u \$tr)$ is *CRR*
 proof –
 have $CRR(CRR(P) \wedge \$tr' =_u \$tr) = (CRR(P) \wedge \$tr' =_u \$tr)$
 by (rel-auto, blast+)
 thus ?thesis
 by (metis *Healthy-def* *assms*)
 qed

3.3 Introduction laws

Extensionality principles for introducing refinement and equality of Circus reactive relations. It is necessary only to consider a subset of the variables that are present.

lemma *CRR-refine-ext*:

assumes
 P is *CRR* Q is *CRR*
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] \sqsubseteq Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 shows $P \sqsubseteq Q$
 proof –
 have $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 $\sqsubseteq (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 by (simp add: *assms* *Healthy-if*)
 hence $CRR P \sqsubseteq CRR Q$
 by (rel-auto)
 thus ?thesis
 by (metis *Healthy-if* *assms*(1) *assms*(2))
 qed

lemma *CRR-eq-ext*:

assumes
 P is *CRR* Q is *CRR*
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] = Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 shows $P = Q$
 proof –
 have $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 $= (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 by (simp add: *assms* *Healthy-if*)
 hence $CRR P = CRR Q$
 by (rel-auto)
 thus ?thesis
 by (metis *Healthy-if* *assms*(1) *assms*(2))
 qed

lemma *CRR-refine-impl-prop*:

assumes P is *CRR* Q is *CRR*
 $\bigwedge t s s' r'. 'Q[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']' \implies 'P[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']'$
 shows $P \sqsubseteq Q$
 by (rule *CRR-refine-ext*, simp-all add: *assms* *closure* *unrest* *usubst*)

(rule refine-prop-intro, simp-all add: unrest unrest-all-circus-vars closure assms)

3.4 Weakest Precondition

lemma *nil-least* [simp]:

$\langle \rangle \leq_u x = \text{true}$ **by** *rel-auto*

lemma *minus-nil* [simp]:

$xs - \langle \rangle = xs$ **by** *rel-auto*

lemma *wp-rea-circus-lemma-1*:

assumes P is CRR $\$ref' \# P$

shows $out\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

proof –

have $out\alpha \# (CRR (\exists \$ref' \cdot P))[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

by (*rel-auto*)

thus *?thesis*

by (*simp add: Healthy-if assms(1) assms(2) ex-unrest*)

qed

lemma *wp-rea-circus-lemma-2*:

assumes P is CRR

shows $in\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

proof –

have $in\alpha \# (CRR P)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

by (*rel-auto*)

thus *?thesis*

by (*simp add: Healthy-if assms ex-unrest*)

qed

The meaning of reactive weakest precondition for Circus. $P \text{ wp}_r Q$ means that, whenever P terminates in a state s_0 having done the interaction trace t_0 , which is a prefix of the overall trace, then Q must be satisfied. This in particular means that the remainder of the trace after t_0 must not be a divergent behaviour of Q .

lemma *wp-rea-circus-form*:

assumes P is CRR $\$ref' \# P$ Q is CRC

shows $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \ll t_0 \gg \leq_u \$tr' \wedge P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \Rightarrow_r Q[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr])$

proof –

have $(P \text{ wp}_r Q) = (\neg_r (\exists t_0 \cdot P[\ll t_0 \gg / \$tr']) ; (\neg_r Q)[\ll t_0 \gg / \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

by (*simp-all add: wp-rea-def R2-tr-middle closure assms*)

also have $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) ; (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

by (*rel-blast*)

also have $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \wedge (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

by (*simp add: seqr-to-conj add: wp-rea-circus-lemma-1 wp-rea-circus-lemma-2 assms closure conj-assoc*)

also have $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

by (*rel-auto*)

also have $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

by (*simp add: Healthy-if assms closure*)

also have $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee (RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

by (*rel-auto*)

also have ... = $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$
 by (rel-auto)
 also have ... = $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r Q[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$
 by (simp add: Healthy-if assms closure)
 finally show ?thesis .
 qed

lemma wp-rea-circus-form-alt:

assumes P is CRR $\$ref' \# P$ Q is CRC

shows $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

proof –

have $(P \text{ wp}_r Q) = R2(P \text{ wp}_r Q)$

by (simp add: CRC-implies-RR CRR-implies-RR Healthy-if RR-implies-R2 assms wp-rea-R2-closed)

also have ... = $R2(\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st, \$tr]$

by (simp add: wp-rea-circus-form assms closure Healthy-if)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle, \langle tt_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (simp add: R2-form, rel-auto)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \langle tt_0 - tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto, (metis list-concat-minus-list-concat)+)

also have ... = $(\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1((RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr'])$

by (rel-auto, blast+)

also have ... = $(\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

by (simp add: Healthy-if assms closure)

finally show ?thesis .

qed

lemma wp-rea-circus-form-alt:

assumes P is CRR $\$ref' \# P$ Q is CRC

shows $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

oops

3.5 Trace Substitution

definition trace-subst $(-[_])_t$ [999, 0] 999

where [upred-defs]: $P[v]_t = (P[\&tt - [v]_{S<} / \&tt] \wedge \$tr + [v]_{S<} \leq_u \$tr')$

lemma unrest-trace-subst [unrest]:

$\llbracket \text{mwb-lens } x; x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \# P \rrbracket \Longrightarrow x \# P[v]_t$

by (simp add: trace-subst-def lens-indep-sym unrest)

lemma trace-subst-RR-closed [closure]:

assumes P is RR

shows $P[v]_t$ is RR

proof –

```

have (RR P)⟦v⟧t is RR
  apply (rel-auto)
  apply (metis diff-add-cancel-left' trace-class.add-left-mono)
  apply (metis le-add minus-cancel-le trace-class.add-diff-cancel-left)
  using le-add order-trans apply blast
done
thus ?thesis
  by (simp add: Healthy-if assms)
qed

```

```

lemma trace-subst-CRR-closed [closure]:
  assumes P is CRR
  shows P⟦v⟧t is CRR
  by (rule CRR-intro, simp-all add: closure assms unrest)

```

```

lemma tsubst-nil [usubst]:
  assumes P is CRR
  shows P⟦⟨⟩⟧t = P
proof -
  have (CRR P)⟦⟨⟩⟧t = CRR P
  by (rel-auto)
  thus ?thesis
  by (simp add: Healthy-if assms)
qed

```

```

lemma tsubst-false [usubst]: false⟦y⟧t = false
  by rel-auto

```

```

lemma cond-rea-tt-subst [usubst]:
  (P ◁ b ▷R Q)⟦v⟧t = (P⟦v⟧t ◁ b ▷R Q⟦v⟧t)
  by (rel-auto)

```

```

lemma tsubst-conj [usubst]: (P ∧ Q)⟦v⟧t = (P⟦v⟧t ∧ Q⟦v⟧t)
  by (rel-auto)

```

```

lemma tsubst-disj [usubst]: (P ∨ Q)⟦v⟧t = (P⟦v⟧t ∨ Q⟦v⟧t)
  by (rel-auto)

```

```

lemma rea-subst-R1-closed [closure]: P⟦v⟧t is R1
  apply (rel-auto) using le-add order.trans by blast

```

```

lemma tsubst-UINF-ind [usubst]: (⋀ i • P(i))⟦v⟧t = (⋀ i • (P(i))⟦v⟧t)
  by (rel-auto)

```

3.6 Initial Interaction

definition *rea-init* :: 's upred ⇒ ('t::trace, 's) uexpr ⇒ ('s, 't, 'α, 'β) rel-rsp (I'(-,-)) **where**
[upred-defs]: $\mathcal{I}(s, t) = ([s]_{S<} \wedge \$tr + [t]_{S<} \leq_u \$tr')$

$\mathcal{I}(s, t)$ is a predicate stating that, if the initial state satisfies state predicate s , then the trace t is an initial trace.

```

lemma unrest-rea-init [unrest]:
  ⟦ x ⋈ ($tr)v; x ⋈ ($tr')v; x ⋈ ($st)v ⟧ ⇒ x # I(s, t)
  by (simp add: rea-init-def unrest lens-indep-sym)

```

lemma *rea-init-R1* [closure]: $\mathcal{I}(s, t)$ is *R1*
apply (*rel-auto*) **using** *dual-order.trans* *le-add* **by** *blast*

lemma *rea-init-R2c* [closure]: $\mathcal{I}(s, t)$ is *R2c*
apply (*rel-auto*)
apply (*metis diff-add-cancel-left' trace-class.add-left-mono*)
apply (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)
done

lemma *rea-init-R2* [closure]: $\mathcal{I}(s, t)$ is *R2*
by (*metis Healthy-def R1-R2c-is-R2 rea-init-R1 rea-init-R2c*)

lemma *csp-init-RR* [closure]: $\mathcal{I}(s, t)$ is *RR*
apply (*rel-auto*)
apply (*metis diff-add-cancel-left' trace-class.add-left-mono*)
apply (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)
apply (*metis le-add less-le less-le-trans*)
done

lemma *csp-init-CRR* [closure]: $\mathcal{I}(s, t)$ is *CRR*
by (*rule CRR-intro, simp-all add: unrest closure*)

lemma *rea-init-impl-st* [closure]: $(\mathcal{I}(b, t) \Rightarrow_r [c]_{S<})$ is *RC*
apply (*rule RC-intro*)
apply (*simp add: closure*)
apply (*rel-auto*)
using *order-trans* **by** *auto*

lemma *rea-init-RC1*:
 $\neg_r \mathcal{I}(P, t)$ is *RC1*
apply (*rel-auto*) **using** *dual-order.trans* **by** *blast*

lemma *init-acts-empty* [*rpred*]: $\mathcal{I}(\text{true}, \langle \rangle) = \text{true}_r$
by (*rel-auto*)

lemma *rea-not-init* [*rpred*]:
 $(\neg_r \mathcal{I}(P, \langle \rangle)) = \mathcal{I}(\neg P, \langle \rangle)$
by (*rel-auto*)

lemma *rea-init-conj* [*rpred*]:
 $(\mathcal{I}(P, t) \wedge \mathcal{I}(Q, t)) = \mathcal{I}(P \wedge Q, t)$
by (*rel-auto*)

lemma *rea-init-empty-trace* [*rpred*]: $\mathcal{I}(s, \langle \rangle) = [s]_{S<}$
by (*rel-auto*)

lemma *rea-init-disj-same* [*rpred*]: $(\mathcal{I}(s_1, t) \vee \mathcal{I}(s_2, t)) = \mathcal{I}(s_1 \vee s_2, t)$
by (*rel-auto*)

lemma *rea-init-impl-same* [*rpred*]: $(\mathcal{I}(s_1, t) \Rightarrow_r \mathcal{I}(s_2, t)) = (\mathcal{I}(s_1, t) \Rightarrow_r [s_2]_{S<})$
apply (*rel-auto*) **using** *dual-order.trans* *le-add* **by** *blast+*

lemma *tsubst-st-cond* [*ustbst*]: $[P]_{S<} \llbracket t \rrbracket_t = \mathcal{I}(P, t)$
by (*rel-auto*)

lemma *tsubst-rea-init* [*usubst*]: $(\mathcal{I}(s, x))\llbracket y \rrbracket_t = \mathcal{I}(s, y+x)$
apply (*rel-auto*)
apply (*metis add.assoc diff-add-cancel-left' trace-class.add-le-imp-le-left trace-class.add-left-mono*)
apply (*metis add.assoc diff-add-cancel-left' le-add trace-class.add-le-imp-le-left trace-class.add-left-mono*) +
done

lemma *tsubst-rea-not* [*usubst*]: $(\neg_r P)\llbracket v \rrbracket_t = ((\neg_r P)\llbracket v \rrbracket_t) \wedge \mathcal{I}(\text{true}, v)$
apply (*rel-auto*)
using *le-add order-trans* **by** *blast*

lemma *tsubst-true* [*usubst*]: $\text{true}_r\llbracket v \rrbracket_t = \mathcal{I}(\text{true}, v)$
by (*rel-auto*)

lemma *R4-csp-init* [*rpred*]: $R4(\mathcal{I}(s, \text{bop Cons } x \text{ } xs)) = \mathcal{I}(s, \text{bop Cons } x \text{ } xs)$
using *less-list-def* **by** (*rel-blast*)

lemma *R5-csp-init* [*rpred*]: $R5(\mathcal{I}(s, \text{bop Cons } x \text{ } xs)) = \text{false}$
by (*rel-auto*)

lemma *R4-trace-subst* [*rpred*]:
 $R4(P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t) = P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t$
using *le-imp-less-or-eq* **by** (*rel-blast*)

lemma *R5-trace-subst* [*rpred*]:
 $R5(P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t) = \text{false}$
by (*rel-auto*)

3.7 Enabled Events

definition *csp-enable* :: $'s \text{ upred} \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('e \text{ set}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\mathcal{E}'(-, -, -))$
where
[upred-defs]: $\mathcal{E}(s, t, E) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\ }_u [t]_{S<} \wedge (\forall e \in [E]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref'))$

Predicate $\mathcal{E}(s, t, E)$ states that, if the initial state satisfies predicate s , then t is a possible (failure) trace, such that the events in the set E are enabled after the given interaction.

lemma *csp-enable-R1-closed* [*closure*]: $\mathcal{E}(s, t, E)$ is *R1*
by (*rel-auto*)

lemma *csp-enable-R2-closed* [*closure*]: $\mathcal{E}(s, t, E)$ is *R2c*
by (*rel-auto*)

lemma *csp-enable-RR* [*closure*]: $\mathcal{E}(s, t, E)$ is *CRR*
by (*rel-auto*)

lemma *tsubst-csp-enable* [*usubst*]: $\mathcal{E}(s, t_2, e)\llbracket t_1 \rrbracket_t = \mathcal{E}(s, t_1 \hat{\ }_u t_2, e)$
apply (*rel-auto*)
apply (*metis append.assoc less-eq-list-def prefix-concat-minus*)
apply (*simp add: list-concat-minus-list-concat*)
done

lemma *csp-enable-unrests* [*unrest*]:
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$ref')_v \rrbracket \Longrightarrow x \# \mathcal{E}(s, t, e)$
by (*simp add: csp-enable-def R1-def lens-indep-sym unrest*)

lemma *csp-enable-tr'-eq-tr* [*rpred*]:

$\mathcal{E}(s, \langle \rangle, r) \triangleleft \$tr' =_u \$tr \triangleright false = \mathcal{E}(s, \langle \rangle, r)$
by (*rel-auto*)

lemma *csp-enable-st-pred* [*rpred*]:
 $([s_1]_{S<} \wedge \mathcal{E}(s_2, t, E)) = \mathcal{E}(s_1 \wedge s_2, t, E)$
by (*rel-auto*)

lemma *csp-enable-conj* [*rpred*]:
 $(\mathcal{E}(s, t, E_1) \wedge \mathcal{E}(s, t, E_2)) = \mathcal{E}(s, t, E_1 \cup_u E_2)$
by (*rel-auto*)

lemma *csp-enable-cond* [*rpred*]:
 $\mathcal{E}(s_1, t_1, E_1) \triangleleft b \triangleright_R \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_1 \triangleleft b \triangleright s_2, t_1 \triangleleft b \triangleright t_2, E_1 \triangleleft b \triangleright E_2)$
by (*rel-auto*)

lemma *csp-enable-tr-empty*: $\mathcal{E}(true, \langle \rangle, \{v\}_u) = (\$tr' =_u \$tr \wedge \lceil v \rceil_{S<} \notin_u \$ref')$
by (*rel-auto*)

lemma *csp-enable-nothing*: $\mathcal{E}(true, \langle \rangle, \{\}_u) = (\$tr' =_u \$tr)$
by (*rel-auto*)

lemma *msubst-nil-csp-enable* [*usubst*]:
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \langle \rangle \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow \langle \rangle \rrbracket, t(x) \llbracket x \rightarrow \langle \rangle \rrbracket, E(x) \llbracket x \rightarrow \langle \rangle \rrbracket)$
by (*pred-auto*)

lemma *msubst-csp-enable* [*usubst*]:
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \lceil v \rceil_{S\leftarrow} \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow v \rrbracket, t(x) \llbracket x \rightarrow v \rrbracket, E(x) \llbracket x \rightarrow v \rrbracket)$
by (*rel-auto*)

lemma *csp-enable-false* [*rpred*]: $\mathcal{E}(false, t, E) = false$
by (*rel-auto*)

lemma *conj-csp-enable* [*rpred*]: $(\mathcal{E}(b_1, t, E_1) \wedge \mathcal{E}(b_2, t, E_2)) = \mathcal{E}(b_1 \wedge b_2, t, E_1 \cup_u E_2)$
by (*rel-auto*)

lemma *USUP-csp-enable* [*rpred*]:
 $(\bigsqcup x \cdot \mathcal{E}(s, t, A(x))) = \mathcal{E}(s, t, (\bigvee x \cdot A(x)))$
by (*rel-auto*)

lemma *R4-csp-enable-nil* [*rpred*]:
 $R4(\mathcal{E}(s, \langle \rangle, E)) = false$
by (*rel-auto*)

lemma *R5-csp-enable-nil* [*rpred*]:
 $R5(\mathcal{E}(s, \langle \rangle, E)) = \mathcal{E}(s, \langle \rangle, E)$
by (*rel-auto*)

lemma *R4-csp-enable-Cons* [*rpred*]:
 $R4(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = \mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)$
by (*rel-auto*, *simp add: Prefix-Order.strict-prefixI'*)

lemma *R5-csp-enable-Cons* [*rpred*]:
 $R5(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = false$
by (*rel-auto*)

3.8 Completed Trace Interaction

definition $csp\text{-}do :: 's \text{ upred} \Rightarrow ('s \Rightarrow 's) \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\Phi'(-,-,-))$ **where**
 $[upred\text{-}defs]: \Phi(s, \sigma, t) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\ }_u [t]_{S<} \wedge [\langle \sigma \rangle_a]_S)$

Predicate $\Phi(s, \sigma, t)$ states that if the initial state satisfies s , and the trace t is performed, then afterwards the state update σ is executed.

lemma $unrest\text{-}csp\text{-}do$ $[unrest]:$

$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$st')_v \rrbracket \Longrightarrow x \# \Phi(s, \sigma, t)$

by ($simp\text{-}all$ $add: csp\text{-}do\text{-}def$ $alpha\text{-}in\text{-}var$ $alpha\text{-}out\text{-}var$ $prod\text{-}as\text{-}plus$ $unrest$ $lens\text{-}indep\text{-}sym$)

lemma $csp\text{-}do\text{-}CRR$ $[closure]: \Phi(s, \sigma, t)$ *is CRR*

by ($rel\text{-}auto$)

lemma $csp\text{-}do\text{-}R4\text{-}closed$ $[closure]:$

$\Phi(b, \sigma, bop \text{ Cons } x \ xs)$ *is R4*

by ($rel\text{-}auto$, $simp$ $add: Prefix\text{-}Order.strict\text{-}prefixI'$)

lemma $st\text{-}pred\text{-}conj\text{-}csp\text{-}do$ $[rpred]:$

$([b]_{S<} \wedge \Phi(s, \sigma, t)) = \Phi(b \wedge s, \sigma, t)$

by ($rel\text{-}auto$)

lemma $trea\text{-}subst\text{-}csp\text{-}do$ $[usubst]:$

$(\Phi(s, \sigma, t_2)) \llbracket t_1 \rrbracket_t = \Phi(s, \sigma, t_1 \hat{\ }_u t_2)$

apply ($rel\text{-}auto$)

apply ($metis$ $append.assoc$ $less\text{-}eq\text{-}list\text{-}def$ $prefix\text{-}concat\text{-}minus$)

apply ($simp$ $add: list\text{-}concat\text{-}minus\text{-}list\text{-}concat$)

done

lemma $st\text{-}subst\text{-}csp\text{-}do$ $[usubst]:$

$[\sigma]_{S\sigma} \dagger \Phi(s, \sigma, t) = \Phi(\sigma \dagger s, \sigma \circ \sigma, \sigma \dagger t)$

by ($rel\text{-}auto$)

lemma $csp\text{-}init\text{-}do$ $[rpred]: (\mathcal{I}(s1, t) \wedge \Phi(s2, \sigma, t)) = \Phi(s1 \wedge s2, \sigma, t)$

by ($rel\text{-}auto$)

lemma $csp\text{-}do\text{-}false$ $[rpred]: \Phi(false, s, t) = false$

by ($rel\text{-}auto$)

lemma $csp\text{-}do\text{-}assign$ $[rpred]:$

assumes P *is CRR*

shows $\Phi(s, \sigma, t) ;; P = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger P) \llbracket t \rrbracket_t)$

proof –

have $\Phi(s, \sigma, t) ;; CRR(P) = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger CRR(P)) \llbracket t \rrbracket_t)$

by ($rel\text{-}blast$)

thus *?thesis*

by ($simp$ $add: Healthy\text{-}if$ $assms$)

qed

lemma $subst\text{-}state\text{-}csp\text{-}enable$ $[usubst]:$

$[\sigma]_{S\sigma} \dagger \mathcal{E}(s, t_2, e) = \mathcal{E}(\sigma \dagger s, \sigma \dagger t_2, \sigma \dagger e)$

by ($rel\text{-}auto$)

lemma $csp\text{-}do\text{-}assign\text{-}enable$ $[rpred]:$

$\Phi(s1, \sigma, t1) ;; \mathcal{E}(s2, t2, e) = \mathcal{E}(s1 \wedge \sigma \dagger s2, t1 \hat{\ }_u (\sigma \dagger t2), (\sigma \dagger e))$

by ($simp$ $add: rpred$ $closure$ $usubst$)

lemma *csp-do-assign-do* [*rpred*]:

$\Phi(s_1, \sigma, t_1) ;; \Phi(s_2, \varrho, t_2) = \Phi(s_1 \wedge (\sigma \dagger s_2), \varrho \circ \sigma, t_1 \hat{^u} (\sigma \dagger t_2))$
by (*rel-auto*)

lemma *csp-do-cond* [*rpred*]:

$\Phi(s_1, \sigma, t_1) \triangleleft b \triangleright_R \Phi(s_2, \varrho, t_2) = \Phi(s_1 \triangleleft b \triangleright s_2, \sigma \triangleleft b \triangleright_s \varrho, t_1 \triangleleft b \triangleright t_2)$
by (*rel-auto*)

lemma *csp-do-skip* [*rpred*]:

assumes *P* is *CRR*
shows $\Phi(\text{true}, \text{id}, t) ;; P = P \llbracket t \rrbracket_t$

proof –

have $\Phi(\text{true}, \text{id}, t) ;; \text{CRR}(P) = (\text{CRR } P) \llbracket t \rrbracket_t$
by (*rel-auto*)

thus *?thesis*

by (*simp add: Healthy-if assms*)

qed

lemma *wp-rea-csp-do-lemma*:

fixes *P* :: (*'σ*, *'φ*) *action*

assumes $\$ok \# P \$wait \# P \$ref \# P$

shows $(\llbracket \langle \sigma \rangle_a \rrbracket_S \wedge \$tr' =_u \$tr \hat{^u} \llbracket t \rrbracket_{S<}) ;; P = (\llbracket \sigma \rrbracket_{S\sigma} \dagger P) \llbracket \$tr \hat{^u} \llbracket t \rrbracket_{S<} / \$tr \rrbracket$
using *assms* **by** (*rel-auto, meson*)

lemma *wp-rea-csp-do* [*wp*]:

fixes *P* :: (*'σ*, *'φ*) *action*

assumes *P* is *CRR*

shows $\Phi(s, \sigma, t) \text{wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r (\llbracket \sigma \rrbracket_{S\sigma} \dagger P) \llbracket t \rrbracket_t)$

proof –

have $\Phi(s, \sigma, t) \text{wp}_r \text{CRR}(P) = (\mathcal{I}(s, t) \Rightarrow_r (\llbracket \sigma \rrbracket_{S\sigma} \dagger \text{CRR}(P)) \llbracket t \rrbracket_t)$
by (*rel-blast*)

thus *?thesis*

by (*simp add: assms Healthy-if*)

qed

lemma *csp-do-power-Suc* [*rpred*]:

$\Phi(\text{true}, \text{id}, t) \hat{^u} (\text{Suc } i) = \Phi(\text{true}, \text{id}, \text{iter}[\text{Suc } i](t))$

by (*induct i, (rel-auto)+*)

lemma *csp-power-do-comp* [*rpred*]:

assumes *P* is *CRR*

shows $\Phi(\text{true}, \text{id}, t) \hat{^u} i ;; P = \Phi(\text{true}, \text{id}, \text{iter}[i](t)) ;; P$

apply (*cases i*)

apply (*simp-all add: rpred usubst assms closure*)

done

lemma *wp-rea-csp-do-skip* [*wp*]:

fixes *Q* :: (*'σ*, *'φ*) *action*

assumes *P* is *CRR*

shows $\Phi(s, \text{id}, t) \text{wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r P \llbracket t \rrbracket_t)$

proof –

have $\Phi(s, \text{id}, t) \text{wp}_r P = \Phi(s, \text{id}, t) \text{wp}_r P$

by (*simp add: skip-r-def*)

thus *?thesis* **by** (*simp add: wp assms usubst alpha*)

qed

lemma *msubst-csp-do* [*usubst*]:

$\Phi(s(x), \sigma, t(x)) \llbracket x \rightarrow [v]_{S \leftarrow} \rrbracket = \Phi(s(x) \llbracket x \rightarrow v \rrbracket, \sigma, t(x) \llbracket x \rightarrow v \rrbracket)$
 by (*rel-auto*)

end

4 Circus and CSP Healthiness Conditions

theory *utp-circus-healths*

imports *utp-circus-rel*

begin

5 Definitions

We here define extra healthiness conditions for Circus / CSP processes.

abbreviation *CSP1* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$

where $CSP1(P) \equiv RD1(P)$

abbreviation *CSP2* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$

where $CSP2(P) \equiv RD2(P)$

abbreviation *CSP* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$

where $CSP(P) \equiv SRD(P)$

definition *STOP* :: $\sigma \text{ rel-csp}$ **where**

[*upred-defs*]: $STOP = CSP1(\$ok' \wedge R3c(\$tr' =_u \$tr \wedge \$wait'))$

definition *SKIP* :: $\sigma \text{ rel-csp}$ **where**

[*upred-defs*]: $SKIP = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$

definition *Stop* :: $(\sigma, \varphi) \text{ action}$ **where**

[*upred-defs*]: $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \$wait'))$

definition *Skip* :: $(\sigma, \varphi) \text{ action}$ **where**

[*upred-defs*]: $Skip = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st))$

definition *CSP3* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$ **where**

[*upred-defs*]: $CSP3(P) = (Skip \mathrel{;;} P)$

definition *CSP4* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$ **where**

[*upred-defs*]: $CSP4(P) = (P \mathrel{;;} Skip)$

definition *NCSP* :: $((\sigma, \varphi) \text{ st-csp} \times (\sigma, \varphi) \text{ st-csp}) \text{ health}$ **where**

[*upred-defs*]: $NCSP = CSP3 \circ CSP4 \circ CSP$

Productive and normal processes

abbreviation *PCSP* $\equiv Productive \circ NCSP$

Instantaneous and normal processes

abbreviation *ICSP* $\equiv ISR1 \circ NCSP$

5.1 Healthiness condition properties

SKIP is the same as *Skip*, and *STOP* is the same as *Stop*, when we consider stateless CSP processes. This is because any reference to the *st* variable degenerates when the alphabet type coerces its type to be empty. We therefore need not consider *SKIP* and *STOP* actions.

theorem *SKIP-is-Skip*: $SKIP = Skip$

by (*rel-auto*)

theorem *STOP-is-Stop*: $STOP = Stop$

by (*rel-auto*)

theorem *Skip-UTP-form*: $Skip = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$

by (*rel-auto*)

lemma *Skip-is-CSP* [*closure*]:

Skip is CSP

by (*simp add: Skip-def RHS-design-is-SRD unrest*)

lemma *Skip-RHS-tri-design*:

$Skip = \mathbf{R}_s(true \vdash (false \diamond (\$tr' =_u \$tr \wedge \$st' =_u \$st)))$

by (*rel-auto*)

lemma *Skip-RHS-tri-design'* [*rdes-def*]:

$Skip = \mathbf{R}_s(true_r \vdash (false \diamond \Phi(true, id, \langle \rangle)))$

by (*rel-auto*)

lemma *Stop-is-CSP* [*closure*]:

Stop is CSP

by (*simp add: Stop-def RHS-design-is-SRD unrest*)

lemma *Stop-RHS-tri-design*: $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr) \diamond false)$

by (*rel-auto*)

lemma *Stop-RHS-rdes-def* [*rdes-def*]: $Stop = \mathbf{R}_s(true_r \vdash \mathcal{E}(true, \langle \rangle, \{\}_u) \diamond false)$

by (*rel-auto*)

lemma *preR-Skip* [*rdes*]: $pre_R(Skip) = true_r$

by (*rel-auto*)

lemma *periR-Skip* [*rdes*]: $peri_R(Skip) = false$

by (*rel-auto*)

lemma *postR-Skip* [*rdes*]: $post_R(Skip) = \Phi(true, id, \langle \rangle)$

by (*rel-auto*)

lemma *Productive-Stop* [*closure*]:

Stop is Productive

by (*simp add: Stop-RHS-tri-design Healthy-def Productive-RHS-design-form unrest*)

lemma *Skip-left-lemma*:

assumes *P is CSP*

shows $Skip \;; P = \mathbf{R}_s((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$

proof –

have $Skip \;; P =$

$\mathbf{R}_s((\$tr' =_u \$tr \wedge \$st' =_u \$st) \wp_r pre_R P \vdash$

$(\$tr' =_u \$tr \wedge \$st' =_u \$st) ;; \text{peri}_R P \diamond$
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) ;; \text{post}_R P$
 by (simp add: SRD-composition-wp alpha rdes closure wp assms rpred C1, rel-auto)
 also have ... = $\mathbf{R}_s ((\forall \$ref \cdot \text{pre}_R P) \vdash$
 $(\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright \text{cmt}_R P))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $\mathbf{R}_s ((\forall \$ref \cdot \text{pre}_R P) \vdash (\exists \$ref \cdot \text{cmt}_R P))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 finally show ?thesis .
 qed

lemma Skip-left-unit-ref-unrest:

assumes P is CSP $\$ref \# P \llbracket \text{false} / \$wait \rrbracket$

shows $\text{Skip} ;; P = P$

using assms

by (simp add: Skip-left-lemma)

(metis SRD-reactive-design-alt all-unrest cmt-unrest-ref cmt-wait-false ex-unrest pre-unrest-ref pre-wait-false)

lemma CSP3-intro:

$\llbracket P \text{ is CSP}; \$ref \# P \llbracket \text{false} / \$wait \rrbracket \rrbracket \implies P \text{ is CSP3}$

by (simp add: CSP3-def Healthy-def' Skip-left-unit-ref-unrest)

lemma ref-unrest-RHS-design:

assumes $\$ref \# P \ \$ref \# Q_1 \ \$ref \# Q_2$

shows $\$ref \# (\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)) \text{ }_f$

by (simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst assms)

lemma CSP3-SRD-intro:

assumes P is CSP $\$ref \# \text{pre}_R(P) \ \$ref \# \text{peri}_R(P) \ \$ref \# \text{post}_R(P)$

shows P is CSP3

proof –

have $P: \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) = P$

by (simp add: SRD-reactive-design-alt assms(1) wait'-cond-peri-post-cmt[THEN sym])

have $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))$ is CSP3

by (rule CSP3-intro, simp add: assms P, simp add: ref-unrest-RHS-design assms)

thus ?thesis

by (simp add: P)

qed

lemma Skip-unrest-ref [unrest]: $\$ref \# \text{Skip} \llbracket \text{false} / \$wait \rrbracket$

by (simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)

lemma Skip-unrest-ref' [unrest]: $\$ref' \# \text{Skip} \llbracket \text{false} / \$wait \rrbracket$

by (simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)

lemma CSP3-iff:

assumes P is CSP

shows $P \text{ is CSP3} \longleftrightarrow (\$ref \# P \llbracket \text{false} / \$wait \rrbracket)$

proof

assume 1: P is CSP3

have $\$ref \# (\text{Skip} ;; P) \llbracket \text{false} / \$wait \rrbracket$

by (simp add: usubst unrest)

with 1 show $\$ref \# P \llbracket \text{false} / \$wait \rrbracket$

by (metis CSP3-def Healthy-def)

next

assume $1:\$ref \# P[\![false/\$wait]\!]$
show P is CSP3
by (*simp add: 1 CSP3-intro assms*)
qed

lemma *CSP3-unrest-ref* [*unrest*]:
assumes P is CSP P is CSP3
shows $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

proof –
have $a: (\$ref \# P[\![false/\$wait]\!])$
using *CSP3-iff assms* **by** *blast*
from a **show** $\$ref \# pre_R(P)$
by (*rel-blast*)
from a **show** $\$ref \# peri_R(P)$
by (*rel-blast*)
from a **show** $\$ref \# post_R(P)$
by (*rel-blast*)
qed

lemma *CSP3-rdes*:
assumes P is RR Q is RR R is RR
shows $CSP3(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\forall \$ref \cdot P) \vdash (\exists \$ref \cdot Q) \diamond (\exists \$ref \cdot R))$
by (*simp add: CSP3-def Skip-left-lemma closure assms rdes, rel-auto*)

lemma *CSP3-form*:
assumes P is CSP
shows $CSP3(P) = \mathbf{R}_s((\forall \$ref \cdot pre_R(P)) \vdash (\exists \$ref \cdot peri_R(P)) \diamond (\exists \$ref \cdot post_R(P)))$
by (*simp add: CSP3-def Skip-left-lemma assms, rel-auto*)

lemma *CSP3-Skip* [*closure*]:
 $Skip$ is CSP3
by (*rule CSP3-intro, simp add: Skip-is-CSP, simp add: Skip-def unrest*)

lemma *CSP3-Stop* [*closure*]:
 $Stop$ is CSP3
by (*rule CSP3-intro, simp add: Stop-is-CSP, simp add: Stop-def unrest*)

lemma *CSP3-Idempotent* [*closure*]: *Idempotent CSP3*
by (*metis (no-types, lifting) CSP3-Skip CSP3-def Healthy-if Idempotent-def seqr-assoc*)

lemma *CSP3-Continuous*: *Continuous CSP3*
by (*simp add: Continuous-def CSP3-def seq-Sup-distl*)

lemma *Skip-right-lemma*:
assumes P is CSP
shows $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$
proof –
have $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash (\exists \$st' \cdot peri_R P) \diamond post_R P ;; (\$tr' =_u \$tr \wedge \$st' =_u \$st))$
by (*simp add: SRD-composition-wp closure assms wp rdes rpred, rel-auto*)
also have $\dots = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((cmt_R P ;; (\exists \$st \cdot [II]_D)) \triangleleft \$wait' \triangleright (cmt_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$
by (*rule cong[of $\mathbf{R}_s \ \mathbf{R}_s$], simp, rel-auto*)
also have $\dots = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash$

$((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (cmt_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $\mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 finally show ?thesis .
 qed

lemma Skip-right-tri-lemma:

assumes P is CSP
 shows $P ;; Skip = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R P) \diamond (\exists \$ref' \cdot post_R P)))$
 proof –
 have $((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)) = ((\exists \$st' \cdot peri_R P) \diamond (\exists \$ref' \cdot post_R P))$
 by (rel-auto)
 thus ?thesis by (simp add: Skip-right-lemma[OF assms])
 qed

lemma CSP4-intro:

assumes P is CSP $(\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P))$
 $\$st' \# (cmt_R P) \llbracket true/\$wait' \rrbracket \$ref' \# (cmt_R P) \llbracket false/\$wait' \rrbracket$
 shows P is CSP4
 proof –
 have $CSP4(P) = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$
 by (simp add: CSP4-def Skip-right-lemma assms(1))
 also have ... = $\mathbf{R}_s (pre_R(P) \vdash ((\exists \$st' \cdot cmt_R P) \llbracket true/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P) \llbracket false/\$wait' \rrbracket))$
 by (simp add: wp-rea-def assms(2) rpred closure cond-var-subst-left cond-var-subst-right)
 also have ... = $\mathbf{R}_s (pre_R(P) \vdash ((\exists \$st' \cdot (cmt_R P) \llbracket true/\$wait' \rrbracket) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot (cmt_R P) \llbracket false/\$wait' \rrbracket)))$
 by (simp add: usubst unrest)
 also have ... = $\mathbf{R}_s (pre_R P \vdash ((cmt_R P) \llbracket true/\$wait' \rrbracket \triangleleft \$wait' \triangleright (cmt_R P) \llbracket false/\$wait' \rrbracket))$
 by (simp add: ex-unrest assms)
 also have ... = $\mathbf{R}_s (pre_R P \vdash cmt_R P)$
 by (simp add: cond-var-split)
 also have ... = P
 by (simp add: SRD-reactive-design-alt assms(1))
 finally show ?thesis
 by (simp add: Healthy-def')
 qed

lemma CSP4-RC-intro:

assumes P is CSP $pre_R(P)$ is RC
 $\$st' \# (cmt_R P) \llbracket true/\$wait' \rrbracket \$ref' \# (cmt_R P) \llbracket false/\$wait' \rrbracket$
 shows P is CSP4
 proof –
 have $(\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P))$
 by (metis (no-types, lifting) R1-seqr-closure assms(2) rea-not-R1 rea-not-false rea-not-not wp-rea-RC-false wp-rea-def)
 thus ?thesis
 by (simp add: CSP4-intro assms)
 qed

lemma CSP4-rdes:

assumes P is RR Q is RR R is RR
 shows $CSP4(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s ((\neg_r P) wp_r false \vdash ((\exists \$st' \cdot Q) \diamond (\exists \$ref' \cdot R)))$

by (simp add: CSP4-def Skip-right-lemma closure assms rdes, rel-auto, blast+)

lemma CSP4-form:

assumes P is CSP

shows $CSP_4(P) = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R P) \diamond (\exists \$ref' \cdot post_R P)))$

by (simp add: CSP4-def Skip-right-tri-lemma assms)

lemma Skip-srdes-right-unit:

$(Skip :: ('\sigma, '\varphi) \text{ action}) ;; II_R = Skip$

by (rdes-simp)

lemma Skip-srdes-left-unit:

$II_R ;; (Skip :: ('\sigma, '\varphi) \text{ action}) = Skip$

by (rdes-eq)

lemma CSP4-right-subsumes-RD3: $RD3(CSP_4(P)) = CSP_4(P)$

by (metis (no-types, hide-lams) CSP4-def RD3-def Skip-srdes-right-unit seqr-assoc)

lemma CSP4-implies-RD3: $P \text{ is } CSP_4 \implies P \text{ is } RD3$

by (metis CSP4-right-subsumes-RD3 Healthy-def)

lemma CSP4-tri-intro:

assumes $P \text{ is } CSP (\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P)) \$st' \# peri_R(P) \$ref' \# post_R(P)$

shows $P \text{ is } CSP_4$

using assms

by (rule-tac CSP4-intro, simp-all add: pre_R-def peri_R-def post_R-def usubst cmt_R-def)

lemma CSP4-NSRD-intro:

assumes $P \text{ is } NSRD \$ref' \# post_R(P)$

shows $P \text{ is } CSP_4$

by (simp add: CSP4-tri-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri assms)

lemma CSP3-commutes-CSP4: $CSP3(CSP_4(P)) = CSP_4(CSP3(P))$

by (simp add: CSP3-def CSP4-def seqr-assoc)

lemma NCSP-implies-CSP [closure]: $P \text{ is } NCSP \implies P \text{ is } CSP$

by (metis (no-types, hide-lams) CSP3-def CSP4-def Healthy-def NCSP-def SRD-idem SRD-seqr-closure Skip-is-CSP comp-apply)

lemma NCSP-elim [RD-elim]:

$\llbracket X \text{ is } NCSP; P(\mathbf{R}_s(pre_R(X) \vdash peri_R(X) \diamond post_R(X))) \rrbracket \implies P(X)$

by (simp add: SRD-reactive-tri-design closure)

lemma NCSP-implies-CSP3 [closure]:

$P \text{ is } NCSP \implies P \text{ is } CSP3$

by (metis (no-types, lifting) CSP3-def Healthy-def' NCSP-def Skip-is-CSP Skip-left-unit-ref-unrest Skip-unrest-ref comp-apply seqr-assoc)

lemma NCSP-implies-CSP4 [closure]:

$P \text{ is } NCSP \implies P \text{ is } CSP4$

by (metis (no-types, hide-lams) CSP3-commutes-CSP4 Healthy-def NCSP-def NCSP-implies-CSP NCSP-implies-CSP3 comp-apply)

lemma NCSP-implies-RD3 [closure]: $P \text{ is } NCSP \implies P \text{ is } RD3$

by (metis CSP3-commutes-CSP4 CSP4-right-subsumes-RD3 Healthy-def NCSP-def comp-apply)

lemma *NCSP-implies-NSRD* [closure]: P is NCSP $\implies P$ is NSRD
 by (simp add: NCSP-implies-CSP NCSP-implies-RD3 SRD-RD3-implies-NSRD)

lemma *NCSP-subset-implies-CSP* [closure]:
 $A \subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{CSP} \rrbracket_H$
 using NCSP-implies-CSP by blast

lemma *NCSP-subset-implies-NSRD* [closure]:
 $A \subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{NSRD} \rrbracket_H$
 using NCSP-implies-NSRD by blast

lemma *CSP-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket \text{CSP} \rrbracket_H \rrbracket \implies P$ is CSP
 by (simp add: is-Healthy-subset-member)

lemma *CSP3-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket \text{CSP3} \rrbracket_H \rrbracket \implies P$ is CSP3
 by (simp add: is-Healthy-subset-member)

lemma *CSP4-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket \text{CSP4} \rrbracket_H \rrbracket \implies P$ is CSP4
 by (simp add: is-Healthy-subset-member)

lemma *NCSP-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket \text{NCSP} \rrbracket_H \rrbracket \implies P$ is NCSP
 by (simp add: is-Healthy-subset-member)

lemma *NCSP-intro*:
 assumes P is CSP P is CSP3 P is CSP4
 shows P is NCSP
 by (metis Healthy-def NCSP-def assms comp-eq-dest-lhs)

lemma *Skip-left-unit*: P is NCSP $\implies \text{Skip} ;; P = P$
 by (metis (full-types) CSP3-def Healthy-if NCSP-implies-CSP3)

lemma *Skip-right-unit*: P is NCSP $\implies P ;; \text{Skip} = P$
 by (metis (full-types) CSP4-def Healthy-if NCSP-implies-CSP4)

lemma *NCSP-NSRD-intro*:
 assumes P is NSRD $\$ref \# pre_R(P) \$ref \# peri_R(P) \$ref \# post_R(P) \$ref' \# post_R(P)$
 shows P is NCSP
 by (simp add: CSP3-SRD-intro CSP4-NSRD-intro NCSP-intro NSRD-is-SRD assms)

lemma *CSP4-neg-pre-unit*:
 assumes P is CSP P is CSP4
 shows $(\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P))$
 by (simp add: CSP4-implies-RD3 NSRD-neg-pre-unit SRD-RD3-implies-NSRD assms(1) assms(2))

lemma *NSRD-CSP4-intro*:
 assumes P is CSP P is CSP4
 shows P is NSRD
 by (simp add: CSP4-implies-RD3 SRD-RD3-implies-NSRD assms(1) assms(2))

lemma *NCSP-form*:
 $\text{NCSP } P = \mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R(P)) wp_r false) \vdash ((\exists \$ref \cdot \exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref \cdot \exists \$ref' \cdot post_R(P))))$
proof –
 have $\text{NCSP } P = \text{CSP3 } (\text{CSP4 } (\text{NSRD } P))$

by (metis (no-types, hide-lams) CSP4-def NCSP-def NSRD-alt-def RA1 RD3-def Skip-srdes-left-unit
 o-apply)
 also
 have ... = $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NSRD P)) wp_r false) \vdash$
 $(\exists \$ref \cdot \exists \$st' \cdot peri_R (NSRD P)) \diamond$
 $(\exists \$ref \cdot \exists \$ref' \cdot post_R (NSRD P)))$
 by (simp add: CSP3-form CSP4-form closure unrest rdes, rel-auto)
 also have ... = $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R(P)) wp_r false) \vdash ((\exists \$ref \cdot \exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref \cdot \exists$
 $\$ref' \cdot post_R(P))))$
 by (simp add: NSRD-form rdes closure, rel-blast)
 finally show ?thesis .
 qed

lemma CSP4-st'-unrest-peri [unrest]:
 assumes P is CSP P is CSP4
 shows $\$st' \not\# peri_R(P)$
 by (simp add: NSRD-CSP4-intro NSRD-st'-unrest-peri assms)

lemma CSP4-healthy-form:
 assumes P is CSP P is CSP4
 shows $P = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$
 proof -
 have $P = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$
 by (metis CSP4-def Healthy-def Skip-right-lemma assms(1) assms(2))
 also have ... = $\mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \llbracket true/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot$
 $cmt_R P) \llbracket false/\$wait' \rrbracket)))$
 by (metis (no-types, hide-lams) subst-wait'-left-subst subst-wait'-right-subst wait'-cond-def)
 also have ... = $\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$
 by (simp add: wait'-cond-def usubst peri_R-def post_R-def cmt_R-def unrest)
 finally show ?thesis .
 qed

lemma CSP4-ref'-unrest-pre [unrest]:
 assumes P is CSP P is CSP4
 shows $\$ref' \not\# pre_R(P)$
 proof -
 have $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$
 using CSP4-healthy-form assms(1) assms(2) by fastforce
 also have ... = $(\neg_r pre_R P) wp_r false$
 by (simp add: rea-pre-RHS-design wp-rea-def usubst unrest
 CSP4-neg-pre-unit R1-rea-not R2c-preR R2c-rea-not assms)
 also have $\$ref' \not\# \dots$
 by (simp add: wp-rea-def unrest)
 finally show ?thesis .
 qed

lemma NCSP-set-unrest-pre-wait':
 assumes $A \subseteq \llbracket NCSP \rrbracket_H$
 shows $\bigwedge P. P \in A \implies \$wait' \not\# pre_R(P)$
 proof -
 fix P
 assume $P \in A$
 hence P is NSRD
 using NCSP-implies-NSRD assms by auto
 thus $\$wait' \not\# pre_R(P)$

using *NSRD-wait'-unrest-pre* by *blast*
qed

lemma *CSP4-set-unrest-pre-st'*:
 assumes $A \subseteq \llbracket CSP \rrbracket_H$ $A \subseteq \llbracket CSP4 \rrbracket_H$
 shows $\bigwedge P. P \in A \implies \$st' \# pre_R(P)$
 proof –
 fix P
 assume $P \in A$
 hence P is *NSRD*
 using *NSRD-CSP4-intro* *assms(1)* *assms(2)* by *blast*
 thus $\$st' \# pre_R(P)$
 using *NSRD-st'-unrest-pre* by *blast*
 qed

lemma *CSP4-ref'-unrest-post* [*unrest*]:
 assumes P is *CSP* P is *CSP4*
 shows $\$ref' \# post_R(P)$
 proof –
 have $post_R(P) = post_R(\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P)))))$
 using *CSP4-healthy-form* *assms(1)* *assms(2)* by *fastforce*
 also have $\dots = R1 (R2c ((\neg_r pre_R P) wp_r false \Rightarrow_r (\exists \$ref' \cdot post_R P)))$
 by (*simp add: rea-post-RHS-design* *usubst unrest wp-rea-def*)
 also have $\$ref' \# \dots$
 by (*simp add: R1-def R2c-def wp-rea-def unrest*)
 finally show ?thesis .
 qed

lemma *CSP3-Chaos* [*closure*]: *Chaos is CSP3*
 by (*simp add: Chaos-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

lemma *CSP4-Chaos* [*closure*]: *Chaos is CSP4*
 by (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

lemma *NCSP-Chaos* [*closure*]: *Chaos is NCSP*
 by (*simp add: NCSP-intro closure*)

lemma *CSP3-Miracle* [*closure*]: *Miracle is CSP3*
 by (*simp add: Miracle-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

lemma *CSP4-Miracle* [*closure*]: *Miracle is CSP4*
 by (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

lemma *NCSP-Miracle* [*closure*]: *Miracle is NCSP*
 by (*simp add: NCSP-intro closure*)

lemma *NCSP-seqr-closure* [*closure*]:
 assumes P is *NCSP* Q is *NCSP*
 shows $P ;; Q$ is *NCSP*
 by (*metis (no-types, lifting) CSP3-def CSP4-def Healthy-def' NCSP-implies-CSP NCSP-implies-CSP3*
NCSP-implies-CSP4 NCSP-intro SRD-seqr-closure assms(1) assms(2) seqr-assoc)

lemma *CSP4-Skip* [*closure*]: *Skip is CSP4*
 apply (*rule CSP4-intro, simp-all add: Skip-is-CSP*)
 apply (*simp-all add: Skip-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)

done

lemma *NCSP-Skip* [closure]: *Skip is NCSP*

by (metis *CSP3-Skip CSP4-Skip Healthy-def NCSP-def Skip-is-CSP comp-apply*)

lemma *CSP4-Stop* [closure]: *Stop is CSP4*

apply (rule *CSP4-intro*, simp-all add: *Stop-is-CSP*)

apply (simp-all add: *Stop-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)

done

lemma *NCSP-Stop* [closure]: *Stop is NCSP*

by (metis *CSP3-Stop CSP4-Stop Healthy-def NCSP-def Stop-is-CSP comp-apply*)

lemma *CSP4-Idempotent*: *Idempotent CSP4*

by (metis (no-types, lifting) *CSP3-Skip CSP3-def CSP4-def Healthy-if Idempotent-def segr-assoc*)

lemma *CSP4-Continuous*: *Continuous CSP4*

by (simp add: *Continuous-def CSP4-def seq-Sup-distr*)

lemma *preR-Stop* [rdes]: $\text{pre}_R(\text{Stop}) = \text{true}_r$

by (simp add: *Stop-def Stop-is-CSP rea-pre-RHS-design unrest usubst R2c-true*)

lemma *periR-Stop* [rdes]: $\text{peri}_R(\text{Stop}) = \mathcal{E}(\text{true}, \langle \rangle, \{\}_u)$

by (rel-auto)

lemma *postR-Stop* [rdes]: $\text{post}_R(\text{Stop}) = \text{false}$

by (rel-auto)

lemma *cmtR-Stop* [rdes]: $\text{cmt}_R(\text{Stop}) = (\$tr' =_u \$tr \wedge \$wait')$

by (rel-auto)

lemma *NCSP-Idempotent* [closure]: *Idempotent NCSP*

by (clarsimp simp add: *NCSP-def Idempotent-def*)

(metis (no-types, hide-lams) *CSP3-Idempotent CSP3-def CSP4-Idempotent CSP4-def Healthy-def Idempotent-def SRD-idem SRD-segr-closure Skip-is-CSP segr-assoc*)

lemma *NCSP-Continuous* [closure]: *Continuous NCSP*

by (simp add: *CSP3-Continuous CSP4-Continuous Continuous-comp NCSP-def SRD-Continuous*)

lemma *preR-CRR* [closure]: $P \text{ is NCSP} \implies \text{pre}_R(P) \text{ is CRR}$

by (rule *CRR-intro*, simp-all add: *closure unrest*)

lemma *periR-CRR* [closure]: $P \text{ is NCSP} \implies \text{peri}_R(P) \text{ is CRR}$

by (rule *CRR-intro*, simp-all add: *closure unrest*)

lemma *postR-CRR* [closure]: $P \text{ is NCSP} \implies \text{post}_R(P) \text{ is CRR}$

by (rule *CRR-intro*, simp-all add: *closure unrest*)

lemma *NCSP-rdes-intro* [closure]:

assumes $P \text{ is CRC } Q \text{ is CRR } R \text{ is CRR}$

$\$st' \# Q \$ref' \# R$

shows $\mathbf{R}_s(P \vdash Q \diamond R) \text{ is NCSP}$

apply (rule *NCSP-intro*)

apply (simp-all add: *closure assms*)

apply (rule *CSP3-SRD-intro*)

```

    apply (simp-all add: rdes closure assms unrest)
  apply (rule CSP4-tri-intro)
    apply (simp-all add: rdes closure assms unrest)
  apply (metis (no-types, lifting) CRC-implies-RC R1-seqr-closure assms(1) rea-not-R1 rea-not-false
    rea-not-not wp-rea-RC-false wp-rea-def)
  done

lemma NCSP-preR-CRC [closure]:
  assumes P is NCSP
  shows preR(P) is CRC
  by (rule CRC-intro, simp-all add: closure assms unrest)

lemma CSP3-Sup-closure [closure]:
  A ⊆ ⟦CSP3⟧H ⟹ (⋂ A) is CSP3
  apply (auto simp add: CSP3-def Healthy-def seq-Sup-distl)
  apply (rule cong[of Sup])
  apply (simp)
  using image-iff apply force
  done

lemma CSP4-Sup-closure [closure]:
  A ⊆ ⟦CSP4⟧H ⟹ (⋂ A) is CSP4
  apply (auto simp add: CSP4-def Healthy-def seq-Sup-distr)
  apply (rule cong[of Sup])
  apply (simp)
  using image-iff apply force
  done

lemma NCSP-Sup-closure [closure]: ⟦ A ⊆ ⟦NCSP⟧H; A ≠ {} ⟧ ⟹ (⋂ A) is NCSP
  apply (rule NCSP-intro, simp-all add: closure)
  apply (metis (no-types, lifting) Ball-Collect CSP3-Sup-closure NCSP-implies-CSP3)
  apply (metis (no-types, lifting) Ball-Collect CSP4-Sup-closure NCSP-implies-CSP4)
  done

lemma NCSP-SUP-closure [closure]: ⟦ ⋀ i. P(i) is NCSP; A ≠ {} ⟧ ⟹ (⋂ i∈A. P(i)) is NCSP
  by (metis (mono-tags, lifting) Ball-Collect NCSP-Sup-closure image-iff image-is-empty)

lemma PCSP-implies-NCSP [closure]:
  assumes P is PCSP
  shows P is NCSP
proof -
  have P = Productive(NCSP(NCSP P))
    by (metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply)

  also have ... = Rs ((∀ $ref • (¬r preR(NCSP P)) wpr false) ⊢
    (∃ $ref • ∃ $st' • periR(NCSP P)) ◇
    ((∃ $ref • ∃ $ref' • postR(NCSP P)) ∧ $tr <u $tr'))
    by (simp add: NCSP-form Productive-RHS-design-form unrest closure)
  also have ... is NCSP
    apply (rule NCSP-rdes-intro)
    apply (rule CRC-intro)
    apply (simp-all add: unrest ex-unrest all-unrest closure)
  done
  finally show ?thesis .
qed

```

lemma *PCSP-elim* [RD-elim]:
assumes X is PCSP P (\mathbf{R}_s ($((pre_R X) \vdash peri_R X \diamond (R4(post_R X))))$)
shows $P X$
by (*metis* *R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms comp-apply*)

lemma *ICSP-implies-NCSP* [closure]:

assumes P is ICSP
shows P is NCSP

proof –

have $P = ISRD1(NCSP(NCSP P))$
by (*metis* (*no-types, hide-lams*) *Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)
also have $\dots = ISRD1(\mathbf{R}_s((\forall \$ref \cdot (\neg_r pre_R(NCSP P)) wp_r false) \vdash$
 $(\exists \$ref \cdot \exists \$st' \cdot peri_R(NCSP P)) \diamond$
 $(\exists \$ref \cdot \exists \$ref' \cdot post_R(NCSP P))))$
by (*simp add: NCSP-form*)
also have $\dots = \mathbf{R}_s((\forall \$ref \cdot (\neg_r pre_R(NCSP P)) wp_r false) \vdash$
 $false \diamond$
 $((\exists \$ref \cdot \exists \$ref' \cdot post_R(NCSP P)) \wedge \$tr' =_u \$tr))$
by (*simp-all add: ISRD1-RHS-design-form closure rdes unrest*)
also have \dots is NCSP
apply (*rule NCSP-rdes-intro*)
apply (*rule CRC-intro*)
apply (*simp-all add: unrest ex-unrest all-unrest closure*)
done
finally show *?thesis* .

qed

lemma *ICSP-implies-ISR* [closure]:

assumes P is ICSP
shows P is ISR

by (*metis* (*no-types, hide-lams*) *Healthy-def ICSP-implies-NCSP ISR-def NCSP-implies-ISR assms comp-apply*)

lemma *ICSP-elim* [RD-elim]:

assumes X is ICSP P (\mathbf{R}_s ($((pre_R X) \vdash false \diamond (post_R X \wedge \$tr' =_u \$tr)))$)
shows $P X$
by (*metis* *Healthy-if NCSP-implies-CSP ICSP-implies-NCSP ISR1-form assms comp-apply*)

lemma *ICSP-Stop-right-zero-lemma*:

$(P \wedge (\$tr' =_u \$tr)) ;; true_r = true_r \implies (P \wedge (\$tr' =_u \$tr)) ;; (\$tr' =_u \$tr) = (\$tr' =_u \$tr)$
by (*rel-blast*)

lemma *ICSP-Stop-right-zero*:

assumes P is ICSP $pre_R(P) = true_r post_R(P) ;; true_r = true_r$
shows $P ;; Stop = Stop$

proof –

from *assms(3)* **have** $1:(post_R P \wedge \$tr' =_u \$tr) ;; true_r = true_r$
by (*rel-auto, metis* (*full-types, hide-lams*) *dual-order.antisym order-refl*)
show *?thesis*
by (*rdes-simp cls: assms(1), simp add: csp-enable-nothing assms(2) ICSP-Stop-right-zero-lemma[OF 1]*)

qed

lemma *ICSP-intro*: $\llbracket P \text{ is NCSP}; P \text{ is ISR1} \rrbracket \implies P \text{ is ICSP}$

using *Healthy-comp* by *blast*

lemma *seq-ICSP-closed* [closure]:

assumes *P* is ICSP *Q* is ICSP

shows *P* ;; *Q* is ICSP

by (*meson* ICSP-implies-ISRDCSP-implies-NCSP ICSP-intro ISRDCSP-implies-ISRDCSP1 NCSP-seqr-closure
assms seq-ISRDCSP-closed)

lemma *Miracle-ICSP* [closure]: *Miracle* is ICSP

by (*rule* ICSP-intro, *simp* add: closure, *simp* add: ISRDCSP1-rdes-intro rdes-def closure)

5.2 CSP theories

typeddecl *TCSP*

abbreviation *TCSP* \equiv *UTHY*(*TCSP*, (σ, φ) *st-csp*)

overloading

tcsp-hcond == *utp-hcond* :: (*TCSP*, (σ, φ) *st-csp*) *uthy* \Rightarrow ((σ, φ) *st-csp* \times (σ, φ) *st-csp*) *health*

tcsp-unit == *utp-unit* :: (*TCSP*, (σ, φ) *st-csp*) *uthy* \Rightarrow (σ, φ) *action*

begin

definition *tcsp-hcond* :: (*TCSP*, (σ, φ) *st-csp*) *uthy* \Rightarrow ((σ, φ) *st-csp* \times (σ, φ) *st-csp*) *health* **where**

[*upred-defs*]: *tcsp-hcond* *T* = *NCSP*

definition *tcsp-unit* :: (*TCSP*, (σ, φ) *st-csp*) *uthy* \Rightarrow (σ, φ) *action* **where**

[*upred-defs*]: *tcsp-unit* *T* = *Skip*

end

interpretation *csp-theory*: *utp-theory-kleene* *UTHY*(*TCSP*, (σ, φ) *st-csp*)

rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } TCSP) \longleftrightarrow P \text{ is } NCSP$

and $P \text{ is } \mathcal{H}_{TCSP} \longleftrightarrow P \text{ is } NCSP$

and $\mathcal{IL}_{TCSP} = \text{Skip}$

and $\top_{TCSP} = \text{Miracle}$

and $\text{carrier } (\text{uthy-order } TCSP) \rightarrow \text{carrier } (\text{uthy-order } TCSP) \equiv \llbracket NCSP \rrbracket_H \rightarrow \llbracket NCSP \rrbracket_H$

and $A \subseteq \text{carrier } (\text{uthy-order } TCSP) \longleftrightarrow A \subseteq \llbracket NCSP \rrbracket_H$

and $\text{le } (\text{uthy-order } TCSP) = \text{op} \sqsubseteq$

proof –

interpret *lat*: *utp-theory-continuous* *UTHY*(*TCSP*, (σ, φ) *st-csp*)

by (*unfold-locales*, *simp-all* add: *tcsp-hcond-def* closure *Healthy-if*)

show 1: $\top_{TCSP} = (\text{Miracle} :: (\sigma, \varphi) \text{ action})$

by (*metis* *NCSP-Miracle* *NCSP-implies-CSP* *lat.top-healthy* *lat.utp-theory-continuous-axioms* *srdes-theory-continuous*.
tcsp-hcond-def *upred-semiring.add-commute* *utp-theory-continuous.meet-top*)

thus *utp-theory-kleene* *UTHY*(*TCSP*, (σ, φ) *st-csp*)

by (*unfold-locales*, *simp-all* add: *tcsp-hcond-def* *tcsp-unit-def* *Skip-left-unit* *Skip-right-unit* closure
Healthy-if *Miracle-left-zero*)

qed (*simp-all* add: *tcsp-hcond-def* *tcsp-unit-def* closure *Healthy-if*)

declare *csp-theory.top-healthy* [*simp del*]

declare *csp-theory.bottom-healthy* [*simp del*]

abbreviation *TestC* (*test_C*) **where**

test_C *P* \equiv *utest* *TCSP* *P*

abbreviation *StarC* :: (σ, φ) *action* \Rightarrow (σ, φ) *action* ($-^{*C}$ [999] 999) **where**

StarC *P* $\equiv P \star_{TCSP}$

lemma *csp-bottom-Chaos*: $\perp_{TCSP} = \text{Chaos}$
using *NCSP-Chaos NCSP-implies-CSP* **by** *auto*

lemma *csp-top-Miracle*: $\top_{TCSP} = \text{Miracle}$
by (*simp add: csp-theory.healthy-top csp-theory.utp-theory-mono-axioms utp-theory-mono.healthy-top*)

5.3 Algebraic laws

lemma *Stop-left-zero*:
assumes *P is CSP*
shows *Stop ; P = Stop*
by (*simp add: NSRD-seq-post-false assms NCSP-implies-NSRD NCSP-Stop postR-Stop*)

end

6 Reactive Contracts for CSP/Circus with refusals

theory *utp-circus-contracts*
imports *utp-circus-healths*
begin

definition *mk-CRD* :: $'s \text{ upred} \Rightarrow ('e \text{ list} \Rightarrow 'e \text{ set} \Rightarrow 's \text{ upred}) \Rightarrow ('e \text{ list} \Rightarrow 's \text{ hrel}) \Rightarrow ('s, 'e) \text{ action}$
where
 $[rdes\text{-}def]: mk\text{-}CRD\ P\ Q\ R = \mathbf{R}_s([P]_{S<} \vdash [Q\ x\ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref\ ' \rrbracket \diamond [R(x)]_S' \llbracket x \rightarrow \&tt \rrbracket)$

syntax
-ref-var :: *logic*
-mk-CRD :: *logic* \Rightarrow *logic* \Rightarrow *logic* \Rightarrow *logic* ($[-/\vdash -/ \mid -]_C$)

parse-translation \ll
let
fun *ref-var-tr* [] = *Syntax.free refs*
| *ref-var-tr* - = *raise Match*;
in
 $[(\text{@}\{\text{syntax-const } -ref\text{-var}\}, K\ \text{ref-var-tr})]$
end
 \gg

translations
 $[P \vdash Q \mid R]_C \Rightarrow CONST\ mk\text{-}CRD\ P\ (\lambda\ -trace\text{-}var\ -ref\text{-}var.\ Q)\ (\lambda\ -trace\text{-}var.\ R)$
 $[P \vdash Q \mid R]_C \Leftarrow CONST\ mk\text{-}CRD\ P\ (\lambda\ x\ r.\ Q)\ (\lambda\ y.\ R)$

lemma *CSP-mk-CRD [closure]*: $[P \vdash Q\ trace\ refs \mid R(trace)]_C$ *is CSP*
by (*simp add: mk-CRD-def closure unrest*)

lemma *preR-mk-CRD [rdes]*: $pre_R([P \vdash Q\ trace\ refs \mid R(trace)]_C) = [P]_{S<}$
by (*simp add: mk-CRD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre rea-st-cond-def, rel-auto*)

lemma *periR-mk-CRD [rdes]*: $peri_R([P \vdash Q\ trace\ refs \mid R(trace)]_C) = ([P]_{S<} \Rightarrow_r ([Q\ trace\ refs]_{S<} \llbracket (trace, refs) \rightarrow (\&tt, \$ref\ ')) \rrbracket)$
by (*simp add: mk-CRD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto*)

lemma *postR-mk-CRD [rdes]*: $post_R([P \vdash Q\ trace\ refs \mid R(trace)]_C) = ([P]_{S<} \Rightarrow_r ([R(trace)]_S' \llbracket trace \rightarrow \&tt \rrbracket))$
by (*simp add: mk-CRD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto)

Refinement introduction law for contracts

lemma *CRD-contract-refine*:

assumes
 $Q \text{ is CSP } \langle [P_1]_{S<} \Rightarrow \text{pre}_R Q \rangle$
 $\langle [P_1]_{S<} \wedge \text{peri}_R Q \Rightarrow [P_2 \ t \ r]_{S<} \llbracket t \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \rrbracket \rangle$
 $\langle [P_1]_{S<} \wedge \text{post}_R Q \Rightarrow [P_3 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket \rangle$
shows $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$
proof –
have $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$
using *assms* **by** (*simp add: mk-CRD-def, rule-tac srdes-tri-refine-intro, rel-auto+*)
thus *?thesis*
by (*simp add: SRD-reactive-tri-design assms(1)*)
qed

lemma *CRD-contract-refine'*:

assumes
 $Q \text{ is CSP } \langle [P_1]_{S<} \Rightarrow \text{pre}_R Q \rangle$
 $[P_2 \ t \ r]_{S<} \llbracket t \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \rrbracket \sqsubseteq ([P_1]_{S<} \wedge \text{peri}_R Q)$
 $[P_3 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket \sqsubseteq ([P_1]_{S<} \wedge \text{post}_R Q)$
shows $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$
using *assms* **by** (*rule-tac CRD-contract-refine, simp-all add: refBy-order*)

lemma *CRD-refine-CRD*:

assumes
 $\langle [P_1]_{S<} \Rightarrow ([Q_1]_{S<} :: ('e, 's) \text{ action}) \rangle$
 $([P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge [Q_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \rrbracket :: ('e, 's) \text{ action})$
 $[P_3 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket \sqsubseteq ([P_1]_{S<} \wedge [Q_3 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket :: ('e, 's) \text{ action})$
shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq [Q_1 \vdash Q_2 \text{ trace refs} \mid Q_3 \text{ trace}]_C$
using *assms*
by (*simp add: mk-CRD-def, rule-tac srdes-tri-refine-intro, rel-auto+*)

lemma *CRD-refine-rdes*:

assumes
 $\langle [P_1]_{S<} \Rightarrow Q_1 \rangle$
 $([P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2)$
 $[P_3 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket \sqsubseteq ([P_1]_{S<} \wedge Q_3)$
shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
using *assms*
by (*simp add: mk-CRD-def, rule-tac srdes-tri-refine-intro, rel-auto+*)

lemma *CRD-refine-rdes'*:

assumes
 $Q_2 \text{ is RR}$
 $Q_3 \text{ is RR}$
 $\langle [P_1]_{S<} \Rightarrow Q_1 \rangle$
 $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2[\langle \rangle, \langle t \rangle / \$tr, \$tr'])$
 $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge Q_3[\langle \rangle, \langle t \rangle / \$tr, \$tr'])$
shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
proof (*simp add: mk-CRD-def, rule srdes-tri-refine-intro*)
show $\langle [P_1]_{S<} \Rightarrow Q_1 \rangle$ **by** (*fact assms(3)*)

```

have  $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_2) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$ 
  by (simp add: assms Healthy-if)
hence  $'[P_1]_{S<} \wedge RR(Q_2) \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket'$ 
  by (rel-simp; meson)
thus  $'[P_1]_{S<} \wedge Q_2 \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket'$ 
  by (simp add: Healthy-if assms)

have  $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_3) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$ 
  by (simp add: assms Healthy-if)
hence  $'[P_1]_{S<} \wedge (RR \ Q_3) \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket'$ 
  by (rel-simp; meson)
thus  $'[P_1]_{S<} \wedge Q_3 \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket'$ 
  by (simp add: Healthy-if assms)
qed

end

```

7 External Choice

```

theory utp-circus-extchoice
imports
  utp-circus-healths
  utp-circus-rel
begin

```

7.1 Definitions and syntax

definition *ExtChoice* ::
 $(\sigma, \varphi) \text{ action set} \Rightarrow (\sigma, \varphi) \text{ action where}$
 $[upred-defs]: \text{ExtChoice } A = \mathbf{R}_s((\bigsqcup P \in A \cdot pre_R(P)) \vdash ((\bigsqcup P \in A \cdot cmt_R(P)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigsqcup P \in A \cdot cmt_R(P))))$

syntax

```

-ExtChoice :: pttm  $\Rightarrow$  'a set  $\Rightarrow$  'b  $\Rightarrow$  'b  $((\exists \square \text{ -- } \cdot / \cdot) [0, 0, 10] 10)$ 
-ExtChoice-simp :: pttm  $\Rightarrow$  'b  $\Rightarrow$  'b  $((\exists \square \text{ -- } \cdot / \cdot) [0, 10] 10)$ 

```

translations

```

 $\square P \in A \cdot B \equiv \text{CONST ExtChoice } ((\lambda P. B) \text{ ' } A)$ 
 $\square P \cdot B \equiv \text{CONST ExtChoice } (\text{CONST range } (\lambda P. B))$ 

```

definition *extChoice* ::

$(\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action (infixl } \square 65) \text{ where}$
 $[upred-defs]: P \square Q \equiv \text{ExtChoice } \{P, Q\}$

Small external choice as an indexed big external choice.

lemma *extChoice-alt-def*:

```

 $P \square Q = (\square i::nat \in \{0,1\} \cdot P \triangleleft \ll i = 0 \gg \triangleright Q)$ 
  by (simp add: extChoice-def ExtChoice-def, unliteralise, simp)

```

7.2 Basic laws

7.3 Algebraic laws

lemma *ExtChoice-empty*: $\text{ExtChoice } \{\} = \text{Stop}$
 by (simp add: ExtChoice-def cond-def Stop-def)

lemma *ExtChoice-single*:

P is CSP \implies ExtChoice $\{P\} = P$

by (simp add: ExtChoice-def usup-and uinf-or SRD-reactive-design-alt)

7.4 Reactive design calculations

lemma *ExtChoice-rdes*:

assumes $\bigwedge i. \$ok' \nmid P(i) \ A \neq \{\}$

shows $(\Box i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\Box i \in A \cdot P(i)) \vdash ((\Box i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\Box i \in A \cdot Q(i))))$

proof –

have $(\Box i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) =$

$\mathbf{R}_s((\Box i \in A \cdot pre_R(\mathbf{R}_s(P(i) \vdash Q(i)))) \vdash$
 $((\Box i \in A \cdot cmt_R(\mathbf{R}_s(P(i) \vdash Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot cmt_R(\mathbf{R}_s(P(i) \vdash Q(i))))))$

by (simp add: ExtChoice-def)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $((\Box i \in A \cdot R1(R2c(cmt_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot R1(R2c(cmt_s \dagger (P(i) \Rightarrow Q(i))))))$

by (simp add: rea-pre-RHS-design rea-cmt-RHS-design)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $R1(R2c$
 $((\Box i \in A \cdot R1(R2c(cmt_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot R1(R2c(cmt_s \dagger (P(i) \Rightarrow Q(i))))))$

by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $R1(R2c$
 $((\Box i \in A \cdot (cmt_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot (cmt_s \dagger (P(i) \Rightarrow Q(i))))$

by (simp add: R2c-UIINF R2c-cond R1-cond R1-idem R1-R2c-commute R2c-idem R1-UIINF assms R1-USUP R2c-USUP)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $cmt_s \dagger$
 $((\Box i \in A \cdot (cmt_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot (cmt_s \dagger (P(i) \Rightarrow Q(i))))$

by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c rdes-export-cmt)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $cmt_s \dagger$
 $((\Box i \in A \cdot (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\Box i \in A \cdot (P(i) \Rightarrow Q(i))))$

by (simp add: usubst)

also have ... =

$\mathbf{R}_s((\Box i \in A \cdot R1(R2c(pre_s \dagger P(i)))) \vdash$
 $((\Box i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\Box i \in A \cdot (P(i) \Rightarrow Q(i))))$

```

    by (simp add: rdes-export-cmt)
  also have ... =
     $\mathbf{R}_s ((R1(R2c(\bigwedge i \in A \cdot (pre_s \uparrow P(i)))))) \vdash$ 
     $((\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))))$ 
    by (simp add: not-UINF R1-UINF R2c-UINF assms)
  also have ... =
     $\mathbf{R}_s ((R2c(\bigwedge i \in A \cdot (pre_s \uparrow P(i)))) \vdash$ 
     $((\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))))$ 
    by (simp add: R1-design-R1-pre)
  also have ... =
     $\mathbf{R}_s (((\bigwedge i \in A \cdot (pre_s \uparrow P(i)))) \vdash$ 
     $((\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))))$ 
    by (metis (no-types, lifting) RHS-design-R2c-pre)
  also have ... =
     $\mathbf{R}_s ([\$ok \mapsto_s true, \$wait \mapsto_s false] \uparrow (\bigwedge i \in A \cdot P(i))) \vdash$ 
     $((\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))))$ 
  proof -
    from assms have  $\bigwedge i. pre_s \uparrow P(i) = [\$ok \mapsto_s true, \$wait \mapsto_s false] \uparrow P(i)$ 
    by (rel-auto)
    thus ?thesis
    by (simp add: usubst)
  qed
  also have ... =
     $\mathbf{R}_s ((\bigwedge i \in A \cdot P(i)) \vdash ((\bigwedge i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot (P(i) \Rightarrow$ 
     $Q(i))))$ 
    by (simp add: rdes-export-pre not-UINF)
  also have ... =  $\mathbf{R}_s ((\bigwedge i \in A \cdot P(i)) \vdash ((\bigwedge i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot Q(i))))$ 
    by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto, blast+)

  finally show ?thesis .
qed

lemma ExtChoice-tri-rdes:
  assumes  $\bigwedge i. \$ok' \nmid P_1(i) \ A \neq \{\}$ 
  shows  $(\bigwedge i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$ 
     $\mathbf{R}_s ((\bigwedge i \in A \cdot P_1(i)) \vdash (((\bigwedge i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge i \in A \cdot P_2(i))) \diamond (\bigwedge i \in A \cdot$ 
     $P_3(i))))$ 
  proof -
    have  $(\bigwedge i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$ 
     $\mathbf{R}_s ((\bigwedge i \in A \cdot P_1(i)) \vdash ((\bigwedge i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge i \in A \cdot P_2(i) \diamond$ 
     $P_3(i))))$ 
    by (simp add: ExtChoice-rdes assms)
  also
  have ... =
     $\mathbf{R}_s ((\bigwedge i \in A \cdot P_1(i)) \vdash ((\bigwedge i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (\bigwedge i \in A \cdot P_2(i) \diamond$ 
     $P_3(i))))$ 
    by (simp add: conj-comm)
  also
  have ... =
     $\mathbf{R}_s ((\bigwedge i \in A \cdot P_1(i)) \vdash (((\bigwedge i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge i \in A \cdot P_2(i) \diamond P_3(i))) \diamond$ 
     $(\bigwedge i \in A \cdot P_2(i) \diamond P_3(i))))$ 
    by (simp add: cond-conj wait'-cond-def)
  also
  have ... =  $\mathbf{R}_s ((\bigwedge i \in A \cdot P_1(i)) \vdash (((\bigwedge i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge i \in A \cdot P_2(i))) \diamond (\bigwedge i \in A \cdot$ 
     $P_3(i))))$ 

```

by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 finally show ?thesis .
 qed

lemma ExtChoice-tri-rdes' [rdes-def]:
 assumes $\bigwedge i. \$ok' \# P_1(i) \ A \neq \{\}$
 shows $(\bigwedge i \in A. \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$
 $\mathbf{R}_s((\bigwedge i \in A. P_1(i)) \vdash (((\bigwedge i \in A. R5(P_2(i))) \vee (\bigwedge i \in A. R4(P_2(i)))) \diamond (\bigwedge i \in A. P_3(i))))$
 by (simp add: ExtChoice-tri-rdes assms, rel-auto, simp-all add: less-le assms)

lemma ExtChoice-tri-rdes-def [rdes-def]:
 assumes $A \subseteq \llbracket CSP \rrbracket_H$
 shows $ExtChoice\ A = \mathbf{R}_s((\bigwedge P \in A. pre_R\ P) \vdash (((\bigwedge P \in A. peri_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A. \text{post}_R\ P)) \diamond (\bigwedge P \in A. post_R\ P)))$
 proof -
 have $((\bigwedge P \in A. cmt_R\ P) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigwedge P \in A. cmt_R\ P)) =$
 $((\bigwedge P \in A. cmt_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A. cmt_R\ P)) \diamond (\bigwedge P \in A. cmt_R\ P)$
 by (rel-auto)
 also have ... = $((\bigwedge P \in A. peri_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A. peri_R\ P)) \diamond (\bigwedge P \in A. post_R\ P)$
 by (rel-auto)
 finally show ?thesis
 by (simp add: ExtChoice-def)
 qed

lemma extChoice-rdes:
 assumes $\$ok' \# P_1 \ \$ok' \# Q_1$
 shows $\mathbf{R}_s(P_1 \vdash P_2) \square \mathbf{R}_s(Q_1 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$
 proof -
 have $(\square i::nat \in \{0, 1\}. \mathbf{R}_s(P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright \mathbf{R}_s(Q_1 \vdash Q_2)) = (\square i::nat \in \{0, 1\}. \mathbf{R}_s((P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright (Q_1 \vdash Q_2)))$
 by (simp only: RHS-cond R2c-lit)
 also have ... = $(\square i::nat \in \{0, 1\}. \mathbf{R}_s((P_1 \triangleleft \ll i = 0 \gg \triangleright Q_1) \vdash (P_2 \triangleleft \ll i = 0 \gg \triangleright Q_2)))$
 by (simp add: design-condr)
 also have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$
 apply (subst ExtChoice-rdes, simp-all add: assms unrest)
 apply unliteralise
 apply (simp add: uinf-or usup-and)
 done
 finally show ?thesis by (simp add: extChoice-alt-def)
 qed

lemma extChoice-tri-rdes:
 assumes $\$ok' \# P_1 \ \$ok' \# Q_1$
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 proof -
 have $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$
 by (simp add: extChoice-rdes assms)
 also
 have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$
 by (simp add: conj-comm)
 also
 have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash$

$$(((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)) \diamond (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$$
 by (*simp add: cond-conj wait'-cond-def*)
 also
 have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (*rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto*)
 finally show ?thesis .
 qed

lemma *extChoice-rdes-def*:
 assumes P_1 is RR Q_1 is RR
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (*subst extChoice-tri-rdes, simp-all add: assms unrest*)

lemma *extChoice-rdes-def' [rdes-def]*:
 assumes P_1 is RR Q_1 is RR
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((R5(P_2 \wedge Q_2) \vee R4(P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (*simp add: extChoice-rdes-def assms, rel-auto, simp-all add: less-le*)

lemma *CSP-ExtChoice [closure]*:
ExtChoice A is CSP
 by (*simp add: ExtChoice-def RHS-design-is-SRD unrest*)

lemma *CSP-extChoice [closure]*:
 $P \sqcap Q$ is CSP
 by (*simp add: CSP-ExtChoice extChoice-def*)

lemma *preR-ExtChoice [rdes]*:
 assumes $A \neq \{\}$ $A \subseteq \llbracket CSP \rrbracket_H$
 shows $pre_R(ExtChoice A) = (\bigsqcup_{P \in A} pre_R(P))$
proof –
 have $pre_R(ExtChoice A) = (R1 (R2c ((\bigsqcup_{P \in A} pre_R(P))))$
 by (*simp add: ExtChoice-def rea-pre-RHS-design usubst unrest*)
 also from *assms* have ... = $(R1 (R2c (\bigsqcup_{P \in A} (pre_R(CSP(P)))))$
 by (*metis USUP-healthy*)
 also from *assms* have ... = $(\bigsqcup_{P \in A} (pre_R(CSP(P))))$
 by (*rel-auto*)
 also from *assms* have ... = $(\bigsqcup_{P \in A} pre_R(P))$
 by (*metis USUP-healthy*)
 finally show ?thesis .
 qed

lemma *preR-ExtChoice-ind [rdes]*:
 assumes $A \neq \{\} \bigwedge P. P \in A \implies F(P)$ is CSP
 shows $pre_R(\bigsqcup_{P \in A} F(P)) = (\bigsqcup_{P \in A} pre_R(F(P)))$
 using *assms* by (*subst preR-ExtChoice, auto*)

lemma *periR-ExtChoice [rdes]*:
 assumes $A \subseteq \llbracket NCSP \rrbracket_H$ $A \neq \{\}$
 shows $peri_R(ExtChoice A) = ((\bigsqcup_{P \in A} pre_R(P)) \Rightarrow_r (\bigsqcup_{P \in A} peri_R(P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap_{P \in A} peri_R(P)))$
proof –
 have $peri_R(ExtChoice A) = peri_R(\mathbf{R}_s((\bigsqcup_{P \in A} pre_R(P)) \vdash$
 $((\bigsqcup_{P \in A} peri_R(P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap_{P \in A} peri_R(P))) \diamond$

$(\bigwedge P \in A \cdot \text{post}_R P))$
by (*simp add: ExtChoice-tri-rdes-def assms closure*)

also have ... = $\text{peri}_R (\mathbf{R}_s ((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \vdash$
 $((\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P))) \diamond$
 $(\bigwedge P \in A \cdot \text{post}_R P)))$
by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have ... = $R1 (R2c ((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r$
 $(\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P))))$
proof –
have $(\bigwedge P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{true}] \dagger \text{pre}_R (\text{NCSP } P))$
 $= (\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P))$
by (*rule USUP-cong, simp add: closure usubst unrest assms*)
thus ?thesis
by (*simp add: rea-peri-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)
qed

also have ... = $R1 ((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r$
 $(\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
by (*simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-periR R2c-tr'-minus-tr R2c-USUP closure*)

also have ... = $((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure, rel-auto*)

also have ... = $((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r \text{peri}_R (\text{NCSP } P)))$
by (*simp add: UINF-rea-impl[THEN sym]*)

also have ... = $((\bigwedge P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\bigwedge P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have ... = $((\bigwedge P \in A \cdot \text{pre}_R P) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\bigwedge P \in A \cdot \text{peri}_R P))$
by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)
finally show ?thesis .
qed

lemma *periR-ExtChoice'*:

assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ $A \neq \{\}$
shows $\text{peri}_R(\text{ExtChoice } A) = (R5((\bigwedge P \in A \cdot \text{pre}_R(P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R P)) \vee (\bigwedge P \in A \cdot R4(\text{peri}_R P)))$
using *assms(2)*
by (*simp add: periR-ExtChoice assms(1), rel-auto*)

lemma *periR-ExtChoice-ind [rdes]*:

assumes $\bigwedge P. P \in A \implies F(P)$ *is NCSP* $A \neq \{\}$
shows $\text{peri}_R(\bigwedge P \in A \cdot F(P)) = ((\bigwedge P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R (F P))) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A \cdot \text{peri}_R (F P))$

using *assms* **by** (*subst periR-ExtChoice*, *auto simp add: closure unrest*)

lemma *periR-ExtChoice-ind'*:

assumes $\bigwedge P. P \in A \implies F(P)$ *is NCSP* $A \neq \{\}$

shows $\text{peri}_R(\bigwedge P \in A \cdot F(P)) = (R5((\bigwedge P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{peri}_R(F P))) \vee (\bigwedge P \in A \cdot R4(\text{peri}_R(F P))))$

using *assms* **by** (*subst periR-ExtChoice'*, *auto simp add: closure unrest*)

lemma *postR-ExtChoice [rdes]*:

assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ $A \neq \{\}$

shows $\text{post}_R(\text{ExtChoice } A) = (\bigwedge P \in A \cdot \text{post}_R P)$

proof –

have $\text{post}_R(\text{ExtChoice } A) = \text{post}_R(\mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R P) \vdash (\bigwedge P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A \cdot \text{peri}_R P)) \diamond (\bigwedge P \in A \cdot \text{post}_R P)))$

by (*simp add: ExtChoice-tri-rdes-def closure assms*)

also have $\dots = \text{post}_R(\mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P)) \vdash (\bigwedge P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\bigwedge P \in A \cdot \text{peri}_R P)) \diamond (\bigwedge P \in A \cdot \text{post}_R(\text{NCSP } P))))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have $\dots = R1(R2c((\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{post}_R(\text{NCSP } P))))$

proof –

have $(\bigwedge P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{false}] \dagger \text{pre}_R(\text{NCSP } P)) = (\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P))$

by (*rule USUP-cong, simp add: usubst closure unrest assms*)

thus *?thesis*

by (*simp add: rea-post-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)

qed

also have $\dots = R1((\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{post}_R(\text{NCSP } P)))$

by (*simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-postR R2c-tr'-minus-tr R2c-USUP closure*)

also from *assms(2)* **have** $\dots = ((\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigwedge P \in A \cdot \text{post}_R(\text{NCSP } P)))$

by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure*)

also have $\dots = (\bigwedge P \in A \cdot \text{pre}_R(\text{NCSP } P) \Rightarrow_r \text{post}_R(\text{NCSP } P))$

by (*simp add: UINF-rea-impl*)

also have $\dots = (\bigwedge P \in A \cdot \text{post}_R(\text{NCSP } P))$

by (*simp add: SRD-post-under-pre closure assms unrest*)

finally show *?thesis*

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

qed

lemma *postR-ExtChoice-ind [rdes]*:

assumes $\bigwedge P. P \in A \implies F(P)$ *is NCSP* $A \neq \{\}$

shows $\text{post}_R(\bigwedge P \in A \cdot F(P)) = (\bigwedge P \in A \cdot \text{post}_R(F(P)))$

using *assms* **by** (*subst postR-ExtChoice*, *auto simp add: closure unrest*)

lemma *preR-extChoice*:

assumes P *is CSP* Q *is CSP* $\$wait' \nVdash \text{pre}_R(P)$ $\$wait' \nVdash \text{pre}_R(Q)$

shows $\text{pre}_R(P \sqcap Q) = (\text{pre}_R(P) \wedge \text{pre}_R(Q))$

by (*simp add: extChoice-def preR-ExtChoice assms usup-and*)

lemma *preR-extChoice' [rdes]*:

assumes P *is NCSP* Q *is NCSP*

shows $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$
by (*simp add: preR-extChoice closure assms unrest*)

lemma *periR-extChoice [rdes]*:
assumes P is NCSP Q is NCSP
shows $peri_R(P \sqcap Q) = ((pre_R(P) \wedge pre_R(Q) \Rightarrow_r peri_R(P) \wedge peri_R(Q)) \triangleleft \$tr' =_u \$tr \triangleright (peri_R(P) \vee peri_R(Q)))$
using *assms*
by (*simp add: extChoice-def, subst periR-ExtChoice, auto simp add: usup-and uinf-or*)

lemma *postR-extChoice [rdes]*:
assumes P is NCSP Q is NCSP
shows $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$
using *assms*
by (*simp add: extChoice-def, subst postR-ExtChoice, auto simp add: usup-and uinf-or*)

lemma *ExtChoice-cong*:
assumes $\bigwedge P. P \in A \implies F(P) = G(P)$
shows $(\sqcap P \in A \cdot F(P)) = (\sqcap P \in A \cdot G(P))$
using *assms image-cong* **by** *force*

lemma *ref-unrest-ExtChoice*:
assumes
 $\bigwedge P. P \in A \implies \$ref \# pre_R(P)$
 $\bigwedge P. P \in A \implies \$ref \# cmt_R(P)$
shows $\$ref \# (ExtChoice A) \llbracket false / \$wait \rrbracket$
proof –
have $\bigwedge P. P \in A \implies \$ref \# pre_R(P \llbracket 0 / \$tr \rrbracket)$
using *assms* **by** (*rel-blast*)
with *assms* **show** *?thesis*
by (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)
qed

lemma *CSP4-ExtChoice*:
assumes $A \subseteq \llbracket NCSP \rrbracket_H$
shows *ExtChoice A* is CSP4
proof (*cases A = {}*)
case *True* **thus** *?thesis*
by (*simp add: ExtChoice-empty Healthy-def CSP4-def, simp add: Skip-is-CSP Stop-left-zero*)
next
case *False*
have $1: (\neg_r (\neg_r pre_R (ExtChoice A)) ;;_h R1 \text{ true}) = pre_R (ExtChoice A)$
proof –
have $\bigwedge P. P \in A \implies (\neg_r pre_R(P)) ;; R1 \text{ true} = (\neg_r pre_R(P))$
by (*simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-neg-pre-unit assms*)
thus *?thesis*
apply (*simp add: False preR-ExtChoice closure NCSP-set-unrest-pre-wait' assms not-UINF seq-UINF-distr not-USUP*)
apply (*rule USUP-cong*)
apply (*simp add: rpred assms closure*)
done
qed
have $2: \$st' \# peri_R (ExtChoice A)$
proof –
have $a: \bigwedge P. P \in A \implies \$st' \# pre_R(P)$

```

    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-pre assms)
  have b:  $\bigwedge P. P \in A \implies \$st' \# \text{peri}_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-peri assms)
  from a b show ?thesis
    apply (subst periR-ExtChoice)
    apply (simp-all add: assms closure unrest CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
False)
  done
qed
have 3:  $\$ref' \# \text{post}_R(\text{ExtChoice } A)$ 
proof -
  have a:  $\bigwedge P. P \in A \implies \$ref' \# \text{pre}_R(P)$ 
    by (simp add: CSP4-ref'-unrest-pre CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  have b:  $\bigwedge P. P \in A \implies \$ref' \# \text{post}_R(P)$ 
    by (simp add: CSP4-ref'-unrest-post CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  from a b show ?thesis
    by (subst postR-ExtChoice, simp-all add: assms CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
unrest False)
  qed
show ?thesis
  by (rule CSP4-tri-intro, simp-all add: 1 2 3 assms closure)
  (metis 1 R1-seqr-closure rea-not-R1 rea-not-not rea-true-R1)
qed

```

lemma *CSP4-extChoice* [closure]:
assumes *P* is NCSP *Q* is NCSP
shows $P \sqsubseteq Q$ is CSP4
by (simp add: extChoice-def, rule CSP4-ExtChoice, simp-all add: assms)

lemma *NCSP-ExtChoice* [closure]:
assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H$
shows *ExtChoice* *A* is NCSP
proof (cases $A = \{\}$)
 case True
 then show ?thesis **by** (simp add: ExtChoice-empty closure)
next
 case False
 show ?thesis
proof (rule NCSP-intro)
 from assms **have** $cls: A \subseteq \llbracket \text{CSP} \rrbracket_H \ A \subseteq \llbracket \text{CSP3} \rrbracket_H \ A \subseteq \llbracket \text{CSP4} \rrbracket_H$
using NCSP-implies-CSP NCSP-implies-CSP3 NCSP-implies-CSP4 **by** blast+
have $wu: \bigwedge P. P \in A \implies \$wait' \# \text{pre}_R(P)$
using NCSP-implies-NSRD NSRD-wait'-unrest-pre assms **by** force
show 1: *ExtChoice* *A* is CSP
by (metis (mono-tags) Ball-Collect CSP-ExtChoice NCSP-implies-CSP assms)
 from cls **show** *ExtChoice* *A* is CSP3
by (rule-tac CSP3-SRD-intro, simp-all add: CSP-Healthy-subset-member CSP3-Healthy-subset-member
closure rdes unrest wu assms 1 False)
 from cls **show** *ExtChoice* *A* is CSP4
by (simp add: CSP4-ExtChoice assms)
 qed
qed

lemma *ExtChoice-NCSP-closed* [closure]:
assumes $\bigwedge i. i \in I \implies P(i)$ *is NCSP*
shows $(\Box i \in I \cdot P(i))$ *is NCSP*
by (*simp add: NCSP-ExtChoice assms image-subset-iff*)

lemma *NCSP-extChoice* [closure]:
assumes P *is NCSP* Q *is NCSP*
shows $P \Box Q$ *is NCSP*
by (*simp add: NCSP-ExtChoice assms extChoice-def*)

7.5 Productivity and Guardedness

lemma *Productive-ExtChoice* [closure]:
assumes $A \neq \{\}$ $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ $A \subseteq \llbracket \text{Productive} \rrbracket_H$
shows *ExtChoice* A *is Productive*

proof –

have $1: \bigwedge P. P \in A \implies \$wait' \nmid pre_R(P)$
using *NCSP-implies-NSRD NSRD-wait'-unrest-pre assms(2)* **by** *blast*
show *?thesis*

proof (*rule Productive-intro, simp-all add: assms closure rdes 1 unrest*)

have $((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot post_R P)) =$
 $((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot (pre_R P \wedge post_R P)))$
by (*rel-auto*)

moreover have $(\Box P \in A \cdot (pre_R P \wedge post_R P)) = (\Box P \in A \cdot ((pre_R P \wedge post_R P) \wedge \$tr <_u \$tr'))$

by (*rule UINF-cong, metis (no-types, lifting) 1 Ball-Collect NCSP-implies-CSP Productive-post-refines-tr-increase assms utp-pred-laws.inf.absorb1*)

ultimately show $(\$tr' >_u \$tr) \sqsubseteq ((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot post_R P))$
by (*rel-auto*)

qed

qed

lemma *Productive-extChoice* [closure]:
assumes P *is NCSP* Q *is NCSP* P *is Productive* Q *is Productive*
shows $P \Box Q$ *is Productive*
by (*simp add: extChoice-def Productive-ExtChoice assms*)

lemma *ExtChoice-Guarded* [closure]:
assumes $\bigwedge P. P \in A \implies \text{Guarded } P$
shows *Guarded* $(\lambda X. \Box P \in A \cdot P(X))$

proof (*rule GuardedI*)

fix $X n$

have $\bigwedge Y. ((\Box P \in A \cdot P Y) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

proof –

fix Y

let $?lhs = ((\Box P \in A \cdot P Y) \wedge gvirt(n+1))$ **and** $?rhs = ((\Box P \in A \cdot (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

have $a: ?lhs \llbracket false/\$ok \rrbracket = ?rhs \llbracket false/\$ok \rrbracket$

by (*rel-auto*)

have $b: ?lhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket$

by (*rel-auto*)

have $c: ?lhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket$

by (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest, rel-blast*)

show $?lhs = ?rhs$

using $a b c$

```

    by (rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all)
  qed
  moreover have  $((\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)) \wedge gvirt(n+1))) \wedge gvirt(n+1))$ 
  proof -
    have  $(\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))) = (\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)) \wedge gvirt(n+1)))$ 
    proof (rule ExtChoice-cong)
      fix P assume  $P \in A$ 
      thus  $(P \ X \ \wedge \ gvirt(n+1)) = (P \ (X \ \wedge \ gvirt(n)) \wedge gvirt(n+1))$ 
      using Guarded-def assms by blast
    qed
    thus ?thesis by simp
  qed
  ultimately show  $((\Box P \in A \cdot P \ X) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)))) \wedge gvirt(n+1))$ 
  by simp
qed

```

```

lemma extChoice-Guarded [closure]:
  assumes Guarded P Guarded Q
  shows Guarded  $(\lambda X. P(X) \Box Q(X))$ 
proof -
  have Guarded  $(\lambda X. \Box F \in \{P, Q\} \cdot F(X))$ 
  by (rule ExtChoice-Guarded, auto simp add: assms)
  thus ?thesis
  by (simp add: extChoice-def)
qed

```

7.6 Algebraic laws

```

lemma extChoice-comm:
   $P \Box Q = Q \Box P$ 
  by (unfold extChoice-def, simp add: insert-commute)

```

```

lemma extChoice-idem:
   $P \text{ is CSP} \implies P \Box P = P$ 
  by (unfold extChoice-def, simp add: ExtChoice-single)

```

```

lemma extChoice-assoc:
  assumes  $P \text{ is CSP } Q \text{ is CSP } R \text{ is CSP}$ 
  shows  $P \Box Q \Box R = P \Box (Q \Box R)$ 
proof -
  have  $P \Box Q \Box R = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \Box \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)) \Box \mathbf{R}_s(\text{pre}_R(R) \vdash \text{cmt}_R(R))$ 
  by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))
  also have ... =
     $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \wedge \text{cmt}_R R)$ 
       $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \vee \text{cmt}_R R)))$ 
  by (simp add: extChoice-rdes unrest)
  also have ... =
     $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \wedge \text{cmt}_R R)$ 
       $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$ 
       $((\text{cmt}_R P \vee \text{cmt}_R Q) \vee \text{cmt}_R R)))$ 
  by (rule cong[of  $\mathbf{R}_s \ \mathbf{R}_s$ ], simp, rel-auto)
  also have ... =

```

$\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R P \vee (cmt_R Q \vee cmt_R R))))$
 by (simp add: conj-assoc disj-assoc)
 also have ... =
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R P \vee (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap (\mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)) \sqcap \mathbf{R}_s(pre_R(R) \vdash cmt_R(R)))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $P \sqcap (Q \sqcap R)$
 by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))
 finally show ?thesis .
 qed

lemma extChoice-Stop:

assumes Q is CSP
 shows $Stop \sqcap Q = Q$
 using assms

proof –

have $Stop \sqcap Q = \mathbf{R}_s (true \vdash (\$tr' =_u \$tr \wedge \$wait')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$
 by (simp add: Stop-def SRD-reactive-design-alt assms)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash (((\$tr' =_u \$tr \wedge \$wait') \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\$tr' =_u \$tr \wedge \$wait' \vee cmt_R Q)))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright cmt_R Q))$
 by (metis (no-types, lifting) cond-def eq-upred-sym neg-conj-cancel1 utp-pred-laws.inf.left-idem)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash cmt_R Q)$
 by (simp add: cond-idem)
 also have ... = Q
 by (simp add: SRD-reactive-design-alt assms)
 finally show ?thesis .
 qed

lemma extChoice-Chaos:

assumes Q is CSP
 shows $Chaos \sqcap Q = Chaos$

proof –

have $Chaos \sqcap Q = \mathbf{R}_s (false \vdash true) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$
 by (simp add: Chaos-def SRD-reactive-design-alt assms)
 also have ... = $\mathbf{R}_s (false \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright true))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $\mathbf{R}_s (false \vdash true)$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $Chaos$
 by (simp add: Chaos-def)
 finally show ?thesis .
 qed

lemma extChoice-Dist:

assumes P is CSP $S \subseteq \llbracket CSP \rrbracket_H S \neq \{\}$
 shows $P \sqcap (\bigcap S) = (\bigcap_{Q \in S} P \sqcap Q)$

proof –

let $?S1 = pre_R \text{ ‘ } S$ and $?S2 = cmt_R \text{ ‘ } S$
 have $P \sqcap (\bigsqcap S) = P \sqcap (\bigsqcap_{Q \in S} \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)))$
 by (simp add: SRD-as-reactive-design[THEN sym] Healthy-SUPREMUM UINF-as-Sup-collect assms)
 also have $\dots = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap \mathbf{R}_s(\bigsqcap_{Q \in S} pre_R(Q) \vdash (\bigsqcap_{Q \in S} cmt_R(Q)))$
 by (simp add: RHS-design-USUP SRD-reactive-design-alt assms)
 also have $\dots = \mathbf{R}_s((pre_R(P) \wedge (\bigsqcap_{Q \in S} pre_R(Q))) \vdash$
 $((cmt_R(P) \wedge (\bigsqcap_{Q \in S} cmt_R(Q)))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R(P) \vee (\bigsqcap_{Q \in S} cmt_R(Q))))$
 by (simp add: extChoice-rdes unrest)
 also have $\dots = \mathbf{R}_s(\bigsqcap_{Q \in S} pre_R P \wedge pre_R Q \vdash$
 $(\bigsqcap_{Q \in S} (cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q)))$
 by (simp add: conj-USUP-dist conj-UINF-dist disj-UINF-dist cond-UINF-dist assms)
 also have $\dots = (\bigsqcap_{Q \in S} \mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash$
 $((cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q))))$
 by (simp add: assms RHS-design-USUP)
 also have $\dots = (\bigsqcap_{Q \in S} \mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)))$
 by (simp add: extChoice-rdes unrest)
 also have $\dots = (\bigsqcap_{Q \in S} P \sqcap CSP(Q))$
 by (simp add: UINF-as-Sup-collect, metis (no-types, lifting) Healthy-if SRD-as-reactive-design
 assms(1))
 also have $\dots = (\bigsqcap_{Q \in S} P \sqcap Q)$
 by (rule SUP-cong, simp-all add: Healthy-subset-member[OF assms(2)])
 finally show $?thesis$.
qed

lemma *extChoice-dist*:

assumes P is CSP Q is CSP R is CSP
 shows $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap (P \sqcap R)$
 using assms extChoice-Dist[of $P \{Q, R\}$] by simp

lemma *ExtChoice-seq-distr*:

assumes $\bigwedge i. i \in A \implies P i$ is PCSP Q is NCSP
 shows $(\bigsqcap_{i \in A} P i) ;; Q = (\bigsqcap_{i \in A} P i ;; Q)$

proof (cases $A = \{\}$)

case True

then show $?thesis$

by (simp add: ExtChoice-empty NCSP-implies-CSP Stop-left-zero assms(2))

next

case False

show $?thesis$

proof –

have $1: (\bigsqcap_{i \in A} P i) = (\bigsqcap_{i \in A} (\mathbf{R}_s((pre_R(P i)) \vdash peri_R(P i) \diamond (R4(post_R(P i)))))$
 (is $?X = ?Y$)

by (rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms(1) comp-apply)

have $2: (\bigsqcap_{i \in A} P i ;; Q) = (\bigsqcap_{i \in A} (\mathbf{R}_s((pre_R(P i)) \vdash peri_R(P i) \diamond (R4(post_R(P i))))) ;; Q)$
 (is $?X = ?Y$)

by (rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms(1) comp-apply)

show $?thesis$

by (simp add: 1 2, rdes-eq cls: assms False cong: ExtChoice-cong USUP-cong)

qed

qed

lemma *extChoice-seq-distr*:
assumes P is PCSP Q is PCSP R is NCSP
shows $(P \sqcap Q) ;; R = (P ;; R \sqcap Q ;; R)$
by (*rdes-eq cls: assms*)

lemma *extChoice-seq-distl*:
assumes P is ICSP Q is ICSP R is NCSP
shows $P ;; (Q \sqcap R) = (P ;; Q \sqcap P ;; R)$
by (*rdes-eq cls: assms*)

end

8 Circus and CSP Actions

theory *utp-circus-actions*
imports
utp-circus-extchoice
begin

8.1 Conditionals

lemma *NCSP-cond-srea* [*closure*]:
assumes P is NCSP Q is NCSP
shows $P \triangleleft b \triangleright_R Q$ is NCSP
by (*rule NCSP-NSRD-intro, simp-all add: closure rdes assms unrest*)

8.2 Guarded commands

lemma *GuardedCommR-NCSP-closed* [*closure*]:
assumes P is NCSP
shows $g \rightarrow_R P$ is NCSP
by (*simp add: gcmd-def closure assms*)

8.3 Alternation

lemma *AlternateR-NCSP-closed* [*closure*]:
assumes $\bigwedge i. i \in A \implies P(i)$ is NCSP Q is NCSP
shows $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi})$ is NCSP
proof (*cases* $A = \{\}$)
case *True*
then show *?thesis*
by (*simp add: assms*)
next
case *False*
then show *?thesis*
by (*simp add: AlternateR-def closure assms*)
qed

8.4 While Loops

lemma *NSRD-coerce-NCSP*:
 P is NSRD \implies $\text{Skip} ;; P ;; \text{Skip}$ is NCSP
by (*metis (no-types, hide-lams) CSP3-Skip CSP3-def CSP4-def Healthy-def NCSP-Skip NCSP-implies-CSP NCSP-intro NSRD-is-SRD RA1 SRD-seqr-closure*)

definition $While_C :: 's \text{ upred} \Rightarrow ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action} \text{ (while}_C \text{ - do - od)}$ **where**
 $while_C \ b \text{ do } P \text{ od} = Skip ;; while_R \ b \text{ do } P \text{ od} ;; Skip$

lemma $WhileC\text{-}NCSP\text{-closed}$ [closure]:
assumes $P \text{ is } NCSP \ P \text{ is Productive}$
shows $while_C \ b \text{ do } P \text{ od is } NCSP$
by (simp add: WhileC-def NSRD-coerce-NCSP assms closure)

8.5 Assignment

definition $AssignsCSP :: 'σ \text{ usubst} \Rightarrow ('σ, 'φ) \text{ action} \text{ (}\langle \cdot \rangle_C \text{)}$ **where**
 $[upred\text{-defs}]: AssignsCSP \ σ = \mathbf{R}_s(true \vdash false \diamond (\$tr' =_u \$tr \wedge [\langle \sigma \rangle_a]_s))$

syntax

$\text{-assigns-csp} :: svids \Rightarrow uexprs \Rightarrow logic \text{ (}'(-)' :=_C '(-) \text{'})$
 $\text{-assigns-csp} :: svids \Rightarrow uexprs \Rightarrow logic \text{ (infixr :=}_C \ 90 \text{)}$

translations

$\text{-assigns-csp} \ xs \ vs \Rightarrow \text{CONST } AssignsCSP \ (-mk\text{-usubst} \ (CONST \ id) \ xs \ vs)$
 $\text{-assigns-csp} \ x \ v \leq \text{CONST } AssignsCSP \ (CONST \ subst\text{-upd} \ (CONST \ id) \ x \ v)$
 $\text{-assigns-csp} \ x \ v \leq \text{-assigns-csp} \ (-spvar \ x) \ v$
 $x, y :=_C \ u, v \leq \text{CONST } AssignsCSP \ (CONST \ subst\text{-upd} \ (CONST \ subst\text{-upd} \ (CONST \ id) \ (CONST \ svar \ x) \ u) \ (CONST \ svar \ y) \ v)$

lemma $preR\text{-}AssignsCSP$ [rdes]: $pre_R(\langle \sigma \rangle_C) = true_r$
by (rel-auto)

lemma $periR\text{-}AssignsCSP$ [rdes]: $peri_R(\langle \sigma \rangle_C) = false$
by (rel-auto)

lemma $postR\text{-}AssignsCSP$ [rdes]: $post_R(\langle \sigma \rangle_C) = \Phi(true, \sigma, \langle \rangle)$
by (rel-auto)

lemma $AssignsCSP\text{-}rdes\text{-def}$ [rdes-def]: $\langle \sigma \rangle_C = \mathbf{R}_s(true_r \vdash false \diamond \Phi(true, \sigma, \langle \rangle))$
by (rel-auto)

lemma $AssignsCSP\text{-}CSP$ [closure]: $\langle \sigma \rangle_C \text{ is } CSP$
by (simp add: AssignsCSP-def RHS-tri-design-is-SRD unrest)

lemma $AssignsCSP\text{-}CSP3$ [closure]: $\langle \sigma \rangle_C \text{ is } CSP3$
by (rule CSP3-intro, simp add: closure, rel-auto)

lemma $AssignsCSP\text{-}CSP4$ [closure]: $\langle \sigma \rangle_C \text{ is } CSP4$
by (rule CSP4-intro, simp add: closure, rel-auto+)

lemma $AssignsCSP\text{-}NCSP$ [closure]: $\langle \sigma \rangle_C \text{ is } NCSP$
by (simp add: AssignsCSP-CSP AssignsCSP-CSP3 AssignsCSP-CSP4 NCSP-intro)

lemma $AssignsCSP\text{-}ICSP$ [closure]: $\langle \sigma \rangle_C \text{ is } ICSP$
apply (rule ICSP-intro, simp add: closure, simp add: rdes-def)
apply (rule ISRDI-rdes-intro)
apply (simp-all add: closure)
apply (rel-auto)
done

8.6 Assignment with update

There are different collections that we would like to assign to, but they all have different types and perhaps more importantly different conditions on the update being well defined. For example, for a list well-definedness equates to the index being less than the length etc. Thus we here set up a polymorphic constant for CSP assignment updates that can be specialised to different types.

definition *AssignCSP-update* ::

$(f \Rightarrow 'k \text{ set}) \Rightarrow (f \Rightarrow 'k \Rightarrow 'v \Rightarrow 'f) \Rightarrow (f \Rightarrow 'f) \Rightarrow (f \Rightarrow 'f) \Rightarrow$
 $(k, 'f) \text{ uexpr} \Rightarrow (v, 'f) \text{ uexpr} \Rightarrow (f, 'f) \text{ action where}$
 $[upred-defs, rdes-def]: \text{AssignCSP-update domf updatef } x \ k \ v =$
 $\mathbf{R}_s([k \in_u \text{uop domf } (\&x)]_{S<} \vdash \text{false} \diamond \Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) \ k \ v], \langle \rangle))$

All different assignment updates have the same syntax; the type resolves which implementation to use.

syntax

$\text{-csp-assign-upd} :: \text{svld} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (-[-] :=_C - [0, 0, 72] \ 72)$

translations

$x[k] :=_C v == \text{CONST AssignCSP-update } (\text{CONST udom}) (\text{CONST uupd}) \ x \ k \ v$

lemma *AssignCSP-update-CSP [closure]*:

AssignCSP-update domf updatef } x \ k \ v \text{ is CSP}
by (*simp add: AssignCSP-update-def RHS-tri-design-is-SRD unrest*)

lemma *preR-AssignCSP-update [rdes]*:

$\text{pre}_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \in_u \text{uop domf } (\&x)]_{S<}$
by (*rel-auto*)

lemma *periR-AssignCSP-update [rdes]*:

$\text{peri}_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \notin_u \text{uop domf } (\&x)]_{S<}$
by (*rel-simp*)

lemma *post-AssignCSP-update [rdes]*:

$\text{post}_R(\text{AssignCSP-update domf updatef } x \ k \ v) =$
 $(\Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) \ k \ v], \langle \rangle) \triangleleft k \in_u \text{uop domf } (\&x) \triangleright_R R1(\text{true}))$
by (*rel-auto*)

lemma *AssignCSP-update-NCSP [closure]*:

(AssignCSP-update domf updatef } x \ k \ v) \text{ is NCSP}

proof (*rule NCSP-intro*)

show *(AssignCSP-update domf updatef } x \ k \ v) \text{ is CSP}*

by (*simp add: closure*)

show *(AssignCSP-update domf updatef } x \ k \ v) \text{ is CSP3}*

by (*rule CSP3-SRD-intro, simp-all add: csp-do-def closure rdes unrest*)

show *(AssignCSP-update domf updatef } x \ k \ v) \text{ is CSP4}*

by (*rule CSP4-tri-intro, simp-all add: csp-do-def closure rdes unrest, rel-auto*)

qed

8.7 State abstraction

lemma *ref-unrest-abs-st [unrest]*:

$\$ref \# P \Longrightarrow \$ref \# \langle P \rangle_S$

$\$ref' \# P \Longrightarrow \$ref' \# \langle P \rangle_S$

by (*rel-simp*)**+**

lemma *NCSP-state-srea* [*closure*]: *P* is NCSP \implies state '*a* · *P* is NCSP
apply (*rule* NCSP-NSRD-intro)
apply (*simp-all* add: *closure rdes*)
apply (*simp-all* add: *state-srea-def unrest closure*)
done

8.8 Assumptions

definition *AssumeCircus* ($\{-\}_C$) **where**
[rdes-def]: $\{b\}_C = \mathbf{R}_s(\mathcal{I}(b, \langle \rangle) \vdash (false \diamond \Phi(true, id, \langle \rangle)))$

8.9 Guards

definition *GuardCSP* ::

' σ cond \Rightarrow
 (' σ , ' φ) action \Rightarrow
 (' σ , ' φ) action (infixr $\&_u$ 70) **where**
[upred-defs]: $g \&_u A = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R(A)) \vdash ((\lceil g \rceil_{S<} \wedge cmt_R(A)) \vee (\lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$

lemma *Guard-tri-design*:

$g \&_u P = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R(P)) \vdash (peri_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge post_R(P)))$
proof –
have $(\lceil g \rceil_{S<} \wedge cmt_R(P) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait') = (peri_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge post_R(P))$
by (*rel-auto*)
thus ?thesis **by** (*simp* add: *GuardCSP-def*)
qed

lemma *csp-do-cond-conj*:

assumes *P* is CRR
shows $(\lceil b \rceil_{S<} \wedge P) = \Phi(b, id, \langle \rangle) ;; P$
proof –
have $(\lceil b \rceil_{S<} \wedge CRR(P)) = \Phi(b, id, \langle \rangle) ;; CRR(P)$
by (*rel-auto*)
thus ?thesis
by (*simp* add: *Healthy-if assms*)
qed

lemma *Guard-rdes-def* [*rdes-def*]:

assumes *P* is RR *Q* is CRR *R* is CRR
shows $g \&_u \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$
 (is ?lhs = ?rhs)
proof –
have ?lhs = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash ((P \Rightarrow_r Q) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge (P \Rightarrow_r R)))$
by (*simp* add: *Guard-tri-design rdes assms closure*)
also have ... = $\mathbf{R}_s((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\lceil g \rceil_{S<} \wedge Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\lceil g \rceil_{S<} \wedge R))$
by (*rel-auto*)
also have ... = $\mathbf{R}_s((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$
by (*simp* add: *assms(2) assms(3) csp-do-cond-conj*)
finally show ?thesis .
qed

lemma *Guard-rdes-def'*:

assumes $\$ok' \# P$
shows $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge Q \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
proof –
have $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R (\mathbf{R}_s(P \vdash Q))) \vdash (\lceil g \rceil_{S<} \wedge cmt_R (\mathbf{R}_s(P \vdash Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: GuardCSP-def*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q))) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: rea-pre-RHS-design rea-cmt-RHS-design*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c(\lceil g \rceil_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q)))) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c(\lceil g \rceil_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: R1-R2c-commute R1-disj R1-extend-conj' R1-idem R2c-and R2c-disj R2c-idem*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger (\lceil g \rceil_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: rdes-export-cmt*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: usubst*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: rdes-export-cmt*)
also from *assms* **have** $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r (pre_s \dagger P)) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*rel-auto*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_s \dagger P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: rdes-export-pre*)
also from *assms* **have** $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*rel-auto*)
also from *assms* **have** $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*simp add: rdes-export-pre*)
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge Q \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by (*rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto*)
finally show *?thesis* .
qed

lemma *CSP-Guard [closure]: $b \&_u P$ is CSP*

by (*simp add: GuardCSP-def, rule RHS-design-is-SRD, simp-all add: unrest*)

lemma *preR-Guard [rdes]: P is CSP $\implies pre_R(b \&_u P) = (\lceil b \rceil_{S<} \Rightarrow_r pre_R P)$*

by (*simp add: Guard-tri-design rea-pre-RHS-design usubst unrest R2c-preR R2c-lift-state-pre R2c-rea-impl R1-rea-impl R1-preR Healthy-if, rel-auto*)

lemma *periR-Guard [rdes]:*

assumes P is NCSP

shows $peri_R(b \&_u P) = (peri_R P \triangleleft b \triangleright_R \mathcal{E}(true, \langle \rangle, \{ \}_u))$

proof –

have $\text{peri}_R(b \&_u P) = (([b]_{S<} \Rightarrow_r \text{pre}_R P) \Rightarrow_r (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)))$
by (*simp add: assms Guard-tri-design rea-peri-RHS-design usubst unrest R1-rea-impl R2c-rea-not R2c-rea-impl R2c-preR R2c-periR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr closure Healthy-if R1-cond R1-tr'-eq-tr*)
also have $\dots = ((\text{pre}_R P \Rightarrow_r \text{peri}_R P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$
by (*rel-auto*)
also have $\dots = (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$
by (*simp add: SRD-peri-under-pre add: unrest closure assms*)
finally show *?thesis*
by *rel-auto*
qed

lemma *postR-Guard [rdes]:*

assumes *P is NCSP*
shows $\text{post}_R(b \&_u P) = ([b]_{S<} \wedge \text{post}_R P)$
proof –
have $\text{post}_R(b \&_u P) = (([b]_{S<} \Rightarrow_r \text{pre}_R P) \Rightarrow_r ([b]_{S<} \wedge \text{post}_R P))$
by (*simp add: Guard-tri-design rea-post-RHS-design usubst unrest R2c-rea-not R2c-and R2c-rea-impl R2c-preR R2c-postR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr R1-rea-impl R1-extend-conj' R1-post-SRD closure assms*)
also have $\dots = ([b]_{S<} \wedge (\text{pre}_R P \Rightarrow_r \text{post}_R P))$
by (*rel-auto*)
also have $\dots = ([b]_{S<} \wedge \text{post}_R P)$
by (*simp add: SRD-post-under-pre add: unrest closure assms*)
also have $\dots = ([b]_{S<} \wedge \text{post}_R P)$
by (*metis CSP-Guard R1-extend-conj R1-post-SRD calculation rea-st-cond-def*)
finally show *?thesis* .
qed

lemma *CSP3-Guard [closure]:*

assumes *P is CSP P is CSP3*
shows $b \&_u P \text{ is CSP3}$
proof –
from *assms* **have** $1:\$ref \# P\llbracket \text{false}/\$wait \rrbracket$
by (*simp add: CSP-Guard CSP3-iff*)
hence $\$ref \# \text{pre}_R (P\llbracket 0/\$tr \rrbracket) \ \$ref \# \text{pre}_R P \ \$ref \# \text{cmt}_R P$
by (*pred-blast*) +
hence $\$ref \# (b \&_u P)\llbracket \text{false}/\$wait \rrbracket$
by (*simp add: CSP3-iff GuardCSP-def RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst*)
thus *?thesis*
by (*metis CSP3-intro CSP-Guard*)
qed

lemma *CSP4-Guard [closure]:*

assumes *P is NCSP*
shows $b \&_u P \text{ is CSP4}$
proof (*rule CSP4-tri-intro[OF CSP-Guard]*)
show $(\neg_r \text{pre}_R (b \&_u P)) ;; R1 \text{ true} = (\neg_r \text{pre}_R (b \&_u P))$
proof –
have $a:(\neg_r \text{pre}_R P) ;; R1 \text{ true} = (\neg_r \text{pre}_R P)$
by (*simp add: CSP4-neg-pre-unit assms closure*)
have $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) ;; R1 \text{ true} = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P))$
proof –
have $1:(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) = ([b]_{S<} \wedge (\neg_r \text{pre}_R P))$

```

    by (rel-auto)
  also have 2:... = ( $[b]_{S<} \wedge ((\neg_r \text{pre}_R P) \mathbin{;;} R1 \text{ true}))$ 
    by (simp add: a)
  also have 3:... = ( $\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)$ )  $;; R1 \text{ true}$ 
    by (rel-auto)
  finally show ?thesis ..
qed
thus ?thesis
  by (simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest)
qed
show  $\$st' \# \text{peri}_R (b \ \&_u \ P)$ 
  by (simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest)
show  $\$ref' \# \text{post}_R (b \ \&_u \ P)$ 
  by (simp add: preR-Guard postR-Guard NSRD-CSP4-intro closure assms unrest)
qed

```

lemma *NCSP-Guard [closure]*:

assumes P is NCSP
 shows $b \ \&_u \ P$ is NCSP

proof –

have P is CSP
 using NCSP-implies-CSP assms by blast
 then show ?thesis

by (metis (no-types) CSP3-Guard CSP3-commutes-CSP4 CSP4-Guard CSP4-Idempotent CSP-Guard
 Healthy-Idempotent Healthy-def NCSP-def assms comp-apply)
 qed

lemma *Productive-Guard [closure]*:

assumes P is CSP P is Productive $\$wait' \# \text{pre}_R(P)$
 shows $b \ \&_u \ P$ is Productive

proof –

have $b \ \&_u \ P = b \ \&_u \ \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$
 by (metis Healthy-def Productive-form assms(1) assms(2))

also have ... =

$\mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash$
 $((\text{pre}_R P \Rightarrow_r \text{peri}_R P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ([b]_{S<} \wedge (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u$
 $\$tr)))$

by (simp add: Guard-tri-design rea-pre-RHS-design rea-peri-RHS-design rea-post-RHS-design unrest
 assms

usubst R1-preR Healthy-if R1-rea-impl R1-peri-SRD R1-extend-conj' R2c-preR R2c-not R2c-rea-impl

$R2c\text{-peri}_R \ R2c\text{-post}_R \ R2c\text{-and} \ R2c\text{-tr-less-tr}' \ R1\text{-tr-less-tr}'$)

also have ... = $\mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (([b]_{S<} \wedge \text{post}_R P)$
 $\wedge \$tr' >_u \$tr))$

by (rel-auto)

also have ... = $\text{Productive}(b \ \&_u \ P)$

by (simp add: Productive-def Guard-tri-design RHS-tri-design-par unrest)

finally show ?thesis

by (simp add: Healthy-def')

qed

8.10 Basic events

definition $do_u ::$

$(\varphi, \sigma) \text{ uexpr} \Rightarrow (\sigma, \varphi) \text{ action}$ **where**

$[upred\text{-defs}]: do_u \ e = ((\$tr' =_u \$tr \wedge [e]_{S<} \notin_u \$ref') \triangleleft \$wait' \triangleright (\$tr' =_u \$tr \hat{^}_u \langle [e]_{S<} \rangle \wedge \$st' =_u$

$\$st))$

definition $DoCSP :: ('\varphi, '\sigma) uexpr \Rightarrow (''\sigma, '\varphi) action (do_C)$ **where**
 $[upred-defs]: DoCSP\ a = \mathbf{R}_s(true \vdash do_u\ a)$

lemma $R1-DoAct: R1(do_u(a)) = do_u(a)$
by $(rel-auto)$

lemma $R2c-DoAct: R2c(do_u(a)) = do_u(a)$
by $(rel-auto)$

lemma $DoCSP-alt-def: do_C(a) = R3h(CSP1(\$ok' \wedge do_u(a)))$
apply $(simp\ add: DoCSP-def\ RHS-def\ design-def\ impl-alt-def\ R1-R3h-commute\ R2c-R3h-commute\ R2c-disj\ R2c-not\ R2c-ok\ R2c-ok'\ R2c-and\ R2c-DoAct\ R1-disj\ R1-extend-conj'\ R1-DoAct)$
apply $(rel-auto)$
done

lemma $DoAct-unrests [unrest]:$
 $\$ok \# do_u(a)\ \$wait \# do_u(a)$
by $(pred-auto)+$

lemma $DoCSP-RHS-tri [rdes-def]:$
 $do_C(a) = \mathbf{R}_s(true_r \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \diamond \Phi(true, id, \langle a \rangle)))$
by $(simp\ add: DoCSP-def\ do_u-def\ wait'-cond-def, rel-auto)$

lemma $CSP-DoCSP [closure]: do_C(a)$ *is* CSP
by $(simp\ add: DoCSP-def\ do_u-def\ RHS-design-is-SRD\ unrest)$

lemma $preR-DoCSP [rdes]: pre_R(do_C(a)) = true_r$
by $(simp\ add: DoCSP-def\ rea-pre-RHS-design\ unrest\ usubst\ R2c-true)$

lemma $periR-DoCSP [rdes]: peri_R(do_C(a)) = \mathcal{E}(true, \langle \rangle, \{a\}_u)$
by $(rel-auto)$

lemma $postR-DoCSP [rdes]: post_R(do_C(a)) = \Phi(true, id, \langle a \rangle)$
by $(rel-auto)$

lemma $CSP3-DoCSP [closure]: do_C(a)$ *is* $CSP3$
by $(rule\ CSP3-intro[OF\ CSP-DoCSP])$
 $(simp\ add: DoCSP-def\ do_u-def\ RHS-def\ design-def\ R1-def\ R2c-def\ R2s-def\ R3h-def\ unrest\ usubst)$

lemma $CSP4-DoCSP [closure]: do_C(a)$ *is* $CSP4$
by $(rule\ CSP4-tri-intro[OF\ CSP-DoCSP], simp-all\ add: preR-DoCSP\ periR-DoCSP\ postR-DoCSP\ unrest)$

lemma $NCSP-DoCSP [closure]: do_C(a)$ *is* $NCSP$
by $(metis\ CSP3-DoCSP\ CSP4-DoCSP\ CSP-DoCSP\ Healthy-def\ NCSP-def\ comp-apply)$

lemma $Productive-DoCSP [closure]:$
 $(do_C\ a :: (''\sigma, '\psi) action)$ *is* $Productive$

proof –
have $((\Phi(true, id, \langle a \rangle) \wedge \$tr' >_u \$tr) :: (''\sigma, '\psi) action)$
 $= (\Phi(true, id, \langle a \rangle))$
by $(rel-auto, simp\ add: Prefix-Order.strict-prefixI')$

hence $Productive(do_C a) = do_C a$
 by (simp add: Productive-RHS-design-form DoCSP-RHS-tri unrest)
 thus ?thesis
 by (simp add: Healthy-def)
 qed

lemma *PCSP-DoCSP [closure]:*
($do_C a :: ('σ, 'ψ)$ action) is PCSP
 by (simp add: Healthy-comp NCSP-DoCSP Productive-DoCSP)

lemma *wp-rea-DoCSP-lemma:*
 fixes $P :: ('σ, 'φ)$ action
 assumes $ok \# P \ \$wait \# P$
 shows $(\$tr' =_u \$tr \hat{ }_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) ;; P = (\exists \$ref \cdot P[\$tr \hat{ }_u \langle [a]_{S<} \rangle / \$tr])$
 using *assms*
 by (rel-auto, meson)

lemma *wp-rea-DoCSP:*
 assumes P is NCSP
 shows $(\$tr' =_u \$tr \hat{ }_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) \ wp_r \ pre_R P =$
 $(\neg_r (\neg_r \ pre_R P)[\$tr \hat{ }_u \langle [a]_{S<} \rangle / \$tr])$
 by (simp add: wp-rea-def wp-rea-DoCSP-lemma unrest usubst ex-unrest assms closure)

lemma *wp-rea-DoCSP-alt:*
 assumes P is NCSP
 shows $(\$tr' =_u \$tr \hat{ }_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) \ wp_r \ pre_R P =$
 $(\$tr' \geq_u \$tr \hat{ }_u \langle [a]_{S<} \rangle \Rightarrow_r (pre_R P)[\$tr \hat{ }_u \langle [a]_{S<} \rangle / \$tr])$
 by (simp add: wp-rea-DoCSP assms rea-not-def R1-def usubst unrest, rel-auto)

8.11 Event prefix

definition *PrefixCSP ::*
 $(\varphi, \sigma) \ uexpr \Rightarrow$
 (σ, φ) action \Rightarrow
 (σ, φ) action $(- \rightarrow_C - [81, 80] \ 80)$ **where**
[upred-defs]: $PrefixCSP \ a \ P = (do_C(a) ;; CSP(P))$

abbreviation $OutputCSP \ c \ v \ P \equiv PrefixCSP \ (c.v)_u \ P$

lemma *CSP-PrefixCSP [closure]:* $PrefixCSP \ a \ P$ is CSP
 by (simp add: PrefixCSP-def closure)

lemma *CSP3-PrefixCSP [closure]:*
 $PrefixCSP \ a \ P$ is CSP3
 by (metis (no-types, hide-lams) CSP3-DoCSP CSP3-def Healthy-def PrefixCSP-def seqr-assoc)

lemma *CSP4-PrefixCSP [closure]:*
 assumes P is CSP P is CSP4
 shows $PrefixCSP \ a \ P$ is CSP4
 by (metis (no-types, hide-lams) CSP4-def Healthy-def PrefixCSP-def assms(1) assms(2) seqr-assoc)

lemma *NCSP-PrefixCSP [closure]:*
 assumes P is NCSP
 shows $PrefixCSP \ a \ P$ is NCSP
 by (metis (no-types, hide-lams) CSP3-PrefixCSP CSP3-commutes-CSP4 CSP4-Idempotent CSP4-PrefixCSP
 CSP-PrefixCSP Healthy-Idempotent Healthy-def NCSP-def NCSP-implies-CSP assms comp-apply)

lemma *Productive-PrefixCSP [closure]: P is NCSP \implies PrefixCSP a P is Productive*
by (*simp add: Healthy-if NCSP-DoCSP NCSP-implies-NSRD NSRD-is-SRD PrefixCSP-def Productive-DoCSP Productive-seq-1*)

lemma *PCSP-PrefixCSP [closure]: P is NCSP \implies PrefixCSP a P is PCSP*
by (*simp add: Healthy-comp NCSP-PrefixCSP Productive-PrefixCSP*)

lemma *PrefixCSP-Guarded [closure]: Guarded (PrefixCSP a)*

proof –

have *PrefixCSP a = ($\lambda X. do_C(a) ;; CSP(X)$)*

by (*simp add: fun-eq-iff PrefixCSP-def*)

thus *?thesis*

using *Guarded-if-Productive NCSP-DoCSP NCSP-implies-NSRD Productive-DoCSP* **by** *auto*

qed

lemma *PrefixCSP-type [closure]: PrefixCSP a $\in \llbracket H \rrbracket_H \rightarrow \llbracket CSP \rrbracket_H$*
using *CSP-PrefixCSP* **by** *blast*

lemma *PrefixCSP-Continuous [closure]: Continuous (PrefixCSP a)*
by (*simp add: Continuous-def PrefixCSP-def ContinuousD[OF SRD-Continuous] seq-Sup-distl*)

lemma *PrefixCSP-RHS-tri-lemma1:*

R1 (R2s ($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \wedge \lceil II \rceil_R$)) = ($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \wedge \lceil II \rceil_R$))

by (*rel-auto*)

lemma *PrefixCSP-RHS-tri-lemma2:*

fixes *P :: (' σ , ' φ) action*

assumes *$\$ok \# P \$wait \# P$*

shows (*($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \wedge \$st' =_u \$st \wedge \neg \$wait'$) ;; $P = (\exists \$ref \cdot P[\$tr \hat{\ }_u \langle \lceil a \rceil_{S<} / \$tr]$)*)

using *assms*

by (*rel-auto, meson, fastforce*)

lemma *tr-extend-seqr:*

fixes *P :: (' σ , ' φ) action*

assumes *$\$ok \# P \$wait \# P \$ref \# P$*

shows (*($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \wedge \$st' =_u \$st$) ;; $P = P[\$tr \hat{\ }_u \langle \lceil a \rceil_{S<} / \$tr]$*)

using *assms* **by** (*simp add: wp-rea-DoCSP-lemma assms unrest ex-unrest*)

lemma *trace-ext-R1-closed [closure]: P is R1 $\implies P[\$tr \hat{\ }_u e / \$tr]$ is R1*
by (*rel-blast*)

lemma *preR-PrefixCSP-NCSP [rdes]:*

assumes *P is NCSP*

shows *$pre_R(PrefixCSP a P) = (\mathcal{I}(true, \langle a \rangle) \Rightarrow_r (pre_R P) \llbracket \langle a \rangle \rrbracket_t)$*

by (*simp add: PrefixCSP-def assms closure rdes rpred Healthy-if wp usubst unrest*)

lemma *periR-PrefixCSP [rdes]:*

assumes *P is NCSP*

shows *$peri_R(PrefixCSP a P) = (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee (peri_R P) \llbracket \langle a \rangle \rrbracket_t)$*

proof –

have *$peri_R(PrefixCSP a P) = peri_R(do_C a ;; P)$*

by (*simp add: PrefixCSP-def closure assms Healthy-if*)

also have *$\dots = ((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r pre_R P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r \$tr' =_u \$tr \wedge \lceil a \rceil_{S<} \notin_u \$ref' \vee peri_R P \llbracket \langle a \rangle \rrbracket_t)$*

by (*simp add: assms NSRD-CSP4-intro csp-enable-tr-empty closure rdes unrest ex-unrest usubst rpred*)

$wp)$
also have $\dots = (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee ((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r pre_R P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r peri_R P \llbracket \langle a \rangle \rrbracket_t))$
by (*rel-auto*)
also have $\dots = (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee ((pre_R(P) \Rightarrow_r peri_R P) \llbracket \langle a \rangle \rrbracket_t))$
by (*rel-auto*)
also have $\dots = (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee (peri_R P) \llbracket \langle a \rangle \rrbracket_t)$
by (*simp add: SRD-peri-under-pre assms closure unrest*)
finally show *?thesis* .
qed

lemma *postR-PrefixCSP* [*rdes*]:

assumes P is NCSP

shows $post_R(PrefixCSP\ a\ P) = (post_R\ P) \llbracket \langle a \rangle \rrbracket_t$

proof –

have $post_R(PrefixCSP\ a\ P) = ((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r (pre_R\ P) \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r (post_R\ P) \llbracket \langle a \rangle \rrbracket_t)$

by (*simp add: PrefixCSP-def assms Healthy-if*)

(*simp add: assms Healthy-if wp closure rdes rpred wp-rea-DoCSP-lemma unrest ex-unrest usubst*)

also have $\dots = (\mathcal{I}(true, \langle a \rangle) \wedge (pre_R\ P \Rightarrow_r post_R\ P) \llbracket \langle a \rangle \rrbracket_t)$

by (*rel-auto*)

also have $\dots = (\mathcal{I}(true, \langle a \rangle) \wedge (post_R\ P) \llbracket \langle a \rangle \rrbracket_t)$

by (*simp add: SRD-post-under-pre assms closure unrest*)

also have $\dots = (post_R\ P) \llbracket \langle a \rangle \rrbracket_t$

by (*rel-auto*)

finally show *?thesis* .

qed

lemma *PrefixCSP-RHS-tri*:

assumes P is NCSP

shows $PrefixCSP\ a\ P = \mathbf{R}_s((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r pre_R P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee peri_R P \llbracket \langle a \rangle \rrbracket_t) \diamond post_R P \llbracket \langle a \rangle \rrbracket_t)$

by (*simp add: PrefixCSP-def Healthy-if unrest assms closure NSRD-composition-wp rdes rpred usubst wp*)

For prefix, we can chose whether to propagate the assumptions or not, hence there are two laws.

lemma *PrefixCSP-rdes-def-1* [*rdes-def*]:

assumes P is CRC Q is CRR R is CRR

$\$st' \# Q \$ref' \# R$

shows $PrefixCSP\ a\ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee Q \llbracket \langle a \rangle \rrbracket_t) \diamond R \llbracket \langle a \rangle \rrbracket_t)$

apply (*subst PrefixCSP-RHS-tri*)

apply (*rule NCSP-rdes-intro*)

apply (*simp-all add: assms rdes closure*)

apply (*rel-auto*)

done

lemma *PrefixCSP-rdes-def-2*:

assumes P is CRC Q is CRR R is CRR

$\$st' \# Q \$ref' \# R$

shows $PrefixCSP\ a\ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee (P \wedge Q) \llbracket \langle a \rangle \rrbracket_t) \diamond (P \wedge R) \llbracket \langle a \rangle \rrbracket_t)$

apply (*subst PrefixCSP-RHS-tri*)

apply (*rule NCSP-rdes-intro*)

apply (*simp-all add: assms rdes closure*)

apply (*rel-auto*)

done

8.12 Guarded external choice

abbreviation $\text{GuardedChoiceCSP} :: 'v \text{ set} \Rightarrow ('v \Rightarrow (' \sigma, 'v) \text{ action}) \Rightarrow (' \sigma, 'v) \text{ action}$ **where**
 $\text{GuardedChoiceCSP } A \ P \equiv (\Box x \in A \cdot \text{PrefixCSP } \ll x \gg (P(x)))$

syntax

$\text{-GuardedChoiceCSP} :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\Box - \in - \rightarrow - [0,0,85] \ 86)$

translations

$\Box x \in A \rightarrow P == \text{CONST } \text{GuardedChoiceCSP } A \ (\lambda x. P)$

lemma $\text{GuardedChoiceCSP} \ [\text{rdes-def}]$:

assumes $\bigwedge x. P(x) \text{ is NCSP } A \neq \{\}$

shows $(\Box x \in A \rightarrow P(x)) =$

$\mathbf{R}_s \ ((\Box x \in A \cdot \mathcal{I}(\text{true}, \langle \ll x \gg \rangle)) \Rightarrow_r \text{pre}_R (P \ x) [\langle \ll x \gg \rangle]_t) \vdash$
 $((\Box x \in A \cdot \mathcal{E}(\text{true}, \langle \rangle, \{\ll x \gg\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\Box x \in A \cdot \text{peri}_R (P \ x) [\langle \ll x \gg \rangle]_t)) \diamond$
 $(\Box x \in A \cdot \text{post}_R (P \ x) [\langle \ll x \gg \rangle]_t))$

by (*simp add: PrefixCSP-RHS-tri assms ExtChoice-tri-rdes closure unrest, rel-auto*)

8.13 Input prefix

definition $\text{InputCSP} ::$

$('a, 'v) \text{ chan} \Rightarrow ('a \Rightarrow ' \sigma \text{ upred}) \Rightarrow ('a \Rightarrow (' \sigma, 'v) \text{ action}) \Rightarrow (' \sigma, 'v) \text{ action}$ **where**
 $[\text{upred-defs}]: \text{InputCSP } c \ A \ P = (\Box x \in \text{UNIV} \cdot A(x) \ \&_u \ \text{PrefixCSP } (c \cdot \ll x \gg)_u (P \ x))$

definition $\text{InputVarCSP} :: ('a, 'v) \text{ chan} \Rightarrow ('a \Rightarrow ' \sigma \text{ upred}) \Rightarrow ('a \Longrightarrow ' \sigma) \Rightarrow (' \sigma, 'v) \text{ action} \Rightarrow (' \sigma, 'v) \text{ action}$ **where**

$\text{InputVarCSP } c \ A \ x \ P = \text{InputCSP } c \ A \ (\lambda v. \langle [x \mapsto_s \ll v \gg] \rangle_C) ;; \text{CSP}(P)$

definition $\text{do}_I ::$

$('a, 'v) \text{ chan} \Rightarrow$

$('a \Longrightarrow (' \sigma, 'v) \text{ st-csp}) \Rightarrow$

$('a \Rightarrow (' \sigma, 'v) \text{ action}) \Rightarrow$

$(' \sigma, 'v) \text{ action}$ **where**

$\text{do}_I \ c \ x \ P = ($

$(\$tr' =_u \$tr \wedge \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot (c \cdot \ll e \gg)_u\}_u \cap_u \$ref' =_u \{\}_u)$

$\triangleleft \$wait' \triangleright$

$((\$tr' - \$tr) \in_u \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot \langle (c \cdot \ll e \gg)_u \rangle_u \wedge (c \cdot \$x')_u =_u \text{last}_u(\$tr')\}))$

lemma $\text{InputCSP-CSP} \ [\text{closure}]: \text{InputCSP } c \ A \ P \text{ is CSP}$

by (*simp add: CSP-ExtChoice InputCSP-def*)

lemma $\text{InputCSP-NCSP} \ [\text{closure}]: \ll \bigwedge v. P(v) \text{ is NCSP} \gg \Longrightarrow \text{InputCSP } c \ A \ P \text{ is NCSP}$

apply (*simp add: InputCSP-def*)

apply (*rule NCSP-ExtChoice*)

apply (*simp add: NCSP-Guard NCSP-PrefixCSP image-Collect-subsetI top-set-def*)

done

lemma $\text{Productive-InputCSP} \ [\text{closure}]:$

$\ll \bigwedge v. P(v) \text{ is NCSP} \gg \Longrightarrow \text{InputCSP } x \ A \ P \text{ is Productive}$

by (*auto simp add: InputCSP-def unrest closure intro: Productive-ExtChoice*)

lemma $\text{preR-InputCSP} \ [\text{rdes}]:$

assumes $\bigwedge v. P(v) \text{ is NCSP}$

shows $\text{pre}_R(\text{InputCSP } a \ A \ P) = (\Box v \cdot [A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle)) \Rightarrow_r (\text{pre}_R (P(v))) [\langle (a \cdot \ll v \gg)_u \rangle]_t$

by (*simp add: InputCSP-def rdes closure assms alpha usubst unrest*)

lemma *periR-InputCSP* [rdes]:
assumes $\bigwedge v. P(v)$ is NCSP
shows $\text{peri}_R(\text{InputCSP } a \ A \ P) =$
 $((\bigsqcup x \cdot [A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\bigsqcap x \cdot [A(x)]_{S<} \wedge (\text{peri}_R(P \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t)$
by (*simp add: InputCSP-def rdes closure assms, rel-auto*)

lemma *postR-InputCSP* [rdes]:
assumes $\bigwedge v. P(v)$ is NCSP
shows $\text{post}_R(\text{InputCSP } a \ A \ P) =$
 $(\bigsqcap x \cdot [A \ x]_{S<} \wedge \text{post}_R(P \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t)$
using *assms* **by** (*simp add: InputCSP-def rdes closure assms usubst unrest*)

lemma *InputCSP-rdes-def* [rdes-def]:
assumes $\bigwedge v. P(v)$ is CRC $\bigwedge v. Q(v)$ is CRR $\bigwedge v. R(v)$ is CRR
 $\bigwedge v. \$st' \# Q(v) \bigwedge v. \$ref' \# R(v)$
shows $\text{InputCSP } a \ A \ (\lambda v. \mathbf{R}_s(P(v) \vdash Q(v) \diamond R(v))) =$
 $\mathbf{R}_s((\bigsqcup v \cdot ([A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r (P \ v))[\langle (a \cdot \ll v \gg)_u \rangle]_t))$
 $\vdash (((\bigsqcup x \cdot [A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\bigsqcap x \cdot [A(x)]_{S<} \wedge (P \ x \wedge Q \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t))$
 $\diamond (\bigsqcap x \cdot [A \ x]_{S<} \wedge (P \ x \wedge R \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t))$ (**is** *?lhs = ?rhs*)

proof –

have $1:\text{pre}_R(?lhs) = (\bigsqcup v \cdot [A \ v]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r P \ v)[\langle (a \cdot \ll v \gg)_u \rangle]_t)$ (**is** $- = ?A$)
by (*simp add: rdes NCSP-rdes-intro assms conj-comm closure*)
have $2:\text{peri}_R(?lhs) = (\bigsqcup x \cdot [A \ x]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap x \cdot [A \ x]_{S<} \wedge (P \ x \Rightarrow_r Q \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t)$
 $\wedge (P \ x \Rightarrow_r Q \ x)[\langle (a \cdot \ll x \gg)_u \rangle]_t)$
(is $- = ?B$)
by (*simp add: rdes NCSP-rdes-intro assms closure*)
have $3:\text{post}_R(?lhs) = (\bigsqcap x \cdot [A \ x]_{S<} \wedge (P \ x \Rightarrow_r R \ x))[\langle (a \cdot \ll x \gg)_u \rangle]_t)$
(is $- = ?C$)
by (*simp add: rdes NCSP-rdes-intro assms closure*)
have $?lhs = \mathbf{R}_s(?A \vdash ?B \diamond ?C)$
by (*subst SRD-reactive-tri-design[THEN sym], simp-all add: closure 1 2 3*)
also have $\dots = ?rhs$
by (*rel-auto*)
finally show *?thesis* .
qed

8.14 Algebraic laws

lemma *AssignCSP-conditional*:
assumes *vwb-lens* *x*
shows $x :=_C e \triangleleft b \triangleright_R x :=_C f = x :=_C (e \triangleleft b \triangleright f)$
by (*rdes-eq cls: assms*)

lemma *AssignsCSP-id*: $\langle id \rangle_C = \text{Skip}$
by (*rel-auto*)

lemma *Guard-comp*:
 $g \&_u h \&_u P = (g \wedge h) \&_u P$
by (*rule antisym, rel-blast, rel-blast*)

lemma *Guard-false* [*simp*]: $\text{false} \&_u P = \text{Stop}$

by (*simp add: GuardCSP-def Stop-def rpred closure alpha R1-design-R1-pre*)

lemma *Guard-true* [*simp*]:

P is CSP $\implies \text{true} \ \&_u \ P = P$

by (*simp add: GuardCSP-def alpha SRD-reactive-design-alt closure rpred*)

lemma *Guard-conditional*:

assumes *P is NCSP*

shows *$b \ \&_u \ P = P \triangleleft b \triangleright_R \text{Stop}$*

by (*rdes-eq cls: assms*)

lemma *Guard-expansion*:

$(g_1 \vee g_2) \ \&_u \ P = (g_1 \ \&_u \ P) \sqcap (g_2 \ \&_u \ P)$

by (*rel-auto*)

lemma *Conditional-as-Guard*:

assumes *P is NCSP Q is NCSP*

shows *$P \triangleleft b \triangleright_R \ Q = b \ \&_u \ P \sqcap (\neg b) \ \&_u \ Q$*

by (*rdes-eq cls: assms; simp add: le-less*)

lemma *PrefixCSP-dist*:

$\text{PrefixCSP } a \ (P \sqcap Q) = (\text{PrefixCSP } a \ P) \sqcap (\text{PrefixCSP } a \ Q)$

using *Continuous-Disjunctuous Disjunctuous-def PrefixCSP-Continuous* **by** *auto*

lemma *DoCSP-is-Prefix*:

$\text{do}_C(a) = \text{PrefixCSP } a \ \text{Skip}$

by (*simp add: PrefixCSP-def Healthy-if closure,metis CSP4-DoCSP CSP4-def Healthy-def*)

lemma *PrefixCSP-seq*:

assumes *P is CSP Q is CSP*

shows *$(\text{PrefixCSP } a \ P) ;; Q = (\text{PrefixCSP } a \ (P ;; Q))$*

by (*simp add: PrefixCSP-def seqr-assoc Healthy-if assms closure*)

lemma *PrefixCSP-extChoice-dist*:

assumes *P is NCSP Q is NCSP R is NCSP*

shows *$((a \rightarrow_C P) \sqcap (b \rightarrow_C Q)) ;; R = (a \rightarrow_C P ;; R) \sqcap (b \rightarrow_C Q ;; R)$*

by (*simp add: PCSP-PrefixCSP assms(1) assms(2) assms(3) extChoice-seq-distr*)

lemma *GuardedChoiceCSP-dist*:

assumes $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$

shows $\square x \in A \rightarrow P(x) ;; Q = \square x \in A \rightarrow (P(x) ;; Q)$

by (*simp add: ExtChoice-seq-distr PrefixCSP-seq closure assms cong: ExtChoice-cong*)

Alternation can be re-expressed as an external choice when the guards are disjoint

declare *ExtChoice-tri-rdes* [*rdes-def*]

declare *ExtChoice-tri-rdes'* [*rdes-def del*]

declare *extChoice-rdes-def* [*rdes-def*]

declare *extChoice-rdes-def'* [*rdes-def del*]

lemma *AlternateR-as-ExtChoice*:

assumes

$\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$

$\bigwedge i j. \llbracket i \in A; j \in A; i \neq j \rrbracket \implies (g \ i \wedge g \ j) = \text{false}$

shows *$(\text{if}_R \ i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) =$*

```

      ( $\Box i \in A \cdot g(i) \&_u P(i)$ )  $\Box (\bigwedge i \in A \cdot \neg g(i) \&_u Q$ 
proof (cases  $A = \{\}$ )
  case True
    then show ?thesis by (simp add: ExtChoice-empty extChoice-Stop closure assms)
next
  case False
  show ?thesis

proof –
  have 1: ( $\bigwedge i \in A \cdot g i \rightarrow_R P i$ ) = ( $\bigwedge i \in A \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i))$ )
    by (rule UINF-cong, simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1))
  have 2: ( $\Box i \in A \cdot g(i) \&_u P(i)$ ) = ( $\Box i \in A \cdot g(i) \&_u \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i))$ )
    by (rule ExtChoice-cong, simp add: NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-tri-design
assms(1))
  from assms(3) show ?thesis
    by (simp add: AlternateR-def 1 2)
    (rdes-eq cls: assms(1-2)_simps: False cong: UINF-cong ExtChoice-cong)
qed
qed

declare ExtChoice-tri-rdes [rdes-def del]
declare ExtChoice-tri-rdes' [rdes-def]

declare extChoice-rdes-def [rdes-def del]
declare extChoice-rdes-def' [rdes-def]

end

```

9 Syntax and Translations for Event Prefix

```

theory utp-circus-prefix
imports utp-circus-actions
begin

syntax
  -simple-prefix :: logic  $\Rightarrow$  logic  $\Rightarrow$  logic ( $- \rightarrow -$  [81, 80] 80)

translations
   $a \rightarrow P == \text{CONST PrefixCSP} \ll a \gg P$ 

```

We next configure a syntax for mixed prefixes.

```

nonterminal prefix-elem' and mixed-prefix'

syntax -end-prefix :: prefix-elem'  $\Rightarrow$  mixed-prefix' (-)

```

Input Prefix: $\dots?(x)$

```

syntax -simple-input-prefix :: id  $\Rightarrow$  prefix-elem' (?'(-))

```

Input Prefix with Constraint: $\dots?(x : P)$

```

syntax -input-prefix :: id  $\Rightarrow$  ('σ, 'ε) action  $\Rightarrow$  prefix-elem' (?'(- :/ -'))

```

Output Prefix: $\dots![v]e$

A variable name must currently be provided for outputs, too. Fix?!

syntax *-output-prefix* :: ('a, 'σ) uexpr ⇒ prefix-elem' (!'(-'))

syntax *-output-prefix* :: ('a, 'σ) uexpr ⇒ prefix-elem' (.'(-'))

syntax (**output**) *-output-prefix-pp* :: ('a, 'σ) uexpr ⇒ prefix-elem' (!'(-'))

syntax

-prefix-aux :: pttrn ⇒ logic ⇒ prefix-elem'

Mixed-Prefix Action: $c \dots (prefix) \rightarrow A$

syntax *-mixed-prefix* :: prefix-elem' ⇒ mixed-prefix' ⇒ mixed-prefix' (--)

syntax

-prefix-action ::

('a, 'ε) chan ⇒ mixed-prefix' ⇒ ('σ, 'ε) action ⇒ ('σ, 'ε) action

((-- →/ -) [81, 81, 80] 80)

Syntax translations

definition *lconj* :: ('a ⇒ 'α upred) ⇒ ('b ⇒ 'α upred) ⇒ ('a × 'b ⇒ 'α upred) (**infixr** \wedge_l 35)

where [*upred-defs*]: $(P \wedge_l Q) \equiv (\lambda (x,y). P \wedge Q y)$

definition *outp-constraint* (**infix** $=_o$ 60) **where**

[*upred-defs*]: *outp-constraint* $v \equiv (\lambda x. \ll x \gg =_u v)$

translations

-simple-input-prefix $x \rightleftharpoons$ *-input-prefix* x *true*

-mixed-prefix (*-input-prefix* x P) (*-prefix-aux* y Q) \rightarrow

-prefix-aux (*-pattern* x y) $((\lambda x. P) \wedge_l Q)$

-mixed-prefix (*-output-prefix* P) (*-prefix-aux* y Q) \rightarrow

-prefix-aux (*-pattern* *-idtdummy* y) $((CONST \text{ outp-constraint } P) \wedge_l Q)$

-end-prefix (*-input-prefix* x P) \rightarrow *-prefix-aux* x $(\lambda x. P)$

-end-prefix (*-output-prefix* P) \rightarrow *-prefix-aux* *-idtdummy* $(CONST \text{ outp-constraint } P)$

-prefix-action c (*-prefix-aux* x P) $A == (CONST \text{ InputCSP}) c P (\lambda x. A)$

Basic print translations; more work needed

translations

-simple-input-prefix $x \leq$ *-input-prefix* x *true*

-output-prefix $v \leq$ *-prefix-aux* p $(CONST \text{ outp-constraint } v)$

-output-prefix u (*-output-prefix* v)

\leq *-prefix-aux* p $(\lambda(x1, y1). CONST \text{ outp-constraint } u \ x2 \wedge CONST \text{ outp-constraint } v \ y2)$

-input-prefix x $P \leq$ *-prefix-aux* v $(\lambda x. P)$

$x!(v) \rightarrow P \leq CONST \text{ OutputCSP } x \ v \ P$

term $x!(1)!(y) \rightarrow P$

term $x?(v) \rightarrow P$

term $x?(v:false) \rightarrow P$

term $x!(\langle 1 \rangle) \rightarrow P$

term $x?(v)!(1) \rightarrow P$

term $x!(\langle 1 \rangle)!(2)?(v:true) \rightarrow P$

Basic translations for state variable communications

syntax

-csp-input-var :: logic ⇒ id ⇒ logic ⇒ logic ⇒ logic (-?%- → - [81, 0, 0, 80] 80)

-csp-inputu-var :: logic ⇒ id ⇒ logic ⇒ logic (-?%- → - [81, 0, 80] 80)

translations

$c?\$x:A \rightarrow P \Rightarrow \text{CONST InputVarCSP } c \ x \ A \ P$
 $c?\$x \rightarrow P \rightarrow \text{CONST InputVarCSP } c \ x \ (\text{CONST UNIV}) \ P$
 $c?\$x \rightarrow P \leq c?\$x:\text{true} \rightarrow P$

lemma *outp-constraint-prod*:

$(\text{outp-constraint } \ll a \gg x \wedge \text{outp-constraint } \ll b \gg y) =$
 $\text{outp-constraint } \ll (a, b) \gg (x, y)$
by (*simp add: outp-constraint-def, pred-auto*)

lemma *subst-outp-constraint* [*usubst*]:

$\sigma \uparrow (v =_o x) = (\sigma \uparrow v =_o x)$
by (*rel-auto*)

lemma *UINF-one-point-simp* [*rpred*]:

$\ll \bigwedge i. P \ i \text{ is } R1 \gg \implies (\bigcap x \cdot \ll i \gg =_o x)_{S<} \wedge P(x) = P(i)$
by (*rel-blast*)

lemma *USUP-one-point-simp* [*rpred*]:

$\ll \bigwedge i. P \ i \text{ is } R1 \gg \implies (\bigcup x \cdot \ll i \gg =_o x)_{S<} \Rightarrow_r P(x) = P(i)$
by (*rel-blast*)

lemma *USUP-eq-event-eq* [*rpred*]:

assumes $\bigwedge y. P(y) \text{ is } RR$
shows $(\bigcup y \cdot [v =_o y]_{S<} \Rightarrow_r P(y)) = P(y) \ll y \rightarrow [v]_{S\leftarrow} \gg$

proof –

have $(\bigcup y \cdot [v =_o y]_{S<} \Rightarrow_r RR(P(y))) = RR(P(y)) \ll y \rightarrow [v]_{S\leftarrow} \gg$
apply (*rel-simp, safe*)
apply *metis*
apply *blast*
apply *simp*
done
thus *?thesis*
by (*simp add: Healthy-if assms*)

qed

lemma *UINF-eq-event-eq* [*rpred*]:

assumes $\bigwedge y. P(y) \text{ is } RR$
shows $(\bigcap y \cdot [v =_o y]_{S<} \wedge P(y)) = P(y) \ll y \rightarrow [v]_{S\leftarrow} \gg$

proof –

have $(\bigcap y \cdot [v =_o y]_{S<} \wedge RR(P(y))) = RR(P(y)) \ll y \rightarrow [v]_{S\leftarrow} \gg$
by (*rel-simp, safe, metis*)
thus *?thesis*
by (*simp add: Healthy-if assms*)

qed

Proofs that the input constrained parser versions of output is the same as the regular definition.

lemma *output-prefix-is-OutputCSP* [*simp*]:

assumes $A \text{ is } NCSP$
shows $x!(P) \rightarrow A = \text{OutputCSP } x \ P \ A \ (\text{is } ?lhs = ?rhs)$
by (*rule SRD-eq-intro, simp-all add: assms closure rdes, rel-auto+*)

lemma *OutputCSP-pair-simp* [*simp*]:

$P \text{ is } NCSP \implies a.(\ll i \gg).(\ll j \gg) \rightarrow P = \text{OutputCSP } a \ \ll (i, j) \gg P$
using *output-prefix-is-OutputCSP* [*of P a*]
by (*simp add: outp-constraint-prod lconj-def InputCSP-def closure del: output-prefix-is-OutputCSP*)

lemma *OutputCSP-triple-simp* [simp]:
 $P \text{ is NCSP} \implies a.(\ll i \gg).(\ll j \gg).(\ll k \gg) \rightarrow P = \text{OutputCSP } a \ll (i, j, k) \gg P$
using *output-prefix-is-OutputCSP*[of $P \ a$]
by (simp add: *outp-constraint-prod lconj-def InputCSP-def closure del: output-prefix-is-OutputCSP*)
end

10 Recursion in Circus

theory *utp-circus-recursion*
imports *utp-circus-prefix utp-circus-contracts*
begin

10.1 Fixed-points

The CSP weakest fixed-point is obtained simply by precomposing the body with the CSP healthiness condition.

abbreviation *mu-CSP* :: $((\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action}) \Rightarrow (\sigma, \varphi) \text{ action} \ (\mu_C)$ **where**
 $\mu_C F \equiv \mu (F \circ \text{CSP})$

syntax
 $\text{-mu-CSP} :: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\mu_C \cdot \cdot \cdot [\theta, 10] \ 10)$

translations
 $\mu_C X \cdot P == \text{CONST } \text{mu-CSP } (\lambda X. P)$

lemma *mu-CSP-equiv*:
assumes *Monotonic* $F \ F \in \llbracket \text{CSP} \rrbracket_H \rightarrow \llbracket \text{CSP} \rrbracket_H$
shows $(\mu_R F) = (\mu_C F)$
by (simp add: *srd-mu-equiv assms comp-def*)

lemma *mu-CSP-unfold*:
 $P \text{ is CSP} \implies (\mu_C X \cdot P ;; X) = P ;; (\mu_C X \cdot P ;; X)$
apply (*subst gfp-unfold*)
apply (*simp-all add: closure Healthy-if*)
done

lemma *mu-csp-expand* [rdes]: $(\mu_C (op ;; Q)) = (\mu X \cdot Q ;; \text{CSP } X)$
by (simp add: *comp-def*)

lemma *mu-csp-basic-refine*:
assumes
 $P \text{ is CSP } Q \text{ is NCSP } Q \text{ is Productive } \text{pre}_R(P) = \text{true}_r \ \text{pre}_R(Q) = \text{true}_r$
 $\text{peri}_R P \sqsubseteq \text{peri}_R Q$
 $\text{peri}_R P \sqsubseteq \text{post}_R Q ;; \text{peri}_R P$
shows $P \sqsubseteq (\mu_C X \cdot Q ;; X)$
proof (rule *SRD-refine-intro'*, *simp-all add: closure usubst alpha rpred rdes unrest wp seq-UINF-distr assms*)
show $\text{peri}_R P \sqsubseteq (\bigcap i \cdot \text{post}_R Q \wedge i ;; \text{peri}_R Q)$
proof (rule *UINF-refines'*)
fix i
show $\text{peri}_R P \sqsubseteq \text{post}_R Q \wedge i ;; \text{peri}_R Q$
proof (*induct i*)


```

    case 0
    then show ?case by (simp add: assms)
next
case (Suc i)
then show ?case
  by (meson assms(6) assms(7) semilattice-sup-class.le-sup-iff upower-inductl)
qed
qed
qed

lemma CRD-mu-basic-refine:
  fixes P :: 'e list  $\Rightarrow$  'e set  $\Rightarrow$  's upred
  assumes
    Q is NCSP Q is Productive  $pre_R(Q) = true_r$ 
     $[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq peri_R \ Q$ 
     $[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq post_R \ Q \ ;;_h [P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket$ 
  shows  $[true \vdash P \ trace \ refs \mid R]_C \sqsubseteq (\mu_C \ X \cdot Q \ ;; \ X)$ 
proof (rule mu-csp-basic-refine, simp-all add: msubst-pair assms closure alpha rdes rpred Healthy-if R1-false)
  show  $[P \ trace \ refs]_{S<} \llbracket trace \rightarrow \&tt \rrbracket \llbracket refs \rightarrow \$ref' \rrbracket \sqsubseteq peri_R \ Q$ 
  using assms by (simp add: msubst-pair)
  show  $[P \ trace \ refs]_{S<} \llbracket trace \rightarrow \&tt \rrbracket \llbracket refs \rightarrow \$ref' \rrbracket \sqsubseteq post_R \ Q \ ;; [P \ trace \ refs]_{S<} \llbracket trace \rightarrow \&tt \rrbracket \llbracket refs \rightarrow \$ref' \rrbracket$ 
  using assms by (simp add: msubst-pair)
qed

```

10.2 Example action expansion

```

lemma mu-example1:  $(\mu \ X \cdot a \rightarrow X) = (\bigcap i \cdot do_C(\ll a \gg) \wedge (i+1)) \ ;; \text{Miracle}$ 
  by (simp add: PrefixCSP-def mu-csp-form-1 closure)

```

```

lemma preR-mu-example1 [rdes]:  $pre_R(\mu \ X \cdot a \rightarrow X) = true_r$ 
  by (simp add: mu-example1 rdes closure unrest wp)

```

```

lemma periR-mu-example1 [rdes]:
   $peri_R(\mu \ X \cdot a \rightarrow X) = (\bigcap i \cdot \mathcal{E}(true, iter[i](\ll a \gg), \{\ll a \gg\}_u))$ 
  by (simp add: mu-example1 rdes rpred closure unrest wp seq-UNF-distr alpha usubst)

```

```

lemma postR-mu-example1 [rdes]:
   $post_R(\mu \ X \cdot a \rightarrow X) = false$ 
  by (simp add: mu-example1 rdes closure unrest wp)

```

end

11 Circus Trace Merge

```

theory utp-circus-traces
  imports utp-circus-core
begin

```

11.1 Function Definition

```

fun tr-par ::
  'v set  $\Rightarrow$  'v list  $\Rightarrow$  'v list  $\Rightarrow$  'v list set where
  tr-par cs [] = {} |
  tr-par cs (e # t) = (if e  $\in$  cs then {} else {e})  $\cap$  (tr-par cs t) |

```

$tr\text{-}par\ cs\ []\ (e\ \# \ t) = (if\ e \in cs\ then\ \{\}\ else\ \{e\}) \frown (tr\text{-}par\ cs\ []\ t) \mid$
 $tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ (e_2\ \# \ t_2) =$
 $(if\ e_1 = e_2$
 $\quad then$
 $\quad if\ e_1 \in cs\ (* \wedge e_2 \in cs\ *)$
 $\quad then\ \{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ t_2)$
 $\quad else$
 $\quad (\{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2))) \cup$
 $\quad (\{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2))$
 $else$
 $if\ e_1 \in cs\ then$
 $if\ e_2 \in cs\ then\ \{\}$
 $else$
 $\{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2)$
 $else$
 $if\ e_2 \in cs\ then$
 $\{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2))$
 $else$
 $\{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2)) \cup$
 $\{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2))$

abbreviation $tr\text{-}inter :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list\ set$ (**infixr** $|||_t\ 100$) **where**
 $x\ |||_t\ y \equiv tr\text{-}par\ \{\}\ x\ y$

11.2 Lifted Trace Merge

syntax $-utr\text{-}par ::$
 $logic \Rightarrow logic \Rightarrow logic \Rightarrow logic\ ((- \star_- / -)\ [100, 0, 101]\ 100)$

The function $trop$ is used to lift ternary operators.

translations

$t1\ \star_{cs}\ t2 == (CONST\ trop)\ (CONST\ tr\text{-}par)\ cs\ t1\ t2$

11.3 Trace Merge Lemmas

lemma $tr\text{-}par\text{-}empty:$

$tr\text{-}par\ cs\ t1\ [] = \{takeWhile\ (\lambda x. x \notin cs)\ t1\}$

$tr\text{-}par\ cs\ []\ t2 = \{takeWhile\ (\lambda x. x \notin cs)\ t2\}$

— Subgoal 1

apply ($induct\ t1; simp$)

— Subgoal 2

apply ($induct\ t2; simp$)

done

lemma $tr\text{-}par\text{-}sym:$

$tr\text{-}par\ cs\ t1\ t2 = tr\text{-}par\ cs\ t2\ t1$

apply ($induct\ t1\ arbitrary; t2$)

— Subgoal 1

apply ($simp\ add: tr\text{-}par\text{-}empty$)

— Subgoal 2

apply ($induct\text{-}tac\ t2$)

— Subgoal 2.1

apply ($clarsimp$)

— Subgoal 2.2

apply ($clarsimp$)

apply ($blast$)

done

lemma *tr-inter-sym*: $x \parallel_t y = y \parallel_t x$
 by (*simp add: tr-par-sym*)

lemma *trace-merge-nil* [*simp*]: $x \star_{\{\}} \langle \rangle = \{x\}_u$
 by (*pred-auto, simp-all add: tr-par-empty, metis takeWhile-eq-all-conv*)

lemma *trace-merge-empty* [*simp*]:
 $(\langle \rangle \star_{cs} \langle \rangle) = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-single-empty* [*simp*]:
 $a \in cs \implies \langle \ll a \gg \star_{\ll cs \gg} \langle \rangle = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-empty-single* [*simp*]:
 $a \in cs \implies \langle \rangle \star_{\ll cs \gg} \langle \ll a \gg \rangle = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-commute*: $t_1 \star_{cs} t_2 = t_2 \star_{cs} t_1$
 by (*rel-simp, simp add: tr-par-sym*)

lemma *csp-trace-simps* [*simp*]:
 $v \hat{^}_u \langle \rangle = v \langle \rangle \hat{^}_u v = v$
 $v + \langle \rangle = v \langle \rangle + v = v$
 $bop (op \#) x xs \hat{^}_u ys = bop (op \#) x (xs \hat{^}_u ys)$
 by (*rel-auto*)+

end

12 Circus Parallel Composition

theory *utp-circus-parallel*

imports

utp-circus-prefix

utp-circus-traces

utp-circus-recursion

begin

12.1 Merge predicates

definition *CSPInnerMerge* :: $(' \alpha \implies ' \sigma) \Rightarrow ' \psi \text{ set} \Rightarrow (' \beta \implies ' \sigma) \Rightarrow ((' \sigma, ' \psi) \text{ st-csp}) \text{ merge } (N_C)$ **where**
 [*upred-defs*]:

$CSPInnerMerge \ ns1 \ cs \ ns2 = ($
 $\ \$ref' \subseteq_u ((\$0-ref \cup_u \$1-ref) \cap_u \ll cs \gg) \cup_u ((\$0-ref \cap_u \$1-ref) - \ll cs \gg) \wedge$
 $\ \$tr_{<} \leq_u \$tr' \wedge$
 $\ (\$tr' - \$tr_{<}) \in_u (\$0-tr - \$tr_{<}) \star_{\ll cs \gg} (\$1-tr - \$tr_{<}) \wedge$
 $\ (\$0-tr - \$tr_{<}) \upharpoonright_u \ll cs \gg =_u (\$1-tr - \$tr_{<}) \upharpoonright_u \ll cs \gg \wedge$
 $\ \$st' =_u (\$st_{<} \oplus \$0-st \text{ on } \&ns1) \oplus \$1-st \text{ on } \&ns2)$

definition *CSPInnerInterleave* :: $(' \alpha \implies ' \sigma) \Rightarrow (' \beta \implies ' \sigma) \Rightarrow ((' \sigma, ' \psi) \text{ st-csp}) \text{ merge } (N_I)$ **where**
 [*upred-defs*]:

$N_I \ ns1 \ ns2 = ($
 $\ \$ref' \subseteq_u (\$0-ref \cap_u \$1-ref) \wedge$

$$\begin{aligned}
& \$tr_{<} \leq_u \$tr' \wedge \\
& (\$tr' - \$tr_{<}) \in_u (\$0-tr - \$tr_{<}) \star_{\{u\}} (\$1-tr - \$tr_{<}) \wedge \\
& \$st' =_u (\$st_{<} \oplus \$0-st \text{ on } \&ns1) \oplus \$1-st \text{ on } \&ns2)
\end{aligned}$$

An intermediate merge hides the state, whilst a final merge hides the refusals.

definition *CSPInterMerge* **where**

[upred-defs]: *CSPInterMerge* $P \ ns1 \ cs \ ns2 \ Q = (P \parallel_{(\exists \ \$st' \cdot N_C \ ns1 \ cs \ ns2)} Q)$

definition *CSPFinalMerge* **where**

[upred-defs]: *CSPFinalMerge* $P \ ns1 \ cs \ ns2 \ Q = (P \parallel_{(\exists \ \$ref' \cdot N_C \ ns1 \ cs \ ns2)} Q)$

syntax

-*cinter-merge* :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \ (- \llbracket -|- \rrbracket^I - [85,0,0,0,86] \ 86)$
-*cfinal-merge* :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \ (- \llbracket -|- \rrbracket^F - [85,0,0,0,86] \ 86)$
-*wrC* :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \ (- \ wr[-|-]_C - [85,0,0,0,86] \ 86)$

translations

-*cinter-merge* $P \ ns1 \ cs \ ns2 \ Q == CONST \ CSPInterMerge \ P \ ns1 \ cs \ ns2 \ Q$
-*cfinal-merge* $P \ ns1 \ cs \ ns2 \ Q == CONST \ CSPFinalMerge \ P \ ns1 \ cs \ ns2 \ Q$
-*wrC* $P \ ns1 \ cs \ ns2 \ Q == P \ wr_R(N_C \ ns1 \ cs \ ns2) \ Q$

lemma *CSPInnerMerge-R2m* [closure]: $N_C \ ns1 \ cs \ ns2$ is *R2m*
by (rel-auto)

lemma *CSPInnerMerge-RDM* [closure]: $N_C \ ns1 \ cs \ ns2$ is *RDM*
by (rule *RDM-intro*, *simp add: closure*, *simp-all add: CSPInnerMerge-def unrest*)

lemma *ex-ref'-R2m-closed* [closure]:

assumes P is *R2m*
shows $(\exists \ \$ref' \cdot P)$ is *R2m*

proof –

have $R2m(\exists \ \$ref' \cdot R2m \ P) = (\exists \ \$ref' \cdot R2m \ P)$
by (rel-auto)
thus ?thesis
by (metis *Healthy-def' assms*)

qed

lemma *CSPInnerMerge-unrests* [unrest]:

$\$ok_{<} \nmid N_C \ ns1 \ cs \ ns2$
 $\$wait_{<} \nmid N_C \ ns1 \ cs \ ns2$
by (rel-auto)+

lemma *CSPInterMerge-RR-closed* [closure]:

assumes P is *RR* Q is *RR*
shows $P \llbracket ns1|cs|ns2 \rrbracket^I \ Q$ is *RR*
by (*simp add: CSPInterMerge-def parallel-RR-closed assms closure unrest*)

lemma *CSPInterMerge-unrest-st'* [unrest]:

$\$st' \nmid P \llbracket ns1|cs|ns2 \rrbracket^I \ Q$
by (rel-auto)

lemma *CSPFinalMerge-RR-closed* [closure]:

assumes P is *RR* Q is *RR*
shows $P \llbracket ns1|cs|ns2 \rrbracket^F \ Q$ is *RR*
by (*simp add: CSPFinalMerge-def parallel-RR-closed assms closure unrest*)

lemma *CSPInnerMerge-empty-Interleave*:

$N_C \text{ ns1 } \{ \} \text{ ns2 } = N_I \text{ ns1 ns2 }$
by (*rel-auto*)

definition *CSPMerge* :: $('α \implies 'σ) \Rightarrow 'ψ \text{ set} \Rightarrow ('β \implies 'σ) \Rightarrow (('σ, 'ψ) \text{ st-csp}) \text{ merge } (M_C)$ **where**
[upred-defs]: $M_C \text{ ns1 cs ns2 } = M_R(N_C \text{ ns1 cs ns2 }) ;; \text{Skip}$

definition *CSPInterleave* :: $('α \implies 'σ) \Rightarrow ('β \implies 'σ) \Rightarrow (('σ, 'ψ) \text{ st-csp}) \text{ merge } (M_I)$ **where**
[upred-defs]: $M_I \text{ ns1 ns2 } = M_R(N_I \text{ ns1 ns2 }) ;; \text{Skip}$

lemma *swap-CSPInnerMerge*:

$\text{ns1} \bowtie \text{ns2} \implies \text{swap}_m ;; (N_C \text{ ns1 cs ns2 }) = (N_C \text{ ns2 cs ns1 })$
apply (*rel-auto*)
using *tr-par-sym* **apply** *blast*
apply (*simp add: lens-indep-comm*)
using *tr-par-sym* **apply** *blast*
apply (*simp add: lens-indep-comm*)

done

lemma *SymMerge-CSPInnerMerge-NS* [*closure*]: $N_C \text{ 0}_L \text{ cs 0}_L$ is *SymMerge*
by (*simp add: Healthy-def swap-CSPInnerMerge*)

lemma *SymMerge-CSPInnerInterleave* [*closure*]:

$N_I \text{ 0}_L \text{ 0}_L$ is *SymMerge*
by (*metis CSPInnerMerge-empty-Interleave SymMerge-CSPInnerMerge-NS*)

lemma *SymMerge-CSPInnerInterleave* [*closure*]:

AssocMerge ($N_I \text{ 0}_L \text{ 0}_L$)
apply (*rel-auto*)
apply (*rename-tac tr tr2' ref0 tr0' ref0' tr1' ref1' tr' ref2' tr_i' ref3'*)

oops

lemma *CSPInterMerge-false* [*rpred*]: $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^I \text{ false} = \text{false}$
by (*simp add: CSPInterMerge-def*)

lemma *CSPFinalMerge-false* [*rpred*]: $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^F \text{ false} = \text{false}$
by (*simp add: CSPFinalMerge-def*)

lemma *CSPInterMerge-commute*:

assumes $\text{ns1} \bowtie \text{ns2}$
shows $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^I Q = Q \llbracket \text{ns2} | \text{cs} | \text{ns1} \rrbracket^I P$

proof –

have $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^I Q = P \parallel_{\exists \$st' \cdot N_C \text{ ns1 cs ns2 }} Q$
by (*simp add: CSPInterMerge-def*)
also have $\dots = P \parallel_{\exists \$st' \cdot (\text{swap}_m ;; N_C \text{ ns2 cs ns1 })} Q$
by (*simp add: swap-CSPInnerMerge lens-indep-sym assms*)
also have $\dots = P \parallel_{\text{swap}_m ;; (\exists \$st' \cdot N_C \text{ ns2 cs ns1 })} Q$
by (*simp add: seqr-exists-right*)
also have $\dots = Q \parallel_{(\exists \$st' \cdot N_C \text{ ns2 cs ns1 })} P$
by (*simp add: par-by-merge-commute-swap[THEN sym]*)
also have $\dots = Q \llbracket \text{ns2} | \text{cs} | \text{ns1} \rrbracket^I P$
by (*simp add: CSPInterMerge-def*)
finally show *?thesis* .

qed

lemma *CSPFinalMerge-commute*:

assumes $ns1 \bowtie ns2$

shows $P \llbracket ns1 | cs | ns2 \rrbracket^F Q = Q \llbracket ns2 | cs | ns1 \rrbracket^F P$

proof –

have $P \llbracket ns1 | cs | ns2 \rrbracket^F Q = P \parallel_{\exists \$ref' \cdot N_C ns1 cs ns2} Q$

by (*simp add: CSPFinalMerge-def*)

also have $\dots = P \parallel_{\exists \$ref' \cdot (swap_m ;; N_C ns2 cs ns1)} Q$

by (*simp add: swap-CSPInnerMerge lens-indep-sym assms*)

also have $\dots = P \parallel_{swap_m ;; (\exists \$ref' \cdot N_C ns2 cs ns1)} Q$

by (*simp add: segr-exists-right*)

also have $\dots = Q \parallel_{(\exists \$ref' \cdot N_C ns2 cs ns1)} P$

by (*simp add: par-by-merge-commute-swap[THEN sym]*)

also have $\dots = Q \llbracket ns2 | cs | ns1 \rrbracket^F P$

by (*simp add: CSPFinalMerge-def*)

finally show *?thesis* .

qed

Important theorem that shows the form of a parallel process

lemma *CSPInnerMerge-form*:

fixes $P Q :: ('\sigma, '\varphi)$ *action*

assumes *vwb-lens ns1 vwb-lens ns2 P is RR Q is RR*

shows

$P \parallel_{N_C ns1 cs ns2} Q =$

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$

$P \llbracket \langle ref_0 \rangle, \langle st_0 \rangle, \langle \rangle, \langle tt_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q \llbracket \langle ref_1 \rangle, \langle st_1 \rangle, \langle \rangle, \langle tt_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$

$\wedge \$ref' \subseteq_u ((\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle) \cup_u ((\langle ref_0 \rangle \cap_u \langle ref_1 \rangle) - \langle cs \rangle)$

$\wedge \$tr \leq_u \tr'

$\wedge \&tt \in_u \langle tt_0 \rangle \star_{\langle cs \rangle} \langle tt_1 \rangle$

$\wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$

$\wedge \$st' =_u (\$st \oplus \langle st_0 \rangle \text{ on } \&ns1) \oplus \langle st_1 \rangle \text{ on } \&ns2)$

(*is ?lhs = ?rhs*)

proof –

have $P;(\exists \{\$ok', \$wait'\} \cdot R2(P)) = P$ (*is ?P' = -*)

by (*simp add: ex-unrest ex-plus Healthy-if assms RR-implies-R2 unrest closure*)

have $Q;(\exists \{\$ok', \$wait'\} \cdot R2(Q)) = Q$ (*is ?Q' = -*)

by (*simp add: ex-unrest ex-plus Healthy-if assms RR-implies-R2 unrest closure*)

from *assms(1,2)*

have $?P' \parallel_{N_C ns1 cs ns2} ?Q' =$

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$

$?P' \llbracket \langle ref_0 \rangle, \langle st_0 \rangle, \langle \rangle, \langle tt_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge ?Q' \llbracket \langle ref_1 \rangle, \langle st_1 \rangle, \langle \rangle, \langle tt_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$

$\wedge \$ref' \subseteq_u ((\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle) \cup_u ((\langle ref_0 \rangle \cap_u \langle ref_1 \rangle) - \langle cs \rangle)$

$\wedge \$tr \leq_u \tr'

$\wedge \&tt \in_u \langle tt_0 \rangle \star_{\langle cs \rangle} \langle tt_1 \rangle$

$\wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$

$\wedge \$st' =_u (\$st \oplus \langle st_0 \rangle \text{ on } \&ns1) \oplus \langle st_1 \rangle \text{ on } \&ns2)$

apply (*simp add: par-by-merge-alt-def, rel-auto, blast*)

apply (*rename-tac ok wait tr st ref tr' ref' ref_0 ref_1 st_0 st_1 tr_0 ok_0 tr_1 wait_0 ok_1 wait_1*)

apply (*rule-tac x=ok in exI*)

apply (*rule-tac x=wait in exI*)

apply (*rule-tac x=tr in exI*)

apply (*rule-tac x=st in exI*)

apply (*rule-tac x=ref in exI*)

apply (*rule-tac x=tr @ tr_0 in exI*)

```

  apply (rule-tac x=st0 in exI)
  apply (rule-tac x=ref0 in exI)
  apply (auto)
  apply (metis Prefix-Order.prefixI append-minus)
done
thus ?thesis
  by (simp add: P Q)
qed

```

lemma *CSPInterMerge-form*:

fixes $P\ Q :: ('σ, 'ϕ)\ action$

assumes $vwb\text{-}lens\ ns1\ vwb\text{-}lens\ ns2\ P\ is\ RR\ Q\ is\ RR$

shows

$P\ \llbracket ns1\ |\ cs\ |\ ns2 \rrbracket^I\ Q =$

$(\exists\ (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P\llbracket \langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q\llbracket \langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle)$

(is ?lhs = ?rhs)

proof –

have ?lhs = $(\exists\ \$st' \cdot P\ \parallel_{NC}\ ns1\ cs\ ns2\ Q)$

by (simp add: CSPInterMerge-def par-by-merge-def segr-exists-right)

also have ... =

$(\exists\ \$st' \cdot$

$(\exists\ (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P\llbracket \langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q\llbracket \langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle\ on\ \&ns1) \oplus \langle\langle st_1 \rangle\rangle\ on\ \&ns2))$

by (simp add: CSPInnerMerge-form assms)

also have ... = ?rhs

by (rel-blast)

finally show ?thesis .

qed

lemma *CSPFinalMerge-form*:

fixes $P\ Q :: ('σ, 'ϕ)\ action$

assumes $vwb\text{-}lens\ ns1\ vwb\text{-}lens\ ns2\ P\ is\ RR\ Q\ is\ RR\ \$ref' \# P\ \$ref' \# Q$

shows

$(P\ \llbracket ns1\ |\ cs\ |\ ns2 \rrbracket^F\ Q) =$

$(\exists\ (st_0, st_1, tt_0, tt_1) \cdot$
 $P\llbracket \langle\langle st_0 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_0 \rangle\rangle / \$st', \$tr, \$tr' \rrbracket \wedge Q\llbracket \langle\langle st_1 \rangle\rangle, \langle\cdot\rangle, \langle\langle tt_1 \rangle\rangle / \$st', \$tr, \$tr' \rrbracket$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle\ on\ \&ns1) \oplus \langle\langle st_1 \rangle\rangle\ on\ \&ns2)$

(is ?lhs = ?rhs)

proof –

have ?lhs = $(\exists\ \$ref' \cdot P\ \parallel_{NC}\ ns1\ cs\ ns2\ Q)$

by (simp add: CSPFinalMerge-def par-by-merge-def segr-exists-right)

also have ... =

$(\exists \$ref' \cdot$
 $(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P[\langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr'] \wedge Q[\langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr']$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2))$
by (*simp add: CSPInnerMerge-form assms*)
also have ... =
 $(\exists \$ref' \cdot$
 $(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $(\exists \$ref' \cdot P)[\langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr'] \wedge (\exists \$ref' \cdot Q)[\langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr']$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2))$
by (*simp add: ex-unrest assms*)
also have ... =
 $(\exists (st_0, st_1, tt_0, tt_1) \cdot$
 $(\exists \$ref' \cdot P)[\langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$st', \$tr, \$tr'] \wedge (\exists \$ref' \cdot Q)[\langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$st', \$tr, \$tr']$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2))$
by (*rel-blast*)
also have ... = ?*rhs*
by (*simp add: ex-unrest assms*)
finally show ?*thesis* .
qed

lemma *merge-csp-do-left:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2 P is RR*

shows $\Phi(s_0, \sigma_0, t_0) \parallel_{N_C} ns1 \text{ cs } ns2 \ P =$

$(\exists (ref_1, st_1, tt_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\langle \rangle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \upharpoonright P \wedge$
 $\$ref' \subseteq_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

(**is** ?*lhs* = ?*rhs*)

proof –

have ?*lhs* =

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\langle \rangle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \upharpoonright \Phi(s_0, \sigma_0, t_0) \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\langle \rangle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \upharpoonright P \wedge$
 $\$ref' \subseteq_u (\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle \wedge \$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1$
 $\oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*simp add: CSPInnerMerge-form assms closure*)

also have ... =

$(\exists (ref_1, st_1, tt_1) \cdot$
 $[s_0]_{S<} \wedge$

$$[\$ref' \mapsto_s \langle ref_1 \rangle, \$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger P \wedge$$

$$\$ref' \subseteq_u \langle cs \rangle \cup_u (\langle ref_1 \rangle - \langle cs \rangle) \wedge$$

$$[\langle trace \rangle \in_u t_0 \star \langle cs \rangle \langle tt_1 \rangle \wedge t_0 \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle \sigma_0 \rangle (\$st)_a \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$
 by (rel-blast)
 finally show ?thesis .
 qed

lemma merge-csp-do-right:

assumes vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR

shows $P \parallel_{N_C} ns1 \text{ cs } ns2 \Phi(s_1, \sigma_1, t_1) =$

$$(\exists (ref_0, st_0, tt_0) \cdot$$

$$[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger P \wedge$$

$$[s_1]_{S<} \wedge$$

$$\$ref' \subseteq_u \langle cs \rangle \cup_u (\langle ref_0 \rangle - \langle cs \rangle) \wedge$$

$$[\langle trace \rangle \in_u \langle tt_0 \rangle \star \langle cs \rangle t_1 \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u t_1 \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle \sigma_1 \rangle (\$st)_a \text{ on } \&ns2)$$

(is ?lhs = ?rhs)

proof –

have ?lhs =

$$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$$

$$[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger P \wedge$$

$$[\$ref' \mapsto_s \langle ref_1 \rangle, \$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger \Phi(s_1, \sigma_1, t_1) \wedge$$

$$\$ref' \subseteq_u (\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle \cup_u (\langle ref_0 \rangle \cap_u \langle ref_1 \rangle - \langle cs \rangle) \wedge$$

$$\$tr \leq_u \$tr' \wedge$$

$$\&tt \in_u \langle tt_0 \rangle \star \langle cs \rangle \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle \wedge \$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$

by (simp add: CSPInnerMerge-form assms closure)

also have ... = ?rhs

by (rel-blast)

finally show ?thesis .

qed

The result of merge two terminated stateful traces is to (1) require both state preconditions hold, (2) merge the traces using, and (3) merge the state using a parallel assignment.

lemma FinalMerge-csp-do-left:

assumes vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR $\$ref' \# P$

shows $\Phi(s_0, \sigma_0, t_0) \llbracket ns1 | cs | ns2 \rrbracket^F P =$

$$(\exists (st_1, t_1) \cdot$$

$$[s_0]_{S<} \wedge$$

$$[\$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \dagger P \wedge$$

$$[\langle trace \rangle \in_u t_0 \star \langle cs \rangle \langle t_1 \rangle \wedge t_0 \upharpoonright_u \langle cs \rangle =_u \langle t_1 \rangle \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle \sigma_0 \rangle (\$st)_a \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$

(is ?lhs = ?rhs)

proof –

have ?lhs =

$$(\exists (st_0, st_1, tt_0, tt_1) \cdot$$

$$[\$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger \Phi(s_0, \sigma_0, t_0) \wedge$$

$$[\$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger RR(\exists \$ref' \cdot P) \wedge$$

$$\$tr \leq_u \$tr' \wedge \&tt \in_u \langle tt_0 \rangle \star \langle cs \rangle \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle \wedge$$

$$\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$

by (simp add: CSPFinalMerge-form ex-unrest Healthy-if unrest closure assms)

also have ... =

$$(\exists (st_1, tt_1) \cdot$$

$$[s_0]_{S<} \wedge$$

$$[\$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger RR(\exists \$ref' \cdot P) \wedge$$

$$[\langle \text{trace} \rangle \in_u t_0 \star \langle cs \rangle \langle tt_1 \rangle \wedge t_0 \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle \sigma_0 \rangle (\$st)_a \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$
 by (*rel-blast*)
 also have ... =
 (
$$\exists (st_1, t_1) \cdot$$

$$[s_0]_{S<} \wedge$$

$$[\$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \dagger P \wedge$$

$$[\langle \text{trace} \rangle \in_u t_0 \star \langle cs \rangle \langle t_1 \rangle \wedge t_0 \upharpoonright_u \langle cs \rangle =_u \langle t_1 \rangle \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle \sigma_0 \rangle (\$st)_a \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$
)
 by (*simp add: ex-unrest Healthy-if unrest closure assms*)
 finally show ?thesis .
 qed

lemma *FinalMerge-csp-do-right:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2* *P is RR \$ref' $\#$ P*
 shows $P \llbracket ns1 | cs | ns2 \rrbracket^F \Phi(s_1, \sigma_1, t_1) =$
 (
$$\exists (st_0, t_0) \cdot$$

$$[\$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger P \wedge$$

$$[s_1]_{S<} \wedge$$

$$[\langle \text{trace} \rangle \in_u \langle t_0 \rangle \star \langle cs \rangle t_1 \wedge \langle t_0 \rangle \upharpoonright_u \langle cs \rangle =_u t_1 \upharpoonright_u \langle cs \rangle]_t \wedge$$

$$\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle \sigma_1 \rangle (\$st)_a \text{ on } \&ns2)$$
)
 (is ?lhs = ?rhs)

proof –

have $P \llbracket ns1 | cs | ns2 \rrbracket^F \Phi(s_1, \sigma_1, t_1) = \Phi(s_1, \sigma_1, t_1) \llbracket ns2 | cs | ns1 \rrbracket^F P$
 by (*simp add: assms CSPFinalMerge-commute*)
 also have ... = ?rhs
 apply (*simp add: FinalMerge-csp-do-left assms lens-indep-sym conj-comm*)
 apply (*rel-auto*)
 using *assms(3) lens-indep.lens-put-comm tr-par-sym* apply *fastforce+*
 done
 finally show ?thesis .
 qed

lemma *FinalMerge-csp-do:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^F \Phi(s_2, \sigma_2, t_2) =$

$$([s_1 \wedge s_2]_{S<} \wedge [\langle \text{trace} \rangle \in_u t_1 \star \langle cs \rangle t_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle]_t \wedge [(\sigma_1 \llbracket \&ns1 | \&ns2 \rrbracket_s$$

$$\sigma_2)_a]_{S'})$$

 (is ?lhs = ?rhs)

proof –

have ?lhs =
 (
$$\exists (st_0, st_1, tt_0, tt_1) \cdot$$

$$[\$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger \Phi(s_1, \sigma_1, t_1) \wedge$$

$$[\$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger \Phi(s_2, \sigma_2, t_2) \wedge$$

$$\$tr \leq_u \$tr' \wedge \&tt \in_u \langle tt_0 \rangle \star \langle cs \rangle \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle \wedge$$

$$\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2)$$
)
 by (*simp add: CSPFinalMerge-form unrest closure assms*)
 also have ... =

$$([s_1 \wedge s_2]_{S<} \wedge [\langle \text{trace} \rangle \in_u t_1 \star \langle cs \rangle t_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle]_t \wedge [(\sigma_1 \llbracket \&ns1 | \&ns2 \rrbracket_s$$

$$\sigma_2)_a]_{S'})$$

 by (*rel-auto*)
 finally show ?thesis .
 qed

lemma *FinalMerge-csp-do'* [rpred]:

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2*

shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1|cs|ns2 \rrbracket^F \Phi(s_2, \sigma_2, t_2) =$

$(\sqcap \text{ trace} \mid \llbracket \text{trace} \rrbracket \in_u [t_1 \star \llbracket cs \rrbracket t_2]_{S<} \cdot$

$\Phi(s_1 \wedge s_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket, \sigma_1 [\&ns1|\&ns2]_s \sigma_2, \llbracket \text{trace} \rrbracket))$

by (*simp add: FinalMerge-csp-do assms, rel-auto*)

lemma *CSPFinalMerge-UNF-ind-left* [rpred]:

$(\sqcap i \cdot P(i)) \llbracket ns1|cs|ns2 \rrbracket^F Q = (\sqcap i \cdot P(i) \llbracket ns1|cs|ns2 \rrbracket^F Q)$

by (*simp add: CSPFinalMerge-def par-by-merge-USUP-ind-left*)

lemma *CSPFinalMerge-UNF-ind-right* [rpred]:

$P \llbracket ns1|cs|ns2 \rrbracket^F (\sqcap i \cdot Q(i)) = (\sqcap i \cdot P \llbracket ns1|cs|ns2 \rrbracket^F Q(i))$

by (*simp add: CSPFinalMerge-def par-by-merge-USUP-ind-right*)

lemma *InterMerge-csp-enable*:

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2*

shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1|cs|ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$([s_1 \wedge s_2]_{S<} \wedge$

$(\forall e \in [(E_1 \cap_u E_2 \cap_u \llbracket cs \rrbracket) \cup_u ((E_1 \cup_u E_2) - \llbracket cs \rrbracket)]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref') \wedge$

$[\llbracket \text{trace} \rrbracket \in_u t_1 \star \llbracket cs \rrbracket t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t)$

(**is** ?lhs = ?rhs)

proof –

have ?lhs =

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$

$[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$

$[\$ref' \mapsto_s \llbracket ref_1 \rrbracket, \$st' \mapsto_s \llbracket st_1 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_1 \rrbracket] \dagger \mathcal{E}(s_2, t_2, E_2) \wedge$

$\$ref' \subseteq_u (\llbracket ref_0 \rrbracket \cup_u \llbracket ref_1 \rrbracket) \cap_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket \cap_u \llbracket ref_1 \rrbracket - \llbracket cs \rrbracket) \wedge$

$\$tr \leq_u \$tr' \wedge \&tt \in_u \llbracket tt_0 \rrbracket \star \llbracket cs \rrbracket \llbracket tt_1 \rrbracket \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u \llbracket tt_1 \rrbracket \upharpoonright_u \llbracket cs \rrbracket)$

by (*simp add: CSPInterMerge-form unrest closure assms*)

also have ... =

$(\exists (ref_0, ref_1, tt_0, tt_1) \cdot$

$[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$

$[\$ref' \mapsto_s \llbracket ref_1 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_1 \rrbracket] \dagger \mathcal{E}(s_2, t_2, E_2) \wedge$

$\$ref' \subseteq_u (\llbracket ref_0 \rrbracket \cup_u \llbracket ref_1 \rrbracket) \cap_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket \cap_u \llbracket ref_1 \rrbracket - \llbracket cs \rrbracket) \wedge$

$\$tr \leq_u \$tr' \wedge \&tt \in_u \llbracket tt_0 \rrbracket \star \llbracket cs \rrbracket \llbracket tt_1 \rrbracket \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u \llbracket tt_1 \rrbracket \upharpoonright_u \llbracket cs \rrbracket)$

by (*rel-auto*)

also have ... =

$([s_1 \wedge s_2]_{S<} \wedge$

$(\forall e \in [(E_1 \cap_u E_2 \cap_u \llbracket cs \rrbracket) \cup_u ((E_1 \cup_u E_2) - \llbracket cs \rrbracket)]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref') \wedge$

$[\llbracket \text{trace} \rrbracket \in_u t_1 \star \llbracket cs \rrbracket t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t)$

)

apply (*rel-auto*)

apply (*rename-tac tr st tr' ref'*)

apply (*rule-tac x = [E1]_e st in exI*)

apply (*simp*)

apply (*rule-tac x = [E2]_e st in exI*)

apply (*auto*)

done

finally show ?thesis .

qed

lemma *InterMerge-csp-enable'* [rpred]:

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2*

shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1|cs|ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$(\sqcap \text{ trace} \mid \llbracket \text{trace} \rrbracket \in_u \lceil t_1 \star_{\llbracket cs \rrbracket} t_2 \rceil_{S<} \cdot$
 $\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket$
 $, \llbracket \text{trace} \rrbracket$
 $, (E_1 \cap_u E_2 \cap_u \llbracket cs \rrbracket) \cup_u ((E_1 \cup_u E_2) - \llbracket cs \rrbracket))$
 by (simp add: InterMerge-csp-enable assms, rel-auto)

lemma *InterMerge-csp-enable-csp-do:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1 | cs | ns2 \rrbracket^I \Phi(s_2, \sigma_2, t_2) =$
 $([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_1 - \llbracket cs \rrbracket)]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref')) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u t_1 \star_{\llbracket cs \rrbracket} t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t$
 (is ?lhs = ?rhs)

proof –

have ?lhs =

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$
 $[\$ref' \mapsto_s \llbracket ref_1 \rrbracket, \$st' \mapsto_s \llbracket st_1 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_1 \rrbracket] \dagger \Phi(s_2, \sigma_2, t_2) \wedge$
 $\$ref' \subseteq_u (\llbracket ref_0 \rrbracket \cup_u \llbracket ref_1 \rrbracket) \cap_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket \cap_u \llbracket ref_1 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $\$tr \leq_u \$tr' \wedge \&tt \in_u \llbracket tt_0 \rrbracket \star_{\llbracket cs \rrbracket} \llbracket tt_1 \rrbracket \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u \llbracket tt_1 \rrbracket \upharpoonright_u \llbracket cs \rrbracket)$

by (simp add: CSPInterMerge-form unrest closure assms)

also have ... =

$(\exists (ref_0, ref_1, tt_0) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$
 $[s_2]_{S<} \wedge$
 $\$ref' \subseteq_u (\llbracket ref_0 \rrbracket \cup_u \llbracket ref_1 \rrbracket) \cap_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket \cap_u \llbracket ref_1 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u t_1 \star_{\llbracket cs \rrbracket} t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t)$

by (rel-auto)

also have ... = $([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_1 - \llbracket cs \rrbracket)]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref')) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u t_1 \star_{\llbracket cs \rrbracket} t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t$

by (rel-auto)

finally show ?thesis .

qed

lemma *InterMerge-csp-enable-csp-do' [rpred]:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1 | cs | ns2 \rrbracket^I \Phi(s_2, \sigma_2, t_2) =$
 $(\sqcap \text{ trace} \mid \llbracket \text{trace} \rrbracket \in_u \lceil t_1 \star_{\llbracket cs \rrbracket} t_2 \rceil_{S<} \cdot$
 $\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket, \llbracket \text{trace} \rrbracket, E_1 - \llbracket cs \rrbracket))$
 by (simp add: InterMerge-csp-enable-csp-do assms, rel-auto)

lemma *InterMerge-csp-do-csp-enable:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$
 $([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_2 - \llbracket cs \rrbracket)]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref')) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u t_1 \star_{\llbracket cs \rrbracket} t_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket]_t$
 (is ?lhs = ?rhs)

proof –

have $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_2, t_2, E_2) \llbracket ns2 | cs | ns1 \rrbracket^I \Phi(s_1, \sigma_1, t_1)$
 by (simp add: CSPInterMerge-commute assms)

also have ... = ?rhs

by (simp add: InterMerge-csp-enable-csp-do assms lens-indep-sym trace-merge-commute conj-comm eq-upred-sym)

finally show ?thesis .

qed

lemma *InterMerge-csp-do-csp-enable'* [rpred]:

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2*

shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$(\sqcap \text{ trace} \mid \llbracket \text{trace} \rrbracket \in_u [t_1 \star \llbracket cs \rrbracket t_2]_{S<} \cdot$

$\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \upharpoonright_u \llbracket cs \rrbracket =_u t_2 \upharpoonright_u \llbracket cs \rrbracket, \llbracket \text{trace} \rrbracket, E_2 - \llbracket cs \rrbracket))$

by (*simp add: InterMerge-csp-do-csp-enable assms, rel-auto*)

lemma *CSPInterMerge-or-left* [rpred]:

$(P \vee Q) \llbracket ns1 | cs | ns2 \rrbracket^I R = (P \llbracket ns1 | cs | ns2 \rrbracket^I R \vee Q \llbracket ns1 | cs | ns2 \rrbracket^I R)$

by (*simp add: CSPInterMerge-def par-by-merge-or-left*)

lemma *CSPInterMerge-or-right* [rpred]:

$P \llbracket ns1 | cs | ns2 \rrbracket^I (Q \vee R) = (P \llbracket ns1 | cs | ns2 \rrbracket^I Q \vee P \llbracket ns1 | cs | ns2 \rrbracket^I R)$

by (*simp add: CSPInterMerge-def par-by-merge-or-right*)

lemma *CSPInterMerge-UINF-ind-left* [rpred]:

$(\sqcap i \cdot P(i)) \llbracket ns1 | cs | ns2 \rrbracket^I Q = (\sqcap i \cdot P(i) \llbracket ns1 | cs | ns2 \rrbracket^I Q)$

by (*simp add: CSPInterMerge-def par-by-merge-USUP-ind-left*)

lemma *CSPInterMerge-UINF-ind-right* [rpred]:

$P \llbracket ns1 | cs | ns2 \rrbracket^I (\sqcap i \cdot Q(i)) = (\sqcap i \cdot P \llbracket ns1 | cs | ns2 \rrbracket^I Q(i))$

by (*simp add: CSPInterMerge-def par-by-merge-USUP-ind-right*)

lemma *par-by-merge-seq-remove*: $(P \parallel_M ; R \ Q) = (P \parallel_M Q) ; R$

by (*simp add: par-by-merge-seq-add[THEN sym]*)

lemma *merge-csp-do-right*:

assumes *vwb-lens ns1 vwb-lens ns2 ns1* \bowtie *ns2* *P is RC*

shows $\Phi(s_1, \sigma_1, t_1) \text{ wr}[ns1 | cs | ns2]_C P = \text{undefined}$

(is ?lhs = ?rhs)

proof –

have *?lhs* =

$(\neg_r (\exists (ref_0, st_0, tt_0) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger (\neg_r RC(P)) \wedge$
 $[s_1]_{S<} \wedge$
 $\$ref' \subseteq_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u \llbracket tt_0 \rrbracket \star \llbracket cs \rrbracket t_1 \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u t_1 \upharpoonright_u \llbracket cs \rrbracket]_t \wedge$
 $\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket \sigma_1 \rrbracket (\$st)_a \text{ on } \&ns2) ; R1 \text{ true})$

by (*simp add: wrR-def par-by-merge-seq-remove merge-csp-do-right closure assms Healthy-if rpred*)

also have ... =

$(\neg_r (\exists (ref_0, st_0, tt_0) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger (\neg_r RC(P)) \wedge$
 $[s_1]_{S<} \wedge$
 $\$ref' \subseteq_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $[\llbracket \text{trace} \rrbracket \in_u \llbracket tt_0 \rrbracket \star \llbracket cs \rrbracket t_1 \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u t_1 \upharpoonright_u \llbracket cs \rrbracket]_t ; true_r \wedge$
 $\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket \sigma_1 \rrbracket (\$st)_a \text{ on } \&ns2))$

apply (*rel-auto*)

oops

12.2 Parallel operator

syntax

-par-circus :: *logic* \Rightarrow *salpha* \Rightarrow *logic* \Rightarrow *salpha* \Rightarrow *logic* \Rightarrow *logic* (*-* $\llbracket - \parallel - \rrbracket$ - [75,0,0,0,76] 76)

-par-csp :: *logic* \Rightarrow *logic* \Rightarrow *logic* \Rightarrow *logic* (*-* $\llbracket - \rrbracket_C$ - [75,0,76] 76)

-inter-circus :: logic \Rightarrow salpha \Rightarrow salpha \Rightarrow logic \Rightarrow logic (- $\llbracket - \rrbracket$ - [75,0,0,76] 76)
-inter-csp :: logic \Rightarrow logic \Rightarrow logic (**infixr** $\llbracket \rrbracket$ 75)

translations

-par-circus $P \text{ ns1 cs ns2 } Q == P \parallel_{M_C} \text{ ns1 cs ns2 } Q$
-par-csp $P \text{ cs } Q == \text{-par-circus } P \text{ } 0_L \text{ cs } 0_L \text{ } Q$
-inter-circus $P \text{ ns1 ns2 } Q == \text{-par-circus } P \text{ ns1 } \{\} \text{ ns2 } Q$
-inter-csp $P \text{ } Q == \text{-par-csp } P \text{ } \{\} \text{ } Q$

definition CSP5 :: (σ, φ) action \Rightarrow (σ, φ) action **where**
[upred-defs]: CSP5(P) = ($P \llbracket \rrbracket \text{ Skip}$)

definition C2 :: (σ, φ) action \Rightarrow (σ, φ) action **where**
[upred-defs]: C2(P) = ($P \llbracket \Sigma \rrbracket \{\} \llbracket \emptyset \rrbracket \text{ Skip}$)

lemma Skip-right-form:

assumes P_1 is RC P_2 is RR P_3 is RR $\$st' \# P_2$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) ;; \text{Skip} = \mathbf{R}_s(P_1 \vdash P_2 \diamond (\exists \$ref' \cdot P_3))$

proof -

have $1:RR(P_3) ;; \Phi(\text{true}, id, \langle \rangle) = (\exists \$ref' \cdot RR(P_3))$
by (rel-auto)
show ?thesis
by (rdes-simp cls: assms, metis 1 Healthy-if assms(3))

qed

lemma ParCSP-rdes-def [rdes-def]:

fixes $P_1 :: (\sigma, \varphi)$ action
assumes P_1 is CRC Q_1 is CRC P_2 is CRR Q_2 is CRR P_3 is CRR Q_3 is CRR
 $\$st' \# P_2 \ \$st' \# Q_2$
 $ns1 \bowtie ns2$

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \llbracket ns1 | cs | ns2 \rrbracket \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1 | cs | ns2]_C P_1 \wedge$
 $(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1 | cs | ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2 | cs | ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2 | cs | ns1]_C Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $((P_1 \Rightarrow_r P_3) \llbracket ns1 | cs | ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3)))$
(is ?P $\llbracket ns1 | cs | ns2 \rrbracket$?Q = ?rhs)

proof -

have ?P $\llbracket ns1 | cs | ns2 \rrbracket$?Q = (?P $\parallel_{M_R(N_C \text{ ns1 cs ns2})}$?Q) ;;_h Skip
by (simp add: CSPMerge-def par-by-merge-seq-add)

also

have ... = $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1 | cs | ns2]_C P_1 \wedge$
 $(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1 | cs | ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2 | cs | ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2 | cs | ns1]_C Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1 | cs | ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1 | cs | ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3)))$;;_h Skip
by (simp add: parallel-rdes-def swap-CSPInnerMerge CSPInterMerge-def closure assms)

also

have ... = $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1 | cs | ns2]_C P_1 \wedge$

$(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1 \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $(\exists \$ref' \cdot ((P_1 \Rightarrow_r P_3) \parallel_{N_C ns1 cs ns2} (Q_1 \Rightarrow_r Q_3))))$
by (*simp add: Skip-right-form closure parallel-RR-closed assms unrest*)
also
have ... = $\mathbf{R}_s (((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $((P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3)))$
proof –
have $(\exists \$ref' \cdot ((P_1 \Rightarrow_r P_3) \parallel_{N_C ns1 cs ns2} (Q_1 \Rightarrow_r Q_3))) = ((P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3))$
by (*rel-blast*)
thus *?thesis* **by** *simp*
qed
finally show *?thesis* .
qed

12.3 Parallel Laws

lemma *ParCSP-expand*:

$P \llbracket ns1|cs|ns2 \rrbracket Q = (P \parallel_{R N_C ns1 cs ns2} Q) ;; \text{Skip}$
by (*simp add: CSPMerge-def par-by-merge-seq-add*)

lemma *parallel-is-CSP [closure]*:

assumes P is CSP Q is CSP
shows $(P \llbracket ns1|cs|ns2 \rrbracket Q)$ is CSP

proof –

have $(P \parallel_{M_R(N_C ns1 cs ns2)} Q)$ is CSP
by (*simp add: closure assms*)
hence $(P \parallel_{M_R(N_C ns1 cs ns2)} Q) ;; \text{Skip}$ is CSP
by (*simp add: closure*)
thus *?thesis*
by (*simp add: CSPMerge-def par-by-merge-seq-add*)

qed

lemma *parallel-is-CSP3 [closure]*:

assumes P is CSP P is CSP3 Q is CSP Q is CSP3
shows $(P \llbracket ns1|cs|ns2 \rrbracket Q)$ is CSP3

proof –

have $(P \parallel_{M_R(N_C ns1 cs ns2)} Q)$ is CSP
by (*simp add: closure assms*)
hence $(P \parallel_{M_R(N_C ns1 cs ns2)} Q) ;; \text{Skip}$ is CSP
by (*simp add: closure*)
thus *?thesis*
oops

theorem *parallel-commutative*:

assumes $ns1 \bowtie ns2$

shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = (Q \llbracket ns2 \parallel cs \parallel ns1 \rrbracket P)$

proof –

have $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = P \parallel_{\text{swap}_m} ;; (M_C \text{ } ns2 \text{ } cs \text{ } ns1) \text{ } Q$

by (*simp add: CSPMerge-def seqr-assoc [THEN sym] swap-merge-rd swap-CSPInnerMerge lens-indep-sym assms*)

also have $\dots = Q \llbracket ns2 \parallel cs \parallel ns1 \rrbracket P$

by (*metis par-by-merge-commute-swap*)

finally show *?thesis* .

qed

lemma *interleave-commute*:

$P \parallel Q = Q \parallel P$

using *parallel-commutative zero-lens-indep* **by** *blast*

The form of C2 tells us that a normal CSP process has a downward closed set of refusals

lemma *C2-form*:

assumes P is NCSP

shows $C2(P) = \mathbf{R}_s (pre_R P \vdash (\exists \text{ } ref_0 \cdot peri_R P \llbracket \langle ref_0 \rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle ref_0 \rangle) \diamond post_R P)$

proof –

have $1: \Phi(true, id, \langle \rangle) \text{ wr}[\Sigma | \{\} | \emptyset]_C pre_R P = pre_R P$ (**is** *?lhs = ?rhs*)

proof –

have $?lhs = (\neg_r (\exists (ref_0, st_0, tt_0) \cdot$

$[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger (\exists \$ref'; \$st' \cdot RR(\neg_r$

$pre_R P)) \wedge$

$\$ref' \subseteq_u \langle ref_0 \rangle \wedge [\langle trace \rangle =_u \langle tt_0 \rangle]_t \wedge$

$\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \Sigma \oplus \langle id \rangle (\$st)_a \text{ on } \emptyset) ;; R1 \text{ true})$

by (*simp add: wrR-def par-by-merge-seq-remove rpred merge-csp-do-right ex-unrest Healthy-if pr-var-def closure assms unrest usubst*)

also have $\dots = (\neg_r (\exists \$ref'; \$st' \cdot RR(\neg_r pre_R P))) ;; R1 \text{ true})$

by (*rel-auto*)

also have $\dots = (\neg_r (\neg_r pre_R P)) ;; R1 \text{ true})$

by (*simp add: Healthy-if closure ex-unrest unrest assms*)

also have $\dots = pre_R P$

by (*simp add: NCSP-implies-NSRD NSRD-neg-pre-unit R1-preR assms rea-not-not*)

finally show *?thesis* .

qed

have $2: (pre_R P \Rightarrow_r peri_R P) \llbracket \Sigma | \{\} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle) =$

$(\exists ref_0 \cdot (peri_R P) \llbracket \langle ref_0 \rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle ref_0 \rangle) \text{ (is } ?lhs = ?rhs)$

proof –

have $?lhs = peri_R P \llbracket \Sigma | \{\} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle)$

by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have $\dots = (\exists \$st' \cdot (peri_R P \parallel_{N_C} 1_L \{\} 0_L \Phi(true, id, \langle \rangle)))$

by (*simp add: CSPInterMerge-def par-by-merge-def seqr-exists-right*)

also have $\dots =$

$(\exists \$st' \cdot \exists (ref_0, st_0, tt_0) \cdot$

$[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger (\exists \$st' \cdot RR(peri_R P)) \wedge$

$\$ref' \subseteq_u \langle ref_0 \rangle \wedge [\langle trace \rangle =_u \langle tt_0 \rangle]_t \wedge \$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \Sigma \oplus \langle id \rangle (\$st)_a \text{ on } \emptyset)$

by (*simp add: merge-csp-do-right pr-var-def assms Healthy-if assms closure rpred unrest ex-unrest*)

also have $\dots =$

$(\exists ref_0 \cdot (\exists \$st' \cdot RR(peri_R P)) \llbracket \langle ref_0 \rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle ref_0 \rangle)$

by (*rel-auto*)

also have $\dots = ?rhs$

by (*simp add: closure ex-unrest Healthy-if unrest assms*)

finally show *?thesis* .
qed
have $3: (pre_R P \Rightarrow_r post_R P) \llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle) = post_R(P)$ (**is** *?lhs = ?rhs*)
proof –
have $?lhs = post_R P \llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle)$
by (*simp add: SRD-post-under-pre closure assms unrest*)
also have $\dots = (\exists (st_0, t_0) \cdot$
 $\quad [\$st' \mapsto_s \ll st_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger RR(post_R P) \wedge$
 $\quad [\ll trace \gg =_u \ll t_0 \gg]_t \wedge \$st' =_u \$st \oplus \ll st_0 \gg \text{ on } \Sigma \oplus \ll id \gg (\$st)_a \text{ on } \emptyset)$
by (*simp add: FinalMerge-csp-do-right pr-var-def assms closure unrest rpred Healthy-if*)
also have $\dots = RR(post_R(P))$
by (*rel-auto*)
finally show *?thesis*
by (*simp add: Healthy-if assms closure*)
qed
show *?thesis*
proof –
have $C2(P) = \mathbf{R}_s (\Phi(true, id, \langle \rangle) \text{ wr } [\Sigma | \{ \} | \emptyset]_C pre_R P \vdash$
 $(pre_R P \Rightarrow_r peri_R P) \llbracket \Sigma | \{ \} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle) \diamond (pre_R P \Rightarrow_r post_R P) \llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle))$
by (*simp add: C2-def, rdes-simp cls: assms, simp add: id-def pr-var-def*)
also have $\dots = \mathbf{R}_s (pre_R P \vdash (\exists ref_0 \cdot peri_R P \llbracket \ll ref_0 \gg / \$ref' \rrbracket \wedge \$ref' \subseteq_u \ll ref_0 \gg) \diamond post_R P)$
by (*simp add: 1 2 3*)
finally show *?thesis* .
qed
qed

We define downward closure of the pericondition by the following healthiness condition

definition $CDC :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ **where**
 $[upred-defs]: CDC(P) = (\exists ref_0 \cdot P \llbracket \ll ref_0 \gg / \$ref' \rrbracket \wedge \$ref' \subseteq_u \ll ref_0 \gg)$

lemma $CDC\text{-idem}: CDC(CDC(P)) = CDC(P)$
by (*rel-auto*)

lemma $CDC\text{-RR-commute}: CDC(RR(P)) = RR(CDC(P))$
by (*rel-blast*)

lemma $CDC\text{-RR-closed}$ [*closure*]: $P \text{ is } RR \Rightarrow CDC(P) \text{ is } RR$
by (*metis CDC-RR-commute Healthy-def*)

lemma $CDC\text{-unrest}$ [*unrest*]: $\ll vwb\text{-lens } x; (\$ref')_v \bowtie x; x \# P \rrbracket \Rightarrow x \# CDC(P)$
by (*simp add: CDC-def unrest usubst lens-indep-sym*)

lemma $CDC\text{-R4-commute}: CDC(R4(P)) = R4(CDC(P))$
by (*rel-auto*)

lemma $R4\text{-CDC-closed}$ [*closure*]: $P \text{ is } CDC \Rightarrow R4(P) \text{ is } CDC$
by (*simp add: CDC-R4-commute Healthy-def*)

lemma $CDC\text{-R5-commute}: CDC(R5(P)) = R5(CDC(P))$
by (*rel-auto*)

lemma $R5\text{-CDC-closed}$ [*closure*]: $P \text{ is } CDC \Rightarrow R5(P) \text{ is } CDC$
by (*simp add: CDC-R5-commute Healthy-def*)

lemma $rea\text{-true}\text{-CDC}$ [*closure*]: $true_r \text{ is } CDC$

by (rel-auto)

lemma false-CDC [closure]: false is CDC
by (rel-auto)

lemma CDC-UNF-closed [closure]:
assumes $\bigwedge i. i \in I \implies P\ i$ is CDC
shows $(\bigcap i \in I \cdot P\ i)$ is CDC
using assms by (rel-blast)

lemma CDC-disj-closed [closure]:
assumes P is CDC Q is CDC
shows $(P \vee Q)$ is CDC
proof –
have $CDC(P \vee Q) = (CDC(P) \vee CDC(Q))$
by (rel-auto)
thus ?thesis
by (metis Healthy-def assms(1) assms(2))
qed

lemma CDC-USUP-closed [closure]:
assumes $\bigwedge i. i \in I \implies P\ i$ is CDC
shows $(\bigcup i \in I \cdot P\ i)$ is CDC
using assms by (rel-blast)

lemma CDC-conj-closed [closure]:
assumes P is CDC Q is CDC
shows $(P \wedge Q)$ is CDC
using assms by (rel-auto, blast, meson)

lemma CDC-rea-impl [rpred]:
 $\$ref' \# P \implies CDC(P \Rightarrow_r Q) = (P \Rightarrow_r CDC(Q))$
by (rel-auto)

lemma rea-impl-CDC-closed [closure]:
assumes $\$ref' \# P\ Q$ is CDC
shows $(P \Rightarrow_r Q)$ is CDC
using assms by (simp add: CDC-rea-impl Healthy-def)

lemma seq-CDC-closed [closure]:
assumes Q is CDC
shows $(P ;; Q)$ is CDC
proof –
have $CDC(P ;; Q) = P ;; CDC(Q)$
by (rel-blast)
thus ?thesis
by (metis Healthy-def assms)
qed

lemma csp-enable-CDC [closure]: $\mathcal{E}(s, t, E)$ is CDC
by (rel-auto)

lemma C2-CDC-form:
assumes P is NCSP
shows $C2(P) = \mathbf{R}_s (pre_R P \vdash CDC(peri_R P) \diamond post_R P)$

by (simp add: C2-form assms CDC-def)

lemma C2-rdes-def:

assumes P_1 is CRC P_2 is CRR P_3 is CRR $\$st' \# P_2 \$ref' \# P_3$
 shows $C2(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3)) = \mathbf{R}_s(P_1 \vdash CDC(P_2) \diamond P_3)$
 by (simp add: C2-form assms closure rdes unrest usubst, rel-auto)

lemma C2-NCSP-intro:

assumes P is NCSP $peri_R(P)$ is CDC
 shows P is C2

proof –

have $C2(P) = \mathbf{R}_s(pre_R P \vdash CDC(peri_R P) \diamond post_R P)$
 by (simp add: C2-CDC-form assms(1))
 also have $\dots = \mathbf{R}_s(pre_R P \vdash peri_R P \diamond post_R P)$
 by (simp add: Healthy-if assms)
 also have $\dots = P$
 by (simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1))
 finally show ?thesis
 by (simp add: Healthy-def)

qed

lemma C2-rdes-intro:

assumes P_1 is CRC P_2 is CRR P_2 is CDC P_3 is CRR $\$st' \# P_2 \$ref' \# P_3$
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3)$ is C2
 unfolding Healthy-def
 by (simp add: C2-rdes-def assms unrest closure Healthy-if)

lemma C2-implies-CDC-peri [closure]:

assumes P is NCSP P is C2
 shows $peri_R(P)$ is CDC

proof –

have $peri_R(P) = peri_R(\mathbf{R}_s(pre_R P \vdash CDC(peri_R P) \diamond post_R P))$
 by (metis C2-CDC-form Healthy-if assms(1) assms(2))
 also have $\dots = CDC(pre_R P \Rightarrow_r peri_R P)$
 by (simp add: rdes rpred assms closure unrest)
 also have $\dots = CDC(peri_R P)$
 by (simp add: SRD-peri-under-pre closure unrest assms)
 finally show ?thesis
 by (simp add: Healthy-def)

qed

lemma Miracle-C2-closed [closure]: Miracle is C2

by (rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest)

lemma Chaos-C2-closed [closure]: Chaos is C2

by (rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest)

lemma Skip-C2-closed [closure]: Skip is C2

by (rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest)

lemma Stop-C2-closed [closure]: Stop is C2

by (rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest)

lemma wp-rea-CRC [closure]: $\llbracket P \text{ is CRR}; Q \text{ is CRC} \rrbracket \implies P \text{ wp}_r Q \text{ is CRC}$

by (rule CRC-intro, simp-all add: unrest closure)

lemma *seq-C2-closed* [closure]:
assumes P is NCSP P is C2 Q is NCSP Q is C2
shows $P \parallel Q$ is C2
by (*rdes-simp* *cls*: *assms*(1,3), *rule* C2-rdes-intro, *simp*-all *add*: *closure* *assms* *unrest*)

lemma *DoCSP-C2* [closure]:
 $do_C(a)$ is C2
by (*rdes-simp*, *rule* C2-rdes-intro, *simp*-all *add*: *closure* *unrest*)

lemma *PrefixCSP-C2-closed* [closure]:
assumes P is NCSP P is C2
shows $a \rightarrow_C P$ is C2
unfolding *PrefixCSP-def* **by** (*metis* *DoCSP-C2* *Healthy-def* *NCSP-DoCSP* *NCSP-implies-CSP* *assms* *seq-C2-closed*)

lemma *ExtChoice-C2-closed* [closure]:
assumes $\bigwedge i. i \in I \implies P(i)$ is NCSP $\bigwedge i. i \in I \implies P(i)$ is C2
shows $(\square i \in I \cdot P(i))$ is C2
proof (*cases* $I = \{\}$)
case *True*
then show *?thesis* **by** (*simp* *add*: *closure* *ExtChoice-empty*)
next
case *False*
show *?thesis*
by (*rule* C2-NCSP-intro, *simp*-all *add*: *assms* *closure* *unrest* *periR-ExtChoice-ind'* *False*)
qed

lemma *extChoice-C2-closed* [closure]:
assumes P is NCSP P is C2 Q is NCSP Q is C2
shows $P \sqcap Q$ is C2
proof –
have $P \sqcap Q = (\square I \in \{P, Q\} \cdot I)$
by (*simp* *add*: *extChoice-def*)
also have ... is C2
by (*rule* *ExtChoice-C2-closed*, *auto* *simp* *add*: *assms*)
finally show *?thesis* .
qed

lemma *CDC-CRR-closed* [closure]:
assumes P is CRR
shows $CDC(P)$ is CRR
by (*rule* CRR-intro, *simp* *add*: *CDC-def* *unrest* *assms* *closure*, *simp* *add*: *unrest* *assms* *closure*)

lemma *C2-idem*:
assumes P is NCSP
shows $C2(C2(P)) = C2(P)$ (*is* *?lhs* = *?rhs*)
proof –
have *?lhs* = $\mathbf{R}_s(\text{pre}_R P \vdash (\text{pre}_R P \Rightarrow_r CDC(\text{peri}_R P)) \diamond (\text{pre}_R P \Rightarrow_r \text{post}_R P))$
by (*simp* *add*: *C2-CDC-form* *assms* *closure* *unrest* *rdes* *rpred* *CDC-idem*)
also have ... = $\mathbf{R}_s(\text{pre}_R P \vdash CDC(\text{pre}_R P \Rightarrow_r \text{peri}_R P) \diamond \text{post}_R P)$
by (*simp* *add*: *rpred* *unrest* *SRD-post-under-pre* *assms* *closure*)
also have ... = $\mathbf{R}_s(\text{pre}_R P \vdash CDC(\text{peri}_R P) \diamond \text{post}_R P)$
by (*simp* *add*: *unrest* *SRD-peri-under-pre* *assms* *closure*)
also have ... = $C2(P)$

by (simp add: C2-CDC-form assms)
 finally show ?thesis .
 qed

lemma

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR*
 shows *P wr[ns1|cs|ns2]_C false = undefined (is ?lhs = ?rhs)*

proof –

have ?lhs = (\neg_r (\exists (*ref*₀, *ref*₁, *st*₀, *st*₁, *tt*₀, *tt*₁) .
 $[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger R1 \text{ true} \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger P \wedge$
 $\$ref' \subseteq_u (\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle \wedge$
 $\$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2) ;;$
R1 true)
 by (simp add: wrR-def par-by-merge-seq-remove CSPInnerMerge-form assms closure usubst unrest)
 also have ... = (\neg_r (\exists (*tt*₀, *tt*₁) .
 $[\$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger P \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle) ;;$
R1 true)
 by (rel-blast)
 also have ... = (\neg_r (\exists (*tt*₀, *tt*₁) .
 $[\$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger RR(P) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{\langle\langle cs \rangle\rangle} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle) ;;$
R1 true)
 by (simp add: Healthy-if assms)
 oops

end

13 Linking to the Failures-Divergences Model

theory *utp-circus-fdsem*

imports *utp-circus-parallel utp-circus-recursion*

begin

13.1 Failures-Divergences Semantics

The following functions play a similar role to those in Roscoe's CSP semantics, and are calculated from the Circus reactive design semantics. A major difference is that these three functions account for state. Each divergence, trace, and failure is subject to an initial state. Moreover, the traces are terminating traces, and therefore also provide a final state following the given interaction. A more subtle difference from the Roscoe semantics is that the set of traces do not include the divergences. The same semantic information is present, but we construct a direct analogy with the pre-, peri- and postconditions of our reactive designs.

definition *divergences* :: (σ, φ) *action* $\Rightarrow \sigma \Rightarrow \varphi$ *list set* (*dv*[-] - [0,100] 100) **where**
 $[upred-defs]: \text{divergences } P \ s = \{t \mid t. '(\neg_r \text{pre}_R(P))[\langle\langle s \rangle\rangle, \langle\rangle, \langle\langle t \rangle\rangle / \$st, \$tr, \$tr']'\}$

definition *traces* :: ($'\sigma, '\varphi$) action $\Rightarrow ' \sigma \Rightarrow (' \varphi$ list $\times ' \sigma$) set ($tr[-] - [0,100] 100$) **where**
 $[upred-defs]: traces\ P\ s = \{(t, s') \mid t\ s'.\ ' (pre_R(P) \wedge post_R(P)) \llbracket \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle \rrbracket / \$st, \$st', \$tr, \$tr' \} \}$

definition *failures* :: ($'\sigma, '\varphi$) action $\Rightarrow ' \sigma \Rightarrow (' \varphi$ list $\times ' \varphi$ set) set ($fl[-] - [0,100] 100$) **where**
 $[upred-defs]: failures\ P\ s = \{(t, r) \mid t\ r.\ ' (pre_R(P) \wedge peri_R(P)) \llbracket \langle r \rangle, \langle s \rangle, \langle \rangle, \langle t \rangle \rrbracket / \$ref', \$st, \$tr, \$tr' \} \}$

lemma *trace-divergence-disj*:

assumes P is NCSP $(t, s') \in tr[P]s\ t \in dv[P]s$
shows *False*
using *assms(2,3)*
by (*simp add: traces-def divergences-def, rdes-simp cls:assms, rel-auto*)

lemma *preR-refine-divergences*:

assumes P is NCSP Q is NCSP $\wedge s. dv[P]s \subseteq dv[Q]s$
shows $pre_R(P) \sqsubseteq pre_R(Q)$

proof (*rule CRR-refine-impl-prop, simp-all add: assms closure usubst unrest*)

fix $t\ s$

assume $a: '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R\ Q'$

with a **show** $'[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R\ P'$

proof (*rule-tac ccontr*)

from *assms(3)[of s]* **have** $b: t \in dv[P]s \implies t \in dv[Q]s$

by (*auto*)

assume $\neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R\ P'$

hence $\neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger CRC(pre_R\ P)'$

by (*simp add: assms closure Healthy-if*)

hence $'[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (\neg_r\ CRC(pre_R\ P))'$

by (*rel-auto*)

hence $'[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (\neg_r\ pre_R\ P)'$

by (*simp add: assms closure Healthy-if*)

with $a\ b$ **show** *False*

by (*rel-auto*)

qed

qed

lemma *preR-eq-divergences*:

assumes P is NCSP Q is NCSP $\wedge s. dv[P]s = dv[Q]s$

shows $pre_R(P) = pre_R(Q)$

by (*metis assms dual-order.antisym order-refl preR-refine-divergences*)

lemma *periR-refine-failures*:

assumes P is NCSP Q is NCSP $\wedge s. fl[Q]s \subseteq fl[P]s$

shows $(pre_R(P) \wedge peri_R(P)) \sqsubseteq (pre_R(Q) \wedge peri_R(Q))$

proof (*rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-3*)

fix $t\ s\ r'$

assume $a: '[$ref' \mapsto_s \langle r' \rangle, \$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (pre_R\ Q \wedge peri_R\ Q)'$

from *assms(3)[of s]* **have** $b: (t, r') \in fl[Q]s \implies (t, r') \in fl[P]s$

by (*auto*)

with a **show** $'[$ref' \mapsto_s \langle r' \rangle, \$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (pre_R\ P \wedge peri_R\ P)'$

by (*simp add: failures-def*)

qed

lemma *periR-eq-failures*:

assumes P is NCSP Q is NCSP $\wedge s. fl[P]s = fl[Q]s$

shows $(pre_R(P) \wedge peri_R(P)) = (pre_R(Q) \wedge peri_R(Q))$

by (*metis (full-types) assms dual-order.antisym order-refl periR-refine-failures*)

lemma *postR-refine-traces*:

assumes P is NCSP Q is NCSP $\wedge s. tr\llbracket Q \rrbracket s \subseteq tr\llbracket P \rrbracket s$

shows $(pre_R(P) \wedge post_R(P)) \sqsubseteq (pre_R(Q) \wedge post_R(Q))$

proof (rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-5)

fix $t s s'$

assume a : $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R Q \wedge post_R Q)'$

from $assms(3)[of\ s]$ **have** b : $(t, s') \in tr\llbracket Q \rrbracket s \implies (t, s') \in tr\llbracket P \rrbracket s$

by (auto)

with a **show** $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R P \wedge post_R P)'$

by (simp add: traces-def)

qed

lemma *postR-eq-traces*:

assumes P is NCSP Q is NCSP $\wedge s. tr\llbracket P \rrbracket s = tr\llbracket Q \rrbracket s$

shows $(pre_R(P) \wedge post_R(P)) = (pre_R(Q) \wedge post_R(Q))$

by (metis assms dual-order.antisym order-refl postR-refine-traces)

lemma *circus-fd-refine-intro*:

assumes P is NCSP Q is NCSP $\wedge s. dv\llbracket Q \rrbracket s \subseteq dv\llbracket P \rrbracket s \wedge s. fl\llbracket Q \rrbracket s \subseteq fl\llbracket P \rrbracket s \wedge s. tr\llbracket Q \rrbracket s \subseteq tr\llbracket P \rrbracket s$

shows $P \sqsubseteq Q$

proof (rule SRD-refine-intro', simp-all add: closure assms)

show a : $pre_R P \Rightarrow pre_R Q$

using $assms(1)$ $assms(2)$ $assms(3)$ *preR-refine-divergences refBy-order* **by** blast

show $peri_R P \sqsubseteq (pre_R P \wedge peri_R Q)$

proof –

have $peri_R P \sqsubseteq (pre_R Q \wedge peri_R Q)$

by (metis (no-types) $assms(1)$ $assms(2)$ $assms(4)$ *periR-refine-failures utp-pred-laws.le-inf-iff*)

then show *?thesis*

by (metis a *refBy-order utp-pred-laws.inf.order-iff utp-pred-laws.inf-assoc*)

qed

show $post_R P \sqsubseteq (pre_R P \wedge post_R Q)$

proof –

have $post_R P \sqsubseteq (pre_R Q \wedge post_R Q)$

by (meson $assms(1)$ $assms(2)$ $assms(5)$ *postR-refine-traces utp-pred-laws.le-inf-iff*)

then show *?thesis*

by (metis a *refBy-order utp-pred-laws.inf.absorb-iff1 utp-pred-laws.inf-assoc*)

qed

qed

13.2 Circus Operators

lemma *traces-Skip*:

$tr\llbracket Skip \rrbracket s = \{(\llbracket \cdot \rrbracket, s)\}$

by (simp add: traces-def rdes alpha closure, rel-simp)

lemma *failures-Skip*:

$fl\llbracket Skip \rrbracket s = \{\}$

by (simp add: failures-def, rdes-calc)

lemma *divergences-Skip*:

$dv\llbracket Skip \rrbracket s = \{\}$

by (simp add: divergences-def, rdes-calc)

lemma *traces-Stop*:

$tr\llbracket Stop \rrbracket s = \{\}$

by (simp add: traces-def, rdes-calc)

lemma failures-Stop:
 $fl\llbracket Stop \rrbracket s = \{(\llbracket \cdot \rrbracket, E) \mid E. True\}$
 by (simp add: failures-def, rdes-calc, rel-auto)

lemma divergences-Stop:
 $dv\llbracket Stop \rrbracket s = \{\}$
 by (simp add: divergences-def, rdes-calc)

lemma traces-AssignsCSP:
 $tr\llbracket \langle \sigma \rangle_C \rrbracket s = \{(\llbracket \cdot \rrbracket, \sigma(s))\}$
 by (simp add: traces-def rdes closure usubst alpha, rel-auto)

lemma failures-AssignsCSP:
 $fl\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$
 by (simp add: failures-def, rdes-calc)

lemma divergences-AssignsCSP:
 $dv\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$
 by (simp add: divergences-def, rdes-calc)

lemma failures-Miracle: $fl\llbracket Miracle \rrbracket s = \{\}$
 by (simp add: failures-def rdes closure usubst)

lemma divergences-Miracle: $dv\llbracket Miracle \rrbracket s = \{\}$
 by (simp add: divergences-def rdes closure usubst)

lemma failures-Chaos: $fl\llbracket Chaos \rrbracket s = \{\}$
 by (simp add: failures-def rdes, rel-auto)

lemma divergences-Chaos: $dv\llbracket Chaos \rrbracket s = UNIV$
 by (simp add: divergences-def rdes, rel-auto)

lemma traces-Chaos: $tr\llbracket Chaos \rrbracket s = \{\}$
 by (simp add: traces-def rdes closure usubst)

lemma divergences-cond:
 assumes P is NCSP Q is NCSP
 shows $dv\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then dv\llbracket P \rrbracket s else dv\llbracket Q \rrbracket s)$
 by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

lemma traces-cond:
 assumes P is NCSP Q is NCSP
 shows $tr\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then tr\llbracket P \rrbracket s else tr\llbracket Q \rrbracket s)$
 by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

lemma failures-cond:
 assumes P is NCSP Q is NCSP
 shows $fl\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then fl\llbracket P \rrbracket s else fl\llbracket Q \rrbracket s)$
 by (rdes-simp cls: assms, simp add: divergences-def failures-def rdes closure rpred assms, rel-auto)

lemma divergences-guard:
 assumes P is NCSP
 shows $dv\llbracket g \&_u P \rrbracket s = (if (\llbracket g \rrbracket_e s) then dv\llbracket g \&_u P \rrbracket s else \{\})$

by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

lemma traces-do: $tr\llbracket do_C(e) \rrbracket s = \{(\llbracket e \rrbracket_e s, s)\}$

by (rdes-simp, simp add: traces-def rdes closure rpred, rel-auto)

lemma failures-do: $fl\llbracket do_C(e) \rrbracket s = \{(\llbracket \cdot \rrbracket, E) \mid E. \llbracket e \rrbracket_e s \notin E\}$

by (rdes-simp, simp add: failures-def rdes closure rpred usubst, rel-auto)

lemma divergences-do: $dv\llbracket do_C(e) \rrbracket s = \{\}$

by (rel-auto)

lemma divergences-seq:

fixes $P :: ('s, 'e) \text{ action}$

assumes $P \text{ is NCSP } Q \text{ is NCSP}$

shows $dv\llbracket P ;; Q \rrbracket s = dv\llbracket P \rrbracket s \cup \{t_1 @ t_2 \mid t_1 \ t_2 \ s_0. (t_1, s_0) \in tr\llbracket P \rrbracket s \wedge t_2 \in dv\llbracket Q \rrbracket s_0\}$

(is ?lhs = ?rhs)

oops

lemma traces-seq:

fixes $P :: ('s, 'e) \text{ action}$

assumes $P \text{ is NCSP } Q \text{ is NCSP}$

shows $tr\llbracket P ;; Q \rrbracket s =$

$\{(t_1 @ t_2, s') \mid t_1 \ t_2 \ s_0 \ s'. (t_1, s_0) \in tr\llbracket P \rrbracket s \wedge (t_2, s') \in tr\llbracket Q \rrbracket s_0$
 $\wedge (t_1 @ t_2) \notin dv\llbracket P \rrbracket s$
 $\wedge (\forall (t, s_1) \in tr\llbracket P \rrbracket s. t \leq t_1 @ t_2 \longrightarrow (t_1 @ t_2) - t \notin dv\llbracket Q \rrbracket s_1) \}$

(is ?lhs = ?rhs)

proof

show ?lhs \subseteq ?rhs

proof (rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto)

fix $t :: 'e \text{ list}$ and $s' :: 's$

let $\sigma = [\$st \mapsto_s \llbracket s \rrbracket, \$st' \mapsto_s \llbracket s' \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket]$

assume

a1: $\sigma \vdash (post_R P ;; post_R Q)'$ and

a2: $[\$st \mapsto_s \llbracket s \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket] \vdash pre_R P'$ and

a3: $[\$st \mapsto_s \llbracket s \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket] \vdash (post_R P \text{ wp}_r pre_R Q)'$

from a1 have $\sigma \vdash (\exists tr_0. ((post_R P)\llbracket \llbracket tr_0 \rrbracket / \$tr' \rrbracket ;; (post_R Q)\llbracket \llbracket tr_0 \rrbracket / \$tr \rrbracket) \wedge \llbracket tr_0 \rrbracket \leq_u \$tr')$
by (simp add: R2-tr-middle assms closure)

then obtain tr_0 where $p1: \sigma \vdash ((post_R P)\llbracket \llbracket tr_0 \rrbracket / \$tr' \rrbracket ;; (post_R Q)\llbracket \llbracket tr_0 \rrbracket / \$tr \rrbracket)'$ and $tr0: tr_0$

$\leq t$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

from p1 have $\sigma \vdash (\exists st_0. (post_R P)\llbracket \llbracket tr_0 \rrbracket / \$tr' \rrbracket \llbracket \llbracket st_0 \rrbracket / \$st' \rrbracket ;; (post_R Q)\llbracket \llbracket tr_0 \rrbracket / \$tr \rrbracket \llbracket \llbracket st_0 \rrbracket / \$st \rrbracket)'$
by (simp add: seqr-middle[of st, THEN sym])

then obtain s_0 where $\sigma \vdash ((post_R P)\llbracket \llbracket s_0 \rrbracket, \llbracket tr_0 \rrbracket / \$st', \$tr' \rrbracket ;; (post_R Q)\llbracket \llbracket s_0 \rrbracket, \llbracket tr_0 \rrbracket / \$st, \$tr \rrbracket)'$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

hence $\sigma \vdash (([\$st \mapsto_s \llbracket s \rrbracket, \$st' \mapsto_s \llbracket s_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tr_0 \rrbracket] \vdash post_R P) ;;$
 $([\$st \mapsto_s \llbracket s_0 \rrbracket, \$st' \mapsto_s \llbracket s' \rrbracket, \$tr \mapsto_s \llbracket tr_0 \rrbracket, \$tr' \mapsto_s \llbracket t \rrbracket] \vdash post_R Q))'$

by (*rel-auto*)
 hence ‘ $([\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger post_R P) \wedge$
 $([\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger post_R Q))'$
 by (*simp add: segr-to-conj unrest-any-circus-var assms closure unrest*)
 hence *postP*: ‘ $([\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger post_R P)'$ and
postQ': ‘ $([\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger post_R Q)'$
 by (*rel-auto*) +
 from *postQ'* have ‘ $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \dagger [\$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle tr_0 \rangle + (\langle t \rangle -$
 $\langle tr_0 \rangle)] \dagger post_R Q'$
 using *tr0* by (*rel-auto*)
 hence ‘ $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \langle t \rangle - \langle tr_0 \rangle] \dagger post_R Q'$
 by (*simp add: R2-subst-tr closure assms*)
 hence *postQ*: ‘ $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - tr_0 \rangle] \dagger post_R Q'$
 by (*rel-auto*)
 have *preP*: ‘ $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger pre_R P'$
 proof –
 have $(pre_R P)[[0, \langle tr_0 \rangle / \$tr, \$tr']] \sqsubseteq (pre_R P)[[0, \langle t \rangle / \$tr, \$tr']]$
 by (*simp add: RC-prefix-refine closure assms tr0*)
 hence $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s$
 $\langle t \rangle] \dagger pre_R P$
 by (*rel-auto*)
 thus ?thesis
 by (*simp add: taut-refine-impl a2*)
 qed
 have *preQ*: ‘ $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - tr_0 \rangle] \dagger pre_R Q'$
 proof –
 from *postP a3* have ‘ $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R Q'$
 apply (*simp add: wp-rea-def*)
 apply (*rel-auto*)
 using *tr0* apply *blast* +
 done
 hence ‘ $[\$st \mapsto_s \langle s_0 \rangle] \dagger [\$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle tr_0 \rangle + (\langle t \rangle - \langle tr_0 \rangle)] \dagger pre_R Q'$
 by (*rel-auto*)
 hence ‘ $[\$st \mapsto_s \langle s_0 \rangle] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \langle t \rangle - \langle tr_0 \rangle] \dagger pre_R Q'$
 by (*simp add: R2-subst-tr closure assms*)
 thus ?thesis
 by (*rel-auto*)
 qed
 from *a2* have *ndiv*: $\neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (\neg_r pre_R P)'$
 by (*rel-auto*)
 have *t-minus-tr0*: $tr_0 @ (t - tr_0) = t$
 using *append-minus tr0* by *blast*
 from *a3*
 have *wpr*: $\bigwedge t_0 s_1.$
 ‘ $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P' \implies$
 ‘ $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P' \implies$
 $t_0 \leq t \implies '[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - t_0 \rangle] \dagger (\neg_r pre_R Q)' \implies False$
 proof –
 fix $t_0 s_1$
 assume *b*:

$\begin{aligned} & \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P\text{' } \\ & \text{'}[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P\text{' } \\ & t_0 \leq t \\ & \text{'}[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - t_0 \gg] \dagger (\neg_r pre_R Q)\text{' } \end{aligned}$

from $a3$ **have** c : $\forall (s_0, t_0) \cdot \ll t_0 \gg \leq_u \ll t \gg$
 $\wedge [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P$
 $\Rightarrow [\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg - \ll t_0 \gg] \dagger pre_R Q\text{'}$
by (*simp add: wp-rea-circus-form-alt[of post_R P pre_R Q] closure assms unrest usubst*)
(rel-simp)

from c $b(2-4)$ **show** *False*
by (*rel-auto*)

qed

show $\exists t_1 t_2$.

$t = t_1 @ t_2 \wedge$
 $(\exists s_0. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger pre_R P \wedge$
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger post_R P\text{' } \wedge$
 $\text{'}[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger pre_R Q \wedge$
 $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger post_R Q\text{' } \wedge$
 $\neg \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\neg_r pre_R P)\text{' } \wedge$
 $(\forall t_0 s_1. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P \wedge$
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P\text{' } \longrightarrow$
 $t_0 \leq t_1 @ t_2 \longrightarrow \neg \text{'}[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger (\neg_r$
 $pre_R Q)\text{' }))$
apply (*rule-tac x=tr₀ in exI*)
apply (*rule-tac x=(t - tr₀) in exI*)
apply (*auto*)
using *tr0* **apply** *auto[1]*
apply (*rule-tac x=s₀ in exI*)
apply (*auto intro:wpr simp add: taut-conj preP preQ postP postQ ndiv wpr t-minus-tr0*)
done
qed

show $?rhs \subseteq ?lhs$

proof (*rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto*)

fix $t_1 t_2 :: 'e \text{ list}$ **and** $s_0 s' :: 's$

assume

$a1: \neg \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\neg_r pre_R P)\text{'}$ **and**
 $a2: \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger pre_R P\text{'}$ **and**
 $a3: \text{'}[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger post_R P\text{'}$ **and**
 $a4: \text{'}[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger pre_R Q\text{'}$ **and**
 $a5: \text{'}[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger post_R Q\text{'}$ **and**
 $a6: \forall t s_1. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P \wedge$
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger post_R P\text{' } \longrightarrow$
 $t \leq t_1 @ t_2 \longrightarrow \neg \text{'}[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg] \dagger (\neg_r pre_R Q)\text{'}$

from $a1$ **have** $preP$: $\text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (pre_R P)\text{'}$
by (*simp add: taut-not unrest-all-circus-vars-st assms closure unrest, rel-auto*)

have $\text{'}[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger post_R Q\text{'}$

proof –

have $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger post_R Q =$

$$[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \dagger [\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger post_R Q$$
 by *rel-auto*
 also have ... = $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \dagger [\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger post_R Q$
 by (*simp add: R2-subst-tr assms closure, rel-auto*)
 finally show *?thesis* using *a5*
 by (*rel-auto*)
 qed
 with *a3*
 have *postPQ*: ' $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P ;; post_R Q)'$
 by (*rel-auto, meson Prefix-Order.prefixI*)

 have ' $[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger pre_R Q'$
 proof –
 have $[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger pre_R Q =$
 $[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger pre_R Q$
 by *rel-auto*
 also have ... = $[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t_2 \gg] \dagger pre_R Q$
 by (*simp add: R2-subst-tr assms closure*)
 finally show *?thesis* using *a4*
 by (*rel-auto*)
 qed

 from *a6*
 have *a6'*: $\bigwedge t s_1. \ll t \leq t_1 @ t_2; [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P'; [\$st \mapsto_s \ll s \gg,$
 $\$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger post_R P' \rrbracket \implies$
 $[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg] \dagger pre_R Q'$
 apply (*subst (asm) taut-not*)
 apply (*simp add: unrest-all-circus-vars-st assms closure unrest*)
 apply (*rel-auto*)
 done

 have *wpR*: ' $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)'$
 proof –
 have $\bigwedge s_1 t_0. \ll t_0 \leq t_1 @ t_2; [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P'$
 $\rrbracket \implies [\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R Q'$
 proof –
 fix $s_1 t_0$
 assume $c: t_0 \leq t_1 @ t_2$ ' $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P'$

 have *preP'*: ' $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P'$
 proof –
 have $(pre_R P) \llbracket 0, \ll t_0 \gg / \$tr, \$tr' \rrbracket \sqsubseteq (pre_R P) \llbracket 0, \ll t_1 @ t_2 \gg / \$tr, \$tr' \rrbracket$
 by (*simp add: RC-prefix-refine closure assms c*)
 hence $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s$
 $\ll t_1 @ t_2 \gg] \dagger pre_R P$
 by (*rel-auto*)
 thus *?thesis*
 by (*simp add: taut-refine-impl preP*)
 qed

 with c *a3 preP a6'* [of $t_0 s_1$] show ' $[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R$
 Q'

```

    by (simp)
qed

thus ?thesis
  apply (simp-all add: wp-rea-circus-form-alt assms closure unrest usubst rea-impl-alt-def)
  apply (simp add: R1-def usubst tcontr-alt-def)
  apply (auto intro!: taut-shAll-intro-2)
  apply (rule taut-impl-intro)
  apply (simp add: unrest-all-circus-vars-st-st' unrest closure assms)
  apply (rel-simp)
done
qed
show '([ $\$st \mapsto_s \ll s \gg$ ,  $\$tr \mapsto_s \langle \rangle$ ,  $\$tr' \mapsto_s \ll t_1 @ t_2 \gg$ ]  $\dagger$   $pre_R P \wedge$ 
  [ $\$st \mapsto_s \ll s \gg$ ,  $\$tr \mapsto_s \langle \rangle$ ,  $\$tr' \mapsto_s \ll t_1 @ t_2 \gg$ ]  $\dagger$  ( $post_R P \wp_r pre_R Q$ ))  $\wedge$ 
  [ $\$st \mapsto_s \ll s \gg$ ,  $\$st' \mapsto_s \ll s' \gg$ ,  $\$tr \mapsto_s \langle \rangle$ ,  $\$tr' \mapsto_s \ll t_1 @ t_2 \gg$ ]  $\dagger$  ( $post_R P ;; post_R Q$ )'
  by (auto simp add: taut-conj preP postPQ wpR)
qed
qed

lemma Cons-minus [simp]:  $(a \# t) - [a] = t$ 
  by (metis append-Cons append-Nil append-minus)

lemma traces-prefix:
  assumes  $P$  is NCSP
  shows  $tr[a \rightarrow P]s = \{(a \# t, s') \mid t \ s'. (t, s') \in tr[P]s\}$ 
  apply (auto simp add: PrefixCSP-def traces-seq traces-do divergences-do lit.rep-eq assms closure
    Healthy-if trace-divergence-disj)
  apply (meson assms trace-divergence-disj)
done

13.3 Deadlock Freedom

definition DF :: 'e set  $\Rightarrow$  ('s, 'e) action where
DF(A) =  $(\mu_C X \cdot (\bigcap_{a \in A} a \rightarrow Skip) ;; X)$ 

lemma DF-CSP [closure]:  $A \neq \{\} \implies DF(A)$  is CSP
  by (simp add: DF-def closure unrest)

end

```

14 Meta theory for Circus

```

theory utp-circus
imports
  utp-circus-core
  utp-circus-rel
  utp-circus-healths
  utp-circus-contracts
  utp-circus-extchoice
  utp-circus-actions
  utp-circus-prefix
  utp-circus-recursion
  utp-circus-traces
  utp-circus-parallel
  utp-circus-fdsem

```

begin end

References

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