Reactive Designs

Simon Foster James Baxter Ana Cavalcanti Jim Woodcock
Samuel Canham

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Contents

1 Introduction

This document contains a mechanisation in Isabelle/UTP [2] of our theory of reactive designs. Reactive designs form an important semantic foundation for reactive modelling languages such as Circus [3]. For more details of this work, please see our recent paper [1].

2 Reactive Designs Healthiness Conditions

theory *utp-rdes-healths*
  imports *UTP−Reactive.utp-reactive*
begin

2.1 Preliminaries

named-theorems *rdes* **and** *rdes-def* **and** *RD-elim*

type-synonym $('s,'t)$ *rdes* = $('s,'t,unit)$ *hrel-rsp*

translations
  $(type)$ $('s,'t)$ *rdes* <= $(type)$ $('s, 't, unit)$ *hrel-rsp*

lemma *R2-st-ex*: $R2\ (\exists\ \$st \cdot P) = (\exists\ \$st \cdot R2(P))$
  by (*rel-auto*)

lemma *R2s-st'-eq-st*:
  $R2s(\$st' =_u \$st) = (\$st' =_u \$st)$
  by (*rel-auto*)

lemma *R2c-st'-eq-st*:
  $R2c(\$st' =_u \$st) = (\$st' =_u \$st)$
  by (*rel-auto*)

lemma *R1-des-lift-skip*: $R1(\lceil II \rceil_D) = \lceil II \rceil_D$
  by (*rel-auto*)

lemma *R2-des-lift-skip*:
  $R2(\lceil II \rceil_D) = \lceil II \rceil_D$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *R1-R2c-ex-st*: $R1\ (R2c\ (\exists\ \$st' \cdot Q_1)) = (\exists\ \$st' \cdot R1\ (R2c\ Q_1))$
  by (*rel-auto*)

2.2 Identities

We define two identities fro reactive designs, which correspond to the regular and state-sensitive versions of reactive designs, respectively. The former is the one used in the UTP book and related publications for CSP.

definition *skip-rea* :: $('t{::}trace, '\alpha)$ *hrel-rp* (II_c) **where**
skip-rea-def [*urel-defs*]: $II_c = (II \lor (\neg\ \$ok \land \$tr \leq_u \$tr'))$

definition *skip-srea* :: $('s, 't{::}trace, '\alpha)$ *hrel-rsp* (II_R) **where**
skip-srea-def [*urel-defs*]: $II_R = ((\exists\ \$st \cdot II_c) \triangleleft \$wait \triangleright II_c)$

lemma *skip-rea-R1-lemma*: $II_c = R1(\$ok \Rightarrow II)$
  by (*rel-auto*)

lemma *skip-rea-form*: $II_c = (II \triangleleft \$ok \triangleright R1(true))$
  by (*rel-auto*)

lemma *skip-srea-form*: $II_R = ((\exists\ \$st \cdot II) \triangleleft \$wait \triangleright II) \triangleleft \$ok \triangleright R1(true)$
  by (*rel-auto*)

lemma *R1-skip-rea*: $R1(II_c) = II_c$
  by (*rel-auto*)

lemma *R2c-skip-rea*: $R2c\ II_c = II_c$
  by (*simp add*: *skip-rea-def R2c-and R2c-disj R2c-skip-r R2c-not R2c-ok R2c-tr′-ge-tr*)

lemma *R2-skip-rea*: $R2(II_c) = II_c$
  by (*metis R1-R2c-is-R2 R1-skip-rea R2c-skip-rea*)

lemma *R2c-skip-srea*: $R2c(II_R) = II_R$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *skip-srea-R1* [*closure*]: II_R *is R1*
  by (*rel-auto*)

lemma *skip-srea-R2c* [*closure*]: II_R *is R2c*
  by (*simp add*: *Healthy-def R2c-skip-srea*)

lemma *skip-srea-R2* [*closure*]: II_R *is R2*
  by (*metis Healthy-def′ R1-R2c-is-R2 R2c-skip-srea skip-srea-R1*)

2.3 RD1: Divergence yields arbitrary traces

definition *RD1* :: $('t{::}trace,'\alpha,'\beta)$ *rel-rp* $\Rightarrow$ $('t,'\alpha,'\beta)$ *rel-rp* **where**
[*upred-defs*]: $RD1(P) = (P \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

$RD1$ is essentially $H1$ from the theory of designs, but viewed through the prism of reactive processes.

lemma *RD1-idem*: $RD1(RD1(P)) = RD1(P)$
  by (*rel-auto*)

lemma *RD1-Idempotent*: *Idempotent RD1*
  by (*simp add*: *Idempotent-def RD1-idem*)

lemma *RD1-mono*: $P \sqsubseteq Q \Longrightarrow RD1(P) \sqsubseteq RD1(Q)$
  by (*rel-auto*)

lemma *RD1-Monotonic*: *Monotonic RD1*
  using *mono-def RD1-mono* **by** *blast*

lemma *RD1-Continuous*: *Continuous RD1*
  by (*rel-auto*)

lemma *R1-true-RD1-closed* [*closure*]: $R1(true)$ *is RD1*
  by (*rel-auto*)

lemma *RD1-wait-false* [*closure*]: P *is RD1* $\Longrightarrow P[\![false/\$wait]\!]$ *is RD1*
  by (*rel-auto*)

lemma *RD1-wait′-false* [*closure*]: P *is RD1* $\Longrightarrow P[\![false/\$wait']\!]$ *is RD1*
  by (*rel-auto*)

lemma *RD1-seq*: $RD1(RD1(P)\ ;;\ RD1(Q)) = RD1(P)\ ;;\ RD1(Q)$
  by (*rel-auto*)

lemma *RD1-seq-closure* [*closure*]: $[\![P \text{ is } RD1;\ Q \text{ is } RD1]\!] \Longrightarrow P \ ;;\ Q \text{ is } RD1$
  by (*metis Healthy-def' RD1-seq*)

lemma *RD1-R1-commute*: $RD1(R1(P)) = R1(RD1(P))$
  by (*rel-auto*)

lemma *RD1-R2c-commute*: $RD1(R2c(P)) = R2c(RD1(P))$
  by (*rel-auto*)

lemma *RD1-via-R1*: $R1(H1(P)) = RD1(R1(P))$
  by (*rel-auto*)

lemma *RD1-R1-cases*: $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$
  by (*rel-auto*)

lemma *skip-rea-RD1-skip*: $II_c = RD1(II)$
  by (*rel-auto*)

lemma *skip-srea-RD1* [*closure*]: II_R *is RD1*
  by (*rel-auto*)

lemma *RD1-algebraic-intro*:
  assumes
    $P \text{ is } R1\ (R1(true_h) \ ;;\ P) = R1(true_h)\ (II_c \ ;;\ P) = P$
  shows *P is RD1*
proof –
  have $P = (II_c \ ;;\ P)$
    by (*simp add*: *assms*(*3*))
  also have ... $= (R1(\$ok \Rightarrow II) \ ;;\ P)$
    by (*simp add*: *skip-rea-R1-lemma*)
  also have ... $= (((\neg \$ok \wedge R1(true)) \ ;;\ P) \vee P)$
   by (*metis* (*no-types*, *lifting*) *R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred-laws.inf-top-left utp-pred-laws.sup-commute*)
  also have ... $= ((R1(\neg \$ok) \ ;;\ (R1(true_h) \ ;;\ P)) \vee P)$
    using *dual-order.trans* **by** (*rel-blast*)
  also have ... $= ((R1(\neg \$ok) \ ;;\ R1(true_h)) \vee P)$
    by (*simp add*: *assms*(*2*))
  also have ... $= (R1(\neg \$ok) \vee P)$
    by (*rel-auto*)
  also have ... $= RD1(P)$
    by (*rel-auto*)
  finally show *?thesis*
    by (*simp add*: *Healthy-def*)
qed

theorem *RD1-left-zero*:
  assumes *P is R1 P is RD1*
  shows $(R1(true) \ ;;\ P) = R1(true)$
proof –
  have $(R1(true) \ ;;\ R1(RD1(P))) = R1(true)$
    by (*rel-auto*)
  thus *?thesis*
    by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

theorem *RD1-left-unit*:
  assumes P *is* $R1$ P *is* $RD1$
  shows $(II_c \;;; P) = P$
proof −
  have $(II_c \;;; R1(RD1(P))) = R1(RD1(P))$
    by (*rel-auto*)
  thus *?thesis*
    by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

lemma *RD1-alt-def*:
  assumes P *is* $R1$
  shows $RD1(P) = (P \triangleleft \$ok \triangleright R1(true))$
proof −
  have $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$
    by (*rel-auto*)
  thus *?thesis*
    by (*simp add*: *Healthy-if assms*)
qed

theorem *RD1-algebraic*:
  assumes P *is* $R1$
  shows P *is* $RD1 \longleftrightarrow (R1(true_h) \;;; P) = R1(true_h) \wedge (II_c \;;; P) = P$
  using *RD1-algebraic-intro RD1-left-unit RD1-left-zero assms* **by** *blast*

2.4 R3c and R3h: Reactive design versions of R3

definition $R3c$ :: $('t{::}trace, '\alpha)$ *hrel-rp* $\Rightarrow$ $('t, '\alpha)$ *hrel-rp* **where**
[*upred-defs*]: $R3c(P) = (II_c \triangleleft \$wait \triangleright P)$

definition $R3h$ :: $('s, 't{::}trace, '\alpha)$ *hrel-rsp* $\Rightarrow$ $('s, 't, '\alpha)$ *hrel-rsp* **where**
R3h-def [*upred-defs*]: $R3h(P) = ((\exists\ \$st \cdot II_c) \triangleleft \$wait \triangleright P)$

lemma *R3c-idem*: $R3c(R3c(P)) = R3c(P)$
  by (*rel-auto*)

lemma *R3c-Idempotent*: *Idempotent R3c*
  by (*simp add*: *Idempotent-def R3c-idem*)

lemma *R3c-mono*: $P \sqsubseteq Q \Longrightarrow R3c(P) \sqsubseteq R3c(Q)$
  by (*rel-auto*)

lemma *R3c-Monotonic*: *Monotonic R3c*
  by (*simp add*: *mono-def R3c-mono*)

lemma *R3c-Continuous*: *Continuous R3c*
  by (*rel-auto*)

lemma *R3h-idem*: $R3h(R3h(P)) = R3h(P)$
  by (*rel-auto*)

lemma *R3h-Idempotent*: *Idempotent R3h*
  by (*simp add*: *Idempotent-def R3h-idem*)

lemma *R3h-mono*: $P \sqsubseteq Q \Longrightarrow R3h(P) \sqsubseteq R3h(Q)$

by (*rel-auto*)

lemma *R3h-Monotonic*: *Monotonic R3h*
by (*simp add*: *mono-def R3h-mono*)

lemma *R3h-Continuous*: *Continuous R3h*
by (*rel-auto*)

lemma *R3h-inf*: $R3h(P \sqcap Q) = R3h(P) \sqcap R3h(Q)$
by (*rel-auto*)

lemma *R3h-UINF*:
$A \neq \{\} \Longrightarrow R3h(\prod\ i \in A \cdot P(i)) = (\prod\ i \in A \cdot R3h(P(i)))$
by (*rel-auto*)

lemma *R3h-cond*: $R3h(P \triangleleft b \triangleright Q) = (R3h(P) \triangleleft b \triangleright R3h(Q))$
by (*rel-auto*)

lemma *R3c-via-RD1-R3*: $RD1(R3(P)) = R3c(RD1(P))$
by (*rel-auto*)

lemma *R3c-RD1-def*: P *is* $RD1 \Longrightarrow R3c(P) = RD1(R3(P))$
by (*simp add*: *Healthy-if R3c-via-RD1-R3*)

lemma *RD1-R3c-commute*: $RD1(R3c(P)) = R3c(RD1(P))$
by (*rel-auto*)

lemma *R1-R3c-commute*: $R1(R3c(P)) = R3c(R1(P))$
by (*rel-auto*)

lemma *R2c-R3c-commute*: $R2c(R3c(P)) = R3c(R2c(P))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *R1-R3h-commute*: $R1(R3h(P)) = R3h(R1(P))$
by (*rel-auto*)

lemma *R2c-R3h-commute*: $R2c(R3h(P)) = R3h(R2c(P))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *RD1-R3h-commute*: $RD1(R3h(P)) = R3h(RD1(P))$
by (*rel-auto*)

lemma *R3c-cancels-R3*: $R3c(R3(P)) = R3c(P)$
by (*rel-auto*)

lemma *R3-cancels-R3c*: $R3(R3c(P)) = R3(P)$
by (*rel-auto*)

lemma *R3h-cancels-R3c*: $R3h(R3c(P)) = R3h(P)$
by (*rel-auto*)

lemma *R3c-semir-form*:
$(R3c(P)\ ;;\ R3c(R1(Q))) = R3c(P\ ;;\ R3c(R1(Q)))$
by (*rel-simp*, *safe*, *auto intro*: *order-trans*)

lemma *R3h-semir-form*:
$(R3h(P) \;;\; R3h(R1(Q))) = R3h(P \;;\; R3h(R1(Q)))$
by (*rel-simp*, *safe*, *auto intro*: *order-trans*, *blast+*)

lemma *R3c-seq-closure*:
assumes *P is R3c Q is R3c Q is R1*
shows $(P \;;\; Q)$ *is R3c*
by (*metis Healthy-def′ R3c-semir-form assms*)

lemma *R3h-seq-closure* [*closure*]:
assumes *P is R3h Q is R3h Q is R1*
shows $(P \;;\; Q)$ *is R3h*
by (*metis Healthy-def′ R3h-semir-form assms*)

lemma *R3c-R3-left-seq-closure*:
assumes *P is R3 Q is R3c*
shows $(P \;;\; Q)$ *is R3c*
proof −
have $(P \;;\; Q) = ((P \;;\; Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens*)
also have ... $= (((II \triangleleft \$wait \triangleright P) \;;\; Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis Healthy-def′ R3-def assms*(*1*))
also have ... $= ((II[\![true/\$wait]\!] \;;\; Q) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*subst-tac*)
also have ... $= (((II \wedge \$wait\acute{}) \;;\; Q) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred-laws.inf-left-idem wait-vwb-lens*)
also have ... $= ((II[\![true/\$wait\acute{}]\!] \;;\; Q[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... $= ((II[\![true/\$wait\acute{}]\!] \;;\; (II_c \triangleleft \$wait \triangleright Q)[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis Healthy-def′ R3c-def assms*(*2*))
also have ... $= ((II[\![true/\$wait\acute{}]\!] \;;\; II_c[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*subst-tac*)
also have ... $= (((II \wedge \$wait\acute{}) \;;\; II_c) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... $= ((II \;;\; II_c) \triangleleft \$wait \triangleright (P \;;\; Q))$
by (*simp add*: *cond-def seqr-pre-transfer utp-rel.unrest-ouvar*)
also have ... $= (II_c \triangleleft \$wait \triangleright (P \;;\; Q))$
by *simp*
also have ... $= R3c(P \;;\; Q)$
by (*simp add*: *R3c-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def′*)
qed

lemma *R3c-cases*: $R3c(P) = ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$
by (*rel-auto*)

lemma *R3h-cases*: $R3h(P) = (((\exists\ \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$
by (*rel-auto*)

lemma *R3h-form*: $R3h(P) = II_R \triangleleft \$wait \triangleright P$
by (*rel-auto*)

lemma *R3c-subst-wait*: $R3c(P) = R3c(P\ _f)$
  by (*simp add*: *R3c-def cond-var-subst-right*)

lemma *R3h-subst-wait*: $R3h(P) = R3h(P\ _f)$
  by (*simp add*: *R3h-cases cond-var-subst-right*)

lemma *skip-srea-R3h* [*closure*]: II_R *is R3h*
  by (*rel-auto*)

lemma *R3h-wait-true*:
  assumes *P is R3h*
  shows $P\ _t = II_R\ _t$
proof −
  have $P\ _t = (II_R \triangleleft \$wait \triangleright P)\ _t$
    by (*metis Healthy-if R3h-form assms*)
  also have ... $= II_R\ _t$
    by (*simp add*: *usubst*)
  finally show *?thesis* **.**
qed

2.5 RD2: A reactive specification cannot require non-termination

definition *RD2* **where**
[*upred-defs*]: $RD2(P) = H2(P)$

$RD2$ is just $H2$ since the type system will automatically have J identifying the reactive variables as required.

lemma *RD2-idem*: $RD2(RD2(P)) = RD2(P)$
  by (*simp add*: *H2-idem RD2-def*)

lemma *RD2-Idempotent*: *Idempotent RD2*
  by (*simp add*: *Idempotent-def RD2-idem*)

lemma *RD2-mono*: $P \sqsubseteq Q \Longrightarrow RD2(P) \sqsubseteq RD2(Q)$
  by (*simp add*: *H2-def RD2-def seqr-mono*)

lemma *RD2-Monotonic*: *Monotonic RD2*
  using *mono-def RD2-mono* **by** *blast*

lemma *RD2-Continuous*: *Continuous RD2*
  by (*rel-auto*)

lemma *RD1-RD2-commute*: $RD1(RD2(P)) = RD2(RD1(P))$
  by (*rel-auto*)

lemma *RD2-R3c-commute*: $RD2(R3c(P)) = R3c(RD2(P))$
  by (*rel-auto*)

lemma *RD2-R3h-commute*: $RD2(R3h(P)) = R3h(RD2(P))$
  by (*rel-auto*)

2.6 Major healthiness conditions

definition *RH* :: $('t{::}trace,'\alpha)$ *hrel-rp* $\Rightarrow$ $('t,'\alpha)$ *hrel-rp* ($\mathbf{R}$)
where [*upred-defs*]: $RH(P) = R1(R2c(R3c(P)))$

definition *RHS* :: $('s,'t::trace,'\alpha)$ *hrel-rsp* $\Rightarrow$ $('s,'t,'\alpha)$ *hrel-rsp* $(\mathbf{R}_s)$
where [*upred-defs*]: $RHS(P) = R1(R2c(R3h(P)))$

definition *RD* :: $('t::trace,'\alpha)$ *hrel-rp* $\Rightarrow$ $('t,'\alpha)$ *hrel-rp*
where [*upred-defs*]: $RD(P) = RD1(RD2(RP(P)))$

definition *SRD* :: $('s,'t::trace,'\alpha)$ *hrel-rsp* $\Rightarrow$ $('s,'t,'\alpha)$ *hrel-rsp*
where [*upred-defs*]: $SRD(P) = RD1(RD2(RHS(P)))$

lemma *RH-comp*: $RH = R1 \circ R2c \circ R3c$
by (*auto simp add*: *RH-def*)

lemma *RHS-comp*: $RHS = R1 \circ R2c \circ R3h$
by (*auto simp add*: *RHS-def*)

lemma *RD-comp*: $RD = RD1 \circ RD2 \circ RP$
by (*auto simp add*: *RD-def*)

lemma *SRD-comp*: $SRD = RD1 \circ RD2 \circ RHS$
by (*auto simp add*: *SRD-def*)

lemma *RH-idem*: $\mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$
by (*simp add*: *R1-R2c-commute R1-R3c-commute R1-idem R2c-R3c-commute R2c-idem R3c-idem RH-def*)

lemma *RH-Idempotent*: *Idempotent* $\mathbf{R}$
by (*simp add*: *Idempotent-def RH-idem*)

lemma *RH-Monotonic*: *Monotonic* $\mathbf{R}$
by (*metis* (*no-types*, *lifting*) *R1-Monotonic R2c-Monotonic R3c-mono RH-def mono-def*)

lemma *RH-Continuous*: *Continuous* $\mathbf{R}$
by (*simp add*: *Continuous-comp R1-Continuous R2c-Continuous R3c-Continuous RH-comp*)

lemma *RHS-idem*: $\mathbf{R}_s(\mathbf{R}_s(P)) = \mathbf{R}_s(P)$
by (*simp add*: *R1-R2c-is-R2 R1-R3h-commute R2-idem R2c-R3h-commute R3h-idem RHS-def*)

lemma *RHS-Idempotent* [*closure*]: *Idempotent* $\mathbf{R}_s$
by (*simp add*: *Idempotent-def RHS-idem*)

lemma *RHS-Monotonic*: *Monotonic* $\mathbf{R}_s$
by (*simp add*: *mono-def R1-R2c-is-R2 R2-mono R3h-mono RHS-def*)

lemma *RHS-mono*: $P \sqsubseteq Q \Longrightarrow \mathbf{R}_s(P) \sqsubseteq \mathbf{R}_s(Q)$
using *mono-def RHS-Monotonic* **by** *blast*

lemma *RHS-Continuous* [*closure*]: *Continuous* $\mathbf{R}_s$
by (*simp add*: *Continuous-comp R1-Continuous R2c-Continuous R3h-Continuous RHS-comp*)

lemma *RHS-inf*: $\mathbf{R}_s(P \sqcap Q) = \mathbf{R}_s(P) \sqcap \mathbf{R}_s(Q)$
using *Continuous-Disjunctous Disjunctuous-def RHS-Continuous* **by** *auto*

lemma *RHS-INF*:
$A \neq \{\} \Longrightarrow \mathbf{R}_s(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot \mathbf{R}_s(P(i)))$

by (*simp add*: *RHS-def R3h-UINF R2c-USUP R1-USUP*)

lemma *RHS-sup*: $\mathbf{R}_s(P \sqcup Q) = \mathbf{R}_s(P) \sqcup \mathbf{R}_s(Q)$
by (*rel-auto*)

lemma *RHS-SUP*:
$A \neq \{\} \Longrightarrow \mathbf{R}_s(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot \mathbf{R}_s(P(i)))$
by (*rel-auto*)

lemma *RHS-cond*: $\mathbf{R}_s(P \triangleleft b \triangleright Q) = (\mathbf{R}_s(P) \triangleleft R2c\ b \triangleright \mathbf{R}_s(Q))$
by (*simp add*: *RHS-def R3h-cond R2c-condr R1-cond*)

lemma *RD-alt-def*: $RD(P) = RD1(RD2(\mathbf{R}(P)))$
by (*simp add*: *R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute RH-def RD-def RP-def*)

lemma *RD1-RH-commute*: $RD1(\mathbf{R}(P)) = \mathbf{R}(RD1(P))$
by (*simp add*: *RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RH-def*)

lemma *RD2-RH-commute*: $RD2(\mathbf{R}(P)) = \mathbf{R}(RD2(P))$
by (*metis R1-H2-commute R2c-H2-commute RD2-R3c-commute RD2-def RH-def*)

lemma *RD-idem*: $RD(RD(P)) = RD(P)$
by (*simp add*: *RD-alt-def RD1-RH-commute RD2-RH-commute RD1-RD2-commute RD2-idem RD1-idem RH-idem*)

lemma *RD-Monotonic*: *Monotonic RD*
by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic RD-comp RP-Monotonic*)

lemma *RD-Continuous*: *Continuous RD*
by (*simp add*: *Continuous-comp RD1-Continuous RD2-Continuous RD-comp RP-Continuous*)

lemma *R3-RD-RP*: $R3(RD(P)) = RP(RD1(RD2(P)))$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R2-R3-commute R3-cancels-R3c RD1-RH-commute RD2-RH-commute RD-alt-def RH-def RP-def*)

lemma *RD1-RHS-commute*: $RD1(\mathbf{R}_s(P)) = \mathbf{R}_s(RD1(P))$
by (*simp add*: *RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RHS-def*)

lemma *RD2-RHS-commute*: $RD2(\mathbf{R}_s(P)) = \mathbf{R}_s(RD2(P))$
by (*metis R1-H2-commute R2c-H2-commute RD2-R3h-commute RD2-def RHS-def*)

lemma *SRD-idem*: $SRD(SRD(P)) = SRD(P)$
by (*simp add*: *RD1-RD2-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RHS-idem SRD-def*)

lemma *SRD-Idempotent* [*closure*]: *Idempotent SRD*
by (*simp add*: *Idempotent-def SRD-idem*)

lemma *SRD-Monotonic*: *Monotonic SRD*
by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic RHS-Monotonic SRD-comp*)

lemma *SRD-Continuous* [*closure*]: *Continuous SRD*
by (*simp add*: *Continuous-comp RD1-Continuous RD2-Continuous RHS-Continuous SRD-comp*)

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*rel-auto*)

lemma *SRD-healths* [*closure*]:
assumes *P is SRD*
shows *P is R1 P is R2 P is R3h P is RD1 P is RD2*
apply (*metis Healthy-def R1-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def R1-R2c-is-R2 R2-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def R1-R3h-commute R2c-R3h-commute R3h-idem RD1-R3h-commute RD2-R3h-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def′ RD1-idem SRD-def assms*)
apply (*metis Healthy-def′ RD1-RD2-commute RD2-idem SRD-def assms*)
done

lemma *SRD-intro*:
assumes *P is R1 P is R2 P is R3h P is RD1 P is RD2*
shows *P is SRD*
by (*metis Healthy-def R1-R2c-is-R2 RHS-def SRD-def assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*))

lemma *SRD-ok-false* [*usubst*]: P *is SRD* $\Longrightarrow P[\![false/\$ok]\!] = R1(true)$
by (*metis* (*no-types*, *hide-lams*) *H1-H2-eq-design Healthy-def R1-ok-false RD1-R1-commute RD1-via-R1 RD2-def SRD-def SRD-healths*(*1*) *design-ok-false*)

lemma *SRD-ok-true-wait-true* [*usubst*]:
assumes *P is SRD*
shows $P[\![true,true/\$ok,\$wait]\!] = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
proof −
have $P = (\exists\ \$st \cdot II) \triangleleft \$ok \triangleright R1\ true \triangleleft \$wait \triangleright P$
by (*metis Healthy-def R3h-cases SRD-healths*(*3*) *assms*)
moreover have $((\exists\ \$st \cdot II) \triangleleft \$ok \triangleright R1\ true \triangleleft \$wait \triangleright P)[\![true,true/\$ok,\$wait]\!] = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *usubst*)
ultimately show *?thesis*
by (*simp*)
qed

lemma *SRD-left-zero-1*: P *is SRD* $\Longrightarrow R1(true)\ ;;\ P = R1(true)$
by (*simp add*: *RD1-left-zero SRD-healths*(*1*) *SRD-healths*(*4*))

lemma *SRD-left-zero-2*:
assumes *P is SRD*
shows $(\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]\ ;;\ P = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
proof −
have $(\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]\ ;;\ R3h(P) = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if SRD-healths*(*3*) *assms*)
qed

2.7 UTP theories

We create two theory objects: one for reactive designs and one for stateful reactive designs.

typedecl *RDES*
typedecl *SRDES*

abbreviation $RDES \equiv UTHY(RDES, ('t::trace,'\alpha)\ rp)$
abbreviation $SRDES \equiv UTHY(SRDES, ('s,'t::trace,'\alpha)\ rsp)$

overloading
  rdes-hcond == *utp-hcond* :: $(RDES, ('t::trace,'\alpha)\ rp)\ uthy \Rightarrow (('t,'\alpha)\ rp \times ('t,'\alpha)\ rp)\ health$
  srdes-hcond == *utp-hcond* :: $(SRDES, ('s,'t::trace,'\alpha)\ rsp)\ uthy \Rightarrow (('s,'t,'\alpha)\ rsp \times ('s,'t,'\alpha)\ rsp)$ *health*
begin
  definition *rdes-hcond* :: $(RDES, ('t::trace,'\alpha)\ rp)\ uthy \Rightarrow (('t,'\alpha)\ rp \times ('t,'\alpha)\ rp)\ health$ **where**
  [*upred-defs*]: *rdes-hcond* $T = RD$
  definition *srdes-hcond* :: $(SRDES, ('s,'t::trace,'\alpha)\ rsp)\ uthy \Rightarrow (('s,'t,'\alpha)\ rsp \times ('s,'t,'\alpha)\ rsp)\ health$ **where**
  [*upred-defs*]: *srdes-hcond* $T = SRD$
end

interpretation *rdes-theory*: *utp-theory* $UTHY(RDES, ('t::trace,'\alpha)\ rp)$
  by (*unfold-locales*, *simp-all add*: *rdes-hcond-def RD-idem*)

interpretation *rdes-theory-continuous*: *utp-theory-continuous* $UTHY(RDES, ('t::trace,'\alpha)\ rp)$
  rewrites $\bigwedge P.\ P \in carrier\ (uthy\text{-}order\ RDES) \longleftrightarrow P\ is\ RD$
  and $carrier\ (uthy\text{-}order\ RDES) \to carrier\ (uthy\text{-}order\ RDES) \equiv [\![RD]\!]_H \to [\![RD]\!]_H$
  and $le\ (uthy\text{-}order\ RDES) = op \sqsubseteq$
  and $eq\ (uthy\text{-}order\ RDES) = op =$
  by (*unfold-locales*, *simp-all add*: *rdes-hcond-def RD-Continuous*)

interpretation *rdes-rea-galois*:
  galois-connection $(RDES \leftarrow\langle RD1 \circ RD2, R3\rangle\rightarrow REA)$
proof (*simp add*: *mk-conn-def*, *rule galois-connectionI'*, *simp-all add*: *utp-partial-order rdes-hcond-def rea-hcond-def*)
  show $R3 \in [\![RD]\!]_H \to [\![RP]\!]_H$
    by (*metis* (*no-types*, *lifting*) *Healthy-def' Pi-I R3-RD-RP RP-idem mem-Collect-eq*)
  show $RD1 \circ RD2 \in [\![RP]\!]_H \to [\![RD]\!]_H$
    by (*simp add*: *Pi-iff Healthy-def*, *metis RD-def RD-idem*)
  show *isotone* (*utp-order RD*) (*utp-order RP*) *R3*
    by (*simp add*: *R3-Monotonic isotone-utp-orderI*)
  show *isotone* (*utp-order RP*) (*utp-order RD*) $(RD1 \circ RD2)$
    by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic isotone-utp-orderI*)
  fix P :: $('a, 'b)$ *hrel-rp*
  assume P *is RD*
  thus $P \sqsubseteq RD1\ (RD2\ (R3\ P))$
    by (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-iff*)
next
  fix P :: $('a, 'b)$ *hrel-rp*
  assume *a*: P *is RP*
  thus $R3\ (RD1\ (RD2\ P)) \sqsubseteq P$
  proof −
    have $R3\ (RD1\ (RD2\ P)) = RP\ (RD1\ (RD2(P)))$
      by (*metis Healthy-if R3-RD-RP RD-def a*)
    moreover have $RD1(RD2(P)) \sqsubseteq P$
      by (*rel-auto*)
    ultimately show *?thesis*
      by (*metis Healthy-if RP-mono a*)
  qed
qed

interpretation *rdes-rea-retract*:
  retract $(RDES \leftarrow\langle RD1 \circ RD2, R3\rangle\rightarrow REA)$
  by (*unfold-locales*, *simp-all add*: *mk-conn-def utp-partial-order rdes-hcond-def rea-hcond-def*)
    (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-refl*)

interpretation *srdes-theory*: *utp-theory UTHY*(*SRDES*, ('*s*,'*t*::*trace*,'α) *rsp*)
  by (*unfold-locales*, *simp-all add*: *srdes-hcond-def SRD-idem*)

interpretation *srdes-theory-continuous*: *utp-theory-continuous UTHY*(*SRDES*, ('*s*,'*t*::*trace*,'α) *rsp*)
  rewrites $\bigwedge P.\ P \in carrier\ (uthy\text{-}order\ SRDES) \longleftrightarrow P\ is\ SRD$
  and $P\ is\ \mathcal{H}_{SRDES} \longleftrightarrow P\ is\ SRD$
  and $(\mu\ X \cdot F\ (\mathcal{H}_{SRDES}\ X)) = (\mu\ X \cdot F\ (SRD\ X))$
  and $carrier\ (uthy\text{-}order\ SRDES) \rightarrow carrier\ (uthy\text{-}order\ SRDES) \equiv [\![SRD]\!]_H \rightarrow [\![SRD]\!]_H$
  and $[\![\mathcal{H}_{SRDES}]\!]_H \rightarrow [\![\mathcal{H}_{SRDES}]\!]_H \equiv [\![SRD]\!]_H \rightarrow [\![SRD]\!]_H$
  and $le\ (uthy\text{-}order\ SRDES) = op\ \sqsubseteq$
  and $eq\ (uthy\text{-}order\ SRDES) = op\ =$
  by (*unfold-locales*, *simp-all add*: *srdes-hcond-def SRD-Continuous*)

declare *srdes-theory-continuous.top-healthy* [*simp del*]
declare *srdes-theory-continuous.bottom-healthy* [*simp del*]

abbreviation *Chaos* :: ('*s*,'*t*::*trace*,'α) *hrel-rsp* **where**
$Chaos \equiv \bot_{SRDES}$

abbreviation *Miracle* :: ('*s*,'*t*::*trace*,'α) *hrel-rsp* **where**
$Miracle \equiv \top_{SRDES}$

thm *srdes-theory-continuous.weak.bottom-lower*
thm *srdes-theory-continuous.weak.top-higher*
thm *srdes-theory-continuous.meet-bottom*
thm *srdes-theory-continuous.meet-top*

abbreviation *srd-lfp* (μ_R) **where** $\mu_R\ F \equiv \boldsymbol{\mu}_{SRDES}\ F$

abbreviation *srd-gfp* (ν_R) **where** $\nu_R\ F \equiv \boldsymbol{\nu}_{SRDES}\ F$

syntax
  -srd-mu :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $(\mu_R\ \text{-} \cdot \text{-}\ [0,\ 10]\ 10)$
  -srd-nu :: *pttrn* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $(\nu_R\ \text{-} \cdot \text{-}\ [0,\ 10]\ 10)$

translations
  $\mu_R\ X \cdot P == \mu_R\ (\lambda\ X.\ P)$
  $\nu_R\ X \cdot P == \mu_R\ (\lambda\ X.\ P)$

The reactive design weakest fixed-point can be defined in terms of relational calculus one.

lemma *srd-mu-equiv*:
  assumes $Monotonic\ F\ F \in [\![SRD]\!]_H \rightarrow [\![SRD]\!]_H$
  shows $(\mu_R\ X \cdot F(X)) = (\mu\ X \cdot F(SRD(X)))$
  by (*metis assms srdes-hcond-def srdes-theory-continuous.utp-lfp-def*)

end

3 Reactive Design Specifications

theory *utp-rdes-designs*
  imports *utp-rdes-healths*
begin

3.1 Reactive design forms

lemma *srdes-skip-def*: $II_R = \mathbf{R}_s(true \vdash (\$tr\acute{} =_u \$tr \land \neg \$wait\acute{} \land \lceil II \rceil_R))$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *Chaos-def*: $Chaos = \mathbf{R}_s(false \vdash true)$
proof −
  have $Chaos = SRD(true)$
    by (*metis srdes-hcond-def srdes-theory-continuous.healthy-bottom*)
  also have ... $= \mathbf{R}_s(\mathbf{H}(true))$
    by (*simp add*: *SRD-RHS-H1-H2*)
  also have ... $= \mathbf{R}_s(false \vdash true)$
    by (*metis H1-design H2-true design-false-pre*)
  finally show *?thesis* **.**
qed

lemma *Miracle-def*: $Miracle = \mathbf{R}_s(true \vdash false)$
proof −
  have $Miracle = SRD(false)$
    by (*metis srdes-hcond-def srdes-theory-continuous.healthy-top*)
  also have ... $= \mathbf{R}_s(\mathbf{H}(false))$
    by (*simp add*: *SRD-RHS-H1-H2*)
  also have ... $= \mathbf{R}_s(true \vdash false)$
    by (*metis* (*no-types*, *lifting*) *H1-H2-eq-design p-imp-p subst-impl subst-not utp-pred-laws.compl-bot-eq utp-pred-laws.compl-top-eq*)
  finally show *?thesis* **.**
qed

lemma *RD1-reactive-design*: $RD1(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
  by (*rel-auto*)

lemma *RD2-reactive-design*:
  assumes $\$ok\acute{} \sharp P$ $\$ok\acute{} \sharp Q$
  shows $RD2(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
  using *assms*
  by (*metis H2-design RD2-RH-commute RD2-def*)

lemma *RD1-st-reactive-design*: $RD1(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
  by (*rel-auto*)

lemma *RD2-st-reactive-design*:
  assumes $\$ok\acute{} \sharp P$ $\$ok\acute{} \sharp Q$
  shows $RD2(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
  using *assms*
  by (*metis H2-design RD2-RHS-commute RD2-def*)

lemma *wait-false-design*:
  $(P \vdash Q)_{\ f} = ((P_{\ f}) \vdash (Q_{\ f}))$
  by (*rel-auto*)

lemma *RD-RH-design-form*:
  $RD(P) = \mathbf{R}((\neg\ P^f{}_f) \vdash P^t{}_f)$
proof −
  have $RD(P) = RD1(RD2(R1(R2c(R3c(P)))))$
    by (*simp add*: *RD-alt-def RH-def*)
  also have ... $= RD1(H2(R1(R2s(R3c(P)))))$
    by (*simp add*: *R1-R2s-R2c RD2-def*)
  also have ... $= RD1(R1(H2(R2s(R3c(P)))))$
    by (*simp add*: *R1-H2-commute*)
  also have ... $= R1(H1(R1(H2(R2s(R3c(P))))))$
    by (*simp add*: *R1-idem RD1-via-R1*)
  also have ... $= R1(H1(H2(R2s(R3c(R1(P))))))$
    by (*simp add*: *R1-H2-commute R1-R2c-commute R1-R2s-R2c R1-R3c-commute RD1-via-R1*)
  also have ... $= R1(R2s(H1(H2(R3c(R1(P))))))$
    by (*simp add*: *R2s-H1-commute R2s-H2-commute*)
  also have ... $= R1(R2s(H1(R3c(H2(R1(P))))))$
    by (*metis RD2-R3c-commute RD2-def*)
  also have ... $= R2(R1(H1(R3c(H2(R1(P))))))$
    by (*metis R1-R2-commute R1-idem R2-def*)
  also have ... $= R2(R3c(R1(\mathbf{H}(R1(P)))))$
    by (*simp add*: *R1-R3c-commute RD1-R3c-commute RD1-via-R1*)
  also have ... $= RH(\mathbf{H}(R1(P)))$
    by (*metis R1-R2s-R2c R1-R3c-commute R2-R1-form RH-def*)
  also have ... $= RH(\mathbf{H}(P))$
    by (*simp add*: *R1-H2-commute R1-R2c-commute R1-R3c-commute R1-idem RD1-via-R1 RH-def*)
  also have ... $= RH((\neg\ P^f) \vdash P^t)$
    by (*simp add*: *H1-H2-eq-design*)
  also have ... $= \mathbf{R}((\neg\ P^f{}_f) \vdash P^t{}_f)$
    by (*metis* (*no-types*, *lifting*) *R3c-subst-wait RH-def subst-not wait-false-design*)
  finally show *?thesis* **.**
qed

lemma *RD-reactive-design*:
  assumes *P is RD*
  shows $\mathbf{R}((\neg\ P^f{}_f) \vdash P^t{}_f) = P$
  by (*metis RD-RH-design-form Healthy-def′ assms*)

lemma *RD-RH-design*:
  assumes $\$ok\acute{}\ \sharp\ P$ $\$ok\acute{}\ \sharp\ Q$
  shows $RD(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
  by (*simp add*: *RD1-reactive-design RD2-reactive-design RD-alt-def RH-idem assms*(*1*) *assms*(*2*))

lemma *RH-design-is-RD*:
  assumes $\$ok\acute{}\ \sharp\ P$ $\$ok\acute{}\ \sharp\ Q$
  shows $\mathbf{R}(P \vdash Q)$ *is RD*
  by (*simp add*: *RD-RH-design Healthy-def′ assms*(*1*) *assms*(*2*))

lemma *SRD-RH-design-form*:
  $SRD(P) = \mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f)$
proof −
  have $SRD(P) = R1(R2c(R3h(RD1(RD2(R1(P))))))$
   by (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-commute R1-R3h-commute R1-idem R2c-H2-commute RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RD2-R3h-commute RD2-def RHS-def SRD-def*)
  also have ... $= R1(R2s(R3h(\mathbf{H}(P))))$
   by (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-is-R2 R1-R3h-commute R2-R1-form RD1-via-R1*

RD2-def)
also have ... = $\mathbf{R}_s(\mathbf{H}(P))$
by (*simp add*: *R1-R2s-R2c RHS-def*)
also have ... = $\mathbf{R}_s((\neg\ P^f) \vdash P^t)$
by (*simp add*: *H1-H2-eq-design*)
also have ... = $\mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f)$
by (*metis* (*no-types*, *lifting*) *R3h-subst-wait RHS-def subst-not wait-false-design*)
finally show *?thesis* **.**
qed

lemma *SRD-reactive-design*:
assumes *P is SRD*
shows $\mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f) = P$
by (*metis SRD-RH-design-form Healthy-def′ assms*)

lemma *SRD-RH-design*:
assumes $\$ok´ \sharp P$ $\$ok´ \sharp Q$
shows $SRD(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
by (*simp add*: *RD1-st-reactive-design RD2-st-reactive-design RHS-idem SRD-def assms*(*1*) *assms*(*2*))

lemma *RHS-design-is-SRD*:
assumes $\$ok´ \sharp P$ $\$ok´ \sharp Q$
shows $\mathbf{R}_s(P \vdash Q)$ *is SRD*
by (*simp add*: *Healthy-def′ SRD-RH-design assms*(*1*) *assms*(*2*))

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*metis* (*no-types*, *lifting*) *H1-H2-eq-design R3h-subst-wait RHS-def SRD-RH-design-form subst-not wait-false-design*)

3.2 Auxiliary healthiness conditions

definition [*upred-defs*]: *R3c-pre*$(P) = (true \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: *R3c-post*$(P) = (\lceil II \rceil_D \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: *R3h-post*$(P) = ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright P)$

lemma *R3c-pre-conj*: *R3c-pre*$(P \wedge Q)$ = (*R3c-pre*$(P) \wedge$ *R3c-pre*(Q))
by (*rel-auto*)

lemma *R3c-pre-seq*:
$(true\ ;;\ Q) = true \Longrightarrow$ *R3c-pre*$(P\ ;;\ Q)$ = (*R3c-pre*$(P)\ ;;\ Q$)
by (*rel-auto*)

lemma *unrest-ok-R3c-pre* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp$ *R3c-pre*(P)
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok′-R3c-pre* [*unrest*]: $\$ok´ \sharp P \Longrightarrow \$ok´ \sharp$ *R3c-pre*(P)
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok-R3c-post* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp$ *R3c-post*(P)
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3c-post′* [*unrest*]: $\$ok´ \sharp P \Longrightarrow \$ok´ \sharp$ *R3c-post*(P)
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3h\text{-}post(P)$
by (*simp add*: *R3h-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post'* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3h\text{-}post(P)$
by (*simp add*: *R3h-post-def cond-def unrest*)

3.3 Composition laws

theorem *R1-design-composition*:
fixes $P\ Q$ :: ('t::trace,'α,'β) *rel-rp*
and $R\ S$ :: ('t,'β,'γ) *rel-rp*
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) =$
$R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$
proof −
have $(R1(P \vdash Q) ;; R1(R \vdash S)) = (\exists\ ok_0 \cdot (R1(P \vdash Q))[\![\ll ok_0 \gg/\$ok']\!] ;; (R1(R \vdash S))[\![\ll ok_0 \gg/\$ok]\!])$
using *seqr-middle ok-vwb-lens* **by** *blast*
also from *assms* **have** ... $= (\exists\ ok_0 \cdot R1((\$ok \wedge P) \Rightarrow (\ll ok_0 \gg \wedge Q)) ;; R1((\ll ok_0 \gg \wedge R) \Rightarrow (\$ok' \wedge S)))$
by (*simp add*: *design-def R1-def usubst unrest*)
also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) ;; R1((true \wedge R) \Rightarrow (\$ok' \wedge S)))$
$\vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) ;; R1((false \wedge R) \Rightarrow (\$ok' \wedge S))))$
by (*simp add*: *false-alt-def true-alt-def*)
also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow Q) ;; R1(R \Rightarrow (\$ok' \wedge S)))$
$\vee (R1(\neg (\$ok \wedge P)) ;; R1(true)))$
by *simp*
also from *assms* **have** ... $= ((R1(\neg \$ok \vee \neg P \vee Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$
by (*simp add*: *impl-alt-def utp-pred-laws.sup.assoc*)
also from *assms* **have** ... $= (((R1(\neg \$ok \vee \neg P) \vee R1(Q)) ;; R1(\neg R \vee (\$ok' \wedge S)))$
$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$
by (*simp add*: *R1-disj utp-pred-laws.disj-assoc*)
also from *assms* **have** ... $= ((R1(\neg \$ok \vee \neg P) ;; R1(\neg R \vee (\$ok' \wedge S)))$
$\vee (R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$
by (*simp add*: *seqr-or-distl utp-pred-laws.sup.assoc*)
also from *assms* **have** ... $= ((R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$
by (*rel-blast*)
also from *assms* **have** ... $= ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$
by (*simp add*: *R1-disj R1-extend-conj utp-pred-laws.inf-commute*)
also have ... $= ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
$\vee ((R1(\neg \$ok)$ :: ('t,'α,'β) *rel-rp*) $;; R1(true))$
$\vee (R1(\neg P) ;; R1(true)))$
by (*simp add*: *R1-disj seqr-or-distl*)
also have ... $= ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
$\vee (R1(\neg \$ok))$
$\vee (R1(\neg P) ;; R1(true)))$
proof −
have $((R1(\neg \$ok)$ :: ('t,'α,'β) *rel-rp*) $;; R1(true)) =$
$(R1(\neg \$ok)$ :: ('t,'α,'γ) *rel-rp*)
by (*rel-auto*)
thus *?thesis*
by *simp*

qed
also have ... = $((R1(Q) \;;\; (R1(\neg\ R) \lor (R1(S \land \$ok\acute{}))))$
$\lor\ R1(\neg\ \$ok)$
$\lor\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-extend-conj*)
also have ... = $(\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\lor\ (R1(Q) \;;\; (R1(S \land \$ok\acute{})))$
$\lor\ R1(\neg\ \$ok)$
$\lor\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *seqr-or-distr utp-pred-laws.sup.assoc*)
also have ... = $R1(\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\lor\ (R1(Q) \;;\; (R1(S \land \$ok\acute{})))$
$\lor\ (\neg\ \$ok)$
$\lor\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-disj R1-seqr*)
also have ... = $R1(\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\lor\ ((R1(Q) \;;\; R1(S)) \land \$ok\acute{})$
$\lor\ (\neg\ \$ok)$
$\lor\ (R1(\neg\ P) \;;\; R1(true)))$
by (*rel-blast*)
also have ... = $R1(\neg(\$ok \land \neg\ (R1(\neg\ P) \;;\; R1(true)) \land \neg\ (R1(Q) \;;\; (R1\ (\neg\ R))))$
$\lor\ \ ((R1(Q) \;;\; R1(S)) \land \$ok\acute{}))$
by (*rel-blast*)
also have ... = $R1((\$ok \land \neg\ (R1(\neg\ P) \;;\; R1(true)) \land \neg\ (R1(Q) \;;\; (R1\ (\neg\ R))))$
$\Rightarrow (\$ok\acute{} \land (R1(Q) \;;\; R1(S))))$
by (*simp add*: *impl-alt-def utp-pred-laws.inf-commute*)
also have ... = $R1((\neg\ (R1(\neg\ P) \;;\; R1(true)) \land \neg\ (R1(Q) \;;\; R1(\neg\ R))) \vdash (R1(Q) \;;\; R1(S)))$
by (*simp add*: *design-def*)
finally show *?thesis* **.**
qed

theorem *R1-design-composition-RR*:
assumes *P is RR Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) \;;\; R1(R \vdash S)) = R1(((\neg_r\ P)\ wp_r\ false \land Q\ wp_r\ R) \vdash (Q \;;\; S))$
apply (*subst R1-design-composition*)
apply (*simp-all add*: *assms unrest wp-rea-def Healthy-if closure*)
apply (*rel-auto*)
done

theorem *R1-design-composition-RC*:
assumes *P is RC Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) \;;\; R1(R \vdash S)) = R1((P \land Q\ wp_r\ R) \vdash (Q \;;\; S))$
by (*simp add*: *R1-design-composition-RR assms unrest Healthy-if closure wp*)

lemma *R2s-design*: $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$
by (*simp add*: *R2s-def design-def usubst*)

lemma *R2c-design*: $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$
by (*simp add*: *design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok'*)

lemma *R1-R3c-design*:
$R1(R3c(P \vdash Q)) = R1(R3c\text{-}pre(P) \vdash R3c\text{-}post(Q))$
by (*rel-auto*)

lemma *R1-R3h-design*:
$R1(R3h(P \vdash Q)) = R1(R3c\text{-}pre(P) \vdash R3h\text{-}post(Q))$
by (*rel-auto*)

lemma *R3c-R1-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(R3c(R1(P \vdash Q)) ;; R3c(R1(R \vdash S))) =$
$R3c(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$
$\vdash (R1(Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1(S)))))$
proof −
have *1*:$(\neg (R1 (\neg R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre (\neg (R1 (\neg P) ;; R1\ true)))$
by (*rel-auto*)
have *2*:$(\neg (R1 (R3c\text{-}post\ Q) ;; R1 (\neg R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg ((R1\ Q \wedge \neg \$wait') ;; R1 (\neg R)))$
by (*rel-auto*, *blast+*)
have *3*:$(R1 (R3c\text{-}post\ Q) ;; R1 (R3c\text{-}post\ S)) = R3c\text{-}post (R1\ Q ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ S))$
by (*rel-auto*)
show *?thesis*
apply (*simp add*: *R3c-semir-form R1-R3c-commute*[*THEN sym*] *R1-R3c-design unrest*)
apply (*subst R1-design-composition*)
apply (*simp-all add*: *unrest assms R3c-pre-conj 1 2 3*)
done
qed

lemma *R3h-R1-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(R3h(R1(P \vdash Q)) ;; R3h(R1(R \vdash S))) =$
$R3h(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$
$\vdash (R1(Q) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(S)))))$
proof −
have *1*:$(\neg (R1 (\neg R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre (\neg (R1 (\neg P) ;; R1\ true)))$
by (*rel-auto*)
have *2*:$(\neg (R1 (R3h\text{-}post\ Q) ;; R1 (\neg R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg ((R1\ Q \wedge \neg \$wait') ;; R1 (\neg R)))$
by (*rel-auto*, *blast+*)
have *3*:$(R1 (R3h\text{-}post\ Q) ;; R1 (R3h\text{-}post\ S)) = R3h\text{-}post (R1\ Q ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ S))$
by (*rel-auto*, *blast+*)
show *?thesis*
apply (*simp add*: *R3h-semir-form R1-R3h-commute*[*THEN sym*] *R1-R3h-design unrest*)
apply (*subst R1-design-composition*)
apply (*simp-all add*: *unrest assms R3c-pre-conj 1 2 3*)
done
qed

lemma *R2-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(R2(P \vdash Q) ;; R2(R \vdash S)) =$
$R2((\neg (R1 (\neg R2c\ P) ;; R1\ true) \wedge \neg (R1 (R2c\ Q) ;; R1 (\neg R2c\ R))) \vdash (R1 (R2c\ Q) ;; R1 (R2c\ S)))$
apply (*simp add*: *R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj R1-R2c-commute*[*THEN sym*] *R2c-idem R2c-R1-seq*)
apply (*metis* (*no-types*, *lifting*) *R2c-R1-seq R2c-not R2c-true*)
done

lemma *RH-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$

shows $(RH(P \vdash Q) \;;\; RH(R \vdash S)) =$
$RH((\neg\ (R1\ (\neg\ R2s\ P) \;;\; R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge (\neg\ \$wait')) \;;\; R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q) \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))))$
proof $-$
have *1*: $R2c\ (R1\ (\neg\ R2s\ P) \;;\; R1\ true) = (R1\ (\neg\ R2s\ P) \;;\; R1\ true)$
proof $-$
have *1*:$(R1\ (\neg\ R2s\ P) \;;\; R1\ true) = (R1(R2\ (\neg\ P) \;;\; R2\ true))$
by (*rel-auto*)
have $R2c(R1(R2\ (\neg\ P) \;;\; R2\ true)) = R2c(R1(R2\ (\neg\ P) \;;\; R2\ true))$
using *R2c-not* **by** *blast*
also have ... $= R2(R2\ (\neg\ P) \;;\; R2\ true)$
by (*metis R1-R2c-commute R1-R2c-is-R2*)
also have ... $= (R2\ (\neg\ P) \;;\; R2\ true)$
by (*simp add*: *R2-seqr-distribute*)
also have ... $= (R1\ (\neg\ R2s\ P) \;;\; R1\ true)$
by (*simp add*: *R2-def R2s-not R2s-true*)
finally show *?thesis*
by (*simp add*: *1*)
qed

have *2*:$R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') \;;\; R1\ (\neg\ R2s\ R)) = ((R1\ (R2s\ Q) \wedge \neg\ \$wait') \;;\; R1\ (\neg\ R2s\ R))$
proof $-$
have $((R1\ (R2s\ Q) \wedge \neg\ \$wait') \;;\; R1\ (\neg\ R2s\ R)) = R1\ (R2\ (Q \wedge \neg\ \$wait') \;;\; R2\ (\neg\ R))$
by (*rel-auto*)
hence $R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') \;;\; R1\ (\neg\ R2s\ R)) = (R2\ (Q \wedge \neg\ \$wait') \;;\; R2\ (\neg\ R))$
by (*metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)
also have ... $= ((R1\ (R2s\ Q) \wedge \neg\ \$wait') \;;\; R1\ (\neg\ R2s\ R))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

have *3*:$R2c((R1\ (R2s\ Q) \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))) = (R1\ (R2s\ Q) \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))$
proof $-$
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![true/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
proof $-$
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])) =$
$R2c(R1\ (R2s\ (Q[\![true/\$wait']\!])) \;;\; \lceil II \rceil_D[\![true/\$wait]\!])$
by (*simp add*: *usubst cond-unit-T R1-def R2s-def*)
also have ... $= R2c(R2(Q[\![true/\$wait']\!]) \;;\; R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
by (*metis R2-def R2-des-lift-skip R2-subst-wait-true*)
also have ... $= (R2(Q[\![true/\$wait']\!]) \;;\; R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
using *R2c-seq* **by** *blast*
also have ... $= ((R1\ (R2s\ Q))[\![true/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
apply (*simp add*: *usubst R2-des-lift-skip*)
apply (*metis R2-def R2-des-lift-skip R2-subst-wait'-true R2-subst-wait-true*)
done
finally show *?thesis* **.**
qed
moreover have $R2c(((R1\ (R2s\ Q))[\![false/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![false/\$wait']\!] \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!])$
by (*simp add*: *usubst cond-unit-F*)
(*metis* (*no-types*, *hide-lams*) *R1-wait'-false R1-wait-false R2-def R2-subst-wait'-false R2-subst-wait-false*

R2c-seq)
  ultimately show *?thesis*
  proof –
   have $\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S) = R2\ (\lceil II \rceil_D \triangleleft \$wait \triangleright S)$
    by (*simp add*: *R1-R2c-is-R2 R1-R2s-R2c R2-condr′ R2-des-lift-skip R2s-wait*)
   then show *?thesis*
    by (*simp add*: *R1-R2c-is-R2 R1-R2s-R2c R2c-seq*)
  qed
 qed

 have $(R1(R2s(R3c(P \vdash Q))) ;; R1(R2s(R3c(R \vdash S)))) =$
   $((R3c(R1(R2s(P) \vdash R2s(Q)))) ;; R3c(R1(R2s(R) \vdash R2s(S))))$
  by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design*)
 also have ... $= R3c\ (R1\ ((\neg\ (R1\ (\neg\ R2s\ P) ;; R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R))) \vdash$
      $(R1\ (R2s\ Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))))$
  by (*simp add*: *R3c-R1-design-composition assms unrest*)
 also have ... $= R3c(R1(R2c((\neg\ (R1\ (\neg\ R2s\ P) ;; R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R))) \vdash$
       $(R1\ (R2s\ Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))))))$
  by (*simp add*: *R2c-design R2c-and R2c-not 1 2 3*)
 finally show *?thesis*
  by (*simp add*: *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-def*)
qed

lemma *RHS-design-composition*:
 assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
 shows $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$
   $\mathbf{R}_s((\neg\ (R1\ (\neg\ R2s\ P) ;; R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge (\neg\ \$wait')) ;; R1\ (\neg\ R2s\ R))) \vdash$
      $(R1\ (R2s\ Q) ;; ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))))$
proof –
 have *1*: $R2c\ (R1\ (\neg\ R2s\ P) ;; R1\ true) = (R1\ (\neg\ R2s\ P) ;; R1\ true)$
 proof –
  have *1*:$(R1\ (\neg\ R2s\ P) ;; R1\ true) = (R1(R2\ (\neg\ P) ;; R2\ true))$
   by (*rel-auto*, *blast*)
  have $R2c(R1(R2\ (\neg\ P) ;; R2\ true)) = R2c(R1(R2\ (\neg\ P) ;; R2\ true))$
   using *R2c-not* **by** *blast*
  also have ... $= R2(R2\ (\neg\ P) ;; R2\ true)$
   by (*metis R1-R2c-commute R1-R2c-is-R2*)
  also have ... $= (R2\ (\neg\ P) ;; R2\ true)$
   by (*simp add*: *R2-seqr-distribute*)
  also have ... $= (R1\ (\neg\ R2s\ P) ;; R1\ true)$
   by (*simp add*: *R2-def R2s-not R2s-true*)
  finally show *?thesis*
   by (*simp add*: *1*)
 qed

 have *2*:$R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R)) = ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R))$
 proof –
  have $((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R)) = R1\ (R2\ (Q \wedge \neg\ \$wait') ;; R2\ (\neg\ R))$
   by (*rel-auto*, *blast+*)
  hence $R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R)) = (R2\ (Q \wedge \neg\ \$wait') ;; R2\ (\neg\ R))$
   by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)
  also have ... $= ((R1\ (R2s\ Q) \wedge \neg\ \$wait') ;; R1\ (\neg\ R2s\ R))$

by (*rel-auto*, *blast+*)
finally show *?thesis* **.**
qed

have $3{:}R2c((R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))) =$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])) =$
$R2c(R1\ (R2s\ (Q[\![true/\$wait']\!]))\ ;;\ (\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!])$
by (*simp add*: *usubst cond-unit-T R1-def R2s-def*)
also have ... $= R2c(R2(Q[\![true/\$wait']\!])\ ;;\ R2((\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!]))$
by (*metis* (*no-types*, *lifting*) *R2-def R2-des-lift-skip R2-subst-wait-true R2-st-ex*)
also have ... $= (R2(Q[\![true/\$wait']\!])\ ;;\ R2((\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!]))$
using *R2c-seq* **by** *blast*
also have ... $= ((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
apply (*simp add*: *usubst R2-des-lift-skip*)
apply (*metis* (*no-types*) *R2-def R2-des-lift-skip R2-st-ex R2-subst-wait′-true R2-subst-wait-true*)
done
finally show *?thesis* **.**
qed
moreover have $R2c(((R1\ (R2s\ Q))[\![false/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![false/\$wait']\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!])$
by (*simp add*: *usubst*)
(*metis* (*no-types*, *lifting*) *R1-wait′-false R1-wait-false R2-R1-form R2-subst-wait′-false R2-subst-wait-false R2c-seq*)
ultimately show *?thesis*
by (*smt R2-R1-form R2-condr′ R2-des-lift-skip R2-st-ex R2c-seq R2s-wait*)
qed

have $(R1(R2s(R3h(P \vdash Q)))\ ;;\ R1(R2s(R3h(R \vdash S)))) =$
$((R3h(R1(R2s(P) \vdash R2s(Q))))\ ;;\ R3h(R1(R2s(R) \vdash R2s(S))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-R3h-commute R2c-R3h-commute R2s-design*)
also have ... $= R3h\ (R1\ ((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))))$
by (*simp add*: *R3h-R1-design-composition assms unrest*)
also have ... $= R3h(R1(R2c((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))))))$
by (*simp add*: *R2c-design R2c-and R2c-not 1 2 3*)
finally show *?thesis*
by (*simp add*: *R1-R2s-R2c R1-R3h-commute R2c-R3h-commute RHS-def*)
qed

lemma *RHS-R2s-design-composition*:
assumes
$\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
P is R2s Q is R2s R is R2s S is R2s
shows $(\mathbf{R}_s(P \vdash Q)\ ;;\ \mathbf{R}_s(R \vdash S)) =$
$\mathbf{R}_s((\neg\ (R1\ (\neg\ P)\ ;;\ R1\ true) \wedge \neg\ ((R1\ Q \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R))) \vdash$
$(R1\ Q\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ S)))$
proof −

have *f1*: $R2s\ P = P$
by (*meson Healthy-def assms*(*5*))
have *f2*: $R2s\ Q = Q$
by (*meson Healthy-def assms*(*6*))
have *f3*: $R2s\ R = R$
by (*meson Healthy-def assms*(*7*))
have $R2s\ S = S$
by (*meson Healthy-def assms*(*8*))
then show *?thesis*
using *f3 f2 f1* **by** (*simp add*: *RHS-design-composition assms*(*1*) *assms*(*2*) *assms*(*3*) *assms*(*4*))
qed

lemma *RH-design-export-R1*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R1(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2s*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2s(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2c*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2c(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R1*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R1(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2s*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2s(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2c*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2c(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2(Q))$
by (*rel-auto*)

lemma *R1-design-R1-pre*:
$\mathbf{R}_s(R1(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

lemma *RHS-design-ok-wait*: $\mathbf{R}_s(P[\![true,false/\$ok,\$wait]\!] \vdash Q[\![true,false/\$ok,\$wait]\!]) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

lemma *RHS-design-neg-R1-pre*:
$\mathbf{R}_s\ ((\neg\ R1\ P) \vdash R) = \mathbf{R}_s\ ((\neg\ P) \vdash R)$
by (*rel-auto*)

lemma *RHS-design-conj-neg-R1-pre*:
$\mathbf{R}_s\ (((\neg\ R1\ P) \wedge Q) \vdash R) = \mathbf{R}_s\ (((\neg\ P) \wedge Q) \vdash R)$
by (*rel-auto*)

lemma *RHS-pre-lemma*: $(\mathbf{R}_s\ P)^f{}_f = R1(R2c(P^f{}_f))$
by (*rel-auto*)

lemma *RHS-design-R2c-pre*:
$\mathbf{R}_s(R2c(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

3.4 Refinement introduction laws

lemma *R1-design-refine*:
assumes
P_1 *is R1* P_2 *is R1* Q_1 *is R1* Q_2 *is R1*
$\$ok \sharp P_1$ $\$ok' \sharp P_1$ $\$ok \sharp P_2$ $\$ok' \sharp P_2$
$\$ok \sharp Q_1$ $\$ok' \sharp Q_1$ $\$ok \sharp Q_2$ $\$ok' \sharp Q_2$
shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \Rightarrow P_2`$
proof –
have $R1((\exists\ \$ok;\$ok' \cdot P_1) \vdash (\exists\ \$ok;\$ok' \cdot P_2)) \sqsubseteq R1((\exists\ \$ok;\$ok' \cdot Q_1) \vdash (\exists\ \$ok;\$ok' \cdot Q_2))$
$\longleftrightarrow `R1(\exists\ \$ok;\$ok' \cdot P_1) \Rightarrow R1(\exists\ \$ok;\$ok' \cdot Q_1)` \wedge `R1(\exists\ \$ok;\$ok' \cdot P_1) \wedge R1(\exists\ \$ok;\$ok'$
$\cdot\ Q_2) \Rightarrow R1(\exists\ \$ok;\$ok' \cdot P_2)`$
by (*rel-auto*, *meson+*)
thus *?thesis*
by (*simp-all add*: *ex-unrest ex-plus Healthy-if assms*)
qed

lemma *R1-design-refine-RR*:
assumes P_1 *is RR* P_2 *is RR* Q_1 *is RR* Q_2 *is RR*
shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \Rightarrow P_2`$
by (*simp add*: *R1-design-refine assms unrest closure*)

lemma *RHS-design-refine*:
assumes
P_1 *is R1* P_2 *is R1* Q_1 *is R1* Q_2 *is R1*
P_1 *is R2c* P_2 *is R2c* Q_1 *is R2c* Q_2 *is R2c*
$\$ok \sharp P_1$ $\$ok' \sharp P_1$ $\$ok \sharp P_2$ $\$ok' \sharp P_2$
$\$ok \sharp Q_1$ $\$ok' \sharp Q_1$ $\$ok \sharp Q_2$ $\$ok' \sharp Q_2$
$\$wait \sharp P_1$ $\$wait \sharp P_2$ $\$wait \sharp Q_1$ $\$wait \sharp Q_2$
shows $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \Rightarrow P_2`$
proof –
have $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow R1(R3h(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3h(R2c(Q_1 \vdash Q_2)))$
by (*simp add*: *R2c-R3h-commute RHS-def*)
also have ... $\longleftrightarrow R1(R3h(P_1 \vdash P_2)) \sqsubseteq R1(R3h(Q_1 \vdash Q_2))$
by (*simp add*: *Healthy-if R2c-design assms*)
also have ... $\longleftrightarrow R1(R3h(P_1 \vdash P_2))[\![false/\$wait]\!] \sqsubseteq R1(R3h(Q_1 \vdash Q_2))[\![false/\$wait]\!]$
by (*rel-auto*, *metis+*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2)[\![false/\$wait]\!] \sqsubseteq R1(Q_1 \vdash Q_2)[\![false/\$wait]\!]$
by (*rel-auto*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$
by (*simp add*: *usubst assms closure unrest*)
also have ... $\longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \Rightarrow P_2`$
by (*simp add*: *R1-design-refine assms*)
finally show *?thesis* **.**
qed

lemma *srdes-refine-intro*:
assumes $`P_1 \Rightarrow P_2`$ $`P_1 \wedge Q_2 \Rightarrow Q_1`$
shows $\mathbf{R}_s(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2)$
by (*simp add*: *RHS-mono assms design-refine-intro*)

3.5 Distribution laws

lemma *RHS-design-choice*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcap \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \vee Q_2))$
by (*metis RHS-inf design-choice*)

lemma *RHS-design-sup*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcup \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \vee P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
by (*metis RHS-sup design-inf*)

lemma *RHS-design-USUP*:
assumes $A \neq \{\}$
shows $(\bigsqcap\ i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\bigsqcup\ i \in A \cdot P(i)) \vdash (\bigsqcap\ i \in A \cdot Q(i)))$
by (*subst RHS-INF*[*OF assms*, *THEN sym*], *simp add*: *design-UINF-mem assms*)

end

4 Reactive Design Triples

theory *utp-rdes-triples*
imports *utp-rdes-designs*
begin

4.1 Diamond notation

definition *wait′-cond* ::
$('t::trace,'\alpha,'\beta)$ *rel-rp* $\Rightarrow$ $('t,'\alpha,'\beta)$ *rel-rp* $\Rightarrow$ $('t,'\alpha,'\beta)$ *rel-rp* (**infixr** $\diamond$ *65*) **where**
[*upred-defs*]: $P \diamond Q = (P \triangleleft \$wait' \triangleright Q)$

lemma *wait′-cond-unrest* [*unrest*]:
$[\![$ *out-var wait* $\bowtie x$; $x \sharp P$; $x \sharp Q$ $]\!] \Longrightarrow x \sharp (P \diamond Q)$
by (*simp add*: *wait′-cond-def unrest*)

lemma *wait′-cond-subst* [*usubst*]:
$\$wait' \sharp \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$
by (*simp add*: *wait′-cond-def usubst unrest*)

lemma *wait′-cond-left-false*: $false \diamond P = (\neg \$wait' \wedge P)$
by (*rel-auto*)

lemma *wait′-cond-seq*: $((P \diamond Q)\ ;;\ R) = ((P\ ;;\ (\$wait \wedge R)) \vee (Q\ ;;\ (\neg\$wait \wedge R)))$
by (*simp add*: *wait′-cond-def cond-def seqr-or-distl*, *rel-blast*)

lemma *wait′-cond-true*: $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$
by (*rel-auto*)

lemma *wait′-cond-false*: $(P \diamond Q \wedge (\neg\$wait')) = (Q \wedge (\neg\$wait'))$
by (*rel-auto*)

lemma *wait′-cond-idem*: $P \diamond P = P$
by (*rel-auto*)

lemma *wait′-cond-conj-exchange*:
$((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$
by (*rel-auto*)

lemma *subst-wait′-cond-true* [*usubst*]: $(P \diamond Q)[\![true/\$wait']\!] = P[\![true/\$wait']\!]$
by (*rel-auto*)

lemma *subst-wait′-cond-false* [*usubst*]: $(P \diamond Q)[\![false/\$wait']\!] = Q[\![false/\$wait']\!]$
by (*rel-auto*)

lemma *subst-wait′-left-subst*: $(P⟦true/\$wait´⟧ \diamond Q) = (P \diamond Q)$
 by (*rel-auto*)

lemma *subst-wait′-right-subst*: $(P \diamond Q⟦false/\$wait´⟧) = (P \diamond Q)$
 by (*rel-auto*)

lemma *wait′-cond-split*: $P⟦true/\$wait´⟧ \diamond P⟦false/\$wait´⟧ = P$
 by (*simp add*: *wait′-cond-def cond-var-split*)

lemma *wait-cond′-assoc* [*simp*]: $P \diamond Q \diamond R = P \diamond R$
 by (*rel-auto*)

lemma *wait-cond′-shadow*: $(P \diamond Q) \diamond R = P \diamond Q \diamond R$
 by (*rel-auto*)

lemma *wait-cond′-conj* [*simp*]: $P \diamond (Q \wedge (R \diamond S)) = P \diamond (Q \wedge S)$
 by (*rel-auto*)

lemma *R1-wait′-cond*: $R1(P \diamond Q) = R1(P) \diamond R1(Q)$
 by (*rel-auto*)

lemma *R2s-wait′-cond*: $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$
 by (*simp add*: *wait′-cond-def R2s-def R2s-def usubst*)

lemma *R2-wait′-cond*: $R2(P \diamond Q) = R2(P) \diamond R2(Q)$
 by (*simp add*: *R2-def R2s-wait′-cond R1-wait′-cond*)

lemma *wait′-cond-R1-closed* [*closure*]:
 $⟦ P\ is\ R1;\ Q\ is\ R1 ⟧ \Longrightarrow P \diamond Q\ is\ R1$
 by (*simp add*: *Healthy-def R1-wait′-cond*)

lemma *wait′-cond-R2c-closed* [*closure*]: $⟦ P\ is\ R2c;\ Q\ is\ R2c ⟧ \Longrightarrow P \diamond Q\ is\ R2c$
 by (*simp add*: *R2c-condr wait′-cond-def Healthy-def*, *rel-auto*)

4.2 Export laws

lemma *RH-design-peri-R1*: $\mathbf{R}(P \vdash R1(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
 by (*metis* (*no-types*, *lifting*) *R1-idem R1-wait′-cond RH-design-export-R1*)

lemma *RH-design-post-R1*: $\mathbf{R}(P \vdash Q \diamond R1(R)) = \mathbf{R}(P \vdash Q \diamond R)$
 by (*metis R1-wait′-cond RH-design-export-R1 RH-design-peri-R1*)

lemma *RH-design-peri-R2s*: $\mathbf{R}(P \vdash R2s(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
 by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-post-R2s*: $\mathbf{R}(P \vdash Q \diamond R2s(R)) = \mathbf{R}(P \vdash Q \diamond R)$
 by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-peri-R2c*: $\mathbf{R}(P \vdash R2c(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
 by (*metis R1-R2s-R2c RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-peri-R1*: $\mathbf{R}_s(P \vdash R1(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
 by (*metis* (*no-types*, *lifting*) *R1-idem R1-wait′-cond RHS-design-export-R1*)

lemma *RHS-design-post-R1*: $\mathbf{R}_s(P \vdash Q \diamond R1(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
 by (*metis R1-wait′-cond RHS-design-export-R1 RHS-design-peri-R1*)

lemma *RHS-design-peri-R2s*: $\mathbf{R}_s(P \vdash R2s(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait'-cond RHS-design-export-R2s*)

lemma *RHS-design-post-R2s*: $\mathbf{R}_s(P \vdash Q \diamond R2s(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R2s-wait'-cond RHS-design-export-R2s RHS-design-peri-R2s*)

lemma *RHS-design-peri-R2c*: $\mathbf{R}_s(P \vdash R2c(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R1-R2s-R2c RHS-design-peri-R1 RHS-design-peri-R2s*)

lemma *RH-design-lemma1*:
$RH(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RH(P \vdash (Q \vee R) \diamond S)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-lemma1*:
$RHS(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RHS(P \vdash (Q \vee R) \diamond S)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RHS-design-peri-R1 RHS-design-peri-R2s*)

4.3 Pre-, peri-, and postconditions

4.3.1 Definitions

abbreviation $pre_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s false, \$wait \mapsto_s false]$
abbreviation $cmt_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false]$
abbreviation $peri_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true]$
abbreviation $post_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s false]$

abbreviation $npre_R(P) \equiv pre_s \dagger P$

definition [*upred-defs*]: $pre_R(P) = (\neg_r\ npre_R(P))$
definition [*upred-defs*]: $cmt_R(P) = R1(cmt_s \dagger P)$
definition [*upred-defs*]: $peri_R(P) = R1(peri_s \dagger P)$
definition [*upred-defs*]: $post_R(P) = R1(post_s \dagger P)$

4.3.2 Unrestriction laws

lemma *ok-pre-unrest* [*unrest*]: $\$ok \sharp pre_R\ P$
by (*simp add*: pre_R-*def unrest usubst*)

lemma *ok-peri-unrest* [*unrest*]: $\$ok \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *ok-post-unrest* [*unrest*]: $\$ok \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

lemma *ok-cmt-unrest* [*unrest*]: $\$ok \sharp cmt_R\ P$
by (*simp add*: cmt_R-*def unrest usubst*)

lemma *ok'-pre-unrest* [*unrest*]: $\$ok' \sharp pre_R\ P$
by (*simp add*: pre_R-*def unrest usubst*)

lemma *ok'-peri-unrest* [*unrest*]: $\$ok' \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *ok′-post-unrest* [*unrest*]: $\$ok´ \sharp\ post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

lemma *ok′-cmt-unrest* [*unrest*]: $\$ok´ \sharp\ cmt_R\ P$
by (*simp add*: cmt_R*-def unrest usubst*)

lemma *wait-pre-unrest* [*unrest*]: $\$wait \sharp\ pre_R\ P$
by (*simp add*: pre_R*-def unrest usubst*)

lemma *wait-peri-unrest* [*unrest*]: $\$wait \sharp\ peri_R\ P$
by (*simp add*: $peri_R$*-def unrest usubst*)

lemma *wait-post-unrest* [*unrest*]: $\$wait \sharp\ post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

lemma *wait-cmt-unrest* [*unrest*]: $\$wait \sharp\ cmt_R\ P$
by (*simp add*: cmt_R*-def unrest usubst*)

lemma *wait′-peri-unrest* [*unrest*]: $\$wait´ \sharp\ peri_R\ P$
by (*simp add*: $peri_R$*-def unrest usubst*)

lemma *wait′-post-unrest* [*unrest*]: $\$wait´ \sharp\ post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

4.3.3 Substitution laws

lemma pre_s*-design*: $pre_s \dagger (P \vdash Q) = (\neg\ pre_s \dagger P)$
by (*simp add*: *design-def* pre_R*-def usubst*)

lemma $peri_s$*-design*: $peri_s \dagger (P \vdash Q \diamond R) = peri_s \dagger (P \Rightarrow Q)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma $post_s$*-design*: $post_s \dagger (P \vdash Q \diamond R) = post_s \dagger (P \Rightarrow R)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma cmt_s*-design*: $cmt_s \dagger (P \vdash Q) = cmt_s \dagger (P \Rightarrow Q)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma pre_s*-R1* [*usubst*]: $pre_s \dagger R1(P) = R1(pre_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma pre_s*-R2c* [*usubst*]: $pre_s \dagger R2c(P) = R2c(pre_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma $peri_s$*-R1* [*usubst*]: $peri_s \dagger R1(P) = R1(peri_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma $peri_s$*-R2c* [*usubst*]: $peri_s \dagger R2c(P) = R2c(peri_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma $post_s$*-R1* [*usubst*]: $post_s \dagger R1(P) = R1(post_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma $post_s$*-R2c* [*usubst*]: $post_s \dagger R2c(P) = R2c(post_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma cmt_s-*R1* [*usubst*]: $cmt_s \dagger R1(P) = R1(cmt_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma cmt_s-*R2c* [*usubst*]: $cmt_s \dagger R2c(P) = R2c(cmt_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma *pre-wait-false*:
$pre_R(P[\![false/\$wait]\!]) = pre_R(P)$
by (*rel-auto*)

lemma *cmt-wait-false*:
$cmt_R(P[\![false/\$wait]\!]) = cmt_R(P)$
by (*rel-auto*)

lemma *rea-pre-RHS-design*: $pre_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(pre_s \dagger P))$
by (*simp add*: *RHS-def usubst R3h-def* pre_R*-def* pre_s*-design R1-negate-R1 R2c-not rea-not-def*)

lemma *rea-cmt-RHS-design*: $cmt_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$
by (*simp add*: *RHS-def usubst R3h-def* cmt_R*-def* cmt_s*-design R1-idem*)

lemma *rea-peri-RHS-design*: $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$
by (*simp add*:*RHS-def usubst* $peri_R$*-def R3h-def* $peri_s$*-design*, *rel-auto*)

lemma *rea-post-RHS-design*: $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$
by (*simp add*:*RHS-def usubst* $post_R$*-def R3h-def* $post_s$*-design*, *rel-auto*)

lemma *peri-cmt-def*: $peri_R(P) = (cmt_R(P))[\![true/\$wait']\!]$
by (*rel-auto*)

lemma *post-cmt-def*: $post_R(P) = (cmt_R(P))[\![false/\$wait']\!]$
by (*rel-auto*)

lemma *rdes-export-cmt*: $\mathbf{R}_s(P \vdash cmt_s \dagger Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

lemma *rdes-export-pre*: $\mathbf{R}_s((P[\![true,false/\$ok,\$wait]\!]) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

4.3.4 Healthiness laws

lemma *wait'-unrest-pre-SRD* [*unrest*]:
$\$wait' \sharp pre_R(P) \Longrightarrow \$wait' \sharp pre_R (SRD\ P)$
apply (*rel-auto*)
using *least-zero* **apply** *blast+*
done

lemma *R1-R2s-cmt-SRD*:
assumes *P is SRD*
shows $R1(R2s(cmt_R(P))) = cmt_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2s-R2c R1-idem R2c-idem SRD-reactive-design assms rea-cmt-RHS-design*)

lemma *R1-R2s-peri-SRD*:
assumes *P is SRD*
shows $R1(R2s(peri_R(P))) = peri_R(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1-R2s-R2c R2-def R2-idem RHS-def SRD-RH-design-form*

assms R1-idem $peri_R$*-def* $peri_s$*-R1* $peri_s$*-R2c*)

lemma *R1-peri-SRD*:
assumes *P is SRD*
shows $R1(peri_R(P)) = peri_R(P)$
proof –
have $R1(peri_R(P)) = R1(R1(R2s(peri_R(P))))$
by (*simp add*: *R1-R2s-peri-SRD assms*)
also have ... $= peri_R(P)$
by (*simp add*: *R1-idem*, *simp add*: *R1-R2s-peri-SRD assms*)
finally show *?thesis* **.**
qed

lemma *periR-SRD-R1* [*closure*]: *P is SRD* $\Longrightarrow$ $peri_R(P)$ *is R1*
by (*simp add*: *Healthy-def' R1-peri-SRD*)

lemma *R1-R2c-peri-RHS*:
assumes *P is SRD*
shows $R1(R2c(peri_R(P))) = peri_R(P)$
by (*metis R1-R2s-R2c R1-R2s-peri-SRD assms*)

lemma *R1-R2s-post-SRD*:
assumes *P is SRD*
shows $R1(R2s(post_R(P))) = post_R(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1-R2s-R2c R1-idem R2-def R2-idem RHS-def SRD-RH-design-form assms* $post_R$*-def* $post_s$*-R1* $post_s$*-R2c*)

lemma *R2c-peri-SRD*:
assumes *P is SRD*
shows $R2c(peri_R(P)) = peri_R(P)$
by (*metis R1-R2c-commute R1-R2c-peri-RHS R1-peri-SRD assms*)

lemma *R1-post-SRD*:
assumes *P is SRD*
shows $R1(post_R(P)) = post_R(P)$
proof –
have $R1(post_R(P)) = R1(R1(R2s(post_R(P))))$
by (*simp add*: *R1-R2s-post-SRD assms*)
also have ... $= post_R(P)$
by (*simp add*: *R1-idem*, *simp add*: *R1-R2s-post-SRD assms*)
finally show *?thesis* **.**
qed

lemma *R2c-post-SRD*:
assumes *P is SRD*
shows $R2c(post_R(P)) = post_R(P)$
by (*metis R1-R2c-commute R1-R2s-R2c R1-R2s-post-SRD R1-post-SRD assms*)

lemma *postR-SRD-R1* [*closure*]: *P is SRD* $\Longrightarrow$ $post_R(P)$ *is R1*
by (*simp add*: *Healthy-def' R1-post-SRD*)

lemma *R1-R2c-post-RHS*:
assumes *P is SRD*
shows $R1(R2c(post_R(P))) = post_R(P)$
by (*metis R1-R2s-R2c R1-R2s-post-SRD assms*)

lemma *R2-cmt-conj-wait′*:
P *is SRD* $\Longrightarrow R2(cmt_R\ P \wedge \neg\ \$wait') = (cmt_R\ P \wedge \neg\ \$wait')$
by (*simp add*: *R2-def R2s-conj R2s-not R2s-wait′ R1-extend-conj R1-R2s-cmt-SRD*)

lemma *R2c-preR*:
P *is SRD* $\Longrightarrow R2c(pre_R(P)) = pre_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R2c-idem SRD-reactive-design rea-pre-RHS-design*)

lemma *preR-R2c-closed* [*closure*]: P *is SRD* $\Longrightarrow pre_R(P)$ *is R2c*
by (*simp add*: *Healthy-def′ R2c-preR*)

lemma *R2c-periR*:
P *is SRD* $\Longrightarrow R2c(peri_R(P)) = peri_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2s-R2c R1-R2s-peri-SRD R2c-idem*)

lemma *periR-R2c-closed* [*closure*]: P *is SRD* $\Longrightarrow peri_R(P)$ *is R2c*
by (*simp add*: *Healthy-def R2c-peri-SRD*)

lemma *R2c-postR*:
P *is SRD* $\Longrightarrow R2c(post_R(P)) = post_R(P)$
by (*metis* (*no-types*, *hide-lams*) *R1-R2c-commute R1-R2c-is-R2 R1-R2s-post-SRD R2-def R2s-idem*)

lemma *postR-R2c-closed* [*closure*]: P *is SRD* $\Longrightarrow post_R(P)$ *is R2c*
by (*simp add*: *Healthy-def R2c-post-SRD*)

lemma *periR-RR* [*closure*]: P *is SRD* $\Longrightarrow peri_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *postR-RR* [*closure*]: P *is SRD* $\Longrightarrow post_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *wpR-trace-ident-pre* [*wp*]:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R)\ wp_r\ pre_R\ P = pre_R\ P$
by (*rel-auto*)

lemma *R1-preR* [*closure*]:
$pre_R(P)$ *is R1*
by (*rel-auto*)

lemma *trace-ident-left-periR*:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; peri_R(P) = peri_R(P)$
by (*rel-auto*)

lemma *trace-ident-left-postR*:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; post_R(P) = post_R(P)$
by (*rel-auto*)

lemma *trace-ident-right-postR*:
$post_R(P) ;; (\$tr' =_u \$tr \wedge \lceil II \rceil_R) = post_R(P)$
by (*rel-auto*)

lemma *preR-R2-closed* [*closure*]: P *is SRD* $\Longrightarrow pre_R(P)$ *is R2*
by (*simp add*: *R2-comp-def Healthy-comp closure*)

lemma *periR-R2-closed* [*closure*]: P *is SRD* $\Longrightarrow$ $peri_R(P)$ *is R2*
by (*simp add*: *Healthy-def′ R1-R2c-peri-RHS R2-R2c-def*)

lemma *postR-R2-closed* [*closure*]: P *is SRD* $\Longrightarrow$ $post_R(P)$ *is R2*
by (*simp add*: *Healthy-def′ R1-R2c-post-RHS R2-R2c-def*)

4.3.5 Calculation laws

lemma *wait′-cond-peri-post-cmt* [*rdes*]:
$cmt_R\ P = peri_R\ P \diamond post_R\ P$
by (*rel-auto*)

lemma *preR-rdes* [*rdes*]:
assumes P *is RR*
shows $pre_R(\mathbf{R}_s(P \vdash Q \diamond R)) = P$
by (*simp add*: *rea-pre-RHS-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1*)

lemma *periR-rdes* [*rdes*]:
assumes P *is RR* Q *is RR*
shows $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$
by (*simp add*: *rea-peri-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *postR-rdes* [*rdes*]:
assumes P *is RR* R *is RR*
shows $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$
by (*simp add*: *rea-post-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *preR-Chaos* [*rdes*]: $pre_R(Chaos) = false$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *periR-Chaos* [*rdes*]: $peri_R(Chaos) = true_r$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *postR-Chaos* [*rdes*]: $post_R(Chaos) = true_r$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *preR-Miracle* [*rdes*]: $pre_R(Miracle) = true_r$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *periR-Miracle* [*rdes*]: $peri_R(Miracle) = false$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *postR-Miracle* [*rdes*]: $post_R(Miracle) = false$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *preR-srdes-skip* [*rdes*]: $pre_R(II_R) = true_r$
by (*rel-auto*)

lemma *periR-srdes-skip* [*rdes*]: $peri_R(II_R) = false$
by (*rel-auto*)

lemma *postR-srdes-skip* [*rdes*]: $post_R(II_R) = (\$tr' =_u \$tr \wedge \lceil II \rceil_R)$
by (*rel-auto*)

lemma *preR-INF* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ A) = (\bigwedge\ P{\in}A \cdot pre_R(P))$
by (*rel-auto*)

lemma *periR-INF* [*rdes*]: $peri_R(\bigsqcap\ A) = (\bigvee\ P{\in}A \cdot peri_R(P))$
by (*rel-auto*)

lemma *postR-INF* [*rdes*]: $post_R(\bigsqcap\ A) = (\bigvee\ P{\in}A \cdot post_R(P))$
by (*rel-auto*)

lemma *preR-UINF* [*rdes*]: $pre_R(\bigsqcap\ i \cdot P(i)) = (\bigsqcup\ i \cdot pre_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF* [*rdes*]: $peri_R(\bigsqcap\ i \cdot P(i)) = (\bigsqcap\ i \cdot peri_R(P(i)))$
by (*rel-auto*)

lemma *postR-UINF* [*rdes*]: $post_R(\bigsqcap\ i \cdot P(i)) = (\bigsqcap\ i \cdot post_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcup\ i{\in}A \cdot pre_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member-2* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcup\ (i,j){\in}A \cdot pre_R(P\ i\ j))$
by (*rel-auto*)

lemma *preR-UINF-member-3* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcup\ (i,j,k){\in}A \cdot pre_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *periR-UINF-member* [*rdes*]: $peri_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcap\ i{\in}A \cdot peri_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF-member-2* [*rdes*]: $peri_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcap\ (i,j){\in}A \cdot peri_R(P\ i\ j))$
by (*rel-auto*)

lemma *periR-UINF-member-3* [*rdes*]: $peri_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcap\ (i,j,k){\in}A \cdot peri_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *postR-UINF-member* [*rdes*]: $post_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcap\ i{\in}A \cdot post_R(P(i)))$
by (*rel-auto*)

lemma *postR-UINF-member-2* [*rdes*]: $post_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcap\ (i,j){\in}A \cdot post_R(P\ i\ j))$
by (*rel-auto*)

lemma *postR-UINF-member-3* [*rdes*]: $post_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcap\ (i,j,k){\in}A \cdot post_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *preR-inf* [*rdes*]: $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$
by (*rel-auto*)

lemma *periR-inf* [*rdes*]: $peri_R(P \sqcap Q) = (peri_R(P) \vee peri_R(Q))$
by (*rel-auto*)

lemma *postR-inf* [*rdes*]: $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$
by (*rel-auto*)

lemma *preR-SUP* [*rdes*]: $pre_R(\bigsqcup\ A) = (\bigvee\ P{\in}A \cdot pre_R(P))$
by (*rel-auto*)

lemma *periR-SUP* [*rdes*]: $A \neq \{\} \Longrightarrow peri_R(\bigsqcup\ A) = (\bigwedge\ P{\in}A \cdot peri_R(P))$
  by (*rel-auto*)

lemma *postR-SUP* [*rdes*]: $A \neq \{\} \Longrightarrow post_R(\bigsqcup\ A) = (\bigwedge\ P{\in}A \cdot post_R(P))$
  by (*rel-auto*)

4.4 Formation laws

lemma *srdes-skip-tri-design* [*rdes-def*]: $II_R = \mathbf{R}_s(true_r \vdash false \diamond II_r)$
  by (*simp add*: *srdes-skip-def*, *rel-auto*)

lemma *Chaos-tri-def* [*rdes-def*]: $Chaos = \mathbf{R}_s(false \vdash false \diamond false)$
  by (*simp add*: *Chaos-def design-false-pre*)

lemma *Miracle-tri-def* [*rdes-def*]: $Miracle = \mathbf{R}_s(true_r \vdash false \diamond false)$
  by (*simp add*: *Miracle-def R1-design-R1-pre wait′-cond-idem*)

lemma *RHS-tri-design-form*:
  assumes P_1 *is RR* P_2 *is RR* P_3 *is RR*
  shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$
proof −
  have $\mathbf{R}_s(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3)))))$
    apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*
  thus *?thesis*
    by (*simp add*: *Healthy-if assms*)
qed

lemma *RHS-design-pre-post-form*:
  $\mathbf{R}_s((\neg P^f{}_f) \vdash P^t{}_f) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
proof −
  have $\mathbf{R}_s((\neg P^f{}_f) \vdash P^t{}_f) = \mathbf{R}_s((\neg P^f{}_f)[\![true/\$ok]\!] \vdash P^t{}_f[\![true/\$ok]\!])$
    by (*simp add*: *design-subst-ok*)
  also have ... $= \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
    by (*simp add*: pre_R*-def* cmt_R*-def usubst*, *rel-auto*)
  finally show *?thesis* **.**
qed

lemma *SRD-as-reactive-design*:
  $SRD(P) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
  by (*simp add*: *RHS-design-pre-post-form SRD-RH-design-form*)

lemma *SRD-reactive-design-alt*:
  assumes P *is SRD*
  shows $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = P$
proof −
  have $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = \mathbf{R}_s((\neg P^f{}_f) \vdash P^t{}_f)$
    by (*simp add*: *RHS-design-pre-post-form*)
  thus *?thesis*
    by (*simp add*: *SRD-reactive-design assms*)
qed

lemma *SRD-reactive-tri-design-lemma*:
  $SRD(P) = \mathbf{R}_s((\neg P^f{}_f) \vdash P^t{}_f[\![true/\$wait']\!] \diamond P^t{}_f[\![false/\$wait']\!])$
  by (*simp add*: *SRD-RH-design-form wait′-cond-split*)

lemma *SRD-as-reactive-tri-design*:
$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
proof −
have $SRD(P) = \mathbf{R}_s((\neg P^f{}_f) \vdash P^t{}_f[\![true/\$wait']\!] \diamond P^t{}_f[\![false/\$wait']\!])$
by (*simp add*: *SRD-RH-design-form wait′-cond-split*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
apply (*simp add*: *usubst*)
apply (*subst design-subst-ok-ok′*[*THEN sym*])
apply (*simp add*: pre_R*-def* $peri_R$*-def* $post_R$*-def usubst unrest*)
apply (*rel-auto*)
done
finally show *?thesis* **.**
qed

lemma *SRD-reactive-tri-design*:
assumes *P is SRD*
shows $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$
by (*metis Healthy-if SRD-as-reactive-tri-design assms*)

lemma *SRD-elim* [*RD-elim*]: $[\![$ *P is SRD*; $Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$ $]\!] \Longrightarrow Q(P)$
by (*simp add*: *SRD-reactive-tri-design*)

lemma *RHS-tri-design-is-SRD* [*closure*]:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*
by (*rule RHS-design-is-SRD*, *simp-all add*: *unrest assms*)

lemma *SRD-rdes-intro* [*closure*]:
assumes *P is RR Q is RR R is RR*
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*
by (*rule RHS-tri-design-is-SRD*, *simp-all add*: *unrest closure assms*)

lemma *USUP-R1-R2s-cmt-SRD*:
assumes $A \subseteq [\![SRD]\!]_H$
shows $(\bigsqcup\ P \in A \cdot R1\ (R2s\ (cmt_R\ P))) = (\bigsqcup\ P \in A \cdot cmt_R\ P)$
by (*rule USUP-cong*[*of A*], *metis* (*mono-tags*, *lifting*) *Ball-Collect R1-R2s-cmt-SRD assms*)

lemma *UINF-R1-R2s-cmt-SRD*:
assumes $A \subseteq [\![SRD]\!]_H$
shows $(\bigsqcap\ P \in A \cdot R1\ (R2s\ (cmt_R\ P))) = (\bigsqcap\ P \in A \cdot cmt_R\ P)$
by (*rule UINF-cong*[*of A*], *metis* (*mono-tags*, *lifting*) *Ball-Collect R1-R2s-cmt-SRD assms*)

4.4.1 Order laws

lemma *preR-antitone*: $P \sqsubseteq Q \Longrightarrow pre_R(Q) \sqsubseteq pre_R(P)$
by (*rel-auto*)

lemma *periR-monotone*: $P \sqsubseteq Q \Longrightarrow peri_R(P) \sqsubseteq peri_R(Q)$
by (*rel-auto*)

lemma *postR-monotone*: $P \sqsubseteq Q \Longrightarrow post_R(P) \sqsubseteq post_R(Q)$
by (*rel-auto*)

4.5 Composition laws

theorem *RH-tri-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q_1$ $\$ok' \sharp Q_2$ $\$ok \sharp R$ $\$ok \sharp S_1$ $\$ok \sharp S_2$
$\$wait' \sharp Q_2$ $\$wait \sharp S_1$ $\$wait \sharp S_2$
shows $(RH(P \vdash Q_1 \diamond Q_2) ;; RH(R \vdash S_1 \diamond S_2)) =$
$RH((\neg (R1\ (\neg R2s\ P) ;; R1\ true) \wedge \neg ((R1\ (R2s\ Q_2) \wedge \neg \$wait') ;; R1\ (\neg R2s\ R))) \vdash$
$((Q_1 \vee (R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_2)))))$
proof −
have *1*: $(\neg ((R1\ (R2s\ (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1\ (\neg R2s\ R))) =$
$(\neg ((R1\ (R2s\ Q_2) \wedge \neg \$wait') ;; R1\ (\neg R2s\ R)))$
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)
have *2*: $(R1\ (R2s\ (Q_1 \diamond Q_2)) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ (S_1 \diamond S_2)))) =$
$((R1\ (R2s\ Q_1) \vee (R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_1))) \diamond (R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_1) ;; (\$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \wedge \$wait')$
proof −
have $(R1\ (R2s\ Q_1) ;; (\$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) ;; (\$wait \wedge \lceil II \rceil_D))$
by (*rel-auto*)
also have ... $= ((R1\ (R2s\ Q_1) ;; \lceil II \rceil_D) \wedge \$wait')$
by (*rel-auto*)
also from *assms*(*2*) **have** ... $= ((R1\ (R2s\ Q_1)) \wedge \$wait')$
by (*simp add*: *lift-des-skip-dr-unit-unrest unrest*)
finally show *?thesis* **.**
qed

moreover have $(R1\ (R2s\ Q_2) ;; (\neg \$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= ((R1\ (R2s\ Q_2)) ;; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_2) ;; (\neg \$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_2) ;; (\neg \$wait \wedge (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

also have ... $= ((R1\ (R2s\ Q_2))[\![false/\$wait']\!] ;; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))[\![false/\$wait]\!])$
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have ... $= ((R1\ (R2s\ Q_2)) ;; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
by (*simp add*: *wait'-cond-def usubst unrest assms*)

finally show *?thesis* **.**
qed

moreover
have $((R1\ (R2s\ Q_1) \wedge \$wait') \vee ((R1\ (R2s\ Q_2)) ;; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \vee (R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2) ;; R1\ (R2s\ S_2)))$
by (*simp add*: *wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*
by (*simp add*: *R2s-wait'-cond R1-wait'-cond wait'-cond-seq*)
qed

show *?thesis*
apply (*subst RH-design-composition*)

apply (*simp-all add*: *assms*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *1 2*)
apply (*simp add*: *R1-R2s-R2c RH-design-lemma1*)
done
qed

theorem *R1-design-composition-RR*:
assumes *P is RR Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P)\ wp_r\ false \wedge Q\ wp_r\ R) \vdash (Q ;; S))$
apply (*subst R1-design-composition*)
apply (*simp-all add*: *assms unrest wp-rea-def Healthy-if closure*)
apply (*rel-auto*)
done

theorem *R1-design-composition-RC*:
assumes *P is RC Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q\ wp_r\ R) \vdash (Q ;; S))$
by (*simp add*: *R1-design-composition-RR assms unrest Healthy-if closure wp*)

theorem *RHS-tri-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q_1$ $\$ok' \sharp Q_2$ $\$ok \sharp R$ $\$ok \sharp S_1$ $\$ok \sharp S_2$
$\$wait \sharp R$ $\$wait' \sharp Q_2$ $\$wait \sharp S_1$ $\$wait \sharp S_2$
shows $(\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)) =$
$\mathbf{R}_s((\neg (R1 (\neg R2s\ P) ;; R1\ true) \wedge \neg (R1(R2s\ Q_2) ;; R1 (\neg R2s\ R))) \vdash$
$(((\exists \$st' \cdot Q_1) \vee (R1 (R2s\ Q_2) ;; R1 (R2s\ S_1))) \diamond ((R1 (R2s\ Q_2) ;; R1 (R2s\ S_2)))))$
proof −
have *1*:$(\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s\ R))) =$
$(\neg ((R1 (R2s\ Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s\ R)))$
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait′ wait′-cond-false*)
have *2*: $(R1 (R2s (Q_1 \diamond Q_2)) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$
$(((\exists \$st' \cdot R1 (R2s\ Q_1)) \vee (R1 (R2s\ Q_2) ;; R1 (R2s\ S_1))) \diamond (R1 (R2s\ Q_2) ;; R1 (R2s$
$S_2)))$
proof −
have $(R1 (R2s\ Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s\ S_1) \diamond R1 (R2s\ S_2))))$
$= (\exists \$st' \cdot ((R1 (R2s\ Q_1)) \wedge \$wait'))$
proof −
have $(R1 (R2s\ Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s\ S_1) \diamond R1 (R2s\ S_2))))$
$= (R1 (R2s\ Q_1) ;; (\$wait \wedge (\exists \$st \cdot \lceil II \rceil_D)))$
by (*rel-auto*, *blast+*)
also have ... $= ((R1 (R2s\ Q_1) ;; (\exists \$st \cdot \lceil II \rceil_D)) \wedge \$wait')$
by (*rel-auto*)
also from *assms*(*2*) **have** ... $= (\exists \$st' \cdot ((R1 (R2s\ Q_1)) \wedge \$wait'))$
by (*rel-auto*, *blast*)
finally show *?thesis* **.**
qed

moreover have $(R1 (R2s\ Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s\ S_1) \diamond R1 (R2s$
$S_2))))$
$= ((R1 (R2s\ Q_2)) ;; (R1 (R2s\ S_1) \diamond R1 (R2s\ S_2)))$
proof −
have $(R1 (R2s\ Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s\ S_1) \diamond R1 (R2s\ S_2))))$

$= (R1\ (R2s\ Q_2)\ ;;\ (\neg\ \$wait \wedge (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

also have ... $= ((R1\ (R2s\ Q_2))\llbracket false/\$wait'\rrbracket\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))\llbracket false/\$wait\rrbracket)$
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have ... $= ((R1\ (R2s\ Q_2))\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
by (*simp add*: *wait′-cond-def usubst unrest assms*)

finally show *?thesis* **.**
qed

moreover
have $((R1\ (R2s\ Q_1) \wedge \$wait') \vee ((R1\ (R2s\ Q_2))\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \vee (R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_2)))$
by (*simp add*: *wait′-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*
by (*simp add*: *R2s-wait′-cond R1-wait′-cond wait′-cond-seq ex-conj-contr-right unrest*)
(*simp add*: *cond-and-T-integrate cond-seq-right-distr unrest-var wait′-cond-def*)
qed

from *assms*(*7*,*8*) **have** *3*: $(R1\ (R2s\ Q_2) \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R2s\ R) = R1\ (R2s\ Q_2)\ ;;\ R1\ (\neg\ R2s\ R)$
by (*rel-auto*, *blast*, *meson*)

show *?thesis*
apply (*subst RHS-design-composition*)
apply (*simp-all add*: *assms*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *1 2 3*)
apply (*simp add*: *R1-R2s-R2c RHS-design-lemma1*)
apply (*metis R1-R2c-ex-st RHS-design-lemma1*)
done
qed

theorem *RHS-tri-design-composition-wp*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q_1$ $\$ok' \sharp Q_2$ $\$ok \sharp R$ $\$ok \sharp S_1$ $\$ok \sharp S_2$
$\$wait \sharp R$ $\$wait' \sharp Q_2$ $\$wait \sharp S_1$ $\$wait \sharp S_2$
P *is* $R2c$ Q_1 *is* $R1$ Q_1 *is* $R2c$ Q_2 *is* $R1$ Q_2 *is* $R2c$
R *is* $R2c$ S_1 *is* $R1$ S_1 *is* $R2c$ S_2 *is* $R1$ S_2 *is* $R2c$
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)\ ;;\ \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash (((\exists\ \$st' \cdot Q_1) \sqcap (Q_2\ ;;\ S_1)) \diamond (Q_2\ ;;\ S_2)))$ (**is** *?lhs* = *?rhs*)
proof −
have $?lhs = \mathbf{R}_s\ ((\neg\ R1\ (\neg\ P)\ ;;\ R1\ true \wedge \neg\ Q_2\ ;;\ R1\ (\neg\ R)) \vdash ((\exists\ \$st' \cdot Q_1) \sqcap Q_2\ ;;\ S_1) \diamond Q_2\ ;;\ S_2)$
by (*simp add*: *RHS-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def*)
(*metis* (*no-types*, *hide-lams*) *R1-negate-R1 R2c-healthy-R2s assms*(*11*,*16*))
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

theorem *RHS-tri-design-composition-RR-wp*:
assumes P *is RR* Q_1 *is RR* Q_2 *is RR*
R *is RR* S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash (((\exists\ \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$ (**is** *?lhs* = *?rhs*)
by (*simp add*: *RHS-tri-design-composition-wp add*: *closure assms unrest RR-implies-R2c*)

lemma *RHS-tri-normal-design-composition*:
assumes
$\$ok' \sharp P$ $\$ok' \sharp Q_1$ $\$ok' \sharp Q_2$ $\$ok \sharp R$ $\$ok \sharp S_1$ $\$ok \sharp S_2$
$\$wait \sharp R$ $\$wait' \sharp Q_2$ $\$wait \sharp S_1$ $\$wait \sharp S_2$
P *is R2c* Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R *is R2c* S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
$R1\ (\neg\ P) ;; R1(true) = R1(\neg\ P)$ $\$st' \sharp Q_1$
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof –
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s\ ((R1\ (\neg\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((\exists\ \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp-all add*: *RHS-tri-design-composition-wp rea-not-def assms unrest*)
also have ... $= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp add*: *assms wp-rea-def ex-unrest*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-normal-design-composition'* [*rdes-def*]:
assumes P *is RC* Q_1 *is RR* $\$st' \sharp Q_1$ Q_2 *is RR* R *is RR* S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof –
have $R1\ (\neg\ P) ;; R1\ true = R1(\neg\ P)$
using *RC-implies-RC1*[*OF assms*(*1*)]
by (*simp add*: *Healthy-def RC1-def rea-not-def*)
(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)
thus *?thesis*
by (*simp add*: *RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)
qed

lemma *RHS-tri-design-right-unit-lemma*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$ $\$wait' \sharp R$
shows $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r\ (\neg_r\ P) ;; true_r) \vdash ((\exists\ \$st' \cdot Q) \diamond R))$
proof –
have $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(true \vdash false \diamond (\$tr' =_u \$tr \wedge \lceil II \rceil_R))$
by (*simp add*: *srdes-skip-tri-design*, *rel-auto*)
also have ... $= \mathbf{R}_s\ ((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond (R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R))))$
by (*simp-all add*: *RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)
also have ... $= \mathbf{R}_s\ ((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond R1\ (R2s\ R))$
proof –
from *assms*(*3*,*4*) **have** $(R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R))) = R1\ (R2s\ R)$
by (*rel-auto*, *metis* (*no-types*, *lifting*) *minus-zero-eq*, *meson order-refl trace-class.diff-cancel*)
thus *?thesis*
by *simp*

qed
also have ... = $\mathbf{R}_s((\neg\ (\neg\ P)\ ;;\ R1\ true) \vdash ((\exists\ \$st\acute{}\ \cdot\ Q) \diamond R))$
by (*metis* (*no-types*, *lifting*) *R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre RHS-design-post-R1 RHS-design-post-R2s*)
also have ... = $\mathbf{R}_s((\neg_r\ (\neg_r\ P)\ ;;\ true_r) \vdash ((\exists\ \$st\acute{}\ \cdot\ Q) \diamond R))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *SRD-composition-wp*:
assumes *P is SRD Q is SRD*
shows $(P\ ;;\ Q) = \mathbf{R}_s\ (((\neg_r\ pre_R\ P)\ wp_r\ false \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash$
$((\exists\ \$st\acute{}\ \cdot\ peri_R\ P) \vee (post_R\ P\ ;;\ peri_R\ Q)) \diamond (post_R\ P\ ;;\ post_R\ Q))$
(**is** *?lhs* = *?rhs*)
proof −
have $(P\ ;;\ Q) = (\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))\ ;;\ \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q)))$
by (*simp add*: *SRD-reactive-tri-design assms*(*1*) *assms*(*2*))
also from *assms*
have ... = *?rhs*
by (*simp add*: *RHS-tri-design-composition-wp disj-upred-def unrest assms closure*)
finally show *?thesis* **.**
qed

4.6 Refinement introduction laws

lemma *RHS-tri-design-refine*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \Rightarrow P_2` \wedge `P_1 \wedge Q_3 \Rightarrow P_3`$
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $\longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3`$
by (*simp add*: *RHS-design-refine assms closure RR-implies-R2c unrest ex-unrest*)
also have ... $\longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3`$
by (*rel-auto*)
also have ... $\longleftrightarrow `P_1 \Rightarrow Q_1` \wedge `((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![true/\$wait\acute{}]\!]` \wedge `((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![false/\$wait\acute{}]\!]`$
by (*rel-auto*, *metis*)
also have ... $\longleftrightarrow$ *?rhs*
by (*simp add*: *usubst unrest assms*)
finally show *?thesis* **.**
qed

lemma *srdes-tri-refine-intro*:
assumes $`P_1 \Rightarrow P_2`\ `P_1 \wedge Q_2 \Rightarrow Q_1`\ `P_1 \wedge R_2 \Rightarrow R_1`$
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac srdes-refine-intro*, *simp-all*, *rel-auto*)

lemma *srdes-tri-eq-intro*:
assumes $P_1 = Q_1\ P_2 = Q_2\ P_3 = Q_3$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
using *assms* **by** (*simp*)

lemma *srdes-tri-refine-intro'*:
assumes $P_2 \sqsubseteq P_1\ Q_1 \sqsubseteq (P_1 \wedge Q_2)\ R_1 \sqsubseteq (P_1 \wedge R_2)$

shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac srdes-tri-refine-intro*, *simp-all add*: *refBy-order*)

lemma *SRD-peri-under-pre*:
assumes P *is SRD* $\$wait' \sharp pre_R(P)$
shows $(pre_R(P) \Rightarrow_r peri_R(P)) = peri_R(P)$
proof −
have $peri_R(P) =$
$peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design assms*)
also have ... $= (pre_R\ P \Rightarrow_r peri_R\ P)$
by (*simp add*: *rea-pre-RHS-design rea-peri-RHS-design assms*
unrest usubst R1-peri-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-periR)
finally show *?thesis* **..**
qed

lemma *SRD-post-under-pre*:
assumes P *is SRD* $\$wait' \sharp pre_R(P)$
shows $(pre_R(P) \Rightarrow_r post_R(P)) = post_R(P)$
proof −
have $post_R(P) =$
$post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design assms*)
also have ... $= (pre_R\ P \Rightarrow_r post_R\ P)$
by (*simp add*: *rea-pre-RHS-design rea-post-RHS-design assms*
unrest usubst R1-post-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-postR)
finally show *?thesis* **..**
qed

lemma *SRD-refine-intro*:
assumes
P *is SRD* Q *is SRD*
`$pre_R(P) \Rightarrow pre_R(Q)$` `$pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)$` `$pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)$`
shows $P \sqsubseteq Q$
by (*metis SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*) *srdes-tri-refine-intro*)

lemma *SRD-refine-intro'*:
assumes
P *is SRD* Q *is SRD*
`$pre_R(P) \Rightarrow pre_R(Q)$` $peri_R(P) \sqsubseteq (pre_R(P) \wedge peri_R(Q))$ $post_R(P) \sqsubseteq (pre_R(P) \wedge post_R(Q))$
shows $P \sqsubseteq Q$
using *assms* **by** (*rule-tac SRD-refine-intro*, *simp-all add*: *refBy-order*)

lemma *SRD-eq-intro*:
assumes
P *is SRD* Q *is SRD* $pre_R(P) = pre_R(Q)$ $peri_R(P) = peri_R(Q)$ $post_R(P) = post_R(Q)$
shows $P = Q$
by (*metis SRD-reactive-tri-design assms*)

4.7 Closure laws

lemma *SRD-srdes-skip* [*closure*]: II_R *is SRD*
by (*simp add*: *srdes-skip-def RHS-design-is-SRD unrest*)

lemma *SRD-seqr-closure* [*closure*]:

assumes P *is SRD* Q *is SRD*
shows $(P$;; $Q)$ *is SRD*
proof −
have $(P$;; $Q) = \mathbf{R}_s$ $(((\neg_r$ pre_R $P)$ wp_r $false \wedge post_R$ P wp_r pre_R $Q) \vdash$
$((\exists$ $\$st'$ $\cdot$ $peri_R$ $P) \vee post_R$ P ;; $peri_R$ $Q) \diamond post_R$ P ;; $post_R$ $Q)$
by (*simp add*: *SRD-composition-wp assms*(*1*) *assms*(*2*))
also have ... *is SRD*
by (*rule RHS-design-is-SRD*, *simp-all add*: *wp-rea-def unrest*)
finally show *?thesis* **.**
qed

lemma *SRD-power-Suc* [*closure*]: P *is SRD* $\Longrightarrow$ $P\hat{}(Suc\ n)$ *is SRD*
proof (*induct n*)
case *0*
then show *?case*
by (*simp*)
next
case (*Suc n*)
then show *?case*
using *SRD-seqr-closure* **by** (*simp add*: *SRD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *SRD-power-comp* [*closure*]: P *is SRD* $\Longrightarrow$ P ;; $P\hat{}n$ *is SRD*
by (*metis SRD-power-Suc upred-semiring.power-Suc*)

lemma *uplus-SRD-closed* [*closure*]: P *is SRD* $\Longrightarrow$ P^+ *is SRD*
by (*simp add*: *uplus-power-def closure*)

lemma *SRD-Sup-closure* [*closure*]:
assumes $A \subseteq [\![SRD]\!]_H$ $A \neq \{\}$
shows $(\bigsqcap\ A)$ *is SRD*
proof −
have SRD $(\bigsqcap\ A) = (\bigsqcap\ (SRD\ `A))$
by (*simp add*: *ContinuousD SRD-Continuous assms*(*2*))
also have ... $= (\bigsqcap\ A)$
by (*simp only*: *Healthy-carrier-image assms*)
finally show *?thesis* **by** (*simp add*: *Healthy-def*)
qed

4.8 Distribution laws

lemma *RHS-tri-design-choice* [*rdes-def*]:
$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$
apply (*simp add*: *RHS-design-choice*)
apply (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$])
apply (*simp*)
apply (*rel-auto*)
done

lemma *RHS-tri-design-sup* [*rdes-def*]:
$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcup \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$
by (*simp add*: *RHS-design-sup*, *rel-auto*)

lemma *RHS-tri-design-conj* [*rdes-def*]:
$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1$

$\Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$
by (*simp add*: *RHS-tri-design-sup conj-upred-def*)

lemma *SRD-UINF* [*rdes-def*]:
assumes $A \neq \{\}$ $A \subseteq [\![SRD]\!]_H$
shows $\bigsqcap A = \mathbf{R}_s((\bigwedge P{\in}A \cdot pre_R(P)) \vdash (\bigvee P{\in}A \cdot peri_R(P)) \diamond (\bigvee P{\in}A \cdot post_R(P)))$
proof –
have $\bigsqcap A = \mathbf{R}_s(pre_R(\bigsqcap A) \vdash peri_R(\bigsqcap A) \diamond post_R(\bigsqcap A))$
by (*metis SRD-as-reactive-tri-design assms srdes-hcond-def srdes-theory-continuous.healthy-inf srdes-theory-continuous.healthy-inf-def*)
also have ... $= \mathbf{R}_s((\bigwedge P{\in}A \cdot pre_R(P)) \vdash (\bigvee P{\in}A \cdot peri_R(P)) \diamond (\bigvee P{\in}A \cdot post_R(P)))$
by (*simp add*: *preR-INF periR-INF postR-INF assms*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-USUP* [*rdes-def*]:
assumes $A \neq \{\}$
shows $(\bigsqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i) \diamond R(i))) = \mathbf{R}_s((\bigsqcup i \in A \cdot P(i)) \vdash (\bigsqcap i \in A \cdot Q(i)) \diamond (\bigsqcap i \in A \cdot R(i)))$
by (*subst RHS-INF*[*OF assms*, *THEN sym*], *simp add*: *design-UINF-mem assms*, *rel-auto*)

lemma *SRD-UINF-mem*:
assumes $A \neq \{\}$ $\bigwedge i.\ P\ i$ *is SRD*
shows $(\bigsqcap i{\in}A \cdot P\ i) = \mathbf{R}_s((\bigwedge i{\in}A \cdot pre_R(P\ i)) \vdash (\bigvee i{\in}A \cdot peri_R(P\ i)) \diamond (\bigvee i{\in}A \cdot post_R(P\ i)))$
(**is** *?lhs* = *?rhs*)
proof –
have *?lhs* $= (\bigsqcap (P\ `\ A))$
by (*rel-auto*)
also have ... $= \mathbf{R}_s\ ((\bigsqcup Pa \in P\ `\ A \cdot pre_R\ Pa) \vdash (\bigsqcap Pa \in P\ `\ A \cdot peri_R\ Pa) \diamond (\bigsqcap Pa \in P\ `\ A \cdot post_R\ Pa))$
by (*subst rdes-def*, *simp-all add*: *assms image-subsetI*)
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-UINF-ind* [*rdes-def*]:
$(\bigsqcap i \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) = \mathbf{R}_s((\bigwedge i \cdot P_1\ i) \vdash (\bigvee i \cdot P_2(i)) \diamond (\bigvee i \cdot P_3(i)))$
by (*rel-auto*)

lemma *cond-srea-form* [*rdes-def*]:
$\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$
proof –
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft R2c(\lceil b \rceil_{S<}) \triangleright \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
by (*pred-auto*)
also have ... $= \mathbf{R}_s\ (P \vdash Q_1 \diamond Q_2 \triangleleft b \triangleright_R R \vdash S_1 \diamond S_2)$
by (*simp add*: *RHS-cond lift-cond-srea-def*)
also have ... $= \mathbf{R}_s\ ((P \triangleleft b \triangleright_R R) \vdash (Q_1 \diamond Q_2 \triangleleft b \triangleright_R S_1 \diamond S_2))$
by (*simp add*: *design-condr lift-cond-srea-def*)
also have ... $= \mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *SRD-cond-srea* [*closure*]:
  assumes P *is SRD* Q *is SRD*
  shows $P \triangleleft b \triangleright_R Q$ *is SRD*
proof −
  have $P \triangleleft b \triangleright_R Q = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \triangleleft b \triangleright_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
    by (*simp add*: *SRD-reactive-tri-design assms*)
  also have ... $= \mathbf{R}_s$ $((pre_R\ P \triangleleft b \triangleright_R pre_R\ Q) \vdash (peri_R\ P \triangleleft b \triangleright_R peri_R\ Q) \diamond (post_R\ P \triangleleft b \triangleright_R post_R\ Q))$
    by (*simp add*: *cond-srea-form*)
  also have ... *is SRD*
    by (*simp add*: *RHS-tri-design-is-SRD lift-cond-srea-def unrest*)
  finally show *?thesis* **.**
qed

4.9 Algebraic laws

lemma *SRD-left-unit*:
  assumes P *is SRD*
  shows II_R ;; $P = P$
  by (*simp add*: *SRD-composition-wp closure rdes wp C1 R1-negate-R1 R1-false rpred trace-ident-left-periR trace-ident-left-postR SRD-reactive-tri-design assms*)

lemma *skip-srea-self-unit* [*simp*]:
  II_R ;; $II_R = II_R$
  by (*simp add*: *SRD-left-unit closure*)

lemma *SRD-right-unit-tri-lemma*:
  assumes P *is SRD*
  shows P ;; $II_R = \mathbf{R}_s$ $((\neg_r\ pre_R\ P)\ wp_r\ false \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P)$
  by (*simp add*: *SRD-composition-wp closure rdes wp rpred trace-ident-right-postR assms*)

lemma *Miracle-left-zero*:
  assumes P *is SRD*
  shows *Miracle* ;; P = *Miracle*
proof −
  have *Miracle* ;; $P = \mathbf{R}_s(true \vdash false)$;; $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
    by (*simp add*: *Miracle-def SRD-reactive-design-alt assms*)
  also have ... $= \mathbf{R}_s(true \vdash false)$
    by (*simp add*: *RHS-design-composition unrest R1-false R2s-false R2s-true*)
  also have ... = *Miracle*
    by (*simp add*: *Miracle-def*)
  finally show *?thesis* **.**
qed

lemma *Chaos-left-zero*:
  assumes P *is SRD*
  shows (*Chaos* ;; P) = *Chaos*
proof −
  have *Chaos* ;; $P = \mathbf{R}_s(false \vdash true)$;; $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
    by (*simp add*: *Chaos-def SRD-reactive-design-alt assms*)
  also have ... $= \mathbf{R}_s$ $((\neg\ R1\ true \wedge \neg\ (R1\ true \wedge \neg\ \$wait')$;; $R1\ (\neg\ R2s\ (pre_R\ P))) \vdash$
      $R1\ true$;; $((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ (cmt_R\ P))))$
    by (*simp add*: *RHS-design-composition unrest R2s-false R2s-true R1-false*)
  also have ... $= \mathbf{R}_s$ $((false \wedge \neg\ (R1\ true \wedge \neg\ \$wait')$;; $R1\ (\neg\ R2s\ (pre_R\ P))) \vdash$
      $R1\ true$;; $((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ (cmt_R\ P))))$

by (*simp add*: *RHS-design-conj-neg-R1-pre*)
also have ... = $\mathbf{R}_s(true)$
by (*simp add*: *design-false-pre*)
also have ... = $\mathbf{R}_s(false \vdash true)$
by (*simp add*: *design-def*)
also have ... = *Chaos*
by (*simp add*: *Chaos-def*)
finally show *?thesis* **.**
qed

lemma *SRD-right-Chaos-tri-lemma*:
assumes *P is SRD*
shows $P\ ;;\ Chaos = \mathbf{R}_s\ (((\neg_r\ pre_R\ P)\ wp_r\ false \wedge post_R\ P\ wp_r\ false) \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond false)$
by (*simp add*: *SRD-composition-wp closure rdes assms wp*, *rel-auto*)

lemma *SRD-right-Miracle-tri-lemma*:
assumes *P is SRD*
shows $P\ ;;\ Miracle = \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond false)$
by (*simp add*: *SRD-composition-wp closure rdes assms wp*, *rel-auto*)

Stateful reactive designs are left unital

overloading
srdes-unit == *utp-unit* :: (*SRDES*, ('s,'t::trace,'α) *rsp*) *uthy* ⇒ ('s,'t,'α) *hrel-rsp*
begin
definition *srdes-unit* :: (*SRDES*, ('s,'t::trace,'α) *rsp*) *uthy* ⇒ ('s,'t,'α) *hrel-rsp* **where**
srdes-unit $T = II_R$
end

interpretation *srdes-left-unital*: *utp-theory-left-unital SRDES*
by (*unfold-locales*, *simp-all add*: *srdes-hcond-def srdes-unit-def SRD-seqr-closure SRD-srdes-skip SRD-left-unit*)

4.10 Recursion laws

lemma *mono-srd-iter*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows *mono* $(\lambda X.\ \mathbf{R}_s(pre_R(F\ X) \vdash peri_R(F\ X) \diamond post_R\ (F\ X)))$
apply (*rule monoI*)
apply (*rule srdes-tri-refine-intro'*)
apply (*meson assms*(*1*) *monoE preR-antitone utp-pred-laws.le-infI2*)
apply (*meson assms*(*1*) *monoE periR-monotone utp-pred-laws.le-infI2*)
apply (*meson assms*(*1*) *monoE postR-monotone utp-pred-laws.le-infI2*)
done

lemma *mu-srd-SRD*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows $(\mu\ X \cdot \mathbf{R}_s\ (pre_R\ (F\ X) \vdash peri_R\ (F\ X) \diamond post_R\ (F\ X)))$ *is SRD*
apply (*subst gfp-unfold*)
apply (*simp add*: *mono-srd-iter assms*)
apply (*rule RHS-tri-design-is-SRD*)
apply (*simp-all add*: *unrest*)
done

lemma *mu-srd-iter*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows $(\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X)))) = F(\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X))))$

apply (*subst gfp-unfold*)
apply (*simp add*: *mono-srd-iter assms*)
apply (*subst SRD-as-reactive-tri-design*[*THEN sym*])
using *Healthy-func assms*(*1*) *assms*(*2*) *mu-srd-SRD* **apply** *blast*
done

lemma *mu-srd-form*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows $\mu_R\ F = (\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X))))$
proof −
have *1*: $F\ (\mu\ X \cdot \mathbf{R}_s(pre_R\ (F\ X) \vdash peri_R(F\ X) \diamond post_R\ (F\ X)))$ *is SRD*
by (*simp add*: *Healthy-apply-closed assms*(*1*) *assms*(*2*) *mu-srd-SRD*)
have *2*:$Mono_{uthy\text{-}order\ SRDES}\ F$
by (*simp add*: *assms*(*1*) *mono-Monotone-utp-order*)
hence *3*:$\mu_R\ F = F\ (\mu_R\ F)$
by (*simp add*: *srdes-theory-continuous.LFP-unfold*[*THEN sym*] *assms*)
hence $\mathbf{R}_s(pre_R\ (F\ (F\ (\mu_R\ F))) \vdash peri_R\ (F\ (F\ (\mu_R\ F))) \diamond post_R\ (F\ (F\ (\mu_R\ F)))) = \mu_R\ F$
using *SRD-reactive-tri-design* **by** *force*
hence $(\mu\ X \cdot \mathbf{R}_s(pre_R\ (F\ X) \vdash peri_R(F\ X) \diamond post_R\ (F\ X))) \sqsubseteq F\ (\mu_R\ F)$
by (*simp add*: *2 srdes-theory-continuous.weak.LFP-lemma3 gfp-upperbound assms*)
thus *?thesis*
using *assms 1 3 srdes-theory-continuous.weak.LFP-lowerbound eq-iff mu-srd-iter*
by (*metis* (*mono-tags*, *lifting*))
qed

lemma *Monotonic-SRD-comp* [*closure*]: *Monotonic* (*op* ;; $P \circ$ *SRD*)
by (*simp add*: *mono-def R1-R2c-is-R2 R2-mono R3h-mono RD1-mono RD2-mono RHS-def SRD-def seqr-mono*)

end

5 Normal Reactive Designs

theory *utp-rdes-normal*
imports
utp-rdes-triples
UTP−KAT.utp-kleene
begin

This additional healthiness condition is analogous to H3

definition *RD3* **where**
[*upred-defs*]: $RD3(P) = P\ ;;\ II_R$

lemma *RD3-idem*: $RD3(RD3(P)) = RD3(P)$
proof −
have *a*: $II_R\ ;;\ II_R = II_R$
by (*simp add*: *SRD-left-unit SRD-srdes-skip*)
show *?thesis*
by (*simp add*: *RD3-def seqr-assoc a*)
qed

lemma *RD3-Idempotent* [*closure*]: *Idempotent RD3*
by (*simp add*: *Idempotent-def RD3-idem*)

lemma *RD3-continuous*: $RD3(\bigsqcap A) = (\bigsqcap P{\in}A.\ RD3(P))$

by (*simp add*: *RD3-def seq-Sup-distr*)

lemma *RD3-Continuous* [*closure*]: *Continuous RD3*
by (*simp add*: *Continuous-def RD3-continuous*)

lemma *RD3-right-subsumes-RD2*: $RD2(RD3(P)) = RD3(P)$
proof −
have $a{:}II_R \;;;\; J = II_R$
by (*rel-auto*)
show *?thesis*
by (*metis* (*no-types*, *hide-lams*) *H2-def RD2-def RD3-def a seqr-assoc*)
qed

lemma *RD3-left-subsumes-RD2*: $RD3(RD2(P)) = RD3(P)$
proof −
have $a{:}J \;;;\; II_R = II_R$
by (*rel-simp*, *safe*, *blast+*)
show *?thesis*
by (*metis* (*no-types*, *hide-lams*) *H2-def RD2-def RD3-def a seqr-assoc*)
qed

lemma *RD3-implies-RD2*: P *is RD3* $\Longrightarrow$ P *is RD2*
by (*metis Healthy-def RD3-right-subsumes-RD2*)

lemma *RD3-intro-pre*:
assumes P *is SRD* $(\neg_r\ pre_R(P)) \;;;\; true_r = (\neg_r\ pre_R(P))$ $\$st' \sharp peri_R(P)$
shows P *is RD3*
proof −
have $RD3(P) = \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P)$
by (*simp add*: *RD3-def SRD-right-unit-tri-lemma assms*)
also have ... $= \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *assms*(*3*) *ex-unrest*)
also have ... $= \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash cmt_R\ P)$
by (*simp add*: *wait'-cond-peri-post-cmt*)
also have ... $= \mathbf{R}_s\ (pre_R\ P \vdash cmt_R\ P)$
by (*simp add*: *assms*(*2*) *rpred wp-rea-def R1-preR*)
finally show *?thesis*
by (*metis Healthy-def SRD-as-reactive-design assms*(*1*))
qed

lemma *RHS-tri-design-right-unit-lemma*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$ $\$wait' \sharp R$
shows $\mathbf{R}_s(P \vdash Q \diamond R) \;;;\; II_R = \mathbf{R}_s((\neg_r\ (\neg_r\ P) \;;;\; true_r) \vdash ((\exists\ \$st' \cdot Q) \diamond R))$
proof −
have $\mathbf{R}_s(P \vdash Q \diamond R) \;;;\; II_R = \mathbf{R}_s(P \vdash Q \diamond R) \;;;\; \mathbf{R}_s(true \vdash false \diamond (\$tr' =_u \$tr \wedge \lceil II \rceil_R))$
by (*simp add*: *srdes-skip-tri-design*, *rel-auto*)
also have ... $= \mathbf{R}_s\ ((\neg\ R1\ (\neg\ R2s\ P) \;;;\; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond (R1\ (R2s\ R) \;;;\; R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R))))$
by (*simp-all add*: *RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)
also have ... $= \mathbf{R}_s\ ((\neg\ R1\ (\neg\ R2s\ P) \;;;\; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond R1\ (R2s\ R))$
proof −
from *assms*(*3*,*4*) **have** $(R1\ (R2s\ R) \;;;\; R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R))) = R1\ (R2s\ R)$
by (*rel-auto*, *metis* (*no-types*, *lifting*) *minus-zero-eq*, *meson order-refl trace-class.diff-cancel*)
thus *?thesis*
by *simp*

qed
also have ... = $\mathbf{R}_s((\neg\ (\neg\ P)\ ;;\ R1\ true) \vdash ((\exists\ \$st\acute{}\ \cdot\ Q) \diamond R))$
by (*metis* (*no-types*, *lifting*) *R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre RHS-design-post-R1 RHS-design-post-R2s*)
also have ... = $\mathbf{R}_s((\neg_r\ (\neg_r\ P)\ ;;\ true_r) \vdash ((\exists\ \$st\acute{}\ \cdot\ Q) \diamond R))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-RD3-intro*:
assumes
$\$ok\acute{}\ \sharp\ P$ $\$ok\acute{}\ \sharp\ Q$ $\$ok\acute{}\ \sharp\ R$ $\$st\acute{}\ \sharp\ Q$ $\$wait\acute{}\ \sharp\ R$
P *is R1* $(\neg_r\ P)\ ;;\ true_r = (\neg_r\ P)$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is RD3*
apply (*simp add*: *Healthy-def RD3-def*)
apply (*subst RHS-tri-design-right-unit-lemma*)
apply (*simp-all add*:*assms ex-unrest rpred*)
done

RD3 reactive designs are those whose assumption can be written as a conjunction of a precondition on (undashed) program variables, and a negated statement about the trace. The latter allows us to state that certain events must not occur in the trace – which are effectively safety properties.

lemma *R1-right-unit-lemma*:
$[\![\ out\alpha\ \sharp\ b;\ out\alpha\ \sharp\ e\ ]\!] \Longrightarrow (\neg_r\ b \vee \$tr\ \hat{}_u\ e \leq_u \$tr\acute{})\ ;;\ R1(true) = (\neg_r\ b \vee \$tr\ \hat{}_u\ e \leq_u \$tr\acute{})$
by (*rel-auto*, *blast*, *metis* (*no-types*, *lifting*) *dual-order.trans*)

lemma *RHS-tri-design-RD3-intro-form*:
assumes
$out\alpha\ \sharp\ b$ $out\alpha\ \sharp\ e$ $\$ok\acute{}\ \sharp\ Q$ $\$st\acute{}\ \sharp\ Q$ $\$ok\acute{}\ \sharp\ R$ $\$wait\acute{}\ \sharp\ R$
shows $\mathbf{R}_s((b \wedge \neg_r\ \$tr\ \hat{}_u\ e \leq_u \$tr\acute{}) \vdash Q \diamond R)$ *is RD3*
apply (*rule RHS-tri-design-RD3-intro*)
apply (*simp-all add*: *assms unrest closure rpred*)
apply (*subst R1-right-unit-lemma*)
apply (*simp-all add*: *assms unrest*)
done

definition *NSRD* :: $('s,'t::trace,'\alpha)$ *hrel-rsp* $\Rightarrow$ $('s,'t,'\alpha)$ *hrel-rsp*
where [*upred-defs*]: *NSRD* = *RD1* $\circ$ *RD3* $\circ$ *RHS*

lemma *RD1-RD3-commute*: *RD1*(*RD3*(*P*)) = *RD3*(*RD1*(*P*))
by (*rel-auto*, *blast+*)

lemma *NSRD-is-SRD* [*closure*]: *P is NSRD* $\Longrightarrow$ *P is SRD*
by (*simp add*: *Healthy-def NSRD-def SRD-def*, *metis Healthy-def RD1-RD3-commute RD2-RHS-commute RD3-def RD3-right-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip*)

lemma *NSRD-elim* [*RD-elim*]:
$[\![\ P$ *is NSRD*; $Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))\ ]\!] \Longrightarrow Q(P)$
by (*simp add*: *RD-elim closure*)

lemma *NSRD-Idempotent* [*closure*]: *Idempotent NSRD*
by (*clarsimp simp add*: *Idempotent-def NSRD-def*, *metis* (*no-types*, *hide-lams*) *Healthy-def RD1-RD3-commute RD3-def RD3-idem RD3-left-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip*)

lemma *NSRD-Continuous* [*closure*]: *Continuous NSRD*
by (*simp add*: *Continuous-comp NSRD-def RD1-Continuous RD3-Continuous RHS-Continuous*)

lemma *NSRD-form*:
$NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))$
proof −
have $NSRD(P) = RD3(SRD(P))$
by (*metis* (*no-types*, *lifting*) *NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-def*)
also have ... $= RD3(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-as-reactive-tri-design*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))\ ;;\ II_R$
by (*simp add*: *RD3-def*)
also have ... $= \mathbf{R}_s((\neg_r (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))$
by (*simp add*: *RHS-tri-design-right-unit-lemma unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-healthy-form*:
assumes *P is NSRD*
shows $\mathbf{R}_s((\neg_r (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))) = P$
by (*metis Healthy-def NSRD-form assms*)

lemma *NSRD-Sup-closure* [*closure*]:
assumes $A \subseteq [\![NSRD]\!]_H$ $A \neq \{\}$
shows $\bigsqcap\ A$ *is NSRD*
proof −
have $NSRD\ (\bigsqcap\ A) = (\bigsqcap\ (NSRD\ `A))$
by (*simp add*: *ContinuousD NSRD-Continuous assms*(*2*))
also have ... $= (\bigsqcap\ A)$
by (*simp only*: *Healthy-carrier-image assms*)
finally show *?thesis* **by** (*simp add*: *Healthy-def*)
qed

lemma *intChoice-NSRD-closed* [*closure*]:
assumes *P is NSRD Q is NSRD*
shows $P \sqcap Q$ *is NSRD*
using *NSRD-Sup-closure*[*of* $\{P,\ Q\}$] **by** (*simp add*: *assms*)

lemma *NRSD-SUP-closure* [*closure*]:
$[\![\bigwedge i.\ i \in A \Longrightarrow P(i)\ is\ NSRD;\ A \neq \{\}\ ]\!] \Longrightarrow (\bigsqcap i{\in}A.\ P(i))\ is\ NSRD$
by (*rule NSRD-Sup-closure*, *auto*)

lemma *NSRD-neg-pre-unit*:
assumes *P is NSRD*
shows $(\neg_r\ pre_R(P))\ ;;\ true_r = (\neg_r\ pre_R(P))$
proof −
have $(\neg_r\ pre_R(P)) = (\neg_r\ pre_R(\mathbf{R}_s((\neg_r (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= R1\ (R2c\ ((\neg_r\ pre_R\ P)\ ;;\ R1\ true))$
by (*simp add*: *rea-pre-RHS-design R1-negate-R1 R1-idem R1-rea-not′ R2c-rea-not usubst rpred unrest closure*)
also have ... $= (\neg_r\ pre_R\ P)\ ;;\ R1\ true$
by (*simp add*: *R1-R2c-seqr-distribute closure assms*)
finally show *?thesis*
by (*simp add*: *rea-not-def*)

qed

lemma *NSRD-neg-pre-left-zero*:
assumes P *is NSRD* Q *is R1* Q *is RD1*
shows $(\neg_r\ pre_R(P))\ ;;\ Q = (\neg_r\ pre_R(P))$
by (*metis* (*no-types*, *hide-lams*) *NSRD-neg-pre-unit RD1-left-zero assms*(*1*) *assms*(*2*) *assms*(*3*) *seqr-assoc*)

lemma *NSRD-st′-unrest-peri* [*unrest*]:
assumes P *is NSRD*
shows $\$st' \sharp\ peri_R(P)$
proof −
have $peri_R(P) = peri_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true \Rightarrow_r (\exists\ \$st' \cdot peri_R\ P)))$
by (*simp add*: *rea-peri-RHS-design usubst unrest*)
also have $\$st' \sharp$ ...
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-wait′-unrest-pre* [*unrest*]:
assumes P *is NSRD*
shows $\$wait' \sharp\ pre_R(P)$
proof −
have $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= (R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true)))$
by (*simp add*: *rea-pre-RHS-design usubst unrest*)
also have $\$wait' \sharp$ ...
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-st′-unrest-pre* [*unrest*]:
assumes P *is NSRD*
shows $\$st' \sharp\ pre_R(P)$
proof −
have $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true))$
by (*simp add*: *rea-pre-RHS-design usubst unrest*)
also have $\$st' \sharp$ ...
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-alt-def*: $NSRD(P) = RD3(SRD(P))$
by (*metis NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-eq-dest-lhs*)

lemma *preR-RR* [*closure*]: P *is NSRD* $\Longrightarrow pre_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *NSRD-neg-pre-RC* [*closure*]:
assumes P *is NSRD*
shows $pre_R(P)$ *is RC*

by (*rule RC-intro*, *simp-all add*: *closure assms NSRD-neg-pre-unit rpred*)

lemma *NSRD-intro*:
assumes P *is SRD* $(\neg_r\ pre_R(P))$;; $true_r = (\neg_r\ pre_R(P))$ $\$st\acute{}\ \sharp\ peri_R(P)$
shows P *is NSRD*
proof −
have $NSRD(P) = \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))$;; $R1\ true) \vdash ((\exists\ \$st\acute{}\ \cdot\ peri_R(P)) \diamond post_R(P)))$
by (*simp add*: *NSRD-form*)
also have ... $= \mathbf{R}_s(pre_R\ P \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *assms ex-unrest rpred closure*)
also have ... $= P$
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
finally show *?thesis*
using *Healthy-def* **by** *blast*
qed

lemma *NSRD-intro′*:
assumes P *is R2* P *is R3h* P *is RD1* P *is RD3*
shows P *is NSRD*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def NSRD-def R1-R2c-is-R2 RHS-def assms comp-apply*)

lemma *NSRD-RC-intro*:
assumes P *is SRD* $pre_R(P)$ *is RC* $\$st\acute{}\ \sharp\ peri_R(P)$
shows P *is NSRD*
by (*metis Healthy-def NSRD-form SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*)
ex-unrest rea-not-false wp-rea-RC-false wp-rea-def)

lemma *NSRD-rdes-intro* [*closure*]:
assumes P *is RC* Q *is RR* R *is RR* $\$st\acute{}\ \sharp\ Q$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is NSRD*
by (*rule NSRD-RC-intro*, *simp-all add*: *rdes closure assms unrest*)

lemma *SRD-RD3-implies-NSRD*:
$[\![$ P *is SRD*; P *is RD3* $]\!] \Longrightarrow P$ *is NSRD*
by (*metis* (*no-types*, *lifting*) *Healthy-def NSRD-def RHS-idem SRD-healths*(*4*) *SRD-reactive-design comp-apply*)

lemma *NSRD-iff*:
P *is NSRD* $\longleftrightarrow ((P$ *is SRD*$) \wedge (\neg_r\ pre_R(P))$;; $R1(true) = (\neg_r\ pre_R(P)) \wedge (\$st\acute{}\ \sharp\ peri_R(P)))$
by (*meson NSRD-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st′-unrest-peri*)

lemma *NSRD-is-RD3* [*closure*]:
assumes P *is NSRD*
shows P *is RD3*
by (*simp add*: *NSRD-is-SRD NSRD-neg-pre-unit NSRD-st′-unrest-peri RD3-intro-pre assms*)

lemma *NSRD-refine-elim*:
assumes
$P \sqsubseteq Q$ P *is NSRD* Q *is NSRD*
$[\![$ ‘$pre_R(P) \Rightarrow pre_R(Q)$‘; ‘$pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)$‘; ‘$pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)$‘ $]\!]$
$\Longrightarrow R$
shows R
proof −
have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$

by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*))
hence *1*:‘pre_R $P \Rightarrow pre_R$ Q‘ **and** *2*:‘pre_R $P \wedge peri_R$ $Q \Rightarrow peri_R$ P‘ **and** *3*:‘pre_R $P \wedge post_R$ $Q \Rightarrow post_R$ P‘
by (*simp-all add*: *RHS-tri-design-refine assms closure*)
with *assms*(*4*) **show** *?thesis*
by *simp*
qed

lemma *NSRD-right-unit*: P *is NSRD* $\Longrightarrow P$;; $II_R = P$
by (*metis Healthy-if NSRD-is-RD3 RD3-def*)

lemma *NSRD-composition-wp*:
assumes P *is NSRD* Q *is SRD*
shows P ;; Q =
$\mathbf{R}_s$ ((pre_R $P \wedge post_R$ P wp_r pre_R $Q) \vdash (peri_R$ $P \vee (post_R$ P ;; $peri_R$ $Q)) \diamond (post_R$ P ;; $post_R$ Q))
by (*simp add*: *SRD-composition-wp assms NSRD-is-SRD wp-rea-def NSRD-neg-pre-unit NSRD-st′-unrest-peri R1-negate-R1 R1-preR ex-unrest rpred*)

lemma *preR-NSRD-seq-lemma*:
assumes P *is NSRD* Q *is SRD*
shows *R1* (*R2c* ($post_R$ P ;; $(\neg_r$ pre_R Q))) = $post_R$ P ;; $(\neg_r$ pre_R Q)
proof −
have $post_R$ P ;; $(\neg_r$ pre_R Q) = *R1*(*R2c*($post_R$ P)) ;; *R1*(*R2c*($\neg_r$ pre_R Q))
by (*simp add*: *NSRD-is-SRD R1-R2c-post-RHS R1-rea-not R2c-preR R2c-rea-not assms*(*1*) *assms*(*2*))
also have ... = *R1* (*R2c* ($post_R$ P ;; $(\neg_r$ pre_R Q)))
by (*simp add*: *R1-seqr R2c-R1-seq calculation*)
finally show *?thesis* **..**
qed

lemma *preR-NSRD-seq* [*rdes*]:
assumes P *is NSRD* Q *is SRD*
shows pre_R(P ;; Q) = (pre_R $P \wedge post_R$ P wp_r pre_R Q)
by (*simp add*: *NSRD-composition-wp assms rea-pre-RHS-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute*[*THEN sym*] *R1-extend-conj′ R1-idem R2c-not closure*)
(*metis* (*no-types*, *lifting*) *Healthy-def Healthy-if NSRD-is-SRD R1-R2c-commute R1-R2c-seqr-distribute R1-seqr-closure assms*(*1*) *assms*(*2*) *postR-R2c-closed postR-SRD-R1 preR-R2c-closed rea-not-R1 rea-not-R2c*)

lemma *periR-NSRD-seq* [*rdes*]:
assumes P *is NSRD* Q *is NSRD*
shows $peri_R$(P ;; Q) = ((pre_R $P \wedge post_R$ P wp_r pre_R $Q) \Rightarrow_r (peri_R$ $P \vee (post_R$ P ;; $peri_R$ Q)))
by (*simp add*: *NSRD-composition-wp assms closure rea-peri-RHS-design usubst unrest wp-rea-def R1-extend-conj′ R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl′ R2c-preR R2c-periR R1-rea-not′ R2c-rea-not R1-peri-SRD*)

lemma *postR-NSRD-seq* [*rdes*]:
assumes P *is NSRD* Q *is NSRD*
shows $post_R$(P ;; Q) = ((pre_R $P \wedge post_R$ P wp_r pre_R $Q) \Rightarrow_r (post_R$ P ;; $post_R$ Q))
by (*simp add*: *NSRD-composition-wp assms closure rea-post-RHS-design usubst unrest wp-rea-def R1-extend-conj′ R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl′ R2c-preR R2c-periR R1-rea-not′ R2c-rea-not*)

lemma *NSRD-seqr-closure* [*closure*]:
assumes P *is NSRD* Q *is NSRD*

shows $(P \;;; Q)$ *is NSRD*
proof –
have $(\neg_r\ post_R\ P\ wp_r\ pre_R\ Q) \;;; true_r = (\neg_r\ post_R\ P\ wp_r\ pre_R\ Q)$
by (*simp add*: *wp-rea-def rpred assms closure seqr-assoc NSRD-neg-pre-unit*)
moreover have $\$st' \sharp\ pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r peri_R\ P \vee post_R\ P \;;; peri_R\ Q$
by (*simp add*: *unrest assms wp-rea-def*)
ultimately show *?thesis*
by (*rule-tac NSRD-intro*, *simp-all add*: *seqr-or-distl NSRD-neg-pre-unit assms closure rdes unrest*)
qed

lemma *RHS-tri-normal-design-composition*:
assumes
$\$ok' \sharp P\ \$ok' \sharp Q_1\ \$ok' \sharp Q_2\ \$ok \sharp R\ \$ok \sharp S_1\ \$ok \sharp S_2$
$\$wait \sharp R\ \$wait' \sharp Q_2\ \$wait \sharp S_1\ \$wait \sharp S_2$
P *is R2c* Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R *is R2c* S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
$R1\ (\neg\ P) \;;; R1(true) = R1(\neg\ P)\ \$st' \sharp Q_1$
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \;;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 \;;; S_1)) \diamond (Q_2 \;;; S_2))$
proof –
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \;;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s\ ((R1\ (\neg\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((\exists\ \$st' \cdot Q_1) \sqcap (Q_2 \;;; S_1)) \diamond (Q_2 \;;; S_2))$
by (*simp-all add*: *RHS-tri-design-composition-wp rea-not-def assms unrest*)
also have ... $= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 \;;; S_1)) \diamond (Q_2 \;;; S_2))$
by (*simp add*: *assms wp-rea-def ex-unrest*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-normal-design-composition'* [*rdes-def*]:
assumes P *is RC* Q_1 *is RR* $\$st' \sharp Q_1\ Q_2$ *is RR* R *is RR* S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \;;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 \;;; S_1)) \diamond (Q_2 \;;; S_2))$
proof –
have $R1\ (\neg\ P) \;;; R1\ true = R1(\neg\ P)$
using *RC-implies-RC1*[*OF assms*(*1*)]
by (*simp add*: *Healthy-def RC1-def rea-not-def*)
(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)
thus *?thesis*
by (*simp add*: *RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)
qed

If a normal reactive design has postcondition false, then it is a left zero for sequential composition.

lemma *NSRD-seq-post-false*:
assumes P *is NSRD* Q *is SRD* $post_R(P) = false$
shows $P \;;; Q = P$
apply (*simp add*: *NSRD-composition-wp assms wp rpred closure*)
using *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*,*3*) **apply** *fastforce*
done

lemma *NSRD-srd-skip* [*closure*]: II_R *is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *rdes closure unrest*)

lemma *NSRD-Chaos* [*closure*]: *Chaos is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *closure rdes unrest*)

lemma *NSRD-Miracle* [*closure*]: *Miracle is NSRD*
  by (*rule NSRD-intro*, *simp-all add*: *closure rdes unrest*)

Post-composing a miracle filters out the non-terminating behaviours

lemma *NSRD-right-Miracle-tri-lemma*:
  assumes *P is NSRD*
  shows P ;; *Miracle* = $\mathbf{R}_s$ (pre_R $P \vdash peri_R$ $P \diamond$ *false*)
  by (*simp add*: *NSRD-composition-wp closure assms rdes wp rpred*)

The set of non-terminating behaviours is a subset

lemma *NSRD-right-Miracle-refines*:
  assumes *P is NSRD*
  shows $P \sqsubseteq P$;; *Miracle*
proof −
  have $\mathbf{R}_s$ (pre_R $P \vdash peri_R$ $P \diamond post_R$ P) $\sqsubseteq$ $\mathbf{R}_s$ (pre_R $P \vdash peri_R$ $P \diamond$ *false*)
    by (*rule srdes-tri-refine-intro*, *rel-auto+*)
  thus *?thesis*
    by (*simp add*: *NSRD-elim NSRD-right-Miracle-tri-lemma assms*)
qed

lemma *upower-Suc-NSRD-closed* [*closure*]:
  P is NSRD $\Longrightarrow$ *P* ˆ *Suc n is NSRD*
proof (*induct n*)
  case *0*
  then show *?case*
    by (*simp*)
next
  case (*Suc n*)
  then show *?case*
    by (*simp add*: *NSRD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *NSRD-power-Suc* [*closure*]:
  P is NSRD $\Longrightarrow$ *P* ;; *P* ˆ *n is NSRD*
  by (*metis upower-Suc-NSRD-closed upred-semiring.power-Suc*)

lemma *uplus-NSRD-closed* [*closure*]: *P is NSRD* $\Longrightarrow$ P^+ *is NSRD*
  by (*simp add*: *uplus-power-def closure*)

lemma *preR-power*:
  assumes *P is NSRD*
  shows $pre_R(P$;; Pˆ$n)$ = ($\bigsqcup$ $i \in \{0..n\}$. ($post_R(P)$ ˆ i) wp_r ($pre_R(P)$))
proof (*induct n*)
  case *0*
  then show *?case*
    by (*simp add*: *wp closure*)
next
  case (*Suc n*) **note** *hyp* = *this*
  have pre_R (P ˆ (*Suc n* + *1*)) = pre_R (P ;; P ˆ (n+*1*))
    by (*simp add*: *upred-semiring.power-Suc*)
  also have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ˆ (*Suc n*)))
    using *NSRD-iff assms preR-NSRD-seq upower-Suc-NSRD-closed* **by** *fastforce*
  also have ... = (pre_R $P \wedge post_R$ P wp_r ($\bigsqcup$ $i \in \{0..n\}$. $post_R$ P ˆ i wp_r pre_R P))
    by (*simp add*: *hyp upred-semiring.power-Suc*)

also have ... = $(pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ post_R\ P\ wp_r\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P)))$
  by (*simp add*: *wp*)
also have ... = $(pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P)))$
proof –
  have $\bigwedge i.\ R1\ (post_R\ P\ \hat{}\ i\ ;;\ (\neg_r\ pre_R\ P)) = (post_R\ P\ \hat{}\ i\ ;;\ (\neg_r\ pre_R\ P))$
    by (*induct-tac i*, *simp-all add*: *closure Healthy-if assms*)
  thus *?thesis*
    by (*simp add*: *wp-rea-def upred-semiring.power-Suc seqr-assoc rpred closure assms*)
qed
also have ... = $(post_R\ P\ \hat{}\ 0\ wp_r\ pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P)))$
  by (*simp add*: *wp assms closure*)
also have ... = $(post_R\ P\ \hat{}\ 0\ wp_r\ pre_R\ P \wedge (\bigsqcup i{\in}\{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P)))$
proof –
  have $(\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P)) = (\bigsqcup i{\in}\{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P))$
    by (*rule cong*[*of Inf*], *simp-all add*: *fun-eq-iff*)
      (*metis* (*no-types*, *lifting*) *image-Suc-atLeastAtMost image-cong image-image*)
  thus *?thesis* **by** *simp*
qed
also have ... = $(\bigsqcup i{\in}insert\ 0\ \{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P))$
  by (*simp add*: *conj-upred-def*)
also have ... = $(\bigsqcup i{\in}\{0..Suc\ n\}.\ post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P)$
  by (*simp add*: *atLeast0-atMost-Suc-eq-insert-0*)
finally show *?case* **by** (*simp add*: *upred-semiring.power-Suc*)
qed

lemma *preR-power'* [*rdes*]:
  assumes *P is NSRD*
  shows $pre_R(P\ ;;\ P\hat{}n) = (\bigsqcup\ i{\in}\{0..n\} \cdot (post_R(P)\ \hat{}\ i)\ wp_r\ (pre_R(P)))$
  by (*simp add*: *preR-power assms USUP-as-Inf*[*THEN sym*])

lemma *preR-power-Suc* [*rdes*]:
  assumes *P is NSRD*
  shows $pre_R(P\hat{}(Suc\ n)) = (\bigsqcup\ i{\in}\{0..n\} \cdot (post_R(P)\ \hat{}\ i)\ wp_r\ (pre_R(P)))$
  by (*simp add*: *upred-semiring.power-Suc rdes assms*)

declare *upred-semiring.power-Suc* [*simp*]

lemma *periR-power*:
  assumes *P is NSRD*
  shows $peri_R(P\ ;;\ P\hat{}n) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r (\bigsqcap\ i{\in}\{0..n\}.\ post_R(P)\ \hat{}\ i)\ ;;\ peri_R(P))$
proof (*induct n*)
  case *0*
  then show *?case*
    by (*simp add*: *NSRD-is-SRD NSRD-wait'-unrest-pre SRD-peri-under-pre assms*)
next
  case (*Suc n*) **note** *hyp* = *this*
  have $peri_R\ (P\ \hat{}\ (Suc\ n + 1)) = peri_R\ (P\ ;;\ P\ \hat{}\ (n{+}1))$
    by (*simp*)
  also have ... = $(pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (peri_R\ P \vee post_R\ P\ ;;\ peri_R\ (P\ ;;\ P\ \hat{}\ n)))$
    by (*simp add*: *closure assms rdes*)
  also have ... = $(pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (peri_R\ P \vee post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ (Suc\ n)) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.$
$post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)))$
    by (*simp only*: *hyp*)
  also
  have ... = $(pre_R\ P \Rightarrow_r peri_R\ P \vee (post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r post_R\ P\ ;;\ (pre_R\ (P\ ;;\ P\ \hat{}\ n)$

$\Rightarrow_r$ ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P)))
    by (*simp add*: *rdes closure assms*, *rel-blast*)
  also
  have ... = (pre_R $P \Rightarrow_r peri_R$ $P \vee$ ($post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r post_R$ P ;; (($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P)))
  proof −
    have ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) *is R1*
    by (*simp add*: *NSRD-is-SRD R1-Continuous R1-power Sup-Continuous-closed assms postR-SRD-R1*)
    hence *1*:(($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P) *is R1*
      by (*simp add*: *closure assms*)
    hence (pre_R (P ;; P ^ n) $\Rightarrow_r$ ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P) *is R1*
      by (*simp add*: *closure*)
    hence ($post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r post_R$ P ;; (pre_R (P ;; P ^ n) $\Rightarrow_r$ ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P))
          = ($post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r$ *R1*($post_R$ P) ;; *R1*(pre_R (P ;; P ^ n) $\Rightarrow_r$ ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P))
      by (*simp add*: *Healthy-if R1-post-SRD assms closure*)
    thus *?thesis*
      by (*simp only*: *wp-rea-impl-lemma*, *simp add*: *Healthy-if 1*, *simp add*: *R1-post-SRD assms closure*)
  qed
  also
  have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r peri_R$ $P \vee post_R$ P ;; (($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ i) ;; $peri_R$ P))
    by (*pred-auto*)
  also
  have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r peri_R$ $P \vee$ (($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ (*Suc* i)) ;; $peri_R$ P))
    by (*simp add*: *seq-Sup-distl seqr-assoc*[*THEN sym*])
  also
  have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r peri_R$ $P \vee$ (($\bigsqcap i\in\{1..Suc\ n\}$. $post_R$ P ^ i) ;; $peri_R$ P))
  proof −
    have ($\bigsqcap i\in\{0..n\}$. $post_R$ P ^ *Suc* i) = ($\bigsqcap i\in\{1..Suc\ n\}$. $post_R$ P ^ i)
      apply (*rule cong*[*of Sup*], *auto*)
     apply (*metis atLeast0AtMost atMost-iff image-Suc-atLeastAtMost rev-image-eqI upred-semiring.power-Suc*)
      using *Suc-le-D* **apply** *fastforce*
    done
    thus *?thesis* **by** *simp*
  qed
  also
  have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ;; P ^ n) $\Rightarrow_r$ (($\bigsqcap i\in\{0..Suc\ n\}$. $post_R$ P ^ i)) ;; $peri_R$ P)
    by (*simp add*: *SUP-atLeastAtMost-first uinf-or seqr-or-distl seqr-or-distr*)
  also
  have ... = (pre_R(P^(*Suc* (*Suc* n))) $\Rightarrow_r$ (($\bigsqcap i\in\{0..Suc\ n\}$. $post_R$ P ^ i) ;; $peri_R$ P))
    by (*simp add*: *rdes closure assms*)
  finally show *?case* **by** (*simp*)
qed

lemma *periR-power'* [*rdes*]:
  assumes P *is NSRD*
  shows $peri_R$(P ;; P^n) = (pre_R(P^(*Suc* n)) $\Rightarrow_r$ ($\bigsqcap$ $i\in\{0..n\}$ · $post_R(P)$ ^ i) ;; $peri_R(P)$)
  by (*simp add*: *periR-power assms UINF-as-Sup*[*THEN sym*])

lemma *periR-power-Suc* [*rdes*]:
  assumes P *is NSRD*

shows $peri_R(P\hat{}(Suc\ n)) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r (\prod\ i{\in}\{0..n\} \cdot post_R(P)\ \hat{}\ i)\ ;;\ peri_R(P))$
by (*simp add*: *rdes assms*)

lemma *postR-power* [*rdes*]:
assumes *P is NSRD*
shows $post_R(P\ ;;\ P\hat{}n) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r post_R(P)\ \hat{}\ Suc\ n)$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *NSRD-is-SRD NSRD-wait′-unrest-pre SRD-post-under-pre assms*)
next
case (*Suc n*) **note** *hyp* = *this*
have $post_R\ (P\ \hat{}\ (Suc\ n + 1)) = post_R\ (P\ ;;\ P\ \hat{}\ (n{+}1))$
by (*simp*)
also have ... $= (pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (post_R\ P\ ;;\ post_R\ (P\ ;;\ P\ \hat{}\ n)))$
by (*simp add*: *closure assms rdes*)
also have ... $= (pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ n)))$
by (*simp only*: *hyp*)
also
have ... $= (pre_R\ P \Rightarrow_r (post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R$
$P\ \hat{}\ Suc\ n)))$
by (*simp add*: *rdes closure assms*, *pred-auto*)
also
have ... $= (pre_R\ P \Rightarrow_r (post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ ;;\ post_R\ P\ \hat{}\ Suc\ n))$
by (*metis* (*no-types*, *lifting*) *Healthy-if NSRD-is-SRD NSRD-power-Suc R1-power assms hyp postR-SRD-R1 upred-semiring.power-Suc wp-rea-impl-lemma*)
also
have ... $= (pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ (Suc\ n))$
by (*pred-auto*)
also have ... $= (pre_R(P\hat{}(Suc\ (Suc\ n))) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ (Suc\ n))$
by (*simp add*: *rdes closure assms*)
finally show *?case* **by** (*simp*)
qed

lemma *postR-power-Suc* [*rdes*]:
assumes *P is NSRD*
shows $post_R(P\hat{}(Suc\ n)) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r post_R(P)\ \hat{}\ Suc\ n)$
by (*simp add*: *rdes assms*)

lemma *power-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR* $\$st' \sharp Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))\hat{}(Suc\ n)$
$= \mathbf{R}_s((\bigsqcup\ i{\in}\{0..n\} \cdot (R\ \hat{}\ i)\ wp_r\ P) \vdash ((\prod\ i{\in}\{0..n\} \cdot R\ \hat{}\ i)\ ;;\ Q) \diamond (R\ \hat{}\ Suc\ n))$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *wp assms closure*)
next
case (*Suc n*)

have *1*: $(P \wedge (\bigsqcup\ i \in \{0..n\} \cdot R\ wp_r\ (R\ \hat{}\ i\ wp_r\ P))) = (\bigsqcup\ i \in \{0..Suc\ n\} \cdot R\ \hat{}\ i\ wp_r\ P)$
(**is** *?lhs* = *?rhs*)
proof −
have $?lhs = (P \wedge (\bigsqcup\ i \in \{0..n\} \cdot (R\ \hat{}\ Suc\ i\ wp_r\ P)))$
by (*simp add*: *wp closure assms*)

also have ... $= (P \wedge (\bigsqcup\ i \in \{0..n\}.\ (R\ \hat{}\ Suc\ i\ wp_r\ P)))$
by (*simp only*: *USUP-as-Inf-collect*)
also have ... $= (P \wedge (\bigsqcup\ i \in \{1..Suc\ n\}.\ (R\ \hat{}\ i\ wp_r\ P)))$
by (*metis* (*no-types*, *lifting*) *INF-cong One-nat-def image-Suc-atLeastAtMost image-image*)
also have ... $= (\bigsqcup\ i \in insert\ 0\ \{1..Suc\ n\}.\ (R\ \hat{}\ i\ wp_r\ P))$
by (*simp add*: *wp assms closure conj-upred-def*)
also have ... $= (\bigsqcup\ i \in \{0..Suc\ n\}.\ (R\ \hat{}\ i\ wp_r\ P))$
by (*simp add*: *atLeastAtMost-insertL*)
finally show *?thesis*
by (*simp add*: *USUP-as-Inf-collect*)
qed

have *2*: $(Q \vee R\ ;;\ (\bigsqcap\ i \in \{0..n\} \cdot R\ \hat{}\ i)\ ;;\ Q) = (\bigsqcap\ i \in \{0..Suc\ n\} \cdot R\ \hat{}\ i)\ ;;\ Q$
(**is** *?lhs = ?rhs*)
proof −
have $?lhs = (Q \vee (\bigsqcap\ i \in \{0..n\} \cdot R\ \hat{}\ Suc\ i)\ ;;\ Q)$
by (*simp add*: *seqr-assoc*[*THEN sym*] *seq-UINF-distl*)
also have ... $= (Q \vee (\bigsqcap\ i \in \{0..n\}.\ R\ \hat{}\ Suc\ i)\ ;;\ Q)$
by (*simp only*: *UINF-as-Sup-collect*)
also have ... $= (Q \vee (\bigsqcap\ i \in \{1..Suc\ n\}.\ R\ \hat{}\ i)\ ;;\ Q)$
by (*metis One-nat-def image-Suc-atLeastAtMost image-image*)
also have ... $= ((\bigsqcap\ i \in insert\ 0\ \{1..Suc\ n\}.\ R\ \hat{}\ i)\ ;;\ Q)$
by (*simp add*: *disj-upred-def*[*THEN sym*] *seqr-or-distl*)
also have ... $= ((\bigsqcap\ i \in \{0..Suc\ n\}.\ R\ \hat{}\ i)\ ;;\ Q)$
by (*simp add*: *atLeastAtMost-insertL*)
finally show *?thesis*
by (*simp add*: *UINF-as-Sup-collect*)
qed

have *3*: $(\bigsqcap\ i \in \{0..n\} \cdot R\ \hat{}\ i)\ ;;\ Q\ is\ RR$
proof −
have $(\bigsqcap\ i \in \{0..n\} \cdot R\ \hat{}\ i)\ ;;\ Q = (\bigsqcap\ i \in \{0..n\}.\ R\ \hat{}\ i)\ ;;\ Q$
by (*simp add*: *UINF-as-Sup-collect*)
also have ... $= (\bigsqcap\ i \in insert\ 0\ \{1..n\}.\ R\ \hat{}\ i)\ ;;\ Q$
by (*simp add*: *atLeastAtMost-insertL*)
also have ... $= (Q \vee (\bigsqcap\ i \in \{1..n\}.\ R\ \hat{}\ i)\ ;;\ Q)$
by (*metis* (*no-types*, *lifting*) *SUP-insert disj-upred-def seqr-left-unit seqr-or-distl upred-semiring.power-0*)
also have ... $= (Q \vee (\bigsqcap\ i \in \{0..<n\}.\ R\ \hat{}\ Suc\ i)\ ;;\ Q)$
by (*metis One-nat-def atLeastLessThanSuc-atLeastAtMost image-Suc-atLeastLessThan image-image*)
also have ... $= (Q \vee (\bigsqcap\ i \in \{0..<n\} \cdot R\ \hat{}\ Suc\ i)\ ;;\ Q)$
by (*simp add*: *UINF-as-Sup-collect*)
also have ... *is RR*
by (*simp-all add*: *closure assms*)
finally show *?thesis* **.**
qed
from *1 2 3 Suc* **show** *?case*
by (*simp add*: *Suc RHS-tri-normal-design-composition′ closure assms wp*)
qed

declare *upred-semiring.power-Suc* [*simp del*]

theorem *uplus-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR* $\$st' \sharp Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))^+ = \mathbf{R}_s(R^{\star r}\ wp_r\ P \vdash R^{\star r}\ ;;\ Q \diamond R^+)$
proof −

have *1*:($\bigsqcap$ *i* · *R* ^ *i*) ;; *Q* = $R^{\star r}$;; *Q*
by (*metis* (*no-types*) *RA1 assms*(*2*) *rea-skip-unit*(*2*) *rrel-thy.Star-def ustar-alt-def*)
show *?thesis*
by (*simp add*: *uplus-power-def seq-UINF-distr wp closure assms rdes-def*)
(*metis 1 seq-UINF-distr′*)
qed

5.1 UTP theory

typedecl *NSRDES*

abbreviation *NSRDES* ≡ *UTHY*(*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*)

overloading
nsrdes-hcond == *utp-hcond* :: (*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*) *uthy* ⇒ ((′*s*,′*t*,′α) *rsp* × (′*s*,′*t*,′α) *rsp*) *health*
nsrdes-unit == *utp-unit* :: (*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*) *uthy* ⇒ (′*s*, ′*t*, ′α) *hrel-rsp*
begin
definition *nsrdes-hcond* :: (*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*) *uthy* ⇒ ((′*s*,′*t*,′α) *rsp* × (′*s*,′*t*,′α) *rsp*) *health* **where**
[*upred-defs*]: *nsrdes-hcond T* = *NSRD*
definition *nsrdes-unit* :: (*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*) *uthy* ⇒ (′*s*, ′*t*, ′α) *hrel-rsp* **where**
[*upred-defs*]: *nsrdes-unit T* = II_R
end

interpretation *nsrd-thy*: *utp-theory-kleene UTHY*(*NSRDES*, (′*s*,′*t*::*trace*,′α) *rsp*)
rewrites ⋀ *P*. *P* ∈ *carrier* (*uthy-order NSRDES*) ⟷ *P is NSRD*
and *P is* $\mathcal{H}_{NSRDES}$ ⟷ *P is NSRD*
and (μ *X* · *F* ($\mathcal{H}_{NSRDES}$ *X*)) = (μ *X* · *F* (*NSRD X*))
and *carrier* (*uthy-order NSRDES*) → *carrier* (*uthy-order NSRDES*) ≡ $[\![NSRD]\!]_H$ → $[\![NSRD]\!]_H$
and $[\![\mathcal{H}_{NSRDES}]\!]_H$ → $[\![\mathcal{H}_{NSRDES}]\!]_H$ ≡ $[\![NSRD]\!]_H$ → $[\![NSRD]\!]_H$
and $\top_{NSRDES}$ = *Miracle*
and $\mathcal{II}_{NSRDES}$ = II_R
and *le* (*uthy-order NSRDES*) = *op* ⊑
proof −
interpret *lat*: *utp-theory-continuous UTHY*(*NSRDES*, (′*s*,′*t*,′α) *rsp*)
by (*unfold-locales*, *simp-all add*: *nsrdes-hcond-def nsrdes-unit-def closure Healthy-if*)
show *1*: $\top_{NSRDES}$ = (*Miracle* :: (′*s*,′*t*,′α) *hrel-rsp*)
by (*metis NSRD-Miracle NSRD-is-SRD lat.top-healthy lat.utp-theory-continuous-axioms nsrdes-hcond-def srdes-theory-continuous.meet-top upred-semiring.add-commute utp-theory-continuous.meet-top*)

thus *utp-theory-kleene UTHY*(*NSRDES*, (′*s*,′*t*,′α) *rsp*)
by (*unfold-locales*, *simp-all add*: *nsrdes-hcond-def nsrdes-unit-def closure Healthy-if Miracle-left-zero SRD-left-unit NSRD-right-unit*)
qed (*simp-all add*: *nsrdes-hcond-def nsrdes-unit-def closure Healthy-if*)

declare *nsrd-thy.top-healthy* [*simp del*]
declare *nsrd-thy.bottom-healthy* [*simp del*]

abbreviation *TestR* ($test_R$) **where**
$test_R$ *P* ≡ *utest NSRDES P*

abbreviation *StarR* :: (′*s*, ′*t*::*trace*, ′α) *hrel-rsp* ⇒ (′*s*, ′*t*, ′α) *hrel-rsp* (-$^{\star R}$ [*999*] *999*) **where**
StarR P ≡ $P\star_{NSRDES}$

lemma *StarR-rdes-def* [*rdes-def*]:

assumes P *is RC* Q *is RR* R *is RR* \$*st*´ ♯ Q
shows $(\mathbf{R}_s(P \vdash Q \diamond R))^{\star R} = \mathbf{R}_s((R^{\star r}\ wp_r\ P) \vdash R^{\star r}\ ;;\ Q \diamond R^{\star r})$
by (*simp add*: *rrel-thy.Star-alt-def nsrd-thy.Star-alt-def assms closure rdes-def unrest rpred disj-upred-def*)

end

6 Syntax for reactive design contracts

theory *utp-rdes-contracts*
imports *utp-rdes-normal*
begin

We give an experimental syntax for reactive design contracts $[P \vdash Q|R]_R$, where P is a precondition on undashed state variables only, Q is a pericondition that can refer to the trace and before state but not the after state, and R is a postcondition. Both Q and R can refer only to the trace contribution through a HOL variable *trace* which is bound to &*tt*.

definition *mk-RD* :: *'s upred* ⇒ (*'t*::*trace* ⇒ *'s upred*) ⇒ (*'t* ⇒ *'s hrel*) ⇒ (*'s*, *'t*, *'a*) *hrel-rsp* **where**
$mk\text{-}RD\ P\ Q\ R = \mathbf{R}_s(\lceil P\rceil_{S<} \vdash \lceil Q(x)\rceil_{S<}[\![x \rightarrow \&tt]\!] \diamond \lceil R(x)\rceil_S[\![x \rightarrow \&tt]\!])$

definition *trace-pred* :: (*'t*::*trace* ⇒ *'s upred*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
[*upred-defs*]: $trace\text{-}pred\ P = [(P\ x)]_{S<}[\![x \rightarrow \&tt]\!]$

syntax
-trace-var :: *logic*
-mk-RD :: *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* ([-/ ⊢ -/ | -]$_R$)
-trace-pred :: *logic* ⇒ *logic* ([-]$_t$)

parse-translation ⟪
let
fun trace-var-tr [] = *Syntax.free trace*
| *trace-var-tr* - = *raise Match*;
in
[(@{*syntax-const -trace-var*}, *K trace-var-tr*)]
end
⟫

translations
$[P \vdash Q \mid R]_R$ => *CONST mk-RD P* (λ *-trace-var*. *Q*) (λ *-trace-var*. *R*)
$[P \vdash Q \mid R]_R$ <= *CONST mk-RD P* (λ *x*. *Q*) (λ *y*. *R*)
$[P]_t$ => *CONST trace-pred* (λ *-trace-var*. *P*)
$[P]_t$ <= *CONST trace-pred* (λ *t*. *P*)

lemma *SRD-mk-RD* [*closure*]: $[P \vdash Q(trace) \mid R(trace)]_R$ *is SRD*
by (*simp add*: *mk-RD-def closure unrest*)

lemma *preR-mk-RD* [*rdes*]: $pre_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = R1(\lceil P\rceil_{S<})$
by (*simp add*: *mk-RD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

lemma *trace-pred-RR-closed* [*closure*]:
$[P\ trace]_t$ *is RR*
by (*rel-auto*)

lemma *unrest-trace-pred-st'* [*unrest*]:

$\$st\acute{} \sharp [P\ trace]_t$
by (*rel-auto*)

lemma *R2c-msubst-tt*: $R2c\ (msubst\ (\lambda x.\ \lceil Q\ x\rceil_S)\ \&tt) = (msubst\ (\lambda x.\ \lceil Q\ x\rceil_S)\ \&tt)$
by (*rel-auto*)

lemma *periR-mk-RD* [*rdes*]: $peri_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = (\lceil P\rceil_{S<} \Rightarrow_r R1((\lceil Q(trace)\rceil_{S<})[\![trace \rightarrow \&tt]\!]))$
by (*simp add*: *mk-RD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre R2c-disj R2c-msubst-tt R1-disj R2c-rea-impl R1-rea-impl*)

lemma *postR-mk-RD* [*rdes*]: $post_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = (\lceil P\rceil_{S<} \Rightarrow_r R1((\lceil R(trace)\rceil_{S})[\![trace \rightarrow \&tt]\!]))$
by (*simp add*: *mk-RD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R2c-rea-impl R1-rea-impl*)

Refinement introduction law for contracts

lemma *RD-contract-refine*:
assumes
Q *is SRD* ‘$\lceil P_1\rceil_{S<} \Rightarrow pre_R\ Q$‘
‘$\lceil P_1\rceil_{S<} \wedge peri_R\ Q \Rightarrow \lceil P_2\ x\rceil_{S<}[\![x \rightarrow \&tt]\!]$‘
‘$\lceil P_1\rceil_{S<} \wedge post_R\ Q \Rightarrow \lceil P_3\ x\rceil_{S}[\![x \rightarrow \&tt]\!]$‘
shows $[P_1 \vdash P_2(trace) \mid P_3(trace)]_R \sqsubseteq Q$
proof −
have $[P_1 \vdash P_2(trace) \mid P_3(trace)]_R \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
using *assms*
by (*simp add*: *mk-RD-def*, *rule-tac srdes-tri-refine-intro*, *simp-all*)
thus *?thesis*
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
qed

end

7 Reactive design tactics

theory *utp-rdes-tactics*
imports *utp-rdes-triples*
begin

Theorems for normalisation

lemmas *rdes-rel-norms* =
prod.case-eq-if
conj-assoc
disj-assoc
conj-UINF-dist
conj-UINF-ind-dist
seqr-or-distl
seqr-or-distr
seq-UINF-distl
seq-UINF-distl'
seq-UINF-distr
seq-UINF-distr'

The following tactic can be used to simply and evaluate reactive predicates.

method *rpred-simp* = (*uexpr-simp simps*: *rpred usubst closure unrest*)

Tactic to expand out healthy reactive design predicates into the syntactic triple form.

method *rdes-expand* **uses** *cls* = (*insert cls*, (*erule RD-elim*)+)

Tactic to simplify the definition of a reactive design

method *rdes-simp* **uses** *cls cong simps* =
((*rdes-expand cls*: *cls*)?, (*simp add*: *rdes-def rdes-rel-norms rdes rpred cls closure alpha usubst unrest wp simps cong*: *cong*))

Tactic to split a refinement conjecture into three POs

method *rdes-refine-split* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; *rule-tac srdes-tri-refine-intro'*)

Tactic to split an equality conjecture into three POs

method *rdes-eq-split* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; (*rule-tac srdes-tri-eq-intro*))

Tactic to prove a refinement

method *rdes-refine* **uses** *cls cong simps* =
(*rdes-refine-split cls*: *cls cong*: *cong simps*: *simps*; (*insert cls*; *rel-auto*))

Tactics to prove an equality

method *rdes-eq* **uses** *cls cong simps* =
(*rdes-eq-split cls*: *cls cong*: *cong simps*: *simps*; *rel-auto*)

Via antisymmetry

method *rdes-eq-anti* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; (*rule-tac antisym*; (*rule-tac srdes-tri-refine-intro*; *rel-auto*)))

Tactic to calculate pre/peri/postconditions from reactive designs

method *rdes-calc* = (*simp add*: *rdes rpred closure alpha usubst unrest wp prod.case-eq-if*)

The following tactic attempts to prove a reactive design refinement by calculation of the pre-, peri-, and postconditions and then showing three implications between them using rel-blast.

method *rdspl-refine* =
(*rule-tac SRD-refine-intro*; (*simp add*: *closure rdes unrest usubst* ; *rel-blast?*))

The following tactic combines antisymmetry with the previous tactic to prove an equality.

method *rdspl-eq* =
(*rule-tac antisym*, *rdes-refine*, *rdes-refine*)

end

8 Reactive design parallel-by-merge

theory *utp-rdes-parallel*
imports
utp-rdes-normal
utp-rdes-tactics
begin

R3h implicitly depends on RD1, and therefore it requires that both sides be RD1. We also require that both sides are R3c, and that $wait_m$ is a quasi-unit, and div_m yields divergence.

lemma *st-U0-alpha*: $\lceil \exists\ \$st \cdot II \rceil_0 = (\exists\ \$st \cdot \lceil II \rceil_0)$

by (*rel-auto*)

lemma *st-U1-alpha*: $\lceil \exists\ \$st \cdot II \rceil_1 = (\exists\ \$st \cdot \lceil II \rceil_1)$
by (*rel-auto*)

definition *skip-rm* :: ('*s*,'*t*::*trace*,'α) *rsp merge* (II_{RM}) **where**
[*upred-defs*]: $II_{RM} = (\exists\ \$st_< \cdot skip_m \vee (\neg\ \$ok_< \wedge \$tr_< \leq_u \$tr'))$

definition [*upred-defs*]: $R3hm(M) = (II_{RM} \triangleleft \$wait_< \triangleright M)$

lemma *R3hm-idem*: $R3hm(R3hm(P)) = R3hm(P)$
by (*rel-auto*)

lemma *R3h-par-by-merge* [*closure*]:
assumes *P is R3h Q is R3h M is R3hm*
shows $(P \parallel_M Q)$ *is R3h*
proof −
have $(P \parallel_M Q) = (((P \parallel_M Q)[\![true/\$ok]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!])[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*simp add*: *cond-var-subst-left cond-var-subst-right*)
also have ... $= (((P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false,true/\$ok,\$wait]\!]) \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= (((\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false,true/\$ok,\$wait]\!]) \triangleleft \$wait \triangleright (P \parallel_M Q))$
proof −
have $(P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; R3hm(M))[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)
also have ... $= ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; (\exists\ \$st_< \cdot \$\mathbf{v}' =_u \$\mathbf{v}_<))[\![true,true/\$ok,\$wait]\!]$
by (*rel-blast*)
also have ... $= ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; (\exists\ \$st_< \cdot \$\mathbf{v}' =_u \$\mathbf{v}_<))[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *assms Healthy-if*)
also have ... $= (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
finally show *?thesis* **by** (*simp add*: *closure assms unrest*)
qed
also have ... $= (((\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright (R1(true))[\![false,true/\$ok,\$wait]\!]) \triangleleft \$wait \triangleright (P \parallel_M Q))$
proof −
have $(P \parallel_M Q)[\![false,true/\$ok,\$wait]\!] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; R3hm(M))[\![false,true/\$ok,\$wait]\!]$
by (*simp add*: *par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)
also have ... $= ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; (\$tr_< \leq_u \$tr'))[\![false,true/\$ok,\$wait]\!]$
by (*rel-blast*)
also have ... $= ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$\mathbf{v}_<' =_u \$\mathbf{v}) ;; (\$tr_< \leq_u \$tr'))[\![false,true/\$ok,\$wait]\!]$
by (*simp add*: *assms Healthy-if*)
also have ... $= (R1(true))[\![false,true/\$ok,\$wait]\!]$
by (*rel-blast*)
finally show *?thesis* **by** *simp*
qed
also have ... $= (((\exists\ \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= R3h(P \parallel_M Q)$
by (*simp add*: *R3h-cases*)
finally show *?thesis*
by (*simp add*: *Healthy-def*)

qed

definition [*upred-defs*]: $RD1m(M) = (M \vee \neg \$ok_< \wedge \$tr_< \leq_u \$tr')$

lemma *RD1-par-by-merge* [*closure*]:
  assumes *P is R1 Q is R1 M is R1m P is RD1 Q is RD1 M is RD1m*
  shows $(P \parallel_M Q)$ *is RD1*
proof −
  have *1*: $(RD1(R1(P)) \parallel_{RD1m(R1m(M))} RD1(R1(Q)))[\![false/\$ok]\!] = R1(true)$
    by (*rel-blast*)
  have $(P \parallel_M Q) = (P \parallel_M Q)[\![true/\$ok]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!]$
    by (*simp add*: *cond-var-split*)
  also have ... $= R1(P \parallel_M Q) \triangleleft \$ok \triangleright R1(true)$
    by (*metis 1 Healthy-if R1-par-by-merge assms calculation
        cond-idem cond-var-subst-right in-var-uvar ok-vwb-lens*)
  also have ... $= RD1(P \parallel_M Q)$
    by (*simp add*: *Healthy-if R1-par-by-merge RD1-alt-def assms*(*3*))
  finally show *?thesis*
    by (*simp add*: *Healthy-def*)
qed

lemma *RD2-par-by-merge* [*closure*]:
  assumes *M is RD2*
  shows $(P \parallel_M Q)$ *is RD2*
proof −
  have $(P \parallel_M Q) = ((P \parallel_s Q) ;; M)$
    by (*simp add*: *par-by-merge-def*)
  also from *assms* **have** ... $= ((P \parallel_s Q) ;; (M ;; J))$
    by (*simp add*: *Healthy-def′ RD2-def H2-def*)
  also from *assms* **have** ... $= (((P \parallel_s Q) ;; M) ;; J)$
    by (*simp add*: *seqr-assoc*)
  also from *assms* **have** ... $= RD2(P \parallel_M Q)$
    by (*simp add*: *RD2-def H2-def par-by-merge-def*)
  finally show *?thesis*
    by (*simp add*: *Healthy-def′*)
qed

lemma *SRD-par-by-merge*:
  assumes *P is SRD Q is SRD M is R1m M is R2m M is R3hm M is RD1m M is RD2*
  shows $(P \parallel_M Q)$ *is SRD*
  by (*rule SRD-intro*, *simp-all add*: *assms closure SRD-healths*)

definition *nmerge-rd0* (N_0) **where**
[*upred-defs*]: $N_0(M) = (\$wait' =_u (\$0{-}wait \vee \$1{-}wait) \wedge \$tr_< \leq_u \$tr'$
      $\wedge (\exists\ \$0{-}ok;\$1{-}ok;\$ok_<;\$ok';\$0{-}wait;\$1{-}wait;\$wait_<;\$wait' \cdot M))$

definition *nmerge-rd1* (N_1) **where**
[*upred-defs*]: $N_1(M) = (\$ok' =_u (\$0{-}ok \wedge \$1{-}ok) \wedge N_0(M))$

definition *nmerge-rd* (N_R) **where**
[*upred-defs*]: $N_R(M) = ((\exists\ \$st_< \cdot \$\mathbf{v}' =_u \$\mathbf{v}_<) \triangleleft \$wait_< \triangleright N_1(M)) \triangleleft \$ok_< \triangleright (\$tr_< \leq_u \$tr')$

definition *merge-rd1* (M_1) **where**
[*upred-defs*]: $M_1(M) = (N_1(M) ;; II_R)$

definition *merge-rd* (M_R) **where**
[*upred-defs*]: $M_R(M) = N_R(M) \;;\; II_R$

abbreviation *rdes-par* (- $\|_{R^-}$ - [*85*,*0*,*86*] *85*) **where**
$P \|_{RM} Q \equiv P \|_{M_R(M)} Q$

Healthiness condition for reactive design merge predicates

definition [*upred-defs*]: $RDM(M) = R2m(\exists\ \$0{-}ok;\$1{-}ok;\$ok_<;\$ok';\$0{-}wait;\$1{-}wait;\$wait_<;\$wait' \cdot M)$

lemma *nmerge-rd-is-R1m* [*closure*]:
  $N_R(M)$ *is R1m*
  by (*rel-blast*)

lemma *R2m-nmerge-rd*: $R2m(N_R(R2m(M))) = N_R(R2m(M))$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *nmerge-rd-is-R2m* [*closure*]:
  M *is R2m* $\Longrightarrow N_R(M)$ *is R2m*
  by (*metis Healthy-def′ R2m-nmerge-rd*)

lemma *nmerge-rd-is-R3hm* [*closure*]: $N_R(M)$ *is R3hm*
  by (*rel-blast*)

lemma *nmerge-rd-is-RD1m* [*closure*]: $N_R(M)$ *is RD1m*
  by (*rel-blast*)

lemma *merge-rd-is-RD3*: $M_R(M)$ *is RD3*
  by (*metis Healthy-Idempotent RD3-Idempotent RD3-def merge-rd-def*)

lemma *merge-rd-is-RD2*: $M_R(M)$ *is RD2*
  by (*simp add*: *RD3-implies-RD2 merge-rd-is-RD3*)

lemma *par-rdes-NSRD* [*closure*]:
  assumes *P is SRD Q is SRD M is RDM*
  shows $P \|_{RM} Q$ *is NSRD*
proof −
  have $(P \|_{N_R\ M} Q \;;\; II_R)$ *is NSRD*
    by (*rule NSRD-intro′*, *simp-all add*: *SRD-healths closure assms*)
      (*metis* (*no-types*, *lifting*) *Healthy-def R2-par-by-merge R2-seqr-closure R2m-nmerge-rd RDM-def SRD-healths*(*2*) *assms skip-srea-R2*
      ,*metis Healthy-Idempotent RD3-Idempotent RD3-def*)
  thus *?thesis*
    by (*simp add*: *merge-rd-def par-by-merge-def seqr-assoc*)
qed

lemma *RDM-intro*:
  assumes *M is R2m* $\$0{-}ok \sharp M$ $\$1{-}ok \sharp M$ $\$ok_< \sharp M$ $\$ok' \sharp M$
    $\$0{-}wait \sharp M$ $\$1{-}wait \sharp M$ $\$wait_< \sharp M$ $\$wait' \sharp M$
  shows *M is RDM*
  using *assms*
  by (*simp add*: *Healthy-def RDM-def ex-unrest unrest*)

lemma *RDM-unrests* [*unrest*]:
  assumes *M is RDM*

shows $\$0{-}ok \sharp M$ $\$1{-}ok \sharp M$ $\$ok_< \sharp M$ $\$ok' \sharp M$
$\$0{-}wait \sharp M$ $\$1{-}wait \sharp M$ $\$wait_< \sharp M$ $\$wait' \sharp M$
by (*subst Healthy-if*[*OF assms*, *THEN sym*], *simp-all add*: *RDM-def unrest*, *rel-auto*)+

lemma *RDM-R1m* [*closure*]: *M is RDM* $\Longrightarrow$ *M is R1m*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1m-idem R2m-def RDM-def*)

lemma *RDM-R2m* [*closure*]: *M is RDM* $\Longrightarrow$ *M is R2m*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R2m-idem RDM-def*)

lemma *ex-st'-R2m-closed* [*closure*]:
assumes *P is R2m*
shows $(\exists\ \$st' \cdot P)$ *is R2m*
proof −
have $R2m(\exists\ \$st' \cdot R2m\ P) = (\exists\ \$st' \cdot R2m\ P)$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def' assms*)
qed

lemma *parallel-RR-closed*:
assumes *P is RR Q is RR M is R2m*
$\$ok_< \sharp M$ $\$wait_< \sharp M$ $\$ok' \sharp M$ $\$wait' \sharp M$
shows $P \parallel_M Q$ *is RR*
by (*rule RR-R2-intro*, *simp-all add*: *unrest assms RR-implies-R2 closure*)

lemma *parallel-ok-cases*:
$((P \parallel_s Q) \;;; M) = ($
$((P^t \parallel_s Q^t) \;;; (M[\![true,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^f \parallel_s Q^t) \;;; (M[\![false,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^t \parallel_s Q^f) \;;; (M[\![true,false/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^f \parallel_s Q^f) \;;; (M[\![false,false/\$0{-}ok,\$1{-}ok]\!])))$
proof −
have $((P \parallel_s Q) \;;; M) = (\exists\ ok_0 \cdot (P \parallel_s Q)[\![\ll ok_0\gg/\$0{-}ok']\!] \;;; M[\![\ll ok_0\gg/\$0{-}ok]\!])$
by (*subst seqr-middle*[*of left-uvar ok*], *simp-all*)
also have ... $= (\exists\ ok_0 \cdot \exists\ ok_1 \cdot ((P \parallel_s Q)[\![\ll ok_0\gg/\$0{-}ok']\!][\![\ll ok_1\gg/\$1{-}ok']\!]) \;;; (M[\![\ll ok_0\gg/\$0{-}ok]\!][\![\ll ok_1\gg/\$1{-}ok]\!]))$
by (*subst seqr-middle*[*of right-uvar ok*], *simp-all*)
also have ... $= (\exists\ ok_0 \cdot \exists\ ok_1 \cdot (P[\![\ll ok_0\gg/\$ok']\!] \parallel_s Q[\![\ll ok_1\gg/\$ok']\!]) \;;; (M[\![\ll ok_0\gg,\ll ok_1\gg/\$0{-}ok,\$1{-}ok]\!]))$
by (*rel-auto robust*)
also have ... = (
$((P^t \parallel_s Q^t) \;;; (M[\![true,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^f \parallel_s Q^t) \;;; (M[\![false,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^t \parallel_s Q^f) \;;; (M[\![true,false/\$0{-}ok,\$1{-}ok]\!])) \vee$
$((P^f \parallel_s Q^f) \;;; (M[\![false,false/\$0{-}ok,\$1{-}ok]\!])))$
by (*simp add*: *true-alt-def*[*THEN sym*] *false-alt-def*[*THEN sym*] *disj-assoc*
utp-pred-laws.sup.left-commute utp-pred-laws.sup-commute usubst)
finally show *?thesis* **.**
qed

lemma *skip-srea-ok-f* [*usubst*]:
$II_R{}^f = R1(\neg\$ok)$
by (*rel-auto*)

lemma *nmerge0-rd-unrest* [*unrest*]:
$\$0{-}ok \sharp N_0\ M$ $\$1{-}ok \sharp N_0\ M$

by (*pred-auto*)+

lemma *parallel-assm-lemma*:
assumes P *is RD2*
shows $pre_s \dagger (P \parallel_{M_R(M)} Q) = (((pre_s \dagger P) \parallel_{N_0(M) \;;\; R1(true)} (cmt_s \dagger Q)) \vee ((cmt_s \dagger P) \parallel_{N_0(M) \;;\; R1(true)} (pre_s \dagger Q)))$
proof −
have $pre_s \dagger (P \parallel_{M_R(M)} Q) = pre_s \dagger ((P \parallel_s Q) \;;\; M_R(M))$
by (*simp add*: *par-by-merge-def*)
also have ... $= ((P \parallel_s Q)[\![true,false/\$ok,\$wait]\!] \;;\; N_R\ M \;;\; R1(\neg\ \$ok))$
by (*simp add*: *merge-rd-def usubst*, *rel-auto*)
also have ... $= ((P[\![true,false/\$ok,\$wait]\!] \parallel_s Q[\![true,false/\$ok,\$wait]\!]) \;;\; N_1(M) \;;\; R1(\neg\ \$ok))$
by (*rel-auto robust*, (*metis*)+)
also have ... = ((
$(((P[\![true,false/\$ok,\$wait]\!])^t \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^t) \;;\; ((N_1\ M)[\![true,true/\$0{-}ok,\$1{-}ok]\!] \;;\; R1(\neg\ \$ok))) \vee$
$(((P[\![true,false/\$ok,\$wait]\!])^f \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^t) \;;\; ((N_1\ M)[\![false,true/\$0{-}ok,\$1{-}ok]\!] \;;\; R1(\neg\ \$ok))) \vee$
$(((P[\![true,false/\$ok,\$wait]\!])^t \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^f) \;;\; ((N_1\ M)[\![true,false/\$0{-}ok,\$1{-}ok]\!] \;;\; R1(\neg\ \$ok))) \vee$
$(((P[\![true,false/\$ok,\$wait]\!])^f \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^f) \;;\; ((N_1\ M)[\![false,false/\$0{-}ok,\$1{-}ok]\!] \;;\; R1(\neg\ \$ok))))$)
(**is** - = (*?C1* $\vee_p$ *?C2* $\vee_p$ *?C3* $\vee_p$ *?C4*))
by (*subst parallel-ok-cases*, *subst-tac*)
also have ... = (*?C2* $\vee$ *?C3*)
proof −
have *?C1* = *false*
by (*rel-auto*)
moreover have ‘*?C4* $\Rightarrow$ *?C3*‘ (**is** ‘(*?A* ;; *?B*) $\Rightarrow$ (*?C* ;; *?D*)‘)
proof −
from *assms* **have** ‘$P^f \Rightarrow P^t$‘
by (*metis RD2-def H2-equivalence Healthy-def'*)
hence P: ‘${P^f}_f \Rightarrow {P^t}_f$‘
by (*rel-auto*)
have ‘*?A* $\Rightarrow$ *?C*‘
using P **by** (*rel-auto*)
moreover have ‘*?B* $\Rightarrow$ *?D*‘
by (*rel-auto*)
ultimately show *?thesis*
by (*simp add*: *impl-seqr-mono*)
qed
ultimately show *?thesis*
by (*simp add*: *subsumption2*)
qed
also have ... = (
$(((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; ((N_0\ M \;;\; R1(true)))) \vee$
$(((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; ((N_0\ M \;;\; R1(true)))))$
by (*rel-auto*, *metis*+)
also have ... = (
$((pre_s \dagger P) \parallel_{N_0\ M \;;\; R1(true)} (cmt_s \dagger Q)) \vee$
$((cmt_s \dagger P) \parallel_{N_0\ M \;;\; R1(true)} (pre_s \dagger Q)))$
by (*simp add*: *par-by-merge-def*)
finally show *?thesis* **.**
qed

lemma pre_s-*SRD*:
assumes P *is SRD*
shows $pre_s \dagger P = (\neg_r\ pre_R(P))$
proof –
have $pre_s \dagger P = pre_s \dagger \mathbf{R}_s(pre_R\ P \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *SRD-reactive-tri-design assms*)
also have ... $= R1(R2c(\neg\ pre_s \dagger pre_R\ P))$
by (*simp add*: *RHS-def usubst R3h-def* pre_s-*design*)
also have ... $= R1(R2c(\neg\ pre_R\ P))$
by (*rel-auto*)
also have ... $= (\neg_r\ pre_R\ P)$
by (*simp add*: *R2c-not R2c-preR assms rea-not-def*)
finally show *?thesis* **.**
qed

lemma *parallel-assm*:
assumes P *is SRD* Q *is SRD*
shows $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r\ ((\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)} cmt_R(Q)) \wedge$
$\neg_r\ (cmt_R(P) \parallel_{N_0(M)\ ;;\ R1(true)} (\neg_r\ pre_R(Q))))$
(**is** *?lhs* = *?rhs*)
proof –
have $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r\ (pre_s \dagger P) \parallel_{N_0\ M\ ;;\ R1\ true} (cmt_s \dagger Q) \wedge$
$\neg_r\ (cmt_s \dagger P) \parallel_{N_0\ M\ ;;\ R1\ true} (pre_s \dagger Q))$
by (*simp add*: pre_R-*def parallel-assm-lemma assms SRD-healths R1-conj rea-not-def*[*THEN sym*])
also have ... = *?rhs*
by (*simp add*: pre_s-*SRD assms* cmt_R-*def Healthy-if closure unrest*)
finally show *?thesis* **.**
qed

lemma *parallel-assm-unrest-wait'* [*unrest*]:
$[\![\ P\ is\ SRD;\ Q\ is\ SRD\ ]\!] \Longrightarrow \$wait' \sharp\ pre_R(P \parallel_{M_R(M)} Q)$
by (*simp add*: *parallel-assm*, *simp add*: *par-by-merge-def unrest*)

lemma *JL1*: $(M_1\ M)^t[\![false,true/\$0{-}ok,\$1{-}ok]\!] = N_0(M)\ ;;\ R1(true)$
by (*rel-blast*)

lemma *JL2*: $(M_1\ M)^t[\![true,false/\$0{-}ok,\$1{-}ok]\!] = N_0(M)\ ;;\ R1(true)$
by (*rel-blast*)

lemma *JL3*: $(M_1\ M)^t[\![false,false/\$0{-}ok,\$1{-}ok]\!] = N_0(M)\ ;;\ R1(true)$
by (*rel-blast*)

lemma *JL4*: $(M_1\ M)^t[\![true,true/\$0{-}ok,\$1{-}ok]\!] = (\$ok' \wedge N_0\ M)\ ;;\ {II_R}^t$
by (*simp add*: *merge-rd1-def usubst nmerge-rd1-def unrest*)

lemma *parallel-commitment-lemma-1*:
assumes P *is RD2*
shows $cmt_s \dagger (P \parallel_{M_R(M)} Q) = ($
$((cmt_s \dagger P) \parallel_{(\$ok' \wedge N_0\ M)\ ;;\ {II_R}^t} (cmt_s \dagger Q)) \vee$
$((pre_s \dagger P) \parallel_{N_0(M)\ ;;\ R1(true)} (cmt_s \dagger Q)) \vee$
$((cmt_s \dagger P) \parallel_{N_0(M)\ ;;\ R1(true)} (pre_s \dagger Q)))$
proof –

have $cmt_s \dagger (P \parallel_{M_R(M)} Q) = (P[\![true,false/\$ok,\$wait]\!] \parallel_{(M_1(M))^t} Q[\![true,false/\$ok,\$wait]\!])$
by (*simp add*: *usubst*, *rel-auto*)
also have ... $= ((P[\![true,false/\$ok,\$wait]\!] \parallel_s Q[\![true,false/\$ok,\$wait]\!]) \;;\; (M_1\ M)^t)$
by (*simp add*: *par-by-merge-def*)
also have ... = (
$(((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; ((M_1\ M)^t[\![true,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$(((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; ((M_1\ M)^t[\![false,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$(((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; ((M_1\ M)^t[\![true,false/\$0{-}ok,\$1{-}ok]\!])) \vee$
$(((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; ((M_1\ M)^t[\![false,false/\$0{-}ok,\$1{-}ok]\!])))$
by (*subst parallel-ok-cases*, *subst-tac*)
also have ... = (
$(((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; ((M_1\ M)^t[\![true,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$(((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; (N_0(M) \;;\; R1(true))) \vee$
$(((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; (N_0(M) \;;\; R1(true))) \vee$
$(((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; (N_0(M) \;;\; R1(true))))$
(**is** - = $(?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4)$)
by (*simp add*: *JL1 JL2 JL3*)
also have ... = (
$(((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; ((M_1(M))^t[\![true,true/\$0{-}ok,\$1{-}ok]\!])) \vee$
$(((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) \;;\; (N_0(M) \;;\; R1(true))) \vee$
$(((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) \;;\; (N_0(M) \;;\; R1(true))))$
proof −
from *assms* **have** '$P^f \Rightarrow P^t$'
by (*metis RD2-def H2-equivalence Healthy-def'*)
hence P:'$P^f{}_f \Rightarrow P^t{}_f$'
by (*rel-auto*)
have '$?C4 \Rightarrow ?C3$' (**is** '$(?A \;;\; ?B) \Rightarrow (?C \;;\; ?D)$')
proof −
have '$?A \Rightarrow ?C$'
using P **by** (*rel-auto*)
thus *?thesis*
by (*simp add*: *impl-seqr-mono*)
qed
thus *?thesis*
by (*simp add*: *subsumption2*)
qed
finally show *?thesis*
by (*simp add*: *par-by-merge-def JL4*)
qed

lemma *parallel-commitment-lemma-2*:
assumes *P is RD2*
shows $cmt_s \dagger (P \parallel_{M_R(M)} Q) =$
$(((cmt_s \dagger P) \parallel_{(\$ok' \wedge N_0\ M) \;;\; II_R{}^t} (cmt_s \dagger Q)) \vee pre_s \dagger (P \parallel_{M_R(M)} Q))$
by (*simp add*: *parallel-commitment-lemma-1 assms parallel-assm-lemma*)

lemma *parallel-commitment-lemma-3*:
M *is R1m* $\Longrightarrow (\$ok' \wedge N_0\ M) \;;\; II_R{}^t$ *is R1m*
by (*rel-simp*, *safe*, *metis+*)

lemma *parallel-commitment*:
assumes *P is SRD Q is SRD M is RDM*
shows $cmt_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R(P) \parallel_{(\$ok' \wedge N_0\ M) \;;\; II_R{}^t} cmt_R(Q))$
by (*simp add*: *parallel-commitment-lemma-2 parallel-commitment-lemma-3 Healthy-if assms* cmt_R*-def* pre_s*-SRD closure rea-impl-def disj-comm unrest*)

theorem *parallel-reactive-design*:
  assumes P *is SRD* Q *is SRD* M *is RDM*
  shows $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s($
    $(\neg_r ((\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)} cmt_R(Q)) \wedge$
    $\neg_r (cmt_R(P) \parallel_{N_0(M)\ ;;\ R1(true)} (\neg_r\ pre_R(Q)))) \vdash$
    $(cmt_R(P) \parallel_{(\$ok' \wedge N_0\ M)\ ;;\ II_R{}^t} cmt_R(Q)))$ (**is** *?lhs = ?rhs*)
proof –
  have $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s(pre_R(P \parallel_{M_R(M)} Q) \vdash cmt_R(P \parallel_{M_R(M)} Q))$
   by (*metis Healthy-def NSRD-is-SRD SRD-as-reactive-design assms*(*1*) *assms*(*2*) *assms*(*3*) *par-rdes-NSRD*)
  also have ... = *?rhs*
    by (*simp add*: *parallel-assm parallel-commitment design-export-spec assms*, *rel-auto*)
  finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma1*:
  $(\$ok' \wedge P)\ ;;\ II_R[\![true,true/\$ok', \$wait']\!] = (\exists\ \$st' \cdot P)[\![true,true/\$ok',\$wait']\!]$
  (**is** *?lhs = ?rhs*)
proof –
  have $?lhs = (\$ok' \wedge P)\ ;;\ (\exists\ \$st \cdot II)[\![true,true/\$ok', \$wait']\!]$
    by (*rel-blast*)
  also have ... = *?rhs*
    by (*rel-auto*)
  finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma2*:
  assumes M *is RDM*
  shows $(\exists\ \$st' \cdot N_0(M))[\![true,true/\$ok', \$wait']\!] = ((\$0{-}wait \vee \$1{-}wait) \wedge (\exists\ \$st' \cdot M))$
proof –
  have $(\exists\ \$st' \cdot N_0(M))[\![true,true/\$ok', \$wait']\!] = (\exists\ \$st' \cdot (\$0{-}wait \vee \$1{-}wait) \wedge \$tr' \geq_u \$tr_<$
$\wedge M)$
    by (*simp add*: *usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)
  also have ... = $(\exists\ \$st' \cdot (\$0{-}wait \vee \$1{-}wait) \wedge M)$
   by (*metis* (*no-types*, *hide-lams*) *Healthy-if R1m-def R1m-idem R2m-def RDM-def assms utp-pred-laws.inf-commute*)
  also have ... = $((\$0{-}wait \vee \$1{-}wait) \wedge (\exists\ \$st' \cdot M))$
    by (*rel-auto*)
  finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma3*:
  $((\$0{-}wait \vee \$1{-}wait) \wedge (\exists\ \$st' \cdot M)) = ((\$0{-}wait \wedge \$1{-}wait \wedge (\exists\ \$st' \cdot M)) \vee (\neg\ \$0{-}wait \wedge$
$\$1{-}wait \wedge (\exists\ \$st' \cdot M)) \vee (\$0{-}wait \wedge \neg\ \$1{-}wait \wedge (\exists\ \$st' \cdot M)))$
  by (*rel-auto*)

lemma *parallel-pericondition* [*rdes*]:
  fixes M :: $('s,'t{::}trace,'\alpha)$ *rsp merge*
  assumes P *is SRD* Q *is SRD* M *is RDM*
  shows $peri_R(P \parallel_{M_R(M)} Q) = (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r peri_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
    $\vee\ post_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
    $\vee\ peri_R(P) \parallel_{\exists\ \$st' \cdot M} post_R(Q))$
proof –
  have $peri_R(P \parallel_{M_R(M)} Q) =$
    $(pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r cmt_R\ P \parallel_{(\$ok' \wedge N_0\ M)\ ;;\ II_R[\![true,true/\$ok', \$wait']\!]} cmt_R\ Q)$

by (*simp add*: *peri-cmt-def parallel-commitment SRD-healths assms usubst unrest assms*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\exists\ \$st\acute{}\ \cdot\ N_0\ M)[\![true,true/\$ok\acute{},\ \$wait\acute{}]\!]}\ cmt_R\ Q)$
by (*simp add*: *parallel-pericondition-lemma1*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\$0-wait\ \vee\ \$1-wait)\ \wedge\ (\exists\ \$st\acute{}\ \cdot\ M)}\ cmt_R\ Q)$
by (*simp add*: *parallel-pericondition-lemma2 assms*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ ((\lceil cmt_R\ P\rceil_0 \wedge \lceil cmt_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\$0-wait \wedge \$1-wait \wedge (\exists\ \$st\acute{}\ \cdot\ M))$
$\vee\ (\lceil cmt_R\ P\rceil_0 \wedge \lceil cmt_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\neg\ \$0-wait \wedge \$1-wait \wedge (\exists\ \$st\acute{}\ \cdot\ M))$
$\vee\ (\lceil cmt_R\ P\rceil_0 \wedge \lceil cmt_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\$0-wait \wedge \neg\ \$1-wait \wedge (\exists\ \$st\acute{}\ \cdot\ M))))$
by (*simp add*: *par-by-merge-alt-def parallel-pericondition-lemma3 seqr-or-distr*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ ((\lceil peri_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\exists\ \$st\acute{}\ \cdot\ M)$
$\vee\ (\lceil post_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\exists\ \$st\acute{}\ \cdot\ M)$
$\vee\ (\lceil peri_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (\exists\ \$st\acute{}\ \cdot\ M)))$
by (*simp add*: *seqr-right-one-point-true seqr-right-one-point-false* cmt_R*-def* $post_R$*-def* $peri_R$*-def usubst unrest assms*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ peri_R(P) \parallel_{\exists\ \$st\acute{}\ \cdot\ M}\ peri_R(Q)$
$\vee\ post_R(P) \parallel_{\exists\ \$st\acute{}\ \cdot\ M}\ peri_R(Q)$
$\vee\ peri_R(P) \parallel_{\exists\ \$st\acute{}\ \cdot\ M}\ post_R(Q))$
by (*simp add*: *par-by-merge-alt-def*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition-lemma1*:
$(\$ok\acute{} \wedge P)\ ;;\ II_R[\![true,false/\$ok\acute{},\$wait\acute{}]\!] = P[\![true,false/\$ok\acute{},\$wait\acute{}]\!]$
(**is** *?lhs* = *?rhs*)
proof –
have *?lhs* = $(\$ok\acute{} \wedge P)\ ;;\ II[\![true,false/\$ok\acute{},\ \$wait\acute{}]\!]$
by (*rel-blast*)
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition-lemma2*:
assumes *M is RDM*
shows $(N_0(M))[\![true,false/\$ok\acute{},\$wait\acute{}]\!] = ((\neg\ \$0-wait \wedge \neg\ \$1-wait) \wedge M)$
proof –
have $(N_0(M))[\![true,false/\$ok\acute{},\$wait\acute{}]\!] = ((\neg\ \$0-wait \wedge \neg\ \$1-wait) \wedge \$tr\acute{} \geq_u \$tr_< \wedge M)$
by (*simp add*: *usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)
also have ... = $((\neg\ \$0-wait \wedge \neg\ \$1-wait) \wedge M)$
by (*metis Healthy-if R1m-def RDM-R1m assms utp-pred-laws.inf-commute*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition* [*rdes*]:
fixes M :: $('s,'t::trace,'\alpha)$ *rsp merge*
assumes *P is SRD Q is SRD M is RDM*
shows $post_R(P \parallel_{M_R(M)}\ Q) = (pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ post_R(P) \parallel_M\ post_R(Q))$
proof –
have $post_R(P \parallel_{M_R(M)}\ Q) =$
$(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\$ok\acute{}\ \wedge\ N_0\ M)\ ;;\ II_R[\![true,false/\$ok\acute{},\ \$wait\acute{}]\!]}\ cmt_R\ Q)$
by (*simp add*: *post-cmt-def parallel-commitment assms usubst unrest SRD-healths*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\neg\ \$0-wait\ \wedge\ \neg\ \$1-wait\ \wedge\ M)}\ cmt_R\ Q)$

by (*simp add*: *parallel-postcondition-lemma1 parallel-postcondition-lemma2 assms*, *simp add*: *utp-pred-laws.inf-commute utp-pred-laws.inf-left-commute*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r post_R\ P \parallel_M post_R\ Q)$
by (*simp add*: *par-by-merge-alt-def seqr-right-one-point-false usubst unrest cmt$_R$-def post$_R$-def assms*)
finally show *?thesis* **.**
qed

lemma *parallel-precondition-lemma*:
fixes $M :: ('s,'t::trace,'\alpha)\ rsp\ merge$
assumes *P is NSRD Q is NSRD M is RDM*
shows $(\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)}\ cmt_R(Q) =$
$((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ peri_R\ Q \vee (\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ post_R\ Q)$
proof −
have $((\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)}\ cmt_R(Q)) =$
$((\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)}\ (peri_R(Q) \diamond post_R(Q)))$
by (*simp add*: *wait'-cond-peri-post-cmt*)
also have ... = $((\lceil \neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q) \diamond post_R(Q)\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ N_0(M)\ ;;\ R1(true))$
by (*simp add*: *par-by-merge-alt-def*)
also have ... = $((\lceil \neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q)\rceil_1 \triangleleft \$1{-}wait' \triangleright \lceil post_R(Q)\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ N_0(M)\ ;;\ R1(true))$
by (*simp add*: *wait'-cond-def alpha*)
also have ... = $(((\lceil \neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q)\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}) \triangleleft \$1{-}wait' \triangleright (\lceil \neg_r\ pre_R(P)\rceil_0 \wedge \lceil post_R(Q)\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v}))\ ;;\ N_0(M)\ ;;\ R1(true))$
(**is** (*?P* ;; -) = (*?Q* ;; -))
proof −
have *?P* = *?Q*
by (*rel-auto*)
thus *?thesis* **by** *simp*
qed
also have ... = $((\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})[\![true/\$1{-}wait']\!]\ ;;\ (N_0\ M\ ;;\ R1\ true)[\![true/\$1{-}wait]\!] \vee$
$(\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})[\![false/\$1{-}wait']\!]\ ;;\ (N_0\ M\ ;;\ R1\ true)[\![false/\$1{-}wait]\!])$
by (*simp add*: *cond-inter-var-split*)
also have ... = $((\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ N_0\ M[\![true/\$1{-}wait]\!]\ ;;\ R1\ true \vee$
$(\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ N_0\ M[\![false/\$1{-}wait]\!]\ ;;\ R1\ true)$
by (*simp add*: *usubst unrest*)
also have ... = $((\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ (\$wait' \wedge M)\ ;;\ R1\ true \vee$
$(\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ (\$wait' =_u \$0{-}wait \wedge M)\ ;;\ R1\ true)$
proof −
have $(\$tr' \geq_u \$tr_< \wedge M) = M$
using *RDM-R1m*[*OF assms*(*3*)]
by (*simp add*: *Healthy-def R1m-def conj-comm*)
thus *?thesis*
by (*simp add*: *nmerge-rd0-def unrest assms closure ex-unrest usubst*)
qed
also have ... = $((\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ M\ ;;\ R1\ true \vee$
$(\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ M\ ;;\ R1\ true)$
(**is** $(?P_1 \vee_p ?P_2) = (?Q_1 \vee ?Q_2)$)
proof −
have $?P_1 = (\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ (M \wedge \$wait')\ ;;\ R1\ true$
by (*simp add*: *conj-comm*)
hence *1*: $?P_1 = ?Q_1$
by (*simp add*: *seqr-left-one-point-true seqr-left-one-point-false add*: *unrest usubst closure assms*)
have $?P_2 = ((\lceil \neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$\mathbf{v}_<{}' =_u \$\mathbf{v})\ ;;\ (M \wedge \$wait')\ ;;\ R1\ true \vee$

$(\lceil \neg_r\ pre_R\ P \rceil_0 \wedge \lceil post_R\ Q \rceil_1 \wedge \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ (M \wedge \neg\ \$wait\acute{})\ ;;\ R1\ true)$
by (*subst seqr-bool-split*[*of left-uvar wait*], *simp-all add*: *usubst unrest assms closure conj-comm*)
hence *2*: $?P_2 = ?Q_2$
by (*simp add*: *seqr-left-one-point-true seqr-left-one-point-false unrest usubst closure assms*)
from *1 2* **show** *?thesis* **by** *simp*
qed
also have ... $= ((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ peri_R\ Q \vee (\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ post_R\ Q)$
by (*simp add*: *par-by-merge-alt-def*)
finally show *?thesis* **.**
qed

lemma *swap-nmerge-rd0*:
$swap_m\ ;;\ N_0(M) = N_0(swap_m\ ;;\ M)$
by (*rel-auto*, *meson+*)

lemma *SymMerge-nmerge-rd0* [*closure*]:
M *is SymMerge* $\Longrightarrow N_0(M)$ *is SymMerge*
by (*rel-auto*, *meson+*)

lemma *swap-merge-rd'*:
$swap_m\ ;;\ N_R(M) = N_R(swap_m\ ;;\ M)$
by (*rel-blast*)

lemma *swap-merge-rd*:
$swap_m\ ;;\ M_R(M) = M_R(swap_m\ ;;\ M)$
by (*simp add*: *merge-rd-def seqr-assoc*[*THEN sym*] *swap-merge-rd'*)

lemma *SymMerge-merge-rd* [*closure*]:
M *is SymMerge* $\Longrightarrow M_R(M)$ *is SymMerge*
by (*simp add*: *Healthy-def swap-merge-rd*)

lemma *nmerge-rd1-merge3*:
assumes M *is RDM*
shows $\mathbf{M}3(N_1(M)) = (\$ok\acute{} =_u (\$0{-}ok \wedge \$1{-}0{-}ok \wedge \$1{-}1{-}ok) \wedge$
$\$wait\acute{} =_u (\$0{-}wait \vee \$1{-}0{-}wait \vee \$1{-}1{-}wait) \wedge$
$\mathbf{M}3(M))$
proof –
have $\mathbf{M}3(N_1(M)) = \mathbf{M}3(\$ok\acute{} =_u (\$0{-}ok \wedge \$1{-}ok) \wedge$
$\$wait\acute{} =_u (\$0{-}wait \vee \$1{-}wait) \wedge$
$\$tr_< \le_u \$tr\acute{} \wedge$
$(\exists\ \{\$0{-}ok, \$1{-}ok, \$ok_<, \$ok\acute{}, \$0{-}wait, \$1{-}wait, \$wait_<, \$wait\acute{}\} \cdot RDM(M)))$
by (*simp add*: *nmerge-rd1-def nmerge-rd0-def assms Healthy-if*)
also have ... $= \mathbf{M}3(\$ok\acute{} =_u (\$0{-}ok \wedge \$1{-}ok) \wedge \$wait\acute{} =_u (\$0{-}wait \vee \$1{-}wait) \wedge RDM(M))$
by (*rel-blast*)
also have ... $= (\$ok\acute{} =_u (\$0{-}ok \wedge \$1{-}0{-}ok \wedge \$1{-}1{-}ok) \wedge \$wait\acute{} =_u (\$0{-}wait \vee \$1{-}0{-}wait \vee \$1{-}1{-}wait) \wedge \mathbf{M}3(RDM(M)))$
by (*rel-blast*)
also have ... $= (\$ok\acute{} =_u (\$0{-}ok \wedge \$1{-}0{-}ok \wedge \$1{-}1{-}ok) \wedge \$wait\acute{} =_u (\$0{-}wait \vee \$1{-}0{-}wait \vee \$1{-}1{-}wait) \wedge \mathbf{M}3(M))$
by (*simp add*: *assms Healthy-if*)
finally show *?thesis* **.**
qed

lemma *nmerge-rd-merge3*:
$\mathbf{M}3(N_R(M)) = (\exists\ \$st_< \cdot \$\mathbf{v}\acute{} =_u \$\mathbf{v}_<) \triangleleft \$wait_< \triangleright \mathbf{M}3(N_1\ M) \triangleleft \$ok_< \triangleright (\$tr_< \le_u \$tr\acute{})$

by (*rel-blast*)

lemma *swap-merge-RDM-closed* [*closure*]:
assumes *M is RDM*
shows $swap_m$;; *M is RDM*
proof –
have $RDM(swap_m \;;;\; RDM(M)) = (swap_m \;;;\; RDM(M))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def′ assms*)
qed

lemma *parallel-precondition*:
fixes M :: ($'s,'t$::*trace*,$'\alpha$) *rsp merge*
assumes *P is NSRD Q is NSRD M is RDM*
shows $pre_R(P \parallel_{M_R(M)} Q) =$
$(\neg_r ((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ peri_R\ Q) \wedge$
$\neg_r ((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ post_R\ Q) \wedge$
$\neg_r ((\neg_r\ pre_R\ Q) \parallel_{(swap_m\ ;;\ M)\ ;;\ R1(true)}\ peri_R\ P) \wedge$
$\neg_r ((\neg_r\ pre_R\ Q) \parallel_{(swap_m\ ;;\ M)\ ;;\ R1(true)}\ post_R\ P))$
proof –
have *a*: $(\neg_r\ pre_R(P)) \parallel_{N_0(M)\ ;;\ R1(true)}\ cmt_R(Q) =$
$((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ peri_R\ Q \vee (\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)}\ post_R\ Q)$
by (*simp add*: *parallel-precondition-lemma assms*)

have *b*: $(\neg_r\ cmt_R\ P \parallel_{N_0}\ M\ ;;\ R1\ true\ (\neg_r\ pre_R\ Q)) =$
$(\neg_r\ (\neg_r\ pre_R(Q)) \parallel_{N_0(swap_m\ ;;\ M)\ ;;\ R1(true)}\ cmt_R(P))$
by (*simp add*: *swap-nmerge-rd0*[*THEN sym*] *seqr-assoc*[*THEN sym*] *par-by-merge-def par-sep-swap*)
have *c*: $(\neg_r\ pre_R(Q)) \parallel_{N_0(swap_m\ ;;\ M)\ ;;\ R1(true)}\ cmt_R(P) =$
$((\neg_r\ pre_R\ Q) \parallel_{(swap_m\ ;;\ M)\ ;;\ R1(true)}\ peri_R\ P \vee (\neg_r\ pre_R\ Q) \parallel_{(swap_m\ ;;\ M)\ ;;\ R1(true)}\ post_R$
$P)$
by (*simp add*: *parallel-precondition-lemma closure assms*)

show *?thesis*
by (*simp add*: *parallel-assm closure assms a b c*, *rel-auto*)
qed

Weakest Parallel Precondition

definition *wrR* ::
($'t$::*trace*, $'\alpha$) *hrel-rp* ⇒
($'t$ :: *trace*, $'\alpha$) *rp merge* ⇒
($'t$, $'\alpha$) *hrel-rp* ⇒
($'t$, $'\alpha$) *hrel-rp* (- wr_R′(-′) - [*60*,*0*,*61*] *61*)
where [*upred-defs*]: $Q\ wr_R(M)\ P = (\neg_r\ ((\neg_r\ P) \parallel_{M\ ;;\ R1(true)}\ Q))$

lemma *wrR-R1* [*closure*]:
M *is R1m* $\Longrightarrow Q\ wr_R(M)\ P$ *is R1*
by (*simp add*: *wrR-def closure*)

lemma *R2-rea-not*: $R2(\neg_r\ P) = (\neg_r\ R2(P))$
by (*rel-auto*)

lemma *wrR-R2-lemma*:
assumes *P is R2 Q is R2 M is R2m*

shows $((\neg_r\ P) \parallel_M Q)$;; $R1(true_h)$ *is R2*
proof −
have $(\neg_r\ P) \parallel_M Q$ *is R2*
by (*simp add*: *closure assms*)
thus *?thesis*
by (*simp add*: *closure*)
qed

lemma *wrR-R2* [*closure*]:
assumes *P is R2 Q is R2 M is R2m*
shows $Q\ wr_R(M)\ P$ *is R2*
proof −
have $((\neg_r\ P) \parallel_M Q)$;; $R1(true_h)$ *is R2*
by (*simp add*: *wrR-R2-lemma assms*)
thus *?thesis*
by (*simp add*: *wrR-def wrR-R2-lemma par-by-merge-seq-add closure*)
qed

lemma *wrR-RR* [*closure*]:
assumes *P is RR Q is RR M is RDM*
shows $Q\ wr_R(M)\ P$ *is RR*
apply (*rule RR-intro*)
apply (*simp-all add*: *unrest assms closure wrR-def rpred*)
apply (*metis* (*no-types*, *lifting*) *Healthy-def′ R1-R2c-commute R1-R2c-is-R2 R1-rea-not RDM-R2m RR-implies-R2 assms*(*1*) *assms*(*2*) *assms*(*3*) *par-by-merge-seq-add rea-not-R2-closed wrR-R2-lemma*)
done

lemma *wrR-RC* [*closure*]:
assumes *P is RR Q is RR M is RDM*
shows $(Q\ wr_R(M)\ P)$ *is RC*
apply (*rule RC-intro*)
apply (*simp add*: *closure assms*)
apply (*simp add*: *wrR-def rpred closure assms*)
apply (*simp add*: *par-by-merge-def seqr-assoc*)
done

lemma *wppR-choice* [*wp*]: $(P \lor Q)\ wr_R(M)\ R = (P\ wr_R(M)\ R \land Q\ wr_R(M)\ R)$
proof −
have $(P \lor Q)\ wr_R(M)\ R =$
$(\neg_r\ ((\neg_r\ R)\ ;;\ U0 \land (P\ ;;\ U1 \lor Q\ ;;\ U1) \land \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ M\ ;;\ true_r)$
by (*simp add*: *wrR-def par-by-merge-def seqr-or-distl*)
also have ... $= (\neg_r\ ((\neg_r\ R)\ ;;\ U0 \land P\ ;;\ U1 \land \$\mathbf{v}_<\acute{} =_u \$\mathbf{v} \lor (\neg_r\ R)\ ;;\ U0 \land Q\ ;;\ U1 \land \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ M\ ;;\ true_r)$
by (*simp add*: *conj-disj-distr utp-pred-laws.inf-sup-distrib2*)
also have ... $= (\neg_r\ (((\neg_r\ R)\ ;;\ U0 \land P\ ;;\ U1 \land \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ M\ ;;\ true_r \lor$
$((\neg_r\ R)\ ;;\ U0 \land Q\ ;;\ U1 \land \$\mathbf{v}_<\acute{} =_u \$\mathbf{v})\ ;;\ M\ ;;\ true_r))$
by (*simp add*: *seqr-or-distl*)
also have ... $= (P\ wr_R(M)\ R \land Q\ wr_R(M)\ R)$
by (*simp add*: *wrR-def par-by-merge-def*)
finally show *?thesis* **.**
qed

lemma *wppR-miracle* [*wp*]: *false* $wr_R(M)\ P = true_r$
by (*simp add*: *wrR-def*)

lemma *wppR-true* [*wp*]: $P\ wr_R(M)\ true_r = true_r$
  by (*simp add*: *wrR-def*)

lemma *parallel-precondition-wr* [*rdes*]:
  assumes P *is NSRD* Q *is NSRD* M *is RDM*
  shows $pre_R(P \parallel_{M_R(M)} Q) = (peri_R(Q)\ wr_R(M)\ pre_R(P) \wedge post_R(Q)\ wr_R(M)\ pre_R(P) \wedge$
  $peri_R(P)\ wr_R(swap_m\ ;;\ M)\ pre_R(Q) \wedge post_R(P)\ wr_R(swap_m\ ;;\ M)\ pre_R(Q))$
  by (*simp add*: *assms parallel-precondition wrR-def*)

lemma *parallel-rdes-def* [*rdes-def*]:
  assumes P_1 *is RC* P_2 *is RR* P_3 *is RR* Q_1 *is RC* Q_2 *is RR* Q_3 *is RR*
  $\$st\acute{}\ \sharp\ P_2\ \$st\acute{}\ \sharp\ Q_2$
  M *is RDM*
  shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \parallel_{M_R(M)} \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
  $\mathbf{R}_s\ (((Q_1 \Rightarrow_r Q_2)\ wr_R(M)\ P_1 \wedge (Q_1 \Rightarrow_r Q_3)\ wr_R(M)\ P_1 \wedge$
  $(P_1 \Rightarrow_r P_2)\ wr_R(swap_m\ ;;\ M)\ Q_1 \wedge (P_1 \Rightarrow_r P_3)\ wr_R(swap_m\ ;;\ M)\ Q_1) \vdash$
  $((P_1 \Rightarrow_r P_2) \parallel_{\exists\ \$st\acute{}\ \cdot\ M} (Q_1 \Rightarrow_r Q_2) \vee$
  $(P_1 \Rightarrow_r P_3) \parallel_{\exists\ \$st\acute{}\ \cdot\ M} (Q_1 \Rightarrow_r Q_2) \vee (P_1 \Rightarrow_r P_2) \parallel_{\exists\ \$st\acute{}\ \cdot\ M} (Q_1 \Rightarrow_r Q_3)) \diamond$
  $((P_1 \Rightarrow_r P_3) \parallel_M (Q_1 \Rightarrow_r Q_3)))$ (**is** *?lhs* = *?rhs*)
proof −
  have $?lhs = \mathbf{R}_s\ (pre_R\ ?lhs \vdash peri_R\ ?lhs \diamond post_R\ ?lhs)$
    by (*simp add*: *SRD-reactive-tri-design assms closure*)
  also have ... = *?rhs*
    by (*simp add*: *rdes closure unrest assms*, *rel-auto*)
  finally show *?thesis* **.**
qed

lemma *Miracle-parallel-left-zero*:
  assumes P *is SRD* M *is RDM*
  shows $Miracle \parallel_{RM} P = Miracle$
proof −
  have $pre_R(Miracle \parallel_{RM} P) = true_r$
    by (*simp add*: *parallel-assm wait'-cond-idem rdes closure assms*)
  moreover hence $cmt_R(Miracle \parallel_{RM} P) = false$
    by (*simp add*: *rdes closure wait'-cond-idem SRD-healths assms*)
  ultimately have $Miracle \parallel_{RM} P = \mathbf{R}_s(true_r \vdash false)$
  by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed*)
  thus *?thesis*
    by (*simp add*: *Miracle-def R1-design-R1-pre*)
qed

lemma *Miracle-parallel-right-zero*:
  assumes P *is SRD* M *is RDM*
  shows $P \parallel_{RM} Miracle = Miracle$
proof −
  have $pre_R(P \parallel_{RM} Miracle) = true_r$
    by (*simp add*: *wait'-cond-idem parallel-assm rdes closure assms*)
  moreover hence $cmt_R(P \parallel_{RM} Miracle) = false$
    by (*simp add*: *wait'-cond-idem rdes closure SRD-healths assms*)
  ultimately have $P \parallel_{RM} Miracle = \mathbf{R}_s(true_r \vdash false)$
  by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed*)
  thus *?thesis*
    by (*simp add*: *Miracle-def R1-design-R1-pre*)
qed

8.1 Example basic merge

definition *BasicMerge* :: $(('s, 't::trace, unit)\ rsp)\ merge\ (N_B)$ **where**
[*upred-defs*]: $BasicMerge = (\$tr_< \leq_u \$tr\acute{} \wedge \$tr\acute{} - \$tr_< =_u \$0{-}tr - \$tr_< \wedge \$tr\acute{} - \$tr_< =_u \$1{-}tr - \$tr_< \wedge \$st\acute{} =_u \$st_<)$

abbreviation *rbasic-par* (- $\|_B$ - [*85*,*86*] *85*) **where**
$P \|_B Q \equiv P \|_{M_R(N_B)} Q$

lemma *BasicMerge-RDM* [*closure*]: N_B *is RDM*
  by (*rule RDM-intro*, (*rel-auto*)+)

lemma *BasicMerge-SymMerge* [*closure*]:
  N_B *is SymMerge*
  by (*rel-auto*)

lemma *BasicMerge′-calc*:
  assumes $\$ok\acute{} \sharp P$ $\$wait\acute{} \sharp P$ $\$ok\acute{} \sharp Q$ $\$wait\acute{} \sharp Q$ *P is R2 Q is R2*
  shows $P \|_{N_B} Q = ((\exists \$st\acute{} \cdot P) \wedge (\exists \$st\acute{} \cdot Q) \wedge \$st\acute{} =_u \$st)$
  using *assms*
proof −
  have $P{:}(\exists \{\$ok\acute{},\$wait\acute{}\} \cdot R2(P)) = P$ (**is** $?P' = -$)
    by (*simp add*: *ex-unrest ex-plus Healthy-if assms*)
  have $Q{:}(\exists \{\$ok\acute{},\$wait\acute{}\} \cdot R2(Q)) = Q$ (**is** $?Q' = -$)
    by (*simp add*: *ex-unrest ex-plus Healthy-if assms*)
  have $?P' \|_{N_B} ?Q' = ((\exists \$st\acute{} \cdot ?P') \wedge (\exists \$st\acute{} \cdot ?Q') \wedge \$st\acute{} =_u \$st)$
    by (*simp add*: *par-by-merge-alt-def*, *rel-auto*, *blast*+)
  thus *?thesis*
    by (*simp add*: *P Q*)
qed

8.2 Simple parallel composition

definition *rea-design-par* ::
  $('s, 't::trace, '\alpha)\ hrel\text{-}rsp \Rightarrow ('s, 't, '\alpha)\ hrel\text{-}rsp \Rightarrow ('s, 't, '\alpha)\ hrel\text{-}rsp$ (**infixr** $\|_R$ *85*)
where [*upred-defs*]: $P \|_R Q = \mathbf{R}_s((pre_R(P) \wedge pre_R(Q)) \vdash (cmt_R(P) \wedge cmt_R(Q)))$

lemma *RHS-design-par*:
  assumes
    $\$ok\acute{} \sharp P_1$ $\$ok\acute{} \sharp P_2$
  shows $\mathbf{R}_s(P_1 \vdash Q_1) \|_R \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
proof −
  have $\mathbf{R}_s(P_1 \vdash Q_1) \|_R \mathbf{R}_s(P_2 \vdash Q_2) =$
        $\mathbf{R}_s(P_1[\![true,false/\$ok,\$wait]\!] \vdash Q_1[\![true,false/\$ok,\$wait]\!]) \|_R \mathbf{R}_s(P_2[\![true,false/\$ok,\$wait]\!] \vdash Q_2[\![true,false/\$ok,\$wait]\!])$
    by (*simp add*: *RHS-design-ok-wait*)

  also from *assms*
  have ... =
      $\mathbf{R}_s((R1\ (R2c\ (P_1)) \wedge R1\ (R2c\ (P_2)))[\![true,false/\$ok,\$wait]\!] \vdash$
        $(R1\ (R2c\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2c\ (P_2 \Rightarrow Q_2)))[\![true,false/\$ok,\$wait]\!])$
    apply (*simp add*: *rea-design-par-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest assms*)
    apply (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*)
    using *assms* **apply** (*rel-auto*)
  done
  also have ... =

$\mathbf{R}_s((R2c(P_1) \wedge R2c(P_2)) \vdash$
$(R1\ (R2s\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (P_2 \Rightarrow Q_2))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-conj R1-design-R1-pre RHS-design-ok-wait*)
also have ... =
$\mathbf{R}_s((P_1 \wedge P_2) \vdash (R1\ (R2s\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (P_2 \Rightarrow Q_2))))$
by (*simp add*: *R2c-R3h-commute R2c-and R2c-design R2c-idem R2c-not RHS-def*)
also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
by (*metis* (*no-types*, *lifting*) *R1-conj R2s-conj RHS-design-export-R1 RHS-design-export-R2s*)
also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-par*:
assumes $\$ok' \sharp P_1$ $\$ok' \sharp P_2$
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
by (*simp add*: *RHS-design-par assms unrest wait'-cond-conj-exchange*)

lemma *RHS-tri-design-par-RR* [*rdes-def*]:
assumes P_1 *is RR* P_2 *is RR*
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
by (*simp add*: *RHS-tri-design-par unrest assms*)

lemma *RHS-comp-assoc*:
assumes *P is NSRD Q is NSRD R is NSRD*
shows $(P \parallel_R Q) \parallel_R R = P \parallel_R Q \parallel_R R$
by (*rdes-eq cls*: *assms*)

end

9 Productive Reactive Designs

theory *utp-rdes-productive*
imports *utp-rdes-parallel*
begin

9.1 Healthiness condition

A reactive design is productive if it strictly increases the trace, whenever it terminates. If it does not terminate, it is also classed as productive.

definition *Productive* :: ('s, 't::trace, 'α) *hrel-rsp* ⇒ ('s, 't, 'α) *hrel-rsp* **where**
[*upred-defs*]: $Productive(P) = P \parallel_R \mathbf{R}_s(true \vdash true \diamond (\$tr <_u \$tr'))$

lemma *Productive-RHS-design-form*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$
shows $Productive(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash Q \diamond (R \wedge \$tr <_u \$tr'))$
using *assms* **by** (*simp add*: *Productive-def RHS-tri-design-par unrest*)

lemma *Productive-form*:
$Productive(SRD(P)) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$
proof −
have $Productive(SRD(P)) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \parallel_R \mathbf{R}_s(true \vdash true \diamond (\$tr <_u \$tr'))$
by (*simp add*: *Productive-def SRD-as-reactive-tri-design*)
also have ... = $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

by (*simp add*: *RHS-tri-design-par unrest*)
finally show *?thesis* **.**
qed

A reactive design is productive provided that the postcondition, under the precondition, strictly increases the trace.

lemma *Productive-intro*:
assumes *P is SRD* $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$ $\$wait' \sharp pre_R(P)$
shows *P is Productive*
proof −
have P:$\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) = P$
proof −
have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge post_R(P)))$
by (*metis* (*no-types*, *hide-lams*) *design-export-pre wait′-cond-conj-exchange wait′-cond-idem*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis assms*(*2*) *utp-pred-laws.inf.absorb1 utp-pred-laws.inf.assoc*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$
by (*metis* (*no-types*, *hide-lams*) *design-export-pre wait′-cond-conj-exchange wait′-cond-idem*)
finally show *?thesis*
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
qed
thus *?thesis*
by (*metis Healthy-def RHS-tri-design-par Productive-def ok′-pre-unrest unrest-true utp-pred-laws.inf-right-idem utp-pred-laws.inf-top-right*)
qed

lemma *Productive-post-refines-tr-increase*:
assumes *P is SRD P is Productive* $\$wait' \sharp pre_R(P)$
shows $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$
proof −
have $post_R(P) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def Productive-form assms*(*1*) *assms*(*2*))
also have ... $= R1(R2c(pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
by (*simp add*: *rea-post-RHS-design unrest usubst assms*, *rel-auto*)
also have ... $= R1((pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
by (*simp add*: *R2c-impl R2c-preR R2c-postR R2c-and R2c-tr-less-tr′ assms*)
also have $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge ...)$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *Continuous-Productive* [*closure*]: *Continuous Productive*
by (*simp add*: *Continuous-def Productive-def*, *rel-auto*)

9.2 Reactive design calculations

lemma *preR-Productive* [*rdes*]:
assumes *P is SRD*
shows $pre_R(Productive(P)) = pre_R(P)$
proof −
have $pre_R(Productive(P)) = pre_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def Productive-form assms*)
thus *?thesis*
by (*simp add*: *rea-pre-RHS-design usubst unrest R2c-not R2c-preR R1-preR Healthy-if assms*)

qed

lemma *periR-Productive* [*rdes*]:
  assumes *P is NSRD*
  shows $peri_R(Productive(P)) = peri_R(P)$
proof −
  have $peri_R(Productive(P)) = peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
    by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)
  also have ... = $R1\ (R2c\ (pre_R\ P \Rightarrow_r peri_R\ P))$
    by (*simp add*: *rea-peri-RHS-design usubst unrest R2c-not assms closure*)
  also have ... = $(pre_R\ P \Rightarrow_r peri_R\ P)$
    by (*simp add*: *R1-rea-impl R2c-rea-impl R2c-preR R2c-peri-SRD R1-peri-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)
  finally show *?thesis*
    by (*simp add*: *SRD-peri-under-pre assms unrest closure*)
qed

lemma *postR-Productive* [*rdes*]:
  assumes *P is NSRD*
  shows $post_R(Productive(P)) = (pre_R(P) \Rightarrow_r post_R(P) \wedge \$tr <_u \$tr')$
proof −
  have $post_R(Productive(P)) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
    by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)
  also have ... = $R1\ (R2c\ (pre_R\ P \Rightarrow_r post_R\ P \wedge \$tr' >_u \$tr))$
    by (*simp add*: *rea-post-RHS-design usubst unrest assms closure*)
  also have ... = $(pre_R\ P \Rightarrow_r post_R\ P \wedge \$tr' >_u \$tr)$
    by (*simp add*: *R1-rea-impl R2c-rea-impl R2c-preR R2c-and R1-extend-conj' R2c-post-SRD R1-post-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)
  finally show *?thesis* **.**
qed

lemma *preR-frame-seq-export*:
  assumes *P is NSRD P is Productive Q is NSRD*
  shows $(pre_R\ P \wedge (pre_R\ P \wedge post_R\ P)\ ;;\ Q) = (pre_R\ P \wedge (post_R\ P\ ;;\ Q))$
proof −
  have $(pre_R\ P \wedge (post_R\ P\ ;;\ Q)) = (pre_R\ P \wedge ((pre_R\ P \Rightarrow_r post_R\ P)\ ;;\ Q))$
    by (*simp add*: *SRD-post-under-pre assms closure unrest*)
  also have ... = $(pre_R\ P \wedge (((\neg_r\ pre_R\ P)\ ;;\ Q \vee (pre_R\ P \Rightarrow_r R1(post_R\ P))\ ;;\ Q)))$
    by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*(*1*) *rea-impl-def seqr-or-distl R1-preR Healthy-if*)
  also have ... = $(pre_R\ P \wedge (((\neg_r\ pre_R\ P)\ ;;\ Q \vee (pre_R\ P \wedge post_R\ P)\ ;;\ Q)))$
  proof −
    have $(pre_R\ P \vee \neg_r\ pre_R\ P) = R1\ true$
      by (*simp add*: *R1-preR rea-not-or*)
    then show *?thesis*
      by (*metis* (*no-types*, *lifting*) *R1-def conj-comm disj-comm disj-conj-distr rea-impl-def seqr-or-distl utp-pred-laws.inf-top-left utp-pred-laws.sup.left-idem*)
  qed
  also have ... = $(pre_R\ P \wedge (((\neg_r\ pre_R\ P) \vee (pre_R\ P \wedge post_R\ P)\ ;;\ Q)))$
    by (*simp add*: *NSRD-neg-pre-left-zero assms closure SRD-healths*)
  also have ... = $(pre_R\ P \wedge (pre_R\ P \wedge post_R\ P)\ ;;\ Q)$
    by (*rel-blast*)
  finally show *?thesis* **..**
qed

9.3 Closure laws

lemma *Productive-rdes-intro*:
  assumes $(\$tr <_u \$tr') \sqsubseteq R$ $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$ $\$wait \sharp P$ $\$wait' \sharp P$
  shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ *is Productive*
proof (*rule Productive-intro*)
  show $\mathbf{R}_s$ $(P \vdash Q \diamond R)$ *is SRD*
    by (*simp add*: *RHS-tri-design-is-SRD assms*)

  from *assms*(*1*) **show** $(\$tr' >_u \$tr) \sqsubseteq (pre_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R)) \wedge post_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R)))$
    apply (*simp add*: *rea-pre-RHS-design rea-post-RHS-design usubst assms unrest*)
    using *assms*(*1*) **apply** (*rel-auto*)
    apply *fastforce*
    done

  show $\$wait' \sharp\ pre_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R))$
   by (*simp add*: *rea-pre-RHS-design rea-post-RHS-design usubst R1-def R2c-def R2s-def assms unrest*)
qed

We use the *R4* healthiness condition to characterise that the postcondition must extend the trace for a reactive design to be productive.

lemma *Productive-rdes-RR-intro*:
  assumes *P is RR Q is RR R is RR R is R4*
  shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ *is Productive*
  using *assms* **by** (*simp add*: *Productive-rdes-intro R4-iff-refine unrest*)

lemma *Productive-Miracle* [*closure*]: *Miracle is Productive*
  unfolding *Miracle-tri-def Healthy-def*
  by (*subst Productive-RHS-design-form*, *simp-all add*: *unrest*)

lemma *Productive-Chaos* [*closure*]: *Chaos is Productive*
  unfolding *Chaos-tri-def Healthy-def*
  by (*subst Productive-RHS-design-form*, *simp-all add*: *unrest*)

lemma *Productive-intChoice* [*closure*]:
  assumes *P is SRD P is Productive Q is SRD Q is Productive*
  shows $P \sqcap Q$ *is Productive*
proof −
  have $P \sqcap Q =$
        $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$
    by (*metis Healthy-if Productive-form assms*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \wedge pre_R\ Q) \vdash (peri_R\ P \vee peri_R\ Q) \diamond ((post_R\ P \wedge \$tr' >_u \$tr) \vee (post_R\ Q \wedge \$tr' >_u \$tr)))$
    by (*simp add*: *RHS-tri-design-choice*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \wedge pre_R\ Q) \vdash (peri_R\ P \vee peri_R\ Q) \diamond (((post_R\ P) \vee (post_R\ Q)) \wedge \$tr' >_u \$tr))$
    by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
  also have ... *is Productive*
    by (*simp add*: *Healthy-def Productive-RHS-design-form unrest*)
  finally show *?thesis* **.**
qed

lemma *Productive-cond-rea* [*closure*]:
  assumes *P is SRD P is Productive Q is SRD Q is Productive*
  shows $P \triangleleft b \triangleright_R Q$ *is Productive*

proof −
  have $P \triangleleft b \triangleright_R Q =$
    $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \triangleleft b \triangleright_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$
    by (*metis Healthy-if Productive-form assms*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \triangleleft b \triangleright_R pre_R\ Q) \vdash (peri_R\ P \triangleleft b \triangleright_R peri_R\ Q) \diamond ((post_R\ P \wedge \$tr' >_u \$tr) \triangleleft b \triangleright_R (post_R\ Q \wedge \$tr' >_u \$tr)))$
    by (*simp add*: *cond-srea-form*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \triangleleft b \triangleright_R pre_R\ Q) \vdash (peri_R\ P \triangleleft b \triangleright_R peri_R\ Q) \diamond (((post_R\ P) \triangleleft b \triangleright_R (post_R\ Q)) \wedge \$tr' >_u \$tr))$
    by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
  also have ... *is Productive*
    by (*simp add*: *Healthy-def Productive-RHS-design-form unrest*)
  finally show *?thesis* **.**
qed

lemma *Productive-seq-1* [*closure*]:
  assumes *P is NSRD P is Productive Q is NSRD*
  shows *P* ;; *Q is Productive*
proof −
  have $P\ ;;\ Q = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))\ ;;\ \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q)))$
    by (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*(*1*) *assms*(*2*) *assms*(*3*))
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \wedge (post_R\ P \wedge \$tr' >_u \$tr)\ wp_r\ pre_R\ Q) \vdash$
    $(peri_R\ P \vee ((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ peri_R\ Q)) \diamond ((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ post_R\ Q))$
    by (*simp add*: *RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \wedge (post_R\ P \wedge \$tr' >_u \$tr)\ wp_r\ pre_R\ Q) \vdash$
    $(peri_R\ P \vee ((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ peri_R\ Q)) \diamond ((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ post_R\ Q \wedge \$tr' >_u \$tr))$
  proof −
    have $((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ R1(post_R\ Q)) = ((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ R1(post_R\ Q) \wedge \$tr' >_u \$tr)$
      by (*rel-auto*)
    thus *?thesis*
      by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*)
  qed
  also have ... *is Productive*
    by (*rule Productive-rdes-intro*, *simp-all add*: *unrest assms closure wp-rea-def*)
  finally show *?thesis* **.**
qed

lemma *Productive-seq-2* [*closure*]:
  assumes *P is NSRD Q is NSRD Q is Productive*
  shows *P* ;; *Q is Productive*
proof −
  have $P\ ;;\ Q = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P)))\ ;;\ \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$
    by (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*)
  also have ... $= \mathbf{R}_s\ ((pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash (peri_R\ P \vee (post_R\ P\ ;;\ peri_R\ Q)) \diamond (post_R\ P\ ;;\ (post_R\ Q \wedge \$tr' >_u \$tr)))$
    by (*simp add*: *RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)

also have ... = $\mathbf{R}_s$ $((pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash (peri_R\ P \vee (post_R\ P\ ;;\ peri_R\ Q)) \diamond (post_R\ P\ ;;\ (post_R\ Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr))$
proof −
have $(R1(post_R\ P)\ ;;\ (post_R\ Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr) = (R1(post_R\ P)\ ;;\ (post_R\ Q \wedge \$tr' >_u \$tr))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*)
qed
also have ... *is Productive*
by (*rule Productive-rdes-intro*, *simp-all add*: *unrest assms closure wp-rea-def*)
finally show *?thesis* **.**
qed

end

10 Guarded Recursion

theory *utp-rdes-guarded*
imports *utp-rdes-productive*
begin

10.1 Traces with a size measure

Guarded recursion relies on our ability to measure the trace's size, in order to see if it is decreasing on each iteration. Thus, we here equip the trace algebra with the *ucard* function that provides this.

class *size-trace* = *trace* + *size* +
assumes
size-zero: $size\ 0 = 0$ **and**
size-nzero: $s > 0 \Longrightarrow size(s) > 0$ **and**
size-plus: $size\ (s + t) = size(s) + size(t)$
— These axioms may be stronger than necessary. In particular, $0 < ?s \Longrightarrow 0 < \#_u(?s)$ requires that a non-empty trace have a positive size. But this may not be the case with all trace models and is possibly more restrictive than necessary. In future we will explore weakening.
begin

lemma *size-mono*: $s \leq t \Longrightarrow size(s) \leq size(t)$
by (*metis le-add1 local.diff-add-cancel-left' local.size-plus*)

lemma *size-strict-mono*: $s < t \Longrightarrow size(s) < size(t)$
by (*metis cancel-ab-semigroup-add-class.add-diff-cancel-left' local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero local.size-plus zero-less-diff*)

lemma *trace-strict-prefixE*: $xs < ys \Longrightarrow (\bigwedge zs.\ [\![\ ys = xs + zs;\ size(zs) > 0\ ]\!] \Longrightarrow thesis) \Longrightarrow thesis$
by (*metis local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero*)

lemma *size-minus-trace*: $y \leq x \Longrightarrow size(x - y) = size(x) - size(y)$
by (*metis diff-add-inverse local.diff-add-cancel-left' local.size-plus*)

end

Both natural numbers and lists are measurable trace algebras.

instance *nat* :: *size-trace*

by (*intro-classes*, *simp-all*)

instance *list* :: (*type*) *size-trace*
by (*intro-classes*, *simp-all add*: *zero-list-def less-list-def′ plus-list-def prefix-length-less*)

syntax
-usize :: *logic* ⇒ *logic* ($size_u$'(-'))

translations
$size_u(t)$ == *CONST uop CONST size t*

10.2 Guardedness

definition *gvrt* :: ((′*t*::*size-trace*,′α) *rp* × (′*t*,′α) *rp*) *chain* **where**
[*upred-defs*]: $gvrt(n) \equiv (\$tr \leq_u \$tr' \wedge size_u(\&tt) <_u «n»)$

lemma *gvrt-chain*: *chain gvrt*
apply (*simp add*: *chain-def*, *safe*)
apply (*rel-simp*)
apply (*rel-simp*)+
done

lemma *gvrt-limit*: $\bigsqcap$ (*range gvrt*) = ($\$tr \leq_u \$tr'$)
by (*rel-auto*)

definition *Guarded* :: ((′*t*::*size-trace*,′α) *hrel-rp* ⇒ (′*t*,′α) *hrel-rp*) ⇒ *bool* **where**
[*upred-defs*]: $Guarded(F) = (\forall\ X\ n.\ (F(X) \wedge gvrt(n+1)) = (F(X \wedge gvrt(n)) \wedge gvrt(n+1)))$

lemma *GuardedI*: $[\![\ \bigwedge X\ n.\ (F(X) \wedge gvrt(n+1)) = (F(X \wedge gvrt(n)) \wedge gvrt(n+1))\ ]\!] \Longrightarrow Guarded\ F$
by (*simp add*: *Guarded-def*)

Guarded reactive designs yield unique fixed-points.

theorem *guarded-fp-uniq*:
assumes *mono F* $F \in [\![id]\!]_H \to [\![SRD]\!]_H$ *Guarded F*
shows $\mu\ F = \nu\ F$
proof −
have *constr F gvrt*
using *assms*
by (*auto simp add*: *constr-def gvrt-chain Guarded-def tcontr-alt-def′*)
hence ($\$tr \leq_u \$tr' \wedge \mu\ F$) = ($\$tr \leq_u \$tr' \wedge \nu\ F$)
apply (*rule constr-fp-uniq*)
apply (*simp add*: *assms*)
using *gvrt-limit* **apply** *blast*
done
moreover have ($\$tr \leq_u \$tr' \wedge \mu\ F$) = $\mu\ F$
proof −
have μ *F is R1*
by (*rule SRD-healths*(*1*), *rule Healthy-mu*, *simp-all add*: *assms*)
thus *?thesis*
by (*metis Healthy-def R1-def conj-comm*)
qed
moreover have ($\$tr \leq_u \$tr' \wedge \nu\ F$) = $\nu\ F$
proof −
have ν *F is R1*
by (*rule SRD-healths*(*1*), *rule Healthy-nu*, *simp-all add*: *assms*)
thus *?thesis*

    by (*metis Healthy-def R1-def conj-comm*)
  qed
  ultimately show *?thesis*
    by (*simp*)
qed

lemma *Guarded-const* [*closure*]: *Guarded* (λ *X*. *P*)
  by (*simp add*: *Guarded-def*)

lemma *UINF-Guarded* [*closure*]:
  assumes $\bigwedge$ *P*. $P \in A \Longrightarrow$ *Guarded P*
  shows *Guarded* (λ *X*. $\bigsqcap P{\in}A \cdot P(X)$)
proof (*rule GuardedI*)
  fix *X n*
  have $\bigwedge$ *Y*. $((\bigsqcap P{\in}A \cdot P\ Y) \wedge gvrt(n{+}1)) = ((\bigsqcap P{\in}A \cdot (P\ Y \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
  proof −
    fix *Y*
    let $?lhs = ((\bigsqcap P{\in}A \cdot P\ Y) \wedge gvrt(n{+}1))$ **and** $?rhs = ((\bigsqcap P{\in}A \cdot (P\ Y \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
    have *a*:$?lhs[\![false/\$ok]\!] = ?rhs[\![false/\$ok]\!]$
      by (*rel-auto*)
    have *b*:$?lhs[\![true/\$ok]\!][\![true/\$wait]\!] = ?rhs[\![true/\$ok]\!][\![true/\$wait]\!]$
      by (*rel-auto*)
    have *c*:$?lhs[\![true/\$ok]\!][\![false/\$wait]\!] = ?rhs[\![true/\$ok]\!][\![false/\$wait]\!]$
      by (*rel-auto*)
    show *?lhs* = *?rhs*
      using *a b c*
      by (*rule-tac bool-eq-splitI*[*of in-var ok*], *simp*, *rule-tac bool-eq-splitI*[*of in-var wait*], *simp-all*)
  qed
  moreover have $((\bigsqcap P{\in}A \cdot (P\ X \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1)) = ((\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
  proof −
    have $(\bigsqcap P{\in}A \cdot (P\ X \wedge gvrt(n{+}1))) = (\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1)))$
    proof (*rule UINF-cong*)
      fix *P* **assume** $P \in A$
      thus $(P\ X \wedge gvrt(n{+}1)) = (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1))$
        using *Guarded-def assms* **by** *blast*
    qed
    thus *?thesis* **by** *simp*
  qed
  ultimately show $((\bigsqcap P{\in}A \cdot P\ X) \wedge gvrt(n{+}1)) = ((\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)))) \wedge gvrt(n{+}1))$
    by *simp*
qed

lemma *intChoice-Guarded* [*closure*]:
  assumes *Guarded P Guarded Q*
  shows *Guarded* (λ *X*. $P(X) \sqcap Q(X)$)
proof −
  have *Guarded* (λ *X*. $\bigsqcap F{\in}\{P,Q\} \cdot F(X)$)
    by (*rule UINF-Guarded*, *auto simp add*: *assms*)
  thus *?thesis*
    by (*simp*)
qed

lemma *cond-srea-Guarded* [*closure*]:
  assumes *Guarded P Guarded Q*

shows *Guarded* (λ X. $P(X) \triangleleft b \triangleright_R Q(X)$)
using *assms* **by** (*rel-auto*)

A tail recursive reactive design with a productive body is guarded.

lemma *Guarded-if-Productive* [*closure*]:
fixes P :: (*'s*, *'t::size-trace*,*'α*) *hrel-rsp*
assumes P *is NSRD* P *is Productive*
shows *Guarded* (λ X. P ;; $SRD(X)$)
proof (*clarsimp simp add*: *Guarded-def*)
— We split the proof into three cases corresponding to valuations for ok, wait, and wait' respectively.
fix X n
have *a*:(P ;; $SRD(X) \wedge$ *gvrt* (*Suc n*))⟦*false*/\$*ok*⟧ =
(P ;; $SRD(X \wedge$ *gvrt n*) $\wedge$ *gvrt* (*Suc n*))⟦*false*/\$*ok*⟧
by (*simp add*: *usubst closure SRD-left-zero-1 assms*)
have *b*:((P ;; $SRD(X) \wedge$ *gvrt* (*Suc n*))⟦*true*/\$*ok*⟧)⟦*true*/\$*wait*⟧ =
((P ;; $SRD(X \wedge$ *gvrt n*) $\wedge$ *gvrt* (*Suc n*))⟦*true*/\$*ok*⟧)⟦*true*/\$*wait*⟧
by (*simp add*: *usubst closure SRD-left-zero-2 assms*)
have *c*:((P ;; $SRD(X) \wedge$ *gvrt* (*Suc n*))⟦*true*/\$*ok*⟧)⟦*false*/\$*wait*⟧ =
((P ;; $SRD(X \wedge$ *gvrt n*) $\wedge$ *gvrt* (*Suc n*))⟦*true*/\$*ok*⟧)⟦*false*/\$*wait*⟧
proof −
have *1*:(P⟦*true*/\$*wait*´⟧ ;; (*SRD X*)⟦*true*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧ =
(P⟦*true*/\$*wait*´⟧ ;; (*SRD* ($X \wedge$ *gvrt n*))⟦*true*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
by (*metis* (*no-types*, *lifting*) *Healthy-def R3h-wait-true SRD-healths*(*3*) *SRD-idem*)
have *2*:(P⟦*false*/\$*wait*´⟧ ;; (*SRD X*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧ =
(P⟦*false*/\$*wait*´⟧ ;; (*SRD* ($X \wedge$ *gvrt n*))⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
proof −
have *exp*:$\bigwedge$ Y::(*'s*, *'t*,*'α*) *hrel-rsp*. (P⟦*false*/\$*wait*´⟧ ;; (*SRD Y*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧ =
(((($\neg_r$ *pre*$_R$ P) ;; ($SRD(Y)$)⟦*false*/\$*wait*⟧ $\vee$ (*post*$_R$ P $\wedge$ \$*tr*´ $>_u$ \$*tr*) ;; (*SRD* Y)⟦*true*,*false*/\$*ok*,\$*wait*⟧))
$\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
proof −
fix Y :: (*'s*, *'t*,*'α*) *hrel-rsp*

have (P⟦*false*/\$*wait*´⟧ ;; (*SRD Y*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧ =
(($\mathbf{R}_s$(*pre*$_R$(P) $\vdash$ *peri*$_R$(P) $\diamond$ (*post*$_R$(P) $\wedge$ \$*tr* $<_u$ \$*tr*´)))⟦*false*/\$*wait*´⟧ ;; (*SRD Y*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
by (*metis* (*no-types*) *Healthy-def Productive-form assms*(*1*) *assms*(*2*) *NSRD-is-SRD*)
also have ... =
((*R1*(*R2c*(*pre*$_R$(P) $\Rightarrow$ (\$*ok*´ $\wedge$ *post*$_R$(P) $\wedge$ \$*tr* $<_u$ \$*tr*´))))⟦*false*/\$*wait*´⟧ ;; (*SRD Y*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
by (*simp add*: *RHS-def R1-def R2c-def R2s-def R3h-def RD1-def RD2-def usubst unrest assms closure design-def*)
also have ... =
((($\neg_r$ *pre*$_R$(P) $\vee$ (\$*ok*´ $\wedge$ *post*$_R$(P) $\wedge$ \$*tr* $<_u$ \$*tr*´)))⟦*false*/\$*wait*´⟧ ;; (*SRD Y*)⟦*false*/\$*wait*⟧ $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
by (*simp add*: *impl-alt-def R2c-disj R1-disj R2c-not assms closure R2c-and R2c-preR rea-not-def R1-extend-conj' R2c-ok' R2c-post-SRD R1-tr-less-tr' R2c-tr-less-tr'*)
also have ... =
(((($\neg_r$ *pre*$_R$ P) ;; ($SRD(Y)$)⟦*false*/\$*wait*⟧ $\vee$ (\$*ok*´ $\wedge$ *post*$_R$ P $\wedge$ \$*tr*´ $>_u$ \$*tr*) ;; (*SRD* Y)⟦*false*/\$*wait*⟧)) $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧
by (*simp add*: *usubst unrest assms closure seqr-or-distl NSRD-neg-pre-left-zero SRD-healths*)
also have ... =
(((($\neg_r$ *pre*$_R$ P) ;; ($SRD(Y)$)⟦*false*/\$*wait*⟧ $\vee$ (*post*$_R$ P $\wedge$ \$*tr*´ $>_u$ \$*tr*) ;; (*SRD Y*)⟦*true*,*false*/\$*ok*,\$*wait*⟧)) $\wedge$ *gvrt* (*Suc n*))⟦*true*,*false*/\$*ok*,\$*wait*⟧

proof −
  have $(\$ok\acute{} \wedge post_R\ P \wedge \$tr\acute{} >_u \$tr)\ ;;\ (SRD\ Y)[\![false/\$wait]\!] =$
    $((post_R\ P \wedge \$tr\acute{} >_u \$tr) \wedge \$ok\acute{} =_u true)\ ;;\ (SRD\ Y)[\![false/\$wait]\!]$
   by (*rel-blast*)
  also have ... $= (post_R\ P \wedge \$tr\acute{} >_u \$tr)[\![true/\$ok\acute{}]\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!][\![true/\$ok]\!]$
   using *seqr-left-one-point*[*of ok* $(post_R\ P \wedge \$tr\acute{} >_u \$tr)$ *True* $(SRD\ Y)[\![false/\$wait]\!]$]
   by (*simp add*: *true-alt-def*[*THEN sym*])
  finally show *?thesis* **by** (*simp add*: *usubst unrest*)
qed
finally
show $(P[\![false/\$wait\acute{}]\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!] =$
  $((((\neg_r\ pre_R\ P)\ ;;\ (SRD(Y))[\![false/\$wait]\!] \vee (post_R\ P \wedge \$tr\acute{} >_u \$tr)\ ;;\ (SRD$
$Y)[\![true,false/\$ok,\$wait]\!]))$
  $\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$ **.**
qed

have *1*:$((post_R\ P \wedge \$tr\acute{} >_u \$tr)\ ;;\ (SRD\ X)[\![true,false/\$ok,\$wait]\!] \wedge gvrt\ (Suc\ n)) =$
  $((post_R\ P \wedge \$tr\acute{} >_u \$tr)\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![true,false/\$ok,\$wait]\!] \wedge gvrt\ (Suc\ n))$
 apply (*rel-auto*)
  apply (*rename-tac tr st more ok wait tr′ st′ more′* tr_0 st_0 $more_0$ *ok′*)
  apply (*rule-tac* $x{=}tr_0$ **in** *exI*, *rule-tac* $x{=}st_0$ **in** *exI*, *rule-tac* $x{=}more_0$ **in** *exI*)
  apply (*simp*)
  apply (*erule trace-strict-prefixE*)
  apply (*rename-tac tr st ref ok wait tr′ st′ ref′* tr_0 st_0 ref_0 *ok′ zs*)
  apply (*rule-tac x=False* **in** *exI*)
  apply (*simp add*: *size-minus-trace*)
  apply (*subgoal-tac* $size(tr) < size(tr_0)$)
   apply (*simp add*: *less-diff-conv2 size-mono*)
 using *size-strict-mono* **apply** *blast*
 apply (*rename-tac tr st more ok wait tr′ st′ more′* tr_0 st_0 $more_0$ *ok′*)
 apply (*rule-tac* $x{=}tr_0$ **in** *exI*, *rule-tac* $x{=}st_0$ **in** *exI*, *rule-tac* $x{=}more_0$ **in** *exI*)
 apply (*simp*)
 apply (*erule trace-strict-prefixE*)
 apply (*rename-tac tr st more ok wait tr′ st′ more′* tr_0 st_0 $more_0$ *ok′ zs*)
 apply (*auto simp add*: *size-minus-trace*)
 apply (*subgoal-tac* $size(tr) < size(tr_0)$)
  apply (*simp add*: *less-diff-conv2 size-mono*)
 using *size-strict-mono* **apply** *blast*
 done
have *2*:$(\neg_r\ pre_R\ P)\ ;;\ (SRD\ X)[\![false/\$wait]\!] = (\neg_r\ pre_R\ P)\ ;;\ (SRD(X \wedge gvrt\ n))[\![false/\$wait]\!]$
 by (*simp add*: *NSRD-neg-pre-left-zero closure assms SRD-healths*)
show *?thesis*
 by (*simp add*: *exp 1 2 utp-pred-laws.inf-sup-distrib2*)
qed

show *?thesis*
proof −
 have $(P\ ;;\ (SRD\ X) \wedge gvrt\ (n{+}1))[\![true,false/\$ok,\$wait]\!] =$
  $((P[\![true/\$wait\acute{}]\!]\ ;;\ (SRD\ X)[\![true/\$wait]\!] \wedge gvrt\ (n{+}1))[\![true,false/\$ok,\$wait]\!] \vee$
  $(P[\![false/\$wait\acute{}]\!]\ ;;\ (SRD\ X)[\![false/\$wait]\!] \wedge gvrt\ (n{+}1))[\![true,false/\$ok,\$wait]\!])$
 by (*subst seqr-bool-split*[*of wait*], *simp-all add*: *usubst utp-pred-laws.distrib*(*4*))

 also
 have ... $= ((P[\![true/\$wait\acute{}]\!]\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![true/\$wait]\!] \wedge gvrt\ (n{+}1))[\![true,false/\$ok,\$wait]\!]$
$\vee$

$(P[\![false/\$wait']\!]\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![false/\$wait]\!] \wedge gvrt\ (n+1))[\![true,false/\$ok,\$wait]\!])$
by (*simp add*: *1 2*)

also
have ... $= ((P[\![true/\$wait']\!]\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![true/\$wait]\!] \vee$
$P[\![false/\$wait']\!]\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![false/\$wait]\!]) \wedge gvrt\ (n+1))[\![true,false/\$ok,\$wait]\!]$
by (*simp add*: *usubst utp-pred-laws.distrib*(*4*))

also have ... $= (P\ ;;\ (SRD\ (X \wedge gvrt\ n)) \wedge gvrt\ (n+1))[\![true,false/\$ok,\$wait]\!]$
by (*subst seqr-bool-split*[*of wait*], *simp-all add*: *usubst*)
finally show *?thesis* **by** (*simp add*: *usubst*)
qed

qed
show $(P\ ;;\ SRD(X) \wedge gvrt\ (Suc\ n)) = (P\ ;;\ SRD(X \wedge gvrt\ n) \wedge gvrt\ (Suc\ n))$
apply (*rule-tac bool-eq-splitI*[*of in-var ok*])
apply (*simp-all add*: *a*)
apply (*rule-tac bool-eq-splitI*[*of in-var wait*])
apply (*simp-all add*: *b c*)
done
qed

10.3 Tail recursive fixed-point calculations

declare *upred-semiring.power-Suc* [*simp*]

lemma *mu-csp-form-1* [*rdes*]:
fixes P :: (*'s*, *'t::size-trace*,*'α*) *hrel-rsp*
assumes *P is NSRD P is Productive*
shows $(\mu\ X \cdot P\ ;;\ SRD(X)) = (\bigsqcap i \cdot P\ \hat{}\ (i+1))\ ;;\ Miracle$
proof −
have *1*:*Continuous* $(\lambda X.\ P\ ;;\ SRD\ X)$
using *SRD-Continuous*
by (*clarsimp simp add*: *Continuous-def seq-SUP-distl*[*THEN sym*], *drule-tac x=A* **in** *spec*, *simp*)
have *2*: $(\lambda X.\ P\ ;;\ SRD\ X) \in [\![id]\!]_H \rightarrow [\![SRD]\!]_H$
by (*blast intro*: *funcsetI closure assms*)
with *1 2* **have** $(\mu\ X \cdot P\ ;;\ SRD(X)) = (\nu\ X \cdot P\ ;;\ SRD(X))$
by (*simp add*: *guarded-fp-uniq Guarded-if-Productive*[*OF assms*] *funcsetI closure*)
also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ i)\ false)$
by (*simp add*: *sup-continuous-lfp 1 sup-continuous-Continuous false-upred-def*)
also have ... $= ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ 0)\ false \sqcap (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ (i+1))\ false)$
by (*subst Sup-power-expand*, *simp*)
also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ (i+1))\ false)$
by (*simp*)
also have ... $= (\bigsqcap i.\ P\ \hat{}\ (i+1)\ ;;\ Miracle)$
proof (*rule SUP-cong*, *simp-all*)
fix *i*
show $P\ ;;\ SRD\ (((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ i)\ false) = (P\ ;;\ P\ \hat{}\ i)\ ;;\ Miracle$
proof (*induct i*)
case *0*
then show *?case*
by (*simp*, *metis srdes-hcond-def srdes-theory-continuous.healthy-top*)
next
case (*Suc i*)
then show *?case*
by (*simp add*: *Healthy-if NSRD-is-SRD SRD-power-comp SRD-seqr-closure assms*(*1*) *seqr-assoc*[*THEN*

sym] *srdes-theory-continuous.weak.top-closed*)
    qed
  qed
  also have ... = ($\bigsqcap$ *i*. *P* ^ (*i*+*1*)) ;; *Miracle*
    by (*simp add*: *seq-Sup-distr*)
  finally show *?thesis*
    by (*simp add*: *UINF-as-Sup*[*THEN sym*])
qed

lemma *mu-csp-form-NSRD* [*closure*]:
  fixes *P* :: ('*s*, '*t*::*size-trace*,'α) *hrel-rsp*
  assumes *P is NSRD P is Productive*
  shows (μ *X* · *P* ;; *SRD*(*X*)) *is NSRD*
  by (*simp add*: *mu-csp-form-1 assms closure*)

lemma *mu-csp-form-1'*:
  fixes *P* :: ('*s*, '*t*::*size-trace*,'α) *hrel-rsp*
  assumes *P is NSRD P is Productive*
  shows (μ *X* · *P* ;; *SRD*(*X*)) = (*P* ;; $P^\star$) ;; *Miracle*
proof −
  have (μ *X* · *P* ;; *SRD*(*X*)) = ($\bigsqcap$ *i*∈*UNIV* · *P* ;; *P* ^ *i*) ;; *Miracle*
    by (*simp add*: *mu-csp-form-1 assms closure ustar-def*)
  also have ... = (*P* ;; $P^\star$) ;; *Miracle*
    by (*simp only*: *seq-UINF-distl*[*THEN sym*], *simp add*: *ustar-def*)
  finally show *?thesis* **.**
qed

declare *upred-semiring.power-Suc* [*simp del*]

end

11 Reactive Design Programs

theory *utp-rdes-prog*
  imports
    utp-rdes-normal
    utp-rdes-tactics
    utp-rdes-parallel
    utp-rdes-guarded
    UTP−KAT.utp-kleene
begin

11.1 State substitution

lemma *srd-subst-RHS-tri-design* [*usubst*]:
  $\lceil\sigma\rceil_{S\sigma} \dagger \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\lceil\sigma\rceil_{S\sigma} \dagger P) \vdash (\lceil\sigma\rceil_{S\sigma} \dagger Q) \diamond (\lceil\sigma\rceil_{S\sigma} \dagger R))$
  by (*rel-auto*)

lemma *srd-subst-SRD-closed* [*closure*]:
  assumes *P is SRD*
  shows $\lceil\sigma\rceil_{S\sigma} \dagger P$ *is SRD*
proof −
  have $SRD(\lceil\sigma\rceil_{S\sigma} \dagger (SRD\ P)) = \lceil\sigma\rceil_{S\sigma} \dagger (SRD\ P)$
    by (*rel-auto*)
  thus *?thesis*

by (*metis Healthy-def assms*)
qed

lemma *preR-srd-subst* [*rdes*]:
$pre_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger pre_R(P)$
by (*rel-auto*)

lemma *periR-srd-subst* [*rdes*]:
$peri_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger peri_R(P)$
by (*rel-auto*)

lemma *postR-srd-subst* [*rdes*]:
$post_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger post_R(P)$
by (*rel-auto*)

lemma *srd-subst-NSRD-closed* [*closure*]:
assumes *P is NSRD*
shows $\lceil\sigma\rceil_{S\sigma} \dagger P$ *is NSRD*
by (*rule NSRD-RC-intro*, *simp-all add*: *closure rdes assms unrest*)

11.2 Assignment

definition *assigns-srd* :: *'s usubst* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp* ($\langle\text{-}\rangle_R$) **where**
[*upred-defs*]: $assigns\text{-}srd\ \sigma = \mathbf{R}_s(true \vdash (\$tr\,' =_u \$tr \land \neg \$wait\,' \land \lceil\langle\sigma\rangle_a\rceil_S \land \$\Sigma_S\,' =_u \$\Sigma_S))$

syntax
-assign-srd :: *svids* ⇒ *uexprs* ⇒ *logic* ('(-') $:=_R$ '(-'))
-assign-srd :: *svids* ⇒ *uexprs* ⇒ *logic* (**infixr** $:=_R$ *90*)

translations
-assign-srd xs vs => *CONST assigns-srd* (*-mk-usubst* (*CONST id*) *xs vs*)
-assign-srd x v <= *CONST assigns-srd* (*CONST subst-upd* (*CONST id*) *x v*)
-assign-srd x v <= *-assign-srd* (*-spvar x*) *v*
x,*y* $:=_R$ *u*,*v* <= *CONST assigns-srd* (*CONST subst-upd* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *u*) (*CONST svar y*) *v*)

lemma *assigns-srd-RHS-tri-des* [*rdes-def*]:
$\langle\sigma\rangle_R = \mathbf{R}_s(true_r \vdash false \diamond \langle\sigma\rangle_r)$
by (*rel-auto*)

lemma *assigns-srd-NSRD-closed* [*closure*]: $\langle\sigma\rangle_R$ *is NSRD*
by (*simp add*: *rdes-def closure unrest*)

lemma *preR-assigns-srd* [*rdes*]: $pre_R(\langle\sigma\rangle_R) = true_r$
by (*simp add*: *rdes-def rdes closure*)

lemma *periR-assigns-srd* [*rdes*]: $peri_R(\langle\sigma\rangle_R) = false$
by (*simp add*: *rdes-def rdes closure*)

lemma *postR-assigns-srd* [*rdes*]: $post_R(\langle\sigma\rangle_R) = \langle\sigma\rangle_r$
by (*simp add*: *rdes-def rdes closure rpred*)

11.3 Conditional

lemma *preR-cond-srea* [*rdes*]:
$pre_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \land pre_R(P) \lor [\neg b]_{S<} \land pre_R(Q))$

by (*rel-auto*)

lemma *periR-cond-srea* [*rdes*]:
assumes *P is SRD Q is SRD*
shows $peri_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge peri_R(P) \vee [\neg b]_{S<} \wedge peri_R(Q))$
proof −
have $peri_R(P \triangleleft b \triangleright_R Q) = peri_R(R1(P) \triangleleft b \triangleright_R R1(Q))$
by (*simp add*: *Healthy-if SRD-healths assms*)
thus *?thesis*
by (*rel-auto*)
qed

lemma *postR-cond-srea* [*rdes*]:
assumes *P is SRD Q is SRD*
shows $post_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge post_R(P) \vee [\neg b]_{S<} \wedge post_R(Q))$
proof −
have $post_R(P \triangleleft b \triangleright_R Q) = post_R(R1(P) \triangleleft b \triangleright_R R1(Q))$
by (*simp add*: *Healthy-if SRD-healths assms*)
thus *?thesis*
by (*rel-auto*)
qed

lemma *NSRD-cond-srea* [*closure*]:
assumes *P is NSRD Q is NSRD*
shows $P \triangleleft b \triangleright_R Q$ *is NSRD*
proof (*rule NSRD-RC-intro*)
show $P \triangleleft b \triangleright_R Q$ *is SRD*
by (*simp add*: *closure assms*)
show $pre_R\ (P \triangleleft b \triangleright_R Q)$ *is RC*
proof −
have $1{:}(\lceil \neg\ b \rceil_{S<} \vee \neg_r\ pre_R\ P)\ ;;\ R1(true) = (\lceil \neg\ b \rceil_{S<} \vee \neg_r\ pre_R\ P)$
by (*metis* (*no-types*, *lifting*) *NSRD-neg-pre-unit aext-not assms*(*1*) *seqr-or-distl st-lift-R1-true-right*)
have $2{:}(\lceil b \rceil_{S<} \vee \neg_r\ pre_R\ Q)\ ;;\ R1(true) = (\lceil b \rceil_{S<} \vee \neg_r\ pre_R\ Q)$
by (*simp add*: *NSRD-neg-pre-unit assms seqr-or-distl st-lift-R1-true-right*)
show *?thesis*
by (*simp add*: *rdes closure assms*)
qed
show $\$st' \sharp\ peri_R\ (P \triangleleft b \triangleright_R Q)$
by (*simp add*: *rdes assms closure unrest*)
qed

11.4 Assumptions

definition *AssumeR* :: *'s cond* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp* ($[\text{-}]^\top{}_R$) **where**
[*upred-defs*]: $AssumeR\ b = II_R \triangleleft b \triangleright_R Miracle$

lemma *AssumeR-rdes-def* [*rdes-def*]:
$[b]^\top{}_R = \mathbf{R}_s(true_r \vdash false \diamond [b]^\top{}_r)$
unfolding *AssumeR-def* **by** (*rdes-eq*)

lemma *AssumeR-NSRD* [*closure*]: $[b]^\top{}_R$ *is NSRD*
by (*simp add*: *AssumeR-def closure*)

lemma *AssumeR-false*: $[false]^\top{}_R = Miracle$
by (*rel-auto*)

lemma *AssumeR-true*: $[true]^{\top}{}_R = II_R$
by (*rel-auto*)

lemma *AssumeR-comp*: $[b]^{\top}{}_R$;; $[c]^{\top}{}_R = [b \wedge c]^{\top}{}_R$
by (*rdes-simp*)

lemma *AssumeR-choice*: $[b]^{\top}{}_R \sqcap [c]^{\top}{}_R = [b \vee c]^{\top}{}_R$
by (*rdes-eq*)

lemma *AssumeR-refine-skip*: $II_R \sqsubseteq [b]^{\top}{}_R$
by (*rdes-refine*)

lemma *AssumeR-test* [*closure*]: $test_R\ [b]^{\top}{}_R$
by (*simp add*: *AssumeR-refine-skip nsrd-thy.utest-intro*)

lemma *Star-AssumeR*: $[b]^{\top}{}_R{}^{\star R} = II_R$
by (*simp add*: *AssumeR-NSRD AssumeR-test nsrd-thy.Star-test*)

lemma *AssumeR-choice-skip*: $II_R \sqcap [b]^{\top}{}_R = II_R$
by (*rdes-eq*)

lemma *cond-srea-AssumeR-form*:
assumes *P is NSRD Q is NSRD*
shows $P \triangleleft b \triangleright_R Q = ([b]^{\top}{}_R$;; $P \sqcap [\neg b]^{\top}{}_R$;; $Q)$
by (*rdes-eq cls*: *assms*)

lemma *cond-srea-insert-assume*:
assumes *P is NSRD Q is NSRD*
shows $P \triangleleft b \triangleright_R Q = ([b]^{\top}{}_R$;; $P \triangleleft b \triangleright_R [\neg b]^{\top}{}_R$;; $Q)$
by (*simp add*: *AssumeR-NSRD AssumeR-comp NSRD-seqr-closure RA1 assms cond-srea-AssumeR-form*)

lemma *AssumeR-cond-left*:
assumes *P is NSRD Q is NSRD*
shows $[b]^{\top}{}_R$;; $(P \triangleleft b \triangleright_R Q) = ([b]^{\top}{}_R$;; $P)$
by (*rdes-eq cls*: *assms*)

lemma *AssumeR-cond-right*:
assumes *P is NSRD Q is NSRD*
shows $[\neg b]^{\top}{}_R$;; $(P \triangleleft b \triangleright_R Q) = ([\neg b]^{\top}{}_R$;; $Q)$
by (*rdes-eq cls*: *assms*)

11.5 Guarded commands

definition *GuardedCommR* :: *'s cond* ⇒ (*'s*, *'t::trace*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* (- $\rightarrow_R$ - [*85*, *86*] *85*) **where**
gcmd-def [*rdes-def*]: *GuardedCommR g A* = $A \triangleleft g \triangleright_R$ *Miracle*

lemma *gcmd-false*[*simp*]: $(false \rightarrow_R A) = Miracle$
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-true*[*simp*]: $(true \rightarrow_R A) = A$
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-SRD*:
assumes *A is SRD*
shows $(g \rightarrow_R A)$ *is SRD*

by (*simp add*: *gcmd-def SRD-cond-srea assms srdes-theory-continuous.weak.top-closed*)

lemma *gcmd-NSRD* [*closure*]:
assumes *A is NSRD*
shows $(g \rightarrow_R A)$ *is NSRD*
by (*simp add*: *gcmd-def NSRD-cond-srea assms NSRD-Miracle*)

lemma *gcmd-Productive* [*closure*]:
assumes *A is NSRD A is Productive*
shows $(g \rightarrow_R A)$ *is Productive*
by (*simp add*: *gcmd-def closure assms*)

lemma *gcmd-seq-distr*:
assumes *B is NSRD*
shows $(g \rightarrow_R A) ;; B = (g \rightarrow_R (A ;; B))$
by (*simp add*: *Miracle-left-zero NSRD-is-SRD assms cond-st-distr gcmd-def*)

lemma *gcmd-nondet-distr*:
assumes *A is NSRD B is NSRD*
shows $(g \rightarrow_R (A \sqcap B)) = (g \rightarrow_R A) \sqcap (g \rightarrow_R B)$
by (*rdes-eq cls*: *assms*)

lemma *AssumeR-as-gcmd*:
$[b]^{\top}{}_R = b \rightarrow_R II_R$
by (*rdes-eq*)

12 Generalised Alternation

definition *AlternateR*
:: *'a set* ⇒ (*'a* ⇒ *'s upred*) ⇒ (*'a* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
[*upred-defs*, *rdes-def*]: $AlternateR\ I\ g\ A\ B = (\bigsqcap\ i \in I \cdot ((g\ i) \rightarrow_R (A\ i))) \sqcap ((\neg (\bigvee\ i \in I \cdot g\ i)) \rightarrow_R B)$

definition *AlternateR-list*
:: (*'s upred* × (*'s*, *'t*::*trace*, *'α*) *hrel-rsp*) *list* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
[*upred-defs*, *ndes-simp*]:
AlternateR-list xs P = *AlternateR* {*0..<length xs*} (λ *i*. *map fst xs* ! *i*) (λ *i*. *map snd xs* ! *i*) *P*

syntax
-altindR-els :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (if_R *-*∈*-* · *-* → *-* *else* *-* *fi*)
-altindR :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (if_R *-*∈*-* · *-* → *-* *fi*)

-altgcommR-els :: *gcomms* ⇒ *logic* ⇒ *logic* (if_R *-* *else* *-* *fi*)
-altgcommR :: *gcomms* ⇒ *logic* (if_R *-* *fi*)

translations
if_R *i*∈*I* · *g* → *A else B fi* ⇀ *CONST AlternateR I* (λ*i*. *g*) (λ*i*. *A*) *B*
if_R *i*∈*I* · *g* → *A fi* ⇀ *CONST AlternateR I* (λ*i*. *g*) (λ*i*. *A*) (*CONST Chaos*)
if_R *i*∈*I* · (*g i*) → *A else B fi* ↽ *CONST AlternateR I g* (λ*i*. *A*) *B*
-altgcommR cs ⇀ *CONST AlternateR-list cs* (*CONST Chaos*)
-altgcommR (*-gcomm-show cs*) ↽ *CONST AlternateR-list cs* (*CONST Chaos*)
-altgcommR-els cs P ⇀ *CONST AlternateR-list cs P*
-altgcommR-els (*-gcomm-show cs*) *P* ↽ *CONST AlternateR-list cs P*

lemma *AlternateR-NSRD-closed* [*closure*]:
  assumes $\bigwedge$ *i*. $i \in I \Longrightarrow A\ i$ *is NSRD B is NSRD*
  shows (if_R $i{\in}I \cdot g\ i \rightarrow A\ i$ *else B fi*) *is NSRD*
proof (*cases* $I = \{\}$)
  case *True*
  then show *?thesis* **by** (*simp add*: *AlternateR-def assms*)
next
  case *False*
  then show *?thesis* **by** (*simp add*: *AlternateR-def closure assms*)
qed

lemma *AlternateR-empty* [*simp*]:
  (if_R $i \in \{\} \cdot g\ i \rightarrow A\ i$ *else B fi*) $= B$
  by (*rdes-simp*)

lemma *AlternateR-Productive* [*closure*]:
  assumes
    $\bigwedge$ *i*. $i \in I \Longrightarrow A\ i$ *is NSRD B is NSRD*
    $\bigwedge$ *i*. $i \in I \Longrightarrow A\ i$ *is Productive B is Productive*
  shows (if_R $i{\in}I \cdot g\ i \rightarrow A\ i$ *else B fi*) *is Productive*
proof (*cases* $I = \{\}$)
  case *True*
  then show *?thesis*
    by (*simp add*: *assms*(*4*))
next
  case *False*
  then show *?thesis*
    by (*simp add*: *AlternateR-def closure assms*)
qed

lemma *AlternateR-singleton*:
  assumes *A k is NSRD B is NSRD*
  shows (if_R $i \in \{k\} \cdot g\ i \rightarrow A\ i$ *else B fi*) $= (A(k) \triangleleft g(k) \triangleright_R B)$
  by (*simp add*: *AlternateR-def*, *rdes-eq cls*: *assms*)

Convert an alternation over disjoint guards into a cascading if-then-else

lemma *AlternateR-insert-cascade*:
  assumes
    $\bigwedge$ *i*. $i \in I \Longrightarrow A\ i$ *is NSRD*
    A k is NSRD B is NSRD
    $(g(k) \wedge (\bigvee i{\in}I \cdot g(i))) = false$
  shows (if_R $i \in$ *insert* $k\ I \cdot g\ i \rightarrow A\ i$ *else B fi*) $= (A(k) \triangleleft g(k) \triangleright_R (if_R\ i{\in}I \cdot g(i) \rightarrow A(i)\ else\ B\ fi))$
proof (*cases* $I = \{\}$)
  case *True*
  then show *?thesis* **by** (*simp add*: *AlternateR-singleton assms*)
next
  case *False*
  have *1*: $(\bigsqcap\ i \in I \cdot g\ i \rightarrow_R A\ i) = (\bigsqcap\ i \in I \cdot g\ i \rightarrow_R \mathbf{R}_s(pre_R(A\ i) \vdash peri_R(A\ i) \diamond post_R(A\ i)))$
    by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*) *cong*: *UINF-cong*)
  from *assms*(*4*) **show** *?thesis*
    by (*simp add*: *AlternateR-def 1 False*)
      (*rdes-eq cls*: *assms*(*1*−*3*) *False cong*: *UINF-cong*)
qed

12.1 Choose

definition *choose-srd* :: $('s,'t{::}trace,'\alpha)$ *hrel-rsp* $(choose_R)$ **where**
[*upred-defs*, *rdes-def*]: $choose_R = \mathbf{R}_s(true_r \vdash false \diamond true_r)$

lemma *preR-choose* [*rdes*]: $pre_R(choose_R) = true_r$
by (*rel-auto*)

lemma *periR-choose* [*rdes*]: $peri_R(choose_R) = false$
by (*rel-auto*)

lemma *postR-choose* [*rdes*]: $post_R(choose_R) = true_r$
by (*rel-auto*)

lemma *choose-srd-SRD* [*closure*]: $choose_R$ *is SRD*
by (*simp add*: *choose-srd-def closure unrest*)

lemma *NSRD-choose-srd* [*closure*]: $choose_R$ *is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *closure unrest rdes*)

12.2 State Abstraction

definition *state-srea* ::
$'s$ *itself* $\Rightarrow ('s,'t{::}trace,'\alpha,'\beta)$ *rel-rsp* $\Rightarrow (unit,'t,'\alpha,'\beta)$ *rel-rsp* **where**
[*upred-defs*]: *state-srea t P* $= \langle\exists\ \{\$st,\$st\'\} \cdot P\rangle_S$

syntax
-state-srea :: *type* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (*state* - $\cdot$ - [*0*,*200*] *200*)

translations
state $'a \cdot P$ == *CONST state-srea TYPE*$('a)$ *P*

lemma *R1-state-srea*: $R1(state\ 'a \cdot P) = (state\ 'a \cdot R1(P))$
by (*rel-auto*)

lemma *R2c-state-srea*: $R2c(state\ 'a \cdot P) = (state\ 'a \cdot R2c(P))$
by (*rel-auto*)

lemma *R3h-state-srea*: $R3h(state\ 'a \cdot P) = (state\ 'a \cdot R3h(P))$
by (*rel-auto*)

lemma *RD1-state-srea*: $RD1(state\ 'a \cdot P) = (state\ 'a \cdot RD1(P))$
by (*rel-auto*)

lemma *RD2-state-srea*: $RD2(state\ 'a \cdot P) = (state\ 'a \cdot RD2(P))$
by (*rel-auto*)

lemma *RD3-state-srea*: $RD3(state\ 'a \cdot P) = (state\ 'a \cdot RD3(P))$
by (*rel-auto*, *blast+*)

lemma *SRD-state-srea* [*closure*]: *P is SRD* $\Longrightarrow$ *state* $'a \cdot P$ *is SRD*
by (*simp add*: *Healthy-def R1-state-srea R2c-state-srea R3h-state-srea RD1-state-srea RD2-state-srea RHS-def SRD-def*)

lemma *NSRD-state-srea* [*closure*]: *P is NSRD* $\Longrightarrow$ *state* $'a \cdot P$ *is NSRD*
by (*metis Healthy-def NSRD-is-RD3 NSRD-is-SRD RD3-state-srea SRD-RD3-implies-NSRD SRD-state-srea*)

lemma *preR-state-srea* [*rdes*]: $pre_R(state\ 'a \cdot P) = \langle\forall\ \{\$st,\$st'\} \cdot pre_R(P)\rangle_S$
  by (*simp add*: *state-srea-def*, *rel-auto*)

lemma *periR-state-srea* [*rdes*]: $peri_R(state\ 'a \cdot P) = state\ 'a \cdot peri_R(P)$
  by (*rel-auto*)

lemma *postR-state-srea* [*rdes*]: $post_R(state\ 'a \cdot P) = state\ 'a \cdot post_R(P)$
  by (*rel-auto*)

12.3 While Loop

definition *WhileR* :: $'s\ upred \Rightarrow ('s, 't{::}size\text{-}trace, '\alpha)\ hrel\text{-}rsp \Rightarrow ('s, 't, '\alpha)\ hrel\text{-}rsp$ ($while_R$ - *do* - *od*)
where
$WhileR\ b\ P = (\mu_R\ X \cdot (P\ ;;\ X) \triangleleft b \triangleright_R II_R)$

lemma *Sup-power-false*:
  fixes $F :: '\alpha\ upred \Rightarrow '\alpha\ upred$
  shows $(\bigsqcap i.\ (F\ \hat{}\hat{}\ i)\ false) = (\bigsqcap i.\ (F\ \hat{}\hat{}\ (i+1))\ false)$
proof −
  have $(\bigsqcap i.\ (F\ \hat{}\hat{}\ i)\ false) = (F\ \hat{}\hat{}\ 0)\ false \sqcap (\bigsqcap i.\ (F\ \hat{}\hat{}\ (i+1))\ false)$
    by (*subst Sup-power-expand*, *simp*)
  also have ... $= (\bigsqcap i.\ (F\ \hat{}\hat{}\ (i+1))\ false)$
    by (*simp*)
  finally show *?thesis* **.**
qed

theorem *WhileR-iter-expand*:
  assumes *P is NSRD P is Productive*
  shows $while_R\ b\ do\ P\ od = (\bigsqcap i \cdot (P \triangleleft b \triangleright_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R))$ (**is** *?lhs* = *?rhs*)
proof −
  have *1*:$Continuous\ (\lambda X.\ P\ ;;\ SRD\ X)$
    using *SRD-Continuous*
    by (*clarsimp simp add*: *Continuous-def seq-SUP-distl*[*THEN sym*], *drule-tac x=A* **in** *spec*, *simp*)
  have *2*: $Continuous\ (\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)$
    by (*simp add*: *1 closure assms*)
  have $?lhs = (\mu_R\ X \cdot P\ ;;\ X \triangleleft b \triangleright_R II_R)$
    by (*simp add*: *WhileR-def*)
  also have ... $= (\mu\ X \cdot P\ ;;\ SRD(X) \triangleleft b \triangleright_R II_R)$
    by (*auto simp add*: *srd-mu-equiv closure assms*)
  also have ... $= (\nu\ X \cdot P\ ;;\ SRD(X) \triangleleft b \triangleright_R II_R)$
    by (*auto simp add*: *guarded-fp-uniq Guarded-if-Productive*[*OF assms*] *funcsetI closure assms*)
  also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)\ \hat{}\hat{}\ i)\ false)$
    by (*simp add*: *sup-continuous-lfp 2 sup-continuous-Continuous false-upred-def*)
  also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)\ \hat{}\hat{}\ (i+1))\ false)$
    by (*simp add*: *Sup-power-false*)
  also have ... $= (\bigsqcap i.\ (P \triangleleft b \triangleright_R II_R)\hat{}i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R))$
  proof (*rule SUP-cong*, *simp*)
    fix *i*
    show $((\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)\ \hat{}\hat{}\ (i + 1))\ false = (P \triangleleft b \triangleright_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b$
$\triangleright_R II_R)$
    proof (*induct i*)
      case *0*
      thm *if-eq-cancel*
      then show *?case*
        by (*simp*, *metis srdes-hcond-def srdes-theory-continuous.healthy-top*)

next
case (*Suc i*)
show *?case*
proof −
have $((\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)\ \hat{}\hat{}\ (Suc\ i + 1))\ false =$
$P\ ;;\ SRD\ (((\lambda X.\ P\ ;;\ SRD\ X \triangleleft b \triangleright_R II_R)\ \hat{}\hat{}\ (i + 1))\ false) \triangleleft b \triangleright_R II_R$
by *simp*
also have ... $= P\ ;;\ SRD\ ((P \triangleleft b \triangleright_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$
using *Suc.hyps* **by** *auto*
also have ... $= P\ ;;\ ((P \triangleleft b \triangleright_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$
by (*metis* (*no-types*, *lifting*) *Healthy-if NSRD-cond-srea NSRD-is-SRD NSRD-power-Suc NSRD-srd-skip SRD-cond-srea SRD-seqr-closure assms*(*1*) *power.power-eq-if seqr-left-unit srdes-theory-continuous.top-clo*
also have ... $= (P \triangleleft b \triangleright_R II_R)\ \hat{}\ Suc\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$
proof (*induct i*)
case *0*
then show *?case*
by (*simp add*: *NSRD-is-SRD SRD-cond-srea SRD-left-unit SRD-seqr-closure SRD-srdes-skip assms*(*1*) *cond-L6 cond-st-distr srdes-theory-continuous.top-closed*)
next
case (*Suc i*)
have *1*: $II_R\ ;;\ ((P \triangleleft b \triangleright_R II_R)\ ;;\ (P \triangleleft b \triangleright_R II_R)\ \hat{}\ i) = ((P \triangleleft b \triangleright_R II_R)\ ;;\ (P \triangleleft b \triangleright_R II_R)\ \hat{}\ i)$
by (*simp add*: *NSRD-is-SRD RA1 SRD-cond-srea SRD-left-unit SRD-srdes-skip assms*(*1*))
then show *?case*
proof −
have $\bigwedge u.\ (u\ ;;\ (P \triangleleft b \triangleright_R II_R)\ \hat{}\ Suc\ i)\ ;;\ (P\ ;;\ (Miracle) \triangleleft b \triangleright_R (II_R)) \triangleleft b \triangleright_R (II_R) =$
$((u \triangleleft b \triangleright_R II_R)\ ;;\ (P \triangleleft b \triangleright_R II_R)\ \hat{}\ Suc\ i)\ ;;\ (P\ ;;\ (Miracle) \triangleleft b \triangleright_R (II_R))$
by (*metis* (*no-types*) *Suc.hyps 1 cond-L6 cond-st-distr power.power.power-Suc*)
then show *?thesis*
by (*simp add*: *RA1 upred-semiring.power-Suc*)
qed
qed
finally show *?thesis* **.**
qed
qed
qed
also have ... $= (\bigsqcap i \cdot (P \triangleleft b \triangleright_R II_R)\hat{}i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R))$
by (*simp add*: *UINF-as-Sup-collect'*)
finally show *?thesis* **.**
qed

theorem *WhileR-star-expand*:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ P\ od = (P \triangleleft b \triangleright_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$ (**is** *?lhs* = *?rhs*)
proof −
have $?lhs = (\bigsqcap i \cdot (P \triangleleft b \triangleright_R II_R)\ \hat{}\ i)\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$
by (*simp add*: *WhileR-iter-expand seq-UINF-distr' assms*)
also have ... $= (P \triangleleft b \triangleright_R II_R)^{\star}\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$
by (*simp add*: *ustar-def*)
also have ... $= ((P \triangleleft b \triangleright_R II_R)^{\star}\ ;;\ II_R)\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$
by (*simp add*: *seqr-assoc SRD-left-unit closure assms*)
also have ... $= (P \triangleleft b \triangleright_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R II_R)$
by (*simp add*: *nsrd-thy.Star-def*)
finally show *?thesis* **.**
qed

lemma *WhileR-NSRD-closed* [*closure*]:
  assumes *P is NSRD P is Productive*
  shows $while_R$ *b do P od is NSRD*
  by (*simp add*: *WhileR-star-expand assms closure*)

theorem *WhileR-iter-form-lemma*:
  assumes *P is NSRD*
  shows $(P \triangleleft b \triangleright_R II_R)^{\star R} \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R) = ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [\neg\, b]^\top{}_R$
proof −
  have $(P \triangleleft b \triangleright_R II_R)^{\star R} \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R) = ([b]^\top{}_R \;;\; P \sqcap [\neg b]^\top{}_R)^{\star R} \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R)$
    by (*simp add*: *AssumeR-NSRD NSRD-right-unit NSRD-srd-skip assms*(*1*) *cond-srea-AssumeR-form*)
  also have ... $= (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [\neg\, b]^\top{}_R{}^{\star R})^{\star R} \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R)$
    by (*simp add*: *AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-denest assms*(*1*))
  also have ... $= (([b]^\top{}_R \;;\; P)^{\star R})^{\star R} \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R)$
    by (*metis* (*no-types*, *hide-lams*) *RD3-def RD3-idem Star-AssumeR nsrd-thy.Star-def*)
  also have ... $= (([b]^\top{}_R \;;\; P)^{\star R}) \;;\; (P \;;\; Miracle \triangleleft b \triangleright_R II_R)$
    by (*simp add*: *AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-invol assms*(*1*))
  also have ... $= (([b]^\top{}_R \;;\; P)^{\star R}) \;;\; ([b]^\top{}_R \;;\; P \;;\; Miracle \sqcap [\neg b]^\top{}_R)$
    by (*simp add*: *AssumeR-NSRD NSRD-Miracle NSRD-right-unit NSRD-seqr-closure NSRD-srd-skip assms*(*1*) *cond-srea-AssumeR-form*)
  also have ... $= (([b]^\top{}_R \;;\; P)^{\star R}) \;;\; [b]^\top{}_R \;;\; P \;;\; Miracle \sqcap (([b]^\top{}_R \;;\; P)^{\star R}) \;;\; [\neg b]^\top{}_R$
    by (*simp add*: *upred-semiring.distrib-left*)
  also have ... $= ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [\neg\, b]^\top{}_R$
  proof −
    have $(([b]^\top{}_R \;;\; P)^{\star R}) \;;\; [\neg b]^\top{}_R = (II_R \sqcap ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P) \;;\; [\neg\, b]^\top{}_R$
      by (*simp add*: *AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-unfoldr-eq assms*(*1*))
    also have ... $= [\neg\, b]^\top{}_R \sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P) \;;\; [\neg\, b]^\top{}_R$
      by (*metis* (*no-types*, *lifting*) *AssumeR-NSRD AssumeR-as-gcmd NSRD-srd-skip Star-AssumeR nsrd-thy.Star-slide gcmd-seq-distr skip-srea-self-unit urel-dioid.distrib-right'*)
    also have ... $= [\neg\, b]^\top{}_R \sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; [b \vee \neg\, b]^\top{}_R) \;;\; [\neg\, b]^\top{}_R$
      by (*simp add*: *AssumeR-true NSRD-right-unit assms*(*1*))
    also have ... $= [\neg\, b]^\top{}_R \sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; [b]^\top{}_R) \;;\; [\neg\, b]^\top{}_R$
      $\sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; [\neg\, b]^\top{}_R) \;;\; [\neg\, b]^\top{}_R$
      by (*metis* (*no-types*, *hide-lams*) *AssumeR-choice upred-semiring.add-assoc upred-semiring.distrib-left upred-semiring.distrib-right*)
    also have ... $= [\neg\, b]^\top{}_R \sqcap ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; ([b]^\top{}_R \;;\; [\neg\, b]^\top{}_R)$
      $\sqcap ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; ([\neg\, b]^\top{}_R \;;\; [\neg\, b]^\top{}_R)$
      by (*simp add*: *RA1*)
    also have ... $= [\neg\, b]^\top{}_R \sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; Miracle)$
      $\sqcap (([b]^\top{}_R \;;\; P)^{\star R} \;;\; [b]^\top{}_R \;;\; P \;;\; [\neg\, b]^\top{}_R)$
      by (*simp add*: *AssumeR-comp AssumeR-false*)
    finally have $([b]^\top{}_R \;;\; P)^{\star R} \;;\; [\neg\, b]^\top{}_R \sqsubseteq (([b]^\top{}_R \;;\; P)^{\star R}) \;;\; [b]^\top{}_R \;;\; P \;;\; Miracle$
      by (*simp add*: *semilattice-sup-class.le-supI1*)
    thus *?thesis*
      by (*simp add*: *semilattice-sup-class.le-iff-sup*)
  qed
  finally show *?thesis* **.**
qed

theorem *WhileR-iter-form*:
  assumes *P is NSRD P is Productive*
  shows $while_R$ *b do P od* $= ([b]^\top{}_R \;;\; P)^{\star R} \;;\; [\neg\, b]^\top{}_R$
  by (*simp add*: *WhileR-iter-form-lemma WhileR-star-expand assms*)

theorem *WhileR-false*:
  assumes *P is NSRD*
  shows $while_R$ *false do P od* = II_R
  by (*simp add*: *WhileR-def rpred closure srdes-theory-continuous.LFP-const*)

theorem *WhileR-true*:
  assumes *P is NSRD P is Productive*
  shows $while_R$ *true do P od* = $P^{\star R}$;; *Miracle*
  by (*simp add*: *WhileR-iter-form AssumeR-true AssumeR-false SRD-left-unit assms closure*)

lemma *WhileR-insert-assume*:
  assumes *P is NSRD P is Productive*
  shows $while_R$ *b do* ($[b]^\top{}_R$;; *P*) *od* = $while_R$ *b do P od*
  by (*simp add*: *AssumeR-NSRD AssumeR-comp NSRD-seqr-closure Productive-seq-2 RA1 WhileR-iter-form assms*)

theorem *WhileR-rdes-def* [*rdes-def*]:
  assumes *P is RC Q is RR R is RR $st´ ♯ Q R is R4*
  shows $while_R$ *b do* $\mathbf{R}_s(P \vdash Q \diamond R)$ *od* =
    $\mathbf{R}_s$ $(([b]^\top{}_r ;; R)^{\star r}\ wp_r\ ([b]_{S<} \Rightarrow_r P) \vdash ([b]^\top{}_r ;; R)^{\star r} ;; [b]^\top{}_r ;; Q \diamond ([b]^\top{}_r ;; R)^{\star r} ;; [\neg\ b]^\top{}_r)$
  (**is** *?lhs* = *?rhs*)
proof –
  have *?lhs* = $([b]^\top{}_R ;; \mathbf{R}_s\ (P \vdash Q \diamond R))^{\star R} ;; [\neg\ b]^\top{}_R$
    by (*simp add*: *WhileR-iter-form Productive-rdes-RR-intro assms closure*)
  also have ... = *?rhs*
    by (*simp add*: *rdes-def assms closure unrest rpred wp del*: *rea-star-wp*)
  finally show *?thesis* **.**
qed

12.4 Iteration Construction

definition *IterateR*
  :: *'a set* ⇒ (*'a* ⇒ *'s upred*) ⇒ (*'a* ⇒ (*'s*, *'t::size-trace*, *'α*) *hrel-rsp*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp*
where *IterateR A g P* = $while_R$ (⋁ *i*∈*A* · *g*(*i*)) *do* (if_R *i*∈*A* · *g*(*i*) → *P*(*i*) *fi*) *od*

syntax
  -iter-srd :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (do_R -∈- · - → - *fi*)

translations
  -iter-srd x A g P => *CONST IterateR A* (λ *x*. *g*) (λ *x*. *P*)
  -iter-srd x A g P <= *CONST IterateR A* (λ *x*. *g*) (λ *x'*. *P*)

lemma *IterateR-NSRD-closed* [*closure*]:
  assumes
    ⋀ *i*. *i* ∈ *I* ⟹ *P*(*i*) *is NSRD*
    ⋀ *i*. *i* ∈ *I* ⟹ *P*(*i*) *is Productive*
  shows do_R *i*∈*I* · *g*(*i*) → *P*(*i*) *fi is NSRD*
  by (*simp add*: *IterateR-def closure assms*)

lemma *IterateR-empty*:
  do_R *i*∈{} · *g*(*i*) → *P*(*i*) *fi* = II_R
  by (*simp add*: *IterateR-def srd-mu-equiv closure rpred gfp-const WhileR-false*)

lemma *IterateR-singleton*:
  assumes *P k is NSRD P k is Productive*
  shows do_R *i*∈{*k*} · *g*(*i*) → *P*(*i*) *fi* = $while_R$ *g*(*k*) *do P*(*k*) *od* (**is** *?lhs* = *?rhs*)

proof −
  have *?lhs* $=$ *while*$_R$ *g k do P k* $\triangleleft$ *g k* $\triangleright_R$ *Chaos od*
    by (*simp add*: *IterateR-def AlternateR-singleton assms closure*)
  also have ... $=$ *while*$_R$ *g k do* $[g\ k]^\top{}_R$;; $(P\ k \triangleleft g\ k \triangleright_R Chaos)$ *od*
    by (*simp add*: *WhileR-insert-assume closure assms*)
  also have ... $=$ *while*$_R$ *g k do P k od*
    by (*simp add*: *AssumeR-cond-left NSRD-Chaos WhileR-insert-assume assms*)
  finally show *?thesis* **.**
qed

12.5 Substitution Laws

lemma *srd-subst-Chaos* [*usubst*]:
  $\sigma \dagger_S Chaos = Chaos$
  by (*rdes-simp*)

lemma *srd-subst-Miracle* [*usubst*]:
  $\sigma \dagger_S Miracle = Miracle$
  by (*rdes-simp*)

lemma *srd-subst-skip* [*usubst*]:
  $\sigma \dagger_S II_R = \langle\sigma\rangle_R$
  by (*rdes-eq*)

lemma *srd-subst-assigns* [*usubst*]:
  $\sigma \dagger_S \langle\varrho\rangle_R = \langle\varrho \circ \sigma\rangle_R$
  by (*rdes-eq*)

12.6 Algebraic Laws

theorem *assigns-srd-id*: $\langle id\rangle_R = II_R$
  by (*rdes-eq*)

theorem *assigns-srd-comp*: $\langle\sigma\rangle_R$;; $\langle\varrho\rangle_R = \langle\varrho \circ \sigma\rangle_R$
  by (*rdes-eq*)

theorem *assigns-srd-Miracle*: $\langle\sigma\rangle_R$;; $Miracle = Miracle$
  by (*rdes-eq*)

theorem *assigns-srd-Chaos*: $\langle\sigma\rangle_R$;; $Chaos = Chaos$
  by (*rdes-eq*)

theorem *assigns-srd-cond* : $\langle\sigma\rangle_R \triangleleft b \triangleright_R \langle\varrho\rangle_R = \langle\sigma \triangleleft b \triangleright_s \varrho\rangle_R$
  by (*rdes-eq*)

theorem *assigns-srd-left-seq*:
  assumes *P is NSRD*
  shows $\langle\sigma\rangle_R$;; $P = \sigma \dagger_S P$
  by (*rdes-simp cls*: *assms*)

lemma *AlternateR-seq-distr*:
  assumes $\bigwedge$ *i*. *A i is NSRD B is NSRD C is NSRD*
  shows (*if*$_R$ $i \in I \cdot g\ i \rightarrow A\ i$ *else B fi*) ;; $C =$ (*if*$_R$ $i \in I \cdot g\ i \rightarrow A\ i$;; *C else B* ;; *C fi*)
proof (*cases* $I = \{\}$)
  case *True*
  then show *?thesis* **by** (*simp*)

next
  case *False*
  then show *?thesis*
    by (*simp add*: *AlternateR-def upred-semiring.distrib-right seq-UINF-distr gcmd-seq-distr assms*(*3*))
qed

lemma *AlternateR-is-cond-srea*:
  assumes *A is NSRD B is NSRD*
  shows (if_R $i \in \{a\} \cdot g \rightarrow A$ *else B fi*) $= (A \triangleleft g \triangleright_R B)$
  by (*rdes-eq cls*: *assms*)

lemma *AlternateR-Chaos*:
  if_R $i \in A \cdot g(i) \rightarrow$ *Chaos fi* = *Chaos*
  by (*cases* $A = \{\}$, *simp*, *rdes-eq*)

lemma *choose-srd-par*:
  $choose_R \parallel_R choose_R = choose_R$
  by (*rdes-eq*)

12.7 Lifting designs to reactive designs

definition *des-rea-lift* :: *'s hrel-des* ⇒ (*'s,'t::trace,'α*) *hrel-rsp* ($\mathbf{R}_D$) **where**
[*upred-defs*]: $\mathbf{R}_D(P) = \mathbf{R}_s(\lceil pre_D(P)\rceil_S \vdash (false \diamond (\$tr' =_u \$tr \wedge \lceil post_D(P)\rceil_S)))$

definition *des-rea-drop* :: (*'s,'t::trace,'α*) *hrel-rsp* ⇒ *'s hrel-des* ($\mathbf{D}_R$) **where**
[*upred-defs*]: $\mathbf{D}_R(P) = \lfloor (pre_R(P))[\![\$tr/\$tr']\!] \restriction_v \$st\rfloor_{S<}$
$\vdash_n \lfloor (post_R(P))[\![\$tr/\$tr']\!] \restriction_v \{\$st,\$st'\}\rfloor_S$

lemma *ndesign-rea-lift-inverse*: $\mathbf{D}_R(\mathbf{R}_D(p \vdash_n Q)) = p \vdash_n Q$
  apply (*simp add*: *des-rea-lift-def des-rea-drop-def rea-pre-RHS-design rea-post-RHS-design*)
  apply (*simp add*: *R1-def R2c-def R2s-def usubst unrest*)
  apply (*rel-auto*)
  done

lemma *ndesign-rea-lift-injective*:
  assumes P *is* $\mathbf{N}$ Q *is* $\mathbf{N}$ $\mathbf{R}_D$ $P = \mathbf{R}_D$ Q (**is** $?RP(P) = ?RQ(Q)$)
  shows $P = Q$
proof −
  have $?RP(\lfloor pre_D(P)\rfloor_< \vdash_n post_D(P)) = ?RQ(\lfloor pre_D(Q)\rfloor_< \vdash_n post_D(Q))$
    by (*simp add*: *ndesign-form assms*)
  hence $\lfloor pre_D(P)\rfloor_< \vdash_n post_D(P) = \lfloor pre_D(Q)\rfloor_< \vdash_n post_D(Q)$
    by (*metis ndesign-rea-lift-inverse*)
  thus *?thesis*
    by (*simp add*: *ndesign-form assms*)
qed

lemma *des-rea-lift-closure* [*closure*]: $\mathbf{R}_D(P)$ *is SRD*
  by (*simp add*: *des-rea-lift-def RHS-design-is-SRD unrest*)

lemma *preR-des-rea-lift* [*rdes*]:
  $pre_R(\mathbf{R}_D(P)) = R1(\lceil pre_D(P)\rceil_S)$
  by (*rel-auto*)

lemma *periR-des-rea-lift* [*rdes*]:
  $peri_R(\mathbf{R}_D(P)) = (false \triangleleft \lceil pre_D(P)\rceil_S \triangleright (\$tr \le_u \$tr'))$
  by (*rel-auto*)

lemma *postR-des-rea-lift* [*rdes*]:
  $post_R(\mathbf{R}_D(P)) = ((true \triangleleft \lceil pre_D(P)\rceil_S \triangleright (\neg \$tr \leq_u \$tr')) \Rightarrow (\$tr' =_u \$tr \wedge \lceil post_D(P)\rceil_S))$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *ndes-rea-lift-closure* [*closure*]:
  assumes P *is* $\mathbf{N}$
  shows $\mathbf{R}_D(P)$ *is NSRD*
proof −
  obtain p Q **where** P: $P = (p \vdash_n Q)$
    by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def assms*)
  show *?thesis*
    apply (*rule NSRD-intro*)
      apply (*simp-all add*: *closure rdes unrest P*)
    apply (*rel-auto*)
    done
qed

lemma *R-D-mono*:
  assumes P *is* $\mathbf{H}$ Q *is* $\mathbf{H}$ $P \sqsubseteq Q$
  shows $\mathbf{R}_D(P) \sqsubseteq \mathbf{R}_D(Q)$
  apply (*simp add*: *des-rea-lift-def*)
  apply (*rule srdes-tri-refine-intro'*)
    apply (*auto intro*: *H1-H2-refines assms aext-mono*)
   apply (*rel-auto*)
  apply (*metis* (*no-types*, *hide-lams*) *aext-mono assms*(*3*) *design-post-choice*
    semilattice-sup-class.sup.orderE utp-pred-laws.inf.coboundedI1 utp-pred-laws.inf.commute utp-pred-laws.sup.order-iff)
  done

Homomorphism laws

lemma *R-D-Miracle*:
  $\mathbf{R}_D(\top_D) = Miracle$
  by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *R-D-Chaos*:
  $\mathbf{R}_D(\bot_D) = Chaos$
proof −
  have $\mathbf{R}_D(\bot_D) = \mathbf{R}_D(false \vdash_r true)$
    by (*rel-auto*)
  also have ... $= \mathbf{R}_s\ (false \vdash false \diamond (\$tr' =_u \$tr))$
    by (*simp add*: *Chaos-def des-rea-lift-def alpha*)
  also have ... $= \mathbf{R}_s\ (true)$
    by (*rel-auto*)
  also have ... = *Chaos*
    by (*simp add*: *Chaos-def design-false-pre*)
  finally show *?thesis* **.**
qed

lemma *R-D-inf*:
  $\mathbf{R}_D(P \sqcap Q) = \mathbf{R}_D(P) \sqcap \mathbf{R}_D(Q)$
  by (*rule antisym*, *rel-auto+*)

lemma *R-D-cond*:
  $\mathbf{R}_D(P \triangleleft \lceil b\rceil_{D<} \triangleright Q) = \mathbf{R}_D(P) \triangleleft b \triangleright_R \mathbf{R}_D(Q)$
  by (*rule antisym*, *rel-auto+*)

lemma *R-D-seq-ndesign*:
$\mathbf{R}_D(p_1 \vdash_n Q_1) ;; \mathbf{R}_D(p_2 \vdash_n Q_2) = \mathbf{R}_D((p_1 \vdash_n Q_1) ;; (p_2 \vdash_n Q_2))$
apply (*rule antisym*)
apply (*rule SRD-refine-intro*)
apply (*simp-all add*: *closure rdes ndesign-composition-wp*)
using *dual-order.trans* **apply** (*rel-blast*)
using *dual-order.trans* **apply** (*rel-blast*)
apply (*rel-auto*)
apply (*rule SRD-refine-intro*)
apply (*simp-all add*: *closure rdes ndesign-composition-wp*)
apply (*rel-auto*)
apply (*rel-auto*)
apply (*rel-auto*)
done

lemma *R-D-seq*:
assumes P *is* **N** Q *is* **N**
shows $\mathbf{R}_D(P) ;; \mathbf{R}_D(Q) = \mathbf{R}_D(P ;; Q)$
by (*metis R-D-seq-ndesign assms ndesign-form*)

Thes laws are applicable only when there is no further alphabet extension

lemma *R-D-skip*:
$\mathbf{R}_D(II_D) = (II_R :: ('s,'t::trace,unit)\ hrel\text{-}rsp)$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *R-D-assigns*:
$\mathbf{R}_D(\langle\sigma\rangle_D) = (\langle\sigma\rangle_R :: ('s,'t::trace,unit)\ hrel\text{-}rsp)$
by (*simp add*: *assigns-d-def des-rea-lift-def alpha assigns-srd-RHS-tri-des*, *rel-auto*)

end

13 Instantaneous Reactive Designs

theory *utp-rdes-instant*
imports *utp-rdes-prog*
begin

definition *ISRD1* :: $('s,'t::trace,'\alpha)\ hrel\text{-}rsp \Rightarrow ('s,'t,'\alpha)\ hrel\text{-}rsp$ **where**
[*upred-defs*]: $ISRD1(P) = P \parallel_R \mathbf{R}_s(true_r \vdash false \diamond (\$tr\' =_u \$tr))$

definition *ISRD* :: $('s,'t::trace,'\alpha)\ hrel\text{-}rsp \Rightarrow ('s,'t,'\alpha)\ hrel\text{-}rsp$ **where**
[*upred-defs*]: $ISRD = ISRD1 \circ NSRD$

lemma *ISRD1-idem*: $ISRD1(ISRD1(P)) = ISRD1(P)$
by (*rel-auto*)

lemma *ISRD1-monotonic*: $P \sqsubseteq Q \Longrightarrow ISRD1(P) \sqsubseteq ISRD1(Q)$
by (*rel-auto*)

lemma *ISRD1-RHS-design-form*:
assumes $\$ok\' \sharp P$ $\$ok\' \sharp Q$ $\$ok\' \sharp R$
shows $ISRD1(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash false \diamond (R \wedge \$tr\' =_u \$tr))$
using *assms* **by** (*simp add*: *ISRD1-def choose-srd-def RHS-tri-design-par unrest*, *rel-auto*)

lemma *ISRD1-form*:
  $ISRD1(SRD(P)) = \mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr\' =_u \$tr))$
  by (*simp add*: *ISRD1-RHS-design-form SRD-as-reactive-tri-design unrest*)

lemma *ISRD1-rdes-def* [*rdes-def*]:
  $\llbracket$ *P is RR*; *R is RR* $\rrbracket \Longrightarrow ISRD1(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash false \diamond (R \wedge \$tr\' =_u \$tr))$
  by (*simp add*: *ISRD1-def rdes-def closure rpred*)

lemma *ISRD-intro*:
  assumes *P is NSRD* $peri_R(P) = (\neg_r\ pre_R(P))$ $(\$tr\' =_u \$tr) \sqsubseteq post_R(P)$
  shows *P is ISRD*
proof –
  have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$ *is ISRD1*
    apply (*simp add*: *Healthy-def rdes-def closure assms*(*1*−*2*))
    using *assms*(*3*) *least-zero* **apply** (*rel-blast*)
    done
  hence *P is ISRD1*
    by (*simp add*: *SRD-reactive-tri-design closure assms*(*1*))
  thus *?thesis*
    by (*simp add*: *ISRD-def Healthy-comp assms*(*1*))
qed

lemma *ISRD1-rdes-intro*:
  assumes *P is RR Q is RR* $(\$tr\' =_u \$tr) \sqsubseteq Q$
  shows $\mathbf{R}_s(P \vdash false \diamond Q)$ *is ISRD1*
  unfolding *Healthy-def*
  by (*simp add*: *ISRD1-rdes-def assms closure unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-rdes-intro* [*closure*]:
  assumes *P is RC Q is RR* $(\$tr\' =_u \$tr) \sqsubseteq Q$
  shows $\mathbf{R}_s(P \vdash false \diamond Q)$ *is ISRD*
  unfolding *Healthy-def*
  by (*simp add*: *ISRD-def closure Healthy-if ISRD1-rdes-def assms unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-implies-ISRD1*:
  assumes *P is ISRD*
  shows *P is ISRD1*
proof –
  have *ISRD*(*P*) *is ISRD1*
    by (*simp add*: *ISRD-def Healthy-def ISRD1-idem*)
  thus *?thesis*
    by (*simp add*: *assms Healthy-if*)
qed

lemma *ISRD-implies-SRD*:
  assumes *P is ISRD*
  shows *P is SRD*
proof –
  have $1{:}ISRD(P) = \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true \wedge R1\ true) \vdash false \diamond (post_R\ P \wedge \$tr\' =_u \$tr))$
    by (*simp add*: *NSRD-form ISRD1-def ISRD-def RHS-tri-design-par rdes-def unrest closure*)
  moreover have ... *is SRD*
    by (*simp add*: *closure unrest*)
  ultimately have *ISRD*(*P*) *is SRD*
    by (*simp*)
  with *assms* **show** *?thesis*

by (*simp add*: *Healthy-def*)
qed

lemma *ISRD-implies-NSRD* [*closure*]:
assumes *P is ISRD*
shows *P is NSRD*
proof −
have *1*:$ISRD(P) = ISRD1(RD3(SRD(P)))$
by (*simp add*: *ISRD-def NSRD-def SRD-def*, *metis RD1-RD3-commute RD3-left-subsumes-RD2*)
also have ... = $ISRD1(RD3(P))$
by (*simp add*: *assms ISRD-implies-SRD Healthy-if*)
also have ... = $ISRD1\ (\mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false_h \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P))$
by (*simp add*: *RD3-def*, *subst SRD-right-unit-tri-lemma*, *simp-all add*: *assms ISRD-implies-SRD*)
also have ... = $\mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false_h \vdash false \diamond (post_R\ P \wedge \$tr' =_u \$tr))$
by (*simp add*: *RHS-tri-design-par ISRD1-def unrest choose-srd-def rpred closure ISRD-implies-SRD assms*)
also have ... = (... ;; II_R)
by (*rdes-simp*, *simp add*: *RHS-tri-normal-design-composition′ closure assms unrest ISRD-implies-SRD wp rpred wp-rea-false-RC*)
also have ... *is RD3*
by (*simp add*: *Healthy-def RD3-def seqr-assoc*)
finally show *?thesis*
by (*simp add*: *SRD-RD3-implies-NSRD Healthy-if assms ISRD-implies-SRD*)
qed

lemma *ISRD-form*:
assumes *P is ISRD*
shows $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr)) = P$
proof −
have $P = ISRD1(P)$
by (*simp add*: *ISRD-implies-ISRD1 assms Healthy-if*)
also have ... = $ISRD1(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design ISRD-implies-SRD assms*)
also have ... = $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr))$
by (*simp add*: *ISRD1-rdes-def closure assms*)
finally show *?thesis* **..**
qed

lemma *ISRD-elim* [*RD-elim*]:
$[\![\ P\ is\ ISRD;\ Q(\mathbf{R}_s\ (pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr)))\ ]\!] \Longrightarrow Q(P)$
by (*simp add*: *ISRD-form*)

lemma *skip-srd-ISRD* [*closure*]: II_R *is ISRD*
by (*rule ISRD-intro*, *simp-all add*: *rdes closure*)

lemma *assigns-srd-ISRD* [*closure*]: $\langle\sigma\rangle_R$ *is ISRD*
by (*rule ISRD-intro*, *simp-all add*: *rdes closure*, *rel-auto*)

lemma *seq-ISRD-closed*:
assumes *P is ISRD Q is ISRD*
shows *P* ;; *Q is ISRD*
apply (*insert assms*)
apply (*erule ISRD-elim*)+
apply (*simp add*: *rdes-def closure assms unrest*)
apply (*rule ISRD-rdes-intro*)

apply (*simp-all add*: *rdes-def closure assms unrest*)
apply (*rel-auto*)
done

lemma *ISRD-Miracle-right-zero*:
assumes P *is ISRD* $pre_R(P) = true_r$
shows P ;; *Miracle* = *Miracle*
by (*rdes-simp cls*: *assms*)

A recursion whose body does not extend the trace results in divergence

lemma *ISRD-recurse-Chaos*:
assumes P *is ISRD* $post_R$ P ;; $true_r = true_r$
shows $(\mu_R\ X \cdot P\ ;;\ X) = Chaos$
proof −
have *1*: $(\mu_R\ X \cdot P\ ;;\ X) = (\mu\ X \cdot P\ ;;\ SRD(X))$
by (*auto simp add*: *srdes-theory-continuous.utp-lfp-def closure assms*)
have $(\mu\ X \cdot P\ ;;\ SRD(X)) \sqsubseteq Chaos$
proof (*rule gfp-upperbound*)
have P ;; *Chaos* $\sqsubseteq$ *Chaos*
apply (*rdes-refine-split cls*: *assms*)
using *assms*(*2*) **apply** (*rel-auto*, *metis* (*no-types*, *lifting*) *dual-order.antisym order-refl*)
apply (*rel-auto*)+
done
thus P ;; *SRD Chaos* $\sqsubseteq$ *Chaos*
by (*simp add*: *Healthy-if srdes-theory-continuous.bottom-closed*)
qed
thus *?thesis*
by (*metis 1 dual-order.antisym srdes-theory-continuous.LFP-closed srdes-theory-continuous.bottom-lower*)
qed

lemma *recursive-assign-Chaos*:
$(\mu_R\ X \cdot \langle\sigma\rangle_R\ ;;\ X) = Chaos$
by (*rule ISRD-recurse-Chaos*, *simp-all add*: *closure rdes*, *rel-auto*)

end

14 Meta-theory for Reactive Designs

theory *utp-rea-designs*
imports
utp-rdes-healths
utp-rdes-designs
utp-rdes-triples
utp-rdes-normal
utp-rdes-contracts
utp-rdes-tactics
utp-rdes-parallel
utp-rdes-prog
utp-rdes-instant
utp-rdes-guarded
begin end

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