

Circus in Isabelle/UTP

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1 Introduction

This document contains a mechanisation in Isabelle/UTP [1] of Circus [2].

2 Circus Trace Merge

```
theory utp-circus-traces
  imports UTP-Stateful-Failures.utp-sf-rdes
begin
```

2.1 Function Definition

```
fun tr-par ::
  'a set ⇒ 'a list ⇒ 'a list ⇒ 'a list set where
tr-par cs [] = {} |
tr-par cs (e # t) [] = (if e ∈ cs then {} else {[e]} ∩ (tr-par cs t [])) |
```

$tr\text{-}par\ cs\ []\ (e\ \# \ t) = (if\ e \in cs\ then\ \{\}\ else\ \{e\}) \frown (tr\text{-}par\ cs\ []\ t) \mid$
 $tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ (e_2\ \# \ t_2) =$
 $(if\ e_1 = e_2$
 $\quad then$
 $\quad if\ e_1 \in cs$
 $\quad \quad then\ \{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ t_2)$
 $\quad \quad else$
 $\quad \quad (\{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2))) \cup$
 $\quad \quad (\{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2))$
 $\quad else$
 $\quad if\ e_1 \in cs\ then$
 $\quad \quad if\ e_2 \in cs\ then\ \{\}$
 $\quad \quad else$
 $\quad \quad \{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2)$
 $\quad else$
 $\quad \quad if\ e_2 \in cs\ then$
 $\quad \quad \quad \{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2))$
 $\quad \quad \quad else$
 $\quad \quad \quad (\{[e_1]\} \frown (tr\text{-}par\ cs\ t_1\ (e_2\ \# \ t_2))) \cup$
 $\quad \quad \quad (\{[e_2]\} \frown (tr\text{-}par\ cs\ (e_1\ \# \ t_1)\ t_2))$

abbreviation $tr\text{-}inter :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list\ set$ (**infixr** $|||_t\ 100$) **where**
 $x\ |||_t\ y \equiv tr\text{-}par\ \{\}\ x\ y$

2.2 Lifted Trace Merge

syntax $-utr\text{-}par ::$
 $logic \Rightarrow logic \Rightarrow logic \Rightarrow logic\ ((- \ \star_- / \ -)\ [100, 0, 101]\ 100)$

The function $trop$ is used to lift ternary operators.

translations

$t1\ \star_{cs}\ t2 == (CONST\ bop)\ (CONST\ tr\text{-}par\ cs)\ t1\ t2$

2.3 Trace Merge Lemmas

lemma $tr\text{-}par\text{-}empty:$

$tr\text{-}par\ cs\ t1\ [] = \{takeWhile\ (\lambda x. x \notin cs)\ t1\}$

$tr\text{-}par\ cs\ []\ t2 = \{takeWhile\ (\lambda x. x \notin cs)\ t2\}$

— Subgoal 1

apply $(induct\ t1;\ simp)$

— Subgoal 2

apply $(induct\ t2;\ simp)$

done

lemma $tr\text{-}par\text{-}sym:$

$tr\text{-}par\ cs\ t1\ t2 = tr\text{-}par\ cs\ t2\ t1$

apply $(induct\ t1\ arbitrary;\ t2)$

— Subgoal 1

apply $(simp\ add;\ tr\text{-}par\text{-}empty)$

— Subgoal 2

apply $(induct\text{-}tac\ t2)$

— Subgoal 2.1

apply $(clarsimp)$

— Subgoal 2.2

apply $(clarsimp)$

apply $(blast)$

done

lemma *tr-inter-sym*: $x \parallel_t y = y \parallel_t x$
 by (*simp add: tr-par-sym*)

lemma *trace-merge-nil* [*simp*]: $x \star_{\{\}} \langle \rangle = \{x\}_u$
 by (*pred-auto, simp-all add: tr-par-empty, metis takeWhile-eq-all-conv*)

lemma *trace-merge-empty* [*simp*]:
 $(\langle \rangle \star_{cs} \langle \rangle) = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-single-empty* [*simp*]:
 $a \in cs \implies \langle \ll a \gg \rangle \star_{cs} \langle \rangle = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-empty-single* [*simp*]:
 $a \in cs \implies \langle \rangle \star_{cs} \langle \ll a \gg \rangle = \{\langle \rangle\}_u$
 by (*rel-auto*)

lemma *trace-merge-commute*: $t_1 \star_{cs} t_2 = t_2 \star_{cs} t_1$
 by (*rel-simp, simp add: tr-par-sym*)

lemma *csp-trace-simps* [*simp*]:
 $v + \langle \rangle = v \langle \rangle + v = v$
 $bop (\#) x xs \hat{^}_u ys = bop (\#) x (xs \hat{^}_u ys)$
 by (*rel-auto*)+

end

3 Syntax and Translations for Event Prefix

theory *utp-circus-prefix*
imports *UTP-Stateful-Failures.utp-sf-rdes*
begin

syntax
-simple-prefix :: *logic* \Rightarrow *logic* \Rightarrow *logic* ($- \rightarrow -$ [63, 62] 62)

translations
 $a \rightarrow P == \text{CONST PrefixCSP } \ll a \gg P$

We next configure a syntax for mixed prefixes.

nonterminal *prefix-elem'* **and** *mixed-prefix'*

syntax *-end-prefix* :: *prefix-elem'* \Rightarrow *mixed-prefix'* (-)

Input Prefix: $\dots?(x)$

syntax *-simple-input-prefix* :: *id* \Rightarrow *prefix-elem'* ($?(-')$)

Input Prefix with Constraint: $\dots?(x : P)$

syntax *-input-prefix* :: *id* \Rightarrow ($'\sigma, '\varepsilon$) *action* \Rightarrow *prefix-elem'* ($?(- : / -')$)

Output Prefix: $\dots![v]e$

A variable name must currently be provided for outputs, too. Fix?!

syntax *-output-prefix* :: ('a, 'σ) *uexpr* ⇒ *prefix-elim'* (!'(-'))

syntax *-output-prefix* :: ('a, 'σ) *uexpr* ⇒ *prefix-elim'* (.'(-'))

syntax (**output**) *-output-prefix-pp* :: ('a, 'σ) *uexpr* ⇒ *prefix-elim'* (!'(-'))

syntax

-prefix-aux :: *pttrn* ⇒ *logic* ⇒ *prefix-elim'*

Mixed-Prefix Action: *c... (prefix)* → *A*

syntax *-mixed-prefix* :: *prefix-elim'* ⇒ *mixed-prefix'* ⇒ *mixed-prefix'* (--)

syntax

-prefix-action ::

('a, 'ε) *chan* ⇒ *mixed-prefix'* ⇒ ('σ, 'ε) *action* ⇒ ('σ, 'ε) *action*

((-- →/ -) [63, 63, 62] 62)

Syntax translations

definition *lconj* :: ('a ⇒ 'α *upred*) ⇒ ('b ⇒ 'α *upred*) ⇒ ('a × 'b ⇒ 'α *upred*) (**infixr** ∧_l 35)

where [*upred-defs*]: (*P* ∧_l *Q*) ≡ (λ (*x*, *y*). *P* *x* ∧ *Q* *y*)

definition *outp-constraint* (**infix** =_o 60) **where**

[*upred-defs*]: *outp-constraint* *v* ≡ (λ *x*. «*x*» =_u *v*)

translations

-simple-input-prefix *x* ≡ *-input-prefix* *x* *true*

-mixed-prefix (*-input-prefix* *x* *P*) (*-prefix-aux* *y* *Q*) →

-prefix-aux (*-pattern* *x* *y*) ((λ *x*. *P*) ∧_l *Q*)

-mixed-prefix (*-output-prefix* *P*) (*-prefix-aux* *y* *Q*) →

-prefix-aux (*-pattern* *-idtdummy* *y*) ((*CONST* *outp-constraint* *P*) ∧_l *Q*)

-end-prefix (*-input-prefix* *x* *P*) → *-prefix-aux* *x* (λ *x*. *P*)

-end-prefix (*-output-prefix* *P*) → *-prefix-aux* *-idtdummy* (*CONST* *outp-constraint* *P*)

-prefix-action *c* (*-prefix-aux* *x* *P*) *A* == (*CONST* *InputCSP*) *c* *P* (λ *x*. *A*)

Basic print translations; more work needed

translations

-simple-input-prefix *x* <= *-input-prefix* *x* *true*

-output-prefix *v* <= *-prefix-aux* *p* (*CONST* *outp-constraint* *v*)

-output-prefix *u* (*-output-prefix* *v*)

<= *-prefix-aux* *p* (λ(*x1*, *y1*). *CONST* *outp-constraint* *u* *x2* ∧ *CONST* *outp-constraint* *v* *y2*)

-input-prefix *x* *P* <= *-prefix-aux* *v* (λ *x*. *P*)

x!(*v*) → *P* <= *CONST* *OutputCSP* *x* *v* *P*

term *x*!(1)!(*y*) → *P*

term *x*?(*v*) → *P*

term *x*?(*v*:*false*) → *P*

term *x*!((1)) → *P*

term *x*?(*v*)!(1) → *P*

term *x*!((1))!(2)?(*v*:*true*) → *P*

Basic translations for state variable communications

syntax

-csp-input-var :: *logic* ⇒ *id* ⇒ *logic* ⇒ *logic* (-?%- [63, 0, 60] 62)

-csp-inputu-var :: *logic* ⇒ *id* ⇒ *logic* (-?%- [63, 60] 62)

translations

$c?\$x:A \rightarrow \text{CONST InputVarCSP } c \ x \ A$
 $c?\$x \rightarrow \text{CONST InputVarCSP } c \ x \ (\lambda \ x. \text{true})$
 $c?\$x:A \leq \text{CONST InputVarCSP } c \ x \ (\lambda \ x'. A)$
 $c?\$x \leq c?\$x:\text{true}$

lemma outp-constraint-prod:

$(\text{outp-constraint } \ll a \gg x \wedge \text{outp-constraint } \ll b \gg y) =$
 $\text{outp-constraint } \ll (a, b) \gg (x, y)$
by (*simp add: outp-constraint-def, pred-auto*)

lemma subst-outp-constraint [usubst]:

$\sigma \uparrow (v =_o x) = (\sigma \uparrow v =_o x)$
by (*rel-auto*)

lemma UINF-one-point-simp [rpred]:

$\ll \bigwedge i. P \ i \text{ is } R1 \gg \implies (\bigcap x \cdot \ll i \gg =_o x)_{S<} \wedge P(x) = P(i)$
by (*rel-blast*)

lemma USUP-one-point-simp [rpred]:

$\ll \bigwedge i. P \ i \text{ is } R1 \gg \implies (\bigcup x \cdot \ll i \gg =_o x)_{S<} \Rightarrow_r P(x) = P(i)$
by (*rel-blast*)

lemma USUP-eq-event-eq [rpred]:

assumes $\bigwedge y. P(y) \text{ is } RR$
shows $(\bigcup y \cdot [v =_o y]_{S<} \Rightarrow_r P(y)) = P(y) \ll y \rightarrow [v]_{S\leftarrow} \gg$

proof –

have $(\bigcup y \cdot [v =_o y]_{S<} \Rightarrow_r RR(P(y))) = RR(P(y)) \ll y \rightarrow [v]_{S\leftarrow} \gg$
apply (*rel-simp, safe*)
apply *metis*
apply *blast*
apply *simp*
done

thus *?thesis*

by (*simp add: Healthy-if assms*)

qed

lemma UINF-eq-event-eq [rpred]:

assumes $\bigwedge y. P(y) \text{ is } RR$
shows $(\bigcap y \cdot [v =_o y]_{S<} \wedge P(y)) = P(y) \ll y \rightarrow [v]_{S\leftarrow} \gg$

proof –

have $(\bigcap y \cdot [v =_o y]_{S<} \wedge RR(P(y))) = RR(P(y)) \ll y \rightarrow [v]_{S\leftarrow} \gg$
by (*rel-simp, safe, metis*)

thus *?thesis*

by (*simp add: Healthy-if assms*)

qed

Proofs that the input constrained parser versions of output is the same as the regular definition.

lemma output-prefix-is-OutputCSP [simp]:

assumes $A \text{ is } NCSP$
shows $x!(P) \rightarrow A = \text{OutputCSP } x \ P \ A$ (**is** *?lhs = ?rhs*)
by (*rule SRD-eq-intro, simp-all add: assms closure rdes, rel-auto+*)

lemma OutputCSP-pair-simp [simp]:

P is NCSP $\implies a.(\llbracket i \rrbracket).(\llbracket j \rrbracket) \rightarrow P = \text{OutputCSP } a \llbracket (i,j) \rrbracket P$
using *output-prefix-is-OutputCSP*[of P a]
by (*simp add: outp-constraint-prod lconj-def InputCSP-def closure del: output-prefix-is-OutputCSP*)

lemma *OutputCSP-triple-simp* [*simp*]:

P is NCSP $\implies a.(\llbracket i \rrbracket).(\llbracket j \rrbracket).(\llbracket k \rrbracket) \rightarrow P = \text{OutputCSP } a \llbracket (i,j,k) \rrbracket P$
using *output-prefix-is-OutputCSP*[of P a]
by (*simp add: outp-constraint-prod lconj-def InputCSP-def closure del: output-prefix-is-OutputCSP*)

end

4 Circus Parallel Composition

theory *utp-circus-parallel*

imports

utp-circus-prefix

utp-circus-traces

begin

4.1 Merge predicates

definition *CSPInnerMerge* :: $(\alpha \implies \sigma) \Rightarrow \psi \text{ set} \Rightarrow (\beta \implies \sigma) \Rightarrow ((\sigma, \psi) \text{ st-csp}) \text{ merge } (N_C)$ **where**
[*upred-defs*]:

$\text{CSPInnerMerge } ns1 \text{ cs } ns2 =$
 $\$ref' \subseteq_u ((\$0-ref \cup_u \$1-ref) \cap_u \llbracket cs \rrbracket) \cup_u ((\$0-ref \cap_u \$1-ref) - \llbracket cs \rrbracket) \wedge$
 $\$tr_{<} \leq_u \$tr' \wedge$
 $(\$tr' - \$tr_{<}) \in_u (\$0-tr - \$tr_{<}) \star_{cs} (\$1-tr - \$tr_{<}) \wedge$
 $(\$0-tr - \$tr_{<}) \upharpoonright_u \llbracket cs \rrbracket =_u (\$1-tr - \$tr_{<}) \upharpoonright_u \llbracket cs \rrbracket \wedge$
 $\$st' =_u (\$st_{<} \oplus \$0-st \text{ on } \&ns1) \oplus \$1-st \text{ on } \&ns2)$

definition *CSPInnerInterleave* :: $(\alpha \implies \sigma) \Rightarrow (\beta \implies \sigma) \Rightarrow ((\sigma, \psi) \text{ st-csp}) \text{ merge } (N_I)$ **where**
[*upred-defs*]:

$N_I \text{ ns1 } ns2 =$
 $\$ref' \subseteq_u (\$0-ref \cap_u \$1-ref) \wedge$
 $\$tr_{<} \leq_u \$tr' \wedge$
 $(\$tr' - \$tr_{<}) \in_u (\$0-tr - \$tr_{<}) \star_{\{\}} (\$1-tr - \$tr_{<}) \wedge$
 $\$st' =_u (\$st_{<} \oplus \$0-st \text{ on } \&ns1) \oplus \$1-st \text{ on } \&ns2)$

An intermediate merge hides the state, whilst a final merge hides the refusals.

definition *CSPInterMerge* **where**

[*upred-defs*]: $\text{CSPInterMerge } P \text{ ns1 cs ns2 } Q = (P \parallel_{(\exists \$st' \cdot N_C \text{ ns1 cs ns2})} Q)$

definition *CSPFinalMerge* **where**

[*upred-defs*]: $\text{CSPFinalMerge } P \text{ ns1 cs ns2 } Q = (P \parallel_{(\exists \$ref' \cdot N_C \text{ ns1 cs ns2})} Q)$

syntax

-cinter-merge :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \text{ (- } \llbracket -|- \rrbracket^I \text{ - } [85,0,0,0,86] \text{ } 86)$
-cfinal-merge :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \text{ (- } \llbracket -|- \rrbracket^F \text{ - } [85,0,0,0,86] \text{ } 86)$
-wrC :: $logic \Rightarrow salpha \Rightarrow logic \Rightarrow salpha \Rightarrow logic \Rightarrow logic \text{ (- wr[-|-]_C - } [85,0,0,0,86] \text{ } 86)$

translations

-cinter-merge $P \text{ ns1 cs ns2 } Q == \text{CONST CSPInterMerge } P \text{ ns1 cs ns2 } Q$
-cfinal-merge $P \text{ ns1 cs ns2 } Q == \text{CONST CSPFinalMerge } P \text{ ns1 cs ns2 } Q$
-wrC $P \text{ ns1 cs ns2 } Q == P \text{ wr}_R(N_C \text{ ns1 cs ns2}) Q$

lemma *CSPInnerMerge-R2m* [closure]: $N_C \text{ } ns1 \text{ } cs \text{ } ns2 \text{ is } R2m$
 by (rel-auto)

lemma *CSPInnerMerge-RDM* [closure]: $N_C \text{ } ns1 \text{ } cs \text{ } ns2 \text{ is } RDM$
 by (rule *RDM-intro*, simp add: closure, simp-all add: *CSPInnerMerge-def unrest*)

lemma *ex-ref'-R2m-closed* [closure]:

assumes $P \text{ is } R2m$
 shows $(\exists \text{ } \$ref' \cdot P) \text{ is } R2m$

proof –

have $R2m(\exists \text{ } \$ref' \cdot R2m \text{ } P) = (\exists \text{ } \$ref' \cdot R2m \text{ } P)$
 by (rel-auto)

thus ?thesis

by (metis *Healthy-def' assms*)

qed

lemma *CSPInnerMerge-unrests* [unrest]:

$\$ok_{<} \# N_C \text{ } ns1 \text{ } cs \text{ } ns2$
 $\$wait_{<} \# N_C \text{ } ns1 \text{ } cs \text{ } ns2$
 by (rel-auto)+

lemma *CSPInterMerge-RR-closed* [closure]:

assumes $P \text{ is } RR \text{ } Q \text{ is } RR$
 shows $P \llbracket ns1|cs|ns2 \rrbracket^I Q \text{ is } RR$
 by (simp add: *CSPInterMerge-def parallel-RR-closed assms closure unrest*)

lemma *CSPInterMerge-unrest-ref* [unrest]:

assumes $P \text{ is } CRR \text{ } Q \text{ is } CRR$
 shows $\$ref \# P \llbracket ns1|cs|ns2 \rrbracket^I Q$

proof –

have $\$ref \# CRR(P) \llbracket ns1|cs|ns2 \rrbracket^I CRR(Q)$
 by (rel-blast)

thus ?thesis

by (simp add: *Healthy-if assms*)

qed

lemma *CSPInterMerge-unrest-st'* [unrest]:

$\$st' \# P \llbracket ns1|cs|ns2 \rrbracket^I Q$
 by (rel-auto)

lemma *CSPInterMerge-CRR-closed* [closure]:

assumes $P \text{ is } CRR \text{ } Q \text{ is } CRR$
 shows $P \llbracket ns1|cs|ns2 \rrbracket^I Q \text{ is } CRR$
 by (simp add: *CRR-implies-RR CRR-intro CSPInterMerge-RR-closed CSPInterMerge-unrest-ref assms*)

lemma *CSPFinalMerge-RR-closed* [closure]:

assumes $P \text{ is } RR \text{ } Q \text{ is } RR$
 shows $P \llbracket ns1|cs|ns2 \rrbracket^F Q \text{ is } RR$
 by (simp add: *CSPFinalMerge-def parallel-RR-closed assms closure unrest*)

lemma *CSPFinalMerge-unrest-ref* [unrest]:

assumes $P \text{ is } CRR \text{ } Q \text{ is } CRR$
 shows $\$ref \# P \llbracket ns1|cs|ns2 \rrbracket^F Q$

proof –

have $\$ref \# CRR(P) \llbracket ns1|cs|ns2 \rrbracket^F CRR(Q)$

by (rel-blast)
 thus ?thesis
 by (simp add: Healthy-if assms)
 qed

lemma *CSPFinalMerge-CRR-closed* [closure]:
 assumes P is CRR Q is CRR
 shows $P \llbracket ns1 | cs | ns2 \rrbracket^F Q$ is CRR
 by (simp add: CRR-implies-RR CRR-intro CSPFinalMerge-RR-closed CSPFinalMerge-unrest-ref assms)

lemma *CSPInnerMerge-empty-Interleave*:
 $N_C ns1 \{\} ns2 = N_I ns1 ns2$
 by (rel-auto)

definition *CSPMerge* :: $(\alpha \Rightarrow \sigma) \Rightarrow \psi \text{ set} \Rightarrow (\beta \Rightarrow \sigma) \Rightarrow ((\sigma, \psi) \text{ st-csp}) \text{ merge } (M_C)$ **where**
 $[upred-defs]: M_C ns1 cs ns2 = M_R(N_C ns1 cs ns2) ;; Skip$

definition *CSPInterleave* :: $(\alpha \Rightarrow \sigma) \Rightarrow (\beta \Rightarrow \sigma) \Rightarrow ((\sigma, \psi) \text{ st-csp}) \text{ merge } (M_I)$ **where**
 $[upred-defs]: M_I ns1 ns2 = M_R(N_I ns1 ns2) ;; Skip$

lemma *swap-CSPInnerMerge*:
 $ns1 \bowtie ns2 \Rightarrow swap_m ;; (N_C ns1 cs ns2) = (N_C ns2 cs ns1)$
 apply (rel-auto)
 using tr-par-sym apply blast
 apply (simp add: lens-indep-comm)
 using tr-par-sym apply blast
 apply (simp add: lens-indep-comm)
 done

lemma *SymMerge-CSPInnerMerge-NS* [closure]: $N_C 0_L cs 0_L$ is SymMerge
 by (simp add: Healthy-def swap-CSPInnerMerge)

lemma *SymMerge-CSPInnerInterleave* [closure]:
 $N_I 0_L 0_L$ is SymMerge
 by (metis CSPInnerMerge-empty-Interleave SymMerge-CSPInnerMerge-NS)

lemma *SymMerge-CSPInnerInterleave* [closure]:
 $AssocMerge (N_I 0_L 0_L)$
 apply (rel-auto)
 apply (rename-tac tr tr₂' ref₀ tr₀' ref₀' tr₁' ref₁' tr' ref₂' tr_i' ref₃')
 oops

lemma *CSPInterMerge-false* [rpred]: $P \llbracket ns1 | cs | ns2 \rrbracket^I false = false$
 by (simp add: CSPInterMerge-def)

lemma *CSPFinalMerge-false* [rpred]: $P \llbracket ns1 | cs | ns2 \rrbracket^F false = false$
 by (simp add: CSPFinalMerge-def)

lemma *CSPInterMerge-commute*:
 assumes $ns1 \bowtie ns2$
 shows $P \llbracket ns1 | cs | ns2 \rrbracket^I Q = Q \llbracket ns2 | cs | ns1 \rrbracket^I P$
proof –
 have $P \llbracket ns1 | cs | ns2 \rrbracket^I Q = P \parallel_{\exists} \$st' \cdot N_C ns1 cs ns2 Q$
 by (simp add: CSPInterMerge-def)
 also have $\dots = P \parallel_{\exists} \$st' \cdot (swap_m ;; N_C ns2 cs ns1) Q$

by (simp add: swap-CSPInnerMerge lens-indep-sym assms)
 also have ... = $P \parallel_{\text{swap}_m} ; (\exists \$st' \cdot N_C \text{ ns2 cs ns1}) Q$
 by (simp add: seqr-exists-right)
 also have ... = $Q \parallel_{(\exists \$st' \cdot N_C \text{ ns2 cs ns1})} P$
 by (simp add: par-by-merge-commute-swap[THEN sym])
 also have ... = $Q \llbracket \text{ns2} | \text{cs} | \text{ns1} \rrbracket^I P$
 by (simp add: CSPInterMerge-def)
 finally show ?thesis .
 qed

lemma CSPFinalMerge-commute:

assumes $\text{ns1} \bowtie \text{ns2}$
 shows $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^F Q = Q \llbracket \text{ns2} | \text{cs} | \text{ns1} \rrbracket^F P$
 proof -
 have $P \llbracket \text{ns1} | \text{cs} | \text{ns2} \rrbracket^F Q = P \parallel_{\exists \$ref' \cdot N_C \text{ ns1 cs ns2}} Q$
 by (simp add: CSPFinalMerge-def)
 also have ... = $P \parallel_{\exists \$ref' \cdot (\text{swap}_m ; N_C \text{ ns2 cs ns1})} Q$
 by (simp add: swap-CSPInnerMerge lens-indep-sym assms)
 also have ... = $P \parallel_{\text{swap}_m} ; (\exists \$ref' \cdot N_C \text{ ns2 cs ns1}) Q$
 by (simp add: seqr-exists-right)
 also have ... = $Q \parallel_{(\exists \$ref' \cdot N_C \text{ ns2 cs ns1})} P$
 by (simp add: par-by-merge-commute-swap[THEN sym])
 also have ... = $Q \llbracket \text{ns2} | \text{cs} | \text{ns1} \rrbracket^F P$
 by (simp add: CSPFinalMerge-def)
 finally show ?thesis .
 qed

Important theorem that shows the form of a parallel process

lemma CSPInnerMerge-form:

fixes $P Q :: ('s, 'v) \text{ action}$
 assumes $\text{vwb-lens ns1 vwb-lens ns2 } P \text{ is } RR \text{ } Q \text{ is } RR$
 shows

$$P \parallel_{N_C \text{ ns1 cs ns2}} Q =$$

$$(\exists (\text{ref}_0, \text{ref}_1, \text{st}_0, \text{st}_1, \text{tt}_0, \text{tt}_1) \cdot$$

$$P \llbracket \langle \text{ref}_0 \rangle, \langle \text{st}_0 \rangle, \langle \rangle, \langle \text{tt}_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q \llbracket \langle \text{ref}_1 \rangle, \langle \text{st}_1 \rangle, \langle \rangle, \langle \text{tt}_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$$

$$\wedge \$ref' \sqsubseteq_u ((\langle \text{ref}_0 \rangle \cup_u \langle \text{ref}_1 \rangle) \cap_u \langle \text{cs} \rangle) \cup_u ((\langle \text{ref}_0 \rangle \cap_u \langle \text{ref}_1 \rangle) - \langle \text{cs} \rangle)$$

$$\wedge \$tr \leq_u \$tr'$$

$$\wedge \& \text{tt} \in_u \langle \text{tt}_0 \rangle \star_{cs} \langle \text{tt}_1 \rangle$$

$$\wedge \langle \text{tt}_0 \rangle \upharpoonright_u \langle \text{cs} \rangle =_u \langle \text{tt}_1 \rangle \upharpoonright_u \langle \text{cs} \rangle$$

$$\wedge \$st' =_u (\$st \oplus \langle \text{st}_0 \rangle \text{ on } \& \text{ns1}) \oplus \langle \text{st}_1 \rangle \text{ on } \& \text{ns2})$$

$$(\text{is } ?lhs = ?rhs)$$

proof -

have $P : (\exists \{ \$ok', \$wait' \} \cdot R2(P)) = P$ (is ?P' = -)
 by (simp add: ex-unrest ex-plus Healthy-if assms RR-implyes-R2 unrest closure)
 have $Q : (\exists \{ \$ok', \$wait' \} \cdot R2(Q)) = Q$ (is ?Q' = -)
 by (simp add: ex-unrest ex-plus Healthy-if assms RR-implyes-R2 unrest closure)
 from assms(1,2)
 have $?P' \parallel_{N_C \text{ ns1 cs ns2}} ?Q' =$

$$(\exists (\text{ref}_0, \text{ref}_1, \text{st}_0, \text{st}_1, \text{tt}_0, \text{tt}_1) \cdot$$

$$?P' \llbracket \langle \text{ref}_0 \rangle, \langle \text{st}_0 \rangle, \langle \rangle, \langle \text{tt}_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge ?Q' \llbracket \langle \text{ref}_1 \rangle, \langle \text{st}_1 \rangle, \langle \rangle, \langle \text{tt}_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$$

$$\wedge \$ref' \sqsubseteq_u ((\langle \text{ref}_0 \rangle \cup_u \langle \text{ref}_1 \rangle) \cap_u \langle \text{cs} \rangle) \cup_u ((\langle \text{ref}_0 \rangle \cap_u \langle \text{ref}_1 \rangle) - \langle \text{cs} \rangle)$$

$$\wedge \$tr \leq_u \$tr'$$

$$\wedge \& \text{tt} \in_u \langle \text{tt}_0 \rangle \star_{cs} \langle \text{tt}_1 \rangle$$

$$\wedge \langle \text{tt}_0 \rangle \upharpoonright_u \langle \text{cs} \rangle =_u \langle \text{tt}_1 \rangle \upharpoonright_u \langle \text{cs} \rangle$$

$\wedge \$st' =_u (\$st \oplus \langle st_0 \rangle \text{ on } \&ns1) \oplus \langle st_1 \rangle \text{ on } \&ns2)$
apply (*simp add: par-by-merge-alt-def, rel-auto, blast*)
apply (*rename-tac ok wait tr st ref tr' ref' ref₀ ref₁ st₀ st₁ tr₀ ok₀ tr₁ wait₀ ok₁ wait₁*)
apply (*rule-tac x=ok in exI*)
apply (*rule-tac x=wait in exI*)
apply (*rule-tac x=tr in exI*)
apply (*rule-tac x=st in exI*)
apply (*rule-tac x=ref in exI*)
apply (*rule-tac x=tr @ tr₀ in exI*)
apply (*rule-tac x=st₀ in exI*)
apply (*rule-tac x=ref₀ in exI*)
apply (*auto*)
apply (*metis Prefix-Order.prefixI append-minus*)
done
thus *?thesis*
by (*simp add: P Q*)
qed

lemma *CSPInterMerge-form:*

fixes $P\ Q :: ('σ, 'φ)$ *action*

assumes *vwb-lens ns1 vwb-lens ns2 P is RR Q is RR*

shows

$P \llbracket ns1 | cs | ns2 \rrbracket^I Q =$

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P \llbracket \langle ref_0 \rangle, \langle st_0 \rangle, \langle \rangle, \langle tt_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q \llbracket \langle ref_1 \rangle, \langle st_1 \rangle, \langle \rangle, \langle tt_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$
 $\wedge \$ref' \subseteq_u ((\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle) \cup_u ((\langle ref_0 \rangle \cap_u \langle ref_1 \rangle) - \langle cs \rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle$
 $\wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle)$

(**is** *?lhs = ?rhs*)

proof –

have *?lhs =* $(\exists \$st' \cdot P \parallel_{N_C} ns1\ cs\ ns2\ Q)$

by (*simp add: CSPInterMerge-def par-by-merge-def seqr-exists-right*)

also have ... =

$(\exists \$st' \cdot$

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P \llbracket \langle ref_0 \rangle, \langle st_0 \rangle, \langle \rangle, \langle tt_0 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket \wedge Q \llbracket \langle ref_1 \rangle, \langle st_1 \rangle, \langle \rangle, \langle tt_1 \rangle / \$ref', \$st', \$tr, \$tr' \rrbracket$
 $\wedge \$ref' \subseteq_u ((\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle) \cup_u ((\langle ref_0 \rangle \cap_u \langle ref_1 \rangle) - \langle cs \rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle$
 $\wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$
 $\wedge \$st' =_u (\$st \oplus \langle st_0 \rangle \text{ on } \&ns1) \oplus \langle st_1 \rangle \text{ on } \&ns2))$

by (*simp add: CSPInnerMerge-form assms*)

also have ... = *?rhs*

by (*rel-blast*)

finally show *?thesis* .

qed

lemma *CSPFinalMerge-form:*

fixes $P\ Q :: ('σ, 'φ)$ *action*

assumes *vwb-lens ns1 vwb-lens ns2 P is RR Q is RR* $\$ref' \# P\ \$ref' \# Q$

shows

$(P \llbracket ns1 | cs | ns2 \rrbracket^F Q) =$

$(\exists (st_0, st_1, tt_0, tt_1) \cdot$
 $P \llbracket \langle st_0 \rangle, \langle \rangle, \langle tt_0 \rangle / \$st', \$tr, \$tr' \rrbracket \wedge Q \llbracket \langle st_1 \rangle, \langle \rangle, \langle tt_1 \rangle / \$st', \$tr, \$tr' \rrbracket$

$\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$
 (is ?lhs = ?rhs)
proof –
 have ?lhs = $(\exists \$ref' \cdot P \parallel_{N_C} ns1 \ cs \ ns2 \ Q)$
 by (simp add: CSPFinalMerge-def par-by-merge-def seqr-exists-right)
 also have ... =
 $(\exists \$ref' \cdot$
 $(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $P[\langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr'] \wedge Q[\langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr']$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2))$
 by (simp add: CSPInnerMerge-form assms)
 also have ... =
 $(\exists \$ref' \cdot$
 $(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $(\exists \$ref' \cdot P)[\langle\langle ref_0 \rangle\rangle, \langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$ref', \$st', \$tr, \$tr'] \wedge (\exists \$ref' \cdot Q)[\langle\langle ref_1 \rangle\rangle, \langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$ref', \$st', \$tr, \$tr']$
 $\wedge \$ref' \subseteq_u ((\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle) \cup_u ((\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle) - \langle\langle cs \rangle\rangle)$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2))$
 by (simp add: ex-unrest assms)
 also have ... =
 $(\exists (st_0, st_1, tt_0, tt_1) \cdot$
 $(\exists \$ref' \cdot P)[\langle\langle st_0 \rangle\rangle, \langle\langle tt_0 \rangle\rangle / \$st', \$tr, \$tr'] \wedge (\exists \$ref' \cdot Q)[\langle\langle st_1 \rangle\rangle, \langle\langle tt_1 \rangle\rangle / \$st', \$tr, \$tr']$
 $\wedge \$tr \leq_u \tr'
 $\wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle$
 $\wedge \langle\langle tt_0 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \downarrow_u \langle\langle cs \rangle\rangle$
 $\wedge \$st' =_u (\$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1) \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$
 by (rel-blast)
 also have ... = ?rhs
 by (simp add: ex-unrest assms)
 finally show ?thesis .
qed

lemma CSPInterleave-merge: $M_I \ ns1 \ ns2 = M_C \ ns1 \ \{\} \ ns2$

by (rel-auto)

lemma csp-wrR-def:

$P \ wr[ns1|cs|ns2]_C \ Q = (\neg_r ((\neg_r \ Q) ;; U0 \wedge P ;; U1 \wedge \$st_{<}' =_u \$st \wedge \$tr_{<}' =_u \$tr) ;; N_C \ ns1 \ cs \ ns2 ;; R1 \ true)$

by (rel-auto, metis+)

lemma csp-wrR-CRC-closed [closure]:

assumes P is CRR Q is CRR

shows $P \ wr[ns1|cs|ns2]_C \ Q$ is CRC

proof –

have $\$ref \ \sharp \ P \ wr[ns1|cs|ns2]_C \ Q$

by (simp add: csp-wrR-def rpred closure assms unrest)

thus *?thesis*
 by (rule *CRC-intro*, *simp-all add: closure assms*)
 qed

lemma *ref'-unrest-final-merge* [*unrest*]:
 $\$ref' \# P \llbracket ns1 | cs | ns2 \rrbracket^F Q$
 by (*rel-auto*)

lemma *inter-merge-CDC-closed* [*closure*]:
 $P \llbracket ns1 | cs | ns2 \rrbracket^I Q$ is CDC
 using *le-less-trans* by (*rel-blast*)

lemma *merge-csp-do-left*:
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR*
 shows $\Phi(s_0, \sigma_0, t_0) \parallel_{N_C} ns1 \ cs \ ns2 \ P =$

$(\exists (ref_1, st_1, tt_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger P \wedge$
 $\$ref' \subseteq_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

(is *?lhs = ?rhs*)

proof –

have *?lhs =*

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger \Phi(s_0, \sigma_0, t_0) \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger P \wedge$
 $\$ref' \subseteq_u (\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle \wedge \$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1$
 $\oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*simp add: CSPInnerMerge-form assms closure*)

also have ... =

$(\exists (ref_1, st_1, tt_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger P \wedge$
 $\$ref' \subseteq_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*rel-blast*)

finally show *?thesis* .

qed

lemma *merge-csp-do-right*:

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR*

shows $P \parallel_{N_C} ns1 \ cs \ ns2 \ \Phi(s_1, \sigma_1, t_1) =$

$(\exists (ref_0, st_0, tt_0) \cdot$
 $[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger P \wedge$
 $[s_1]_{S<} \wedge$
 $\$ref' \subseteq_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $[\langle\langle trace \rangle\rangle \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} t_1 \wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u t_1 \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1 \oplus \langle\langle \sigma_1 \rangle\rangle (\$st)_a \text{ on } \&ns2)$

(is *?lhs = ?rhs*)

proof –

have *?lhs =*

$(\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger P \wedge$
 $[\$ref' \mapsto_s \langle\langle ref_1 \rangle\rangle, \$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger \Phi(s_1, \sigma_1, t_1) \wedge$
 $\$ref' \subseteq_u (\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle \wedge \$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$
by (*simp add: CSPInnerMerge-form assms closure*)
also have ... = ?rhs
by (*rel-blast*)
finally show ?thesis .
qed

The result of merge two terminated stateful traces is to (1) require both state preconditions hold, (2) merge the traces using, and (3) merge the state using a parallel assignment.

lemma *FinalMerge-csp-do-left:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR $\$ref' \nmid P$*

shows $\Phi(s_0, \sigma_0, t_0) \llbracket ns1 | cs | ns2 \rrbracket^F P =$

$(\exists (st_1, t_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle t_1 \rangle\rangle] \dagger P \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{cs} \langle\langle t_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle t_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

(**is** ?lhs = ?rhs)

proof –

have ?lhs =

$(\exists (st_0, st_1, tt_0, tt_1) \cdot$
 $[\$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger \Phi(s_0, \sigma_0, t_0) \wedge$
 $[\$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger RR(\exists \$ref' \cdot P) \wedge$
 $\$tr \leq_u \$tr' \wedge \&tt \in_u \langle\langle tt_0 \rangle\rangle \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge \langle\langle tt_0 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle \wedge$
 $\$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*simp add: CSPFinalMerge-form ex-unrest Healthy-if unrest closure assms*)

also have ... =

$(\exists (st_1, tt_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_1 \rangle\rangle] \dagger RR(\exists \$ref' \cdot P) \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{cs} \langle\langle tt_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle tt_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*rel-blast*)

also have ... =

$(\exists (st_1, t_1) \cdot$
 $[s_0]_{S<} \wedge$
 $[\$st' \mapsto_s \langle\langle st_1 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle t_1 \rangle\rangle] \dagger P \wedge$
 $[\langle\langle trace \rangle\rangle \in_u t_0 \star_{cs} \langle\langle t_1 \rangle\rangle \wedge t_0 \upharpoonright_u \langle\langle cs \rangle\rangle =_u \langle\langle t_1 \rangle\rangle \upharpoonright_u \langle\langle cs \rangle\rangle]_t \wedge$
 $\$st' =_u \$st \oplus \langle\langle \sigma_0 \rangle\rangle (\$st)_a \text{ on } \&ns1 \oplus \langle\langle st_1 \rangle\rangle \text{ on } \&ns2)$

by (*simp add: ex-unrest Healthy-if unrest closure assms*)

finally show ?thesis .

qed

lemma *FinalMerge-csp-do-right:*

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR $\$ref' \nmid P$*

shows $P \llbracket ns1 | cs | ns2 \rrbracket^F \Phi(s_1, \sigma_1, t_1) =$

$(\exists (st_0, t_0) \cdot$
 $[\$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle t_0 \rangle\rangle] \dagger P \wedge$
 $[s_1]_{S<} \wedge$

$$[\llbracket \text{trace} \rrbracket \in_u \llbracket t_0 \rrbracket \star_{cs} t_1 \wedge \llbracket t_0 \rrbracket \downarrow_u \llbracket cs \rrbracket =_u t_1 \downarrow_u \llbracket cs \rrbracket]_t \wedge$$

$$\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket \sigma_1 \rrbracket (\$st)_a \text{ on } \&ns2)$$
 (is ?lhs = ?rhs)

proof –
 have $P \llbracket ns1|cs|ns2 \rrbracket^F \Phi(s_1, \sigma_1, t_1) = \Phi(s_1, \sigma_1, t_1) \llbracket ns2|cs|ns1 \rrbracket^F P$
 by (simp add: assms CSPFinalMerge-commute)
 also have ... = ?rhs
 apply (simp add: FinalMerge-csp-do-left assms lens-indep-sym conj-comm)
 apply (rel-auto)
 using assms(3) lens-indep.lens-put-comm tr-par-sym apply fastforce+
 done
 finally show ?thesis .
qed

lemma *FinalMerge-csp-do*:
 assumes $vwb\text{-}lens\ ns1\ vwb\text{-}lens\ ns2\ ns1 \bowtie ns2$
 shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1|cs|ns2 \rrbracket^F \Phi(s_2, \sigma_2, t_2) =$

$$([s_1 \wedge s_2]_{S<} \wedge [\llbracket \text{trace} \rrbracket \in_u t_1 \star_{cs} t_2 \wedge t_1 \downarrow_u \llbracket cs \rrbracket =_u t_2 \downarrow_u \llbracket cs \rrbracket]_t \wedge [\langle \sigma_1 [\&ns1|\&ns2]_s \sigma_2 \rangle_a]_{S'})$$
 (is ?lhs = ?rhs)

proof –
 have ?lhs =

$$(\exists (st_0, st_1, tt_0, tt_1) \cdot$$

$$[\$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger \Phi(s_1, \sigma_1, t_1) \wedge$$

$$[\$st' \mapsto_s \llbracket st_1 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_1 \rrbracket] \dagger \Phi(s_2, \sigma_2, t_2) \wedge$$

$$\$tr \leq_u \$tr' \wedge \&tt \in_u \llbracket tt_0 \rrbracket \star_{cs} \llbracket tt_1 \rrbracket \wedge \llbracket tt_0 \rrbracket \downarrow_u \llbracket cs \rrbracket =_u \llbracket tt_1 \rrbracket \downarrow_u \llbracket cs \rrbracket \wedge$$

$$\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket st_1 \rrbracket \text{ on } \&ns2)$$
 by (simp add: CSPFinalMerge-form unrest closure assms)
 also have ... =

$$([s_1 \wedge s_2]_{S<} \wedge [\llbracket \text{trace} \rrbracket \in_u t_1 \star_{cs} t_2 \wedge t_1 \downarrow_u \llbracket cs \rrbracket =_u t_2 \downarrow_u \llbracket cs \rrbracket]_t \wedge [\langle \sigma_1 [\&ns1|\&ns2]_s \sigma_2 \rangle_a]_{S'})$$
 by (rel-auto)
 finally show ?thesis .
qed

lemma *FinalMerge-csp-do' [rpred]*:
 assumes $vwb\text{-}lens\ ns1\ vwb\text{-}lens\ ns2\ ns1 \bowtie ns2$
 shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1|cs|ns2 \rrbracket^F \Phi(s_2, \sigma_2, t_2) =$

$$(\exists \text{ trace} \in [t_1 \star_{cs} t_2]_{S<} \cdot$$

$$\Phi(s_1 \wedge s_2 \wedge t_1 \downarrow_u \llbracket cs \rrbracket =_u t_2 \downarrow_u \llbracket cs \rrbracket, \sigma_1 [\&ns1|\&ns2]_s \sigma_2, \llbracket \text{trace} \rrbracket))$$
 by (simp add: FinalMerge-csp-do assms, rel-auto)

lemma *CSPFinalMerge-UINF-ind-left [rpred]*:
 $(\bigcap i \cdot P(i)) \llbracket ns1|cs|ns2 \rrbracket^F Q = (\bigcap i \cdot P(i) \llbracket ns1|cs|ns2 \rrbracket^F Q)$
 by (simp add: CSPFinalMerge-def par-by-merge-USUP-ind-left)

lemma *CSPFinalMerge-UINF-ind-right [rpred]*:
 $P \llbracket ns1|cs|ns2 \rrbracket^F (\bigcap i \cdot Q(i)) = (\bigcap i \cdot P \llbracket ns1|cs|ns2 \rrbracket^F Q(i))$
 by (simp add: CSPFinalMerge-def par-by-merge-USUP-ind-right)

lemma *InterMerge-csp-enable*:
 assumes $vwb\text{-}lens\ ns1\ vwb\text{-}lens\ ns2\ ns1 \bowtie ns2$
 shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1|cs|ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$$([s_1 \wedge s_2]_{S<} \wedge$$

$(\forall e \in [(E_1 \cap_u E_2 \cap_u \langle cs \rangle) \cup_u ((E_1 \cup_u E_2) - \langle cs \rangle)]_{S<} \cdot \langle e \rangle \notin_u \$ref') \wedge$
 $[\langle trace \rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle]_t$
 (is ?lhs = ?rhs)
proof –
 have ?lhs =
 ($\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$
 $[\$ref' \mapsto_s \langle ref_1 \rangle, \$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger \mathcal{E}(s_2, t_2, E_2) \wedge$
 $\$ref' \subseteq_u (\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle \cup_u (\langle ref_0 \rangle \cap_u \langle ref_1 \rangle - \langle cs \rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge \&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$)
 by (simp add: CSPInterMerge-form unrest closure assms)
 also have ... =
 ($\exists (ref_0, ref_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle ref_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$
 $[\$ref' \mapsto_s \langle ref_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger \mathcal{E}(s_2, t_2, E_2) \wedge$
 $\$ref' \subseteq_u (\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle \cup_u (\langle ref_0 \rangle \cap_u \langle ref_1 \rangle - \langle cs \rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge \&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$)
 by (rel-auto)
 also have ... =
 ($[s_1 \wedge s_2]_{S<} \wedge$
 $(\forall e \in [(E_1 \cap_u E_2 \cap_u \langle cs \rangle) \cup_u ((E_1 \cup_u E_2) - \langle cs \rangle)]_{S<} \cdot \langle e \rangle \notin_u \$ref') \wedge$
 $[\langle trace \rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle]_t$
)
 apply (rel-auto)
 apply (rename-tac tr st tr' ref')
 apply (rule-tac x=– $\llbracket E_1 \rrbracket_e$ st in exI)
 apply (simp)
 apply (rule-tac x=– $\llbracket E_2 \rrbracket_e$ st in exI)
 apply (auto)
 done
 finally show ?thesis .
 qed

lemma *InterMerge-csp-enable' [rpred]:*
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$
 $(\exists trace \in [t_1 \star_{cs} t_2]_{S<} \cdot$
 $\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle$
 $, \langle trace \rangle$
 $, (E_1 \cap_u E_2 \cap_u \langle cs \rangle) \cup_u ((E_1 \cup_u E_2) - \langle cs \rangle)))$
 by (simp add: InterMerge-csp-enable assms, rel-auto)

lemma *InterMerge-csp-enable-csp-do:*
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1 | cs | ns2 \rrbracket^I \Phi(s_2, \sigma_2, t_2) =$
 $([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_1 - \langle cs \rangle)]_{S<} \cdot \langle e \rangle \notin_u \$ref') \wedge$
 $[\langle trace \rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \upharpoonright_u \langle cs \rangle =_u t_2 \upharpoonright_u \langle cs \rangle]_t$
 (is ?lhs = ?rhs)
proof –
 have ?lhs =
 ($\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$
 $[\$ref' \mapsto_s \langle ref_1 \rangle, \$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger \Phi(s_2, \sigma_2, t_2) \wedge$
 $\$ref' \subseteq_u (\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle \cup_u (\langle ref_0 \rangle \cap_u \langle ref_1 \rangle - \langle cs \rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge \&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle$)

by (*simp add: CSPInterMerge-form unrest closure assms*)
 also have ... =

$$(\exists (ref_0, ref_1, tt_0) \cdot$$

$$[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$tr \mapsto_s \langle\rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger \mathcal{E}(s_1, t_1, E_1) \wedge$$

$$[s_2]_{S<} \wedge$$

$$\$ref' \subseteq_u (\langle\langle ref_0 \rangle\rangle \cup_u \langle\langle ref_1 \rangle\rangle) \cap_u \langle\langle cs \rangle\rangle \cup_u (\langle\langle ref_0 \rangle\rangle \cap_u \langle\langle ref_1 \rangle\rangle - \langle\langle cs \rangle\rangle) \wedge$$

$$[\langle\langle trace \rangle\rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \downarrow_u \langle\langle cs \rangle\rangle =_u t_2 \downarrow_u \langle\langle cs \rangle\rangle]_t)$$
 by (*rel-auto*)
 also have ... = $([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_1 - \langle\langle cs \rangle\rangle)]_{S<} \cdot \langle\langle e \rangle\rangle \notin_u \$ref')) \wedge$

$$[\langle\langle trace \rangle\rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \downarrow_u \langle\langle cs \rangle\rangle =_u t_2 \downarrow_u \langle\langle cs \rangle\rangle]_t)$$
 by (*rel-auto*)
 finally show *?thesis* .
 qed

lemma *InterMerge-csp-enable-csp-do' [rpred]*:
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\mathcal{E}(s_1, t_1, E_1) \llbracket ns1 | cs | ns2 \rrbracket^I \Phi(s_2, \sigma_2, t_2) =$

$$(\bigcap trace \mid \langle\langle trace \rangle\rangle \in_u [t_1 \star_{cs} t_2]_{S<} \cdot$$

$$\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \downarrow_u \langle\langle cs \rangle\rangle =_u t_2 \downarrow_u \langle\langle cs \rangle\rangle, \langle\langle trace \rangle\rangle, E_1 - \langle\langle cs \rangle\rangle))$$
 by (*simp add: InterMerge-csp-enable-csp-do assms, rel-auto*)

lemma *InterMerge-csp-do-csp-enable*:
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$$([s_1 \wedge s_2]_{S<} \wedge (\forall e \in [(E_2 - \langle\langle cs \rangle\rangle)]_{S<} \cdot \langle\langle e \rangle\rangle \notin_u \$ref')) \wedge$$

$$[\langle\langle trace \rangle\rangle \in_u t_1 \star_{cs} t_2 \wedge t_1 \downarrow_u \langle\langle cs \rangle\rangle =_u t_2 \downarrow_u \langle\langle cs \rangle\rangle]_t)$$
 (is *?lhs = ?rhs*)
proof –
 have $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_2, t_2, E_2) \llbracket ns2 | cs | ns1 \rrbracket^I \Phi(s_1, \sigma_1, t_1)$
 by (*simp add: CSPInterMerge-commute assms*)
 also have ... = *?rhs*
 by (*simp add: InterMerge-csp-enable-csp-do assms lens-indep-sym trace-merge-commute conj-comm eq-upred-sym*)
 finally show *?thesis* .
 qed

lemma *InterMerge-csp-do-csp-enable' [rpred]*:
 assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2*
 shows $\Phi(s_1, \sigma_1, t_1) \llbracket ns1 | cs | ns2 \rrbracket^I \mathcal{E}(s_2, t_2, E_2) =$

$$(\bigcap trace \mid \langle\langle trace \rangle\rangle \in_u [t_1 \star_{cs} t_2]_{S<} \cdot$$

$$\mathcal{E}(s_1 \wedge s_2 \wedge t_1 \downarrow_u \langle\langle cs \rangle\rangle =_u t_2 \downarrow_u \langle\langle cs \rangle\rangle, \langle\langle trace \rangle\rangle, E_2 - \langle\langle cs \rangle\rangle))$$
 by (*simp add: InterMerge-csp-do-csp-enable assms, rel-auto*)

lemma *CSPInterMerge-or-left [rpred]*:
 $(P \vee Q) \llbracket ns1 | cs | ns2 \rrbracket^I R = (P \llbracket ns1 | cs | ns2 \rrbracket^I R \vee Q \llbracket ns1 | cs | ns2 \rrbracket^I R)$
 by (*simp add: CSPInterMerge-def par-by-merge-or-left*)

lemma *CSPInterMerge-or-right [rpred]*:
 $P \llbracket ns1 | cs | ns2 \rrbracket^I (Q \vee R) = (P \llbracket ns1 | cs | ns2 \rrbracket^I Q \vee P \llbracket ns1 | cs | ns2 \rrbracket^I R)$
 by (*simp add: CSPInterMerge-def par-by-merge-or-right*)

lemma *CSPFinalMerge-or-left [rpred]*:
 $(P \vee Q) \llbracket ns1 | cs | ns2 \rrbracket^F R = (P \llbracket ns1 | cs | ns2 \rrbracket^F R \vee Q \llbracket ns1 | cs | ns2 \rrbracket^F R)$
 by (*simp add: CSPFinalMerge-def par-by-merge-or-left*)

lemma *CSPFinalMerge-or-right* [rpred]:

$P \llbracket ns1 | cs | ns2 \rrbracket^F (Q \vee R) = (P \llbracket ns1 | cs | ns2 \rrbracket^F Q \vee P \llbracket ns1 | cs | ns2 \rrbracket^F R)$
by (*simp add: CSPFinalMerge-def par-by-merge-or-right*)

lemma *CSPInterMerge-UINF-ind-left* [rpred]:

$(\bigcap i \cdot P(i)) \llbracket ns1 | cs | ns2 \rrbracket^I Q = (\bigcap i \cdot P(i) \llbracket ns1 | cs | ns2 \rrbracket^I Q)$
by (*simp add: CSPInterMerge-def par-by-merge-USUP-ind-left*)

lemma *CSPInterMerge-UINF-ind-right* [rpred]:

$P \llbracket ns1 | cs | ns2 \rrbracket^I (\bigcap i \cdot Q(i)) = (\bigcap i \cdot P \llbracket ns1 | cs | ns2 \rrbracket^I Q(i))$
by (*simp add: CSPInterMerge-def par-by-merge-USUP-ind-right*)

lemma *par-by-merge-seq-remove*: $(P \parallel_M ; R \ Q) = (P \parallel_M Q) ; R$

by (*simp add: par-by-merge-seq-add[THEN sym]*)

lemma *merge-csp-do-right*:

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RC*

shows $\Phi(s_1, \sigma_1, t_1) \text{ wr}[ns1 | cs | ns2]_C P = \text{undefined}$

(is ?lhs = ?rhs)

proof –

have *?lhs* =

$(\neg_r (\exists (ref_0, st_0, tt_0) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger (\neg_r RC(P)) \wedge$
 $[s_1]_{S<} \wedge$
 $\$ref' \subseteq_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $\llbracket trace \rrbracket \in_u \llbracket tt_0 \rrbracket \star_{cs} t_1 \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u t_1 \upharpoonright_u \llbracket cs \rrbracket]_t \wedge$
 $\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket \sigma_1 \rrbracket (\$st)_a \text{ on } \&ns2) ; R1 \text{ true})$

by (*simp add: wrR-def par-by-merge-seq-remove merge-csp-do-right closure assms Healthy-if rpred*)

also have ... =

$(\neg_r (\exists (ref_0, st_0, tt_0) \cdot$
 $[\$ref' \mapsto_s \llbracket ref_0 \rrbracket, \$st' \mapsto_s \llbracket st_0 \rrbracket, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket tt_0 \rrbracket] \dagger (\neg_r RC(P)) \wedge$
 $[s_1]_{S<} \wedge$
 $\$ref' \subseteq_u \llbracket cs \rrbracket \cup_u (\llbracket ref_0 \rrbracket - \llbracket cs \rrbracket) \wedge$
 $\llbracket trace \rrbracket \in_u \llbracket tt_0 \rrbracket \star_{cs} t_1 \wedge \llbracket tt_0 \rrbracket \upharpoonright_u \llbracket cs \rrbracket =_u t_1 \upharpoonright_u \llbracket cs \rrbracket]_t ; \text{true}_r \wedge$
 $\$st' =_u \$st \oplus \llbracket st_0 \rrbracket \text{ on } \&ns1 \oplus \llbracket \sigma_1 \rrbracket (\$st)_a \text{ on } \&ns2))$

apply (*rel-auto*)

oops

4.2 Parallel operator

syntax

-par-circus :: *logic* \Rightarrow *salpha* \Rightarrow *logic* \Rightarrow *salpha* \Rightarrow *logic* \Rightarrow *logic* (- $\llbracket - \rrbracket - \llbracket - \rrbracket$ - [75,0,0,0,76] 76)

-par-csp :: *logic* \Rightarrow *logic* \Rightarrow *logic* \Rightarrow *logic* (- $\llbracket - \rrbracket_C$ - [75,0,76] 76)

-inter-circus :: *logic* \Rightarrow *salpha* \Rightarrow *salpha* \Rightarrow *logic* \Rightarrow *logic* (- $\llbracket - \rrbracket - \llbracket - \rrbracket$ - [75,0,0,76] 76)

translations

-par-circus $P \ ns1 \ cs \ ns2 \ Q == P \parallel_{M_C} ns1 \ cs \ ns2 \ Q$

-par-csp $P \ cs \ Q == \text{-par-circus } P \ 0_L \ cs \ 0_L \ Q$

-inter-circus $P \ ns1 \ ns2 \ Q == \text{-par-circus } P \ ns1 \ \{\} \ ns2 \ Q$

abbreviation *InterleaveCSP* :: (*'s*, *'e*) *action* \Rightarrow (*'s*, *'e*) *action* \Rightarrow (*'s*, *'e*) *action* (**infixr** \parallel 75)

where $P \parallel Q \equiv P \llbracket \emptyset \rrbracket \emptyset Q$

abbreviation *SynchroniseCSP* :: (*'s*, *'e*) *action* \Rightarrow (*'s*, *'e*) *action* \Rightarrow (*'s*, *'e*) *action* (**infixr** \parallel 75)

where $P \parallel Q \equiv P \llbracket UNIV \rrbracket_C Q$

definition $CSP5 :: ' \varphi \text{ process} \Rightarrow ' \varphi \text{ process}$ **where**

$[upred-defs]: CSP5(P) = (P \parallel Skip)$

definition $C2 :: (' \sigma, ' \varphi) \text{ action} \Rightarrow (' \sigma, ' \varphi) \text{ action}$ **where**

$[upred-defs]: C2(P) = (P \llbracket \Sigma \parallel \{ \} \parallel \emptyset \rrbracket Skip)$

definition $CACT :: (' s, ' e) \text{ action} \Rightarrow (' s, ' e) \text{ action}$ **where**

$[upred-defs]: CACT(P) = C2(NCSP(P))$

abbreviation $CPROC :: ' e \text{ process} \Rightarrow ' e \text{ process}$ **where**

$CPROC(P) \equiv CACT(P)$

lemma *Skip-right-form*:

assumes $P_1 \text{ is } RC \ P_2 \text{ is } RR \ P_3 \text{ is } RR \ \$st' \# P_2$

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) ;; Skip = \mathbf{R}_s(P_1 \vdash P_2 \diamond (\exists \$ref' \cdot P_3))$

proof –

have $1:RR(P_3) ;; \Phi(true, id, \langle \rangle) = (\exists \$ref' \cdot RR(P_3))$

by (*rel-auto*)

show *?thesis*

by (*rdes-simp cls: assms, metis 1 Healthy-if assms(3)*)

qed

lemma *ParCSP-rdes-def* [*rdes-def*]:

fixes $P_1 :: (' s, ' e) \text{ action}$

assumes $P_1 \text{ is } CRC \ Q_1 \text{ is } CRC \ P_2 \text{ is } CRR \ Q_2 \text{ is } CRR \ P_3 \text{ is } CRR \ Q_3 \text{ is } CRR$

$\$st' \# P_2 \ \$st' \# Q_2$

$ns1 \bowtie ns2$

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \llbracket ns1 \parallel cs \parallel ns2 \rrbracket \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$

$\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $((P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3)))$

(**is** $?P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket ?Q = ?rhs$)

proof –

have $?P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket ?Q = (?P \parallel_{M_R(N_C \ ns1 \ cs \ ns2)} ?Q) ;;_h Skip$

by (*simp add: CSPMerge-def par-by-merge-seq-add*)

also

have $\dots = \mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$

$(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge$
 $(P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond$
 $(P_1 \Rightarrow_r P_3) \parallel_{N_C \ ns1 \ cs \ ns2} (Q_1 \Rightarrow_r Q_3)) ;;_h Skip$

by (*simp add: parallel-rdes-def swap-CSPInnerMerge CSPInterMerge-def closure assms*)

also

have $\dots = \mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$

$(Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge$

$$\begin{aligned}
& (P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge \\
& (P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1 \vdash \\
& ((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee \\
& (P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee \\
& (P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond \\
& (\exists \$ref' \cdot ((P_1 \Rightarrow_r P_3) \parallel_{N_C} ns1 \text{ cs } ns2 (Q_1 \Rightarrow_r Q_3))) \\
& \text{by (simp add: Skip-right-form closure parallel-RR-closed assms unrest)} \\
& \text{also} \\
& \text{have ... = } \mathbf{R}_s (((Q_1 \Rightarrow_r Q_2) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge \\
& (Q_1 \Rightarrow_r Q_3) \text{ wr}[ns1|cs|ns2]_C P_1 \wedge \\
& (P_1 \Rightarrow_r P_2) \text{ wr}[ns2|cs|ns1]_C Q_1 \wedge \\
& (P_1 \Rightarrow_r P_3) \text{ wr}[ns2|cs|ns1]_C Q_1) \vdash \\
& ((P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee \\
& (P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_2) \vee \\
& (P_1 \Rightarrow_r P_2) \llbracket ns1|cs|ns2 \rrbracket^I (Q_1 \Rightarrow_r Q_3)) \diamond \\
& ((P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^F (Q_1 \Rightarrow_r Q_3))) \\
& \text{proof -} \\
& \text{have } (\exists \$ref' \cdot ((P_1 \Rightarrow_r P_3) \parallel_{N_C} ns1 \text{ cs } ns2 (Q_1 \Rightarrow_r Q_3))) = ((P_1 \Rightarrow_r P_3) \llbracket ns1|cs|ns2 \rrbracket^F (Q_1 \Rightarrow_r \\
& Q_3)) \\
& \text{by (rel-blast)} \\
& \text{thus ?thesis by simp} \\
& \text{qed} \\
& \text{finally show ?thesis .} \\
& \text{qed}
\end{aligned}$$

4.3 Parallel Laws

lemma *ParCSP-expand*:

$P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q = (P \parallel_{R N_C} ns1 \text{ cs } ns2 Q) ;; \text{Skip}$
 by (simp add: CSPMerge-def par-by-merge-seq-add)

lemma *parallel-is-CSP [closure]*:

assumes P is CSP Q is CSP
 shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q)$ is CSP

proof –

have $(P \parallel_{M_R(N_C \text{ ns1 cs ns2})} Q)$ is CSP
 by (simp add: closure assms)
 hence $(P \parallel_{M_R(N_C \text{ ns1 cs ns2})} Q) ;; \text{Skip}$ is CSP
 by (simp add: closure)
 thus ?thesis
 by (simp add: CSPMerge-def par-by-merge-seq-add)

qed

lemma *parallel-is-NCSP [closure]*:

assumes $ns1 \bowtie ns2$ P is NCSP Q is NCSP
 shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q)$ is NCSP

proof –

have $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = (\mathbf{R}_s(\text{pre}_R P \vdash \text{peri}_R P \diamond \text{post}_R P) \llbracket ns1 \parallel cs \parallel ns2 \rrbracket \mathbf{R}_s(\text{pre}_R Q \vdash \text{peri}_R Q \diamond \text{post}_R Q))$
 by (metis NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-design-alt assms wait'-cond-peri-post-cmt)
 also have ... is NCSP
 by (simp add: ParCSP-rdes-def assms closure unrest)
 finally show ?thesis .
qed

theorem *parallel-commutative*:

assumes $ns1 \bowtie ns2$

shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = (Q \llbracket ns2 \parallel cs \parallel ns1 \rrbracket P)$

proof –

have $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = P \parallel_{\text{swap}_m} ;; (M_C \text{ } ns2 \text{ } cs \text{ } ns1) \text{ } Q$

by (*simp add: CSPMerge-def seqr-assoc [THEN sym] swap-merge-rd swap-CSPInnerMerge lens-indep-sym assms*)

also have $\dots = Q \llbracket ns2 \parallel cs \parallel ns1 \rrbracket P$

by (*metis par-by-merge-commute-swap*)

finally show *?thesis* .

qed

CSP5 is precisely *C2* when applied to a process

lemma *CSP5-is-C2*:

fixes $P :: 'e \text{ process}$

assumes $P \text{ is NCSP}$

shows $CSP5(P) = C2(P)$

unfolding *CSP5-def C2-def* **by** (*rdes-eq cls: assms*)

The form of *C2* tells us that a normal CSP process has a downward closed set of refusals

lemma *C2-form*:

assumes $P \text{ is NCSP}$

shows $C2(P) = \mathbf{R}_s \text{ } (pre_R \text{ } P \vdash (\exists \text{ } ref_0 \cdot peri_R \text{ } P \llbracket \langle\langle ref_0 \rangle\rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle\langle ref_0 \rangle\rangle) \diamond post_R \text{ } P)$

proof –

have $1: \Phi(true, id, \langle \rangle) \text{ wr}[\Sigma | \{\} | \emptyset]_C \text{ } pre_R \text{ } P = pre_R \text{ } P \text{ (is ?lhs = ?rhs)}$

proof –

have $?lhs = (\neg_r (\exists (ref_0, st_0, tt_0) \cdot$

$[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger (\exists \$ref'; \$st' \cdot RR(\neg_r$

$pre_R \text{ } P)) \wedge$

$\$ref' \subseteq_u \langle\langle ref_0 \rangle\rangle \wedge [\langle\langle trace \rangle\rangle =_u \langle\langle tt_0 \rangle\rangle]_t \wedge$

$\$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \Sigma \oplus \langle\langle id \rangle\rangle(\$st)_a \text{ on } \emptyset ;; R1 \text{ true})$

by (*simp add: wrR-def par-by-merge-seq-remove rpred merge-csp-do-right ex-unrest Healthy-if pr-var-def closure assms unrest usubst*)

also have $\dots = (\neg_r (\exists \$ref'; \$st' \cdot RR(\neg_r pre_R \text{ } P)) ;; R1 \text{ true})$

by (*rel-auto*)

also have $\dots = (\neg_r (\neg_r pre_R \text{ } P) ;; R1 \text{ true})$

by (*simp add: Healthy-if closure ex-unrest unrest assms*)

also have $\dots = pre_R \text{ } P$

by (*simp add: NCSP-implies-NSRD NSRD-neg-pre-unit R1-preR assms rea-not-not*)

finally show *?thesis* .

qed

have $2: (pre_R \text{ } P \Rightarrow_r peri_R \text{ } P) \llbracket \Sigma | \{\} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle) =$

$(\exists ref_0 \cdot (peri_R \text{ } P) \llbracket \langle\langle ref_0 \rangle\rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle\langle ref_0 \rangle\rangle) \text{ (is ?lhs = ?rhs)}$

proof –

have $?lhs = peri_R \text{ } P \llbracket \Sigma | \{\} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle)$

by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have $\dots = (\exists \$st' \cdot (peri_R \text{ } P \parallel_{N_C} 1_L \{\} 0_L \Phi(true, id, \langle \rangle)))$

by (*simp add: CSPInterMerge-def par-by-merge-def seqr-exists-right*)

also have $\dots =$

$(\exists \$st' \cdot \exists (ref_0, st_0, tt_0) \cdot$

$[\$ref' \mapsto_s \langle\langle ref_0 \rangle\rangle, \$st' \mapsto_s \langle\langle st_0 \rangle\rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle\langle tt_0 \rangle\rangle] \dagger (\exists \$st' \cdot RR(peri_R \text{ } P)) \wedge$

$\$ref' \subseteq_u \langle\langle ref_0 \rangle\rangle \wedge [\langle\langle trace \rangle\rangle =_u \langle\langle tt_0 \rangle\rangle]_t \wedge \$st' =_u \$st \oplus \langle\langle st_0 \rangle\rangle \text{ on } \Sigma \oplus \langle\langle id \rangle\rangle(\$st)_a \text{ on } \emptyset)$

by (*simp add: merge-csp-do-right pr-var-def assms Healthy-if assms closure rpred unrest ex-unrest*)

also have $\dots =$

$(\exists ref_0 \cdot (\exists \$st' \cdot RR(peri_R \text{ } P)) \llbracket \langle\langle ref_0 \rangle\rangle / \$ref' \rrbracket \wedge \$ref' \subseteq_u \langle\langle ref_0 \rangle\rangle)$

by (*rel-auto*)
 also have ... = ?rhs
 by (*simp add: closure ex-unrest Healthy-if unrest assms*)
 finally show ?thesis .
 qed
 have 3: ($pre_R P \Rightarrow_r post_R P$) $\llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle) = post_R(P)$ (is ?lhs = ?rhs)
 proof –
 have ?lhs = $post_R P \llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle)$
 by (*simp add: SRD-post-under-pre closure assms unrest*)
 also have ... = $(\exists (st_0, t_0) \cdot$
 $[\$st' \mapsto_s \ll st_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger RR(post_R P) \wedge$
 $[\ll trace \gg =_u \ll t_0 \gg]_t \wedge \$st' =_u \$st \oplus \ll st_0 \gg \text{ on } \Sigma \oplus \ll id \gg (\$st)_a \text{ on } \emptyset)$
 by (*simp add: FinalMerge-csp-do-right pr-var-def assms closure unrest rpred Healthy-if*)
 also have ... = $RR(post_R(P))$
 by (*rel-auto*)
 finally show ?thesis
 by (*simp add: Healthy-if assms closure*)
 qed
 show ?thesis
 proof –
 have $C2(P) = \mathbf{R}_s (\Phi(true, id, \langle \rangle) \text{ wr } [\Sigma | \{ \} | \emptyset]_C pre_R P \vdash$
 $(pre_R P \Rightarrow_r peri_R P) \llbracket \Sigma | \{ \} | \emptyset \rrbracket^I \Phi(true, id, \langle \rangle) \diamond (pre_R P \Rightarrow_r post_R P) \llbracket \Sigma | \{ \} | \emptyset \rrbracket^F \Phi(true, id, \langle \rangle))$
 by (*simp add: C2-def, rdes-simp cls: assms, simp add: id-def pr-var-def*)
 also have ... = $\mathbf{R}_s (pre_R P \vdash (\exists ref_0 \cdot peri_R P \llbracket \ll ref_0 \gg / \$ref' \rrbracket \wedge \$ref' \subseteq_u \ll ref_0 \gg) \diamond post_R P)$
 by (*simp add: 1 2 3*)
 finally show ?thesis .
 qed
 qed

 lemma *C2-CDC-form*:
 assumes P is NCSP
 shows $C2(P) = \mathbf{R}_s (pre_R P \vdash CDC(peri_R P) \diamond post_R P)$
 by (*simp add: C2-form assms CDC-def*)

 lemma *C2-rdes-def*:
 assumes P_1 is CRC P_2 is CRR P_3 is CRR $\$st' \# P_2 \$ref' \# P_3$
 shows $C2(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3)) = \mathbf{R}_s(P_1 \vdash CDC(P_2) \diamond P_3)$
 by (*simp add: C2-form assms closure rdes unrest usubst, rel-auto*)

 lemma *C2-NCSP-intro*:
 assumes P is NCSP $peri_R(P)$ is CDC
 shows P is C2
 proof –
 have $C2(P) = \mathbf{R}_s (pre_R P \vdash CDC(peri_R P) \diamond post_R P)$
 by (*simp add: C2-CDC-form assms(1)*)
 also have ... = $\mathbf{R}_s (pre_R P \vdash peri_R P \diamond post_R P)$
 by (*simp add: Healthy-if assms*)
 also have ... = P
 by (*simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1)*)
 finally show ?thesis
 by (*simp add: Healthy-def*)
 qed
 qed

 lemma *C2-rdes-intro*:
 assumes P_1 is CRC P_2 is CRR P_2 is CDC P_3 is CRR $\$st' \# P_2 \$ref' \# P_3$

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3)$ is $C2$
 unfolding *Healthy-def*
 by (simp add: *C2-rdes-def* *assms unrest closure Healthy-if*)

lemma *C2-implies-CDC-peri* [closure]:

assumes P is NCSP P is $C2$
 shows $\text{peri}_R(P)$ is CDC

proof –

have $\text{peri}_R(P) = \text{peri}_R(\mathbf{R}_s(\text{pre}_R P \vdash \text{CDC}(\text{peri}_R P) \diamond \text{post}_R P))$
 by (metis *C2-CDC-form Healthy-if* *assms(1)* *assms(2)*)
 also have $\dots = \text{CDC}(\text{pre}_R P \Rightarrow_r \text{peri}_R P)$
 by (simp add: *rdes rpred* *assms closure unrest del: NSRD-peri-under-pre*)
 also have $\dots = \text{CDC}(\text{peri}_R P)$
 by (simp add: *SRD-peri-under-pre closure unrest* *assms*)
 finally show ?thesis
 by (simp add: *Healthy-def*)

qed

lemma *CACT-intro*:

assumes P is NCSP P is $C2$
 shows P is CACT
 by (metis *CACT-def Healthy-def* *assms(1)* *assms(2)*)

lemma *CACT-rdes-intro*:

assumes P_1 is CRC P_2 is CRR P_2 is CDC P_3 is CRR $\$st' \# P_2 \$ref' \# P_3$
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3)$ is CACT
 by (rule *CACT-intro*, simp add: *closure* *assms*, rule *C2-rdes-intro*, simp-all add: *assms*)

lemma *C2-NCSP-quasi-commute*:

assumes P is NCSP
 shows $C2(\text{NCSP}(P)) = \text{NCSP}(C2(P))$

proof –

have $1: C2(\text{NCSP}(P)) = C2(P)$
 by (simp add: *assms Healthy-if*)
 also have $\dots = \mathbf{R}_s(\text{pre}_R P \vdash \text{CDC}(\text{peri}_R P) \diamond \text{post}_R P)$
 by (simp add: *C2-CDC-form* *assms*)
 also have \dots is NCSP
 by (rule *NCSP-rdes-intro*, simp-all add: *closure* *assms unrest*)
 finally show ?thesis
 by (simp add: *Healthy-if 1*)

qed

lemma *C2-quasi-idem*:

assumes P is NCSP
 shows $C2(C2(P)) = C2(P)$

proof –

have $C2(C2(P)) = C2(C2(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))))$
 by (simp add: *NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-tri-design* *assms*)
 also have $\dots = \mathbf{R}_s(\text{pre}_R P \vdash \text{CDC}(\text{peri}_R P) \diamond \text{post}_R P)$
 by (simp add: *C2-rdes-def closure* *assms unrest CDC-idem*)
 also have $\dots = C2(P)$
 by (simp add: *C2-CDC-form* *assms*)
 finally show ?thesis .

qed

lemma *CACT-implies-NCSP* [closure]:

assumes *P* is *CACT*

shows *P* is *NCSP*

proof –

have $P = C2(NCSP(NCSP(P)))$

by (*metis CACT-def Healthy-Idempotent Healthy-if NCSP-Idempotent assms*)

also have $\dots = NCSP(C2(NCSP(P)))$

by (*simp add: C2-NCSP-quasi-commute Healthy-Idempotent NCSP-Idempotent*)

also have \dots is *NCSP*

by (*metis CACT-def Healthy-def assms calculation*)

finally show *?thesis* .

qed

lemma *CACT-implies-C2* [closure]:

assumes *P* is *CACT*

shows *P* is *C2*

by (*metis CACT-def CACT-implies-NCSP Healthy-def assms*)

lemma *CACT-idem*: $CACT(CACT(P)) = CACT(P)$

by (*simp add: CACT-def C2-NCSP-quasi-commute[THEN sym] C2-quasi-idem Healthy-Idempotent Healthy-if NCSP-Idempotent*)

lemma *CACT-Idempotent*: *Idempotent CACT*

by (*simp add: CACT-idem Idempotent-def*)

lemma *PACT-elim* [*RD-elim*]:

$\llbracket X \text{ is } CACT; P(\mathbf{R}_s(\text{pre}_R(X) \vdash \text{peri}_R(X) \diamond \text{post}_R(X))) \rrbracket \implies P(X)$

using *CACT-implies-NCSP NCSP-elim* **by** *blast*

lemma *Miracle-C2-closed* [closure]: *Miracle is C2*

by (*rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest*)

lemma *Chaos-C2-closed* [closure]: *Chaos is C2*

by (*rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest*)

lemma *Skip-C2-closed* [closure]: *Skip is C2*

by (*rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest*)

lemma *Stop-C2-closed* [closure]: *Stop is C2*

by (*rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest*)

lemma *Miracle-CACT-closed* [closure]: *Miracle is CACT*

by (*simp add: CACT-intro Miracle-C2-closed csp-theory.top-closed*)

lemma *Chaos-CACT-closed* [closure]: *Chaos is CACT*

by (*simp add: CACT-intro closure*)

lemma *Skip-CACT-closed* [closure]: *Skip is CACT*

by (*simp add: CACT-intro closure*)

lemma *Stop-CACT-closed* [closure]: *Stop is CACT*

by (*simp add: CACT-intro closure*)

lemma *seq-C2-closed* [closure]:

assumes *P* is *NCSP* *P* is *C2* *Q* is *NCSP* *Q* is *C2*

shows $P \;; \; Q$ is $C2$
by (*rdes-simp* *cls*: *assms*(1,3), *rule* $C2$ -*rdes-intro*, *simp-all* *add*: *closure* *assms* *unrest*)

lemma *seq-CACT-closed* [*closure*]:
assumes P is $CACT$ Q is $CACT$
shows $P \;; \; Q$ is $CACT$
by (*meson* $CACT$ -*implies-C2* $CACT$ -*implies-NCSP* $CACT$ -*intro* *assms* *csp-theory.Healthy-Sequence* *seq-C2-closed*)

lemma *AssignsCSP-C2* [*closure*]: $\langle \sigma \rangle_C$ is $C2$
by (*rdes-simp*, *rule* $C2$ -*rdes-intro*, *simp-all* *add*: *closure* *unrest*)

lemma *AssignsCSP-CACT* [*closure*]: $\langle \sigma \rangle_C$ is $CACT$
by (*simp* *add*: $CACT$ -*intro* *closure*)

lemma *map-st-ext-CDC-closed* [*closure*]:
assumes P is CDC
shows $P \oplus_r \text{map-st}_L[a]$ is CDC
proof –
have CDC $P \oplus_r \text{map-st}_L[a]$ is CDC
by (*rel-auto*)
thus *?thesis*
by (*simp* *add*: *assms* *Healthy-if*)
qed

lemma *rdes-frame-ext-C2-closed* [*closure*]:
assumes P is $NCSP$ P is $C2$
shows $a:[P]_R^+$ is $C2$
by (*rdes-simp* *cls*:*assms*(2), *rule* $C2$ -*rdes-intro*, *simp-all* *add*: *closure* *assms* *unrest*)

lemma *rdes-frame-ext-CACT-closed* [*closure*]:
assumes *vwb-lens* a P is $CACT$
shows $a:[P]_R^+$ is $CACT$
by (*rule* $CACT$ -*intro*, *simp-all* *add*: *closure* *assms*)

lemma *UINF-C2-closed* [*closure*]:
assumes $A \neq \{\}$ $\bigwedge i. i \in A \implies P(i)$ is $NCSP$ $\bigwedge i. i \in A \implies P(i)$ is $C2$
shows $(\bigcap i \in A \cdot P(i))$ is $C2$
proof –
have $(\bigcap i \in A \cdot P(i)) = (\bigcap i \in A \cdot \mathbf{R}_s(\text{pre}_R(P(i)) \vdash \text{peri}_R(P(i)) \diamond \text{post}_R(P(i))))$
by (*simp* *add*: *closure* *SRD-reactive-tri-design* *assms* *cong*: *UINF-cong*)
also have ... is $C2$
by (*rdes-simp* *cls*: *assms*, *rule* $C2$ -*rdes-intro*, *simp-all* *add*: *closure* *unrest* *assms*)
finally show *?thesis* .
qed

lemma *UINF-CACT-closed* [*closure*]:
assumes $A \neq \{\}$ $\bigwedge i. i \in A \implies P(i)$ is $CACT$
shows $(\bigcap i \in A \cdot P(i))$ is $CACT$
by (*rule* $CACT$ -*intro*, *simp-all* *add*: *assms* *closure*)

lemma *inf-C2-closed* [*closure*]:
assumes P is $NCSP$ Q is $NCSP$ P is $C2$ Q is $C2$
shows $P \sqcap Q$ is $C2$
by (*rdes-simp* *cls*: *assms*, *rule* $C2$ -*rdes-intro*, *simp-all* *add*: *closure* *unrest* *assms*)

lemma *cond-CDC-closed* [closure]:

assumes P is CDC Q is CDC

shows $P \triangleleft b \triangleright_R Q$ is CDC

proof –

have CDC $P \triangleleft b \triangleright_R CDC Q$ is CDC

by (*rel-auto*)

thus ?thesis

by (*simp add: Healthy-if assms*)

qed

lemma *cond-C2-closed* [closure]:

assumes P is NCSP Q is NCSP P is C2 Q is C2

shows $P \triangleleft b \triangleright_R Q$ is C2

by (*rdes-simp cls: assms, rule C2-rdes-intro, simp-all add: closure unrest assms*)

lemma *cond-CACT-closed* [closure]:

assumes P is CACT Q is CACT

shows $P \triangleleft b \triangleright_R Q$ is CACT

by (*rule CACT-intro, simp-all add: assms closure*)

lemma *gcomm-C2-closed* [closure]:

assumes P is NCSP P is C2

shows $b \rightarrow_R P$ is C2

by (*rdes-simp cls: assms, rule C2-rdes-intro, simp-all add: closure unrest assms*)

lemma *AssumeCircus-CACT* [closure]: $[b]_C$ is CACT

by (*metis AssumeCircus-NCSP AssumeCircus-def CACT-intro NCSP-Skip Skip-C2-closed gcomm-C2-closed*)

lemma *StateInvR-CACT* [closure]: $\text{inv}_R(b)$ is CACT

by (*simp add: CACT-rdes-intro rdes-def closure unrest*)

lemma *AlternateR-C2-closed* [closure]:

assumes

$\bigwedge i. i \in A \implies P(i)$ is NCSP Q is NCSP

$\bigwedge i. i \in A \implies P(i)$ is C2 Q is C2

shows $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi})$ is C2

proof (*cases* $A = \{\}$)

case *True*

then show ?thesis

by (*simp add: assms(4)*)

next

case *False*

then show ?thesis

by (*simp add: AlternateR-def closure assms*)

qed

lemma *AlternateR-CACT-closed* [closure]:

assumes $\bigwedge i. i \in A \implies P(i)$ is CACT Q is CACT

shows $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi})$ is CACT

by (*rule CACT-intro, simp-all add: closure assms*)

lemma *AlternateR-list-C2-closed* [closure]:

assumes

$\bigwedge b P. (b, P) \in \text{set } A \implies P$ is NCSP Q is NCSP

$\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is } C2 \text{ } Q \text{ is } C2$
shows (*AlternateR-list A Q*) *is C2*
apply (*simp add: AlternateR-list-def*)
apply (*rule AlternateR-C2-closed*)
apply (*auto simp add: assms closure*)
apply (*metis assms nth-mem prod.collapse*) +
done

lemma *AlternateR-list-CACT-closed* [closure]:
assumes $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is } CACT \text{ } Q \text{ is } CACT$
shows (*AlternateR-list A Q*) *is CACT*
by (*rule CACT-intro, simp-all add: closure assms*)

lemma *R4-CRR-closed* [closure]: $P \text{ is } CRR \implies R4(P) \text{ is } CRR$
by (*rule CRR-intro, simp-all add: closure unrest R4-def*)

lemma *WhileC-C2-closed* [closure]:
assumes $P \text{ is } NCSP \text{ } P \text{ is } Productive \text{ } P \text{ is } C2$
shows *while_C b do P od is C2*

proof –

have *while_C b do P od = while_C b do Productive(**R_s** (*pre_R P* \vdash *peri_R P* \diamond *post_R P*)) od*
by (*simp add: assms Healthy-if SRD-reactive-tri-design closure*)
also have *... = while_C b do **R_s** (*pre_R P* \vdash *peri_R P* \diamond *R4(post_R P)*) od*
by (*simp add: Productive-RHS-design-form unrest assms rdes closure R4-def*)
also have *... is C2*
by (*simp add: closure assms unrest rdes-def C2-rdes-intro*)
finally show *?thesis* .

qed

lemma *WhileC-CACT-closed* [closure]:
assumes $P \text{ is } CACT \text{ } P \text{ is } Productive$
shows *while_C b do P od is CACT*
using *CACT-implies-C2 CACT-implies-NCSP CACT-intro WhileC-C2-closed WhileC-NCSP-closed*
assms by blast

lemma *IterateC-C2-closed* [closure]:
assumes
 $\bigwedge i. i \in A \implies P(i) \text{ is } NCSP \bigwedge i. i \in A \implies P(i) \text{ is } Productive \bigwedge i. i \in A \implies P(i) \text{ is } C2$
shows (*do_C i \in A \cdot g(i) \rightarrow P(i) od*) *is C2*
unfolding *IterateC-def* **by** (*simp add: closure assms*)

lemma *IterateC-CACT-closed* [closure]:
assumes
 $\bigwedge i. i \in A \implies P(i) \text{ is } CACT \bigwedge i. i \in A \implies P(i) \text{ is } Productive$
shows (*do_C i \in A \cdot g(i) \rightarrow P(i) od*) *is CACT*
by (*metis CACT-implies-C2 CACT-implies-NCSP CACT-intro IterateC-C2-closed IterateC-NCSP-closed*
assms)

lemma *IterateC-list-C2-closed* [closure]:
assumes
 $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is } NCSP$
 $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is } Productive$
 $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is } C2$
shows (*IterateC-list A*) *is C2*
unfolding *IterateC-list-def*

by (rule IterateC-C2-closed, (metis assms atLeastLessThan-iff nth-map nth-mem prod.collapse)+)

lemma *IterateC-list-CACT-closed* [closure]:
assumes
 $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is CACT}$
 $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is Productive}$
shows *(IterateC-list A) is CACT*
by (metis CACT-implies-C2 CACT-implies-NCSP CACT-intro IterateC-list-C2-closed IterateC-list-NCSP-closed assms)

lemma *GuardCSP-C2-closed* [closure]:
assumes *P is NCSP P is C2*
shows $g \&_u P \text{ is C2}$
by (rdes-simp cls: assms(1), rule C2-rdes-intro, simp-all add: closure assms unrest)

lemma *GuardCSP-CACT-closed* [closure]:
assumes *P is CACT*
shows $g \&_u P \text{ is CACT}$
by (rule CACT-intro, simp-all add: closure assms)

lemma *DoCSP-C2* [closure]:
 $do_C(a) \text{ is C2}$
by (rdes-simp, rule C2-rdes-intro, simp-all add: closure unrest)

lemma *DoCSP-CACT* [closure]:
 $do_C(a) \text{ is CACT}$
by (rule CACT-intro, simp-all add: closure)

lemma *PrefixCSP-C2-closed* [closure]:
assumes *P is NCSP P is C2*
shows $a \rightarrow_C P \text{ is C2}$
unfolding *PrefixCSP-def* **by** (metis DoCSP-C2 Healthy-def NCSP-DoCSP NCSP-implies-CSP assms seq-C2-closed)

lemma *PrefixCSP-CACT-closed* [closure]:
assumes *P is CACT*
shows $a \rightarrow_C P \text{ is CACT}$
using *CACT-implies-C2 CACT-implies-NCSP CACT-intro NCSP-PrefixCSP PrefixCSP-C2-closed*
assms **by** blast

lemma *ExtChoice-C2-closed* [closure]:
assumes $\bigwedge i. i \in I \implies P(i) \text{ is NCSP} \bigwedge i. i \in I \implies P(i) \text{ is C2}$
shows $(\square i \in I \cdot P(i)) \text{ is C2}$
proof (cases $I = \{\}$)
case *True*
then show *?thesis* **by** (simp add: closure ExtChoice-empty)
next
case *False*
show *?thesis*
by (rule C2-NCSP-intro, simp-all add: assms closure unrest periR-ExtChoice-ind' False)
qed

lemma *ExtChoice-CACT-closed* [closure]:
assumes $\bigwedge i. i \in I \implies P(i) \text{ is CACT}$
shows $(\square i \in I \cdot P(i)) \text{ is CACT}$

by (rule *CACT-intro*, simp-all add: closure assms)

lemma *extChoice-C2-closed* [closure]:
 assumes P is NCSP P is C2 Q is NCSP Q is C2
 shows $P \sqcap Q$ is C2
proof –
 have $P \sqcap Q = (\sqcap I \in \{P, Q\} \cdot I)$
 by (simp add: *extChoice-def*)
 also have ... is C2
 by (rule *ExtChoice-C2-closed*, auto simp add: assms)
 finally show ?thesis .
qed

lemma *extChoice-CACT-closed* [closure]:
 assumes P is CACT Q is CACT
 shows $P \sqcap Q$ is CACT
 by (rule *CACT-intro*, simp-all add: closure assms)

lemma *state-srea-C2-closed* [closure]:
 assumes P is NCSP P is C2
 shows state ' a · P is C2
 by (rule *C2-NCSP-intro*, simp-all add: closure rdes assms)

lemma *state-srea-CACT-closed* [closure]:
 assumes P is CACT
 shows state ' a · P is CACT
 by (rule *CACT-intro*, simp-all add: closure assms)

lemma *parallel-C2-closed* [closure]:
 assumes $ns1 \bowtie ns2$ P is NCSP Q is NCSP P is C2 Q is C2
 shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q)$ is C2
proof –
 have $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q) = (\mathbf{R}_s(\text{pre}_R P \vdash \text{peri}_R P \diamond \text{post}_R P) \llbracket ns1 \parallel cs \parallel ns2 \rrbracket \mathbf{R}_s(\text{pre}_R Q \vdash \text{peri}_R Q \diamond \text{post}_R Q))$
 by (metis *NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-design-alt* assms wait'-cond-peri-post-cmt)
 also have ... is C2
 by (simp add: *ParCSP-rdes-def C2-rdes-intro* assms closure unrest)
 finally show ?thesis .
qed

lemma *parallel-CACT-closed* [closure]:
 assumes $ns1 \bowtie ns2$ P is CACT Q is CACT
 shows $(P \llbracket ns1 \parallel cs \parallel ns2 \rrbracket Q)$ is CACT
 by (meson *CACT-implies-C2 CACT-implies-NCSP CACT-intro* assms *parallel-C2-closed parallel-is-NCSP*)

lemma *RenameCSP-C2-closed* [closure]:
 assumes P is NCSP P is C2
 shows $P \llbracket f \rrbracket_C$ is C2
 by (simp add: *RenameCSP-def C2-rdes-intro RenameCSP-pre-CRC-closed* closure assms unrest)

lemma *RenameCSP-CACT-closed* [closure]:
 assumes P is CACT
 shows $P \llbracket f \rrbracket_C$ is CACT
 by (rule *CACT-intro*, simp-all add: closure assms)

This property depends on downward closure of the refusals

lemma *rename-extChoice-pre*:

assumes *inj f P is NCSP Q is NCSP P is C2 Q is C2*
shows $(P \sqcap Q) \llbracket f \rrbracket_C = (P \llbracket f \rrbracket_C \sqcap Q \llbracket f \rrbracket_C)$
by (*rdes-eq-split cls: assms*)

lemma *rename-extChoice*:

assumes *inj f P is CACT Q is CACT*
shows $(P \sqcap Q) \llbracket f \rrbracket_C = (P \llbracket f \rrbracket_C \sqcap Q \llbracket f \rrbracket_C)$
by (*simp add: CACT-implies-C2 CACT-implies-NCSP assms rename-extChoice-pre*)

lemma *interleave-commute*:

$P \parallel Q = Q \parallel P$
using *parallel-commutative zero-lens-indep* **by** *blast*

lemma *interleave-unit*:

assumes *P is CPROC*
shows $P \parallel \text{Skip} = P$
by (*metis CACT-implies-C2 CACT-implies-NCSP CSP5-def CSP5-is-C2 Healthy-if assms*)

lemma *parallel-miracle*:

$P \text{ is NCSP} \implies \text{Miracle } \llbracket ns1 \parallel cs \parallel ns2 \rrbracket P = \text{Miracle}$
by (*simp add: CSPMerge-def par-by-merge-seq-add[THEN sym] Miracle-parallel-left-zero Skip-right-unit closure*)

lemma

assumes *vwb-lens ns1 vwb-lens ns2 ns1 \bowtie ns2 P is RR*
shows $P \text{ wr}[ns1|cs|ns2]_C \text{ false} = \text{undefined}$ (**is** *?lhs = ?rhs*)

proof –

have *?lhs* = $(\neg_r (\exists (ref_0, ref_1, st_0, st_1, tt_0, tt_1) \cdot$
 $[\$ref' \mapsto_s \langle ref_0 \rangle, \$st' \mapsto_s \langle st_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_0 \rangle] \dagger R1 \text{ true} \wedge$
 $[\$ref' \mapsto_s \langle ref_1 \rangle, \$st' \mapsto_s \langle st_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger P \wedge$
 $\$ref' \subseteq_u (\langle ref_0 \rangle \cup_u \langle ref_1 \rangle) \cap_u \langle cs \rangle \cup_u (\langle ref_0 \rangle \cap_u \langle ref_1 \rangle - \langle cs \rangle) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle \wedge$
 $\$st' =_u \$st \oplus \langle st_0 \rangle \text{ on } \&ns1 \oplus \langle st_1 \rangle \text{ on } \&ns2) ;;$
 $R1 \text{ true})$

by (*simp add: wrR-def par-by-merge-seq-remove CSPInnerMerge-form assms closure usubst unrest*)

also have ... = $(\neg_r (\exists (tt_0, tt_1) \cdot$

$[\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger P \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle) ;;$
 $R1 \text{ true})$

by (*rel-blast*)

also have ... = $(\neg_r (\exists (tt_0, tt_1) \cdot$

$[\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tt_1 \rangle] \dagger RR(P) \wedge$
 $\$tr \leq_u \$tr' \wedge$
 $\&tt \in_u \langle tt_0 \rangle \star_{cs} \langle tt_1 \rangle \wedge \langle tt_0 \rangle \upharpoonright_u \langle cs \rangle =_u \langle tt_1 \rangle \upharpoonright_u \langle cs \rangle) ;;$
 $R1 \text{ true})$

by (*simp add: Healthy-if assms*)

oops

end

5 Hiding

theory *utp-circus-hiding*
 imports *utp-circus-parallel*
 begin

5.1 Hiding in peri- and postconditions

definition *hide-rea* (*hide_r*) **where**

[*upred-defs*]: $\text{hide}_r P E = (\exists s \cdot (P[\text{tr}^u \ll s \gg, (\ll E \gg \cup_u \text{ref}') / \text{tr}', \text{ref}'] \wedge \text{tr}' =_u \text{tr}^u (\ll s \gg \downarrow_u \ll -E \gg)))$

lemma *hide-rea-CRR-closed* [*closure*]:

assumes *P is CRR*

shows *hide_r P E is CRR*

proof –

have $\text{CRR}(\text{hide}_r (\text{CRR } P) E) = \text{hide}_r (\text{CRR } P) E$

by (*rel-auto*, *fastforce* +)

thus ?thesis

by (*metis Healthy-def' assms*)

qed

lemma *hide-rea-CDC* [*closure*]:

assumes *P is CDC*

shows *hide_r P E is CDC*

proof –

have $\text{CDC}(\text{hide}_r (\text{CDC } P) E) = \text{hide}_r (\text{CDC } P) E$

by (*rel-blast*)

thus ?thesis

by (*simp add: Healthy-if Healthy-intro assms*)

qed

lemma *hide-rea-false* [*rpred*]: *hide_r false E = false*

by (*rel-auto*)

lemma *hide-rea-disj* [*rpred*]: *hide_r (P ∨ Q) E = (hide_r P E ∨ hide_r Q E)*

by (*rel-auto*)

lemma *hide-rea-csp-enable* [*rpred*]:

$\text{hide}_r \mathcal{E}(s, t, E) F = \mathcal{E}(s \wedge E - \ll F \gg =_u E, t \downarrow_u \ll -F \gg, E)$

by (*rel-auto*)

lemma *hide-rea-csp-do* [*rpred*]: *hide_r Φ(s,σ,t) E = Φ(s,σ,t ↓_u <<-E>>)*

by (*rel-auto*)

lemma *filter-eval* [*simp*]:

$(\text{bop } \text{Cons } x \text{ } xs) \downarrow_u E = (\text{bop } \text{Cons } x \text{ } (xs \downarrow_u E) \triangleleft x \in_u E \triangleright xs \downarrow_u E)$

by (*rel-simp*)

lemma *hide-rea-seq* [*rpred*]:

assumes *P is CRR* *\$ref' # P Q is CRR*

shows *hide_r (P ;; Q) E = hide_r P E ;; hide_r Q E*

proof –

have $\text{hide}_r (\text{CRR}(\exists \text{ref}' \cdot P) ;; \text{CRR}(Q)) E = \text{hide}_r (\text{CRR}(\exists \text{ref}' \cdot P)) E ;; \text{hide}_r (\text{CRR } Q) E$

apply (*simp add: hide-rea-def usubst unrest CRR-seqr-form*)

apply (*simp add: CRR-form*)

apply (*rel-auto*)

```

    using seq-filter-append apply fastforce
    apply (metis seq-filter-append)
  done
thus ?thesis
  by (simp add: Healthy-if assms ex-unrest)
qed

```

```

lemma hide-rea-true-R1-true [rpred]:
  hider (R1 true) A ;; R1 true = R1 true
  by (rel-auto, metis append-Nil2 seq-filter-Nil)

```

```

lemma hide-rea-empty [rpred]:
  assumes P is RR
  shows hider P {} = P
proof -
  have hider (RR P) {} = (RR P)
    by (rel-auto; force)
  thus ?thesis
    by (simp add: Healthy-if assms)
qed

```

```

lemma hide-rea-twice [rpred]: hider (hider P A) B = hider P (A ∪ B)
  apply (rel-auto)
  apply (metis (no-types, hide-lams) semilattice-sup-class.sup-assoc)
  apply (metis (no-types, lifting) semilattice-sup-class.sup-assoc seq-filter-twice)
  done

```

```

lemma st'-unrest-hide-rea [unrest]: $st' # P ⇒ $st' # hider P E
  by (simp add: hide-rea-def unrest)

```

```

lemma ref'-unrest-hide-rea [unrest]: $ref' # P ⇒ $ref' # hider P E
  by (simp add: hide-rea-def unrest usubst)

```

5.2 Hiding in preconditions

definition *abs-rea* :: ('s, 'e) action ⇒ 'e set ⇒ ('s, 'e) action (*abs_r*) **where**
[upred-defs]: *abs_r* P E = (¬_r (hide_r (¬_r P) E ;; true_r))

```

lemma abs-rea-false [rpred]: absr false E = false
  by (rel-simp, metis append.right-neutral seq-filter-Nil)

```

```

lemma abs-rea-conj [rpred]: absr (P ∧ Q) E = (absr P E ∧ absr Q E)
  by (rel-blast)

```

```

lemma abs-rea-true [rpred]: absr truer E = truer
  by (rel-auto)

```

```

lemma abs-rea-RC-closed [closure]:
  assumes P is CRR
  shows absr P E is CRC
proof -
  have RC1 (absr (CRR P) E) = absr (CRR P) E
    apply (rel-auto)
    apply (metis order-refl)
    apply blast
  done

```

hence $abs_r P E$ is $RC1$
 by (simp add: assms Healthy-if Healthy-intro closure)
 thus ?thesis
 by (rule-tac CRC-intro'', simp-all add: abs-rea-def closure assms unrest)
 qed

lemma *hide-rea-impl-under-abs*:
 assumes P is CRC Q is CRR
 shows $(abs_r P A \Rightarrow_r hide_r (P \Rightarrow_r Q) A) = (abs_r P A \Rightarrow_r hide_r Q A)$
 by (simp add: RC1-def abs-rea-def rea-impl-def rpred closure assms unrest)
 (rel-auto, metis order-refl)

lemma *abs-rea-not-CRR*: P is $CRR \implies abs_r (\neg_r P) E = (\neg_r hide_r P E ;; R1 \text{ true})$
 by (simp add: abs-rea-def rpred closure)

lemma *abs-rea-wpR [rpred]*:
 assumes P is CRR $\$ref' \# P Q$ is CRC
 shows $abs_r (P wp_r Q) A = (hide_r P A) wp_r (abs_r Q A)$
 by (simp add: wp-rea-def abs-rea-not-CRR hide-rea-seq assms closure)
 (simp add: abs-rea-def rpred closure assms seqr-assoc)

lemma *abs-rea-empty [rpred]*:
 assumes P is RC
 shows $abs_r P \{\} = P$
proof –
 have $abs_r (RC P) \{\} = (RC P)$
 apply (rel-auto)
 apply (metis diff-add-cancel-left' order-refl plus-list-def)
 using dual-order.trans apply blast
 done
 thus ?thesis
 by (simp add: Healthy-if assms)
 qed

lemma *abs-rea-twice [rpred]*:
 assumes P is CRC
 shows $abs_r (abs_r P A) B = abs_r P (A \cup B)$ (is ?lhs = ?rhs)
proof –
 have ?lhs = $abs_r (\neg_r hide_r (\neg_r P) A ;; R1 \text{ true}) B$
 by (simp add: abs-rea-def)
 thus ?thesis
 by (simp add: abs-rea-def rpred closure unrest seqr-assoc assms)
 qed

5.3 Hiding Operator

In common with the UTP book definition of hiding, this definition does not introduce divergence if there is an infinite sequence of events that are hidden. For this, we would need a more complex precondition which is left for future work.

definition *HideCSP* :: $('s, 'e) \text{ action} \Rightarrow 'e \text{ set} \Rightarrow ('s, 'e) \text{ action} (\text{infixl } \setminus_C \ 80)$ **where**
 [upred-defs]:
 $HideCSP P E = \mathbf{R}_s(abs_r(pre_R(P)) E \vdash hide_r(peri_R(P)) E \diamond hide_r(post_R(P)) E)$

lemma *HideCSP-rdes-def [rdes-def]*:

assumes P is CRC Q is CRR R is CRR
shows $\mathbf{R}_s(P \vdash Q \diamond R) \setminus_C A = \mathbf{R}_s(\text{abs}_r(P) A \vdash \text{hide}_r Q A \diamond \text{hide}_r R A)$ (**is** $?lhs = ?rhs$)
proof –
have $?lhs = \mathbf{R}_s(\text{abs}_r P A \vdash \text{hide}_r (P \Rightarrow_r Q) A \diamond \text{hide}_r (P \Rightarrow_r R) A)$
by (*simp add: HideCSP-def rdes assms closure*)
also have $\dots = \mathbf{R}_s(\text{abs}_r P A \vdash (\text{abs}_r P A \Rightarrow_r \text{hide}_r (P \Rightarrow_r Q) A) \diamond (\text{abs}_r P A \Rightarrow_r \text{hide}_r (P \Rightarrow_r R) A))$
by (*metis RHS-tri-design-conj conj-idem utp-pred-laws.sup.idem*)
also have $\dots = ?rhs$
by (*metis RHS-tri-design-conj assms conj-idem hide-rea-impl-under-abs utp-pred-laws.sup.idem*)
finally show $?thesis$.
qed

lemma *HideCSP-NCSP-closed* [*closure*]: P is NCSP $\implies P \setminus_C E$ is NCSP
by (*simp add: HideCSP-def closure unrest*)

lemma *HideCSP-C2-closed* [*closure*]:
assumes P is NCSP P is C2
shows $P \setminus_C E$ is C2
by (*rdes-simp cls: assms, simp add: C2-rdes-intro closure unrest assms*)

lemma *HideCSP-CACT-closed* [*closure*]:
assumes P is CACT
shows $P \setminus_C E$ is CACT
by (*rule CACT-intro, simp-all add: closure assms*)

lemma *HideCSP-Chaos*: $\text{Chaos} \setminus_C E = \text{Chaos}$
by (*rdes-simp*)

lemma *HideCSP-Miracle*: $\text{Miracle} \setminus_C A = \text{Miracle}$
by (*rdes-eq*)

lemma *HideCSP-AssignsCSP*:
 $\langle \sigma \rangle_C \setminus_C A = \langle \sigma \rangle_C$
by (*rdes-eq*)

lemma *HideCSP-cond*:
assumes P is NCSP Q is NCSP
shows $(P \triangleleft b \triangleright_R Q) \setminus_C A = (P \setminus_C A \triangleleft b \triangleright_R Q \setminus_C A)$
by (*rdes-eq cls: assms*)

lemma *HideCSP-int-choice*:
assumes P is NCSP Q is NCSP
shows $(P \sqcap Q) \setminus_C A = (P \setminus_C A \sqcap Q \setminus_C A)$
by (*rdes-eq cls: assms*)

lemma *HideCSP-guard*:
assumes P is NCSP
shows $(b \&_u P) \setminus_C A = b \&_u (P \setminus_C A)$
by (*rdes-eq cls: assms*)

lemma *HideCSP-seq*:
assumes P is NCSP Q is NCSP
shows $(P ;; Q) \setminus_C A = (P \setminus_C A ;; Q \setminus_C A)$
by (*rdes-eq-split cls: assms*)

```

lemma HideCSP-DoCSP [rdes-def]:
   $do_C(a) \setminus_C A = (Skip \triangleleft (a \in_u \ll A \gg) \triangleright_R do_C(a))$ 
  by (rdes-eq)

lemma HideCSP-PrefixCSP:
  assumes P is NCSP
  shows  $(a \rightarrow_C P) \setminus_C A = ((P \setminus_C A) \triangleleft (a \in_u \ll A \gg) \triangleright_R (a \rightarrow_C (P \setminus_C A)))$ 
  apply (simp add: PrefixCSP-def Healthy-if HideCSP-seq HideCSP-DoCSP closure assms rdes rpred)
  apply (simp add: HideCSP-NCSP-closed Skip-left-unit assms cond-st-distr)
  done

lemma HideCSP-empty:
  assumes P is NCSP
  shows  $P \setminus_C \{\} = P$ 
  by (rdes-eq cls: assms)

lemma HideCSP-twice:
  assumes P is NCSP
  shows  $P \setminus_C A \setminus_C B = P \setminus_C (A \cup B)$ 
  by (rdes-simp cls: assms)

lemma HideCSP-Skip:  $Skip \setminus_C A = Skip$ 
  by (rdes-eq)

lemma HideCSP-Stop:  $Stop \setminus_C A = Stop$ 
  by (rdes-eq)

end

```

6 Meta theory for Circus

```

theory utp-circus
imports
  utp-circus-traces
  utp-circus-parallel
  utp-circus-hiding
begin end

```

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