Isabelle/UTP: Mechanised reasoning for the UTP

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no-notation *inner* (**infix** · *70*)

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no-notation
Set.member (*op* :) **and**
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declare *fst-vwb-lens* [*simp*]
declare *snd-vwb-lens* [*simp*]
declare *lens-indep-left-comp* [*simp*]
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This theory describes the foundational structure of UTP variables, upon which the rest of our model rests. We start by defining alphabets, which following [3, 4] in this shallow model are simply represented as types, though by convention usually a record type where each field corresponds to a variable.

type-synonym $'\alpha$ *alphabet* $= '\alpha$

UTP variables carry two type parameters, $'a$ that corresponds to the variable's type and $'\alpha$ that corresponds to alphabet of which the variable is a type. There is a thus a strong link between alphabets and variables in this model. Variable are characterized by two functions, *var-lookup* and *var-update*, that respectively lookup and update the variable's value in some alphabetised state space. These functions can readily be extracted from an Isabelle record type.

type-synonym $('a, '\alpha)$ *uvar* $=$ $('a, '\alpha)$ *lens*

The VAR function [3] is a syntactic translations that allows to retrieve a variable given its name, assuming the variable is a field in a record.

syntax *-VAR* :: *id* $\Rightarrow$ $('a, 'r)$ *uvar* (*VAR* -)
translations *VAR x* => *FLDLENS x*

We also define some lifting functions for variables to create input and output variables. These simply lift the alphabet to a tuple type since relations will ultimately be defined to a tuple alphabet.

definition *in-var* :: $('a, '\alpha)$ *uvar* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uvar* **where**
[*lens-defs*]: *in-var* $x = x\ ;_L\ fst_L$

definition *out-var* :: $('a, '\beta)$ *uvar* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uvar* **where**
[*lens-defs*]: *out-var* $x = x\ ;_L\ snd_L$

definition *pr-var* :: $('a, '\beta)$ *uvar* $\Rightarrow$ $('a, '\beta)$ *uvar* **where**
[*simp*]: *pr-var* $x = x$

lemma *in-var-semi-uvar* [*simp*]:
  mwb-lens $x \Longrightarrow$ *mwb-lens* (*in-var* x)
  by (*simp add*: *comp-mwb-lens fst-vwb-lens in-var-def*)

lemma *in-var-uvar* [*simp*]:
  vwb-lens $x \Longrightarrow$ *vwb-lens* (*in-var* x)
  by (*simp add*: *comp-vwb-lens fst-vwb-lens in-var-def*)

lemma *out-var-semi-uvar* [*simp*]:
  mwb-lens $x \Longrightarrow$ *mwb-lens* (*out-var* x)
  by (*simp add*: *comp-mwb-lens out-var-def snd-vwb-lens*)

lemma *out-var-uvar* [*simp*]:
  vwb-lens $x \Longrightarrow$ *vwb-lens* (*out-var* x)
  by (*simp add*: *comp-vwb-lens out-var-def snd-vwb-lens*)

lemma *in-out-indep* [*simp*]:
  in-var $x \bowtie$ *out-var* y
  by (*simp add*: *lens-indep-def in-var-def out-var-def fst-lens-def snd-lens-def lens-comp-def*)

lemma *out-in-indep* [*simp*]:

$out\text{-}var\ x \bowtie in\text{-}var\ y$
by (*simp add*: *lens-indep-def in-var-def out-var-def fst-lens-def snd-lens-def lens-comp-def*)

lemma *in-var-indep* [*simp*]:
$x \bowtie y \Longrightarrow in\text{-}var\ x \bowtie in\text{-}var\ y$
by (*simp add*: *in-var-def out-var-def fst-vwb-lens lens-indep-left-comp*)

lemma *out-var-indep* [*simp*]:
$x \bowtie y \Longrightarrow out\text{-}var\ x \bowtie out\text{-}var\ y$
by (*simp add*: *lens-indep-left-comp out-var-def snd-vwb-lens*)

We also define some lookup abstraction simplifications.

lemma *var-lookup-in* [*simp*]: $lens\text{-}get\ (in\text{-}var\ x)\ (A,\ A') = lens\text{-}get\ x\ A$
by (*simp add*: *in-var-def fst-lens-def lens-comp-def*)

lemma *var-lookup-out* [*simp*]: $lens\text{-}get\ (out\text{-}var\ x)\ (A,\ A') = lens\text{-}get\ x\ A'$
by (*simp add*: *out-var-def snd-lens-def lens-comp-def*)

lemma *var-update-in* [*simp*]: $lens\text{-}put\ (in\text{-}var\ x)\ (A,\ A')\ v = (lens\text{-}put\ x\ A\ v,\ A')$
by (*simp add*: *in-var-def fst-lens-def lens-comp-def*)

lemma *var-update-out* [*simp*]: $lens\text{-}put\ (out\text{-}var\ x)\ (A,\ A')\ v = (A,\ lens\text{-}put\ x\ A'\ v)$
by (*simp add*: *out-var-def snd-lens-def lens-comp-def*)

Variables can also be used to effectively define sets of variables. Here we define the the universal alphabet (Σ) to be a variable with identity for both the lookup and update functions. Effectively this is just a function directly on the alphabet type.

abbreviation (*input*) *univ-alpha* :: $('\alpha,\ '\alpha)\ uvar\ (\Sigma)$ **where**
$univ\text{-}alpha \equiv 1_L$

nonterminal *svid* **and** *svar* **and** *salpha*

syntax
-*salphaid* :: *id* ⇒ *salpha* (- [*998*] *998*)
-*salphavar* :: *svar* ⇒ *salpha* (- [*998*] *998*)

-*salphacomp* :: *salpha* ⇒ *salpha* ⇒ *salpha* (**infixr** ; *75*)
-*svid* :: *id* ⇒ *svid* (- [*999*] *999*)
-*svid-alpha* :: *svid* (Σ)
-*svid-empty* :: *svid* (∅)
-*svid-dot* :: *svid* ⇒ *svid* ⇒ *svid* (-:- [*999*,*998*] *999*)
-*spvar* :: *svid* ⇒ *svar* (&- [*998*] *998*)
-*sinvar* :: *svid* ⇒ *svar* ($- [*998*] *998*)
-*soutvar* :: *svid* ⇒ *svar* ($-´ [*998*] *998*)

consts
$svar :: 'v \Rightarrow 'e$
$ivar :: 'v \Rightarrow 'e$
$ovar :: 'v \Rightarrow 'e$

adhoc-overloading
svar pr-var **and** *ivar in-var* **and** *ovar out-var*

translations
-*salphaid* x => x

-salphacomp x y => x $+_L$ *y*
-salphavar x => x
-svid-alpha == Σ
-svid-empty == 0_L
-svid-dot x y => y $;_L$ *x*
-svid x => x
-sinvar (-svid-dot x y) <= CONST ivar (CONST lens-comp y x)
-soutvar (-svid-dot x y) <= CONST ovar (CONST lens-comp y x)
-spvar x == CONST svar x
-sinvar x == CONST ivar x
-soutvar x == CONST ovar x

Syntactic function to construct a uvar type given a return type

syntax
-uvar-ty :: *type* ⇒ *type* ⇒ *type*

parse-translation ⟪
let
fun uvar-ty-tr [ty] = Syntax.const @{type-syntax uvar} $ ty $ Syntax.const @{type-syntax dummy}
| *uvar-ty-tr ts = raise TERM (uvar-ty-tr, ts);*
in [(@{syntax-const -uvar-ty}, K uvar-ty-tr)] end
⟫

end

1.1 Deep UTP variables

theory *utp-dvar*
imports *utp-var*
begin

UTP variables represented by record fields are shallow, nameless entities. They are fundamentally static in nature, since a new record field can only be introduced definitionally and cannot be otherwise arbitrarily created. They are nevertheless very useful as proof automation is excellent, and they can fully make use of the Isabelle type system. However, for constructs like alphabet extension that can introduce new variables they are inadequate. As a result we also introduce a notion of deep variables to complement them. A deep variable is not a record field, but rather a key within a store map that records the values of all deep variables. As such the Isabelle type system is agnostic of them, and the creation of a new deep variable does not change the portion of the alphabet specified by the type system.

In order to create a type of stores (or bindings) for variables, we must fix a universe for the variable valuations. This is the major downside of deep variables – they cannot have any type, but only a type whose cardinality is up to $\mathfrak{c}$, the cardinality of the continuum. This is why we need both deep and shallow variables, as the latter are unrestricted in this respect. Each deep variable will therefore specify the cardinality of the type it possesses.

1.2 Cardinalities

We first fix a datatype representing all possible cardinalities for a deep variable. These include finite cardinalities, $\aleph_0$ (countable), and $\mathfrak{c}$ (uncountable up to the continuum).

datatype *ucard = fin nat* | *aleph0* ($\aleph_0$) | *cont* (c)

Our universe is simply the set of natural numbers; this is sufficient for all types up to cardinality $\mathfrak{c}$.

type-synonym *uuniv* = *nat set*

We introduce a function that gives the set of values within our universe of the given cardinality. Since a cardinality of 0 is no proper type, we use finite cardinality 0 to mean cardinality 1, 1 to mean 2 etc.

fun *uuniv* :: *ucard* ⇒ *uuniv set* ($\mathcal{U}$'(-')) **where**
$\mathcal{U}(\mathit{fin}\ n) = \{\{x\} \mid x.\ x \leq n\}$ |
$\mathcal{U}(\aleph_0) = \{\{x\} \mid x.\ \mathit{True}\}$ |
$\mathcal{U}(\mathrm{c}) = \mathit{UNIV}$

We also define the following function that gives the cardinality of a type within the *continuum* type class.

definition *ucard-of* :: '*a*::*continuum itself* ⇒ *ucard* **where**
ucard-of x = (*if* (*finite* (*UNIV* :: '*a set*))
    then fin(*card*(*UNIV* :: '*a set*) − *1*)
  else if (*countable* (*UNIV* :: '*a set*))
    then $\aleph_0$
  else c)

syntax
  -*ucard* :: *type* ⇒ *ucard* (*UCARD*'(-'))

translations
  UCARD('*a*) == *CONST ucard-of* (*TYPE*('*a*))

lemma *ucard-non-empty*:
  $\mathcal{U}(x) \neq \{\}$
  by (*induct x*, *auto*)

lemma *ucard-of-finite* [*simp*]:
  finite (*UNIV* :: '*a*::*continuum set*) ⟹ *UCARD*('*a*) = *fin*(*card*(*UNIV* :: '*a set*) − *1*)
  by (*simp add*: *ucard-of-def*)

lemma *ucard-of-countably-infinite* [*simp*]:
  ⟦ *countable*(*UNIV* :: '*a*::*continuum set*); *infinite*(*UNIV* :: '*a set*) ⟧ ⟹ *UCARD*('*a*) = $\aleph_0$
  by (*simp add*: *ucard-of-def*)

lemma *ucard-of-uncountably-infinite* [*simp*]:
  uncountable (*UNIV* :: '*a set*) ⟹ *UCARD*('*a* :: *continuum*) = c
  apply (*simp add*: *ucard-of-def*)
  using *countable-finite* **apply** *blast*
done

1.3 Injection functions

definition *uinject-finite* :: '*a*::*finite* ⇒ *uuniv* **where**
uinject-finite x = {*to-nat-fin x*}

definition *uinject-aleph0* :: '*a*::{*countable*, *infinite*} ⇒ *uuniv* **where**
uinject-aleph0 x = {*to-nat-bij x*}

definition *uinject-continuum* :: '*a*::{*continuum*, *infinite*} ⇒ *uuniv* **where**

uinject-continuum x = *to-nat-set-bij x*

definition *uinject* :: *'a::continuum* ⇒ *uuniv* **where**
uinject x = (*if* (*finite* (*UNIV* :: *'a set*))
        then {*to-nat-fin x*}
      else if (*countable* (*UNIV* :: *'a set*))
        then {*to-nat-on* (*UNIV* :: *'a set*) *x*}
      else to-nat-set x)

definition *uproject* :: *uuniv* ⇒ *'a::continuum* **where**
uproject = *inv uinject*

lemma *uinject-finite*:
  finite (*UNIV* :: *'a::continuum set*) ⟹ *uinject* = (λ *x* :: *'a*. {*to-nat-fin x*})
  by (*rule ext*, *auto simp add*: *uinject-def*)

lemma *uinject-uncountable*:
  uncountable (*UNIV* :: *'a::continuum set*) ⟹ (*uinject* :: *'a* ⇒ *uuniv*) = *to-nat-set*
  by (*rule ext*, *auto simp add*: *uinject-def countable-finite*)

lemma *card-finite-lemma*:
  assumes *finite* (*UNIV* :: *'a set*)
  shows *x* < *card* (*UNIV* :: *'a set*) ⟷ *x* ≤ *card* (*UNIV* :: *'a set*) − *Suc 0*
proof −
  have *card* (*UNIV* :: *'a set*) > *0*
    by (*simp add*: *assms finite-UNIV-card-ge-0*)
  thus *?thesis*
    by *linarith*
qed

This is a key theorem that shows that the injection function provides a bijection between any continuum type and the subuniverse of types with a matching cardinality.

lemma *uinject-bij*:
  bij-betw (*uinject* :: *'a::continuum* ⇒ *uuniv*) *UNIV* $\mathcal{U}$(*UCARD*(*'a*))
proof (*cases finite* (*UNIV* :: *'a set*))
  case *True* **thus** *?thesis*
    apply (*auto simp add*: *uinject-def bij-betw-def inj-on-def image-def card-finite-lemma*[*THEN sym*])
    apply (*auto simp add*: *inj-eq to-nat-fin-inj to-nat-fin-bounded*)
    using *to-nat-fin-ex* **apply** *blast*
  done
  next
  case *False* **note** *infinite* = *this* **thus** *?thesis*
  proof (*cases countable* (*UNIV* :: *'a set*))
    case *True* **thus** *?thesis*
     apply (*auto simp add*: *uinject-def bij-betw-def inj-on-def infinite image-def card-finite-lemma*[*THEN sym*])
      apply (*meson image-to-nat-on infinite surj-def*)
    done
    next
    case *False* **note** *uncount* = *this* **thus** *?thesis*
      apply (*simp add*: *uinject-uncountable*)
      using *to-nat-set-bij* **apply** *blast*
    done
  qed
qed

lemma *uinject-card* [*simp*]: *uinject* (x :: $'a$::*continuum*) $\in \mathcal{U}(UCARD('a))$
  by (*metis bij-betw-def rangeI uinject-bij*)

lemma *uinject-inv* [*simp*]:
  uproject (*uinject* x) $= x$
  by (*metis UNIV-I bij-betw-def inv-into-f-f uinject-bij uproject-def*)

lemma *uproject-inv* [*simp*]:
  $x \in \mathcal{U}(UCARD('a::continuum)) \Longrightarrow$ *uinject* ((*uproject* :: *nat set* $\Rightarrow$ $'a$) x) $= x$
  by (*metis bij-betw-inv-into-right uinject-bij uproject-def*)

1.4 Deep variables

A deep variable name stores both a name and the cardinality of the type it points to

record *dname* =
  dname-name :: *string*
  dname-card :: *ucard*

declare *dname.splits* [*alpha-splits*]

A vstore is a function mapping deep variable names to corresponding values in the universe, such that the deep variables specified cardinality is matched by the value it points to.

typedef *vstore* = $\{f$ :: *dname* $\Rightarrow$ *uuniv*. $\forall\ x.\ f(x) \in \mathcal{U}$(*dname-card* x)$\}$
  apply (*rule-tac* x=$\lambda\ x.\ \{0\}$ **in** *exI*)
  apply (*auto*)
  apply (*rename-tac* x)
  apply (*case-tac dname-card* x)
  apply (*simp-all*)
done

setup-lifting *type-definition-vstore*

typedef ($'a$::*continuum*) *dvar* = $\{x$ :: *dname*. *dname-card* x = $UCARD('a)\}$
  morphisms *dvar-dname Abs-dvar*
  by (*auto*, *meson dname.select-convs*(*2*))

setup-lifting *type-definition-dvar*

lift-definition *mk-dvar* :: *string* $\Rightarrow$ ($'a$::{*continuum*,*two*}) *dvar* ($\lceil$-$\rceil_d$)
is $\lambda\ n.$ (| *dname-name* = n, *dname-card* = $UCARD('a)$ |)
  by *auto*

lift-definition *dvar-name* :: $'a$::*continuum dvar* $\Rightarrow$ *string* **is** *dname-name* **.**
lift-definition *dvar-card* :: $'a$::*continuum dvar* $\Rightarrow$ *ucard* **is** *dname-card* **.**

lemma *dvar-name* [*simp*]: *dvar-name* $\lceil x\rceil_d = x$
  by (*transfer*, *simp*)

term *fun-lens*

setup-lifting *type-definition-lens-ext*

lift-definition *dvar-get* :: ($'a$::*continuum*) *dvar* $\Rightarrow$ *vstore* $\Rightarrow$ $'a$
is $\lambda\ x\ s.$ (*uproject* :: *uuniv* $\Rightarrow$ $'a$) ($s(x)$) **.**

lift-definition *dvar-put* :: ($'a$::*continuum*) *dvar* $\Rightarrow$ *vstore* $\Rightarrow$ $'a$ $\Rightarrow$ *vstore*
is λ (x :: *dname*) f (v :: $'a$) . $f(x := uinject\ v)$
  by (*auto*)

definition *dvar-lens* :: ($'a$::*continuum*) *dvar* $\Rightarrow$ ($'a \Longrightarrow vstore$) **where**
dvar-lens x = (| *lens-get* = *dvar-get* x, *lens-put* = *dvar-put* x |)

lemma *vstore-vwb-lens* [*simp*]:
  vwb-lens (*dvar-lens* x)
  apply (*unfold-locales*)
  apply (*simp-all add*: *dvar-lens-def*)
  apply (*transfer*, *auto*)
  apply (*transfer*)
  apply (*metis fun-upd-idem uproject-inv*)
  apply (*transfer*, *simp*)
done

lemma *dvar-lens-indep-iff*:
  fixes x :: $'a$::{*continuum*,*two*} *dvar* **and** y :: $'b$::{*continuum*,*two*} *dvar*
  shows *dvar-lens* $x \bowtie$ *dvar-lens* $y \longleftrightarrow$ (*dvar-dname* $x \neq$ *dvar-dname* y)
proof –
  obtain *v1 v2* :: $'b$::{*continuum*,*two*} **where** *v*:*v1* $\neq$ *v2*
    using *two-diff* **by** *auto*
  obtain u :: $'a$::{*continuum*,*two*} **and** v :: $'b$::{*continuum*,*two*}
    where *uv*: *uinject* $u \neq$ *uinject* v
    by (*metis* (*full-types*) *uinject-inv* v)
  show *?thesis*
  proof (*simp add*: *dvar-lens-def lens-indep-def*, *transfer*, *auto simp add*: *fun-upd-twist*)
    fix y :: *dname*
    assume *a1*: *ucard-of* ($TYPE('b)$::$'b$ *itself*) = *ucard-of* ($TYPE('a)$::$'a$ *itself*)
    assume *dname-card* y = *ucard-of* ($TYPE('a)$::$'a$ *itself*)
    assume *a2*:
      $\forall\ \sigma.\ (\forall x.\ \sigma\ x \in \mathcal{U}(dname\text{-}card\ x)) \longrightarrow (\forall\ v\ u.\ \sigma(y := uinject\ (u::'a)) = \sigma(y := uinject\ (v::'b)))$
      $\forall\ \sigma.\ (\forall x.\ \sigma\ x \in \mathcal{U}(dname\text{-}card\ x)) \longrightarrow (\forall\ v.\ (uproject\ (uinject\ v)::'a) = uproject\ (\sigma\ y))$
      $\forall\ \sigma.\ (\forall x.\ \sigma\ x \in \mathcal{U}(dname\text{-}card\ x)) \longrightarrow (\forall\ u.\ (uproject\ (uinject\ u)::'b) = uproject\ (\sigma\ y))$
    obtain *NN* :: *vstore* $\Rightarrow$ *dname* $\Rightarrow$ *nat set* **where**
      $\bigwedge v.\ \forall d.\ NN\ v\ d \in \mathcal{U}(dname\text{-}card\ d)$
      by (*metis* (*lifting*) *Abs-vstore-cases mem-Collect-eq*)
    then show *False*
      using *a2 a1* **by** (*metis fun-upd-same uv*)
  qed
qed

The vst class provides the location of the store in a larger type via a lens

class *vst* =
  fixes *vstore-lens* :: *vstore* $\Longrightarrow$ $'a$ ($\mathcal{V}$)
  assumes *vstore-vwb-lens* [*simp*]: *vwb-lens vstore-lens*

definition *dvar-lift* :: $'a$::*continuum dvar* $\Rightarrow$ ($'a$, $'\alpha$::*vst*) *uvar* (-$\uparrow$ [*999*] *999*) **where**
dvar-lift x = *dvar-lens* x $;_L$ *vstore-lens*

definition [*simp*]: *in-dvar* x = *in-var* ($x\uparrow$)
definition [*simp*]: *out-dvar* x = *out-var* ($x\uparrow$)

adhoc-overloading
ivar in-dvar **and** *ovar out-dvar* **and** *svar dvar-lift*

lemma *uvar-dvar*: *vwb-lens* $(x\uparrow)$
by (*auto intro*: *comp-vwb-lens simp add*: *dvar-lift-def*)

Deep variables with different names are independent

lemma *dvar-lift-indep-iff*:
fixes x :: $'a$::{*continuum*,*two*} *dvar* **and** y :: $'b$::{*continuum*,*two*} *dvar*
shows $x\uparrow \bowtie y\uparrow \longleftrightarrow$ *dvar-dname* $x \neq$ *dvar-dname* y
proof −
have $x\uparrow \bowtie y\uparrow \longleftrightarrow$ *dvar-lens* $x \bowtie$ *dvar-lens* y
by (*metis dvar-lift-def lens-comp-indep-cong-left lens-indep-left-comp vst-class.vstore-vwb-lens vwb-lens-mwb*)
also have ... $\longleftrightarrow$ *dvar-dname* $x \neq$ *dvar-dname* y
by (*simp add*: *dvar-lens-indep-iff*)
finally show *?thesis* **.**
qed

lemma *dvar-indep-diff-name′* [*simp*]:
$x \neq y \Longrightarrow \lceil x \rceil_d\uparrow \bowtie \lceil y \rceil_d\uparrow$
by (*simp add*: *dvar-lift-indep-iff mk-dvar.rep-eq*)

A basic record structure for vstores

record *vstore-d* =
vstore :: *vstore*

instantiation *vstore-d-ext* :: (*type*) *vst*
begin
definition *vstore-lens-vstore-d-ext* = *VAR vstore*
instance
by (*intro-classes*, *unfold-locales*, *simp-all add*: *vstore-lens-vstore-d-ext-def*)
end

syntax
-sin-dvar :: *id* ⇒ *svar* (%- [*999*] *999*)
-sout-dvar :: *id* ⇒ *svar* (%-´ [*999*] *999*)

translations
-sin-dvar x => *CONST in-dvar* (*CONST mk-dvar IDSTR*(*x*))
-sout-dvar x => *CONST out-dvar* (*CONST mk-dvar IDSTR*(*x*))

definition *MkDVar* $x = \lceil x \rceil_d\uparrow$

lemma *uvar-MkDVar* [*simp*]: *vwb-lens* (*MkDVar x*)
by (*simp add*: *MkDVar-def uvar-dvar*)

lemma *MkDVar-indep* [*simp*]: $x \neq y \Longrightarrow$ *MkDVar* $x \bowtie$ *MkDVar* y
apply (*rule lens-indepI*)
apply (*simp-all add*: *MkDVar-def*)
apply (*meson dvar-indep-diff-name′ lens-indep-comm*)
done

lemma *MkDVar-put-comm* [*simp*]:
$m <_l n \Longrightarrow put_{MkDVar\ n}\ (put_{MkDVar\ m}\ s\ u)\ v = put_{MkDVar\ m}\ (put_{MkDVar\ n}\ s\ v)\ u$
by (*simp add*: *lens-indep-comm*)

Set up parsing and pretty printing for deep variables

```
syntax
  -dvar     :: id ⇒ svid (<->)
  -dvar-ty  :: id ⇒ type ⇒ svid (<-::->)
  -dvard    :: id ⇒ logic (<->d)
  -dvar-tyd :: id ⇒ type ⇒ logic (<-::->d)

translations
  -dvar x => CONST MkDVar IDSTR(x)
  -dvar-ty x a => -constrain (CONST MkDVar IDSTR(x)) (-uvar-ty a)
  -dvard x => CONST MkDVar IDSTR(x)
  -dvar-tyd x a => -constrain (CONST MkDVar IDSTR(x)) (-uvar-ty a)

print-translation ⟪
let fun MkDVar-tr' - [name] =
      Const (@{syntax-const -dvar}, dummyT) $
        Name-Utils.mk-id (HOLogic.dest-string (Name-Utils.deep-unmark-const name))
    | MkDVar-tr' - - = raise Match in
  [(@{const-syntax MkDVar}, MkDVar-tr')]
end
⟫

end
```

2 UTP expressions

```
theory utp-expr
imports
  utp-var
  utp-dvar
begin
```

Before building the predicate model, we will build a model of expressions that generalise alphabetised predicates. Expressions are represented semantically as mapping from the alphabet to the expression's type. This general model will allow us to unify all constructions under one type. All definitions in the file are given using the *lifting* package.

Since we have two kinds of variable (deep and shallow) in the model, we will also need two versions of each construct that takes a variable. We make use of adhoc-overloading to ensure the correct instance is automatically chosen, within the user noticing a difference.

```
typedef ('t, 'α) uexpr = UNIV :: ('α alphabet ⇒ 't) set ..

notation Rep-uexpr (⟦-⟧e)

lemma uexpr-eq-iff:
  e = f ⟷ (∀ b. ⟦e⟧e b = ⟦f⟧e b)
  using Rep-uexpr-inject[of e f, THEN sym] by (auto)

named-theorems ueval and lit-simps

setup-lifting type-definition-uexpr
```

Get the alphabet of an expression

```
definition alpha-of :: ('a, 'α) uexpr ⇒ ('α, 'α) lens (α'(-')) where
```

alpha-of e = 1_L

A variable expression corresponds to the lookup function of the variable.

lift-definition *var* :: ('t, 'α) *uvar* ⇒ ('t, 'α) *uexpr* **is** *lens-get* **.**

declare [[*coercion-enabled*]]
declare [[*coercion var*]]

definition *dvar-exp* :: 't::*continuum dvar* ⇒ ('t, 'α::*vst*) *uexpr*
where *dvar-exp x* = *var* (*dvar-lift x*)

A literal is simply a constant function expression, always returning the same value.

lift-definition *lit* :: 't ⇒ ('t, 'α) *uexpr*
  is λ *v b*. *v* **.**

We define lifting for unary, binary, and ternary functions, that simply apply the function to all possible results of the expressions.

lift-definition *uop* :: ('a ⇒ 'b) ⇒ ('a, 'α) *uexpr* ⇒ ('b, 'α) *uexpr*
  is λ *f e b*. *f* (*e b*) **.**
lift-definition *bop* ::
  ('a ⇒ 'b ⇒ 'c) ⇒ ('a, 'α) *uexpr* ⇒ ('b, 'α) *uexpr* ⇒ ('c, 'α) *uexpr*
  is λ *f u v b*. *f* (*u b*) (*v b*) **.**
lift-definition *trop* ::
  ('a ⇒ 'b ⇒ 'c ⇒ 'd) ⇒ ('a, 'α) *uexpr* ⇒ ('b, 'α) *uexpr* ⇒ ('c, 'α) *uexpr* ⇒ ('d, 'α) *uexpr*
  is λ *f u v w b*. *f* (*u b*) (*v b*) (*w b*) **.**
lift-definition *qtop* ::
  ('a ⇒ 'b ⇒ 'c ⇒ 'd ⇒ 'e) ⇒
   ('a, 'α) *uexpr* ⇒ ('b, 'α) *uexpr* ⇒ ('c, 'α) *uexpr* ⇒ ('d, 'α) *uexpr* ⇒
   ('e, 'α) *uexpr*
  is λ *f u v w x b*. *f* (*u b*) (*v b*) (*w b*) (*x b*) **.**

We also define a UTP expression version of function abstract

lift-definition *ulambda* :: ('a ⇒ ('b, 'α) *uexpr*) ⇒ ('a ⇒ 'b, 'α) *uexpr*
is λ *f A x*. *f x A* **.**

We define syntax for expressions using adhoc overloading – this allows us to later define operators on different types if necessary (e.g. when adding types for new UTP theories).

consts
  ulit :: 't ⇒ 'e («-»)
  ueq   :: 'a ⇒ 'a ⇒ 'b (**infixl** $=_u$ *50*)

adhoc-overloading
  ulit lit

syntax
  -uuvar :: *svar* ⇒ *logic*

translations
  -uuvar x == *CONST var x*

syntax
  -uuvar :: *svar* ⇒ *logic* (-)

We also set up some useful standard arithmetic operators for Isabelle by lifting the functions to binary operators.

instantiation *uexpr* :: (*plus*, *type*) *plus*
begin
  definition *plus-uexpr-def*: $u + v = bop\ (op\ +)\ u\ v$
instance ..
end

Instantiating uminus also provides negation for predicates later

instantiation *uexpr* :: (*uminus*, *type*) *uminus*
begin
  definition *uminus-uexpr-def*: $-\ u = uop\ uminus\ u$
instance ..
end

instantiation *uexpr* :: (*minus*, *type*) *minus*
begin
  definition *minus-uexpr-def*: $u - v = bop\ (op\ -)\ u\ v$
instance ..
end

instantiation *uexpr* :: (*times*, *type*) *times*
begin
  definition *times-uexpr-def*: $u * v = bop\ (op\ *)\ u\ v$
instance ..
end

instance *uexpr* :: (*Rings.dvd*, *type*) *Rings.dvd* **..**

instantiation *uexpr* :: (*divide*, *type*) *divide*
begin
  definition *divide-uexpr* :: $('a, 'b)$ *uexpr* $\Rightarrow$ $('a, 'b)$ *uexpr* $\Rightarrow$ $('a, 'b)$ *uexpr* **where**
  divide-uexpr u v = bop divide u v
instance ..
end

instantiation *uexpr* :: (*inverse*, *type*) *inverse*
begin
  definition *inverse-uexpr* :: $('a, 'b)$ *uexpr* $\Rightarrow$ $('a, 'b)$ *uexpr*
  where *inverse-uexpr u = uop inverse u*
instance ..
end

instantiation *uexpr* :: (*Divides.div*, *type*) *Divides.div*
begin
  definition *mod-uexpr-def*: *u mod v = bop (op mod) u v*
instance ..
end

instantiation *uexpr* :: (*sgn*, *type*) *sgn*
begin
  definition *sgn-uexpr-def*: *sgn u = uop sgn u*
instance ..
end

instantiation *uexpr* :: (*abs*, *type*) *abs*
begin

definition *abs-uexpr-def*: *abs u* = *uop abs u*
instance ..
end

instantiation *uexpr* :: (*zero*, *type*) *zero*
begin
definition *zero-uexpr-def*: *0* = *lit 0*
instance ..
end

instantiation *uexpr* :: (*one*, *type*) *one*
begin
definition *one-uexpr-def*: *1* = *lit 1*
instance ..

end

instance *uexpr* :: (*semigroup-mult*, *type*) *semigroup-mult*
by (*intro-classes*) (*simp add*: *times-uexpr-def one-uexpr-def*, *transfer*, *simp add*: *mult.assoc*)+

instance *uexpr* :: (*monoid-mult*, *type*) *monoid-mult*
by (*intro-classes*) (*simp add*: *times-uexpr-def one-uexpr-def*, *transfer*, *simp*)+

instance *uexpr* :: (*semigroup-add*, *type*) *semigroup-add*
by (*intro-classes*) (*simp add*: *plus-uexpr-def zero-uexpr-def*, *transfer*, *simp add*: *add.assoc*)+

instance *uexpr* :: (*monoid-add*, *type*) *monoid-add*
by (*intro-classes*) (*simp add*: *plus-uexpr-def zero-uexpr-def*, *transfer*, *simp*)+

instance *uexpr* :: (*ab-semigroup-add*, *type*) *ab-semigroup-add*
by (*intro-classes*) (*simp add*: *plus-uexpr-def*, *transfer*, *simp add*: *add.commute*)+

instance *uexpr* :: (*cancel-semigroup-add*, *type*) *cancel-semigroup-add*
by (*intro-classes*) (*simp add*: *plus-uexpr-def*, *transfer*, *simp add*: *fun-eq-iff*)+

instance *uexpr* :: (*cancel-ab-semigroup-add*, *type*) *cancel-ab-semigroup-add*
by (*intro-classes*, (*simp add*: *plus-uexpr-def minus-uexpr-def*, *transfer*, *simp add*: *fun-eq-iff add.commute cancel-ab-semigroup-add-class.diff-diff-add*)+)

instance *uexpr* :: (*group-add*, *type*) *group-add*
by (*intro-classes*)
(*simp add*: *plus-uexpr-def uminus-uexpr-def minus-uexpr-def zero-uexpr-def*, *transfer*, *simp*)+

instance *uexpr* :: (*ab-group-add*, *type*) *ab-group-add*
by (*intro-classes*)
(*simp add*: *plus-uexpr-def uminus-uexpr-def minus-uexpr-def zero-uexpr-def*, *transfer*, *simp*)+

instantiation *uexpr* :: (*ord*, *type*) *ord*
begin
lift-definition *less-eq-uexpr* :: ($'a$, $'b$) *uexpr* $\Rightarrow$ ($'a$, $'b$) *uexpr* $\Rightarrow$ *bool*
is λ P Q. ($\forall$ A. P A $\leq$ Q A) **.**
definition *less-uexpr* :: ($'a$, $'b$) *uexpr* $\Rightarrow$ ($'a$, $'b$) *uexpr* $\Rightarrow$ *bool*
where *less-uexpr* P Q = (P $\leq$ Q $\wedge$ $\neg$ Q $\leq$ P)
instance ..
end

instance *uexpr* :: (*order*, *type*) *order*
proof
  fix $x\ y\ z$ :: ('*a*, '*b*) *uexpr*
  show $(x < y) = (x \leq y \wedge \neg\ y \leq x)$ **by** (*simp add*: *less-uexpr-def*)
  show $x \leq x$ **by** (*transfer*, *auto*)
  show $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
    by (*transfer*, *blast intro*:*order.trans*)
  show $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
    by (*transfer*, *rule ext*, *simp add*: *eq-iff*)
qed

instance *uexpr* :: (*ordered-ab-group-add*, *type*) *ordered-ab-group-add*
  by (*intro-classes*) (*simp add*: *plus-uexpr-def*, *transfer*, *simp*)

instance *uexpr* :: (*ordered-ab-group-add-abs*, *type*) *ordered-ab-group-add-abs*
  apply (*intro-classes*)
  apply (*simp add*: *abs-uexpr-def zero-uexpr-def plus-uexpr-def uminus-uexpr-def*, *transfer*, *simp add*: *abs-ge-self abs-le-iff abs-triangle-ineq*)+
  apply (*metis ab-group-add-class.ab-diff-conv-add-uminus abs-ge-minus-self abs-ge-self add-mono-thms-linordered-semiri*
done

lemma *uexpr-diff-zero* [*simp*]:
  fixes *a* :: ('α::*ordered-cancel-monoid-diff*, '*a*) *uexpr*
  shows $a - 0 = a$
  by (*simp add*: *minus-uexpr-def zero-uexpr-def*, *transfer*, *auto*)

lemma *uexpr-add-diff-cancel-left* [*simp*]:
  fixes *a b* :: ('α::*ordered-cancel-monoid-diff*, '*a*) *uexpr*
  shows $(a + b) - a = b$
  by (*simp add*: *minus-uexpr-def plus-uexpr-def*, *transfer*, *auto*)

instance *uexpr* :: (*semiring*, *type*) *semiring*
  by (*intro-classes*) (*simp add*: *plus-uexpr-def times-uexpr-def*, *transfer*, *simp add*: *fun-eq-iff add.commute semiring-class.distrib-right semiring-class.distrib-left*)+

instance *uexpr* :: (*ring-1*, *type*) *ring-1*
  by (*intro-classes*) (*simp add*: *plus-uexpr-def uminus-uexpr-def minus-uexpr-def times-uexpr-def zero-uexpr-def one-uexpr-def*, *transfer*, *simp add*: *fun-eq-iff*)+

instance *uexpr* :: (*numeral*, *type*) *numeral*
  by (*intro-classes*, *simp add*: *plus-uexpr-def*, *transfer*, *simp add*: *add.assoc*)

Set up automation for numerals

lemma *numeral-uexpr-rep-eq*: $[\![$*numeral x*$]\!]_e$ *b* = *numeral x*
  by (*induct x*, *simp-all add*: *plus-uexpr-def one-uexpr-def numeral.simps lit.rep-eq bop.rep-eq*)

lemma *numeral-uexpr-simp*: *numeral x* = «*numeral x*»
  by (*simp add*: *uexpr-eq-iff numeral-uexpr-rep-eq lit.rep-eq*)

definition *eq-upred* :: ('*a*, 'α) *uexpr* ⇒ ('*a*, 'α) *uexpr* ⇒ (*bool*, 'α) *uexpr*
where *eq-upred x y* = *bop HOL.eq x y*

adhoc-overloading

ueq eq-upred

definition *fun-apply f x = f x*
declare *fun-apply-def* [*simp*]

consts
uempty :: 'f
uapply :: 'f ⇒ 'k ⇒ 'v
uupd :: 'f ⇒ 'k ⇒ 'v ⇒ 'f
udom :: 'f ⇒ 'a set
uran :: 'f ⇒ 'b set
udomres :: 'a set ⇒ 'f ⇒ 'f
uranres :: 'f ⇒ 'b set ⇒ 'f
ucard :: 'f ⇒ nat

definition *LNil = Nil*
definition *LZero = 0*

adhoc-overloading
uempty LZero **and** *uempty LNil* **and**
uapply fun-apply **and** *uapply nth* **and** *uapply pfun-app* **and**
uapply ffun-app **and** *uapply cgf-apply* **and** *uapply tt-apply* **and**
uupd pfun-upd **and** *uupd ffun-upd* **and** *uupd list-update* **and**
udom Domain **and** *udom pdom* **and** *udom fdom* **and** *udom seq-dom* **and**
udom Range **and** *uran pran* **and** *uran fran* **and** *uran set* **and**
udomres pdom-res **and** *udomres fdom-res* **and**
uranres pran-res **and** *udomres fran-res* **and**
ucard card **and** *ucard pcard* **and** *ucard length*

nonterminal *utuple-args* **and** *umaplet* **and** *umaplets*

syntax
-ucoerce :: ('a, 'α) uexpr ⇒ type ⇒ ('a, 'α) uexpr (**infix** $:_u$ 50)
-unil :: ('a list, 'α) uexpr (⟨⟩)
-ulist :: args => ('a list, 'α) uexpr (⟨(-)⟩)
-uappend :: ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr (**infixr** $\hat{}_u$ 80)
-ulast :: ('a list, 'α) uexpr ⇒ ('a, 'α) uexpr ($last_u$'(-'))
-ufront :: ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr ($front_u$'(-'))
-uhead :: ('a list, 'α) uexpr ⇒ ('a, 'α) uexpr ($head_u$'(-'))
-utail :: ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr ($tail_u$'(-'))
-utake :: (nat, 'α) uexpr ⇒ ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr ($take_u$'(-,/ -'))
-udrop :: (nat, 'α) uexpr ⇒ ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr ($drop_u$'(-,/ -'))
-ucard :: ('a list, 'α) uexpr ⇒ (nat, 'α) uexpr ($\#_u$'(-'))
-ufilter :: ('a list, 'α) uexpr ⇒ ('a set, 'α) uexpr ⇒ ('a list, 'α) uexpr (**infixl** $\restriction_u$ 75)
-uextract :: ('a set, 'α) uexpr ⇒ ('a list, 'α) uexpr ⇒ ('a list, 'α) uexpr (**infixl** $\upharpoonleft_u$ 75)
-uelems :: ('a list, 'α) uexpr ⇒ ('a set, 'α) uexpr ($elems_u$'(-'))
-usorted :: ('a list, 'α) uexpr ⇒ (bool, 'α) uexpr ($sorted_u$'(-'))
-udistinct :: ('a list, 'α) uexpr ⇒ (bool, 'α) uexpr ($distinct_u$'(-'))
-uless :: ('a, 'α) uexpr ⇒ ('a, 'α) uexpr ⇒ (bool, 'α) uexpr (**infix** $<_u$ 50)
-uleq :: ('a, 'α) uexpr ⇒ ('a, 'α) uexpr ⇒ (bool, 'α) uexpr (**infix** $\leq_u$ 50)
-ugreat :: ('a, 'α) uexpr ⇒ ('a, 'α) uexpr ⇒ (bool, 'α) uexpr (**infix** $>_u$ 50)
-ugeq :: ('a, 'α) uexpr ⇒ ('a, 'α) uexpr ⇒ (bool, 'α) uexpr (**infix** $\geq_u$ 50)
-umin :: logic ⇒ logic ⇒ logic (min_u'(-, -'))
-umax :: logic ⇒ logic ⇒ logic (max_u'(-, -'))
-ugcd :: logic ⇒ logic ⇒ logic (gcd_u'(-, -'))

$\text{-}ufinite$ :: $logic \Rightarrow logic$ $(finite_u\text{'}(\text{-}\text{'}))$
$\text{-}uempset$ :: $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr$ $(\{\}_u)$
$\text{-}uset$ :: $args$ => $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr$ $(\{(\text{-})\}_u)$
$\text{-}uunion$ :: $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr$ (**infixl** $\cup_u$ *65*)
$\text{-}uinter$ :: $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr$ (**infixl** $\cap_u$ *70*)
$\text{-}umem$ :: $(\text{'}a,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (bool,\ \text{'}\alpha)\ uexpr$ (**infix** $\in_u$ *50*)
$\text{-}usubset$ :: $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (bool,\ \text{'}\alpha)\ uexpr$ (**infix** $\subset_u$ *50*)
$\text{-}usubseteq$ :: $(\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a\ set,\ \text{'}\alpha)\ uexpr \Rightarrow (bool,\ \text{'}\alpha)\ uexpr$ (**infix** $\subseteq_u$ *50*)
$\text{-}utuple$ :: $(\text{'}a,\ \text{'}\alpha)\ uexpr \Rightarrow utuple\text{-}args \Rightarrow (\text{'}a * \text{'}b,\ \text{'}\alpha)\ uexpr$ $((1\text{'}(\text{-},/\ \text{-}\text{'})_u))$
$\text{-}utuple\text{-}arg$ :: $(\text{'}a,\ \text{'}\alpha)\ uexpr \Rightarrow utuple\text{-}args$ (-)
$\text{-}utuple\text{-}args$ :: $(\text{'}a,\ \text{'}\alpha)\ uexpr$ => $utuple\text{-}args \Rightarrow utuple\text{-}args$ (-,/ -)
$\text{-}uunit$ :: $(\text{'}a,\ \text{'}\alpha)\ uexpr$ $(\text{'}(\text{'})_u)$
$\text{-}ufst$ :: $(\text{'}a \times \text{'}b,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}a,\ \text{'}\alpha)\ uexpr$ $(\pi_1\text{'}(\text{-}\text{'}))$
$\text{-}usnd$ :: $(\text{'}a \times \text{'}b,\ \text{'}\alpha)\ uexpr \Rightarrow (\text{'}b,\ \text{'}\alpha)\ uexpr$ $(\pi_2\text{'}(\text{-}\text{'}))$
$\text{-}uapply$ :: $(\text{'}a \Rightarrow \text{'}b,\ \text{'}\alpha)\ uexpr \Rightarrow utuple\text{-}args \Rightarrow (\text{'}b,\ \text{'}\alpha)\ uexpr$ $(\text{-}(\!|\text{-}|\!)_u\ [999,0]\ 999)$
$\text{-}ulamba$ :: $pttrn \Rightarrow logic \Rightarrow logic$ $(\lambda\ \text{-}\ \bullet\ \text{-}\ [0,\ 10]\ 10)$
$\text{-}udom$ :: $logic \Rightarrow logic$ $(dom_u\text{'}(\text{-}\text{'}))$
$\text{-}uran$ :: $logic \Rightarrow logic$ $(ran_u\text{'}(\text{-}\text{'}))$
$\text{-}uinl$ :: $logic \Rightarrow logic$ $(inl_u\text{'}(\text{-}\text{'}))$
$\text{-}uinr$ :: $logic \Rightarrow logic$ $(inr_u\text{'}(\text{-}\text{'}))$
$\text{-}umap\text{-}empty$ :: $logic$ $([]_u)$
$\text{-}umap\text{-}plus$ :: $logic \Rightarrow logic \Rightarrow logic$ (**infixl** $\oplus_u$ *85*)
$\text{-}umap\text{-}minus$ :: $logic \Rightarrow logic \Rightarrow logic$ (**infixl** $\ominus_u$ *85*)
$\text{-}udom\text{-}res$ :: $logic \Rightarrow logic \Rightarrow logic$ (**infixl** $\lhd_u$ *85*)
$\text{-}uran\text{-}res$ :: $logic \Rightarrow logic \Rightarrow logic$ (**infixl** $\rhd_u$ *85*)
$\text{-}umaplet$ :: $[logic,\ logic]$ => $umaplet$ $(\text{-}\ /{\mapsto}/\ \text{-})$
:: $umaplet$ => $umaplets$ (-)
$\text{-}UMaplets$ :: $[umaplet,\ umaplets]$ => $umaplets$ (-,/ -)
$\text{-}UMapUpd$ :: $[logic,\ umaplets]$ => $logic$ $(\text{-}/\text{'}(\text{-}\text{'})_u\ [900,0]\ 900)$
$\text{-}UMap$ :: $umaplets$ => $logic$ $((1[\text{-}]_u))$

translations
$f(\!|v|\!)_u$ <= $CONST\ uapply\ f\ v$
$dom_u(f)$ <= $CONST\ udom\ f$
$ran_u(f)$ <= $CONST\ uran\ f$
$A \lhd_u f$ <= $CONST\ udomres\ A\ f$
$f \rhd_u A$ <= $CONST\ uranres\ f\ A$
$\#_u(f)$ <= $CONST\ ucard\ f$
$f(k \mapsto v)_u$ <= $CONST\ uupd\ f\ k\ v$

translations
$x :_u \text{'}a$ == $x :: (\text{'}a,\ \text{-})\ uexpr$
$\langle\rangle$ == $\ll[]\gg$
$\langle x,\ xs\rangle$ == $CONST\ bop\ (op\ \#)\ x\ \langle xs\rangle$
$\langle x\rangle$ == $CONST\ bop\ (op\ \#)\ x \ll[]\gg$
$x\ \hat{}_u\ y$ == $CONST\ bop\ (op\ @)\ x\ y$
$last_u(xs)$ == $CONST\ uop\ CONST\ last\ xs$
$front_u(xs)$ == $CONST\ uop\ CONST\ butlast\ xs$
$head_u(xs)$ == $CONST\ uop\ CONST\ hd\ xs$
$tail_u(xs)$ == $CONST\ uop\ CONST\ tl\ xs$
$drop_u(n,xs)$ == $CONST\ bop\ CONST\ drop\ n\ xs$
$take_u(n,xs)$ == $CONST\ bop\ CONST\ take\ n\ xs$
$\#_u(xs)$ == $CONST\ uop\ CONST\ ucard\ xs$
$elems_u(xs)$ == $CONST\ uop\ CONST\ set\ xs$
$sorted_u(xs)$ == $CONST\ uop\ CONST\ sorted\ xs$

$distinct_u(xs)$ == *CONST uop CONST distinct xs*
$xs \upharpoonright_u A$ == *CONST bop CONST seq-filter xs A*
$A \upharpoonleft_u xs$ == *CONST bop* $(op \upharpoonleft_l)$ *A xs*
$x <_u y$ == *CONST bop* $(op <)$ *x y*
$x \leq_u y$ == *CONST bop* $(op \leq)$ *x y*
$x >_u y$ == $y <_u x$
$x \geq_u y$ == $y \leq_u x$
$min_u(x, y)$ == *CONST bop (CONST min) x y*
$max_u(x, y)$ == *CONST bop (CONST max) x y*
$gcd_u(x, y)$ == *CONST bop (CONST gcd) x y*
$finite_u(x)$ == *CONST uop (CONST finite) x*
$\{\}_u$ == «{}»
$\{x, xs\}_u$ == *CONST bop (CONST insert) x* $\{xs\}_u$
$\{x\}_u$ == *CONST bop (CONST insert) x* «{}»
$A \cup_u B$ == *CONST bop* $(op \cup)$ *A B*
$A \cap_u B$ == *CONST bop* $(op \cap)$ *A B*
$f \oplus_u g$ => *(f :: ((-, -) pfun, -) uexpr) + g*
$f \ominus_u g$ => *(f :: ((-, -) pfun, -) uexpr)* $-$ *g*
$x \in_u A$ == *CONST bop* $(op \in)$ *x A*
$A \subset_u B$ == *CONST bop* $(op <)$ *A B*
$A \subset_u B$ <= *CONST bop* $(op \subset)$ *A B*
$f \subset_u g$ <= *CONST bop* $(op \subset_p)$ *f g*
$f \subset_u g$ <= *CONST bop* $(op \subset_f)$ *f g*
$A \subseteq_u B$ == *CONST bop* $(op \leq)$ *A B*
$A \subseteq_u B$ <= *CONST bop* $(op \subseteq)$ *A B*
$f \subseteq_u g$ <= *CONST bop* $(op \subseteq_p)$ *f g*
$f \subseteq_u g$ <= *CONST bop* $(op \subseteq_f)$ *f g*
$()_u$ == «()»
$(x, y)_u$ == *CONST bop (CONST Pair) x y*
-utuple x (-utuple-args y z) == -utuple x (-utuple-arg (-utuple y z))
$\pi_1(x)$ == *CONST uop CONST fst x*
$\pi_2(x)$ == *CONST uop CONST snd x*
$f(\!|x|\!)_u$ == *CONST bop CONST uapply f x*
$\lambda\ x \cdot p$ == *CONST ulambda* $(\lambda\ x.\ p)$
$dom_u(f)$ == *CONST uop CONST udom f*
$ran_u(f)$ == *CONST uop CONST uran f*
$inl_u(x)$ == *CONST uop CONST Inl x*
$inr_u(x)$ == *CONST uop CONST Inr x*
$[]_u$ == «*CONST uempty*»
$A \lhd_u f$ == *CONST bop (CONST udomres) A f*
$f \rhd_u A$ == *CONST bop (CONST uranres) f A*
-UMapUpd m (-UMaplets xy ms) == -UMapUpd (-UMapUpd m xy) ms
-UMapUpd m (-umaplet x y) == CONST trop CONST uupd m x y
-UMap ms == *-UMapUpd* $[]_u$ *ms*
-UMap (-UMaplets ms1 ms2) <= -UMapUpd (-UMap ms1) ms2
-UMaplets ms1 (-UMaplets ms2 ms3) <= -UMaplets (-UMaplets ms1 ms2) ms3
$f(\!|x,y|\!)_u$ == *CONST bop CONST uapply f* $(x,y)_u$

Lifting set intervals

syntax
-uset-atLeastAtMost :: $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a\ set, '\alpha)$ *uexpr* ((*1*{-..-}$_u$))
-uset-atLeastLessThan :: $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a\ set, '\alpha)$ *uexpr* ((*1*{-..<-}$_u$))
-uset-compr :: *id* $\Rightarrow$ $('a\ set, '\alpha)$ *uexpr* $\Rightarrow$ $(bool, '\alpha)$ *uexpr* $\Rightarrow$ $('b, '\alpha)$ *uexpr* $\Rightarrow$ $('b\ set, '\alpha)$ *uexpr* ((*1*{-
:/ - |/ - $\cdot$/ -}$_u$))

lift-definition *ZedSetCompr* ::
  $('a\ set,\ '\alpha)\ uexpr \Rightarrow ('a \Rightarrow (bool,\ '\alpha)\ uexpr \times ('b,\ '\alpha)\ uexpr) \Rightarrow ('b\ set,\ '\alpha)\ uexpr$
is $\lambda\ A\ PF\ b.\ \{\ snd\ (PF\ x)\ b \mid x.\ x \in A\ b \wedge fst\ (PF\ x)\ b\}$ **.**

translations
  $\{x..y\}_u$ == *CONST bop CONST atLeastAtMost x y*
  $\{x..<y\}_u$ == *CONST bop CONST atLeastLessThan x y*
  $\{x : A \mid P \cdot F\}_u$ == *CONST ZedSetCompr A* $(\lambda\ x.\ (P,\ F))$

Lifting limits

definition *ulim-left* = $(\lambda\ p\ f.\ Lim\ (at\text{-}left\ p)\ f)$
definition *ulim-right* = $(\lambda\ p\ f.\ Lim\ (at\text{-}right\ p)\ f)$
definition *ucont-on* = $(\lambda\ f\ A.\ continuous\text{-}on\ A\ f)$

syntax
  -ulim-left :: *id* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(lim_u'(\text{-} \rightarrow \text{-}^{-}')'(\text{-}'))$
  -ulim-right :: *id* ⇒ *logic* ⇒ *logic* ⇒ *logic* $(lim_u'(\text{-} \rightarrow \text{-}^{+}')'(\text{-}'))$
  -ucont-on :: *logic* ⇒ *logic* ⇒ *logic* (**infix** $cont{-}on_u$ *90*)

translations
  $lim_u(x \rightarrow p^-)(e)$ == *CONST bop CONST ulim-left p* $(\lambda\ x \cdot e)$
  $lim_u(x \rightarrow p^+)(e)$ == *CONST bop CONST ulim-right p* $(\lambda\ x \cdot e)$
  $f\ cont{-}on_u\ A$ == *CONST bop CONST continuous-on A f*

lemmas *uexpr-defs* =
  alpha-of-def
  zero-uexpr-def
  one-uexpr-def
  plus-uexpr-def
  uminus-uexpr-def
  minus-uexpr-def
  times-uexpr-def
  inverse-uexpr-def
  divide-uexpr-def
  sgn-uexpr-def
  abs-uexpr-def
  mod-uexpr-def
  eq-upred-def
  numeral-uexpr-simp
  ulim-left-def
  ulim-right-def
  ucont-on-def
  LNil-def
  LZero-def
  plus-list-def

2.1 Evaluation laws for expressions

lemma *lit-ueval* [*ueval*]: $[\![\ll x \gg]\!]_e b = x$
  by (*transfer*, *simp*)

lemma *var-ueval* [*ueval*]: $[\![var\ x]\!]_e b = get_x\ b$
  by (*transfer*, *simp*)

lemma *uop-ueval* [*ueval*]: $[\![uop\ f\ x]\!]_e b = f\ ([\![x]\!]_e b)$
  by (*transfer*, *simp*)

lemma *bop-ueval* [*ueval*]: $[\![bop\ f\ x\ y]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)$
by (*transfer*, *simp*)

lemma *trop-ueval* [*ueval*]: $[\![trop\ f\ x\ y\ z]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)\ ([\![z]\!]_e b)$
by (*transfer*, *simp*)

lemma *qtop-ueval* [*ueval*]: $[\![qtop\ f\ x\ y\ z\ w]\!]_e b = f\ ([\![x]\!]_e b)\ ([\![y]\!]_e b)\ ([\![z]\!]_e b)\ ([\![w]\!]_e b)$
by (*transfer*, *simp*)

declare *uexpr-defs* [*ueval*]

2.2 Misc laws

lemma *tail-cons* [*simp*]: $tail_u(\langle x\rangle\ \hat{}_u\ xs) = xs$
by (*transfer*, *simp*)

2.3 Literalise tactics

The following tactic converts literal HOL expressions to UTP expressions and vice-versa via a collection of simplification rules. The two tactics are called "literalise", which converts UTP to expressions to HOL expressions – i.e. it pushes them into literals – and unliteralise that reverses this. We collect the equations in a theorem attribute called "lit_simps".

lemma *lit-num-simps* [*lit-simps*]: $\ll 0\gg\ = 0$ $\ll 1\gg\ = 1$ $\ll numeral\ n\gg\ = numeral\ n$ $\ll -\ x\gg\ = -\ \ll x\gg$
by (*simp-all add*: *ueval*, *transfer*, *simp*)

lemma *lit-arith-simps* [*lit-simps*]:
$\ll -\ x\gg\ = -\ \ll x\gg$
$\ll x + y\gg\ = \ll x\gg + \ll y\gg$ $\ll x - y\gg\ = \ll x\gg - \ll y\gg$
$\ll x * y\gg\ = \ll x\gg * \ll y\gg$ $\ll x\ /\ y\gg\ = \ll x\gg\ /\ \ll y\gg$
$\ll x\ div\ y\gg\ = \ll x\gg\ div\ \ll y\gg$
by (*simp add*: *uexpr-defs*, *transfer*, *simp*)+

lemma *lit-fun-simps* [*lit-simps*]:
$\ll i\ x\ y\ z\ u\gg\ = qtop\ i\ \ll x\gg\ \ll y\gg\ \ll z\gg\ \ll u\gg$
$\ll h\ x\ y\ z\gg\ = trop\ h\ \ll x\gg\ \ll y\gg\ \ll z\gg$
$\ll g\ x\ y\gg\ = bop\ g\ \ll x\gg\ \ll y\gg$
$\ll f\ x\gg\ = uop\ f\ \ll x\gg$
by (*transfer*, *simp*)+

In general unliteralising converts function applications to corresponding expression liftings. Since some operators, like + and *, have specific operators we also have to use $\alpha(?e) = 1_L$

$0 = \ll 0{::}?'a\gg$

$1 = \ll 1{::}?'a\gg$

$?u + ?v = bop\ op\ +\ ?u\ ?v$

$-\ ?u = uop\ uminus\ ?u$

$?u - ?v = bop\ op\ -\ ?u\ ?v$

$?u \cdot ?v = bop\ op\ \cdot\ ?u\ ?v$

$inverse\ ?u = uop\ inverse\ ?u$

$?u\ div\ ?v = bop\ op\ div\ ?u\ ?v$

$sgn\ ?u = uop\ sgn\ ?u$

$|?u| = uop\ abs\ ?u$

$?u\ mod\ ?v = bop\ op\ mod\ ?u\ ?v$
$(?x =_u ?y) = bop\ op = ?x\ ?y$
$numeral\ ?x = \ll numeral\ ?x \gg$
$ulim\text{-}left = (\lambda p.\ Lim\ (at\text{-}left\ p))$
$ulim\text{-}right = (\lambda p.\ Lim\ (at\text{-}right\ p))$
$ucont\text{-}on = (\lambda f\ A.\ continuous\text{-}on\ A\ f)$
$uempty = []$
$uempty = (0::?'a)$
op + = *op* @ in reverse to correctly interpret these. Moreover, numerals must be handled separately by first simplifying them and then converting them into UTP expression numerals; hence the following two simplification rules.

lemma *lit-numeral-1*: $uop\ numeral\ x = Abs\text{-}uexpr\ (\lambda b.\ numeral\ ([\![x]\!]_e\ b))$
by (*simp add*: *uop-def*)

lemma *lit-numeral-2*: $Abs\text{-}uexpr\ (\lambda\ b.\ numeral\ v) = numeral\ v$
by (*metis lit.abs-eq lit-num-simps*(*3*))

method *literalise* = (*unfold lit-simps*[*THEN sym*])
method *unliteralise* = (*unfold lit-simps uexpr-defs*[*THEN sym*];
(*unfold lit-numeral-1* ; (*unfold ueval*); (*unfold lit-numeral-2*))*?*)+
end

3 Unrestriction

theory *utp-unrest*
imports *utp-expr*
begin

Unrestriction is an encoding of semantic freshness, that allows us to reason about the presence of variables in predicates without being concerned with abstract syntax trees. An expression p is unrestricted by variable x, written $x \sharp p$, if altering the value of x has no effect on the valuation of p. This is a sufficient notion to prove many laws that would ordinarily rely on an *fv* function.

consts
$unrest :: 'a \Rightarrow 'b \Rightarrow bool$

syntax
$\text{-}unrest :: salpha \Rightarrow logic \Rightarrow logic \Rightarrow logic$ (**infix** ♯ *20*)

translations
$\text{-}unrest\ x\ p == CONST\ unrest\ x\ p$

named-theorems *unrest*

method *unrest-tac* = (*simp add*: *unrest*)*?*

lift-definition $unrest\text{-}upred :: ('a, '\alpha)\ uvar \Rightarrow ('b, '\alpha)\ uexpr \Rightarrow bool$
is $\lambda\ x\ e.\ \forall\ b\ v.\ e\ (put_x\ b\ v) = e\ b$.

definition $unrest\text{-}dvar\text{-}upred :: 'a::continuum\ dvar \Rightarrow ('b, '\alpha::vst)\ uexpr \Rightarrow bool$ **where**
$unrest\text{-}dvar\text{-}upred\ x\ P = unrest\text{-}upred\ (x\uparrow)\ P$

adhoc-overloading
unrest unrest-upred

lemma *unrest-var-comp* [*unrest*]:
$\llbracket\ x \sharp P;\ y \sharp P\ \rrbracket \Longrightarrow x;y \sharp P$
by (*transfer*, *simp add*: *lens-defs*)

lemma *unrest-lit* [*unrest*]: $x \sharp \ll v \gg$
by (*transfer*, *simp*)

The following law demonstrates why we need variable independence: a variable expression is unrestricted by another variable only when the two variables are independent.

lemma *unrest-var* [*unrest*]: $\llbracket\ vwb\text{-}lens\ x;\ x \bowtie y\ \rrbracket \Longrightarrow y \sharp\ var\ x$
by (*transfer*, *auto*)

lemma *unrest-iuvar* [*unrest*]: $\llbracket\ vwb\text{-}lens\ x;\ x \bowtie y\ \rrbracket \Longrightarrow \$y \sharp \$x$
by (*metis in-var-indep in-var-uvar unrest-var*)

lemma *unrest-ouvar* [*unrest*]: $\llbracket\ vwb\text{-}lens\ x;\ x \bowtie y\ \rrbracket \Longrightarrow \$y\acute{} \sharp \$x\acute{}$
by (*metis out-var-indep out-var-uvar unrest-var*)

lemma *unrest-iuvar-ouvar* [*unrest*]:
fixes $x :: ('a, '\alpha)\ uvar$
assumes *vwb-lens y*
shows $\$x \sharp \$y\acute{}$
by (*metis prod.collapse unrest-upred.rep-eq var.rep-eq var-lookup-out var-update-in*)

lemma *unrest-ouvar-iuvar* [*unrest*]:
fixes $x :: ('a, '\alpha)\ uvar$
assumes *vwb-lens y*
shows $\$x\acute{} \sharp \$y$
by (*metis prod.collapse unrest-upred.rep-eq var.rep-eq var-lookup-in var-update-out*)

lemma *unrest-uop* [*unrest*]: $x \sharp e \Longrightarrow x \sharp\ uop\ f\ e$
by (*transfer*, *simp*)

lemma *unrest-bop* [*unrest*]: $\llbracket\ x \sharp u;\ x \sharp v\ \rrbracket \Longrightarrow x \sharp\ bop\ f\ u\ v$
by (*transfer*, *simp*)

lemma *unrest-trop* [*unrest*]: $\llbracket\ x \sharp u;\ x \sharp v;\ x \sharp w\ \rrbracket \Longrightarrow x \sharp\ trop\ f\ u\ v\ w$
by (*transfer*, *simp*)

lemma *unrest-qtop* [*unrest*]: $\llbracket\ x \sharp u;\ x \sharp v;\ x \sharp w;\ x \sharp y\ \rrbracket \Longrightarrow x \sharp\ qtop\ f\ u\ v\ w\ y$
by (*transfer*, *simp*)

lemma *unrest-eq* [*unrest*]: $\llbracket\ x \sharp u;\ x \sharp v\ \rrbracket \Longrightarrow x \sharp u =_u v$
by (*simp add*: *eq-upred-def*, *transfer*, *simp*)

lemma *unrest-zero* [*unrest*]: $x \sharp\ 0$
by (*simp add*: *unrest-lit zero-uexpr-def*)

lemma *unrest-one* [*unrest*]: $x \sharp\ 1$
by (*simp add*: *one-uexpr-def unrest-lit*)

lemma *unrest-numeral* [*unrest*]: $x \sharp\ (numeral\ n)$

by (*simp add*: *numeral-uexpr-simp unrest-lit*)

lemma *unrest-sgn* [*unrest*]: $x \sharp u \Longrightarrow x \sharp sgn\ u$
by (*simp add*: *sgn-uexpr-def unrest-uop*)

lemma *unrest-abs* [*unrest*]: $x \sharp u \Longrightarrow x \sharp abs\ u$
by (*simp add*: *abs-uexpr-def unrest-uop*)

lemma *unrest-plus* [*unrest*]: $[\![\ x \sharp u;\ x \sharp v\ ]\!] \Longrightarrow x \sharp u + v$
by (*simp add*: *plus-uexpr-def unrest*)

lemma *unrest-uminus* [*unrest*]: $x \sharp u \Longrightarrow x \sharp - u$
by (*simp add*: *uminus-uexpr-def unrest*)

lemma *unrest-minus* [*unrest*]: $[\![\ x \sharp u;\ x \sharp v\ ]\!] \Longrightarrow x \sharp u - v$
by (*simp add*: *minus-uexpr-def unrest*)

lemma *unrest-times* [*unrest*]: $[\![\ x \sharp u;\ x \sharp v\ ]\!] \Longrightarrow x \sharp u * v$
by (*simp add*: *times-uexpr-def unrest*)

lemma *unrest-divide* [*unrest*]: $[\![\ x \sharp u;\ x \sharp v\ ]\!] \Longrightarrow x \sharp u\ /\ v$
by (*simp add*: *divide-uexpr-def unrest*)

lemma *unrest-ulambda* [*unrest*]:
$[\![\ uvar\ v;\ \bigwedge x.\ v \sharp F\ x\ ]\!] \Longrightarrow v \sharp (\lambda\ x \cdot F\ x)$
by (*transfer*, *simp*)

end

4 Substitution

theory *utp-subst*
imports
utp-expr
utp-unrest
begin

4.1 Substitution definitions

We introduce a polymorphic constant that will be used to represent application of a substitution, and also a set of theorems to represent laws.

consts
usubst :: $'s \Rightarrow 'a \Rightarrow 'b$ (**infixr** † *80*)

named-theorems *usubst*

A substitution is simply a transformation on the alphabet; it shows how variables should be mapped to different values.

type-synonym $('\alpha,'\beta)\ psubst = '\alpha\ alphabet \Rightarrow '\beta\ alphabet$
type-synonym $'\alpha\ usubst = '\alpha\ alphabet \Rightarrow '\alpha\ alphabet$

lift-definition *subst* :: $('\alpha,\ '\beta)\ psubst \Rightarrow ('a,\ '\beta)\ uexpr \Rightarrow ('a,\ '\alpha)\ uexpr$ **is**
$\lambda\ \sigma\ e\ b.\ e\ (\sigma\ b)$ **.**

adhoc-overloading
usubst subst

Update the value of a variable to an expression in a substitution

consts *subst-upd* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $'v$ $\Rightarrow$ $('a,\ '\alpha)$ *uexpr* $\Rightarrow$ $('\alpha,'\beta)$ *psubst*

definition *subst-upd-uvar* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $('a,\ '\beta)$ *uvar* $\Rightarrow$ $('a,\ '\alpha)$ *uexpr* $\Rightarrow$ $('\alpha,'\beta)$ *psubst* **where**
$subst\text{-}upd\text{-}uvar\ \sigma\ x\ v = (\lambda\ b.\ put_x\ (\sigma\ b)\ ([\![v]\!]_e b))$

definition *subst-upd-dvar* :: $('\alpha,'\beta{::}vst)$ *psubst* $\Rightarrow$ $'a{::}continuum$ *dvar* $\Rightarrow$ $('a,\ '\alpha)$ *uexpr* $\Rightarrow$ $('\alpha,'\beta)$ *psubst*
where
$subst\text{-}upd\text{-}dvar\ \sigma\ x\ v = subst\text{-}upd\text{-}uvar\ \sigma\ (x\uparrow)\ v$

adhoc-overloading
subst-upd subst-upd-uvar **and** *subst-upd subst-upd-dvar*

Lookup the expression associated with a variable in a substitution

lift-definition *usubst-lookup* :: $('\alpha,'\beta)$ *psubst* $\Rightarrow$ $('a,\ '\beta)$ *uvar* $\Rightarrow$ $('a,\ '\alpha)$ *uexpr* $(\langle\text{-}\rangle_s)$
is $\lambda\ \sigma\ x\ b.\ get_x\ (\sigma\ b)$ **.**

Relational lifting of a substitution to the first element of the state space

definition *unrest-usubst* :: $('a,\ '\alpha)$ *uvar* $\Rightarrow$ $'\alpha$ *usubst* $\Rightarrow$ *bool*
where $unrest\text{-}usubst\ x\ \sigma = (\forall\ \varrho\ v.\ \sigma\ (put_x\ \varrho\ v) = put_x\ (\sigma\ \varrho)\ v)$

adhoc-overloading
unrest unrest-usubst

nonterminal *smaplet* **and** *smaplets*

syntax
```
  -smaplet  :: [salpha, 'a] => smaplet              (- /↦s/ -)
            :: smaplet => smaplets                 (-)
  -SMaplets :: [smaplet, smaplets] => smaplets (-,/ -)
  -SubstUpd :: ['m usubst, smaplets] => 'm usubst (-/'(-') [900,0] 900)
  -Subst    :: smaplets => 'a ⇀ 'b                ((1[-]))
```

translations
```
  -SubstUpd m (-SMaplets xy ms)      == -SubstUpd (-SubstUpd m xy) ms
  -SubstUpd m (-smaplet x y)         == CONST subst-upd m x y
  -Subst ms                          == -SubstUpd (CONST id) ms
  -Subst (-SMaplets ms1 ms2)         <= -SubstUpd (-Subst ms1) ms2
  -SMaplets ms1 (-SMaplets ms2 ms3) <= -SMaplets (-SMaplets ms1 ms2) ms3
```

Deletion of a substitution maplet

definition *subst-del* :: $'\alpha$ *usubst* $\Rightarrow$ $('a,\ '\alpha)$ *uvar* $\Rightarrow$ $'\alpha$ *usubst* (**infix** $-_s$ *85*) **where**
$subst\text{-}del\ \sigma\ x = \sigma(x \mapsto_s \&x)$

4.2 Substitution laws

We set up a simple substitution tactic that applies substitution and unrestriction laws

method *subst-tac* = (*simp add*: *usubst unrest*)?

lemma *usubst-lookup-id* [*usubst*]: $\langle id\rangle_s\ x = var\ x$
by (*transfer*, *simp*)

lemma *usubst-lookup-upd* [*usubst*]:
  assumes *mwb-lens x*
  shows $\langle\sigma(x \mapsto_s v)\rangle_s\ x = v$
  using *assms*
  by (*simp add*: *subst-upd-uvar-def*, *transfer*) (*simp*)

lemma *usubst-upd-idem* [*usubst*]:
  assumes *mwb-lens x*
  shows $\sigma(x \mapsto_s u, x \mapsto_s v) = \sigma(x \mapsto_s v)$
  by (*simp add*: *subst-upd-uvar-def assms comp-def*)

lemma *usubst-upd-comm*:
  assumes $x \bowtie y$
  shows $\sigma(x \mapsto_s u, y \mapsto_s v) = \sigma(y \mapsto_s v, x \mapsto_s u)$
  using *assms*
  by (*rule-tac ext*, *auto simp add*: *subst-upd-uvar-def assms comp-def lens-indep-comm*)

lemma *usubst-upd-comm2*:
  assumes $z \bowtie y$ **and** *mwb-lens x*
  shows $\sigma(x \mapsto_s u, y \mapsto_s v, z \mapsto_s s) = \sigma(x \mapsto_s u, z \mapsto_s s, y \mapsto_s v)$
  using *assms*
  by (*rule-tac ext*, *auto simp add*: *subst-upd-uvar-def assms comp-def lens-indep-comm*)

lemma *swap-usubst-inj*:
  fixes $x\ y$ :: $('a, '\alpha)$ *uvar*
  assumes *vwb-lens x vwb-lens y* $x \bowtie y$
  shows *inj* $[x \mapsto_s \&y, y \mapsto_s \&x]$
  using *assms*
  apply (*auto simp add*: *inj-on-def subst-upd-uvar-def*)
  apply (*smt lens-indep-get lens-indep-sym var.rep-eq vwb-lens.put-eq vwb-lens-wb wb-lens-weak weak-lens.put-get*)
done

lemma *usubst-upd-var-id* [*usubst*]:
  vwb-lens x $\Longrightarrow [x \mapsto_s$ *var* $x] = id$
  apply (*simp add*: *subst-upd-uvar-def*)
  apply (*transfer*)
  apply (*rule ext*)
  apply (*auto*)
done

lemma *usubst-upd-comm-dash* [*usubst*]:
  fixes x :: $('a, '\alpha)$ *uvar*
  shows $\sigma(\$x\acute{} \mapsto_s v, \$x \mapsto_s u) = \sigma(\$x \mapsto_s u, \$x\acute{} \mapsto_s v)$
  using *out-in-indep usubst-upd-comm* **by** *blast*

lemma *usubst-lookup-upd-indep* [*usubst*]:
  assumes *mwb-lens x* $x \bowtie y$
  shows $\langle\sigma(y \mapsto_s v)\rangle_s\ x = \langle\sigma\rangle_s\ x$
  using *assms*
  by (*simp add*: *subst-upd-uvar-def*, *transfer*, *simp*)

lemma *usubst-apply-unrest* [*usubst*]:
  $[\![$ *vwb-lens x*; $x \sharp \sigma\ ]\!] \Longrightarrow \langle\sigma\rangle_s\ x =$ *var x*
  by (*simp add*: *unrest-usubst-def*, *transfer*, *auto simp add*: *fun-eq-iff*, *metis vwb-lens-wb wb-lens.get-put*

wb-lens-weak weak-lens.put-get)

lemma *subst-del-id* [*usubst*]:
  vwb-lens $x \Longrightarrow id -_s x = id$
  by (*simp add*: *subst-del-def subst-upd-uvar-def*, *transfer*, *auto*)

lemma *subst-del-upd-same* [*usubst*]:
  mwb-lens $x \Longrightarrow \sigma(x \mapsto_s v) -_s x = \sigma -_s x$
  by (*simp add*: *subst-del-def subst-upd-uvar-def*)

lemma *subst-del-upd-diff* [*usubst*]:
  $x \bowtie y \Longrightarrow \sigma(y \mapsto_s v) -_s x = (\sigma -_s x)(y \mapsto_s v)$
  by (*simp add*: *subst-del-def subst-upd-uvar-def lens-indep-comm*)

lemma *subst-unrest* [*usubst*]: $x \sharp P \Longrightarrow \sigma(x \mapsto_s v) \dagger P = \sigma \dagger P$
  by (*simp add*: *subst-upd-uvar-def*, *transfer*, *auto*)

lemma *subst-compose-upd* [*usubst*]: $x \sharp \sigma \Longrightarrow \sigma \circ \varrho(x \mapsto_s v) = (\sigma \circ \varrho)(x \mapsto_s v)$
  by (*simp add*: *subst-upd-uvar-def*, *transfer*, *auto simp add*: *unrest-usubst-def*)

lemma *id-subst* [*usubst*]: $id \dagger v = v$
  by (*transfer*, *simp*)

lemma *subst-lit* [*usubst*]: $\sigma \dagger \ll v \gg = \ll v \gg$
  by (*transfer*, *simp*)

lemma *subst-var* [*usubst*]: $\sigma \dagger var\ x = \langle\sigma\rangle_s\ x$
  by (*transfer*, *simp*)

lemma *unrest-usubst-del* [*unrest*]: $[\![$ *vwb-lens* x; $x \sharp (\langle\sigma\rangle_s\ x)$; $x \sharp \sigma -_s x$ $]\!] \Longrightarrow x \sharp (\sigma \dagger P)$
  by (*simp add*: *subst-del-def subst-upd-uvar-def unrest-upred-def unrest-usubst-def subst.rep-eq usubst-lookup.rep-eq*)
    (*metis vwb-lens.put-eq*)

We set up a purely syntactic order on variable lenses which is useful for the substitution normal form.

definition *var-name-ord* :: $('a, '\alpha)$ *uvar* $\Rightarrow$ $('b, '\alpha)$ *uvar* $\Rightarrow$ *bool* **where**
[*no-atp*]: *var-name-ord* $x\ y$ = *True*

syntax
  -var-name-ord :: *salpha* $\Rightarrow$ *salpha* $\Rightarrow$ *bool* (**infix** $\prec_v$ *65*)

translations
  -var-name-ord $x\ y$ == *CONST var-name-ord* $x\ y$

lemma *usubst-upd-comm-ord* [*usubst*]:
  assumes $x \bowtie y\ y \prec_v x$
  shows $\sigma(x \mapsto_s u, y \mapsto_s v) = \sigma(y \mapsto_s v, x \mapsto_s u)$
  by (*simp add*: *assms*(*1*) *usubst-upd-comm*)

We add the symmetric definition of input and output variables to substitution laws so that the variables are correctly normalised after substitution.

lemma *subst-uop* [*usubst*]: $\sigma \dagger uop\ f\ v = uop\ f\ (\sigma \dagger v)$
  by (*transfer*, *simp*)

lemma *subst-bop* [*usubst*]: $\sigma \dagger bop\ f\ u\ v = bop\ f\ (\sigma \dagger u)\ (\sigma \dagger v)$

by (*transfer*, *simp*)

lemma *subst-trop* [*usubst*]: $\sigma \dagger$ *trop f u v w* = *trop f* ($\sigma \dagger u$) ($\sigma \dagger v$) ($\sigma \dagger w$)
by (*transfer*, *simp*)

lemma *subst-qtop* [*usubst*]: $\sigma \dagger$ *qtop f u v w x* = *qtop f* ($\sigma \dagger u$) ($\sigma \dagger v$) ($\sigma \dagger w$) ($\sigma \dagger x$)
by (*transfer*, *simp*)

lemma *subst-plus* [*usubst*]: $\sigma \dagger (x + y) = \sigma \dagger x + \sigma \dagger y$
by (*simp add*: *plus-uexpr-def subst-bop*)

lemma *subst-times* [*usubst*]: $\sigma \dagger (x * y) = \sigma \dagger x * \sigma \dagger y$
by (*simp add*: *times-uexpr-def subst-bop*)

lemma *subst-mod* [*usubst*]: $\sigma \dagger (x \ mod \ y) = \sigma \dagger x \ mod \ \sigma \dagger y$
by (*simp add*: *mod-uexpr-def usubst*)

lemma *subst-div* [*usubst*]: $\sigma \dagger (x \ div \ y) = \sigma \dagger x \ div \ \sigma \dagger y$
by (*simp add*: *divide-uexpr-def usubst*)

lemma *subst-minus* [*usubst*]: $\sigma \dagger (x - y) = \sigma \dagger x - \sigma \dagger y$
by (*simp add*: *minus-uexpr-def subst-bop*)

lemma *subst-uminus* [*usubst*]: $\sigma \dagger (- x) = - (\sigma \dagger x)$
by (*simp add*: *uminus-uexpr-def subst-uop*)

lemma *usubst-sgn* [*usubst*]: $\sigma \dagger$ *sgn x* = *sgn* ($\sigma \dagger x$)
by (*simp add*: *sgn-uexpr-def subst-uop*)

lemma *usubst-abs* [*usubst*]: $\sigma \dagger$ *abs x* = *abs* ($\sigma \dagger x$)
by (*simp add*: *abs-uexpr-def subst-uop*)

lemma *subst-zero* [*usubst*]: $\sigma \dagger 0 = 0$
by (*simp add*: *zero-uexpr-def subst-lit*)

lemma *subst-one* [*usubst*]: $\sigma \dagger 1 = 1$
by (*simp add*: *one-uexpr-def subst-lit*)

lemma *subst-eq-upred* [*usubst*]: $\sigma \dagger (x =_u y) = (\sigma \dagger x =_u \sigma \dagger y)$
by (*simp add*: *eq-upred-def usubst*)

lemma *subst-subst* [*usubst*]: $\sigma \dagger \varrho \dagger e = (\varrho \circ \sigma) \dagger e$
by (*transfer*, *simp*)

lemma *subst-upd-comp* [*usubst*]:
fixes *x* :: ('*a*, 'α) *uvar*
shows $\varrho(x \mapsto_s v) \circ \sigma = (\varrho \circ \sigma)(x \mapsto_s \sigma \dagger v)$
by (*rule ext*, *simp add*:*uexpr-defs subst-upd-uvar-def*, *transfer*, *simp*)

nonterminal *uexprs* **and** *svars* **and** *salphas*

syntax
-psubst :: [*logic*, *svars*, *uexprs*] ⇒ *logic*
-subst :: *logic* ⇒ *uexprs* ⇒ *salphas* ⇒ *logic* ((-⟦-'/-⟧) [*999*,*0*,*0*] *1000*)
-uexprs :: [*logic*, *uexprs*] => *uexprs* (-,/ -)

:: *logic* => *uexprs* (-)
-svars :: [*svar*, *svars*] => *svars* (-,/ -)
:: *svar* => *svars* (-)
-salphas :: [*salpha*, *salphas*] => *salphas* (-,/ -)
:: *salpha* => *salphas* (-)

translations
-subst P es vs => *CONST subst* (*-psubst* (*CONST id*) *vs es*) *P*
-psubst m (*-salphas x xs*) (*-uexprs v vs*) => *-psubst* (*-psubst m x v*) *xs vs*
-psubst m x v => *CONST subst-upd m x v*
$P[\![v/\$x]\!]$ <= *CONST usubst* (*CONST subst-upd* (*CONST id*) (*CONST ivar x*) *v*) *P*
$P[\![v/\$x']\!]$ <= *CONST usubst* (*CONST subst-upd* (*CONST id*) (*CONST ovar x*) *v*) *P*
$P[\![v/x]\!]$ <= *CONST usubst* (*CONST subst-upd* (*CONST id*) *x v*) *P*

lemma *subst-singleton*:
fixes x :: $('a, '\alpha)$ *uvar*
assumes $x \sharp \sigma$
shows $\sigma(x \mapsto_s v) \dagger P = (\sigma \dagger P)[\![v/x]\!]$
using *assms*
by (*simp add*: *usubst*)

lemmas *subst-to-singleton* = *subst-singleton id-subst*

4.3 Unrestriction laws

lemma *unrest-usubst-single* [*unrest*]:
$[\![$ *mwb-lens* x; $x \sharp v$ $]\!] \Longrightarrow x \sharp P[\![v/x]\!]$
by (*transfer*, *auto simp add*: *subst-upd-uvar-def unrest-upred-def*)

lemma *unrest-usubst-id* [*unrest*]:
mwb-lens $x \Longrightarrow x \sharp id$
by (*simp add*: *unrest-usubst-def*)

lemma *unrest-usubst-upd* [*unrest*]:
$[\![\ x \bowtie y;\ x \sharp \sigma;\ x \sharp v\ ]\!] \Longrightarrow x \sharp \sigma(y \mapsto_s v)$
by (*simp add*: *subst-upd-uvar-def unrest-usubst-def unrest-upred.rep-eq lens-indep-comm*)

lemma *unrest-subst* [*unrest*]:
$[\![\ x \sharp P;\ x \sharp \sigma\ ]\!] \Longrightarrow x \sharp (\sigma \dagger P)$
by (*transfer*, *simp add*: *unrest-usubst-def*)

end

5 Alphabet manipulation

theory *utp-alphabet*
imports
utp-pred
begin

named-theorems *alpha*

method *alpha-tac* = (*simp add*: *alpha unrest*)?

5.1 Alphabet extension

Extend an alphabet by application of a lens that demonstrates how the smaller alphabet (β) injects into the larger alphabet (α).

lift-definition *aext* :: $('a, '\beta)$ *uexpr* $\Rightarrow$ $('\beta, '\alpha)$ *lens* $\Rightarrow$ $('a, '\alpha)$ *uexpr* (**infixr** $\oplus_p$ *95*)
is λ *P x b. P* (get_x *b*) **.**

lemma *aext-id* [*alpha*]: $P \oplus_p 1_L = P$
 by (*pred-auto*)

lemma *aext-lit* [*alpha*]: $\ll v \gg \oplus_p a = \ll v \gg$
 by (*pred-auto*)

lemma *aext-zero* [*alpha*]: $0 \oplus_p a = 0$
 by (*pred-auto*)

lemma *aext-one* [*alpha*]: $1 \oplus_p a = 1$
 by (*pred-auto*)

lemma *aext-numeral* [*alpha*]: $numeral\ n \oplus_p a = numeral\ n$
 by (*pred-auto*)

lemma *aext-uop* [*alpha*]: $uop\ f\ u \oplus_p a = uop\ f\ (u \oplus_p a)$
 by (*pred-auto*)

lemma *aext-bop* [*alpha*]: $bop\ f\ u\ v \oplus_p a = bop\ f\ (u \oplus_p a)\ (v \oplus_p a)$
 by (*pred-auto*)

lemma *aext-trop* [*alpha*]: $trop\ f\ u\ v\ w \oplus_p a = trop\ f\ (u \oplus_p a)\ (v \oplus_p a)\ (w \oplus_p a)$
 by (*pred-auto*)

lemma *aext-qtop* [*alpha*]: $qtop\ f\ u\ v\ w\ x \oplus_p a = qtop\ f\ (u \oplus_p a)\ (v \oplus_p a)\ (w \oplus_p a)\ (x \oplus_p a)$
 by (*pred-auto*)

lemma *aext-plus* [*alpha*]:
 $(x + y) \oplus_p a = (x \oplus_p a) + (y \oplus_p a)$
 by (*pred-auto*)

lemma *aext-minus* [*alpha*]:
 $(x - y) \oplus_p a = (x \oplus_p a) - (y \oplus_p a)$
 by (*pred-auto*)

lemma *aext-uminus* [*simp*]:
 $(- x) \oplus_p a = - (x \oplus_p a)$
 by (*pred-auto*)

lemma *aext-times* [*alpha*]:
 $(x * y) \oplus_p a = (x \oplus_p a) * (y \oplus_p a)$
 by (*pred-auto*)

lemma *aext-divide* [*alpha*]:
 $(x / y) \oplus_p a = (x \oplus_p a) / (y \oplus_p a)$
 by (*pred-auto*)

lemma *aext-var* [*alpha*]:

$var\ x \oplus_p a = var\ (x ;_L a)$
by (*pred-auto*)

lemma *aext-true* [*alpha*]: $true \oplus_p a = true$
by (*pred-auto*)

lemma *aext-false* [*alpha*]: $false \oplus_p a = false$
by (*pred-auto*)

lemma *aext-not* [*alpha*]: $(\neg\ P) \oplus_p x = (\neg\ (P \oplus_p x))$
by (*pred-auto*)

lemma *aext-and* [*alpha*]: $(P \wedge Q) \oplus_p x = (P \oplus_p x \wedge Q \oplus_p x)$
by (*pred-auto*)

lemma *aext-or* [*alpha*]: $(P \vee Q) \oplus_p x = (P \oplus_p x \vee Q \oplus_p x)$
by (*pred-auto*)

lemma *aext-imp* [*alpha*]: $(P \Rightarrow Q) \oplus_p x = (P \oplus_p x \Rightarrow Q \oplus_p x)$
by (*pred-auto*)

lemma *aext-iff* [*alpha*]: $(P \Leftrightarrow Q) \oplus_p x = (P \oplus_p x \Leftrightarrow Q \oplus_p x)$
by (*pred-auto*)

lemma *unrest-aext* [*unrest*]:
$[\![\ mwb\text{-}lens\ a;\ x \sharp p\ ]\!] \Longrightarrow unrest\ (x ;_L a)\ (p \oplus_p a)$
by (*transfer*, *simp add*: *lens-comp-def*)

lemma *unrest-aext-indep* [*unrest*]:
$a \bowtie b \Longrightarrow b \sharp (p \oplus_p a)$
by *pred-auto*

5.2 Alphabet restriction

Restrict an alphabet by application of a lens that demonstrates how the smaller alphabet (β) injects into the larger alphabet (α). Unlike extension, this operation can lose information if the expressions refers to variables in the larger alphabet.

lift-definition *arestr* :: $('a, '\alpha)\ uexpr \Rightarrow ('\beta, '\alpha)\ lens \Rightarrow ('a, '\beta)\ uexpr$ (**infixr** $\restriction_p$ *90*)
is $\lambda\ P\ x\ b.\ P\ (create_x\ b)$ **.**

lemma *arestr-id* [*alpha*]: $P \restriction_p 1_L = P$
by (*pred-auto*)

lemma *arestr-aext* [*simp*]: $mwb\text{-}lens\ a \Longrightarrow (P \oplus_p a) \restriction_p a = P$
by (*pred-auto*)

If an expression's alphabet can be divided into two disjoint sections and the expression does not depend on the second half then restricting the expression to the first half is lossless.

lemma *aext-arestr* [*alpha*]:
assumes $mwb\text{-}lens\ a\ bij\text{-}lens\ (a +_L b)\ a \bowtie b\ b \sharp P$
shows $(P \restriction_p a) \oplus_p a = P$
proof −
from *assms*(*2*) **have** $1_L \subseteq_L a +_L b$
by (*simp add*: *bij-lens-equiv-id lens-equiv-def*)

with *assms*(*1*,*3*,*4*) **show** *?thesis*
apply (*auto simp add*: *alpha-of-def id-lens-def lens-plus-def sublens-def lens-comp-def prod.case-eq-if*)
apply (*pred-auto*)
apply (*metis lens-indep-comm mwb-lens-weak weak-lens.put-get*)
done
qed

lemma *arestr-lit* [*alpha*]: $\ll v \gg \restriction_p a = \ll v \gg$
by (*pred-auto*)

lemma *arestr-zero* [*alpha*]: $0 \restriction_p a = 0$
by (*pred-auto*)

lemma *arestr-one* [*alpha*]: $1 \restriction_p a = 1$
by (*pred-auto*)

lemma *arestr-numeral* [*alpha*]: $numeral\ n \restriction_p a = numeral\ n$
by (*pred-auto*)

lemma *arestr-var* [*alpha*]:
$var\ x \restriction_p a = var\ (x /_L a)$
by (*pred-auto*)

lemma *arestr-true* [*alpha*]: $true \restriction_p a = true$
by (*pred-auto*)

lemma *arestr-false* [*alpha*]: $false \restriction_p a = false$
by (*pred-auto*)

lemma *arestr-not* [*alpha*]: $(\neg P)\restriction_p a = (\neg (P\restriction_p a))$
by (*pred-auto*)

lemma *arestr-and* [*alpha*]: $(P \wedge Q)\restriction_p x = (P\restriction_p x \wedge Q\restriction_p x)$
by (*pred-auto*)

lemma *arestr-or* [*alpha*]: $(P \vee Q)\restriction_p x = (P\restriction_p x \vee Q\restriction_p x)$
by (*pred-auto*)

lemma *arestr-imp* [*alpha*]: $(P \Rightarrow Q)\restriction_p x = (P\restriction_p x \Rightarrow Q\restriction_p x)$
by (*pred-auto*)

5.3 Alphabet lens laws

lemma *alpha-in-var* [*alpha*]: $x ;_L fst_L = in\text{-}var\ x$
by (*simp add*: *in-var-def*)

lemma *alpha-out-var* [*alpha*]: $x ;_L snd_L = out\text{-}var\ x$
by (*simp add*: *out-var-def*)

lemma *in-var-prod-lens* [*alpha*]:
$wb\text{-}lens\ Y \Longrightarrow in\text{-}var\ x ;_L (X \times_L Y) = in\text{-}var\ (x ;_L X)$
by (*simp add*: *in-var-def prod-as-plus lens-comp-assoc fst-lens-plus*)

lemma *out-var-prod-lens* [*alpha*]:
$wb\text{-}lens\ X \Longrightarrow out\text{-}var\ x ;_L (X \times_L Y) = out\text{-}var\ (x ;_L Y)$
apply (*simp add*: *out-var-def prod-as-plus lens-comp-assoc*)

apply (*subst snd-lens-prod*)
using *comp-wb-lens fst-vwb-lens vwb-lens-wb* **apply** *blast*
apply (*simp add*: *alpha-in-var alpha-out-var*)
apply (*simp*)
done

5.4 Alphabet coercion

definition *id-on* :: $('a \Longrightarrow '\alpha) \Rightarrow '\alpha \Rightarrow '\alpha$ **where**
[*upred-defs*]: $id\text{-}on\ x = (\lambda\ s.\ undefined \oplus_L s\ on\ x)$

definition *alpha-coerce* :: $('a \Longrightarrow '\alpha) \Rightarrow '\alpha\ upred \Rightarrow '\alpha\ upred$
where [*upred-defs*]: $alpha\text{-}coerce\ x\ P = id\text{-}on\ x \dagger P$

syntax
-alpha-coerce :: $salpha \Rightarrow logic \Rightarrow logic$ ($!_\alpha$ - • - [*0*, *10*] *10*)

translations
-alpha-coerce P x == *CONST alpha-coerce P x*

5.5 Substitution alphabet extension

definition *subst-ext* :: $'\alpha\ usubst \Rightarrow ('\alpha \Longrightarrow '\beta) \Rightarrow '\beta\ usubst$ (**infix** $\oplus_s$ *65*) **where**
[*upred-defs*]: $\sigma \oplus_s x = (\lambda\ s.\ put_x\ s\ (\sigma\ (get_x\ s)))$

lemma *id-subst-ext* [*usubst*,*alpha*]:
$vwb\text{-}lens\ x \Longrightarrow id \oplus_s x = id$
by *pred-auto*

lemma *upd-subst-ext* [*alpha*]:
$vwb\text{-}lens\ x \Longrightarrow \sigma(y \mapsto_s v) \oplus_s x = (\sigma \oplus_s x)(\&x{:}y \mapsto_s v \oplus_p x)$
by *pred-auto*

lemma *apply-subst-ext* [*alpha*]:
$vwb\text{-}lens\ x \Longrightarrow (\sigma \dagger e) \oplus_p x = (\sigma \oplus_s x) \dagger (e \oplus_p x)$
by (*pred-auto*)

lemma *aext-upred-eq* [*alpha*]:
$((e =_u f) \oplus_p a) = ((e \oplus_p a) =_u (f \oplus_p a))$
by (*pred-auto*)

5.6 Substitution alphabet restriction

definition *subst-res* :: $'\alpha\ usubst \Rightarrow ('\beta \Longrightarrow '\alpha) \Rightarrow '\beta\ usubst$ (**infix** $\restriction_s$ *65*) **where**
[*upred-defs*]: $\sigma \restriction_s x = (\lambda\ s.\ get_x\ (\sigma\ (create_x\ s)))$

lemma *id-subst-res* [*alpha*,*usubst*]:
$mwb\text{-}lens\ x \Longrightarrow id \restriction_s x = id$
by *pred-auto*

lemma *upd-subst-res* [*alpha*]:
$vwb\text{-}lens\ x \Longrightarrow \sigma(\&x{:}y \mapsto_s v) \restriction_s x = (\sigma \restriction_s x)(\&y \mapsto_s v \restriction_p x)$
by (*pred-auto*)

lemma *subst-ext-res* [*alpha*,*usubst*]:
$vwb\text{-}lens\ x \Longrightarrow (\sigma \oplus_s x) \restriction_s x = \sigma$

by (*pred-auto*)

lemma *unrest-subst-alpha-ext* [*unrest*]:
$x \bowtie y \Longrightarrow x \sharp (P \oplus_s y)$
by (*pred-auto*, *metis lens-indep-def*)

end

6 Lifting expressions

theory *utp-lift*
imports
utp-alphabet
begin

6.1 Lifting definitions

We define operators for converting an expression to and from a relational state space

abbreviation *lift-pre* :: $('a, '\alpha)$ *uexpr* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uexpr* $(\lceil \text{-} \rceil_<)$
where $\lceil P \rceil_< \equiv P \oplus_p fst_L$

abbreviation *drop-pre* :: $('a, '\alpha \times '\beta)$ *uexpr* $\Rightarrow$ $('a, '\alpha)$ *uexpr* $(\lfloor \text{-} \rfloor_<)$
where $\lfloor P \rfloor_< \equiv P \restriction_p fst_L$

abbreviation *lift-post* :: $('a, '\beta)$ *uexpr* $\Rightarrow$ $('a, '\alpha \times '\beta)$ *uexpr* $(\lceil \text{-} \rceil_>)$
where $\lceil P \rceil_> \equiv P \oplus_p snd_L$

abbreviation *drop-post* :: $('a, '\alpha \times '\beta)$ *uexpr* $\Rightarrow$ $('a, '\beta)$ *uexpr* $(\lfloor \text{-} \rfloor_>)$
where $\lfloor P \rfloor_> \equiv P \restriction_p snd_L$

6.2 Lifting laws

lemma *lift-pre-var* [*simp*]:
$\lceil var\ x \rceil_< = \$x$
by (*alpha-tac*)

lemma *lift-post-var* [*simp*]:
$\lceil var\ x \rceil_> = \$x´$
by (*alpha-tac*)

6.3 Unrestriction laws

lemma *unrest-dash-var-pre* [*unrest*]:
fixes $x :: ('a, '\alpha)$ *uvar*
shows $\$x´ \sharp \lceil p \rceil_<$
by (*pred-auto*)

end

7 Alphabetised Predicates

theory *utp-pred*

imports
utp-expr
utp-subst
begin

An alphabetised predicate is a simply a boolean valued expression

type-synonym $'\alpha$ *upred* = (*bool*, $'\alpha$) *uexpr*

translations
(*type*) $'\alpha$ *upred* <= (*type*) (*bool*, $'\alpha$) *uexpr*

7.1 Automatic Tactics

named-theorems *upred-defs*

We set up several automatic tactics that recast theorems on UTP predicates into equivalent HOL predicates, eliminating artefacts of the mechanisation as much as this is possible. Our approach is first to unfold all relevant definition of the UTP predicate model, then perform a transfer, and finally simplify by using lens and variable definitions, the split laws of alphabet records, and interpretation laws to convert record-based state spaces into products. The definition of the methods is facilitated by the Eisbach tool.

Without re-interpretation of lens types in state spaces (legacy).

method *pred-simp'* = (
(*unfold upred-defs*)?,
(*transfer*),
(*simp add*: *fun-eq-iff*
lens-defs uvar-defs upred-defs alpha-splits Product-Type.split-beta)?,
(*clarsimp*)?)

Variations that adjoin *pred-simp'* with automatic tactics.

method *pred-auto'* = (*pred-simp'*, *auto?*)
method *pred-blast'* = (*pred-simp'*; *blast*)

With reinterpretation of lens types in state spaces (default).

method *pred-simp* = (
(*unfold upred-defs*)?,
(*transfer*),
(*simp add*: *fun-eq-iff*
lens-defs uvar-defs upred-defs alpha-splits Product-Type.split-beta)?,
(*simp add*: *lens-interp-laws*)?,
(*clarsimp*)?)

Variations that adjoin *pred-simp* with automatic tactics.

method *pred-auto* = (*pred-simp*, *auto?*)
method *pred-blast* = (*pred-simp*; *blast*)

— TODO: Rename *pred-auto* into *pred-auto*.

7.2 Predicate syntax

We want to remain as close as possible to the mathematical UTP syntax, but also want to be conservative with HOL. For this reason we chose not to steal syntax from HOL, but where possible use polymorphism to allow selection of the appropriate operator (UTP vs. HOL). Thus we will first remove the standard syntax for conjunction, disjunction, and negation, and replace these with adhoc overloaded definitions.

no-notation
  conj (**infixr** $\wedge$ *35*) **and**
  disj (**infixr** $\vee$ *30*) **and**
  Not ($\neg$ - [*40*] *40*)

consts
  utrue :: $'a$ (*true*)
  ufalse :: $'a$ (*false*)
  uconj :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\wedge$ *35*)
  udisj :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\vee$ *30*)
  uimpl :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\Rightarrow$ *25*)
  uiff :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\Leftrightarrow$ *25*)
  unot :: $'a \Rightarrow 'a$ ($\neg$ - [*40*] *40*)
  uex :: $('a, '\alpha)$ *uvar* $\Rightarrow 'p \Rightarrow 'p$
  uall :: $('a, '\alpha)$ *uvar* $\Rightarrow 'p \Rightarrow 'p$
  ushEx :: $['a \Rightarrow 'p] \Rightarrow 'p$
  ushAll :: $['a \Rightarrow 'p] \Rightarrow 'p$

adhoc-overloading
  uconj conj **and**
  udisj disj **and**
  unot Not

We set up two versions of each of the quantifiers: *uex* / *uall* and *ushEx* / *ushAll*. The former pair allows quantification of UTP variables, whilst the latter allows quantification of HOL variables. Both varieties will be needed at various points. Syntactically they are distinguish by a boldface quantifier for the HOL versions (achieved by the "bold" escape in Isabelle).

nonterminal *idt-list*

syntax
  -idt-el :: *idt* $\Rightarrow$ *idt-list* (-)
  -idt-list :: *idt* $\Rightarrow$ *idt-list* $\Rightarrow$ *idt-list* ((-,/ -) [*0*, *1*])
  -uex :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\exists$ - $\cdot$ - [*0*, *10*] *10*)
  -uall :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - $\cdot$ - [*0*, *10*] *10*)
  -ushEx :: *idt-list* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\exists$ - $\cdot$ - [*0*, *10*] *10*)
  -ushAll :: *idt-list* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - $\cdot$ - [*0*, *10*] *10*)
  -ushBEx :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\exists$ - $\in$ - $\cdot$ - [*0*, *0*, *10*] *10*)
  -ushBAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - $\in$ - $\cdot$ - [*0*, *0*, *10*] *10*)
  -ushGAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - | - $\cdot$ - [*0*, *0*, *10*] *10*)
  -ushGtAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - > - $\cdot$ - [*0*, *0*, *10*] *10*)
  -ushLtAll :: *idt* $\Rightarrow$ *logic* $\Rightarrow$ *logic* $\Rightarrow$ *logic* ($\forall$ - < - $\cdot$ - [*0*, *0*, *10*] *10*)

translations
  -uex x P == *CONST uex x P*
  -uall x P == *CONST uall x P*
  -ushEx (*-idt-el x*) *P* == *CONST ushEx* (λ *x*. *P*)
  -ushEx (*-idt-list x y*) *P* => *CONST ushEx* (λ *x*. (*-ushEx y P*))
  $\exists\ x \in A \cdot P$ => $\exists\ x \cdot «x» \in_u A \wedge P$
  -ushAll (*-idt-el x*) *P* == *CONST ushAll* (λ *x*. *P*)
  -ushAll (*-idt-list x y*) *P* => *CONST ushAll* (λ *x*. (*-ushAll y P*))
  $\forall\ x \in A \cdot P$ => $\forall\ x \cdot «x» \in_u A \Rightarrow P$

$\forall\ x \mid P \cdot Q$ => $\forall\ x \cdot P \Rightarrow Q$
$\forall\ x > y \cdot P$ => $\forall\ x \cdot «x» >_u y \Rightarrow P$
$\forall\ x < y \cdot P$ => $\forall\ x \cdot «x» <_u y \Rightarrow P$

7.3 Predicate operators

We chose to maximally reuse definitions and laws built into HOL. For this reason, when introducing the core operators we proceed by lifting operators from the polymorphic algebraic hiearchy of HOL. Thus the initial definitions take place in the context of type class instantiations. We first introduce our own class called *refine* that will add the refinement operator syntax to the HOL partial order class.

class *refine* = *order*

abbreviation *refineBy* :: *'a::refine* ⇒ *'a* ⇒ *bool* (**infix** ⊑ *50*) **where**
P ⊑ *Q* ≡ *less-eq Q P*

Since, on the whole, lattices in UTP are the opposite way up to the standard definitions in HOL, we syntactically invert the lattice operators. This is the one exception where we do steal HOL syntax, but I think it makes sense for UTP.

no-notation *inf* (**infixl** ⊓ *70*)
notation *inf* (**infixl** ⊔ *70*)
no-notation *sup* (**infixl** ⊔ *65*)
notation *sup* (**infixl** ⊓ *65*)

no-notation *Inf* (⨅- [*900*] *900*)
notation *Inf* (⨆- [*900*] *900*)
no-notation *Sup* (⨆- [*900*] *900*)
notation *Sup* (⨅- [*900*] *900*)

no-notation *bot* (⊥)
notation *bot* (⊤)
no-notation *top* (⊤)
notation *top* (⊥)

no-syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨅-./ -) [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨅-∈-./ -) [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨆-./ -) [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨆-∈-./ -) [*0*, *0*, *10*] *10*)

syntax
-INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨆-./ -) [*0*, *10*] *10*)
-INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨆-∈-./ -) [*0*, *0*, *10*] *10*)
-SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨅-./ -) [*0*, *10*] *10*)
-SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨅-∈-./ -) [*0*, *0*, *10*] *10*)

We trivially instantiate our refinement class

instance *uexpr* :: (*order*, *type*) *refine* **..**

Next we introduce the lattice operators, which is again done by lifting.

instantiation *uexpr* :: (*lattice*, *type*) *lattice*
begin
lift-definition *sup-uexpr* :: (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr* ⇒ (*'a*, *'b*) *uexpr*

is $\lambda P\ Q\ A.\ sup\ (P\ A)\ (Q\ A)$ **.**
lift-definition *inf-uexpr* :: $('a,\ 'b)\ uexpr \Rightarrow ('a,\ 'b)\ uexpr \Rightarrow ('a,\ 'b)\ uexpr$
is $\lambda P\ Q\ A.\ inf\ (P\ A)\ (Q\ A)$ **.**
instance
by (*intro-classes*) (*transfer*, *auto*)+
end

instantiation *uexpr* :: (*bounded-lattice*, *type*) *bounded-lattice*
begin
lift-definition *bot-uexpr* :: $('a,\ 'b)\ uexpr$ **is** $\lambda\ A.\ bot$ **.**
lift-definition *top-uexpr* :: $('a,\ 'b)\ uexpr$ **is** $\lambda\ A.\ top$ **.**
instance
by (*intro-classes*) (*transfer*, *auto*)+
end

Finally we show that predicates form a Boolean algebra (under the lattice operators).

instance *uexpr* :: (*boolean-algebra*, *type*) *boolean-algebra*
apply (*intro-classes*, *unfold uexpr-defs*; *transfer*, *rule ext*)
apply (*simp-all add*: *sup-inf-distrib1 diff-eq*)
done

instantiation *uexpr* :: (*complete-lattice*, *type*) *complete-lattice*
begin
lift-definition *Inf-uexpr* :: $('a,\ 'b)\ uexpr\ set \Rightarrow ('a,\ 'b)\ uexpr$
is $\lambda\ PS\ A.\ INF\ P{:}PS.\ P(A)$ **.**
lift-definition *Sup-uexpr* :: $('a,\ 'b)\ uexpr\ set \Rightarrow ('a,\ 'b)\ uexpr$
is $\lambda\ PS\ A.\ SUP\ P{:}PS.\ P(A)$ **.**
instance
by (*intro-classes*)
(*transfer*, *auto intro*: *INF-lower SUP-upper simp add*: *INF-greatest SUP-least*)+
end

With the lattice operators defined, we can proceed to give definitions for the standard predicate operators in terms of them.

definition *true-upred* $= (top :: '\alpha\ upred)$
definition *false-upred* $= (bot :: '\alpha\ upred)$
definition *conj-upred* $= (inf :: '\alpha\ upred \Rightarrow '\alpha\ upred \Rightarrow '\alpha\ upred)$
definition *disj-upred* $= (sup :: '\alpha\ upred \Rightarrow '\alpha\ upred \Rightarrow '\alpha\ upred)$
definition *not-upred* $= (uminus :: '\alpha\ upred \Rightarrow '\alpha\ upred)$
definition *diff-upred* $= (minus :: '\alpha\ upred \Rightarrow '\alpha\ upred \Rightarrow '\alpha\ upred)$

notation
conj-upred (**infixr** $\wedge_p$ *35*) **and**
disj-upred (**infixr** $\vee_p$ *30*)

lift-definition *USUP* :: $('a \Rightarrow '\alpha\ upred) \Rightarrow ('a \Rightarrow ('b{::}complete\text{-}lattice,\ '\alpha)\ uexpr) \Rightarrow ('b,\ '\alpha)\ uexpr$
is $\lambda\ P\ F\ b.\ Sup\ \{[\![F\ x]\!]_e b \mid x.\ [\![P\ x]\!]_e b\}$ **.**

lift-definition *UINF* :: $('a \Rightarrow '\alpha\ upred) \Rightarrow ('a \Rightarrow ('b{::}complete\text{-}lattice,\ '\alpha)\ uexpr) \Rightarrow ('b,\ '\alpha)\ uexpr$
is $\lambda\ P\ F\ b.\ Inf\ \{[\![F\ x]\!]_e b \mid x.\ [\![P\ x]\!]_e b\}$ **.**

declare *USUP-def* [*upred-defs*]
declare *UINF-def* [*upred-defs*]

syntax

-$USup$ :: $idt \Rightarrow logic \Rightarrow logic$ ($\bigsqcap$ - $\cdot$ - [0, 10] 10)
-$USup$-mem :: $idt \Rightarrow logic \Rightarrow logic \Rightarrow logic$ ($\bigsqcap$ - $\in$ - $\cdot$ - [0, 10] 10)
-$USUP$ :: $idt \Rightarrow logic \Rightarrow logic \Rightarrow logic$ ($\bigsqcap$ - | - $\cdot$ - [0, 0, 10] 10)
-$UInf$ :: $idt \Rightarrow logic \Rightarrow logic$ ($\bigsqcup$ - $\cdot$ - [0, 10] 10)
-$UInf$-mem :: $idt \Rightarrow logic \Rightarrow logic \Rightarrow logic$ ($\bigsqcup$ - $\in$ - $\cdot$ - [0, 10] 10)
-$UINF$ :: $idt \Rightarrow logic \Rightarrow logic \Rightarrow logic$ ($\bigsqcup$ - | - $\cdot$ - [0, 10] 10)

translations
$\bigsqcap\ x \mid P \cdot F$ => $CONST\ USUP\ (\lambda\ x.\ P)\ (\lambda\ x.\ F)$
$\bigsqcap\ x \cdot F$ == $\bigsqcap\ x \mid true \cdot F$
$\bigsqcap\ x \cdot F$ == $\bigsqcap\ x \mid true \cdot F$
$\bigsqcap\ x \in A \cdot F$ => $\bigsqcap\ x \mid \ll x \gg \in_u \ll A \gg \cdot F$
$\bigsqcap\ x \mid P \cdot F$ <= $CONST\ USUP\ (\lambda\ x.\ P)\ (\lambda\ y.\ F)$
$\bigsqcup\ x \mid P \cdot F$ => $CONST\ UINF\ (\lambda\ x.\ P)\ (\lambda\ x.\ F)$
$\bigsqcup\ x \cdot F$ == $\bigsqcup\ x \mid true \cdot F$
$\bigsqcup\ x \in A \cdot F$ => $\bigsqcup\ x \mid \ll x \gg \in_u \ll A \gg \cdot F$
$\bigsqcup\ x \mid P \cdot F$ <= $CONST\ UINF\ (\lambda\ x.\ P)\ (\lambda\ y.\ F)$

We also define the other predicate operators

lift-definition $impl$::$'\alpha\ upred \Rightarrow\ '\alpha\ upred \Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ P\ Q\ A.\ P\ A \longrightarrow Q\ A$ **.**

lift-definition iff-$upred$ ::$'\alpha\ upred \Rightarrow\ '\alpha\ upred \Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ P\ Q\ A.\ P\ A \longleftrightarrow Q\ A$ **.**

lift-definition ex :: $('a,\ '\alpha)\ uvar \Rightarrow\ '\alpha\ upred \Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ x\ P\ b.\ (\exists\ v.\ P(put_x\ b\ v))$ **.**

lift-definition $shEx$ ::[$'\beta \Rightarrow '\alpha\ upred$] $\Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ P\ A.\ \exists\ x.\ (P\ x)\ A$ **.**

lift-definition all :: $('a,\ '\alpha)\ uvar \Rightarrow\ '\alpha\ upred \Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ x\ P\ b.\ (\forall\ v.\ P(put_x\ b\ v))$ **.**

lift-definition $shAll$ ::[$'\beta \Rightarrow '\alpha\ upred$] $\Rightarrow\ '\alpha\ upred$ **is**
$\lambda\ P\ A.\ \forall\ x.\ (P\ x)\ A$ **.**

We have to add a u subscript to the closure operator as I don't want to override the syntax for HOL lists (we'll be using them later).

lift-definition $closure$::$'\alpha\ upred \Rightarrow\ '\alpha\ upred$ ([-]$_u$) **is**
$\lambda\ P\ A.\ \forall A'.\ P\ A'$ **.**

lift-definition $taut$ :: $'\alpha\ upred \Rightarrow bool$ ('-')
is $\lambda\ P.\ \forall\ A.\ P\ A$ **.**

adhoc-overloading
utrue true-upred **and**
ufalse false-upred **and**
unot not-upred **and**
uconj conj-upred **and**
udisj disj-upred **and**
uimpl impl **and**
uiff iff-upred **and**
uex ex **and**
uall all **and**

ushEx shEx **and**
ushAll shAll

syntax
-uneq :: *logic* ⇒ *logic* ⇒ *logic* (**infixl** $\neq_u$ *50*)
-unmem :: (*'a*, *'α*) *uexpr* ⇒ (*'a set*, *'α*) *uexpr* ⇒ (*bool*, *'α*) *uexpr* (**infix** $\notin_u$ *50*)

translations
$x \neq_u y == CONST\ unot\ (x =_u y)$
$x \notin_u A == CONST\ unot\ (CONST\ bop\ (op \in)\ x\ A)$

declare *true-upred-def* [*upred-defs*]
declare *false-upred-def* [*upred-defs*]
declare *conj-upred-def* [*upred-defs*]
declare *disj-upred-def* [*upred-defs*]
declare *not-upred-def* [*upred-defs*]
declare *diff-upred-def* [*upred-defs*]
declare *subst-upd-uvar-def* [*upred-defs*]
declare *subst-upd-dvar-def* [*upred-defs*]
declare *unrest-usubst-def* [*upred-defs*]
declare *uexpr-defs* [*upred-defs*]

lemma *true-alt-def*: *true* = «*True*»
by (*pred-auto*)

lemma *false-alt-def*: *false* = «*False*»
by (*pred-auto*)

declare *true-alt-def*[*THEN sym*,*lit-simps*]
declare *false-alt-def*[*THEN sym*,*lit-simps*]

7.4 Unrestriction Laws

lemma *unrest-true* [*unrest*]: $x \sharp true$
by (*pred-auto*)

lemma *unrest-false* [*unrest*]: $x \sharp false$
by (*pred-auto*)

lemma *unrest-conj* [*unrest*]: $[\![\ x \sharp (P :: 'α\ upred); x \sharp Q\ ]\!] \Longrightarrow x \sharp P \wedge Q$
by (*pred-auto*)

lemma *unrest-disj* [*unrest*]: $[\![\ x \sharp (P :: 'α\ upred); x \sharp Q\ ]\!] \Longrightarrow x \sharp P \vee Q$
by (*pred-auto*)

lemma *unrest-USUP* [*unrest*]:
$[\![\ (\bigwedge i.\ x \sharp P(i)); (\bigwedge i.\ x \sharp Q(i))\ ]\!] \Longrightarrow x \sharp (\bigsqcap\ i \mid P(i) \cdot Q(i))$
by *pred-auto*

lemma *unrest-UINF* [*unrest*]:
$[\![\ (\bigwedge i.\ x \sharp P(i)); (\bigwedge i.\ x \sharp Q(i))\ ]\!] \Longrightarrow x \sharp (\bigsqcup\ i \mid P(i) \cdot Q(i))$
by *pred-auto*

lemma *unrest-impl* [*unrest*]: $[\![\ x \sharp P; x \sharp Q\ ]\!] \Longrightarrow x \sharp P \Rightarrow Q$
by (*pred-auto*)

lemma *unrest-iff* [*unrest*]: $\llbracket\ x \sharp P;\ x \sharp Q\ \rrbracket \Longrightarrow x \sharp P \Leftrightarrow Q$
  by (*pred-auto*)

lemma *unrest-not* [*unrest*]: $x \sharp (P :: 'α\ upred) \Longrightarrow x \sharp (\neg P)$
  by (*pred-auto*)

The sublens proviso can be thought of as membership below.

lemma *unrest-ex-in* [*unrest*]:
  $\llbracket\ mwb\text{-}lens\ y;\ x \subseteq_L\ y\ \rrbracket \Longrightarrow x \sharp (\exists\ y \cdot P)$
  by (*pred-auto*)

declare *sublens-refl* [*simp*]
declare *lens-plus-ub* [*simp*]
declare *lens-plus-right-sublens* [*simp*]
declare *comp-wb-lens* [*simp*]
declare *comp-mwb-lens* [*simp*]
declare *plus-mwb-lens* [*simp*]

lemma *unrest-ex-diff* [*unrest*]:
  assumes $x \bowtie y\ y \sharp P$
  shows $y \sharp (\exists\ x \cdot P)$
  using *assms*
  apply (*pred-auto*)
  using *lens-indep-comm* **apply** *fastforce+*
done

lemma *unrest-all-in* [*unrest*]:
  $\llbracket\ mwb\text{-}lens\ y;\ x \subseteq_L\ y\ \rrbracket \Longrightarrow x \sharp (\forall\ y \cdot P)$
  by *pred-auto*

lemma *unrest-all-diff* [*unrest*]:
  assumes $x \bowtie y\ y \sharp P$
  shows $y \sharp (\forall\ x \cdot P)$
  using *assms*
  by (*pred-auto*, *simp-all add*: *lens-indep-comm*)

lemma *unrest-shEx* [*unrest*]:
  assumes $\bigwedge\ y.\ x \sharp P(y)$
  shows $x \sharp (\exists\ y \cdot P(y))$
  using *assms* **by** *pred-auto*

lemma *unrest-shAll* [*unrest*]:
  assumes $\bigwedge\ y.\ x \sharp P(y)$
  shows $x \sharp (\forall\ y \cdot P(y))$
  using *assms* **by** *pred-auto*

lemma *unrest-closure* [*unrest*]:
  $x \sharp [P]_u$
  by *pred-auto*

7.5 Substitution Laws

lemma *subst-true* [*usubst*]: $\sigma \dagger true = true$
  by (*pred-auto*)

lemma *subst-false* [*usubst*]: $\sigma \dagger false = false$

by (*pred-auto*)

lemma *subst-not* [*usubst*]: $\sigma \dagger (\neg\ P) = (\neg\ \sigma \dagger P)$
by (*pred-auto*)

lemma *subst-impl* [*usubst*]: $\sigma \dagger (P \Rightarrow\ Q) = (\sigma \dagger P \Rightarrow \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-iff* [*usubst*]: $\sigma \dagger (P \Leftrightarrow\ Q) = (\sigma \dagger P \Leftrightarrow \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-disj* [*usubst*]: $\sigma \dagger (P \vee\ Q) = (\sigma \dagger P \vee \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-conj* [*usubst*]: $\sigma \dagger (P \wedge\ Q) = (\sigma \dagger P \wedge \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-sup* [*usubst*]: $\sigma \dagger (P \sqcap\ Q) = (\sigma \dagger P \sqcap \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-inf* [*usubst*]: $\sigma \dagger (P \sqcup\ Q) = (\sigma \dagger P \sqcup \sigma \dagger Q)$
by (*pred-auto*)

lemma *subst-USUP* [*usubst*]: $\sigma \dagger (\bigsqcap\ i \mid P(i) \cdot Q(i)) = (\bigsqcap\ i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$
by (*simp add*: *USUP-def*, *pred-auto*)

lemma *subst-UINF* [*usubst*]: $\sigma \dagger (\bigsqcup\ i \mid P(i) \cdot Q(i)) = (\bigsqcup\ i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$
by (*simp add*: *UINF-def*, *pred-auto*)

lemma *subst-closure* [*usubst*]: $\sigma \dagger [P]_u = [P]_u$
by (*pred-auto*)

lemma *subst-shEx* [*usubst*]: $\sigma \dagger (\exists\ x \cdot P(x)) = (\exists\ x \cdot \sigma \dagger P(x))$
by *pred-auto*

lemma *subst-shAll* [*usubst*]: $\sigma \dagger (\forall\ x \cdot P(x)) = (\forall\ x \cdot \sigma \dagger P(x))$
by *pred-auto*

TODO: Generalise the quantifier substitution laws to n-ary substitutions

lemma *subst-ex-same* [*usubst*]:
assumes *mwb-lens x*
shows $(\exists\ x \cdot P)[\![v/x]\!] = (\exists\ x \cdot P)$
by (*simp add*: *assms id-subst subst-unrest unrest-ex-in*)

lemma *subst-ex-indep* [*usubst*]:
assumes $x \bowtie y$ $y \sharp v$
shows $(\exists\ y \cdot P)[\![v/x]\!] = (\exists\ y \cdot P[\![v/x]\!])$
using *assms*
apply (*pred-auto*)
using *lens-indep-comm* **apply** *fastforce+*
done

lemma *subst-all-same* [*usubst*]:
assumes *mwb-lens x*
shows $(\forall\ x \cdot P)[\![v/x]\!] = (\forall\ x \cdot P)$

by (*simp add*: *assms id-subst subst-unrest unrest-all-in*)

lemma *subst-all-indep* [*usubst*]:
assumes $x \bowtie y$ $y \sharp v$
shows $(\forall\ y \cdot P)[\![v/x]\!] = (\forall\ y \cdot P[\![v/x]\!])$
using *assms*
by (*pred-auto*, *simp-all add*: *lens-indep-comm*)

7.6 Predicate Laws

Showing that predicates form a Boolean Algebra (under the predicate operators) gives us many useful laws.

interpretation *boolean-algebra diff-upred not-upred conj-upred op* $\leq$ *op* $<$ *disj-upred false-upred true-upred*
by (*unfold-locales*, *pred-auto*+)

lemma *taut-true* [*simp*]: *‘true‘*
by (*pred-auto*)

lemma *refBy-order*: $P \sqsubseteq Q$ = ‘$Q \Rightarrow P$‘
by (*transfer*, *auto*)

lemma *conj-idem* [*simp*]: $((P::'\alpha\ upred) \wedge P) = P$
by *pred-auto*

lemma *disj-idem* [*simp*]: $((P::'\alpha\ upred) \vee P) = P$
by *pred-auto*

lemma *conj-comm*: $((P::'\alpha\ upred) \wedge Q) = (Q \wedge P)$
by *pred-auto*

lemma *disj-comm*: $((P::'\alpha\ upred) \vee Q) = (Q \vee P)$
by *pred-auto*

lemma *conj-subst*: $P = R \Longrightarrow ((P::'\alpha\ upred) \wedge Q) = (R \wedge Q)$
by *pred-auto*

lemma *disj-subst*: $P = R \Longrightarrow ((P::'\alpha\ upred) \vee Q) = (R \vee Q)$
by *pred-auto*

lemma *conj-assoc*:$(((P::'\alpha\ upred) \wedge Q) \wedge S) = (P \wedge (Q \wedge S))$
by *pred-auto*

lemma *disj-assoc*:$(((P::'\alpha\ upred) \vee Q) \vee S) = (P \vee (Q \vee S))$
by *pred-auto*

lemma *conj-disj-abs*:$((P::'\alpha\ upred) \wedge (P \vee Q)) = P$
by *pred-auto*

lemma *disj-conj-abs*:$((P::'\alpha\ upred) \vee (P \wedge Q)) = P$
by *pred-auto*

lemma *conj-disj-distr*:$((P::'\alpha\ upred) \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
by *pred-auto*

lemma *disj-conj-distr*:$((P::'\alpha\ upred) \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$

by *pred-auto*

lemma *true-disj-zero* [*simp*]:
$(P \vee \mathit{true}) = \mathit{true}$ $(\mathit{true} \vee P) = \mathit{true}$
by *pred-auto*

lemma *true-conj-zero* [*simp*]:
$(P \wedge \mathit{false}) = \mathit{false}$ $(\mathit{false} \wedge P) = \mathit{false}$
by *pred-auto*

lemma *imp-vacuous* [*simp*]: $(\mathit{false} \Rightarrow u) = \mathit{true}$
by *pred-auto*

lemma *imp-true* [*simp*]: $(p \Rightarrow \mathit{true}) = \mathit{true}$
by *pred-auto*

lemma *true-imp* [*simp*]: $(\mathit{true} \Rightarrow p) = p$
by *pred-auto*

lemma *p-and-not-p* [*simp*]: $(P \wedge \neg P) = \mathit{false}$
by *pred-auto*

lemma *p-or-not-p* [*simp*]: $(P \vee \neg P) = \mathit{true}$
by *pred-auto*

lemma *p-imp-p* [*simp*]: $(P \Rightarrow P) = \mathit{true}$
by *pred-auto*

lemma *p-iff-p* [*simp*]: $(P \Leftrightarrow P) = \mathit{true}$
by *pred-auto*

lemma *p-imp-false* [*simp*]: $(P \Rightarrow \mathit{false}) = (\neg P)$
by *pred-auto*

lemma *not-conj-deMorgans* [*simp*]: $(\neg ((P::'\alpha\ \mathit{upred}) \wedge Q)) = ((\neg P) \vee (\neg Q))$
by *pred-auto*

lemma *not-disj-deMorgans* [*simp*]: $(\neg ((P::'\alpha\ \mathit{upred}) \vee Q)) = ((\neg P) \wedge (\neg Q))$
by *pred-auto*

lemma *conj-disj-not-abs* [*simp*]: $((P::'\alpha\ \mathit{upred}) \wedge ((\neg P) \vee Q)) = (P \wedge Q)$
by (*pred-auto*)

lemma *subsumption1*:
$`P \Rightarrow Q` \Longrightarrow (P \vee Q) = Q$
by (*pred-auto*)

lemma *subsumption2*:
$`Q \Rightarrow P` \Longrightarrow (P \vee Q) = P$
by (*pred-auto*)

lemma *neg-conj-cancel1*: $(\neg P \wedge (P \vee Q)) = (\neg P \wedge Q :: '\alpha\ \mathit{upred})$
by (*pred-auto*)

lemma *neg-conj-cancel2*: $(\neg Q \wedge (P \vee Q)) = (\neg Q \wedge P :: '\alpha\ \mathit{upred})$

by (*pred-auto*)

lemma *double-negation* [*simp*]: $(\neg\ \neg\ (P::'\alpha\ upred)) = P$
by (*pred-auto*)

lemma *true-not-false* [*simp*]: $true \neq false\ false \neq true$
by *pred-auto+*

lemma *closure-conj-distr*: $([P]_u \wedge [Q]_u) = [P \wedge Q]_u$
by *pred-auto*

lemma *closure-imp-distr*: ‘$[P \Rightarrow Q]_u \Rightarrow [P]_u \Rightarrow [Q]_u$‘
by *pred-auto*

lemma *USUP-cong-eq*:
$\llbracket \bigwedge x.\ P_1(x) = P_2(x);\ \bigwedge x.\ ‘P_1(x) \Rightarrow Q_1(x) =_u Q_2(x)‘\ \rrbracket \Longrightarrow$
$(\bigsqcap\ x \mid P_1(x) \cdot Q_1(x)) = (\bigsqcap\ x \mid P_2(x) \cdot Q_2(x))$
by (*simp add*: *USUP-def*, *pred-auto*, *metis*)

lemma *USUP-as-Sup*: $(\bigsqcap\ P \in \mathcal{P} \cdot P) = \bigsqcap\ \mathcal{P}$
apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def*)
apply (*pred-auto*)
apply (*unfold SUP-def*)
apply (*rule cong*[*of Sup*])
apply (*auto*)
done

lemma *USUP-as-Sup-collect*: $(\bigsqcap P{\in}A \cdot f(P)) = (\bigsqcap P{\in}A.\ f(P))$
apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def*)
apply (*unfold SUP-def*)
apply (*pred-auto*)
apply (*simp add*: *Setcompr-eq-image*)
done

lemma *USUP-as-Sup-image*: $(\bigsqcap\ P \mid \ll P\gg \in_u \ll A\gg \cdot f(P)) = \bigsqcap\ (f\ ‘\ A)$
apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def*)
apply (*pred-auto*)
apply (*unfold SUP-def*)
apply (*rule cong*[*of Sup*])
apply (*auto*)
done

lemma *UINF-as-Inf*: $(\bigsqcup\ P \in \mathcal{P} \cdot P) = \bigsqcup\ \mathcal{P}$
apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Inf-uexpr-def*)
apply (*pred-auto*)
apply (*unfold INF-def*)
apply (*rule cong*[*of Inf*])
apply (*auto*)
done

lemma *UINF-as-Inf-collect*: $(\bigsqcup P{\in}A \cdot f(P)) = (\bigsqcup P{\in}A.\ f(P))$
apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def*)
apply (*unfold INF-def*)
apply (*pred-auto*)
apply (*simp add*: *Setcompr-eq-image*)

done

lemma *UINF-as-Inf-image*: $(\bigsqcup\ P \in \mathcal{P} \cdot f(P)) = \bigsqcup\ (f \text{ ` } \mathcal{P})$
  apply (*simp add*: *upred-defs bop.rep-eq lit.rep-eq Inf-uexpr-def*)
  apply (*pred-auto*)
  apply (*unfold INF-def*)
  apply (*rule cong*[*of Inf*])
  apply (*auto*)
done

lemma *true-iff* [*simp*]: $(P \Leftrightarrow true) = P$
  by *pred-auto*

lemma *impl-alt-def*: $(P \Rightarrow Q) = (\neg\ P \vee Q)$
  by *pred-auto*

lemma *eq-upred-refl* [*simp*]: $(x =_u x) = true$
  by *pred-auto*

lemma *eq-upred-sym*: $(x =_u y) = (y =_u x)$
  by *pred-auto*

lemma *eq-cong-left*:
  assumes *vwb-lens x* $\$x \sharp Q$ $\$x' \sharp Q$ $\$x \sharp R$ $\$x' \sharp R$
  shows $((\$x' =_u \$x \wedge Q) = (\$x' =_u \$x \wedge R)) \longleftrightarrow (Q = R)$
  using *assms*
  by (*pred-auto*, (*meson mwb-lens-def vwb-lens-mwb weak-lens-def*)+)

lemma *conj-eq-in-var-subst*:
  fixes x :: $('a, '\alpha)$ *uvar*
  assumes *vwb-lens x*
  shows $(P \wedge \$x =_u v) = (P⟦v/\$x⟧ \wedge \$x =_u v)$
  using *assms*
  by (*pred-auto*, (*metis vwb-lens-wb wb-lens.get-put*)+)

lemma *conj-eq-out-var-subst*:
  fixes x :: $('a, '\alpha)$ *uvar*
  assumes *vwb-lens x*
  shows $(P \wedge \$x' =_u v) = (P⟦v/\$x'⟧ \wedge \$x' =_u v)$
  using *assms*
  by (*pred-auto*, (*metis vwb-lens-wb wb-lens.get-put*)+)

lemma *conj-pos-var-subst*:
  assumes *vwb-lens x*
  shows $(\$x \wedge Q) = (\$x \wedge Q⟦true/\$x⟧)$
  using *assms*
  by (*pred-auto*, *metis* (*full-types*) *vwb-lens-wb wb-lens.get-put*, *metis* (*full-types*) *vwb-lens-wb wb-lens.get-put*)

lemma *conj-neg-var-subst*:
  assumes *vwb-lens x*
  shows $(\neg\ \$x \wedge Q) = (\neg\ \$x \wedge Q⟦false/\$x⟧)$
  using *assms*
  by (*pred-auto*, *metis* (*full-types*) *vwb-lens-wb wb-lens.get-put*, *metis* (*full-types*) *vwb-lens-wb wb-lens.get-put*)

lemma *le-pred-refl* [*simp*]:

fixes x :: $('a::preorder, '\alpha)$ *uexpr*
shows $(x \leq_u x) = true$
by (*pred-auto*)

lemma *shEx-unbound* [*simp*]: $(\exists\ x \cdot P) = P$
by *pred-auto*

lemma *shEx-bool* [*simp*]: *shEx* $P = (P\ True \vee P\ False)$
by (*pred-auto*, *metis* (*full-types*))

lemma *shEx-commute*: $(\exists\ x \cdot \exists\ y \cdot P\ x\ y) = (\exists\ y \cdot \exists\ x \cdot P\ x\ y)$
by *pred-auto*

lemma *shEx-cong*: $[\![\bigwedge x.\ P\ x = Q\ x]\!] \Longrightarrow shEx\ P = shEx\ Q$
by (*pred-auto*)

lemma *shAll-unbound* [*simp*]: $(\forall\ x \cdot P) = P$
by *pred-auto*

lemma *shAll-bool* [*simp*]: *shAll* $P = (P\ True \wedge P\ False)$
by (*pred-auto*, *metis* (*full-types*))

lemma *shAll-cong*: $[\![\bigwedge x.\ P\ x = Q\ x]\!] \Longrightarrow shAll\ P = shAll\ Q$
by (*pred-auto*)

lemma *upred-eq-true* [*simp*]: $(p =_u true) = p$
by *pred-auto*

lemma *upred-eq-false* [*simp*]: $(p =_u false) = (\neg\ p)$
by *pred-auto*

lemma *conj-var-subst*:
assumes *vwb-lens* x
shows $(P \wedge var\ x =_u v) = (P[\![v/x]\!] \wedge var\ x =_u v)$
using *assms*
by (*pred-auto*, (*metis* (*full-types*) *vwb-lens-def wb-lens.get-put*)+)

lemma *one-point*:
assumes *mwb-lens* x $x \sharp v$
shows $(\exists\ x \cdot P \wedge var\ x =_u v) = P[\![v/x]\!]$
using *assms*
by (*pred-auto*)

lemma *uvar-assign-exists*:
vwb-lens $x \Longrightarrow \exists\ v.\ b = put_x\ b\ v$
by (*rule-tac* x=get_x b **in** *exI*, *simp*)

lemma *uvar-obtain-assign*:
assumes *vwb-lens* x
obtains v **where** $b = put_x\ b\ v$
using *assms*
by (*drule-tac uvar-assign-exists*[*of* - *b*], *auto*)

lemma *eq-split-subst*:
assumes *vwb-lens* x

shows $(P = Q) \longleftrightarrow (\forall\ v.\ P[\![\ll v\gg/x]\!] = Q[\![\ll v\gg/x]\!])$
using *assms*
by (*pred-auto*, *metis uvar-assign-exists*)

lemma *eq-split-substI*:
assumes *vwb-lens x* $\bigwedge v.\ P[\![\ll v\gg/x]\!] = Q[\![\ll v\gg/x]\!]$
shows $P = Q$
using *assms*(*1*) *assms*(*2*) *eq-split-subst* **by** *blast*

lemma *taut-split-subst*:
assumes *vwb-lens x*
shows ‘P‘ $\longleftrightarrow (\forall\ v.$ ‘$P[\![\ll v\gg/x]\!]$‘$)$
using *assms*
by (*pred-auto*, *metis uvar-assign-exists*)

lemma *eq-split*:
assumes ‘$P \Rightarrow Q$‘ ‘$Q \Rightarrow P$‘
shows $P = Q$
using *assms*
by (*pred-auto*)

lemma *subst-bool-split*:
assumes *vwb-lens x*
shows ‘P‘ = ‘$(P[\![false/x]\!] \wedge P[\![true/x]\!])$‘
proof −
from *assms* **have** ‘P‘ = $(\forall\ v.$ ‘$P[\![\ll v\gg/x]\!]$‘$)$
by (*subst taut-split-subst*[*of x*], *auto*)
also have ... = (‘$P[\![\ll True\gg/x]\!]$‘ $\wedge$ ‘$P[\![\ll False\gg/x]\!]$‘)
by (*metis* (*mono-tags*, *lifting*))
also have ... = ‘$(P[\![false/x]\!] \wedge P[\![true/x]\!])$‘
by (*pred-auto*)
finally show *?thesis* **.**
qed

lemma *taut-iff-eq*:
‘$P \Leftrightarrow Q$‘ $\longleftrightarrow (P = Q)$
by *pred-auto*

lemma *subst-eq-replace*:
fixes x :: $('a, '\alpha)$ *uvar*
shows $(p[\![u/x]\!] \wedge u =_u v) = (p[\![v/x]\!] \wedge u =_u v)$
by *pred-auto*

lemma *exists-twice*: *mwb-lens x* $\Longrightarrow (\exists\ x \cdot \exists\ x \cdot P) = (\exists\ x \cdot P)$
by (*pred-auto*)

lemma *all-twice*: *mwb-lens x* $\Longrightarrow (\forall\ x \cdot \forall\ x \cdot P) = (\forall\ x \cdot P)$
by (*pred-auto*)

lemma *exists-sub*: $[\![$ *mwb-lens y*; $x \subseteq_L y$ $]\!] \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot P)$
by *pred-auto*

lemma *all-sub*: $[\![$ *mwb-lens y*; $x \subseteq_L y$ $]\!] \Longrightarrow (\forall\ x \cdot \forall\ y \cdot P) = (\forall\ y \cdot P)$
by *pred-auto*

lemma *ex-commute*:
  assumes $x \bowtie y$
  shows $(\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
  using *assms*
  apply (*pred-auto*)
  using *lens-indep-comm* **apply** *fastforce*+
done

lemma *all-commute*:
  assumes $x \bowtie y$
  shows $(\forall\ x \cdot \forall\ y \cdot P) = (\forall\ y \cdot \forall\ x \cdot P)$
  using *assms*
  apply (*pred-auto*)
  using *lens-indep-comm* **apply** *fastforce*+
done

lemma *ex-equiv*:
  assumes $x \approx_L y$
  shows $(\exists\ x \cdot P) = (\exists\ y \cdot P)$
  using *assms*
  by (*pred-auto*, *metis* (*no-types*, *lifting*) *lens.select-convs*(*2*))

lemma *all-equiv*:
  assumes $x \approx_L y$
  shows $(\forall\ x \cdot P) = (\forall\ y \cdot P)$
  using *assms*
  by (*pred-auto*, *metis* (*no-types*, *lifting*) *lens.select-convs*(*2*))

lemma *ex-zero*:
  $(\exists\ \&\emptyset \cdot P) = P$
  by *pred-auto*

lemma *all-zero*:
  $(\forall\ \&\emptyset \cdot P) = P$
  by *pred-auto*

lemma *ex-plus*:
  $(\exists\ y;x \cdot P) = (\exists\ x \cdot \exists\ y \cdot P)$
  by *pred-auto*

lemma *all-plus*:
  $(\forall\ y;x \cdot P) = (\forall\ x \cdot \forall\ y \cdot P)$
  by *pred-auto*

lemma *closure-all*:
  $[P]_u = (\forall\ \&\Sigma \cdot P)$
  by *pred-auto*

lemma *unrest-as-exists*:
  vwb-lens $x \Longrightarrow (x \sharp P) \longleftrightarrow ((\exists\ x \cdot P) = P)$
  by (*pred-auto*, *metis vwb-lens.put-eq*)

7.7 Cylindric algebra

lemma *C1*: $(\exists\ x \cdot \mathit{false}) = \mathit{false}$
  by (*pred-auto*)

lemma *C2*: *wb-lens* $x \Longrightarrow$ ‘$P \Rightarrow (\exists\ x \cdot P)$‘
  by (*pred-auto*, *metis wb-lens.get-put*)

lemma *C3*: *mwb-lens* $x \Longrightarrow (\exists\ x \cdot (P \wedge (\exists\ x \cdot Q))) = ((\exists\ x \cdot P) \wedge (\exists\ x \cdot Q))$
  by (*pred-auto*)

lemma *C4a*: $x \approx_L y \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
  by (*pred-auto*, *metis* (*no-types*, *lifting*) *lens.select-convs*(*2*))+

lemma *C4b*: $x \bowtie y \Longrightarrow (\exists\ x \cdot \exists\ y \cdot P) = (\exists\ y \cdot \exists\ x \cdot P)$
  using *ex-commute* **by** *blast*

lemma *C5*:
  fixes x :: ($'a$, $'\alpha$) *uvar*
  shows $(\&x =_u \&x) = true$
  by *pred-auto*

lemma *C6*:
  assumes *wb-lens* x $x \bowtie y$ $x \bowtie z$
  shows $(\&y =_u \&z) = (\exists\ x \cdot \&y =_u \&x \wedge \&x =_u \&z)$
  using *assms*
  by (*pred-auto*, (*metis lens-indep-def*)+)

lemma *C7*:
  assumes *weak-lens* x $x \bowtie y$
  shows $((\exists\ x \cdot \&x =_u \&y \wedge P) \wedge (\exists\ x \cdot \&x =_u \&y \wedge \neg\ P)) = false$
  using *assms*
  by (*pred-auto′*, *simp add*: *lens-indep-sym*)

7.8 Quantifier lifting

named-theorems *uquant-lift*

lemma *shEx-lift-conj-1* [*uquant-lift*]:
  $((\exists\ x \cdot P(x)) \wedge Q) = (\exists\ x \cdot P(x) \wedge Q)$
  by *pred-auto*

lemma *shEx-lift-conj-2* [*uquant-lift*]:
  $(P \wedge (\exists\ x \cdot Q(x))) = (\exists\ x \cdot P \wedge Q(x))$
  by *pred-auto*

end

8 Alphabetised relations

theory *utp-rel*
imports
  utp-pred
  utp-lift
begin

default-sort *type*

8.1 Automatic Tactics

named-theorems *urel-defs*

We set up several automatic tactics that recast theorems on UTP predicates into equivalent HOL predicates, eliminating artefacts of the mechanisation as much as this is possible. Our approach is first to unfold all relevant definition of the UTP predicate model, then perform a transfer, and finally simplify by using lens and variable definitions, the split laws of alphabet records, and interpretation laws to convert record-based state spaces into products. The definition of the methods is facilitated by the Eisbach tool.

Without re-interpretation of lens types in state spaces (legacy).

method *rel-simp'* = (
  (*unfold upred-defs urel-defs*)?,
  (*transfer*),
  (*simp add*: *fun-eq-iff relcomp-unfold OO-def*
    lens-defs uvar-defs upred-defs alpha-splits Product-Type.split-beta)?,
  (*clarsimp*)?)

Variations that adjoin *rel-simp'* with automatic tactics.

method *rel-auto'* = (*rel-simp'*, *auto?*)
method *rel-blast'* = (*rel-simp'*; *blast*)

With reinterpretation of lens types in state spaces (default).

method *rel-simp* = (
  (*unfold upred-defs urel-defs*)?,
  (*transfer*),
  (*simp add*: *fun-eq-iff relcomp-unfold OO-def*
    lens-defs uvar-defs upred-defs alpha-splits Product-Type.split-beta)?,
  (*simp add*: *lens-interp-laws*)?,
  (*clarsimp*)?)

Variations that adjoin *rel-simp* with automatic tactics.

method *rel-auto* = (*rel-simp*, *auto?*)
method *rel-blast* = (*rel-simp*; *blast*)

— TODO: Rename *rel-auto* into *rel-auto*.

consts
  useq :: $'a \Rightarrow 'b \Rightarrow 'c$ (**infixr** ;; *15*)
  uskip :: $'a$ (*II*)

definition *in*α :: $('\alpha, '\alpha \times '\beta)$ *uvar* **where**
inα = (| *lens-get* = *fst*, *lens-put* = λ (A, A') v. (v, A') |)

definition *out*α :: $('\beta, '\alpha \times '\beta)$ *uvar* **where**
outα = (| *lens-get* = *snd*, *lens-put* = λ (A, A') v. (A, v) |)

declare *in*α*-def* [*urel-defs*]
declare *out*α*-def* [*urel-defs*]

lemma *var-in-alpha* [*simp*]: $x ;_L$ *in*α = *ivar* x
  by (*simp add*: *fst-lens-def in*α*-def in-var-def*)

lemma *var-out-alpha* [*simp*]: $x ;_L$ *out*α = *ovar* x

by (*simp add*: *outα-def out-var-def snd-lens-def*)

lemma *out-alpha-in-indep* [*simp*]:
outα $\bowtie$ *in-var x in-var x* $\bowtie$ *outα*
by (*simp-all add*: *in-var-def outα-def lens-indep-def fst-lens-def lens-comp-def*)

lemma *in-alpha-out-indep* [*simp*]:
inα $\bowtie$ *out-var x out-var x* $\bowtie$ *inα*
by (*simp-all add*: *in-var-def inα-def lens-indep-def fst-lens-def lens-comp-def*)

The alphabet of a relation consists of the input and output portions

lemma *alpha-in-out*:
$\Sigma \approx_L$ *inα* $+_L$ *outα*
by (*metis fst-lens-def fst-snd-id-lens inα-def lens-equiv-refl outα-def snd-lens-def*)

type-synonym $'\alpha$ *condition* $= '\alpha$ *upred*
type-synonym $('\alpha, '\beta)$ *relation* $= ('\alpha \times '\beta)$ *upred*
type-synonym $'\alpha$ *hrelation* $= ('\alpha \times '\alpha)$ *upred*

translations
(*type*) $('\alpha, '\beta)$ *relation* <= (*type*) $('\alpha \times '\beta)$ *upred*

definition *cond*::$'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred* $\Rightarrow$ $'\alpha$ *upred*
((*3*- $\triangleleft$ - $\triangleright$/ -) [*14*,*0*,*15*] *14*)
where $(P \triangleleft b \triangleright Q) \equiv (b \wedge P) \vee ((\neg\ b) \wedge Q)$

abbreviation *rcond*::$('\alpha, '\beta)$ *relation* $\Rightarrow$ $'\alpha$ *condition* $\Rightarrow$ $('\alpha, '\beta)$ *relation* $\Rightarrow$ $('\alpha, '\beta)$ *relation*
((*3*- $\triangleleft$ - $\triangleright_r$/ -) [*14*,*0*,*15*] *14*)
where $(P \triangleleft b \triangleright_r Q) \equiv (P \triangleleft \lceil b \rceil_< \triangleright Q)$

lift-definition *seqr*::$(('\alpha \times '\beta)$ *upred*$) \Rightarrow (('\beta \times '\gamma)$ *upred*$) \Rightarrow ('\alpha \times '\gamma)$ *upred*
is $\lambda\ P\ Q\ r.\ r \in (\{p.\ P\ p\}\ O\ \{q.\ Q\ q\})$ **.**

lift-definition *conv-r* :: $('a, '\alpha \times '\beta)$ *uexpr* $\Rightarrow$ $('a, '\beta \times '\alpha)$ *uexpr* (-^- [*999*] *999*)
is $\lambda\ e\ (b1, b2).\ e\ (b2, b1)$ **.**

definition *skip-ra* :: $('\beta, '\alpha)$ *lens* $\Rightarrow$$'\alpha$ *hrelation* **where**
[*urel-defs*]: *skip-ra v* $= (\$v' =_u \$v)$

syntax
-skip-ra :: *salpha* $\Rightarrow$ *logic* ($II_{\text{-}}$)

translations
-skip-ra v == *CONST skip-ra v*

abbreviation *usubst-rel-lift* :: $'\alpha$ *usubst* $\Rightarrow$ $('\alpha \times '\beta)$ *usubst* ($\lceil\text{-}\rceil_s$) **where**
$\lceil\sigma\rceil_s \equiv \sigma \oplus_s$ *inα*

abbreviation *usubst-rel-drop* :: $('\alpha \times '\alpha)$ *usubst* $\Rightarrow$ $'\alpha$ *usubst* ($\lfloor\text{-}\rfloor_s$) **where**
$\lfloor\sigma\rfloor_s \equiv \sigma \upharpoonright_s$ *inα*

definition *assigns-ra* :: $'\alpha$ *usubst* $\Rightarrow$ $('\beta, '\alpha)$ *lens* $\Rightarrow$ $'\alpha$ *hrelation* ($\langle\text{-}\rangle_{\text{-}}$) **where**
$\langle\sigma\rangle_a = (\lceil\sigma\rceil_s \dagger II_a)$

lift-definition *assigns-r* :: $'\alpha$ *usubst* $\Rightarrow$ $'\alpha$ *hrelation* ($\langle\text{-}\rangle_a$)

is $\lambda\ \sigma\ (A,\ A').\ A' = \sigma(A)$ **.**

definition *skip-r* :: *'α hrelation* **where**
skip-r = *assigns-r id*

abbreviation *assign-r* :: (*'t*, *'α*) *uvar* ⇒ (*'t*, *'α*) *uexpr* ⇒ *'α hrelation*
where *assign-r x v* ≡ *assigns-r* [$x \mapsto_s v$]

abbreviation *assign-2-r* ::
(*'t1*, *'α*) *uvar* ⇒ (*'t2*, *'α*) *uvar* ⇒ (*'t1*, *'α*) *uexpr* ⇒ (*'t2*, *'α*) *uexpr* ⇒ *'α hrelation*
where *assign-2-r x y u v* ≡ *assigns-r* [$x \mapsto_s u,\ y \mapsto_s v$]

nonterminal
svid-list **and** *uexpr-list*

syntax
-svid-unit :: *svid* ⇒ *svid-list* (-)
-svid-list :: *svid* ⇒ *svid-list* ⇒ *svid-list* (-,/ -)
-uexpr-unit :: (*'a*, *'α*) *uexpr* ⇒ *uexpr-list* (- [*40*] *40*)
-uexpr-list :: (*'a*, *'α*) *uexpr* ⇒ *uexpr-list* ⇒ *uexpr-list* (-,/ - [*40*,*40*] *40*)
-assignment :: *svid-list* ⇒ *uexprs* ⇒ *'α hrelation* (**infixr** := *62*)
-mk-usubst :: *svid-list* ⇒ *uexprs* ⇒ *'α usubst*

translations
-mk-usubst σ (*-svid-unit x*) *v* == σ(&*x* $\mapsto_s$ *v*)
-mk-usubst σ (*-svid-list x xs*) (*-uexprs v vs*) == (*-mk-usubst* (σ(&*x* $\mapsto_s$ *v*)) *xs vs*)
-assignment xs vs => *CONST assigns-r* (*-mk-usubst* (*CONST id*) *xs vs*)
x := *v* <= *CONST assigns-r* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *v*)
x := *v* <= *CONST assigns-r* (*CONST subst-upd* (*CONST id*) *x v*)
x,*y* := *u*,*v* <= *CONST assigns-r* (*CONST subst-upd* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *u*) (*CONST svar y*) *v*)

adhoc-overloading
useq seqr **and**
uskip skip-r

definition *rassume* :: *'α upred* ⇒ *'α hrelation* ($-^{\top}$ [*999*] *999*) **where**
[*urel-defs*]: *rassume c* = (*II* ◃ *c* $\triangleright_r$ *false*)

definition *rassert* :: *'α upred* ⇒ *'α hrelation* ($-_{\perp}$ [*999*] *999*) **where**
[*urel-defs*]: *rassert c* = (*II* ◃ *c* $\triangleright_r$ *true*)

We describe some properties of relations

definition *ufunctional* :: (*'a*, *'b*) *relation* ⇒ *bool*
where *ufunctional R* ⟷ (*II* ⊑ (R^- ;; *R*))

declare *ufunctional-def* [*urel-defs*]

definition *uinj* :: (*'a*, *'b*) *relation* ⇒ *bool*
where *uinj R* ⟷ *II* ⊑ (*R* ;; R^-)

declare *uinj-def* [*urel-defs*]

A test is like a precondition, except that it identifies to the postcondition. It forms the basis for Kleene Algebra with Tests (KAT).

definition *lift-test* :: $'\alpha$ *condition* $\Rightarrow$ $'\alpha$ *hrelation* ($\lceil$-$\rceil_t$)
where $\lceil b \rceil_t = (\lceil b \rceil_< \wedge II)$

declare *cond-def* [*urel-defs*]
declare *skip-r-def* [*urel-defs*]

We implement a poor man's version of alphabet restriction that hides a variable within a relation

definition *rel-var-res* :: $'\alpha$ *hrelation* $\Rightarrow$ ($'a$, $'\alpha$) *uvar* $\Rightarrow$ $'\alpha$ *hrelation* (**infix** $\restriction_\alpha$ *80*) **where**
$P \restriction_\alpha x = (\exists\ \$x \cdot \exists\ \$x\acute{} \cdot P)$

declare *rel-var-res-def* [*urel-defs*]

8.2 Unrestriction Laws

lemma *unrest-iuvar* [*unrest*]: *mwb-lens* $x \Longrightarrow$ *out*α $\sharp$ $\$x$
by (*simp add*: *out*α*-def*, *transfer*, *auto*)

lemma *unrest-ouvar* [*unrest*]: *mwb-lens* $x \Longrightarrow$ *in*α $\sharp$ $\$x\acute{}$
by (*simp add*: *in*α*-def*, *transfer*, *auto*)

lemma *unrest-semir-undash* [*unrest*]:
fixes x :: ($'a$, $'\alpha$) *uvar*
assumes $\$x \sharp P$
shows $\$x \sharp (P\ ;;\ Q)$
using *assms* **by** (*rel-auto*)

lemma *unrest-semir-dash* [*unrest*]:
fixes x :: ($'a$, $'\alpha$) *uvar*
assumes $\$x\acute{} \sharp Q$
shows $\$x\acute{} \sharp (P\ ;;\ Q)$
using *assms* **by** (*rel-auto*)

lemma *unrest-cond* [*unrest*]:
$[\![\ x \sharp P;\ x \sharp b;\ x \sharp Q\ ]\!] \Longrightarrow x \sharp (P \triangleleft b \triangleright Q)$
by (*rel-auto*)

lemma *unrest-in*α*-var* [*unrest*]:
$[\![$ *mwb-lens* x; *in*α $\sharp$ (P :: ($'\alpha$, $'\beta$) *relation*) $]\!] \Longrightarrow \$x \sharp P$
by (*pred-auto*, *simp add*: *in*α*-def*, *blast*, *metis in*α*-def lens.select-convs*(*2*) *old.prod.case*)

lemma *unrest-out*α*-var* [*unrest*]:
$[\![$ *mwb-lens* x; *out*α $\sharp$ (P :: ($'\alpha$, $'\beta$) *relation*) $]\!] \Longrightarrow \$x\acute{} \sharp P$
by (*pred-auto*, *simp add*: *out*α*-def*, *blast*, *metis lens.select-convs*(*2*) *old.prod.case out*α*-def*)

lemma *in*α*-uvar* [*simp*]: *vwb-lens in*α
by (*unfold-locales*, *auto simp add*: *in*α*-def*)

lemma *out*α*-uvar* [*simp*]: *vwb-lens out*α
by (*unfold-locales*, *auto simp add*: *out*α*-def*)

lemma *unrest-pre-out*α [*unrest*]: *out*α $\sharp$ $\lceil b \rceil_<$
by (*transfer*, *auto simp add*: *out*α*-def*)

lemma *unrest-post-in*α [*unrest*]: *in*α $\sharp$ $\lceil b \rceil_>$
by (*transfer*, *auto simp add*: *in*α*-def*)

lemma *unrest-pre-in-var* [*unrest*]:
$x \sharp p1 \Longrightarrow \$x \sharp \lceil p1 \rceil_<$
by (*transfer*, *simp*)

lemma *unrest-post-out-var* [*unrest*]:
$x \sharp p1 \Longrightarrow \$x\acute{} \sharp \lceil p1 \rceil_>$
by (*transfer*, *simp*)

lemma *unrest-convr-outα* [*unrest*]:
$in\alpha \sharp p \Longrightarrow out\alpha \sharp p^-$
by (*transfer*, *auto simp add*: *inα-def outα-def*)

lemma *unrest-convr-inα* [*unrest*]:
$out\alpha \sharp p \Longrightarrow in\alpha \sharp p^-$
by (*transfer*, *auto simp add*: *inα-def outα-def*)

lemma *unrest-in-rel-var-res* [*unrest*]:
$vwb\text{-}lens\ x \Longrightarrow \$x \sharp (P \upharpoonright_\alpha x)$
by (*simp add*: *rel-var-res-def unrest*)

lemma *unrest-out-rel-var-res* [*unrest*]:
$vwb\text{-}lens\ x \Longrightarrow \$x\acute{} \sharp (P \upharpoonright_\alpha x)$
by (*simp add*: *rel-var-res-def unrest*)

8.3 Substitution laws

lemma *subst-seq-left* [*usubst*]:
$out\alpha \sharp \sigma \Longrightarrow \sigma \dagger (P ;; Q) = ((\sigma \dagger P) ;; Q)$
by (*rel-auto*, (*metis* (*no-types*, *lifting*) *Pair-inject surjective-pairing*)+)

lemma *subst-seq-right* [*usubst*]:
$in\alpha \sharp \sigma \Longrightarrow \sigma \dagger (P ;; Q) = (P ;; (\sigma \dagger Q))$
by (*rel-auto*, (*metis* (*no-types*, *lifting*) *Pair-inject surjective-pairing*)+)

The following laws support substitution in heterogeneous relations for polymorphically types literal expressions. These cannot be supported more generically due to limitations in HOL's type system. The laws are presented in a slightly strange way so as to be as general as possible.

lemma *bool-seqr-laws* [*usubst*]:
fixes $x :: (bool \Longrightarrow '\alpha)$
shows
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P[\![true/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P[\![false/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x\acute{} \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![true/\$x\acute{}]\!])$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x\acute{} \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![false/\$x\acute{}]\!])$
by (*rel-auto*)+

lemma *zero-one-seqr-laws* [*usubst*]:
fixes $x :: (\text{-} \Longrightarrow '\alpha)$
shows
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P[\![0/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P[\![1/\$x]\!] ;; Q)$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x\acute{} \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![0/\$x\acute{}]\!])$
$\bigwedge P\ Q\ \sigma.\ \sigma(\$x\acute{} \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P ;; Q[\![1/\$x\acute{}]\!])$
by (*rel-auto*)+

lemma *numeral-seqr-laws* [*usubst*]:
  fixes x :: (- $\Longrightarrow$ $'\alpha$)
  shows
    $\bigwedge P\ Q\ \sigma.\ \sigma(\$x \mapsto_s \mathit{numeral}\ n) \dagger (P\ ;;\ Q) = \sigma \dagger (P[\![\mathit{numeral}\ n/\$x]\!]\ ;;\ Q)$
    $\bigwedge P\ Q\ \sigma.\ \sigma(\$x\acute{} \mapsto_s \mathit{numeral}\ n) \dagger (P\ ;;\ Q) = \sigma \dagger (P\ ;;\ Q[\![\mathit{numeral}\ n/\$x\acute{}]\!])$
  by (*rel-auto*)+

lemma *usubst-condr* [*usubst*]:
  $\sigma \dagger (P \triangleleft b \triangleright Q) = (\sigma \dagger P \triangleleft \sigma \dagger b \triangleright \sigma \dagger Q)$
  by *rel-auto*

lemma *subst-skip-r* [*usubst*]:
  $out\alpha \sharp \sigma \Longrightarrow \sigma \dagger II = \langle\lfloor\sigma\rfloor_s\rangle_a$
  by (*rel-auto*, (*metis* (*mono-tags*, *lifting*) *prod.sel*(*1*) *sndI surjective-pairing*)+)

lemma *usubst-upd-in-comp* [*usubst*]:
  $\sigma(\&in\alpha{:}x \mapsto_s v) = \sigma(\$x \mapsto_s v)$
  by (*simp add*: *fst-lens-def inα-def in-var-def*)

lemma *usubst-upd-out-comp* [*usubst*]:
  $\sigma(\&out\alpha{:}x \mapsto_s v) = \sigma(\$x\acute{} \mapsto_s v)$
  by (*simp add*: *outα-def out-var-def snd-lens-def*)

lemma *subst-lift-upd* [*usubst*]:
  fixes x :: ($'a$, $'\alpha$) *uvar*
  shows $\lceil\sigma(x \mapsto_s v)\rceil_s = \lceil\sigma\rceil_s(\$x \mapsto_s \lceil v\rceil_<)$
  by (*simp add*: *alpha usubst*, *simp add*: *fst-lens-def inα-def in-var-def*)

lemma *subst-drop-upd* [*usubst*]:
  fixes x :: ($'a$, $'\alpha$) *uvar*
  shows $\lfloor\sigma(\$x \mapsto_s v)\rfloor_s = \lfloor\sigma\rfloor_s(x \mapsto_s \lfloor v\rfloor_<)$
  by (*pred-auto*, *simp add*: *inα-def prod.case-eq-if*)

lemma *subst-lift-pre* [*usubst*]: $\lceil\sigma\rceil_s \dagger \lceil b\rceil_< = \lceil\sigma \dagger b\rceil_<$
  by (*metis apply-subst-ext fst-lens-def fst-vwb-lens inα-def*)

lemma *unrest-usubst-lift-in* [*unrest*]:
  $x \sharp P \Longrightarrow \$x \sharp \lceil P\rceil_s$
  by (*pred-auto*, *auto simp add*: *unrest-usubst-def inα-def*)

lemma *unrest-usubst-lift-out* [*unrest*]:
  fixes x :: ($'a$, $'\alpha$) *uvar*
  shows $\$x\acute{} \sharp \lceil P\rceil_s$
  by (*pred-auto*, *auto simp add*: *unrest-usubst-def inα-def*)

8.4 Relation laws

Homogeneous relations form a quantale. This allows us to import a large number of laws from Struth and Armstrong's Kleene Algebra theory [1].

abbreviation *truer* :: $'\alpha$ *hrelation* ($true_h$) **where**
truer $\equiv$ *true*

abbreviation *falser* :: $'\alpha$ *hrelation* ($false_h$) **where**
falser $\equiv$ *false*

interpretation *upred-quantale*: *unital-quantale-plus*
where $times = seqr$ **and** $one = skip\text{-}r$ **and** $Sup = Sup$ **and** $Inf = Inf$ **and** $inf = inf$ **and** $less\text{-}eq = less\text{-}eq$ **and** $less = less$
and $sup = sup$ **and** $bot = bot$ **and** $top = top$
apply (*unfold-locales*)
apply (*rel-auto*)
apply (*unfold SUP-def*, *transfer*, *auto*)
apply (*unfold SUP-def*, *transfer*, *auto*)
apply (*unfold INF-def*, *transfer*, *auto*)
apply (*unfold INF-def*, *transfer*, *auto*)
apply (*rel-auto*)
apply (*rel-auto*)
done

lemma *drop-pre-inv* [*simp*]: $[\![\ out\alpha \sharp p\ ]\!] \Longrightarrow \lceil\lfloor p\rfloor_<\rceil_< = p$
by (*pred-auto*, *auto simp add*: *outα-def lens-create-def fst-lens-def prod.case-eq-if*)

abbreviation *ustar* :: $'\alpha\ hrelation \Rightarrow\ '\alpha\ hrelation$ ($\text{-}^\star{}_u$ [*999*] *999*) **where**
$P^\star{}_u \equiv unital\text{-}quantale.qstar\ II\ op\ ;;\ Sup\ P$

definition *while* :: $'\alpha\ condition \Rightarrow\ '\alpha\ hrelation \Rightarrow\ '\alpha\ hrelation$ (*while - do - od*) **where**
$while\ b\ do\ P\ od = ((\lceil b\rceil_< \wedge P)^\star{}_u \wedge (\neg\ \lceil b\rceil_>))$

declare *while-def* [*urel-defs*]

While loops with invariant decoration

definition *while-inv* :: $'\alpha\ condition \Rightarrow\ '\alpha\ condition \Rightarrow\ '\alpha\ hrelation \Rightarrow\ '\alpha\ hrelation$ (*while - invr - do - od*) **where**
$while\ b\ invr\ p\ do\ S\ od = while\ b\ do\ S\ od$

lemma *cond-idem*:$(P \triangleleft b \triangleright P) = P$ **by** *rel-auto*

lemma *cond-symm*:$(P \triangleleft b \triangleright Q) = (Q \triangleleft \neg\ b \triangleright P)$ **by** *rel-auto*

lemma *cond-assoc*: $((P \triangleleft b \triangleright Q) \triangleleft c \triangleright R) = (P \triangleleft b \wedge c \triangleright (Q \triangleleft c \triangleright R))$ **by** *rel-auto*

lemma *cond-distr*: $(P \triangleleft b \triangleright (Q \triangleleft c \triangleright R)) = ((P \triangleleft b \triangleright Q) \triangleleft c \triangleright (P \triangleleft b \triangleright R))$ **by** *rel-auto*

lemma *cond-unit-T* [*simp*]:$(P \triangleleft true \triangleright Q) = P$ **by** *rel-auto*

lemma *cond-unit-F* [*simp*]:$(P \triangleleft false \triangleright Q) = Q$ **by** *rel-auto*

lemma *cond-and-T-integrate*:
$((P \wedge b) \vee (Q \triangleleft b \triangleright R)) = ((P \vee Q) \triangleleft b \triangleright R)$
by (*rel-auto*)

lemma *cond-L6*: $(P \triangleleft b \triangleright (Q \triangleleft b \triangleright R)) = (P \triangleleft b \triangleright R)$ **by** *rel-auto*

lemma *cond-L7*: $(P \triangleleft b \triangleright (P \triangleleft c \triangleright Q)) = (P \triangleleft b \vee c \triangleright Q)$ **by** *rel-auto*

lemma *cond-and-distr*: $((P \wedge Q) \triangleleft b \triangleright (R \wedge S)) = ((P \triangleleft b \triangleright R) \wedge (Q \triangleleft b \triangleright S))$ **by** *rel-auto*

lemma *cond-or-distr*: $((P \vee Q) \triangleleft b \triangleright (R \vee S)) = ((P \triangleleft b \triangleright R) \vee (Q \triangleleft b \triangleright S))$ **by** *rel-auto*

lemma *cond-imp-distr*:

$((P \Rightarrow Q) \triangleleft b \triangleright (R \Rightarrow S)) = ((P \triangleleft b \triangleright R) \Rightarrow (Q \triangleleft b \triangleright S))$ **by** *rel-auto*

lemma *cond-eq-distr*:
$((P \Leftrightarrow Q) \triangleleft b \triangleright (R \Leftrightarrow S)) = ((P \triangleleft b \triangleright R) \Leftrightarrow (Q \triangleleft b \triangleright S))$ **by** *rel-auto*

lemma *cond-conj-distr*:$(P \wedge (Q \triangleleft b \triangleright S)) = ((P \wedge Q) \triangleleft b \triangleright (P \wedge S))$ **by** *rel-auto*

lemma *cond-disj-distr*:$(P \vee (Q \triangleleft b \triangleright S)) = ((P \vee Q) \triangleleft b \triangleright (P \vee S))$ **by** *rel-auto*

lemma *cond-neg*: $\neg (P \triangleleft b \triangleright Q) = (\neg P \triangleleft b \triangleright \neg Q)$ **by** *rel-auto*

lemma *comp-cond-left-distr*:
$((P \triangleleft b \triangleright_r Q) ;; R) = ((P ;; R) \triangleleft b \triangleright_r (Q ;; R))$
by *rel-auto*

lemma *cond-var-subst-left*:
assumes *vwb-lens x*
shows $(P \triangleleft \$x \triangleright Q) = (P[\![true/\$x]\!] \triangleleft \$x \triangleright Q)$
using *assms* **by** (*metis cond-def conj-pos-var-subst*)

lemma *cond-var-subst-right*:
assumes *vwb-lens x*
shows $(P \triangleleft \$x \triangleright Q) = (P \triangleleft \$x \triangleright Q[\![false/\$x]\!])$
using *assms* **by** (*metis cond-def conj-neg-var-subst*)

lemma *cond-var-split*:
$vwb\text{-}lens\ x \Longrightarrow (P[\![true/x]\!] \triangleleft var\ x \triangleright P[\![false/x]\!]) = P$
by (*rel-auto*, (*metis* (*full-types*) *vwb-lens.put-eq*)+)

lemma *cond-seq-left-distr*:
$out\alpha \sharp b \Longrightarrow ((P \triangleleft b \triangleright Q) ;; R) = ((P ;; R) \triangleleft b \triangleright (Q ;; R))$
by *rel-auto*

lemma *cond-seq-right-distr*:
$in\alpha \sharp b \Longrightarrow (P ;; (Q \triangleleft b \triangleright R)) = ((P ;; Q) \triangleleft b \triangleright (P ;; R))$
by *rel-auto*

These laws may seem to duplicate quantale laws, but they don't – they are applicable to non-homogeneous relations as well, which will become important later.

lemma *seqr-assoc*: $(P ;; (Q ;; R)) = ((P ;; Q) ;; R)$
by *rel-auto*

lemma *seqr-left-unit* [*simp*]:
$(II ;; P) = P$
by *rel-auto*

lemma *seqr-right-unit* [*simp*]:
$(P ;; II) = P$
by *rel-auto*

lemma *seqr-left-zero* [*simp*]:
$(false ;; P) = false$
by *pred-auto*

lemma *seqr-right-zero* [*simp*]:

$(P ;; false) = false$
by *pred-auto*

lemma *impl-seqr-mono*: $\llbracket$ ‘$P \Rightarrow Q$‘; ‘$R \Rightarrow S$‘ $\rrbracket \Longrightarrow$ ‘$(P ;; R) \Rightarrow (Q ;; S)$‘
by (*pred-blast*)

lemma *seqr-mono*:
$\llbracket P_1 \sqsubseteq P_2; Q_1 \sqsubseteq Q_2 \rrbracket \Longrightarrow (P_1 ;; Q_1) \sqsubseteq (P_2 ;; Q_2)$
by (*rel-blast*)

lemma *spec-refine*:
$Q \sqsubseteq (P \wedge R) \Longrightarrow (P \Rightarrow Q) \sqsubseteq R$
by (*rel-auto*)

lemma *cond-skip*: $out\alpha \sharp b \Longrightarrow (b \wedge II) = (II \wedge b^-)$
by (*rel-auto*)

lemma *pre-skip-post*: $(\lceil b \rceil_< \wedge II) = (II \wedge \lceil b \rceil_>)$
by (*rel-auto*)

lemma *skip-var*:
fixes x :: (*bool*, $'\alpha$) *uvar*
shows $(\$x \wedge II) = (II \wedge \$x')$
by (*rel-auto*)

lemma *seqr-exists-left*:
mwb-lens $x \Longrightarrow ((\exists \$x \cdot P) ;; Q) = (\exists \$x \cdot (P ;; Q))$
by (*rel-auto*)

lemma *seqr-exists-right*:
mwb-lens $x \Longrightarrow (P ;; (\exists \$x' \cdot Q)) = (\exists \$x' \cdot (P ;; Q))$
by (*rel-auto*)

lemma *assigns-subst* [*usubst*]:
$\lceil \sigma \rceil_s \dagger \langle \varrho \rangle_a = \langle \varrho \circ \sigma \rangle_a$
by (*rel-auto*)

lemma *assigns-r-comp*: $(\langle \sigma \rangle_a ;; P) = (\lceil \sigma \rceil_s \dagger P)$
by *rel-auto*

lemma *assigns-r-feasible*:
$(\langle \sigma \rangle_a ;; true) = true$
by (*rel-auto*)

lemma *assign-subst* [*usubst*]:
$\llbracket$ *mwb-lens* x; *mwb-lens* y $\rrbracket \Longrightarrow [\$x \mapsto_s \lceil u \rceil_<] \dagger (y := v) = (x, y := u, [x \mapsto_s u] \dagger v)$
by *rel-auto*

lemma *assigns-idem*: *mwb-lens* $x \Longrightarrow (x,x := u,v) = (x := v)$
by (*simp add*: *usubst*)

lemma *assigns-comp*: $(\langle f \rangle_a ;; \langle g \rangle_a) = \langle g \circ f \rangle_a$
by (*simp add*: *assigns-r-comp usubst*)

lemma *assigns-r-conv*:

$bij\ f \Longrightarrow \langle f\rangle_a{}^- = \langle inv\ f\rangle_a$
by (*rel-auto*, *simp-all add*: *bij-is-inj bij-is-surj surj-f-inv-f*)

lemma *assign-pred-transfer*:
fixes $x :: ('a, '\alpha)\ uvar$
assumes $\$x \sharp b\ out\alpha \sharp b$
shows $(b \wedge x := v) = (x := v \wedge b^-)$
using *assms* **by** (*rel-blast*)

lemma *assign-r-comp*: $mwb\text{-}lens\ x \Longrightarrow (x := u\ ;;\ P) = P[\![\lceil u\rceil_</\$x]\!]$
by (*simp add*: *assigns-r-comp usubst*)

lemma *assign-test*: $mwb\text{-}lens\ x \Longrightarrow (x := \ll u\gg\ ;;\ x := \ll v\gg) = (x := \ll v\gg)$
by (*simp add*: *assigns-comp subst-upd-comp subst-lit usubst-upd-idem*)

lemma *assign-twice*: $[\![\ vwb\text{-}lens\ x;\ x \sharp f\ ]\!] \Longrightarrow (x := e\ ;;\ x := f) = (x := f)$
by (*simp add*: *assigns-comp usubst*)

lemma *assign-commute*:
assumes $x \bowtie y\ x \sharp f\ y \sharp e$
shows $(x := e\ ;;\ y := f) = (y := f\ ;;\ x := e)$
using *assms*
by (*rel-auto*, *simp-all add*: *lens-indep-comm*)

lemma *assign-cond*:
fixes $x :: ('a, '\alpha)\ uvar$
assumes $out\alpha \sharp b$
shows $(x := e\ ;;\ (P \triangleleft b \triangleright Q)) = ((x := e\ ;;\ P) \triangleleft (b[\![\lceil e\rceil_</\$x]\!]) \triangleright (x := e\ ;;\ Q))$
by *rel-auto*

lemma *assign-rcond*:
fixes $x :: ('a, '\alpha)\ uvar$
shows $(x := e\ ;;\ (P \triangleleft b \triangleright_r Q)) = ((x := e\ ;;\ P) \triangleleft (b[\![e/x]\!]) \triangleright_r (x := e\ ;;\ Q))$
by *rel-auto*

lemma *assign-r-alt-def*:
fixes $x :: ('a, '\alpha)\ uvar$
shows $x := v = II[\![\lceil v\rceil_</\$x]\!]$
by *rel-auto*

lemma *assigns-r-ufunc*: $ufunctional\ \langle f\rangle_a$
by (*rel-auto*)

lemma *assigns-r-uinj*: $inj\ f \Longrightarrow uinj\ \langle f\rangle_a$
by (*rel-auto*, *simp add*: *inj-eq*)

lemma *assigns-r-swap-uinj*:
$[\![\ vwb\text{-}lens\ x;\ vwb\text{-}lens\ y;\ x \bowtie y\ ]\!] \Longrightarrow uinj\ (x,y := \&y,\&x)$
using *assigns-r-uinj swap-usubst-inj* **by** *auto*

lemma *skip-r-unfold*:
$vwb\text{-}lens\ x \Longrightarrow II = (\$x' =_u \$x \wedge II\restriction_\alpha x)$
by (*rel-auto*, *metis mwb-lens.put-put vwb-lens-mwb vwb-lens-wb wb-lens.get-put*)

lemma *skip-r-alpha-eq*:

$II = (\$\Sigma' =_u \$\Sigma)$
by (*rel-auto*)

lemma *skip-ra-unfold*:
$II_{x;y} = (\$x' =_u \$x \wedge II_y)$
by (*rel-auto*)

lemma *skip-res-as-ra*:
$[\![\textit{vwb-lens } y; x +_L y \approx_L 1_L; x \bowtie y]\!] \Longrightarrow II\restriction_\alpha x = II_y$
apply (*rel-auto*)
apply (*metis* (*no-types*, *lifting*) *lens-indep-def*)
apply (*metis vwb-lens.put-eq*)
done

lemma *assign-unfold*:
$\textit{vwb-lens } x \Longrightarrow (x := v) = (\$x' =_u \lceil v \rceil_< \wedge II\restriction_\alpha x)$
apply (*rel-auto*, *auto simp add*: *comp-def*)
using *vwb-lens.put-eq* **by** *fastforce*

lemma *seqr-or-distl*:
$((P \vee Q) ;; R) = ((P ;; R) \vee (Q ;; R))$
by *rel-auto*

lemma *seqr-or-distr*:
$(P ;; (Q \vee R)) = ((P ;; Q) \vee (P ;; R))$
by *rel-auto*

lemma *seqr-and-distr-ufunc*:
$\textit{ufunctional } P \Longrightarrow (P ;; (Q \wedge R)) = ((P ;; Q) \wedge (P ;; R))$
by *rel-auto*

lemma *seqr-and-distl-uinj*:
$\textit{uinj } R \Longrightarrow ((P \wedge Q) ;; R) = ((P ;; R) \wedge (Q ;; R))$
by (*rel-auto*)

lemma *seqr-unfold*:
$(P ;; Q) = (\exists\ v \cdot P[\![\ll v \gg/\$\Sigma']\!] \wedge Q[\![\ll v \gg/\$\Sigma]\!])$
by *rel-auto*

lemma *seqr-middle*:
assumes *vwb-lens x*
shows $(P ;; Q) = (\exists\ v \cdot P[\![\ll v \gg/\$x']\!] ;; Q[\![\ll v \gg/\$x]\!])$
using *assms*
apply (*rel-auto*)
apply (*rename-tac xa P Q a b y*)
apply (*rule-tac* $x = get_{xa}\ y$ **in** *exI*)
apply (*rule-tac* $x = y$ **in** *exI*)
apply (*simp*)
done

lemma *seqr-left-one-point*:
assumes *vwb-lens x*
shows $(P \wedge (\$x' =_u \ll v \gg) ;; Q) = (P[\![\ll v \gg/\$x']\!] ;; Q[\![\ll v \gg/\$x]\!])$
using *assms*
by (*rel-auto*, *metis vwb-lens-wb wb-lens.get-put*)

lemma *seqr-right-one-point*:
  assumes *vwb-lens x*
  shows $(P\ ;;\ (\$x =_u \ll v \gg) \wedge Q) = (P[\![\ll v\gg/\$x']\!]\ ;;\ Q[\![\ll v\gg/\$x]\!])$
  using *assms*
  by (*rel-auto*, *metis vwb-lens-wb wb-lens.get-put*)

lemma *seqr-insert-ident-left*:
  assumes *vwb-lens x* $\$x' \sharp P$ $\$x \sharp Q$
  shows $((\$x' =_u \$x \wedge P)\ ;;\ Q) = (P\ ;;\ Q)$
  using *assms*
  by (*rel-auto*, *meson vwb-lens-wb wb-lens-weak weak-lens.put-get*)

lemma *seqr-insert-ident-right*:
  assumes *vwb-lens x* $\$x' \sharp P$ $\$x \sharp Q$
  shows $(P\ ;;\ (\$x' =_u \$x \wedge Q)) = (P\ ;;\ Q)$
  using *assms*
  by (*rel-auto*, *metis* (*no-types*, *hide-lams*) *vwb-lens-def wb-lens-def weak-lens.put-get*)

lemma *seq-var-ident-lift*:
  assumes *vwb-lens x* $\$x' \sharp P$ $\$x \sharp Q$
  shows $((\$x' =_u \$x \wedge P)\ ;;\ (\$x' =_u \$x) \wedge Q) = (\$x' =_u \$x \wedge (P\ ;;\ Q))$
  using *assms* **apply** (*rel-auto*)
  by (*metis* (*no-types*, *lifting*) *vwb-lens-wb wb-lens-weak weak-lens.put-get*)

theorem *precond-equiv*:
  $P = (P\ ;;\ true) \longleftrightarrow (out\alpha \sharp P)$
  by (*rel-auto*)

theorem *postcond-equiv*:
  $P = (true\ ;;\ P) \longleftrightarrow (in\alpha \sharp P)$
  by (*rel-auto*)

lemma *precond-right-unit*: $out\alpha \sharp p \Longrightarrow (p\ ;;\ true) = p$
  by (*metis precond-equiv*)

lemma *postcond-left-unit*: $in\alpha \sharp p \Longrightarrow (true\ ;;\ p) = p$
  by (*metis postcond-equiv*)

theorem *precond-left-zero*:
  assumes $out\alpha \sharp p$ $p \neq false$
  shows $(true\ ;;\ p) = true$
  using *assms*
  apply (*simp add*: *out*α*-def upred-defs*)
  apply (*transfer*, *auto simp add*: *relcomp-unfold*, *rule ext*, *auto*)
  apply (*rename-tac p b*)
  apply (*subgoal-tac* $\exists$ *b1 b2*. *p* (*b1*, *b2*))
  apply (*auto*)
done

8.5 Converse laws

lemma *convr-invol* [*simp*]: $p^{--} = p$
  by *pred-auto*

lemma *lit-convr* [*simp*]: $\ll v\gg^{-} = \ll v\gg$

by *pred-auto*

lemma *uivar-convr* [*simp*]:
fixes x :: ($'a$, $'\alpha$) *uvar*
shows $(\$x)^{-} = \$x'$
by *pred-auto*

lemma *uovar-convr* [*simp*]:
fixes x :: ($'a$, $'\alpha$) *uvar*
shows $(\$x')^{-} = \$x$
by *pred-auto*

lemma *uop-convr* [*simp*]: $(uop\ f\ u)^{-} = uop\ f\ (u^{-})$
by (*pred-auto*)

lemma *bop-convr* [*simp*]: $(bop\ f\ u\ v)^{-} = bop\ f\ (u^{-})\ (v^{-})$
by (*pred-auto*)

lemma *eq-convr* [*simp*]: $(p =_u q)^{-} = (p^{-} =_u q^{-})$
by (*pred-auto*)

lemma *not-convr* [*simp*]: $(\neg\ p)^{-} = (\neg\ p^{-})$
by (*pred-auto*)

lemma *disj-convr* [*simp*]: $(p \vee q)^{-} = (q^{-} \vee p^{-})$
by (*pred-auto*)

lemma *conj-convr* [*simp*]: $(p \wedge q)^{-} = (q^{-} \wedge p^{-})$
by (*pred-auto*)

lemma *seqr-convr* [*simp*]: $(p\ ;;\ q)^{-} = (q^{-}\ ;;\ p^{-})$
by *rel-auto*

lemma *pre-convr* [*simp*]: $\lceil p \rceil_{<}{}^{-} = \lceil p \rceil_{>}$
by (*rel-auto*)

lemma *post-convr* [*simp*]: $\lceil p \rceil_{>}{}^{-} = \lceil p \rceil_{<}$
by (*rel-auto*)

theorem *seqr-pre-transfer*: $in\alpha \sharp q \Longrightarrow ((P \wedge q)\ ;;\ R) = (P\ ;;\ (q^{-} \wedge R))$
by (*rel-auto*)

theorem *seqr-pre-transfer'*:
$((P \wedge \lceil q \rceil_{>})\ ;;\ R) = (P\ ;;\ (\lceil q \rceil_{<} \wedge R))$
by (*rel-auto*)

theorem *seqr-post-out*: $in\alpha \sharp r \Longrightarrow (P\ ;;\ (Q \wedge r)) = ((P\ ;;\ Q) \wedge r)$
by (*rel-blast*)

lemma *seqr-post-var-out*:
fixes x :: (*bool*, $'\alpha$) *uvar*
shows $(P\ ;;\ (Q \wedge \$x')) = ((P\ ;;\ Q) \wedge \$x')$
by (*rel-auto*)

theorem *seqr-post-transfer*: $out\alpha \sharp q \Longrightarrow (P\ ;;\ (q \wedge R)) = (P \wedge q^{-}\ ;;\ R)$

by (*simp add*: *seqr-pre-transfer unrest-convr-in*α)

lemma *seqr-pre-out*: *out*α $\sharp$ $p \Longrightarrow ((p \wedge Q) \;;;\; R) = (p \wedge (Q \;;;\; R))$
by (*rel-blast*)

lemma *seqr-pre-var-out*:
fixes x :: (*bool*, $'\alpha$) *uvar*
shows $((\$x \wedge P) \;;;\; Q) = (\$x \wedge (P \;;;\; Q))$
by (*rel-auto*)

lemma *seqr-true-lemma*:
$(P = (\neg (\neg P \;;;\; true))) = (P = (P \;;;\; true))$
by *rel-auto*

lemma *shEx-lift-seq-1* [*uquant-lift*]:
$((\exists\ x \cdot P\ x) \;;;\; Q) = (\exists\ x \cdot (P\ x \;;;\; Q))$
by *pred-auto*

lemma *shEx-lift-seq-2* [*uquant-lift*]:
$(P \;;;\; (\exists\ x \cdot Q\ x)) = (\exists\ x \cdot (P \;;;\; Q\ x))$
by *pred-auto*

8.6 Assertions and assumptions

lemma *assume-twice*: $(b^\top \;;;\; c^\top) = (b \wedge c)^\top$
by (*rel-auto*)

lemma *assert-twice*: $(b_\bot \;;;\; c_\bot) = (b \wedge c)_\bot$
by (*rel-auto*)

8.7 Frame and antiframe

definition *frame* :: ($'a$, $'\alpha$) *lens* $\Rightarrow$ $'\alpha$ *hrelation* $\Rightarrow$ $'\alpha$ *hrelation* **where**
[*urel-defs*]: *frame* x $P = (II_x \wedge P)$

definition *antiframe* :: ($'a$, $'\alpha$) *lens* $\Rightarrow$ $'\alpha$ *hrelation* $\Rightarrow$ $'\alpha$ *hrelation* **where**
[*urel-defs*]: *antiframe* x $P = (II\restriction_\alpha x \wedge P)$

syntax
-frame :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (-:⟦-⟧ [*64*,*0*] *80*)
-antiframe :: *salpha* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (-:[-] [*64*,*0*] *80*)

translations
-frame x P == *CONST frame* x P
-antiframe x P == *CONST antiframe* x P

lemma *frame-disj*: $(x{:}[\![P]\!] \vee x{:}[\![Q]\!]) = x{:}[\![P \vee Q]\!]$
by (*rel-auto*)

lemma *frame-conj*: $(x{:}[\![P]\!] \wedge x{:}[\![Q]\!]) = x{:}[\![P \wedge Q]\!]$
by (*rel-auto*)

lemma *frame-seq*:
$[\![$ *vwb-lens* x; $\$x\acute{}\ \sharp\ P$; $\$x\ \sharp\ Q$ $]\!] \Longrightarrow (x{:}[\![P]\!] \;;;\; x{:}[\![Q]\!]) = x{:}[\![P \;;;\; Q]\!]$
by (*rel-auto*, *metis vwb-lens-def wb-lens-weak weak-lens.put-get*)

lemma *antiframe-to-frame*:
  $[\![x \bowtie y; x +_L y = 1_L]\!] \Longrightarrow x{:}[P] = y{:}[\![P]\!]$
  by (*rel-auto*, *metis lens-indep-def*, *metis lens-indep-def surj-pair*)

While loop laws

lemma *while-cond-true*:
  $((\mathit{while}\ b\ \mathit{do}\ P\ \mathit{od}) \wedge \lceil b \rceil_<) = ((P \wedge \lceil b \rceil_<)\ ;;\ \mathit{while}\ b\ \mathit{do}\ P\ \mathit{od})$
proof −
  have $(\mathit{while}\ b\ \mathit{do}\ P\ \mathit{od} \wedge \lceil b \rceil_<) = (((\lceil b \rceil_< \wedge P)^\star{}_u \wedge (\neg \lceil b \rceil_>)) \wedge \lceil b \rceil_<)$
    by (*simp add*: *while-def*)
  also have ... $= (((II \vee ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge \neg \lceil b \rceil_>) \wedge \lceil b \rceil_<)$
    by (*simp add*: *disj-upred-def*)
  also have ... $= ((\lceil b \rceil_< \wedge (II \vee ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u))) \wedge (\neg \lceil b \rceil_>))$
    by (*simp add*: *conj-comm utp-pred.inf.left-commute*)
  also have ... $= (((\lceil b \rceil_< \wedge II) \vee (\lceil b \rceil_< \wedge ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u))) \wedge (\neg \lceil b \rceil_>))$
    by (*simp add*: *conj-disj-distr*)
  also have ... $= ((((\lceil b \rceil_< \wedge II) \vee ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u))) \wedge (\neg \lceil b \rceil_>))$
    by (*subst seqr-pre-out*[*THEN sym*], *simp add*: *unrest*, *simp add*: *upred-defs urel-defs*)
  also have ... $= ((((II \wedge \lceil b \rceil_>) \vee ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u))) \wedge (\neg \lceil b \rceil_>))$
    by (*simp add*: *pre-skip-post*)
  also have ... $= ((II \wedge \lceil b \rceil_> \wedge \neg \lceil b \rceil_>) \vee (((\lceil b \rceil_< \wedge P)\ ;;\ ((\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge (\neg \lceil b \rceil_>)))$
    by (*simp add*: *utp-pred.inf.assoc utp-pred.inf-sup-distrib2*)
  also have ... $= (((\lceil b \rceil_< \wedge P)\ ;;\ ((\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge (\neg \lceil b \rceil_>))$
    by *simp*
  also have ... $= ((\lceil b \rceil_< \wedge P)\ ;;\ (((\lceil b \rceil_< \wedge P)^\star{}_u) \wedge (\neg \lceil b \rceil_>)))$
    by (*simp add*: *seqr-post-out unrest*)
  also have ... $= ((P \wedge \lceil b \rceil_<)\ ;;\ \mathit{while}\ b\ \mathit{do}\ P\ \mathit{od})$
    by (*simp add*: *utp-pred.inf-commute while-def*)
  finally show *?thesis* **.**
qed

lemma *while-cond-false*:
  $((\mathit{while}\ b\ \mathit{do}\ P\ \mathit{od}) \wedge (\neg \lceil b \rceil_<)) = (II \wedge \neg \lceil b \rceil_<)$
proof −
  have $(\mathit{while}\ b\ \mathit{do}\ P\ \mathit{od} \wedge (\neg \lceil b \rceil_<)) = (((\lceil b \rceil_< \wedge P)^\star{}_u \wedge (\neg \lceil b \rceil_>)) \wedge (\neg \lceil b \rceil_<))$
    by (*simp add*: *while-def*)
  also have ... $= (((II \vee ((\lceil b \rceil_< \wedge P)\ ;;\ (\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge \neg \lceil b \rceil_>) \wedge (\neg \lceil b \rceil_<))$
    by (*simp add*: *disj-upred-def*)
  also have ... $= (((II \wedge \neg \lceil b \rceil_>) \wedge \neg \lceil b \rceil_<) \vee ((\neg \lceil b \rceil_<) \wedge (((\lceil b \rceil_< \wedge P)\ ;;\ ((\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge \neg \lceil b \rceil_>)))$
    by (*simp add*: *conj-disj-distr utp-pred.inf.commute*)
  also have ... $= (((II \wedge \neg \lceil b \rceil_>) \wedge \neg \lceil b \rceil_<) \vee ((((\neg \lceil b \rceil_<) \wedge (\lceil b \rceil_< \wedge P)\ ;;\ ((\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge \neg \lceil b \rceil_>)))$
    by (*simp add*: *seqr-pre-out unrest-not unrest-pre-outα utp-pred.inf.assoc*)
  also have ... $= (((II \wedge \neg \lceil b \rceil_>) \wedge \neg \lceil b \rceil_<) \vee (((\mathit{false}\ ;;\ ((\lceil b \rceil_< \wedge P)^\star{}_u)) \wedge \neg \lceil b \rceil_>)))$
    by (*simp add*: *conj-comm utp-pred.inf.left-commute*)
  also have ... $= ((II \wedge \neg \lceil b \rceil_>) \wedge \neg \lceil b \rceil_<)$
    by *simp*
  also have ... $= (II \wedge \neg \lceil b \rceil_<)$
    by *rel-auto*
  finally show *?thesis* **.**
qed

theorem *while-unfold*:
  $\mathit{while}\ b\ \mathit{do}\ P\ \mathit{od} = ((P\ ;;\ \mathit{while}\ b\ \mathit{do}\ P\ \mathit{od}) \triangleleft b \triangleright_r II)$
  by (*metis* (*no-types*, *hide-lams*) *bounded-semilattice-sup-bot-class.sup-bot.left-neutral comp-cond-left-distr cond-def cond-idem disj-comm disj-upred-def seqr-right-zero upred-quantale.bot-zerol utp-pred.inf-bot-right*

utp-pred.inf-commute while-cond-false while-cond-true)

8.8 Relational unrestriction

Relational unrestriction states that a variable is unchanged by a relation. Eventually I'd also like to have it state that the relation also does not depend on the variable's initial value, but I'm not sure how to state that yet. For now we represent this by the parametric healthiness condition RID.

definition $RID :: ('a, '\alpha)\ uvar \Rightarrow '\alpha\ hrelation \Rightarrow '\alpha\ hrelation$
where $RID\ x\ P = ((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x)$

declare *RID-def* [*urel-defs*]

lemma *RID-idem*:
  $mwb\text{-}lens\ x \Longrightarrow RID(x)(RID(x)(P)) = RID(x)(P)$
  by *rel-auto*

lemma *RID-mono*:
  $P \sqsubseteq Q \Longrightarrow RID(x)(P) \sqsubseteq RID(x)(Q)$
  by *rel-auto*

lemma *RID-skip-r*:
  $vwb\text{-}lens\ x \Longrightarrow RID(x)(II) = II$
  apply *rel-auto* **using** *vwb-lens.put-eq* **by** *fastforce*

lemma *RID-disj*:
  $RID(x)(P \vee Q) = (RID(x)(P) \vee RID(x)(Q))$
  by *rel-auto*

lemma *RID-conj*:
  $vwb\text{-}lens\ x \Longrightarrow RID(x)(RID(x)(P) \wedge RID(x)(Q)) = (RID(x)(P) \wedge RID(x)(Q))$
  by *rel-auto*

lemma *RID-assigns-r-diff*:
  $[\![\ vwb\text{-}lens\ x;\ x \sharp \sigma\ ]\!] \Longrightarrow RID(x)(\langle\sigma\rangle_a) = \langle\sigma\rangle_a$
  apply (*rel-auto*)
  apply (*metis vwb-lens.put-eq*)
  apply (*metis vwb-lens-wb wb-lens.get-put wb-lens-weak weak-lens.put-get*)
done

lemma *RID-assign-r-same*:
  $vwb\text{-}lens\ x \Longrightarrow RID(x)(x := v) = II$
  apply (*rel-auto*)
  using *vwb-lens.put-eq* **apply** *fastforce*
done

lemma *RID-seq-left*:
  assumes *vwb-lens x*
  shows $RID(x)(RID(x)(P)\ ;;\ Q) = (RID(x)(P)\ ;;\ RID(x)(Q))$
proof −
  have $RID(x)(RID(x)(P)\ ;;\ Q) = ((\exists\ \$x \cdot \exists\ \$x' \cdot (\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x\ ;;\ Q) \wedge \$x' =_u \$x)$
    by (*simp add*: *RID-def usubst*)
  also from *assms* **have** ... $= (((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge (\exists\ \$x \cdot \$x' =_u \$x)\ ;;\ (\exists\ \$x' \cdot Q)) \wedge \$x' =_u \$x)$
    by (*rel-auto*)

also from *assms* **have** ... $= (((\exists\ \$x \cdot \exists\ \$x' \cdot P) ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q)) \wedge \$x' =_u \$x)$
  apply (*rel-auto*)
  apply (*metis vwb-lens.put-eq*)
  apply (*metis mwb-lens.put-put vwb-lens-mwb*)
done
also from *assms* **have** ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q)) \wedge \$x' =_u \$x)$
  by (*rel-auto*, *metis* (*full-types*) *mwb-lens.put-put vwb-lens-def wb-lens-weak weak-lens.put-get*)
also have ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge \$x' =_u \$x)) \wedge \$x' =_u \$x)$
  by (*rel-auto*, *fastforce*)
also have ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge \$x' =_u \$x)))$
  by *rel-auto*
also have ... $= (RID(x)(P) ;; RID(x)(Q))$
  by *rel-auto*
finally show *?thesis* **.**
qed

lemma *RID-seq-right*:
assumes *vwb-lens x*
shows $RID(x)(P ;; RID(x)(Q)) = (RID(x)(P) ;; RID(x)(Q))$
proof −
have $RID(x)(P ;; RID(x)(Q)) = ((\exists\ \$x \cdot \exists\ \$x' \cdot P ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge \$x' =_u \$x) \wedge \$x' =_u \$x)$
  by (*simp add*: *RID-def usubst*)
also from *assms* **have** ... $= (((\exists\ \$x \cdot\ P) ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge (\exists\ \$x' \cdot \$x' =_u \$x)) \wedge \$x' =_u \$x)$
  by (*rel-auto*)
also from *assms* **have** ... $= (((\exists\ \$x \cdot \exists\ \$x' \cdot P) ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q)) \wedge \$x' =_u \$x)$
  apply (*rel-auto*)
  apply (*metis vwb-lens.put-eq*)
  apply (*metis mwb-lens.put-put vwb-lens-mwb*)
done
also from *assms* **have** ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; (\exists\ \$x \cdot \exists\ \$x' \cdot Q)) \wedge \$x' =_u \$x)$
  by (*rel-auto*, *metis* (*full-types*) *mwb-lens.put-put vwb-lens-def wb-lens-weak weak-lens.put-get*)
also have ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge \$x' =_u \$x)) \wedge \$x' =_u \$x)$
  by (*rel-auto*, *fastforce*)
also have ... $= ((((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists\ \$x \cdot \exists\ \$x' \cdot Q) \wedge \$x' =_u \$x)))$
  by *rel-auto*
also have ... $= (RID(x)(P) ;; RID(x)(Q))$
  by *rel-auto*
finally show *?thesis* **.**
qed

definition *unrest-relation* :: $('a, '\alpha)$ *uvar* $\Rightarrow$ $'\alpha$ *hrelation* $\Rightarrow$ *bool* (**infix** $\sharp\sharp$ *20*)
where $(x \sharp\sharp P) \longleftrightarrow (P = RID(x)(P))$

declare *unrest-relation-def* [*urel-defs*]

lemma *skip-r-runrest* [*unrest*]:
$vwb\text{-}lens\ x \Longrightarrow x \sharp\sharp II$
by (*simp add*: *RID-skip-r unrest-relation-def*)

lemma *assigns-r-runrest*:
$[\![vwb\text{-}lens\ x; x \sharp \sigma]\!] \Longrightarrow x \sharp\sharp \langle\sigma\rangle_a$

by (*simp add*: *RID-assigns-r-diff unrest-relation-def*)

lemma *seq-r-runrest* [*unrest*]:
assumes *vwb-lens* x x ♯♯ P x ♯♯ Q
shows x ♯♯ $(P ;; Q)$
by (*metis RID-seq-left assms unrest-relation-def*)

lemma *false-runrest* [*unrest*]: x ♯♯ *false*
by (*rel-auto*)

lemma *and-runrest* [*unrest*]: $[\![$ *vwb-lens* x; x ♯♯ P; x ♯♯ Q $]\!] \Longrightarrow x$ ♯♯ $(P \wedge Q)$
by (*metis RID-conj unrest-relation-def*)

lemma *or-runrest* [*unrest*]: $[\![$ x ♯♯ P; x ♯♯ Q $]\!] \Longrightarrow x$ ♯♯ $(P \vee Q)$
by (*simp add*: *RID-disj unrest-relation-def*)

8.9 Alphabet laws

lemma *aext-cond* [*alpha*]:
$(P \triangleleft b \triangleright Q) \oplus_p a = ((P \oplus_p a) \triangleleft (b \oplus_p a) \triangleright (Q \oplus_p a))$
by *rel-auto*

lemma *aext-seq* [*alpha*]:
wb-lens $a \Longrightarrow ((P ;; Q) \oplus_p (a \times_L a)) = ((P \oplus_p (a \times_L a)) ;; (Q \oplus_p (a \times_L a)))$
by (*rel-auto*, *metis wb-lens-weak weak-lens.put-get*)

8.10 Relation algebra laws

theorem *RA1*: $(P ;; (Q ;; R)) = ((P ;; Q) ;; R)$
using *seqr-assoc* **by** *auto*

theorem *RA2*: $(P ;; II) = P$ $(II ;; P) = P$
by *simp-all*

theorem *RA3*: $P^{--} = P$
by *simp*

theorem *RA4*: $(P ;; Q)^{-} = (Q^{-} ;; P^{-})$
by *simp*

theorem *RA5*: $(P \vee Q)^{-} = (P^{-} \vee Q^{-})$
by *rel-auto*

theorem *RA6*: $((P \vee Q) ;; R) = ((P;;R) \vee (Q;;R))$
using *seqr-or-distl* **by** *blast*

theorem *RA7*: $((P^{-} ;; (\neg(P ;; Q))) \vee (\neg Q)) = (\neg Q)$
by (*rel-auto*)

8.11 Relational alphabet extension

lift-definition *rel-alpha-ext* :: $'\beta$ *hrelation* $\Rightarrow ('\beta \Longrightarrow '\alpha) \Rightarrow '\alpha$ *hrelation* (**infix** $\oplus_R$ *65*)
is λ P x $(b1, b2)$. P $(get_x\ b1, get_x\ b2) \wedge (\forall\ b.\ b1 \oplus_L b$ *on* $x = b2 \oplus_L b$ *on* $x)$ **.**

lemma *rel-alpha-ext-alt-def*:
assumes *vwb-lens* y $x +_L y \approx_L 1_L$ $x \bowtie y$

shows $P \oplus_R x = (P \oplus_p (x \times_L x) \wedge \$y' =_u \$y)$
using *assms*
apply (*rel-auto*, *simp-all add*: *lens-override-def*)
apply (*metis lens-indep-get lens-indep-sym*)
apply (*metis vwb-lens-def wb-lens.get-put wb-lens-def weak-lens.put-get*)
done

8.12 Program values

abbreviation *prog-val* :: $'\alpha$ *hrelation* $\Rightarrow$ ($'\alpha$ *hrelation*, $'\alpha$) *uexpr* ($\{\!|$-$|\!\}_u$)
where $\{\!|P|\!\}_u \equiv \ll P \gg$

lift-definition *call* :: ($'\alpha$ *hrelation*, $'\alpha$) *uexpr* $\Rightarrow$ $'\alpha$ *hrelation*
is λ P b. P (*fst* b) b **.**

lemma *call-prog-val*: *call* $\{\!|P|\!\}_u = P$
by (*simp add*: *call-def urel-defs lit.rep-eq Rep-uexpr-inverse*)

end

8.13 Relational Hoare calculus

theory *utp-hoare*
imports *utp-rel*
begin

named-theorems *hoare*

definition *hoare-r* :: $'\alpha$ *condition* $\Rightarrow$ $'\alpha$ *hrelation* $\Rightarrow$ $'\alpha$ *condition* $\Rightarrow$ *bool* ($\{\!|$-$|\!\}$-$\{\!|$-$|\!\}_u$) **where**
$\{\!|p|\!\}Q\{\!|r|\!\}_u = ((\lceil p \rceil_< \Rightarrow \lceil r \rceil_>) \sqsubseteq Q)$

declare *hoare-r-def* [*upred-defs*]

lemma *hoare-r-conj* [*hoare*]: $\llbracket$ $\{\!|p|\!\}Q\{\!|r|\!\}_u$; $\{\!|p|\!\}Q\{\!|s|\!\}_u$ $\rrbracket \Longrightarrow \{\!|p|\!\}Q\{\!|r \wedge s|\!\}_u$
by *rel-auto*

lemma *hoare-r-conseq* [*hoare*]: $\llbracket$ `$p_1 \Rightarrow p_2$`; $\{\!|p_2|\!\}S\{\!|q_2|\!\}_u$; `$q_2 \Rightarrow q_1$` $\rrbracket \Longrightarrow \{\!|p_1|\!\}S\{\!|q_1|\!\}_u$
by *rel-auto*

lemma *assigns-hoare-r* [*hoare*]: `$p \Rightarrow \sigma \dagger q$` $\Longrightarrow \{\!|p|\!\}\langle\sigma\rangle_a\{\!|q|\!\}_u$
by *rel-auto*

lemma *skip-hoare-r* [*hoare*]: $\{\!|p|\!\}II\{\!|p|\!\}_u$
by *rel-auto*

lemma *seq-hoare-r* [*hoare*]: $\llbracket$ $\{\!|p|\!\}Q_1\{\!|s|\!\}_u$; $\{\!|s|\!\}Q_2\{\!|r|\!\}_u$ $\rrbracket \Longrightarrow \{\!|p|\!\}Q_1 \;;;\; Q_2\{\!|r|\!\}_u$
by *rel-auto*

lemma *cond-hoare-r* [*hoare*]: $\llbracket$ $\{\!|b \wedge p|\!\}S\{\!|q|\!\}_u$; $\{\!|\neg b \wedge p|\!\}T\{\!|q|\!\}_u$ $\rrbracket \Longrightarrow \{\!|p|\!\}S \triangleleft b \triangleright_r T\{\!|q|\!\}_u$
by *rel-auto*

lemma *while-hoare-r* [*hoare*]:
assumes $\{\!|p \wedge b|\!\}S\{\!|p|\!\}_u$
shows $\{\!|p|\!\}$*while* b *do* S *od*$\{\!|\neg b \wedge p|\!\}_u$
proof −

from *assms* **have** $(\lceil p \rceil_< \Rightarrow \lceil p \rceil_>) \sqsubseteq (II \sqcap ((\lceil b \rceil_< \wedge S) \;;\; (\lceil p \rceil_< \Rightarrow \lceil p \rceil_>)))$
  by (*simp add*: *hoare-r-def*) (*rel-auto*)
hence p: $(\lceil p \rceil_< \Rightarrow \lceil p \rceil_>) \sqsubseteq (\lceil b \rceil_< \wedge S)^\star{}_u$
  by (*rule upred-quantale.star-inductl-one*[*rule-format*])
have $(\neg \lceil b \rceil_> \wedge \lceil p \rceil_>) \sqsubseteq ((\lceil p \rceil_< \wedge (\lceil p \rceil_< \Rightarrow \lceil p \rceil_>)) \wedge (\neg \lceil b \rceil_>))$
  by (*rel-auto*)
with p **have** $(\neg \lceil b \rceil_> \wedge \lceil p \rceil_>) \sqsubseteq ((\lceil p \rceil_< \wedge (\lceil b \rceil_< \wedge S)^\star{}_u) \wedge (\neg \lceil b \rceil_>))$
  by (*meson order-refl order-trans utp-pred.inf-mono*)
thus *?thesis*
  unfolding *hoare-r-def while-def*
  by (*auto intro*: *spec-refine simp add*: *alpha utp-pred.conj-assoc*)
qed

lemma *while-invr-hoare-r* [*hoare*]:
  assumes $\{\!|p \wedge b|\!\} S \{\!|p|\!\}_u$ *‘pre ⇒ p‘ ‘(¬b ∧ p) ⇒ post‘*
  shows $\{\!|pre|\!\}$ *while b invr p do S od* $\{\!|post|\!\}_u$
  by (*metis assms hoare-r-conseq while-hoare-r while-inv-def*)

end

8.14 Weakest precondition calculus

theory *utp-wp*
imports *utp-hoare*
begin

A very quick implementation of wp – more laws still needed!

named-theorems *wp*

method *wp-tac* = (*simp add*: *wp*)

consts
  uwp :: $'a \Rightarrow 'b \Rightarrow 'c$ (**infix** *wp 60*)

definition *wp-upred* :: $('\alpha, '\beta)$ *relation* $\Rightarrow$ $'\beta$ *condition* $\Rightarrow$ $'\alpha$ *condition* **where**
wp-upred Q r = $\lfloor \neg\, (Q \;;\; \neg\, \lceil r \rceil_<) :: ('\alpha, '\beta)\ relation \rfloor_<$

adhoc-overloading
  uwp wp-upred

declare *wp-upred-def* [*urel-defs*]

theorem *wp-assigns-r* [*wp*]:
  $\langle \sigma \rangle_a$ *wp r* = $\sigma \dagger r$
  by *rel-auto*

theorem *wp-skip-r* [*wp*]:
  II wp r = r
  by *rel-auto*

theorem *wp-true* [*wp*]:
  $r \neq true \Longrightarrow true\ wp\ r = false$
  by *rel-auto*

theorem *wp-conj* [*wp*]:
  $P\ wp\ (q \wedge r) = (P\ wp\ q \wedge P\ wp\ r)$

by *rel-auto*

theorem *wp-seq-r* [*wp*]: $(P ;; Q)$ *wp* $r = P$ *wp* $(Q$ *wp* $r)$
by *rel-auto*

theorem *wp-cond* [*wp*]: $(P \triangleleft b \triangleright_r Q)$ *wp* $r = ((b \Rightarrow P$ *wp* $r) \wedge ((\neg b) \Rightarrow Q$ *wp* $r))$
by *rel-auto*

theorem *wp-hoare-link*:
$\{\!|p|\!\} Q \{\!|r|\!\}_u \longleftrightarrow (Q$ *wp* $r \sqsubseteq p)$
by *rel-auto*

end

9 Relational operational semantics

theory *utp-rel-opsem*
imports *utp-rel*
begin

fun *trel* :: $'\alpha$ *usubst* $\times$ $'\alpha$ *hrelation* $\Rightarrow$ $'\alpha$ *usubst* $\times$ $'\alpha$ *hrelation* $\Rightarrow$ *bool* (**infix** $\rightarrow_u$ *85*) **where**
$(\sigma, P) \rightarrow_u (\varrho, Q) \longleftrightarrow (\langle\sigma\rangle_a ;; P) \sqsubseteq (\langle\varrho\rangle_a ;; Q)$

lemma *trans-trel*:
$[\![(\sigma, P) \rightarrow_u (\varrho, Q); (\varrho, Q) \rightarrow_u (\varphi, R)]\!] \Longrightarrow (\sigma, P) \rightarrow_u (\varphi, R)$
by *auto*

lemma *skip-trel*: $(\sigma, II) \rightarrow_u (\sigma, II)$
by *simp*

lemma *assigns-trel*: $(\sigma, \langle\varrho\rangle_a) \rightarrow_u (\varrho \circ \sigma, II)$
by (*simp add*: *assigns-comp*)

lemma *assign-trel*:
fixes x :: $('a, '\alpha)$ *uvar*
assumes *uvar x*
shows $(\sigma, x := v) \rightarrow_u (\sigma(x \mapsto_s \sigma \dagger v), II)$
by (*simp add*: *assigns-comp subst-upd-comp*)

lemma *seq-trel*:
assumes $(\sigma, P) \rightarrow_u (\varrho, Q)$
shows $(\sigma, P ;; R) \rightarrow_u (\varrho, Q ;; R)$
by (*metis* (*no-types*, *lifting*) *assms seqr-assoc trel.simps upred-quantale.mult-isor*)

lemma *seq-skip-trel*:
$(\sigma, II ;; P) \rightarrow_u (\sigma, P)$
by *simp*

lemma *nondet-left-trel*:
$(\sigma, P \sqcap Q) \rightarrow_u (\sigma, P)$
by (*simp add*: *upred-quantale.subdistl*)

lemma *nondet-right-trel*:
$(\sigma, P \sqcap Q) \rightarrow_u (\sigma, Q)$
using *nondet-left-trel* **by** *force*

lemma *rcond-true-trel*:
assumes $\sigma \dagger b = true$
shows $(\sigma, P \triangleleft b \triangleright_r Q) \rightarrow_u (\sigma, P)$
using *assms*
by (*simp add*: *assigns-r-comp usubst aext-true cond-unit-T*)

lemma *rcond-false-trel*:
assumes $\sigma \dagger b = false$
shows $(\sigma, P \triangleleft b \triangleright_r Q) \rightarrow_u (\sigma, Q)$
using *assms*
by (*simp add*: *assigns-r-comp usubst aext-false cond-unit-F*)

lemma *while-true-trel*:
assumes $\sigma \dagger b = true$
shows $(\sigma, \textit{while } b \textit{ do } P \textit{ od}) \rightarrow_u (\sigma, P \;;;\; \textit{while } b \textit{ do } P \textit{ od})$
by (*metis assms rcond-true-trel while-unfold*)

lemma *while-false-trel*:
assumes $\sigma \dagger b = false$
shows $(\sigma, \textit{while } b \textit{ do } P \textit{ od}) \rightarrow_u (\sigma, II)$
by (*metis assms rcond-false-trel while-unfold*)

declare *trel.simps* [*simp del*]

end

10 UTP Theories

theory *utp-theory*
imports *utp-rel*
begin

10.1 Complete lattice of predicates

definition *upred-lattice* :: $('\alpha\ upred)\ gorder\ (\mathcal{P})$ **where**
$upred\text{-}lattice = (\!|\ carrier = UNIV,\ eq = (op =),\ le = op \sqsubseteq\ |\!)$

interpretation *upred-lattice*: *complete-lattice* $\mathcal{P}$
proof (*unfold-locales*, *simp-all add*: *upred-lattice-def*)
fix $A :: '\alpha\ upred\ set$
show $\exists s.\ is\text{-}lub\ (\!|carrier = UNIV,\ eq = op =,\ le = op \sqsubseteq|\!)\ s\ A$
apply (*rule-tac* $x{=}\bigsqcup A$ **in** *exI*)
apply (*rule least-UpperI*)
apply (*auto intro*: *Inf-greatest simp add*: *Inf-lower Upper-def*)
done
show $\exists i.\ is\text{-}glb\ (\!|carrier = UNIV,\ eq = op =,\ le = op \sqsubseteq|\!)\ i\ A$
apply (*rule-tac* $x{=}\bigsqcap A$ **in** *exI*)
apply (*rule greatest-LowerI*)
apply (*auto intro*: *Sup-least simp add*: *Sup-upper Lower-def*)
done
qed

lemma *upred-weak-complete-lattice* [*simp*]: *weak-complete-lattice* $\mathcal{P}$
by (*simp add*: *upred-lattice.weak.weak-complete-lattice-axioms*)

lemma *upred-lattice-eq* [*simp*]:
  $op\ .=_{\mathcal{P}} = op\ =$
  by (*simp add*: *upred-lattice-def*)

lemma *upred-lattice-le* [*simp*]:
  $le\ \mathcal{P}\ P\ Q = (P \sqsubseteq Q)$
  by (*simp add*: *upred-lattice-def*)

lemma *upred-lattice-carrier* [*simp*]:
  $carrier\ \mathcal{P} = UNIV$
  by (*simp add*: *upred-lattice-def*)

10.2 Healthiness conditions

type-synonym $'\alpha$ *Healthiness-condition* $= {'\alpha}\ upred \Rightarrow {'\alpha}\ upred$

definition
$Healthy{::}{'\alpha}\ upred \Rightarrow {'\alpha}\ Healthiness\text{-}condition \Rightarrow bool$ (**infix** *is 30*)
where $P\ is\ H \equiv (H\ P = P)$

lemma *Healthy-def'*: $P\ is\ H \longleftrightarrow (H\ P = P)$
  unfolding *Healthy-def* **by** *auto*

lemma *Healthy-if*: $P\ is\ H \Longrightarrow (H\ P = P)$
  unfolding *Healthy-def* **by** *auto*

declare *Healthy-def'* [*upred-defs*]

abbreviation *Healthy-carrier* :: $'\alpha$ *Healthiness-condition* $\Rightarrow {'\alpha}\ upred\ set$ ($[\![-]\!]_H$)
where $[\![H]\!]_H \equiv \{P.\ P\ is\ H\}$

definition $Idempotent(H) \longleftrightarrow (\forall\ P.\ H(H(P)) = H(P))$

definition $Monotonic(H) \longleftrightarrow (\forall\ P\ Q.\ Q \sqsubseteq P \longrightarrow (H(Q) \sqsubseteq H(P)))$

definition $IMH(H) \longleftrightarrow Idempotent(H) \land Monotonic(H)$

definition $Antitone(H) \longleftrightarrow (\forall\ P\ Q.\ Q \sqsubseteq P \longrightarrow (H(P) \sqsubseteq H(Q)))$

lemma *Idempotent-id* [*simp*]: *Idempotent id*
  by (*simp add*: *Idempotent-def*)

lemma *Idempotent-comp* [*intro*]:
  $[\![\ Idempotent\ f;\ Idempotent\ g;\ f \circ g = g \circ f\ ]\!] \Longrightarrow Idempotent\ (f \circ g)$
  by (*auto simp add*: *Idempotent-def comp-def*, *metis*)

lemma *Monotonic-id* [*simp*]: *Monotonic id*
  by (*simp add*: *Monotonic-def*)

lemma *Monotonic-comp* [*intro*]:
  $[\![\ Monotonic\ f;\ Monotonic\ g\ ]\!] \Longrightarrow Monotonic\ (f \circ g)$
  by (*auto simp add*: *Monotonic-def*)

definition NM : $NM(P) = (\neg\ P \wedge true)$

lemma $Monotonic(NM)$
  apply $(simp\ add{:}Monotonic\text{-}def)$
  nitpick
  oops

lemma $Antitone(NM)$
  by $(simp\ add{:}Antitone\text{-}def\ NM)$

definition $Conjunctive :: {'\alpha}\ Healthiness\text{-}condition \Rightarrow bool$ **where**
  $Conjunctive(H) \longleftrightarrow (\exists\ Q.\ \forall\ P.\ H(P) = (P \wedge Q))$

lemma $Conjuctive\text{-}Idempotent$:
  $Conjunctive(H) \Longrightarrow Idempotent(H)$
  by $(auto\ simp\ add{:}\ Conjunctive\text{-}def\ Idempotent\text{-}def)$

lemma $Conjunctive\text{-}Monotonic$:
  $Conjunctive(H) \Longrightarrow Monotonic(H)$
  unfolding $Conjunctive\text{-}def\ Monotonic\text{-}def$
  using $dual\text{-}order.trans$ **by** $fastforce$

lemma $Conjunctive\text{-}conj$:
  assumes $Conjunctive(HC)$
  shows $HC(P \wedge Q) = (HC(P) \wedge Q)$
  using $assms$ **unfolding** $Conjunctive\text{-}def$
  by $(metis\ utp\text{-}pred.inf.assoc\ utp\text{-}pred.inf.commute)$

lemma $Conjunctive\text{-}distr\text{-}conj$:
  assumes $Conjunctive(HC)$
  shows $HC(P \wedge Q) = (HC(P) \wedge HC(Q))$
  using $assms$ **unfolding** $Conjunctive\text{-}def$
  by $(metis\ Conjunctive\text{-}conj\ assms\ utp\text{-}pred.inf.assoc\ utp\text{-}pred.inf\text{-}right\text{-}idem)$

lemma $Conjunctive\text{-}distr\text{-}disj$:
  assumes $Conjunctive(HC)$
  shows $HC(P \vee Q) = (HC(P) \vee HC(Q))$
  using $assms$ **unfolding** $Conjunctive\text{-}def$
  using $utp\text{-}pred.inf\text{-}sup\text{-}distrib2$ **by** $fastforce$

lemma $Conjunctive\text{-}distr\text{-}cond$:
  assumes $Conjunctive(HC)$
  shows $HC(P \triangleleft b \triangleright Q) = (HC(P) \triangleleft b \triangleright HC(Q))$
  using $assms$ **unfolding** $Conjunctive\text{-}def$
  by $(metis\ cond\text{-}conj\text{-}distr\ utp\text{-}pred.inf\text{-}commute)$

definition $FunctionalConjunctive :: {'\alpha}\ Healthiness\text{-}condition \Rightarrow bool$ **where**
$FunctionalConjunctive(H) \longleftrightarrow (\exists\ F.\ \forall\ P.\ H(P) = (P \wedge F(P)) \wedge Monotonic(F))$

definition $WeakConjunctive :: {'\alpha}\ Healthiness\text{-}condition \Rightarrow bool$ **where**
$WeakConjunctive(H) \longleftrightarrow (\forall\ P.\ \exists\ Q.\ H(P) = (P \wedge Q))$

lemma $FunctionalConjunctive\text{-}Monotonic$:
  $FunctionalConjunctive(H) \Longrightarrow Monotonic(H)$
  unfolding $FunctionalConjunctive\text{-}def$ **by** $(metis\ Monotonic\text{-}def\ utp\text{-}pred.inf\text{-}mono)$

lemma *WeakConjunctive-Refinement*:
assumes *WeakConjunctive*(*HC*)
shows $P \sqsubseteq HC(P)$
using *assms* **unfolding** *WeakConjunctive-def* **by** (*metis utp-pred.inf.cobounded1*)

lemma *WeakCojunctive-Healthy-Refinement*:
assumes *WeakConjunctive*(*HC*) **and** *P is HC*
shows $HC(P) \sqsubseteq P$
using *assms* **unfolding** *WeakConjunctive-def Healthy-def* **by** *simp*

lemma *WeakConjunctive-implies-WeakConjunctive*:
$Conjunctive(H) \Longrightarrow WeakConjunctive(H)$
unfolding *WeakConjunctive-def Conjunctive-def* **by** *pred-auto*

declare *Conjunctive-def* [*upred-defs*]
declare *Monotonic-def* [*upred-defs*]

lemma *Healthy-fixed-points* [*simp*]: $fps\ \mathcal{P}\ H = [\![H]\!]_H$
by (*simp add*: *fps-def upred-lattice-def Healthy-def*)

lemma *upred-lattice-Idempotent* [*simp*]: $Idem_{\mathcal{P}}\ H = Idempotent\ H$
using *upred-lattice.weak-partial-order-axioms* **by** (*auto simp add*: *idempotent-def Idempotent-def*)

lemma *upred-lattice-Monotonic* [*simp*]: $Mono_{\mathcal{P}}\ H = Monotonic\ H$
using *upred-lattice.weak-partial-order-axioms* **by** (*auto simp add*: *isotone-def Monotonic-def*)

10.3 UTP theory hierarchy

Unfortunately we can currently only characterise UTP theories of homogeneous relations; this is due to restrictions in the instantiation of Isabelle's polymorphic constants.

consts
utp-hcond :: $('\mathcal{T} \times '\alpha)$ *itself* $\Rightarrow$ $('\alpha \times '\alpha)$ *Healthiness-condition* $(\mathcal{H}1)$
utp-unit :: $('\mathcal{T} \times '\alpha)$ *itself* $\Rightarrow$ $'\alpha$ *hrelation* $(\mathcal{II}1)$

definition *utp-order* :: $('\mathcal{T} \times '\alpha)$ *itself* $\Rightarrow$ $'\alpha$ *hrelation gorder* **where**
$utp\text{-}order\ T = (\!|\ carrier = \{P.\ P\ is\ \mathcal{H}_T\},\ eq = (op =),\ le = op \sqsubseteq\ |\!)$

lemma *utp-order-carrier* [*simp*]:
$carrier\ (utp\text{-}order\ T) = [\![\mathcal{H}_T]\!]_H$
by (*simp add*: *utp-order-def*)

lemma *utp-order-eq* [*simp*]:
$eq\ (utp\text{-}order\ T) = op =$
by (*simp add*: *utp-order-def*)

lemma *utp-order-le* [*simp*]:
$le\ (utp\text{-}order\ T) = op \sqsubseteq$
by (*simp add*: *utp-order-def*)

lemma *utp-partial-order*: *partial-order* (*utp-order T*)
by (*unfold-locales*, *simp-all add*: *utp-order-def*)

lemma *utp-weak-partial-order*: *weak-partial-order* (*utp-order T*)
by (*unfold-locales*, *simp-all add*: *utp-order-def*)

lemma *mono-Monotone-utp-order*:
  mono f $\Longrightarrow$ *Monotone* (*utp-order T*) *f*
  apply (*auto simp add*: *isotone-def*)
  apply (*metis partial-order-def utp-partial-order*)
  apply (*metis monoD*)
done

lemma *isotone-utp-orderI*: *Monotonic H* $\Longrightarrow$ *isotone* (*utp-order X*) (*utp-order Y*) *H*
  by (*auto simp add*: *Monotonic-def isotone-def utp-weak-partial-order*)

UTP order is the fixed point lattice

lemma *utp-order-fpl*: *utp-order T* = *fpl* $\mathcal{P}$ ($\mathcal{H}_T$)
  by (*auto simp add*: *utp-order-def upred-lattice-def fps-def Healthy-def*)

locale *utp-theory* =
  fixes $\mathcal{T}$ :: ($'\mathcal{T}$ × $'\alpha$) *itself* (**structure**)
  assumes *HCond-Idem*: $\mathcal{H}(\mathcal{H}(P)) = \mathcal{H}(P)$
begin
  lemma *HCond-Idempotent* [*intro*]: *Idempotent* $\mathcal{H}$
    by (*simp add*: *Idempotent-def HCond-Idem*)

  sublocale *partial-order utp-order* $\mathcal{T}$
    by (*unfold-locales*, *simp-all add*: *utp-order-def*)
end

locale *utp-theory-lattice* = *utp-theory* $\mathcal{T}$ + *complete-lattice utp-order* $\mathcal{T}$ **for** $\mathcal{T}$ :: ($'\mathcal{T}$ × $'\alpha$) *itself* (**structure**)

abbreviation *utp-top* (⊤ı)
where *utp-top* $\mathcal{T}$ ≡ *atop* (*utp-order* $\mathcal{T}$)

abbreviation *utp-bottom* (⊥ı)
where *utp-bottom* $\mathcal{T}$ ≡ *abottom* (*utp-order* $\mathcal{T}$)

lemma *upred-top*: $\top_{\mathcal{P}}$ = *false*
  using *ball-UNIV greatest-def* **by** *fastforce*

lemma *upred-bottom*: $\bot_{\mathcal{P}}$ = *true*
  by *fastforce*

locale *utp-theory-mono* = *utp-theory* +
  assumes *HCond-Mono* [*intro*]: *Monotonic* $\mathcal{H}$
begin
  interpretation *utp-theory-lattice*
  proof –

Use Knaster-Tarski theorem to obtain complete lattice

    interpret *weak-complete-lattice fpl* $\mathcal{P}$ $\mathcal{H}$
      by (*rule Knaster-Tarski*, *auto simp add*: *upred-lattice.weak.weak-complete-lattice-axioms*)

    have *complete-lattice* (*fpl* $\mathcal{P}$ $\mathcal{H}$)
      by (*unfold-locales*, *simp add*: *fps-def sup-exists*, (*blast intro*: *sup-exists inf-exists*)+)

    hence *complete-lattice* (*utp-order* $\mathcal{T}$)

```
    by (simp add: utp-order-def, simp add: upred-lattice-def)

  thus utp-theory-lattice 𝒯
    by (simp add: utp-theory-axioms utp-theory-lattice-def)
 qed
end

sublocale utp-theory-mono ⊆ utp-theory-lattice
proof −
```

Use Knaster-Tarski theorem to obtain complete lattice

```
 interpret weak-complete-lattice fpl 𝒫 ℋ
  by (rule Knaster-Tarski, auto simp add: upred-lattice.weak.weak-complete-lattice-axioms)

 have complete-lattice (fpl 𝒫 ℋ)
  by (unfold-locales, simp add: fps-def sup-exists, (blast intro: sup-exists inf-exists)+)

 hence complete-lattice (utp-order 𝒯)
  by (simp add: utp-order-def, simp add: upred-lattice-def)

 thus utp-theory-lattice 𝒯
  by (simp add: utp-theory-axioms utp-theory-lattice-def)
qed

context utp-theory-mono
begin

 lemma healthy-top: ⊤ = ℋ(false)
 proof −
  have ⊤ = ⊤_{fpl 𝒫 ℋ}
   by (simp add: utp-order-fpl)
  also have ... = ℋ ⊤_𝒫
   using Knaster-Tarski-idem-extremes(1)[of 𝒫 ℋ]
   by (simp add: HCond-Idempotent HCond-Mono)
  also have ... = ℋ false
   by (simp add: upred-top)
  finally show ?thesis .
 qed

 lemma healthy-bottom: ⊥ = ℋ(true)
 proof −
  have ⊥ = ⊥_{fpl 𝒫 ℋ}
   by (simp add: utp-order-fpl)
  also have ... = ℋ ⊥_𝒫
   using Knaster-Tarski-idem-extremes(2)[of 𝒫 ℋ]
   by (simp add: HCond-Idempotent HCond-Mono)
  also have ... = ℋ true
   by (simp add: upred-bottom)
  finally show ?thesis .
 qed

end

locale utp-theory-left-unital =
 utp-theory +
```

assumes *Healthy-Left-Unit*: $\mathcal{II}$ *is* $\mathcal{H}$
and *Left-Unit*: P *is* $\mathcal{H} \Longrightarrow (\mathcal{II} \;;;\; P) = P$

locale *utp-theory-right-unital* =
utp-theory +
assumes *Healthy-Right-Unit*: $\mathcal{II}$ *is* $\mathcal{H}$
and *Right-Unit*: P *is* $\mathcal{H} \Longrightarrow (P \;;;\; \mathcal{II}) = P$

locale *utp-theory-unital* =
utp-theory +
assumes *Healthy-Unit*: $\mathcal{II}$ *is* $\mathcal{H}$
and *Unit-Left*: P *is* $\mathcal{H} \Longrightarrow (\mathcal{II} \;;;\; P) = P$
and *Unit-Right*: P *is* $\mathcal{H} \Longrightarrow (P \;;;\; \mathcal{II}) = P$

locale *utp-theory-mono-unital* = *utp-theory-mono* + *utp-theory-unital*

sublocale *utp-theory-unital* ⊆ *utp-theory-left-unital*
by (*simp add*: *Healthy-Unit Unit-Left utp-theory-axioms utp-theory-left-unital-axioms-def utp-theory-left-unital-def*)

sublocale *utp-theory-unital* ⊆ *utp-theory-right-unital*
by (*simp add*: *Healthy-Unit Unit-Right utp-theory-axioms utp-theory-right-unital-axioms-def utp-theory-right-unital-def*)

typedef *REL* = *UNIV* :: *unit set* **..**

abbreviation $REL \equiv TYPE(REL \times '\alpha)$

overloading
rel-hcond == *utp-hcond* :: $(REL \times '\alpha)$ *itself* ⇒ $('\alpha \times '\alpha)$ *Healthiness-condition*
rel-unit == *utp-unit* :: $(REL \times '\alpha)$ *itself* ⇒ $'\alpha$ *hrelation*
begin
definition *rel-hcond* :: $(REL \times '\alpha)$ *itself* ⇒ $('\alpha \times '\alpha)$ *upred* ⇒ $('\alpha \times '\alpha)$ *upred* **where**
rel-hcond T = *id*

definition *rel-unit* :: $(REL \times '\alpha)$ *itself* ⇒ $'\alpha$ *hrelation* **where**
rel-unit T = *II*
end

interpretation *rel-theory*: *utp-theory-mono-unital REL*
by (*unfold-locales*, *simp-all add*: *rel-hcond-def rel-unit-def Healthy-def*)

lemma *REL-top*: $\top_{REL}$ = *false*
by (*simp add*: *rel-hcond-def rel-theory.healthy-top*)

lemma *REL-bottom*: $\bot_{REL}$ = *true*
by (*simp add*: *rel-hcond-def rel-theory.healthy-bottom*)

10.4 Theory links

definition *mk-conn* (- ←⟨-,-⟩→ - [*90*,*0*,*0*,*91*] *91*) **where**
$T1 \leftarrow\langle\mathcal{H}_1,\mathcal{H}_2\rangle\rightarrow T2 \equiv$ (| *orderA* = *utp-order T1*, *orderB* = *utp-order T2*, *lower* = $\mathcal{H}_2$, *upper* = $\mathcal{H}_1$ |)

lemma *mk-conn-orderA* [*simp*]: $\mathcal{X}_{T1 \leftarrow\langle\mathcal{H}_1,\mathcal{H}_2\rangle\rightarrow T2}$ = *utp-order T1*
by (*simp add*:*mk-conn-def*)

lemma *mk-conn-orderB* [*simp*]: $\mathcal{Y}_{T1 \leftarrow\langle\mathcal{H}_1,\mathcal{H}_2\rangle\rightarrow T2}$ = *utp-order T2*

by (*simp add*:*mk-conn-def*)

lemma *mk-conn-lower* [*simp*]: $\pi_{*\,T1} \leftarrow\langle H_1, H_2\rangle\rightarrow {}_{T2} = H_1$
by (*simp add*: *mk-conn-def*)

lemma *mk-conn-upper* [*simp*]: $\pi^{*}{}_{T1} \leftarrow\langle H_1, H_2\rangle\rightarrow {}_{T2} = H_2$
by (*simp add*: *mk-conn-def*)

end

11 Example UTP theory: Boyle's laws

In order to exemplify the use of Isabelle/UTP, we mechanise a simple theory representing Boyle's law. Boyle's law states that, for an ideal gas at fixed temperature, pressure p is inversely proportional to volume V, or more formally that for $k = p \cdot V$ is invariant, for constant k. We here encode this as a simple UTP theory. We first create a record to represent the alphabet of the theory consisting of the three variables k, p and V.

record *alpha-boyle* =
boyle-k :: *real*
boyle-p :: *real*
boyle-V :: *real*

declare *alpha-boyle.splits* [*alpha-splits*]

The two locale interpretations below are a technicality to improve automatic proof support via the predicate and relational tactics. This is to enable the (re-)interpretation of state spaces to remove any occurrences of lens types after the proof tactics *pred-simp* and *rel-simp*, or any of their derivatives have been applied. Eventually, it would be desirable to automate both interpretations as part of a custom outer command for defining alphabets.

interpretation *alpha-boyle-prd*: — Closed records are sufficient here.
lens-interp λr::*alpha-boyle*. (*boyle-k r*, *boyle-p r*, *boyle-V r*)
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

interpretation *alpha-boyle-rel*: — Closed records are sufficient here.
lens-interp λ(*r*::*alpha-boyle*, *r'*::*alpha-boyle*).
(*boyle-k r*, *boyle-k r'*, *boyle-p r*, *boyle-p r'*, *boyle-V r*, *boyle-V r'*)
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

For now we have to explicitly cast the fields to lenses using the VAR syntactic transformation function [3] – in the future this will be automated. We also have to add the definitional equations for these variables to the simplification set for predicates to enable automated proof through our tactics.

definition k :: *real* $\Longrightarrow$ *alpha-boyle* **where** k = *VAR boyle-k*
definition p :: *real* $\Longrightarrow$ *alpha-boyle* **where** p = *VAR boyle-p*
definition V :: *real* $\Longrightarrow$ *alpha-boyle* **where** V = *VAR boyle-V*

declare *k-def* [*upred-defs*] **and** *p-def* [*upred-defs*] **and** *V-def* [*upred-defs*]

We also prove that our new lenses are well-behaved and independent of each other. A selection of these properties are shown below.

lemma *vwb-lens-k* [*simp*]: *vwb-lens k*
  by (*unfold-locales*, *simp-all add*: *k-def*)
lemma *boyle-indeps* [*simp*]:
  $k \bowtie p$ $p \bowtie k$ $k \bowtie V$ $V \bowtie k$ $p \bowtie V$ $V \bowtie p$
  by (*simp-all add*: *k-def p-def V-def lens-indep-def*)

11.1 Static invariant

We first create a simple UTP theory representing Boyle's laws on a single state, as a static invariant healthiness condition. We state Boyle's law using the function B, which recalculates the value of the constant k based on p and V.

definition $B(\varphi) = ((\exists\ k \cdot \varphi) \wedge (\&k =_u \&p{\cdot}\&V))$

We can then prove that B is both idempotent and monotone simply by application of the predicate tactic. Idempotence means that healthy predicates cannot be made more healthy. Together with idempotence, monotonicity ensures that image of the healthiness functions forms a complete lattice, which is useful to allow the representation of recursive and iterative constructions with the theory.

lemma *B-idempotent*: $B(B(P)) = B(P)$
  by *pred-auto'*

lemma *B-monotone*: $X \sqsubseteq Y \Longrightarrow B(X) \sqsubseteq B(Y)$
  by *pred-auto'*

We also create some example observations; the first (φ_1) satisfies Boyle's law and the second doesn't (φ_2).

definition $\varphi_1 = ((\&p =_u 10) \wedge (\&V =_u 5) \wedge (\&k =_u 50))$
definition $\varphi_2 = ((\&p =_u 10) \wedge (\&V =_u 5) \wedge (\&k =_u 100))$

We first prove an obvious property: that these two predicates are different observations. We must show that there exists a valuation of one which is not of the other. This is achieved through application of *pred-tac*, followed by *sledgehammer* [2] which yields a *metis* proof.

lemma φ_1*-diff-*φ_2: $\varphi_1 \neq \varphi_2$
  by (*pred-auto*, *metis select-convs num.distinct*(*5*) *numeral-eq-iff semiring-norm*(*87*))

We prove that φ_1 satisfies Boyle's law by application of the predicate calculus tactic, *pred-tac*.

lemma *B-*φ_1: φ_1 *is B*
  by (*pred-auto*)

We prove that φ_2 does not satisfy Boyle's law by showing that applying B to it results in φ_1. We prove this using Isabelle's natural proof language, Isar.

lemma *B-*φ_2: $B(\varphi_2) = \varphi_1$
proof −
  have $B(\varphi_2) = B(\&p =_u 10 \wedge \&V =_u 5 \wedge \&k =_u 100)$
    by (*simp add*: φ_2*-def*)
  also have ... $= ((\exists\ k \cdot \&p =_u 10 \wedge \&V =_u 5 \wedge \&k =_u 100) \wedge \&k =_u \&p{\cdot}\&V)$
    by (*simp add*: *B-def*)
  also have ... $= (\&p =_u 10 \wedge \&V =_u 5 \wedge \&k =_u \&p{\cdot}\&V)$

by *pred-auto*
also have ... = $(\&p =_u 10 \wedge \&V =_u 5 \wedge \&k =_u 50)$
by *pred-auto*
also have ... = φ_1
by *(simp add: φ_1-def)*
finally show *?thesis* **.**
qed

11.2 Dynamic invariants

Next we build a relational theory that allows the pressure and volume to be changed, whilst still respecting Boyle's law. We create two dynamic invariants for this purpose.

definition $D1(P) = ((\$k =_u \$p{\cdot}\$V \Rightarrow \$k' =_u \$p'{\cdot}\$V') \wedge P)$
definition $D2(P) = (\$k' =_u \$k \wedge P)$

D1 states that if Boyle's law satisfied in the previous state, then it should be satisfied in the next state. We define this by conjunction of the formal specification of this property with the predicate. The annotations $\$p$ and $\$p'$ refer to relational variables p and p'. *D2* states that the constant k indeed remains constant throughout the evolution of the system, which is also specified as a conjunctive healthiness condition. As before we demonstrate that *D1* and *D2* are both idempotent and monotone.

lemma *D1-idempotent*: $D1(D1(P)) = D1(P)$ **by** *rel-auto*
lemma *D2-idempotent*: $D2(D2(P)) = D2(P)$ **by** *rel-auto*

lemma *D1-monotone*: $X \sqsubseteq Y \Longrightarrow D1(X) \sqsubseteq D1(Y)$ **by** *rel-auto*
lemma *D2-monotone*: $X \sqsubseteq Y \Longrightarrow D2(X) \sqsubseteq D2(Y)$ **by** *rel-auto*

Since these properties are relational, we discharge them using our relational calculus tactic *rel-tac*. Next we specify three operations that make up the signature of the theory.

definition *InitSys ip iV*
$= ((\ll ip\gg >_u 0 \wedge \ll iV\gg >_u 0)^\top ;; p,V,k := \ll ip\gg,\ll iV\gg,(\ll ip\gg{\cdot}\ll iV\gg))$

definition *ChPres dp*
$= ((\&p + \ll dp\gg >_u 0)^\top ;; p := \&p + \ll dp\gg ;; V := (\&k/\&p))$

definition *ChVol dV*
$= ((\&V + \ll dV\gg >_u 0)^\top ;; V := \&V + \ll dV\gg ;; p := (\&k/\&V))$

InitSys initialises the system with a given initial pressure (*ip*) and volume (*iV*). It assumes that both are greater than 0 using the assumption construct $c^\top$ which equates to *II* if c is true and *false* (i.e. errant) otherwise. It then creates a state assignment for p and V, uses the B healthiness condition to make it healthy (by calculating k), and finally turns the predicate into a postcondition using the $\lceil P\rceil_>$ function.

ChPres raises or lowers the pressure based on an input *dp*. It assumes that the resulting pressure change would not result in a zero or negative pressure, i.e. $p + dp > 0$. It assigns the updated value to p and recalculates V using the original value of k. *ChVol* is similar but updates the volume.

lemma *D1-InitSystem*: *D1 (InitSys ip iV) = InitSys ip iV*
by *rel-auto*

InitSys is *D1*, since it establishes the invariant for the system. However, it is not *D2* since it sets the global value of k and thus can change its value. We can however show that both *ChPres* and *ChVol* are healthy relations.

lemma *D1*: *D1* (*ChPres dp*) = *ChPres dp* **and** *D1* (*ChVol dV*) = *ChVol dV*
  by (*rel-auto*, *rel-auto*)

lemma *D2*: *D2* (*ChPres dp*) = *ChPres dp* **and** *D2* (*ChVol dV*) = *ChVol dV*
  by (*rel-auto*, *rel-auto*)

Finally we show a calculation a simple animation of Boyle's law, where the initial pressure and volume are set to 10 and 4, respectively, and then the pressure is lowered by 2.

lemma *ChPres-example*:
  $(\mathit{InitSys}\ 10\ 4\ ;;\ \mathit{ChPres}\ (-2)) = p,V,k := 8,5,40$
proof −
  — *InitSys* yields an assignment to the three variables
  have $\mathit{InitSys}\ 10\ 4 = p,V,k := 10,4,40$
    by (*rel-auto*)
  — This assignment becomes a substitution
  hence $(\mathit{InitSys}\ 10\ 4\ ;;\ \mathit{ChPres}\ (-2))$
      $= (\mathit{ChPres}\ (-2))[\![10,4,40/\$p,\$V,\$k]\!]$
    by (*simp add*: *assigns-r-comp alpha*)
  — Unfold definition of *ChPres*
  also have ... $= ((\&p - 2 >_u 0)^\top[\![10,4,40/\$p,\$V,\$k]\!]$
      $;;\ p := \&p - 2\ ;;\ V := \&k\ /\ \&p)$
    by (*simp add*: *ChPres-def lit-num-simps usubst unrest*)
  — Unfold definition of assumption
  also have ... $= ((p,V,k := 10,4,40 \triangleleft (8 :_u real) >_u 0 \triangleright \mathit{false})$
      $;;\ p := \&p - 2\ ;;\ V := \&k\ /\ \&p)$
    by (*simp add*: *rassume-def usubst alpha unrest*)
  — $(0::'a) < (8::'a)$ is true; simplify conditional
  also have ... $= (p,V,k := 10,4,40\ ;;\ p := \&p - 2\ ;;\ V := \&k\ /\ \&p)$
    by *rel-auto*
  — Application of both assignments
  also have ... $= p,V,k := 8,5,40$
    by *rel-auto*
  finally show *?thesis* **.**
qed

12 Designs

theory *utp-designs*
imports
  utp-rel
  utp-wp
  utp-theory
begin

In UTP, in order to explicitly record the termination of a program, a subset of alphabetized relations is introduced. These relations are called designs and their alphabet should contain the special boolean observational variable ok. It is used to record the start and termination of a program.

12.1 Definitions

In the following, the definitions of designs alphabets, designs and healthiness (well-formedness) conditions are given. The healthiness conditions of designs are defined by $H1$, $H2$, $H3$ and $H4$.

record *alpha-d* = ok_v :: *bool*

declare *alpha-d.splits* [*alpha-splits*]

The two locale interpretations below are a technicality to improve automatic proof support via the predicate and relational tactics. This is to enable the (re-)interpretation of state spaces to remove any occurrences of lens types after the proof tactics *pred-simp* and *rel-simp*, or any of their derivatives have been applied. Eventually, it would be desirable to automate both interpretations as part of a custom outer command for defining alphabets.

interpretation *alpha-d*: *lens-interp* $\lambda r.\ (ok_v\ r,\ more\ r)$
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

interpretation *alpha-d-rel*:
  lens-interp $\lambda(r,\ r').\ (ok_v\ r,\ ok_v\ r',\ more\ r,\ more\ r')$
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

The ok variable is defined using the syntactic translation *VAR*

definition *ok* = *VAR* ok_v

declare *ok-def* [*uvar-defs*]

lemma *vwb-lens-ok* [*simp*]: *vwb-lens ok*
  by (*unfold-locales*, *simp-all add*: *ok-def*)

lemma *ok-ord* [*usubst*]:
  $\$ok \prec_v \$ok\acute{}$
  by (*simp add*: *var-name-ord-def*)

type-synonym $'\alpha$ *alphabet-d* = $'\alpha$ *alpha-d-scheme alphabet*
type-synonym $('a,\ '\alpha)$ *uvar-d* = $('a,\ '\alpha$ *alphabet-d*) *uvar*
type-synonym $('\alpha,\ '\beta)$ *relation-d* = $('\alpha$ *alphabet-d*, $'\beta$ *alphabet-d*) *relation*
type-synonym $'\alpha$ *hrelation-d* = $'\alpha$ *alphabet-d hrelation*

translations
  (*type*) $'\alpha$ *alphabet-d* <= (*type*) $'\alpha$ *alpha-d-scheme*
  (*type*) $'\alpha$ *alphabet-d* <= (*type*) $'\alpha$ *alpha-d-ext*
  (*type*) $('\alpha,\ '\beta)$ *relation-d* <= (*type*) $('\alpha$ *alpha-d-scheme*, $'\beta$ *alpha-d-scheme*) *relation*

definition *des-lens* :: $('\alpha,\ '\alpha$ *alphabet-d*) *lens* (Σ_D) **where**
[*uvar-defs*]: *des-lens* = (| *lens-get* = *more*, *lens-put* = *fld-put more-update* |)

syntax
  -svid-alpha-d :: *svid* (Σ_D)

translations
  -svid-alpha-d => Σ_D

lemma *vwb-des-lens* [*simp*]: *vwb-lens des-lens*
  by (*unfold-locales*, *simp-all add*: *des-lens-def*)

lemma *ok-indep-des-lens* [*simp*]: $ok \bowtie des\text{-}lens$ $des\text{-}lens \bowtie ok$
  by (*rule lens-indepI*, *simp-all add*: *ok-def des-lens-def*)+

lemma *ok-des-bij-lens*: *bij-lens* ($ok +_L des\text{-}lens$)
  by (*unfold-locales*, *simp-all add*: *ok-def des-lens-def lens-plus-def prod.case-eq-if*)

It would be nice to be able to prove some general distributivity properties about these lifting operators. I don't know if that's possible somehow...

abbreviation *lift-desr* :: $('\alpha, '\beta)$ *relation* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* ($\lceil\text{-}\rceil_D$)
where $\lceil P\rceil_D \equiv P \oplus_p (des\text{-}lens \times_L des\text{-}lens)$

abbreviation *lift-pre-desr* :: $'\alpha$ *upred* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* ($\lceil\text{-}\rceil_{D<}$)
where $\lceil p\rceil_{D<} \equiv \lceil\lceil p\rceil_<\rceil_D$

abbreviation *lift-post-desr* :: $'\beta$ *upred* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* ($\lceil\text{-}\rceil_{D>}$)
where $\lceil p\rceil_{D>} \equiv \lceil\lceil p\rceil_>\rceil_D$

abbreviation *drop-desr* :: $('\alpha, '\beta)$ *relation-d* $\Rightarrow$ $('\alpha, '\beta)$ *relation* ($\lfloor\text{-}\rfloor_D$)
where $\lfloor P\rfloor_D \equiv P \upharpoonright_p (des\text{-}lens \times_L des\text{-}lens)$

definition *design*::$('\alpha, '\beta)$ *relation-d* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* (**infixl** $\vdash$ *60*)
where $P \vdash Q = (\$ok \wedge P \Rightarrow \$ok' \wedge Q)$

An rdesign is a design that uses the Isabelle type system to prevent reference to ok in the assumption and commitment.

definition *rdesign*::$('\alpha, '\beta)$ *relation* $\Rightarrow$ $('\alpha, '\beta)$ *relation* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* (**infixl** $\vdash_r$ *60*)
where $(P \vdash_r Q) = \lceil P\rceil_D \vdash \lceil Q\rceil_D$

An ndesign is a normal design, i.e. where the assumption is a condition

definition *ndesign*::$'\alpha$ *condition* $\Rightarrow$ $('\alpha, '\beta)$ *relation* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* (**infixl** $\vdash_n$ *60*)
where $(p \vdash_n Q) = (\lceil p\rceil_< \vdash_r Q)$

definition *skip-d* :: $'\alpha$ *hrelation-d* (II_D)
where $II_D \equiv (true \vdash_r II)$

definition *assigns-d* :: $'\alpha$ *usubst* $\Rightarrow$ $'\alpha$ *hrelation-d* ($\langle\text{-}\rangle_D$)
where *assigns-d* σ = (*true* $\vdash_r$ *assigns-r* σ)

syntax
  -assignmentd :: *svid-list* $\Rightarrow$ *uexprs* $\Rightarrow$ *logic* (**infixr** $:=_D$ *55*)

translations
  -assignmentd xs vs => *CONST assigns-d* (*-mk-usubst* (*CONST id*) *xs vs*)
  $x :=_D v$ <= *CONST assigns-d* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *v*)
  $x :=_D v$ <= *CONST assigns-d* (*CONST subst-upd* (*CONST id*) *x v*)
  $x,y :=_D u,v$ <= *CONST assigns-d* (*CONST subst-upd* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *u*) (*CONST svar y*) *v*)

definition *J* :: $'\alpha$ *hrelation-d*
where $J = ((\$ok \Rightarrow \$ok') \wedge \lceil II\rceil_D)$

definition *H1* (*P*) $\equiv$ $\$ok \Rightarrow P$

definition *H2* (*P*) $\equiv$ *P* ;; *J*

definition *H3* $(P) \;\equiv\; P \;;;\; II_D$

definition *H4* $(P) \;\equiv ((P;;true) \Rightarrow P)$

syntax
  -ok-f :: *logic* $\Rightarrow$ *logic* (-^f [*1000*] *1000*)
  -ok-t :: *logic* $\Rightarrow$ *logic* (-^t [*1000*] *1000*)
  -top-d :: *logic* ($\top_D$)
  -bot-d :: *logic* ($\bot_D$)

translations
  $P^f \rightleftharpoons$ *CONST usubst* (*CONST subst-upd CONST id* (*CONST ovar CONST ok*) *false*) *P*
  $P^t \rightleftharpoons$ *CONST usubst* (*CONST subst-upd CONST id* (*CONST ovar CONST ok*) *true*) *P*
  $\top_D$ => *CONST not-upred* (*CONST var* (*CONST ivar CONST ok*))
  $\bot_D$ => *true*

definition *pre-design* :: ($'\alpha$, $'\beta$) *relation-d* $\Rightarrow$ ($'\alpha$, $'\beta$) *relation* (pre_D'(-')) **where**
$pre_D(P) = \lfloor \neg\ P[\![true,false/\$ok,\$ok']\!] \rfloor_D$

definition *post-design* :: ($'\alpha$, $'\beta$) *relation-d* $\Rightarrow$ ($'\alpha$, $'\beta$) *relation* ($post_D$'(-')) **where**
$post_D(P) = \lfloor P[\![true,true/\$ok,\$ok']\!] \rfloor_D$

definition *wp-design* :: ($'\alpha$, $'\beta$) *relation-d* $\Rightarrow$ $'\beta$ *condition* $\Rightarrow$ $'\alpha$ *condition* (**infix** wp_D *60*) **where**
$Q\ wp_D\ r = (\lfloor pre_D(Q)$;; *true* :: ($'\alpha$, $'\beta$) *relation*$\rfloor_<$ $\wedge$ $(post_D(Q)$ *wp* $r))$

declare *design-def* [*upred-defs*]
declare *rdesign-def* [*upred-defs*]
declare *ndesign-def* [*upred-defs*]
declare *skip-d-def* [*upred-defs*]
declare *J-def* [*upred-defs*]
declare *pre-design-def* [*upred-defs*]
declare *post-design-def* [*upred-defs*]
declare *wp-design-def* [*upred-defs*]
declare *assigns-d-def* [*upred-defs*]

declare *H1-def* [*upred-defs*]
declare *H2-def* [*upred-defs*]
declare *H3-def* [*upred-defs*]
declare *H4-def* [*upred-defs*]

lemma *drop-desr-inv* [*simp*]: $\lfloor \lceil P \rceil_D \rfloor_D = P$
  by (*simp add*: *arestr-aext prod-mwb-lens*)

lemma *lift-desr-inv*:
  fixes *P* :: ($'\alpha$, $'\beta$) *relation-d*
  assumes $\$ok \sharp P$ $\$ok' \sharp P$
  shows $\lceil \lfloor P \rfloor_D \rceil_D = P$
proof –
  have *bij-lens* (*des-lens* $\times_L$ *des-lens* $+_L$ (*in-var ok* $+_L$ *out-var ok*) :: (-, $'\alpha$ *alpha-d-scheme* $\times$ $'\beta$ *alpha-d-scheme*) *lens*)
    (**is** *bij-lens* (*?P*))
  proof –
    have *?P* $\approx_L$ (*ok* $+_L$ *des-lens*) $\times_L$ (*ok* $+_L$ *des-lens*) (**is** *?P* $\approx_L$ *?Q*)
      apply (*simp add*: *in-var-def out-var-def prod-as-plus*)

apply (*simp add*: *prod-as-plus*[*THEN sym*])
apply (*meson lens-equiv-sym lens-equiv-trans lens-indep-prod lens-plus-comm lens-plus-prod-exchange ok-indep-des-lens*)
done
moreover have *bij-lens ?Q*
by (*simp add*: *ok-des-bij-lens prod-bij-lens*)
ultimately show *?thesis*
by (*metis bij-lens-equiv lens-equiv-sym*)
qed

with *assms* **show** *?thesis*
apply (*rule-tac aext-arestr*[*of - in-var ok* $+_L$ *out-var ok*])
apply (*simp add*: *prod-mwb-lens*)
apply (*simp*)
apply (*metis alpha-in-var lens-indep-prod lens-indep-sym ok-indep-des-lens out-var-def prod-as-plus*)
using *unrest-var-comp* **apply** *blast*
done
qed

12.2 Design laws

lemma *prod-lens-indep-in-var* [*simp*]:
$a \bowtie x \Longrightarrow a \times_L b \bowtie$ *in-var* x
by (*metis in-var-def in-var-indep out-in-indep out-var-def plus-pres-lens-indep prod-as-plus*)

lemma *prod-lens-indep-out-var* [*simp*]:
$b \bowtie x \Longrightarrow a \times_L b \bowtie$ *out-var* x
by (*metis in-out-indep in-var-def out-var-def out-var-indep plus-pres-lens-indep prod-as-plus*)

lemma *unrest-out-des-lift* [*unrest*]: $out\alpha \sharp p \Longrightarrow out\alpha \sharp \lceil p \rceil_D$
by (*pred-auto*, *auto simp add*: *outα-def des-lens-def prod-lens-def*)

thm *alpha-d.select-convs*

lemma *lift-dist-seq* [*simp*]:
$\lceil P \;;; Q \rceil_D = (\lceil P \rceil_D \;;; \lceil Q \rceil_D)$
by (*rel-auto*)

lemma *lift-des-skip-dr-unit-unrest*: $\$ok' \sharp P \Longrightarrow (P \;;; \lceil II \rceil_D) = P$
by (*rel-auto*)

lemma *true-is-design*:
$(false \vdash true) = true$
by *rel-auto*

lemma *true-is-rdesign*:
$(false \vdash_r true) = true$
by *rel-auto*

lemma *design-false-pre*:
$(false \vdash P) = true$
by *rel-auto*

lemma *rdesign-false-pre*:
$(false \vdash_r P) = true$
by *rel-auto*

lemma *ndesign-false-pre*:
$(false \vdash_n P) = true$
by *rel-auto*

theorem *design-refinement*:
assumes
$\$ok \sharp P1$ $\$ok' \sharp P1$ $\$ok \sharp P2$ $\$ok' \sharp P2$
$\$ok \sharp Q1$ $\$ok' \sharp Q1$ $\$ok \sharp Q2$ $\$ok' \sharp Q2$
shows $(P1 \vdash Q1 \sqsubseteq P2 \vdash Q2) \longleftrightarrow (`P1 \Rightarrow P2` \wedge `P1 \wedge Q2 \Rightarrow Q1`)$
proof −
have $(P1 \vdash Q1) \sqsubseteq (P2 \vdash Q2) \longleftrightarrow `(\$ok \wedge P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (\$ok \wedge P1 \Rightarrow \$ok' \wedge Q1)`$
by *pred-auto*
also with *assms* **have** ... $= `(P2 \Rightarrow \$ok' \wedge Q2) \Rightarrow (P1 \Rightarrow \$ok' \wedge Q1)`$
by (*subst subst-bool-split*[*of in-var ok*], *simp-all*, *subst-tac*)
also with *assms* **have** ... $= `(\neg P2 \Rightarrow \neg P1) \wedge ((P2 \Rightarrow Q2) \Rightarrow P1 \Rightarrow Q1)`$
by (*subst subst-bool-split*[*of out-var ok*], *simp-all*, *subst-tac*)
also have ... $\longleftrightarrow `(P1 \Rightarrow P2)` \wedge `P1 \wedge Q2 \Rightarrow Q1`$
by (*pred-auto*)
finally show *?thesis* **.**
qed

theorem *rdesign-refinement*:
$(P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2) \longleftrightarrow (`P1 \Rightarrow P2` \wedge `P1 \wedge Q2 \Rightarrow Q1`)$
by *rel-auto*

lemma *design-refine-intro*:
assumes $`P1 \Rightarrow P2`$ $`P1 \wedge Q2 \Rightarrow Q1`$
shows $P1 \vdash Q1 \sqsubseteq P2 \vdash Q2$
using *assms* **unfolding** *upred-defs*
by *pred-auto*

lemma *rdesign-refine-intro*:
assumes $`P1 \Rightarrow P2`$ $`P1 \wedge Q2 \Rightarrow Q1`$
shows $P1 \vdash_r Q1 \sqsubseteq P2 \vdash_r Q2$
using *assms* **unfolding** *upred-defs*
by *pred-auto*

lemma *ndesign-refine-intro*:
assumes $`p1 \Rightarrow p2`$ $`\lceil p1 \rceil_< \wedge Q2 \Rightarrow Q1`$
shows $p1 \vdash_n Q1 \sqsubseteq p2 \vdash_n Q2$
using *assms* **unfolding** *upred-defs*
by *pred-auto*

lemma *design-subst* [*usubst*]:
$[\![\$ok \sharp \sigma; \$ok' \sharp \sigma]\!] \Longrightarrow \sigma \dagger (P \vdash Q) = (\sigma \dagger P) \vdash (\sigma \dagger Q)$
by (*simp add*: *design-def usubst*)

theorem *design-ok-false* [*usubst*]: $(P \vdash Q)[\![false/\$ok]\!] = true$
by (*simp add*: *design-def usubst*)

theorem *design-npre*:
$(P \vdash Q)^f = (\neg \$ok \vee \neg P^f)$
by (*rel-auto*)

theorem *design-pre*:
  $\neg (P \vdash Q)^f = (\$ok \wedge P^f)$
  by (*simp add*: *design-def*, *subst-tac*)
    (*metis* (*no-types*, *hide-lams*) *not-conj-deMorgans true-not-false*(*2*) *utp-pred.compl-top-eq utp-pred.sup.idem utp-pred.sup-compl-top*)

theorem *design-post*:
  $(P \vdash Q)^t = ((\$ok \wedge P^t) \Rightarrow Q^t)$
  by (*rel-auto*)

theorem *rdesign-pre* [*simp*]: $pre_D(P \vdash_r Q) = P$
  by *pred-auto*

theorem *rdesign-post* [*simp*]: $post_D(P \vdash_r Q) = (P \Rightarrow Q)$
  by *pred-auto*

theorem *design-true-left-zero*: $(true\ ;;\ (P \vdash Q)) = true$
proof −
  have $(true\ ;;\ (P \vdash Q)) = (\exists\ ok_0 \cdot true[\![\ll ok_0 \gg/\$ok']\!]\ ;;\ (P \vdash Q)[\![\ll ok_0 \gg/\$ok]\!])$
    by (*subst seqr-middle*[*of ok*], *simp-all*)
  also have ... $= ((true[\![false/\$ok']\!]\ ;;\ (P \vdash Q)[\![false/\$ok]\!]) \vee (true[\![true/\$ok']\!]\ ;;\ (P \vdash Q)[\![true/\$ok]\!]))$
    by (*simp add*: *disj-comm false-alt-def true-alt-def*)
  also have ... $= ((true[\![false/\$ok']\!]\ ;;\ true_h) \vee (true\ ;;\ ((P \vdash Q)[\![true/\$ok]\!])))$
    by (*subst-tac*, *rel-auto*)
  also have ... $= true$
    by (*subst-tac*, *simp add*: *precond-right-unit unrest*)
  finally show *?thesis* **.**
qed

theorem *design-top-left-zero*: $(\top_D\ ;;\ (P \vdash Q)) = \top_D$
  by *rel-auto*

theorem *design-choice*:
  $(P_1 \vdash P_2) \sqcap (Q_1 \vdash Q_2) = ((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2))$
  by *rel-auto*

theorem *design-inf*:
  $(P_1 \vdash P_2) \sqcup (Q_1 \vdash Q_2) = ((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow P_2) \wedge (Q_1 \Rightarrow Q_2)))$
  by *rel-auto*

theorem *rdesign-choice*:
  $(P_1 \vdash_r P_2) \sqcap (Q_1 \vdash_r Q_2) = ((P_1 \wedge Q_1) \vdash_r (P_2 \vee Q_2))$
  by *rel-auto*

theorem *design-condr*:
  $((P_1 \vdash P_2) \triangleleft b \triangleright (Q_1 \vdash Q_2)) = ((P_1 \triangleleft b \triangleright Q_1) \vdash (P_2 \triangleleft b \triangleright Q_2))$
  by *rel-auto*

lemma *design-top*:
  $(P \vdash Q) \sqsubseteq \top_D$
  by *rel-auto*

lemma *design-bottom*:
  $\bot_D \sqsubseteq (P \vdash Q)$
  by *simp*

lemma *design-USUP*:
  assumes $A \neq \{\}$
  shows $(\bigsqcap\ i \in A \cdot P(i) \vdash Q(i)) = (\bigsqcup\ i \in A \cdot P(i)) \vdash (\bigsqcap\ i \in A \cdot Q(i))$
  using *assms* **by** *rel-auto*

lemma *design-UINF*:
  $(\bigsqcup\ i \in A \cdot P(i) \vdash Q(i)) = (\bigsqcap\ i \in A \cdot P(i)) \vdash (\bigsqcup\ i \in A \cdot P(i) \Rightarrow Q(i))$
  by *rel-auto*

theorem *design-composition-subst*:
  assumes
    $\$ok' \sharp P1$ $\$ok \sharp P2$
  shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) =$
    $(((\neg ((\neg P1) ;; true)) \wedge \neg (Q1[\![true/\$ok']\!] ;; (\neg P2))) \vdash (Q1[\![true/\$ok']\!] ;; Q2[\![true/\$ok]\!]))$
proof −
  have $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (\exists\ ok_0 \cdot ((P1 \vdash Q1)[\![\ll ok_0 \gg/\$ok']\!] ;; (P2 \vdash Q2)[\![\ll ok_0 \gg/\$ok]\!]))$
    by (*rule seqr-middle*, *simp*)
  also have ...
    $= (((P1 \vdash Q1)[\![false/\$ok']\!] ;; (P2 \vdash Q2)[\![false/\$ok]\!])$
      $\vee ((P1 \vdash Q1)[\![true/\$ok']\!] ;; (P2 \vdash Q2)[\![true/\$ok]\!]))$
    by (*simp add*: *true-alt-def false-alt-def*, *pred-auto*)
  also from *assms*
  have ... $= (((\$ok \wedge P1 \Rightarrow Q1[\![true/\$ok']\!]) ;; (P2 \Rightarrow \$ok' \wedge Q2[\![true/\$ok]\!])) \vee ((\neg (\$ok \wedge P1)) ;; true))$
    by (*simp add*: *design-def usubst unrest*, *pred-auto*)
  also have ... $= ((\neg\$ok ;; true_h) \vee (\neg P1 ;; true) \vee (Q1[\![true/\$ok']\!] ;; \neg P2) \vee (\$ok' \wedge (Q1[\![true/\$ok']\!] ;; Q2[\![true/\$ok]\!])))$
    by (*rel-auto*)
  also have ... $= (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1[\![true/\$ok']\!] ;; (\neg P2))) \vdash (Q1[\![true/\$ok']\!] ;; Q2[\![true/\$ok]\!]))$
    by (*simp add*: *precond-right-unit design-def unrest*, *rel-auto*)
  finally show *?thesis* **.**
qed

lemma *design-export-ok*:
  $P \vdash Q = (P \vdash (\$ok \wedge Q))$
  by (*rel-auto*)

lemma *design-export-ok'*:
  $P \vdash Q = (P \vdash (\$ok' \wedge Q))$
  by (*rel-auto*)

lemma *design-export-pre*: $P \vdash (P \wedge Q) = P \vdash Q$
  by (*rel-auto*)

theorem *design-composition*:
  assumes
    $\$ok' \sharp P1$ $\$ok \sharp P2$ $\$ok' \sharp Q1$ $\$ok \sharp Q2$
  shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$
  using *assms* **by** (*simp add*: *design-composition-subst usubst*)

lemma *runrest-ident-var*:
  assumes $x \sharp\sharp P$
  shows $(\$x \wedge P) = (P \wedge \$x')$
proof −

have $P = (\$x\acute{} =_u \$x \wedge P)$
  by (*metis* (*no-types*, *lifting*) *RID-def assms conj-idem unrest-relation-def utp-pred.inf.left-commute*)
moreover have $(\$x\acute{} =_u \$x \wedge (\$x \wedge P)) = (\$x\acute{} =_u \$x \wedge (P \wedge \$x\acute{}))$
  by (*rel-auto*)
ultimately show *?thesis*
  by (*metis utp-pred.inf.assoc utp-pred.inf-left-commute*)
qed

theorem *design-composition-runrest*:
assumes
$\$ok\acute{} \sharp P1$ $\$ok \sharp P2$ $ok \sharp\sharp Q1$ $ok \sharp\sharp Q2$
shows $((P1 \vdash Q1) ;; (P2 \vdash Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1^t ;; (\neg P2))) \vdash (Q1 ;; Q2))$
proof –
have $(\$ok \wedge \$ok\acute{} \wedge (Q1^t ;; Q2[\![true/\$ok]\!])) = (\$ok \wedge \$ok\acute{} \wedge (Q1 ;; Q2))$
proof –
have $(\$ok \wedge \$ok\acute{} \wedge (Q1 ;; Q2)) = (\$ok \wedge Q1 ;; Q2 \wedge \$ok\acute{})$
  by (*metis* (*no-types*, *hide-lams*) *seqr-post-out seqr-pre-out utp-pred.inf.commute utp-rel.unrest-iuvar utp-rel.unrest-ouvar vwb-lens-ok vwb-lens-mwb*)
also have ... $= (Q1 \wedge \$ok\acute{} ;; \$ok \wedge Q2)$
  by (*simp add*: *assms*(*3*) *assms*(*4*) *runrest-ident-var*)
also have ... $= (Q1^t ;; Q2[\![true/\$ok]\!])$
  by (*metis seqr-left-one-point seqr-post-transfer true-alt-def uivar-convr upred-eq-true utp-pred.inf.cobounded2 utp-pred.inf.orderE utp-rel.unrest-iuvar vwb-lens-ok vwb-lens-mwb*)
finally show *?thesis*
  by (*metis utp-pred.inf.left-commute utp-pred.inf-left-idem*)
qed
moreover have $(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; \neg P2)) \vdash (Q1^t ;; Q2[\![true/\$ok]\!]) =$
$(\neg (\neg P1 ;; true) \wedge \neg (Q1^t ;; \neg P2)) \vdash (\$ok \wedge \$ok\acute{} \wedge (Q1^t ;; Q2[\![true/\$ok]\!]))$
  by (*metis design-export-ok design-export-ok'*)
ultimately show *?thesis* **using** *assms*
  by (*simp add*: *design-composition-subst usubst*, *metis design-export-ok design-export-ok'*)
qed

theorem *rdesign-composition*:
$((P1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = (((\neg ((\neg P1) ;; true)) \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$
by (*simp add*: *rdesign-def design-composition unrest alpha*)

lemma *skip-d-alt-def*: $II_D = true \vdash II$
by (*rel-auto*)

theorem *design-skip-idem* [*simp*]:
$(II_D ;; II_D) = II_D$
by (*rel-auto*)

theorem *design-composition-cond*:
assumes
$out\alpha \sharp p1$ $\$ok \sharp P2$ $\$ok\acute{} \sharp Q1$ $\$ok \sharp Q2$
shows $((p1 \vdash Q1) ;; (P2 \vdash Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash (Q1 ;; Q2))$
using *assms*
by (*simp add*: *design-composition unrest precond-right-unit*)

theorem *rdesign-composition-cond*:
assumes $out\alpha \sharp p1$
shows $((p1 \vdash_r Q1) ;; (P2 \vdash_r Q2)) = ((p1 \wedge \neg (Q1 ;; (\neg P2))) \vdash_r (Q1 ;; Q2))$
using *assms*

by (*simp add*: *rdesign-def design-composition-cond unrest alpha*)

theorem *design-composition-wp*:
assumes
$ok \sharp p1$ $ok \sharp p2$
$\$ok \sharp Q1$ $\$ok' \sharp Q1$ $\$ok \sharp Q2$ $\$ok' \sharp Q2$
shows $((\lceil p1 \rceil_< \vdash Q1) ;; (\lceil p2 \rceil_< \vdash Q2)) = ((\lceil p1 \wedge Q1\ wp\ p2 \rceil_<) \vdash (Q1 ;; Q2))$
using *assms* **by** (*rel-blast*)

theorem *rdesign-composition-wp*:
$((\lceil p1 \rceil_< \vdash_r Q1) ;; (\lceil p2 \rceil_< \vdash_r Q2)) = ((\lceil p1 \wedge Q1\ wp\ p2 \rceil_<) \vdash_r (Q1 ;; Q2))$
by *rel-blast*

theorem *ndesign-composition-wp*:
$((p1 \vdash_n Q1) ;; (p2 \vdash_n Q2)) = ((p1 \wedge Q1\ wp\ p2) \vdash_n (Q1 ;; Q2))$
by *rel-blast*

theorem *rdesign-wp* [*wp*]:
$(\lceil p \rceil_< \vdash_r Q)\ wp_D\ r = (p \wedge Q\ wp\ r)$
by *rel-auto*

theorem *ndesign-wp* [*wp*]:
$(p \vdash_n Q)\ wp_D\ r = (p \wedge Q\ wp\ r)$
by (*simp add*: *ndesign-def rdesign-wp*)

theorem *wpd-seq-r*:
fixes $Q1$ $Q2$ :: $'\alpha$ *hrelation*
shows $(\lceil p1 \rceil_< \vdash_r Q1 ;; \lceil p2 \rceil_< \vdash_r Q2)\ wp_D\ r = (\lceil p1 \rceil_< \vdash_r Q1)\ wp_D\ ((\lceil p2 \rceil_< \vdash_r Q2)\ wp_D\ r)$
apply (*simp add*: *wp*)
apply (*subst rdesign-composition-wp*)
apply (*simp only*: *wp*)
apply (*rel-auto*)
done

theorem *wpnd-seq-r* [*wp*]:
fixes $Q1$ $Q2$ :: $'\alpha$ *hrelation*
shows $(p1 \vdash_n Q1 ;; p2 \vdash_n Q2)\ wp_D\ r = (p1 \vdash_n Q1)\ wp_D\ ((p2 \vdash_n Q2)\ wp_D\ r)$
by (*simp add*: *ndesign-def wpd-seq-r*)

lemma *design-subst-ok-ok'*:
$(P[\![true/\$ok]\!] \vdash Q[\![true,true/\$ok,\$ok']\!]) = (P \vdash Q)$
proof −
have $(P \vdash Q) = ((\$ok \wedge P) \vdash (\$ok \wedge \$ok' \wedge Q))$
by (*pred-auto*)
also have ... $= ((\$ok \wedge P[\![true/\$ok]\!]) \vdash (\$ok \wedge (\$ok' \wedge Q[\![true/\$ok']\!])[\![true/\$ok]\!]))$
by (*metis conj-eq-out-var-subst conj-pos-var-subst upred-eq-true utp-pred.inf-commute vwb-lens-ok*)
also have ... $= ((\$ok \wedge P[\![true/\$ok]\!]) \vdash (\$ok \wedge \$ok' \wedge Q[\![true,true/\$ok,\$ok']\!]))$
by (*simp add*: *usubst*)
also have ... $= (P[\![true/\$ok]\!] \vdash Q[\![true,true/\$ok,\$ok']\!])$
by (*pred-auto*)
finally show *?thesis* **..**
qed

lemma *design-subst-ok'*:
$(P \vdash Q[\![true/\$ok']\!]) = (P \vdash Q)$

proof −
  have $(P \vdash Q) = (P \vdash (\$ok' \wedge Q))$
    by (*pred-auto*)
  also have ... $= (P \vdash (\$ok' \wedge Q[\![true/\$ok']\!]))$
    by (*metis conj-eq-out-var-subst upred-eq-true utp-pred.inf-commute vwb-lens-ok*)
  also have ... $= (P \vdash Q[\![true/\$ok']\!])$
    by (*pred-auto*)
  finally show *?thesis* **..**
qed

theorem *design-left-unit-hom*:
  fixes P Q :: $'\alpha$ *hrelation-d*
  shows $(II_D \;;\; P \vdash_r Q) = (P \vdash_r Q)$
proof −
  have $(II_D \;;\; P \vdash_r Q) = (true \vdash_r II \;;\; P \vdash_r Q)$
    by (*simp add*: *skip-d-def*)
  also have ... $= (true \wedge \neg (II \;;\; \neg P)) \vdash_r (II \;;\; Q)$
  proof −
    have $out\alpha \sharp true$
      by *unrest-tac*
    thus *?thesis*
      using *rdesign-composition-cond* **by** *blast*
  qed
  also have ... $= (\neg (\neg P)) \vdash_r Q$
    by *simp*
  finally show *?thesis* **by** *simp*
qed

theorem *design-left-unit* [*simp*]:
  $(II_D \;;\; P \vdash_r Q) = (P \vdash_r Q)$
  by *rel-auto*

theorem *design-right-cond-unit* [*simp*]:
  assumes $out\alpha \sharp p$
  shows $(p \vdash_r Q \;;\; II_D) = (p \vdash_r Q)$
  using *assms*
  by (*simp add*: *skip-d-def rdesign-composition-cond*)

lemma *lift-des-skip-dr-unit* [*simp*]:
  $(\lceil P \rceil_D \;;\; \lceil II \rceil_D) = \lceil P \rceil_D$
  $(\lceil II \rceil_D \;;\; \lceil P \rceil_D) = \lceil P \rceil_D$
  by *rel-auto rel-auto*

lemma *assigns-d-id* [*simp*]: $\langle id \rangle_D = II_D$
  by (*rel-auto*)

lemma *assign-d-left-comp*:
  $(\langle f \rangle_D \;;\; (P \vdash_r Q)) = (\lceil f \rceil_s \dagger P \vdash_r \lceil f \rceil_s \dagger Q)$
  by (*simp add*: *assigns-d-def rdesign-composition assigns-r-comp subst-not*)

lemma *assign-d-right-comp*:
  $((P \vdash_r Q) \;;\; \langle f \rangle_D) = ((\neg (\neg P \;;\; true)) \vdash_r (Q \;;\; \langle f \rangle_a))$
  by (*simp add*: *assigns-d-def rdesign-composition*)

lemma *assigns-d-comp*:

$(\langle f \rangle_D ;; \langle g \rangle_D) = \langle g \circ f \rangle_D$
using *assms*
by (*simp add*: *assigns-d-def rdesign-composition assigns-comp*)

12.3 Design preconditions

lemma *design-pre-choice* [*simp*]:
$pre_D(P \sqcap Q) = (pre_D(P) \wedge pre_D(Q))$
by (*rel-auto*)

lemma *design-post-choice* [*simp*]:
$post_D(P \sqcap Q) = (post_D(P) \vee post_D(Q))$
by (*rel-auto*)

lemma *design-pre-condr* [*simp*]:
$pre_D(P \triangleleft \lceil b \rceil_D \triangleright Q) = (pre_D(P) \triangleleft b \triangleright pre_D(Q))$
by (*rel-auto*)

lemma *design-post-condr* [*simp*]:
$post_D(P \triangleleft \lceil b \rceil_D \triangleright Q) = (post_D(P) \triangleleft b \triangleright post_D(Q))$
by (*rel-auto*)

12.4 H1: No observation is allowed before initiation

lemma *H1-idem*:
$H1\ (H1\ P) = H1(P)$
by *pred-auto*

lemma *H1-monotone*:
$P \sqsubseteq Q \Longrightarrow H1(P) \sqsubseteq H1(Q)$
by *pred-auto*

lemma *H1-below-top*:
$H1(P) \sqsubseteq \top_D$
by *pred-auto*

lemma *H1-design-skip*:
$H1(II) = II_D$
by *rel-auto*

The H1 algebraic laws are valid only when $\alpha(R)$ is homogeneous. This should maybe be generalised.

theorem *H1-algebraic-intro*:
assumes
$(true_h ;; R) = true_h$
$(II_D ;; R) = R$
shows *R is H1*
proof −
have $R = (II_D ;; R)$ **by** (*simp add*: *assms*(*2*))
also have ... $= (H1(II) ;; R)$
by (*simp add*: *H1-design-skip*)
also have ... $= ((\$ok \Rightarrow II) ;; R)$
by (*simp add*: *H1-def*)
also have ... $= ((\neg \$ok ;; R) \vee R)$
by (*simp add*: *impl-alt-def seqr-or-distl*)
also have ... $= (((\neg \$ok ;; true_h) ;; R) \vee R)$

```
    by (simp add: precond-right-unit unrest)
  also have ... = ((¬ $ok ;; true_h) ∨ R)
    by (metis assms(1) seqr-assoc)
  also have ... = ($ok ⇒ R)
    by (simp add: impl-alt-def precond-right-unit unrest)
  finally show ?thesis by (metis H1-def Healthy-def')
qed

lemma nok-not-false:
  (¬ $ok) ≠ false
  by pred-auto

theorem H1-left-zero:
  assumes P is H1
  shows (true ;; P) = true
proof –
  from assms have (true ;; P) = (true ;; ($ok ⇒ P))
    by (simp add: H1-def Healthy-def')

  also from assms have ... = (true ;; (¬ $ok ∨ P)) (is - = (?true ;; -))
    by (simp add: impl-alt-def)
  also from assms have ... = ((?true ;; ¬ $ok) ∨ (?true ;; P))
    using seqr-or-distr by blast
  also from assms have ... = (true ∨ (true ;; P))
    by (simp add: nok-not-false precond-left-zero unrest)
  finally show ?thesis
    by (simp add: upred-defs urel-defs)
qed

theorem H1-left-unit:
  fixes P :: 'α hrelation-d
  assumes P is H1
  shows (II_D ;; P) = P
proof –
  have (II_D ;; P) = (($ok ⇒ II) ;; P)
    by (metis H1-def H1-design-skip)
  also have ... = ((¬ $ok ;; P) ∨ P)
    by (simp add: impl-alt-def seqr-or-distl)
  also from assms have ... = (((¬ $ok ;; true_h) ;; P) ∨ P)
    by (simp add: precond-right-unit unrest)
  also have ... = ((¬ $ok ;; (true_h ;; P)) ∨ P)
    by (simp add: seqr-assoc)
  also from assms have ... = ($ok ⇒ P)
    by (simp add: H1-left-zero impl-alt-def precond-right-unit unrest)
  finally show ?thesis using assms
    by (simp add: H1-def Healthy-def')
qed

theorem H1-algebraic:
  P is H1 ⟷ (true_h ;; P) = true_h ∧ (II_D ;; P) = P
  using H1-algebraic-intro H1-left-unit H1-left-zero by blast

theorem H1-nok-left-zero:
  fixes P :: 'α hrelation-d
  assumes P is H1
```

shows $(\neg \$ok \;;\; P) = (\neg \$ok)$
proof $-$
have $(\neg \$ok \;;\; P) = ((\neg \$ok \;;\; true_h) \;;\; P)$
by (*simp add*: *precond-right-unit unrest*)
also have ... $= ((\neg \$ok) \;;\; true_h)$
by (*metis H1-left-zero assms seqr-assoc*)
also have ... $= (\neg \$ok)$
by (*simp add*: *precond-right-unit unrest*)
finally show *?thesis* **.**
qed

lemma *H1-design*:
$H1(P \vdash Q) = (P \vdash Q)$
by (*rel-auto*)

lemma *H1-rdesign*:
$H1(P \vdash_r Q) = (P \vdash_r Q)$
by (*rel-auto*)

lemma *H1-choice-closed*:
$\llbracket P \text{ is } H1;\ Q \text{ is } H1 \rrbracket \Longrightarrow P \sqcap Q \text{ is } H1$
by (*simp add*: *H1-def Healthy-def' disj-upred-def impl-alt-def semilattice-sup-class.sup-left-commute*)

lemma *H1-inf-closed*:
$\llbracket P \text{ is } H1;\ Q \text{ is } H1 \rrbracket \Longrightarrow P \sqcup Q \text{ is } H1$
by *rel-blast*

lemma *H1-USUP*:
assumes $A \neq \{\}$
shows $H1(\prod\ i \in A \cdot P(i)) = (\prod\ i \in A \cdot H1(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *H1-Sup*:
assumes $A \neq \{\}\ \forall\ P \in A.\ P \text{ is } H1$
shows $(\prod\ A) \text{ is } H1$
proof $-$
from *assms*(*2*) **have** $H1\ `\ A = A$
by (*auto simp add*: *Healthy-def rev-image-eqI*)
with *H1-USUP*[*of A id*, *OF assms*(*1*)] **show** *?thesis*
by (*simp add*: *USUP-as-Sup-image Healthy-def*)
qed

lemma *H1-UINF*:
shows $H1(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot H1(P(i)))$
by (*rel-auto*)

lemma *H1-Inf*:
assumes $\forall\ P \in A.\ P \text{ is } H1$
shows $(\bigsqcup\ A) \text{ is } H1$
proof $-$
from *assms* **have** $H1\ `\ A = A$
by (*auto simp add*: *Healthy-def rev-image-eqI*)
with *H1-UINF*[*of A id*] **show** *?thesis*
by (*simp add*: *UINF-as-Inf-image Healthy-def*)
qed

12.5 H2: A specification cannot require non-termination

lemma *J-split*:
shows $(P\ ;;\ J) = (P^f \vee (P^t \wedge \$ok'))$
proof −
have $(P\ ;;\ J) = (P\ ;;\ ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D))$
by (*simp add*: *H2-def J-def design-def*)
also have ... $= (P\ ;;\ ((\$ok \Rightarrow \$ok \wedge \$ok') \wedge \lceil II \rceil_D))$
by *rel-auto*
also have ... $= ((P\ ;;\ (\neg\ \$ok \wedge \lceil II \rceil_D)) \vee (P\ ;;\ (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))))$
by *rel-auto*
also have ... $= (P^f \vee (P^t \wedge \$ok'))$
proof −
have $(P\ ;;\ (\neg\ \$ok \wedge \lceil II \rceil_D)) = P^f$
proof −
have $(P\ ;;\ (\neg\ \$ok \wedge \lceil II \rceil_D)) = ((P \wedge \neg\ \$ok')\ ;;\ \lceil II \rceil_D)$
by *rel-auto*
also have ... $= (\exists\ \$ok' \cdot P \wedge \$ok' =_u \mathit{false})$
by *rel-auto*
also have ... $= P^f$
by (*metis C1 one-point out-var-uvar pr-var-def unrest-as-exists vwb-lens-ok vwb-lens-mwb*)
finally show *?thesis* **.**
qed
moreover have $(P\ ;;\ (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P^t \wedge \$ok')$
proof −
have $(P\ ;;\ (\$ok \wedge (\lceil II \rceil_D \wedge \$ok'))) = (P\ ;;\ (\$ok \wedge II))$
by *rel-auto*
also have ... $= (P^t \wedge \$ok')$
by *rel-auto*
finally show *?thesis* **.**
qed
ultimately show *?thesis*
by *simp*
qed
finally show *?thesis* **.**
qed

lemma *H2-split*:
shows $H2(P) = (P^f \vee (P^t \wedge \$ok'))$
by (*simp add*: *H2-def J-split*)

theorem *H2-equivalence*:
P *is* $H2 \longleftrightarrow$ ‘$P^f \Rightarrow P^t$‘
proof −
have ‘$P \Leftrightarrow (P\ ;;\ J)$‘ $\longleftrightarrow$ ‘$P \Leftrightarrow (P^f \vee (P^t \wedge \$ok'))$‘
by (*simp add*: *J-split*)
also from *assms* **have** ... $\longleftrightarrow$ ‘$(P \Leftrightarrow P^f \vee P^t \wedge \$ok')^f \wedge (P \Leftrightarrow P^f \vee P^t \wedge \$ok')^t$‘
by (*simp add*: *subst-bool-split*)
also from *assms* **have** ... $=$ ‘$(P^f \Leftrightarrow P^f) \wedge (P^t \Leftrightarrow P^f \vee P^t)$‘
by *subst-tac*
also have ... $=$ ‘$P^t \Leftrightarrow (P^f \vee P^t)$‘
by *pred-auto*
also have ... $=$ ‘$(P^f \Rightarrow P^t)$‘
by *pred-auto*
finally show *?thesis* **using** *assms*
by (*metis H2-def Healthy-def' taut-iff-eq*)

qed

lemma *H2-equiv*:
P *is* $H2 \longleftrightarrow P^t \sqsubseteq P^f$
using *H2-equivalence refBy-order* **by** *blast*

lemma *H2-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $H2(P \vdash Q) = P \vdash Q$
using *assms*
by (*simp add*: *H2-split design-def usubst unrest*, *pred-auto*)

lemma *H2-rdesign*:
$H2(P \vdash_r Q) = P \vdash_r Q$
by (*simp add*: *H2-design unrest rdesign-def*)

theorem *J-idem*:
$(J ;; J) = J$
by *rel-auto*

theorem *H2-idem*:
$H2(H2(P)) = H2(P)$
by (*metis H2-def J-idem seqr-assoc*)

theorem *H2-not-okay*: $H2\ (\neg \$ok) = (\neg \$ok)$
proof −
have $H2\ (\neg \$ok) = ((\neg \$ok)^f \vee ((\neg \$ok)^t \wedge \$ok'))$
by (*simp add*: *H2-split*)
also have ... $= (\neg \$ok \vee (\neg \$ok) \wedge \$ok')$
by (*subst-tac*)
also have ... $= (\neg \$ok)$
by *pred-auto*
finally show *?thesis* **.**
qed

lemma *H2-true*: $H2(true) = true$
by (*rel-auto*)

lemma *H2-choice-closed*:
$\llbracket P$ *is* $H2$; Q *is* $H2 \rrbracket \Longrightarrow P \sqcap Q$ *is* $H2$
by (*metis H2-def Healthy-def' disj-upred-def seqr-or-distl*)

lemma *H2-inf-closed*:
assumes P *is* $H2$ Q *is* $H2$
shows $P \sqcup Q$ *is* $H2$
proof −
have $P \sqcup Q = (P^f \vee P^t \wedge \$ok') \sqcup (Q^f \vee Q^t \wedge \$ok')$
by (*metis H2-def Healthy-def J-split assms*(*1*) *assms*(*2*))
moreover have $H2(...) = ...$
by (*simp add*: *H2-split usubst*, *pred-auto*)
ultimately show *?thesis*
by (*simp add*: *Healthy-def*)
qed

lemma *H2-USUP*:

shows $H2(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot H2(P(i)))$
using *assms* **by** (*rel-auto*)

theorem *H1-H2-commute*:
$H1\ (H2\ P) = H2\ (H1\ P)$
proof –
have $H2\ (H1\ P) = ((\$ok \Rightarrow P)\ ;;\ J)$
by (*simp add*: *H1-def H2-def*)
also from *assms* **have** ... $= ((\neg\ \$ok \vee P)\ ;;\ J)$
by *rel-auto*
also have ... $= ((\neg\ \$ok\ ;;\ J) \vee (P\ ;;\ J))$
using *seqr-or-distl* **by** *blast*
also have ... $= ((H2\ (\neg\ \$ok)) \vee H2(P))$
by (*simp add*: *H2-def*)
also have ... $= ((\neg\ \$ok) \vee H2(P))$
by (*simp add*: *H2-not-okay*)
also have ... $= H1(H2(P))$
by *rel-auto*
finally show *?thesis* **by** *simp*
qed

lemma *ok-pre*: $(\$ok \wedge \lceil pre_D(P)\rceil_D) = (\$ok \wedge (\neg\ P^f))$
apply (*pred-auto*)
done

lemma *ok-post*: $(\$ok \wedge \lceil post_D(P)\rceil_D) = (\$ok \wedge (P^t))$
apply (*pred-auto*)
done

theorem *H1-H2-eq-design*:
$H1\ (H2\ P) = (\neg\ P^f) \vdash P^t$
proof –
have $H1\ (H2\ P) = (\$ok \Rightarrow H2(P))$
by (*simp add*: *H1-def*)
also have ... $= (\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$
by (*metis H2-split*)
also have ... $= (\$ok \wedge (\neg\ P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t)$
by *rel-auto*
also have ... $= (\neg\ P^f) \vdash P^t$
by *rel-auto*
finally show *?thesis* **.**
qed

theorem *H1-H2-is-design*:
assumes *P is H1 P is H2*
shows $P = (\neg\ P^f) \vdash P^t$
using *assms* **by** (*metis H1-H2-eq-design Healthy-def*)

theorem *H1-H2-is-rdesign*:
assumes *P is H1 P is H2*
shows $P = pre_D(P) \vdash_r post_D(P)$
proof –
from *assms* **have** $P = (\$ok \Rightarrow H2(P))$
by (*simp add*: *H1-def Healthy-def'*)
also have ... $= (\$ok \Rightarrow (P^f \vee (P^t \wedge \$ok')))$

by (*metis H2-split*)
also have ... = ($\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge P^t$)
by *pred-auto*
also have ... = ($\$ok \wedge (\neg P^f) \Rightarrow \$ok' \wedge \$ok \wedge P^t$)
by *pred-auto*
also have ... = ($\$ok \wedge \lceil pre_D(P) \rceil_D \Rightarrow \$ok' \wedge \$ok \wedge \lceil post_D(P) \rceil_D$)
by (*simp add*: *ok-post ok-pre*)
also have ... = ($\$ok \wedge \lceil pre_D(P) \rceil_D \Rightarrow \$ok' \wedge \lceil post_D(P) \rceil_D$)
by *pred-auto*
also from *assms* have ... = $pre_D(P) \vdash_r post_D(P)$
by (*simp add*: *rdesign-def design-def*)
finally show *?thesis* .
qed

abbreviation *H1-H2 P* ≡ *H1* (*H2 P*)

notation *H1-H2* (**H**)

lemma *H1-H2-idempotent*: **H** (**H** *P*) = **H** *P*
by (*simp add*: *H1-H2-commute H1-idem H2-idem*)

lemma *H1-H2-Idempotent*: *Idempotent* **H**
by (*simp add*: *Idempotent-def H1-H2-idempotent*)

lemma *H1-H2-monotonic*: *Monotonic* **H**
by (*simp add*: *H1-monotone H2-def Monotonic-def seqr-mono*)

lemma *design-is-H1-H2*:
$[\![\$ok' \sharp P; \$ok' \sharp Q]\!] \Longrightarrow (P \vdash Q)$ *is H1-H2*
by (*simp add*: *H1-design H2-design Healthy-def'*)

lemma *rdesign-is-H1-H2*:
$(P \vdash_r Q)$ *is H1-H2*
by (*simp add*: *Healthy-def H1-rdesign H2-rdesign*)

lemma *seq-r-H1-H2-closed*:
assumes *P is H1-H2 Q is H1-H2*
shows (*P* ;; *Q*) *is H1-H2*
proof −
obtain P_1 P_2 **where** $P = P_1 \vdash_r P_2$
by (*metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms*(*1*))
moreover obtain Q_1 Q_2 **where** $Q = Q_1 \vdash_r Q_2$
by (*metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def assms*(*2*))
moreover have $((P_1 \vdash_r P_2) ;; (Q_1 \vdash_r Q_2))$ *is H1-H2*
by (*simp add*: *rdesign-composition rdesign-is-H1-H2*)
ultimately show *?thesis* **by** *simp*
qed

lemma *assigns-d-comp-ext*:
fixes *P* :: *'α hrelation-d*
assumes *P is H1-H2*
shows $(\langle\sigma\rangle_D ;; P) = \lceil \sigma \oplus_s \Sigma_D \rceil_s \dagger P$
proof −
have $(\langle\sigma\rangle_D ;; P) = (\langle\sigma\rangle_D ;; pre_D(P) \vdash_r post_D(P))$
by (*metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms*)

also have ... = $\lceil\sigma\rceil_s$ † $pre_D(P) \vdash_r \lceil\sigma\rceil_s$ † $post_D(P)$
  by (*simp add*: *assign-d-left-comp*)
also have ... = $\lceil\sigma \oplus_s \Sigma_D\rceil_s$ † $(pre_D(P) \vdash_r post_D(P))$
  by (*rel-auto*)
also have ... = $\lceil\sigma \oplus_s \Sigma_D\rceil_s$ † P
  by (*metis H1-H2-commute H1-H2-is-rdesign H2-idem Healthy-def' assms*)
finally show *?thesis* **.**
qed

lemma *USUP-H1-H2-closed*:
assumes $A \neq \{\}$ $\forall$ $P \in A$. P *is H1-H2*
shows ($\bigsqcap$ A) *is H1-H2*
proof –
from *assms* **have** A: A = *H1-H2* ‘ A
  by (*auto simp add*: *Healthy-def rev-image-eqI*)
also have ($\bigsqcap$ ...) = ($\bigsqcap$ $P \in A$. *H1-H2*(P))
  by *auto*
also have ... = ($\bigsqcap$ $P \in A$ · *H1-H2*(P))
  by (*simp add*: *USUP-as-Sup-collect*)
also have ... = ($\bigsqcap$ $P \in A$ · ($\neg$ P^f) $\vdash$ P^t)
  by (*meson H1-H2-eq-design*)
also have ... = ($\bigsqcup$ $P \in A$ · $\neg$ P^f) $\vdash$ ($\bigsqcap$ $P \in A$ · P^t)
  by (*simp add*: *design-USUP assms*)
also have ... *is H1-H2*
  by (*simp add*: *design-is-H1-H2 unrest*)
finally show *?thesis* **.**
qed

definition *design-sup* :: ($'\alpha$, $'\beta$) *relation-d set* $\Rightarrow$ ($'\alpha$, $'\beta$) *relation-d* ($\bigsqcap_D$- [*900*] *900*) **where**
$\bigsqcap_D$ A = (*if* (A = {}) *then* $\top_D$ *else* $\bigsqcap$ A)

lemma *design-sup-H1-H2-closed*:
assumes $\forall$ $P \in A$. P *is H1-H2*
shows ($\bigsqcap_D$ A) *is H1-H2*
apply (*auto simp add*: *design-sup-def*)
apply (*simp add*: *H1-def H2-not-okay Healthy-def impl-alt-def*)
using *USUP-H1-H2-closed assms* **apply** *blast*
done

lemma *design-sup-empty* [*simp*]: $\bigsqcap_D$ {} = $\top_D$
  by (*simp add*: *design-sup-def*)

lemma *design-sup-non-empty* [*simp*]: $A \neq \{\} \Longrightarrow \bigsqcap_D A = \bigsqcap A$
  by (*simp add*: *design-sup-def*)

lemma *UINF-H1-H2-closed*:
assumes $\forall$ $P \in A$. P *is H1-H2*
shows ($\bigsqcup$ A) *is H1-H2*
proof –
from *assms* **have** A: A = *H1-H2* ‘ A
  by (*auto simp add*: *Healthy-def rev-image-eqI*)
also have ($\bigsqcup$ ...) = ($\bigsqcup$ $P \in A$. *H1-H2*(P))
  by *auto*
also have ... = ($\bigsqcup$ $P \in A$ · *H1-H2*(P))
  by (*simp add*: *UINF-as-Inf-collect*)

also have ... $= (\bigsqcup\ P \in A \cdot (\neg\ P^f) \vdash P^t)$
  by (*meson H1-H2-eq-design*)
also have ... $= (\bigsqcap\ P \in A \cdot \neg\ P^f) \vdash (\bigsqcup\ P \in A \cdot \neg\ P^f \Rightarrow P^t)$
  by (*simp add*: *design-UINF*)
also have ... *is H1-H2*
  by (*simp add*: *design-is-H1-H2 unrest*)
finally show *?thesis* **.**
qed

abbreviation *design-inf* :: $('\alpha, '\beta)$ *relation-d set* $\Rightarrow$ $('\alpha, '\beta)$ *relation-d* $(\bigsqcup_D$- [*900*] *900*) **where**
$\bigsqcup_D A \equiv \bigsqcup\ A$

12.6 H3: The design assumption is a precondition

theorem *H3-idem*:
  $H3(H3(P)) = H3(P)$
  by (*metis H3-def design-skip-idem seqr-assoc*)

theorem *design-condition-is-H3*:
  assumes *out*$\alpha \sharp p$
  shows $(p \vdash Q)$ *is H3*
proof −
  have $((p \vdash Q)\ ;;\ II_D) = (\neg\ (\neg\ p\ ;;\ true)) \vdash (Q^t\ ;;\ II[\![true/\$ok]\!])$
    by (*simp add*: *skip-d-alt-def design-composition-subst unrest assms*)
  also have ... $= p \vdash (Q^t\ ;;\ II[\![true/\$ok]\!])$
    using *assms precond-equiv seqr-true-lemma* **by** *force*
  also have ... $= p \vdash Q$
    by (*rel-auto*)
  finally show *?thesis*
    by (*simp add*: *H3-def Healthy-def'*)
qed

theorem *rdesign-H3-iff-pre*:
  $P \vdash_r Q$ *is H3* $\longleftrightarrow P = (P\ ;;\ true)$
proof −
  have $(P \vdash_r Q\ ;;\ II_D) = (P \vdash_r Q\ ;;\ true \vdash_r II)$
    by (*simp add*: *skip-d-def*)
  also from *assms* **have** ... $= (\neg\ (\neg\ P\ ;;\ true) \wedge \neg\ (Q\ ;;\ \neg\ true)) \vdash_r (Q\ ;;\ II)$
    by (*simp add*: *rdesign-composition*)
  also from *assms* **have** ... $= (\neg\ (\neg\ P\ ;;\ true) \wedge \neg\ (Q\ ;;\ \neg\ true)) \vdash_r Q$
    by *simp*
  also have ... $= (\neg\ (\neg\ P\ ;;\ true)) \vdash_r Q$
    by *pred-auto*
  finally have $P \vdash_r Q$ *is H3* $\longleftrightarrow P \vdash_r Q = (\neg\ (\neg\ P\ ;;\ true)) \vdash_r Q$
    by (*metis H3-def Healthy-def'*)
  also have ... $\longleftrightarrow P = (\neg\ (\neg\ P\ ;;\ true))$
    by (*metis rdesign-pre*)
  also have ... $\longleftrightarrow P = (P\ ;;\ true)$
    by (*simp add*: *seqr-true-lemma*)
  finally show *?thesis* **.**
qed

theorem *design-H3-iff-pre*:
  assumes $\$ok \sharp P$ $\$ok' \sharp P$ $\$ok \sharp Q$ $\$ok' \sharp Q$
  shows $P \vdash Q$ *is H3* $\longleftrightarrow P = (P\ ;;\ true)$
proof −

have $P \vdash Q = \lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D$
by (*simp add*: *assms lift-desr-inv rdesign-def*)
moreover hence $\lfloor P \rfloor_D \vdash_r \lfloor Q \rfloor_D$ *is H3* $\longleftrightarrow \lfloor P \rfloor_D = (\lfloor P \rfloor_D \;;;\; true)$
using *rdesign-H3-iff-pre* **by** *blast*
ultimately show *?thesis*
by (*metis assms drop-desr-inv lift-desr-inv lift-dist-seq aext-true*)
qed

theorem *H1-H3-commute*:
$H1\ (H3\ P) = H3\ (H1\ P)$
by *rel-auto*

lemma *skip-d-absorb-J-1*:
$(II_D \;;;\; J) = II_D$
by (*metis H2-def H2-rdesign skip-d-def*)

lemma *skip-d-absorb-J-2*:
$(J \;;;\; II_D) = II_D$
proof −
have $(J \;;;\; II_D) = ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D \;;;\; true \vdash II)$
by (*simp add*: *J-def skip-d-alt-def*)
also have ... $= (\exists\ ok_0 \cdot ((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D)[\![\ll ok_0 \gg/\$ok']\!] \;;;\; (true \vdash II)[\![\ll ok_0 \gg/\$ok]\!])$
by (*subst seqr-middle*[*of ok*], *simp-all*)
also have ... $= ((((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D)[\![false/\$ok']\!] \;;;\; (true \vdash II)[\![false/\$ok]\!])$
$\vee (((\$ok \Rightarrow \$ok') \wedge \lceil II \rceil_D)[\![true/\$ok']\!] \;;;\; (true \vdash II)[\![true/\$ok]\!]))$
by (*simp add*: *disj-comm false-alt-def true-alt-def*)
also have ... $= ((\neg\ \$ok \wedge \lceil II \rceil_D \;;;\; true) \vee (\lceil II \rceil_D \;;;\; \$ok' \wedge \lceil II \rceil_D))$
by *rel-auto*
also have ... $= II_D$
by *rel-auto*
finally show *?thesis* **.**
qed

lemma *H2-H3-absorb*:
$H2\ (H3\ P) = H3\ P$
by (*metis H2-def H3-def seqr-assoc skip-d-absorb-J-1*)

lemma *H3-H2-absorb*:
$H3\ (H2\ P) = H3\ P$
by (*metis H2-def H3-def seqr-assoc skip-d-absorb-J-2*)

theorem *H2-H3-commute*:
$H2\ (H3\ P) = H3\ (H2\ P)$
by (*simp add*: *H2-H3-absorb H3-H2-absorb*)

theorem *H3-design-pre*:
assumes $\$ok \sharp p$ $out\alpha \sharp p$ $\$ok \sharp Q$ $\$ok' \sharp Q$
shows $H3(p \vdash Q) = p \vdash Q$
using *assms*
by (*metis Healthy-def' design-H3-iff-pre precond-right-unit unrest-outα-var vwb-lens-ok vwb-lens-mwb*)

theorem *H3-rdesign-pre*:
assumes $out\alpha \sharp p$
shows $H3(p \vdash_r Q) = p \vdash_r Q$
using *assms*

by (*simp add*: *H3-def*)

theorem *H3-ndesign*:
H3($p \vdash_n Q$) = ($p \vdash_n Q$)
by (*simp add*: *H3-def ndesign-def unrest-pre-out*α)

theorem *H1-H3-is-design*:
assumes *P is H1 P is H3*
shows $P = (\neg P^f) \vdash P^t$
by (*metis H1-H2-eq-design H2-H3-absorb Healthy-def' assms*(*1*) *assms*(*2*))

theorem *H1-H3-is-rdesign*:
assumes *P is H1 P is H3*
shows $P = pre_D(P) \vdash_r post_D(P)$
by (*metis H1-H2-is-rdesign H2-H3-absorb Healthy-def' assms*)

theorem *H1-H3-is-normal-design*:
assumes *P is H1 P is H3*
shows $P = \lfloor pre_D(P) \rfloor_< \vdash_n post_D(P)$
by (*metis H1-H3-is-rdesign assms drop-pre-inv ndesign-def precond-equiv rdesign-H3-iff-pre*)

abbreviation *H1-H3 p* $\equiv$ *H1* (*H3 p*)

lemma *H1-H3-impl-H2*: *P is H1-H3* $\Longrightarrow$ *P is H1-H2*
by (*metis H1-H2-commute H1-idem H2-H3-absorb Healthy-def'*)

lemma *H1-H3-eq-design-d-comp*: *H1* (*H3 P*) = $((\neg P^f) \vdash P^t \;;; II_D)$
by (*metis H1-H2-eq-design H1-H3-commute H3-H2-absorb H3-def*)

lemma *H1-H3-eq-design*: *H1* (*H3 P*) = $(\neg (P^f \;;; true)) \vdash P^t$
apply (*simp add*: *H1-H3-eq-design-d-comp skip-d-alt-def*)
apply (*subst design-composition-subst*)
apply (*simp-all add*: *usubst unrest*)
apply (*rel-auto*)
done

lemma *H3-unrest-out-alpha-nok* [*unrest*]:
assumes *P is H1-H3*
shows *out*α $\sharp$ P^f
proof –
have $P = (\neg (P^f \;;; true)) \vdash P^t$
by (*metis H1-H3-eq-design Healthy-def assms*)
also have *out*α $\sharp$ $(...^f)$
by (*simp add*: *design-def usubst unrest*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *H3-unrest-out-alpha* [*unrest*]: *P is H1-H3* $\Longrightarrow$ *out*α $\sharp$ $pre_D(P)$
by (*metis H1-H3-commute H1-H3-is-rdesign H1-idem Healthy-def' precond-equiv rdesign-H3-iff-pre*)

theorem *wpd-seq-r-H1-H2* [*wp*]:
fixes *P Q* :: *'*α *hrelation-d*
assumes *P is H1-H3 Q is H1-H3*
shows $(P \;;; Q)\ wp_D\ r = P\ wp_D\ (Q\ wp_D\ r)$
by (*smt H1-H3-commute H1-H3-is-rdesign H1-idem Healthy-def' assms*(*1*) *assms*(*2*) *drop-pre-inv*

precond-equiv rdesign-H3-iff-pre wpd-seq-r)

12.7 H4: Feasibility

theorem *H4-idem*:
  $H4(H4(P)) = H4(P)$
  by *pred-auto*

lemma *is-H4-alt-def*:
  P *is* $H4 \longleftrightarrow (P \;;;\; true) = true$
  by (*rel-auto*)

lemma *H4-assigns-d*: $\langle\sigma\rangle_D$ *is* $H4$
proof −
  have $(\langle\sigma\rangle_D \;;;\; (false \vdash_r true_h)) = (false \vdash_r true)$
    by (*simp add*: *assigns-d-def rdesign-composition assigns-r-feasible*)
  moreover have ... = *true*
    by (*rel-auto*)
  ultimately show *?thesis*
    using *is-H4-alt-def* **by** *auto*
qed

12.8 UTP theories

typedef *DES* = *UNIV* :: *unit set* **by** *simp*
typedef *NDES* = *UNIV* :: *unit set* **by** *simp*

abbreviation *DES* ≡ *TYPE*(*DES* × *'α alphabet-d*)
abbreviation *NDES* ≡ *TYPE*(*NDES* × *'α alphabet-d*)

overloading
 des-hcond == *utp-hcond* :: (*DES* × *'α alphabet-d*) *itself* ⇒ (*'α alphabet-d* × *'α alphabet-d*) *Healthiness-condition*
 des-unit == *utp-unit* :: (*DES* × *'α alphabet-d*) *itself* ⇒ *'α hrelation-d*

 ndes-hcond == *utp-hcond* :: (*NDES* × *'α alphabet-d*) *itself* ⇒ (*'α alphabet-d* × *'α alphabet-d*) *Healthiness-condition*
 ndes-unit == *utp-unit* :: (*NDES* × *'α alphabet-d*) *itself* ⇒ *'α hrelation-d*

begin
 definition *des-hcond* :: (*DES* × *'α alphabet-d*) *itself* ⇒ (*'α alphabet-d* × *'α alphabet-d*) *Healthiness-condition*
where
  [*upred-defs*]: *des-hcond t* = *H1-H2*

  definition *des-unit* :: (*DES* × *'α alphabet-d*) *itself* ⇒ *'α hrelation-d* **where**
  [*upred-defs*]: *des-unit t* = II_D

 definition *ndes-hcond* :: (*NDES* × *'α alphabet-d*) *itself* ⇒ (*'α alphabet-d* × *'α alphabet-d*) *Healthiness-condition*
where
  [*upred-defs*]: *ndes-hcond t* = *H1-H3*

  definition *ndes-unit* :: (*NDES* × *'α alphabet-d*) *itself* ⇒ *'α hrelation-d* **where**
  [*upred-defs*]: *ndes-unit t* = II_D

end

interpretation *des-utp-theory*: *utp-theory TYPE*(*DES* × *'α alphabet-d*)

by (*simp add*: *H1-H2-commute H1-idem H2-idem des-hcond-def utp-theory-def*)

interpretation *ndes-utp-theory*: *utp-theory TYPE*(*NDES* $\times$ *'α alphabet-d*)
by (*simp add*: *H1-H3-commute H1-idem H3-idem ndes-hcond-def utp-theory.intro*)

interpretation *des-left-unital*: *utp-theory-left-unital TYPE*(*DES* $\times$ *'α alphabet-d*)
apply (*unfold-locales*)
apply (*simp-all add*: *des-hcond-def des-unit-def*)
apply (*simp add*: *rdesign-is-H1-H2 skip-d-def*)
apply (*metis H1-idem H1-left-unit Healthy-def'*)
done

interpretation *ndes-unital*: *utp-theory-unital TYPE*(*NDES* $\times$ (*'α alphabet-d*))
apply (*unfold-locales*, *simp-all add*: *ndes-hcond-def ndes-unit-def*)
apply (*metis H1-rdesign H3-def Healthy-def' design-skip-idem skip-d-def*)
apply (*metis H1-idem H1-left-unit Healthy-def'*)
apply (*metis H1-H3-commute H3-def H3-idem Healthy-def'*)
done

interpretation *design-theory-mono*: *utp-theory-mono TYPE*(*DES* $\times$ *'α alphabet-d*)
rewrites *carrier* (*utp-order DES*) = $[\![H1\text{-}H2]\!]_H$
by (*unfold-locales*, *simp-all add*: *des-hcond-def H1-H2-monotonic utp-order-def*)

lemma *design-lat-top*: $\top_{DES}$ = $\mathbf{H}$(*false*)
by (*simp add*: *des-hcond-def design-theory-mono.healthy-top*)

lemma *design-lat-bottom*: $\bot_{DES}$ = $\mathbf{H}$(*true*)
by (*simp add*: *des-hcond-def design-theory-mono.healthy-bottom*)

abbreviation *design-lfp* :: - $\Rightarrow$ - (μ_D) **where**
$\mu_D\ F \equiv \mu_{\textit{utp-order DES}}\ F$

abbreviation *design-gfp* :: - $\Rightarrow$ - (ν_D) **where**
$\nu_D\ F \equiv \nu_{\textit{utp-order DES}}\ F$

thm *design-theory-mono.GFP-unfold*
thm *design-theory-mono.LFP-unfold*

Example Galois connection between designs and relations. Based on Jim's example in COMPASS deliverable D23.5.

definition [*upred-defs*]: *Des*(*R*) = $\mathbf{H}(\lceil R \rceil_D \wedge \$ok')$
definition [*upred-defs*]: *Rel*(*D*) = $\lfloor D[\![true,true/\$ok,\$ok']\!] \rfloor_D$

lemma *Des-design*: *Des*(*R*) = *true* $\vdash_r$ *R*
by (*rel-auto*)

lemma *Rel-design*: *Rel*($P \vdash_r Q$) = ($P \Rightarrow Q$)
by (*rel-auto*)

interpretation *Des-Rel-coretract*:
coretract DES $\leftarrow\langle Des,Rel \rangle\rightarrow$ *REL*
rewrites
$\bigwedge x.\ x \in \textit{carrier}\ \mathcal{X}_{DES \leftarrow\langle Des,Rel\rangle\rightarrow REL}$ = (x *is* $\mathbf{H}$) **and**

$\bigwedge x.\ x \in carrier\ \mathcal{Y}_{DES\ \leftarrow\langle Des,Rel\rangle\rightarrow\ REL} = True$ **and**
$\pi_{*DES\ \leftarrow\langle Des,Rel\rangle\rightarrow\ REL} = Des$ **and**
$\pi^{*}{}_{DES\ \leftarrow\langle Des,Rel\rangle\rightarrow\ REL} = Rel$ **and**
$le\ \mathcal{X}_{DES\ \leftarrow\langle Des,Rel\rangle\rightarrow\ REL} = op\ \sqsubseteq$ **and**
$le\ \mathcal{Y}_{DES\ \leftarrow\langle Des,Rel\rangle\rightarrow\ REL} = op\ \sqsubseteq$
proof (*unfold-locales*, *simp-all add*: *utp-order-def rel-hcond-def des-hcond-def*)
show $\bigwedge x.\ x$ *is id*
by (*simp add*: *Healthy-def*)
next
show $Rel \in [\![\mathbf{H}]\!]_H \rightarrow [\![id]\!]_H$
by (*auto simp add*: *Rel-def rel-hcond-def Healthy-def*)
next
show $Des \in [\![id]\!]_H \rightarrow [\![\mathbf{H}]\!]_H$
by (*auto simp add*: *Des-def des-hcond-def Healthy-def H1-H2-commute H1-idem H2-idem*)
next
fix R :: *'a hrelation*
show $R \sqsubseteq Rel\ (Des\ R)$
by (*simp add*: *Des-design Rel-design*)
next
fix R :: *'a hrelation* **and** D :: *'a hrelation-d*
assume a: D *is* **H**
then obtain $D_1\ D_2$ **where** D: $D = D_1 \vdash_r D_2$
by (*metis H1-H2-commute H1-H2-is-rdesign H1-idem Healthy-def'*)
show $(Rel\ D \sqsubseteq R) = (D \sqsubseteq Des\ R)$
proof −
have $(D \sqsubseteq Des\ R) = (D_1 \vdash_r D_2 \sqsubseteq true \vdash_r R)$
by (*simp add*: *D Des-design*)
also have ... = ‘$D_1 \wedge R \Rightarrow D_2$‘
by (*simp add*: *rdesign-refinement*)
also have ... $= ((D_1 \Rightarrow D_2) \sqsubseteq R)$
by (*rel-auto*)
also have ... $= (Rel\ D \sqsubseteq R)$
by (*simp add*: *D Rel-design*)
finally show *?thesis* **..**
qed
qed

From this interpretation we gain many Galois theorems. Some require simplification to remove superfluous assumptions.

thm *Des-Rel-coretract.deflation*[*simplified*]
thm *Des-Rel-coretract.inflation*
thm *Des-Rel-coretract.upper-comp*[*simplified*]
thm *Des-Rel-coretract.lower-comp*

end

13 Concurrent programming

theory *utp-concurrency*
imports *utp-rel*
begin

In parallel-by-merge constructions, a merge predicate defines the behaviour following execution of of parallel processes, P —— Q, as a relation that merges the output of P and Q. In order to achieve this we need to separate the variable values output from P and Q, and in addition the variable values before execution. The following three constructs do these separations.

definition [*upred-defs*]: *left-uvar* $x = x\ ;_L\ fst_L\ ;_L\ snd_L$

definition [*upred-defs*]: *right-uvar* $x = x\ ;_L\ snd_L\ ;_L\ snd_L$

definition [*upred-defs*]: *pre-uvar* $x = x\ ;_L\ fst_L$

lemma *left-uvar-indep-right-uvar* [*simp*]:
  left-uvar $x \bowtie$ *right-uvar* y
  apply (*simp add*: *left-uvar-def right-uvar-def lens-comp-assoc*[*THEN sym*])
  apply (*simp add*: *alpha-in-var alpha-out-var*)
done

lemma *right-uvar-indep-left-uvar* [*simp*]:
  right-uvar $x \bowtie$ *left-uvar* y
  by (*simp add*: *lens-indep-sym*)

lemma *left-uvar* [*simp*]: *vwb-lens* $x \Longrightarrow$ *vwb-lens* (*left-uvar* x)
  by (*simp add*: *left-uvar-def*)

lemma *right-uvar* [*simp*]: *vwb-lens* $x \Longrightarrow$ *vwb-lens* (*right-uvar* x)
  by (*simp add*: *right-uvar-def*)

syntax
  -svarpre :: *svid* $\Rightarrow$ *svid* ($\text{-}_<$ [*999*] *999*)
  -svarleft :: *svid* $\Rightarrow$ *svid* (*0*−- [*999*] *999*)
  -svarright :: *svid* $\Rightarrow$ *svid* (*1*−- [*999*] *999*)

translations
  -svarpre x == *CONST pre-uvar* x
  -svarleft x == *CONST left-uvar* x
  -svarright x == *CONST right-uvar* x

type-synonym $'\alpha$ *merge* = $('\alpha \times ('\alpha \times '\alpha), '\alpha)$ *relation*

U0 and U1 are relations that index all input variables x to 0-x and 1-x, respectively.

definition [*upred-defs*]: $U0 = (\$0{-}\Sigma' =_u \$\Sigma)$

definition [*upred-defs*]: $U1 = (\$1{-}\Sigma' =_u \$\Sigma)$

As shown below, separating simulations can also be expressed using the following two alphabet extrusions

definition $U0\alpha$ **where** [*upred-defs*]: $U0\alpha = (1_L \times_L$ *out-var* $fst_L)$

definition $U1\alpha$ **where** [*upred-defs*]: $U1\alpha = (1_L \times_L$ *out-var* $snd_L)$

abbreviation *U0-alpha-lift* ($\lceil\text{-}\rceil_0$) **where** $\lceil P \rceil_0 \equiv P \oplus_p U0\alpha$

abbreviation *U1-alpha-lift* ($\lceil\text{-}\rceil_1$) **where** $\lceil P \rceil_1 \equiv P \oplus_p U1\alpha$

We implement the following useful abbreviation for separating of two parallel processes and copying of the before variables, all to act as input to the merge predicate.

abbreviation *par-sep* (**infixl** $\|_s$ *85*) **where**

$P \parallel_s Q \equiv (P \;;;\; U0) \wedge (Q \;;;\; U1) \wedge \$\Sigma_{<}' =_u \$\Sigma$

The following implementation of parallel by merge is less general than the book version, in that it does not properly partition the alphabet into two disjoint segments. We could actually achieve this specifying lenses into the larger alphabet, but this would complicate the definition of programs. May reconsider later.

definition *par-by-merge* (- $\parallel_{-}$ - [*85*,*0*,*86*] *85*)
where [*upred-defs*]: $P \parallel_M Q = (P \parallel_s Q \;;;\; M)$

nil is the merge predicate which ignores the output of both parallel predicates

definition [*upred-defs*]: $nil_m = (\$\Sigma' =_u \$\Sigma_{<})$

swap is a predicate that the swaps the left and right indices; it is used to specify commutativity of the parallel operator

definition [*upred-defs*]: $swap_m = (0{-}\Sigma, 1{-}\Sigma := \&1{-}\Sigma, \&0{-}\Sigma)$

lemma *U0-swap*: $(U0 \;;;\; swap_m) = U1$
  by *rel-auto*

lemma *U1-swap*: $(U1 \;;;\; swap_m) = U0$
  by *rel-auto*

We can equivalently express separating simulations using alphabet extrusion

lemma *U0-as-alpha*: $(P \;;;\; U0) = \lceil P \rceil_0$
  by *rel-auto*

lemma *U1-as-alpha*: $(P \;;;\; U1) = \lceil P \rceil_1$
  by *rel-auto*

lemma *U0α-vwb-lens* [*simp*]: *vwb-lens U0α*
  by (*simp add*: *U0α-def id-vwb-lens prod-vwb-lens*)

lemma *U1α-vwb-lens* [*simp*]: *vwb-lens U1α*
  by (*simp add*: *U1α-def id-vwb-lens prod-vwb-lens*)

lemma *U0-alpha-out-var* [*alpha*]: $\lceil \$x' \rceil_0 = \$0{-}x'$
  by (*rel-auto*)

lemma *U1-alpha-out-var* [*alpha*]: $\lceil \$x' \rceil_1 = \$1{-}x'$
  by (*rel-auto*)

lemma *U0α-comp-in-var* [*alpha*]: $(in\text{-}var\ x) ;_L U0\alpha = in\text{-}var\ x$
  by (*simp add*: *U0α-def alpha-in-var in-var-prod-lens pre-uvar-def*)

lemma *U0α-comp-out-var* [*alpha*]: $(out\text{-}var\ x) ;_L U0\alpha = out\text{-}var\ (left\text{-}uvar\ x)$
  by (*simp add*: *U0α-def alpha-out-var id-wb-lens left-uvar-def out-var-prod-lens*)

lemma *U1α-comp-in-var* [*alpha*]: $(in\text{-}var\ x) ;_L U1\alpha = in\text{-}var\ x$
  by (*simp add*: *U1α-def alpha-in-var in-var-prod-lens pre-uvar-def*)

lemma *U1α-comp-out-var* [*alpha*]: $(out\text{-}var\ x) ;_L U1\alpha = out\text{-}var\ (right\text{-}uvar\ x)$
  by (*simp add*: *U1α-def alpha-out-var id-wb-lens right-uvar-def out-var-prod-lens*)

lemma *U0-seq-subst*: $(P \;;;\; U0)[\![\ll v \gg / \$0{-}x']\!] = (P[\![\ll v \gg / \$x']\!] \;;;\; U0)$

by *rel-auto*

lemma *U1-seq-subst*: $(P ;; U1)[\![\ll v\gg/\$1{-}x']\!] = (P[\![\ll v\gg/\$x']\!] ;; U1)$
by *rel-auto*

lemma *par-by-merge-false* [*simp*]:
$P \parallel_{false} Q = false$
by (*rel-auto*)

lemma *par-by-merge-left-false* [*simp*]:
$false \parallel_M Q = false$
by (*rel-auto*)

lemma *par-by-merge-right-false* [*simp*]:
$P \parallel_M false = false$
by (*rel-auto*)

lemma *par-by-merge-commute*:
assumes $(swap_m ;; M) = M$
shows $P \parallel_M Q = Q \parallel_M P$
proof –
have $P \parallel_M Q = (((P ;; U0) \wedge (Q ;; U1) \wedge \$\Sigma_<' =_u \$\Sigma) ;; M)$
by (*simp add*: *par-by-merge-def*)
also have ... $= ((((P ;; U0) \wedge (Q ;; U1) \wedge \$\Sigma_<' =_u \$\Sigma) ;; swap_m) ;; M)$
by (*metis assms seqr-assoc*)
also have ... $= (((P ;; U0 ;; swap_m) \wedge (Q ;; U1 ;; swap_m) \wedge \$\Sigma_<' =_u \$\Sigma) ;; M)$
by *rel-auto*
also have ... $= (((P ;; U1) \wedge (Q ;; U0) \wedge \$\Sigma_<' =_u \$\Sigma) ;; M)$
by (*simp add*: *U0-swap U1-swap*)
also have ... $= Q \parallel_M P$
by (*simp add*: *par-by-merge-def utp-pred.inf.left-commute*)
finally show *?thesis* **.**
qed

lemma *shEx-pbm-left*: $((\exists\ x \cdot P\ x) \parallel_M Q) = (\exists\ x \cdot (P\ x \parallel_M Q))$
by (*rel-auto*)

lemma *shEx-pbm-right*: $(P \parallel_M (\exists\ x \cdot Q\ x)) = (\exists\ x \cdot (P \parallel_M Q\ x))$
by (*rel-auto*)

lemma *par-by-merge-mono-1*:
assumes $P_1 \sqsubseteq P_2$
shows $P_1 \parallel_M Q \sqsubseteq P_2 \parallel_M Q$
using *assms* **by** (*rel-auto*)

lemma *par-by-merge-mono-2*:
assumes $Q_1 \sqsubseteq Q_2$
shows $(P \parallel_M Q_1) \sqsubseteq (P \parallel_M Q_2)$
using *assms* **by** *rel-blast*

lemma *bool-pbm-laws* [*usubst*]:
fixes $x :: (bool \Longrightarrow {}'\alpha)$
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s true) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![true/\$x]\!]) \parallel_{M[\![true/\$x_<]\!]} (Q[\![true/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s false) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![false/\$x]\!]) \parallel_{M[\![false/\$x_<]\!]} (Q[\![false/\$x]\!]))$

$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x\acute{}\ \mapsto_s\ true) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![true/\$x\acute{}]\!]} Q)$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x\acute{}\ \mapsto_s\ false) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![false/\$x\acute{}]\!]} Q)$
by (*rel-auto*)+

lemma *zero-one-pbm-laws* [*usubst*]:
fixes x :: (- $\Longrightarrow$ $'\alpha$)
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![0/\$x]\!]) \parallel_{M[\![0/\$x_<]\!]} (Q[\![0/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![1/\$x]\!]) \parallel_{M[\![1/\$x_<]\!]} (Q[\![1/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x\acute{}\ \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![0/\$x\acute{}]\!]} Q)$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x\acute{}\ \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![1/\$x\acute{}]\!]} Q)$
by (*rel-auto*)+

lemma *numeral-pbm-laws* [*usubst*]:
fixes x :: (- $\Longrightarrow$ $'\alpha$)
shows
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x \mapsto_s numeral\ n) \dagger (P \parallel_M Q) = \sigma \dagger ((P[\![numeral\ n/\$x]\!]) \parallel_{M[\![numeral\ n/\$x_<]\!]} (Q[\![numeral\ n/\$x]\!]))$
$\bigwedge P\ Q\ M\ \sigma.\ \sigma(\$x\acute{}\ \mapsto_s numeral\ n) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M[\![numeral\ n/\$x\acute{}]\!]} Q)$
by (*rel-auto*)+

end

14 Reactive processes

theory *utp-reactive*
imports
utp-designs
utp-concurrency
utp-event
begin

record *'t::ordered-cancel-monoid-diff alpha-rp'* =
$wait_v$ :: *bool*
tr_v :: *'t*

declare *alpha-rp'.splits* [*alpha-splits*]

The two locale interpretations below are a technicality to improve automatic proof support via the predicate and relational tactics. This is to enable the (re-)interpretation of state spaces to remove any occurrences of lens types after the proof tactics *pred-simp* and *rel-simp*, or any of their derivatives have been applied. Eventually, it would be desirable to automate both interpretations as part of a custom outer command for defining alphabets.

interpretation *alphabet-rp*:
lens-interp $\lambda(ok, r).\ (ok, wait_v\ r, tr_v\ r, more\ r)$
apply (*unfold-locales*)
apply (*rule injI*)
apply (*clarsimp*)
done

interpretation *alphabet-rp-rel*: *lens-interp* $\lambda(ok, ok', r, r')$.
$(ok, ok', wait_v\ r, wait_v\ r', tr_v\ r, tr_v\ r', more\ r, more\ r')$
apply (*unfold-locales*)

apply (*rule injI*)
apply (*clarsimp*)
done

type-synonym ($'t$, $'\alpha$) *alpha-rp-scheme* = ($'t$, $'\alpha$) *alpha-rp'-scheme alpha-d-scheme*

type-synonym ($'t$,$'\alpha$) *alphabet-rp* = ($'t$,$'\alpha$) *alpha-rp-scheme alphabet*
type-synonym ($'t$,$'\alpha$,$'\beta$) *relation-rp* = (($'t$,$'\alpha$) *alphabet-rp*, ($'t$,$'\beta$) *alphabet-rp*) *relation*
type-synonym ($'t$,$'\alpha$) *hrelation-rp* = (($'t$,$'\alpha$) *alphabet-rp*, ($'t$,$'\alpha$) *alphabet-rp*) *relation*
type-synonym ($'t$,$'\sigma$) *predicate-rp* = ($'t$,$'\sigma$) *alphabet-rp upred*

translations
(*type*) ($'t$, $'\alpha$) *alphabet-rp* <= (*type*) ($'t$, $'\alpha$) *alpha-rp'-scheme alpha-d-ext*
(*type*) ($'t$, $'\alpha$) *alphabet-rp* <= (*type*) ($'t$, $'\alpha$) *alpha-rp'-ext alpha-d-ext*

definition $wait_r$ = *VAR* $wait_v$
definition tr_r = *VAR* tr_v
definition Σ_r = *VAR more*

declare $wait_r$*-def* [*uvar-defs*]
declare tr_r*-def* [*uvar-defs*]
declare Σ_r*-def* [*uvar-defs*]

lemma $wait_r$*-vwb-lens* [*simp*]: *vwb-lens* $wait_r$
by (*unfold-locales*, *simp-all add*: $wait_r$*-def*)

lemma tr_r*-vwb-lens* [*simp*]: *vwb-lens* tr_r
by (*unfold-locales*, *simp-all add*: tr_r*-def*)

lemma *rea-vwb-lens* [*simp*]: *vwb-lens* Σ_r
by (*unfold-locales*, *simp-all add*: Σ_r*-def*)

definition [*uvar-defs*]: *wait* = ($wait_r$ $;_L$ Σ_D)
definition [*uvar-defs*]: *tr* = (tr_r $;_L$ Σ_D)
definition [*uvar-defs*]: Σ_R = (Σ_r $;_L$ Σ_D)

lemma *wait-vwb-lens* [*simp*]: *vwb-lens wait*
by (*simp add*: *wait-def*)

lemma *tr-vwb-lens* [*simp*]: *vwb-lens tr*
by (*simp add*: *tr-def*)

lemma *rea-lens-vwb-lens* [*simp*]: *vwb-lens* Σ_R
by (*simp add*: Σ_R*-def*)

lemma *rea-lens-under-des-lens*: $\Sigma_R \subseteq_L \Sigma_D$
by (*simp add*: Σ_R*-def lens-comp-lb*)

lemma *rea-lens-indep-ok* [*simp*]: $\Sigma_R \bowtie ok$ $ok \bowtie \Sigma_R$
using *ok-indep-des-lens*(*2*) *rea-lens-under-des-lens sublens-pres-indep* **apply** *blast*
using *lens-indep-sym ok-indep-des-lens*(*2*) *rea-lens-under-des-lens sublens-pres-indep* **apply** *blast*
done

lemma *tr-ok-indep* [*simp*]: $tr \bowtie ok$ $ok \bowtie tr$
by (*simp-all add*: *lens-indep-left-ext lens-indep-sym tr-def*)

lemma *wait-ok-indep* [*simp*]: *wait* $\bowtie$ *ok ok* $\bowtie$ *wait*
by (*simp-all add*: *lens-indep-left-ext lens-indep-sym wait-def*)

lemma tr_r*-wait*$_r$*-indep* [*simp*]: $tr_r \bowtie wait_r$ $wait_r \bowtie tr_r$
by (*auto intro*!:*lens-indepI simp add*: tr_r*-def wait*$_r$*-def*)

lemma *tr-wait-indep* [*simp*]: *tr* $\bowtie$ *wait wait* $\bowtie$ *tr*
by (*auto intro*: *lens-indep-left-comp simp add*: *tr-def wait-def*)

lemma *rea-indep-wait* [*simp*]: $\Sigma_r \bowtie wait_r$ $wait_r \bowtie \Sigma_r$
by (*auto intro*!:*lens-indepI simp add*: *wait*$_r$*-def* Σ_r*-def*)

lemma *rea-lens-indep-wait* [*simp*]: $\Sigma_R \bowtie$ *wait wait* $\bowtie \Sigma_R$
by (*auto intro*: *lens-indep-left-comp simp add*: *wait-def* Σ_R*-def*)

lemma *rea-indep-tr* [*simp*]: $\Sigma_r \bowtie tr_r$ $tr_r \bowtie \Sigma_r$
by (*auto intro*!:*lens-indepI simp add*: tr_r*-def* Σ_r*-def*)

lemma *rea-lens-indep-tr* [*simp*]: $\Sigma_R \bowtie$ *tr tr* $\bowtie \Sigma_R$
by (*auto intro*: *lens-indep-left-comp simp add*: *tr-def* Σ_R*-def*)

lemma *rea-var-ords* [*usubst*]:
$\$tr \prec_v \$tr'$ $\$wait \prec_v \$wait'$
$\$ok \prec_v \$tr$ $\$ok' \prec_v \$tr'$ $\$ok \prec_v \$tr'$ $\$ok' \prec_v \$tr$
$\$ok \prec_v \$wait$ $\$ok' \prec_v \$wait'$ $\$ok \prec_v \$wait'$ $\$ok' \prec_v \$wait$
$\$tr \prec_v \$wait$ $\$tr' \prec_v \$wait'$ $\$tr \prec_v \$wait'$ $\$tr' \prec_v \$wait$
by (*simp-all add*: *var-name-ord-def*)

abbreviation *wait-f*::($'t$::*ordered-cancel-monoid-diff*, $'\alpha$, $'\beta$) *relation-rp* $\Rightarrow$ ($'t$, $'\alpha$, $'\beta$) *relation-rp*
where *wait-f R* $\equiv$ $R[\![false/\$wait]\!]$

abbreviation *wait-t*::($'t$::*ordered-cancel-monoid-diff*, $'\alpha$, $'\beta$) *relation-rp* $\Rightarrow$ ($'t$, $'\alpha$, $'\beta$) *relation-rp*
where *wait-t R* $\equiv$ $R[\![true/\$wait]\!]$

syntax
-wait-f :: *logic* $\Rightarrow$ *logic* (-_f [*1000*] *1000*)
-wait-t :: *logic* $\Rightarrow$ *logic* (-_t [*1000*] *1000*)

translations
$P\ _f \rightleftharpoons$ *CONST usubst* (*CONST subst-upd CONST id* (*CONST ivar CONST wait*) *false*) *P*
$P\ _t \rightleftharpoons$ *CONST usubst* (*CONST subst-upd CONST id* (*CONST ivar CONST wait*) *true*) *P*

abbreviation *lift-rea* :: - $\Rightarrow$ - ($\lceil\text{-}\rceil_R$) **where**
$\lceil P\rceil_R \equiv P \oplus_p (\Sigma_R \times_L \Sigma_R)$

abbreviation *drop-rea* :: ($'t$::*ordered-cancel-monoid-diff*, $'\alpha$, $'\beta$) *relation-rp* $\Rightarrow$ ($'\alpha$, $'\beta$) *relation* ($\lfloor\text{-}\rfloor_R$)
where
$\lfloor P\rfloor_R \equiv P \upharpoonright_p (\Sigma_R \times_L \Sigma_R)$

abbreviation *rea-pre-lift* :: - $\Rightarrow$ - ($\lceil\text{-}\rceil_{R<}$) **where** $\lceil n\rceil_{R<} \equiv \lceil\lceil n\rceil_<\rceil_R$

definition *skip-rea-def* [*urel-defs*]: $II_r = (II \lor (\neg \$ok \land \$tr \leq_u \$tr'))$

instantiation *alpha-rp′-ext* :: (*ordered-cancel-monoid-diff*,*vst*) *vst*
begin
definition *vstore-lens-alpha-rp′-ext* :: *vstore* $\Longrightarrow$ ('*a*, '*b*) *alpha-rp′-scheme* **where**
vstore-lens-alpha-rp′-ext = $\mathcal{V}$ $;_L$ Σ_r
instance
by (*intro-classes*, *simp add*: *vstore-lens-alpha-rp′-ext-def*)
end

14.1 Reactive lemmas

lemma *unrest-ok-lift-rea* [*unrest*]:
$\$ok \sharp \lceil P \rceil_R$ $\$ok' \sharp \lceil P \rceil_R$
by (*pred-auto*)+

lemma *unrest-wait-lift-rea* [*unrest*]:
$\$wait \sharp \lceil P \rceil_R$ $\$wait' \sharp \lceil P \rceil_R$
by (*pred-auto*)+

lemma *unrest-tr-lift-rea* [*unrest*]:
$\$tr \sharp \lceil P \rceil_R$ $\$tr' \sharp \lceil P \rceil_R$
by (*pred-auto*)+

lemma *tr-prefix-as-concat*: $(xs \leq_u ys) = (\exists\ zs \cdot ys =_u xs\ \hat{}_u\ \ll zs \gg)$
by (*rel-auto*, *simp add*: *less-eq-list-def prefixeq-def*)

lemma *seqr-wait-true* [*usubst*]: $(P\ ;;\ Q)\ _t = (P\ _t\ ;;\ Q)$
by *rel-auto*

lemma *seqr-wait-false* [*usubst*]: $(P\ ;;\ Q)\ _f = (P\ _f\ ;;\ Q)$
by *rel-auto*

14.2 R1: Events cannot be undone

definition *R1-def* [*upred-defs*]: $R1\ (P) = \ (P \wedge (\$tr \leq_u \$tr'))$

lemma *R1-idem*: $R1(R1(P)) = R1(P)$
by *pred-auto*

lemma *R1-Idempotent*: *Idempotent R1*
by (*simp add*: *Idempotent-def R1-idem*)

lemma *R1-mono*: $P \sqsubseteq Q \Longrightarrow R1(P) \sqsubseteq R1(Q)$
by *pred-auto*

lemma *R1-Monotonic*: *Monotonic R1*
by (*simp add*: *Monotonic-def R1-mono*)

lemma *R1-unrest* [*unrest*]: $[\![x \bowtie in\text{-}var\ tr;\ x \bowtie out\text{-}var\ tr;\ x \sharp P]\!] \Longrightarrow x \sharp R1(P)$
by (*metis R1-def in-var-uvar lens-indep-sym out-var-uvar tr-vwb-lens unrest-bop unrest-conj unrest-var*)

lemma *R1-false*: $R1(false) = false$
by *pred-auto*

lemma *R1-conj*: $R1(P \wedge Q) = (R1(P) \wedge R1(Q))$
by *pred-auto*

lemma *R1-disj*: $R1(P \vee Q) = (R1(P) \vee R1(Q))$
  by *pred-auto*

lemma *R1-USUP*:
  $R1(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot R1(P(i)))$
  by (*rel-auto*)

lemma *R1-UINF*:
  assumes $A \neq \{\}$
  shows $R1(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot R1(P(i)))$
  using *assms* **by** (*rel-auto*)

lemma *R1-extend-conj*: $R1(P \wedge Q) = (R1(P) \wedge Q)$
  by *pred-auto*

lemma *R1-extend-conj'*: $R1(P \wedge Q) = (P \wedge R1(Q))$
  by *pred-auto*

lemma *R1-cond*: $R1(P \triangleleft b \triangleright Q) = (R1(P) \triangleleft b \triangleright R1(Q))$
  by *rel-auto*

lemma *R1-negate-R1*: $R1(\neg R1(P)) = R1(\neg P)$
  by *pred-auto*

lemma *R1-wait-true*: $(R1\ P)_t = R1(P)_t$
  by *pred-auto*

lemma *R1-wait-false*: $(R1\ P)\ _f = R1(P)\ _f$
  by *pred-auto*

lemma *R1-skip*: $R1(II) = II$
  by *rel-auto*

lemma *R1-skip-rea*: $R1(II_r) = II_r$
  by *rel-auto*

lemma *skip-rea-form*: $II_r = (II \triangleleft \$ok \triangleright R1(true))$
  by *rel-auto*

lemma *R1-by-refinement*:
  $P\ is\ R1 \longleftrightarrow ((\$tr \leq_u \$tr') \sqsubseteq P)$
  by *rel-blast*

lemma *tr-le-trans*:
  $(\$tr \leq_u \$tr'\ ;;\ \$tr \leq_u \$tr') = (\$tr \leq_u \$tr')$
  by (*rel-auto*)

lemma *R1-seqr*:
  $R1(R1(P)\ ;;\ R1(Q)) = (R1(P)\ ;;\ R1(Q))$
  by (*rel-auto*)

lemma *R1-seqr-closure*:
  assumes $P\ is\ R1\ Q\ is\ R1$
  shows $(P\ ;;\ Q)\ is\ R1$

using *assms* **unfolding** *R1-by-refinement*
by (*metis seqr-mono tr-le-trans*)

lemma *R1-true-comp*: $(R1(true) \;;\; R1(true)) = R1(true)$
by (*rel-auto*)

lemma *R1-ok'-true*: $(R1(P))^t = R1(P^t)$
by *pred-auto*

lemma *R1-ok'-false*: $(R1(P))^f = R1(P^f)$
by *pred-auto*

lemma *R1-ok-true*: $(R1(P))[\![true/\$ok]\!] = R1(P[\![true/\$ok]\!])$
by *pred-auto*

lemma *R1-ok-false*: $(R1(P))[\![false/\$ok]\!] = R1(P[\![false/\$ok]\!])$
by *pred-auto*

lemma *seqr-R1-true-right*: $((P \;;\; R1(true)) \vee P) = (P \;;\; (\$tr \leq_u \$tr'))$
by *rel-auto*

lemma *R1-extend-conj-unrest*: $[\![\$tr \sharp Q; \$tr' \sharp Q]\!] \Longrightarrow R1(P \wedge Q) = (R1(P) \wedge Q)$
by *pred-auto*

lemma *R1-extend-conj-unrest'*: $[\![\$tr \sharp P; \$tr' \sharp P]\!] \Longrightarrow R1(P \wedge Q) = (P \wedge R1(Q))$
by *pred-auto*

lemma *R1-tr'-eq-tr*: $R1(\$tr' =_u \$tr) = (\$tr' =_u \$tr)$
by (*rel-auto*)

lemma *R1-H2-commute*: $R1(H2(P)) = H2(R1(P))$
by (*simp add*: *H2-split R1-def usubst*, *rel-auto*)

14.3 R2

definition *R2a-def* [*upred-defs*]: $R2a\ (P) = (\bigsqcap\ s \cdot P[\![\ll s\gg, \ll s\gg + (\$tr' - \$tr)/\$tr, \$tr']\!])$
definition *R2s-def* [*upred-defs*]: $R2s\ (P) = (P[\![0/\$tr]\!][\![(\$tr' - \$tr)/\$tr']\!])$
definition *R2-def* [*upred-defs*]: $R2(P) = R1(R2s(P))$
definition *R2c-def* [*upred-defs*]: $R2c(P) = (R2s(P) \triangleleft R1(true) \triangleright P)$

lemma *R2a-R2s*: $R2a(R2s(P)) = R2s(P)$
by *rel-auto*

lemma *R2s-R2a*: $R2s(R2a(P)) = R2a(P)$
by *rel-auto*

lemma *R2a-equiv-R2s*: P *is* $R2a \longleftrightarrow P$ *is* $R2s$
by (*metis Healthy-def' R2a-R2s R2s-R2a*)

lemma *R2s-idem*: $R2s(R2s(P)) = R2s(P)$
by (*pred-auto*)

lemma *R2s-unrest* [*unrest*]: $[\![vwb\text{-}lens\ x; x \bowtie in\text{-}var\ tr; x \bowtie out\text{-}var\ tr; x \sharp P]\!] \Longrightarrow x \sharp R2s(P)$
by (*simp add*: *R2s-def unrest usubst lens-indep-sym*)

lemma *R2-idem*: $R2(R2(P)) = R2(P)$

by (*pred-auto*)

lemma *R2-mono*: $P \sqsubseteq Q \Longrightarrow R2(P) \sqsubseteq R2(Q)$
by (*pred-auto*)

lemma *R2s-conj*: $R2s(P \wedge Q) = (R2s(P) \wedge R2s(Q))$
by (*pred-auto*)

lemma *R2-conj*: $R2(P \wedge Q) = (R2(P) \wedge R2(Q))$
by (*pred-auto*)

lemma *R2s-disj*: $R2s(P \vee Q) = (R2s(P) \vee R2s(Q))$
by *pred-auto*

lemma *R2s-USUP*:
$R2s(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot R2s(P(i)))$
by (*simp add*: *R2s-def usubst*)

lemma *R2c-USUP*:
$R2c(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot R2c(P(i)))$
by (*rel-auto*)

lemma *R2s-UINF*:
$R2s(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot R2s(P(i)))$
by (*simp add*: *R2s-def usubst*)

lemma *R2c-UINF*:
$R2c(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot R2c(P(i)))$
by (*rel-auto*)

lemma *R2-disj*: $R2(P \vee Q) = (R2(P) \vee R2(Q))$
by (*pred-auto*)

lemma *R2s-not*: $R2s(\neg\ P) = (\neg\ R2s(P))$
by *pred-auto*

lemma *R2s-condr*: $R2s(P \triangleleft b \triangleright Q) = (R2s(P) \triangleleft R2s(b) \triangleright R2s(Q))$
by *rel-auto*

lemma *R2-condr*: $R2(P \triangleleft b \triangleright Q) = (R2(P) \triangleleft R2(b) \triangleright R2(Q))$
by *rel-auto*

lemma *R2-condr'*: $R2(P \triangleleft b \triangleright Q) = (R2(P) \triangleleft R2s(b) \triangleright R2(Q))$
by *rel-auto*

lemma *R2s-ok*: $R2s(\$ok) = \$ok$
by *rel-auto*

lemma *R2s-ok'*: $R2s(\$ok') = \$ok'$
by *rel-auto*

lemma *R2s-wait*: $R2s(\$wait) = \$wait$
by *rel-auto*

lemma *R2s-wait'*: $R2s(\$wait') = \$wait'$

by *rel-auto*

lemma *R2s-true*: *R2s*(*true*) = *true*
  by *pred-auto*

lemma *R2s-false*: *R2s*(*false*) = *false*
  by *pred-auto*

lemma *true-is-R2s*:
  true is R2s
  by (*simp add*: *Healthy-def R2s-true*)

lemma *R2s-lift-rea*: $R2s(\lceil P \rceil_R) = \lceil P \rceil_R$
  by (*simp add*: *R2s-def usubst unrest*)

lemma *R2c-true*: *R2c*(*true*) = *true*
  by *rel-auto*

lemma *R2c-false*: *R2c*(*false*) = *false*
  by *rel-auto*

lemma *R2c-and*: $R2c(P \wedge Q) = (R2c(P) \wedge R2c(Q))$
  by (*rel-auto*)

lemma *R2c-disj*: $R2c(P \vee Q) = (R2c(P) \vee R2c(Q))$
  by (*rel-auto*)

lemma *R2c-not*: $R2c(\neg P) = (\neg R2c(P))$
  by (*rel-auto*)

lemma *R2c-ok*: $R2c(\$ok) = (\$ok)$
  by (*rel-auto*)

lemma *R2c-ok'*: $R2c(\$ok') = (\$ok')$
  by (*rel-auto*)

lemma *R2c-wait*: $R2c(\$wait) = \$wait$
  by (*rel-auto*)

lemma *R2c-tr'-minus-tr*: $R2c(\$tr' =_u \$tr) = (\$tr' =_u \$tr)$
  apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *R2c-tr'-ge-tr*: $R2c(\$tr' \geq_u \$tr) = (\$tr' \geq_u \$tr)$
  by (*rel-auto*)

lemma *R2c-condr*: $R2c(P \triangleleft b \triangleright Q) = (R2c(P) \triangleleft R2c(b) \triangleright R2c(Q))$
  by (*rel-auto*)

lemma *R2c-skip-r*: $R2c(II) = II$
proof −
  have $R2c(II) = R2c(\$tr' =_u \$tr \wedge II\restriction_\alpha tr)$
    by (*subst skip-r-unfold*[*of tr*], *simp-all*)
  also have ... $= (R2c(\$tr' =_u \$tr) \wedge II\restriction_\alpha tr)$
    by (*simp add*: *R2c-and*, *simp add*: *R2c-def R2s-def usubst unrest cond-idem*)
  also have ... $= (\$tr' =_u \$tr \wedge II\restriction_\alpha tr)$

by (*simp add*: *R2c-tr′-minus-tr*)
finally show *?thesis*
by (*subst skip-r-unfold*[*of tr*], *simp-all*)
qed

lemma *R1-R2c-commute*: $R1(R2c(P)) = R2c(R1(P))$
by (*rel-auto*)

lemma *R1-R2c-is-R2*: $R1(R2c(P)) = R2(P)$
by (*rel-auto*)

lemma *R2c-skip-rea*: $R2c\ II_r = II_r$
by (*simp add*: *skip-rea-def R2c-and R2c-disj R2c-skip-r R2c-not R2c-ok R2c-tr′-ge-tr*)

lemma *R1-R2s-R2c*: $R1(R2s(P)) = R1(R2c(P))$
by (*rel-auto*)

lemma *R2-skip-rea*: $R2(II_r) = II_r$
by (*metis R1-R2c-is-R2 R1-skip-rea R2c-skip-rea*)

lemma *R2-tr-prefix*: $R2(\$tr \leq_u \$tr') = (\$tr \leq_u \$tr')$
by (*pred-auto*)

lemma *R2-form*:
$R2(P) = (\exists\ tt \cdot P[\![0/\$tr]\!][\![\ll tt\gg/\$tr']\!] \wedge \$tr' =_u \$tr + \ll tt\gg)$
by (*rel-auto*, *metis ordered-cancel-monoid-diff-class.add-diff-cancel-left ordered-cancel-monoid-diff-class.le-iff-add*)

lemma *R2-seqr-form*:
shows $(R2(P)\ ;;\ R2(Q)) =$
$(\exists\ tt_1 \cdot \exists\ tt_2 \cdot ((P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!])\ ;;\ (Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!]))$
$\wedge (\$tr' =_u \$tr + \ll tt_1\gg + \ll tt_2\gg))$
proof −
have $(R2(P)\ ;;\ R2(Q)) = (\exists\ tr_0 \cdot (R2(P))[\![\ll tr_0\gg/\$tr']\!]\ ;;\ (R2(Q))[\![\ll tr_0\gg/\$tr]\!])$
by (*subst seqr-middle*[*of tr*], *simp-all*)
also have ... =
$(\exists\ tr_0 \cdot \exists\ tt_1 \cdot \exists\ tt_2 \cdot ((P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!] \wedge \ll tr_0\gg =_u \$tr + \ll tt_1\gg)\ ;;$
$(Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!] \wedge \$tr' =_u \ll tr_0\gg + \ll tt_2\gg)))$
by (*simp add*: *R2-form usubst unrest uquant-lift*, *rel-blast*)
also have ... =
$(\exists\ tr_0 \cdot \exists\ tt_1 \cdot \exists\ tt_2 \cdot ((\ll tr_0\gg =_u \$tr + \ll tt_1\gg \wedge P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!])\ ;;$
$(Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!] \wedge \$tr' =_u \ll tr_0\gg + \ll tt_2\gg)))$
by (*simp add*: *conj-comm*)
also have ... =
$(\exists\ tt_1 \cdot \exists\ tt_2 \cdot \exists\ tr_0 \cdot ((P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!])\ ;;\ (Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!]))$
$\wedge \ll tr_0\gg =_u \$tr + \ll tt_1\gg \wedge \$tr' =_u \ll tr_0\gg + \ll tt_2\gg)$
by *rel-blast*
also have ... =
$(\exists\ tt_1 \cdot \exists\ tt_2 \cdot ((P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!])\ ;;\ (Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!]))$
$\wedge (\exists\ tr_0 \cdot \ll tr_0\gg =_u \$tr + \ll tt_1\gg \wedge \$tr' =_u \ll tr_0\gg + \ll tt_2\gg))$
by *rel-auto*
also have ... =
$(\exists\ tt_1 \cdot \exists\ tt_2 \cdot ((P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!])\ ;;\ (Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!]))$
$\wedge (\$tr' =_u \$tr + \ll tt_1\gg + \ll tt_2\gg))$
by *rel-auto*
finally show *?thesis* **.**

qed

lemma *R2-seqr-distribute*:
  fixes P :: $('t$::*ordered-cancel-monoid-diff*$,'\alpha,'\beta)$ *relation-rp* **and** Q :: $('t,'\beta,'\gamma)$ *relation-rp*
  shows $R2(R2(P)$;; $R2(Q)) = (R2(P)$;; $R2(Q))$
proof −
  have $R2(R2(P)$;; $R2(Q)) =$
    $((\exists\ tt_1 \cdot \exists\ tt_2 \cdot (P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!]$;; $Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!])[\![(\$tr' - \$tr)/\$tr']\!]$
      $\wedge\ \$tr' - \$tr =_u \ll tt_1\gg + \ll tt_2\gg) \wedge \$tr' \geq_u \$tr)$
    by (*simp add*: *R2-seqr-form*, *simp add*: *R2s-def usubst unrest*, *rel-auto*)
  also have ... =
    $((\exists\ tt_1 \cdot \exists\ tt_2 \cdot (P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!]$;; $Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!])[\![(\ll tt_1\gg + \ll tt_2\gg)/\$tr']\!]$
      $\wedge\ \$tr' - \$tr =_u \ll tt_1\gg + \ll tt_2\gg) \wedge \$tr' \geq_u \$tr)$
      by (*subst subst-eq-replace*, *simp*)
  also have ... =
    $((\exists\ tt_1 \cdot \exists\ tt_2 \cdot (P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!]$;; $Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!])$
      $\wedge\ \$tr' - \$tr =_u \ll tt_1\gg + \ll tt_2\gg) \wedge \$tr' \geq_u \$tr)$
      by (*rel-auto*)
  also have ... =
    $(\exists\ tt_1 \cdot \exists\ tt_2 \cdot (P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!]$;; $Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!])$
      $\wedge\ (\$tr' - \$tr =_u \ll tt_1\gg + \ll tt_2\gg \wedge \$tr' \geq_u \$tr))$
    by *pred-auto*
  also have ... =
    $((\exists\ tt_1 \cdot \exists\ tt_2 \cdot (P[\![0/\$tr]\!][\![\ll tt_1\gg/\$tr']\!]$;; $Q[\![0/\$tr]\!][\![\ll tt_2\gg/\$tr']\!])$
      $\wedge\ \$tr' =_u \$tr + \ll tt_1\gg + \ll tt_2\gg))$
  proof −
    have $\bigwedge$ tt_1 tt_2. $(((\$tr' - \$tr =_u \ll tt_1\gg + \ll tt_2\gg) \wedge \$tr' \geq_u \$tr)$ :: $('t,'\alpha,'\gamma)$ *relation-rp*)
          $= (\$tr' =_u \$tr + \ll tt_1\gg + \ll tt_2\gg)$
      apply (*rel-auto*)
      apply (*metis add.assoc diff-add-cancel-left'*)
      apply (*simp add*: *add.assoc*)
      apply (*meson le-add order-trans*)
    done
    thus *?thesis* **by** *simp*
  qed
  also have ... = $(R2(P)$;; $R2(Q))$
    by (*simp add*: *R2-seqr-form*)
  finally show *?thesis* **.**
qed

lemma *R2-seqr-closure*:
  assumes P *is* $R2$ Q *is* $R2$
  shows $(P$;; $Q)$ *is* $R2$
  by (*metis Healthy-def' R2-seqr-distribute assms*(*1*) *assms*(*2*))

lemma *R1-R2-commute*:
  $R1(R2(P)) = R2(R1(P))$
  by *pred-auto*

lemma *R2-R1-form*: $R2(R1(P)) = R1(R2s(P))$
  by (*rel-auto*)

lemma *R2s-H1-commute*:
  $R2s(H1(P)) = H1(R2s(P))$
  by *rel-auto*

lemma *R2s-H2-commute*:
$R2s(H2(P)) = H2(R2s(P))$
by (*simp add*: *H2-split R2s-def usubst*)

lemma *R2-R1-seq-drop-left*:
$R2(R1(P) \;;\; R1(Q)) = R2(P \;;\; R1(Q))$
by *rel-auto*

lemma *R2c-idem*: $R2c(R2c(P)) = R2c(P)$
by (*rel-auto*)

lemma *R2c-Idempotent*: *Idempotent R2c*
by (*simp add*: *Idempotent-def R2c-idem*)

lemma *R2c-Monotonic*: *Monotonic R2c*
by (*rel-auto*)

lemma *R2c-H2-commute*: $R2c(H2(P)) = H2(R2c(P))$
by (*simp add*: *H2-split R2c-disj R2c-def R2s-def usubst*, *rel-auto*)

lemma *R2c-seq*: $R2c(R2(P) \;;\; R2(Q)) = (R2(P) \;;\; R2(Q))$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute R2c-idem*)

lemma *R2-R2c-def*: $R2(P) = R1(R2c(P))$
by *rel-auto*

lemma *R2c-R1-seq*: $R2c(R1(R2c(P)) \;;\; R1(R2c(Q))) = (R1(R2c(P)) \;;\; R1(R2c(Q)))$
using *R2c-seq*[*of P Q*] **by** (*simp add*: *R2-R2c-def*)

14.4 R3

definition *R3-def* [*upred-defs*]: $R3\ (P) = (II \triangleleft \$wait \triangleright P)$

definition *R3c-def* [*upred-defs*]: $R3c\ (P) = (II_r \triangleleft \$wait \triangleright P)$

lemma *R3-idem*: $R3(R3(P)) = R3(P)$
by *rel-auto*

lemma *R3-Idempotent*: *Idempotent R3*
by (*simp add*: *Idempotent-def R3-idem*)

lemma *R3-mono*: $P \sqsubseteq Q \Longrightarrow R3(P) \sqsubseteq R3(Q)$
by *rel-auto*

lemma *R3-Monotonic*: *Monotonic R3*
by (*simp add*: *Monotonic-def R3-mono*)

lemma *R3-conj*: $R3(P \wedge Q) = (R3(P) \wedge R3(Q))$
by *rel-auto*

lemma *R3-disj*: $R3(P \vee Q) = (R3(P) \vee R3(Q))$
by *rel-auto*

lemma *R3-USUP*:
assumes $A \neq \{\}$

shows $R3(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot R3(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *R3-UINF*:
assumes $A \neq \{\}$
shows $R3(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot R3(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *R3-condr*: $R3(P \triangleleft b \triangleright Q) = (R3(P) \triangleleft b \triangleright R3(Q))$
by *rel-auto*

lemma *R3-skipr*: $R3(II) = II$
by *rel-auto*

lemma *R3-form*: $R3(P) = ((\$wait \wedge II) \vee (\neg\ \$wait \wedge P))$
by *rel-auto*

lemma *wait-R3*:
$(\$wait \wedge R3(P)) = (II \wedge \$wait')$
by (*rel-auto*)

lemma *nwait-R3*:
$(\neg\$wait \wedge R3(P)) = (\neg\$wait \wedge P)$
by (*rel-auto*)

lemma *R3-semir-form*:
$(R3(P) \ ;; \ R3(Q)) = R3(P \ ;; \ R3(Q))$
by *rel-auto*

lemma *R3-semir-closure*:
assumes *P is R3 Q is R3*
shows $(P \ ;; \ Q)$ *is R3*
using *assms*
by (*metis Healthy-def' R3-semir-form*)

lemma *R3c-semir-form*:
$(R3c(P) \ ;; \ R3c(R1(Q))) = R3c(P \ ;; \ R3c(R1(Q)))$
by (*rel-simp*, *safe*, *auto intro*: *order-trans*)

lemma *R3c-seq-closure*:
assumes *P is R3c Q is R3c Q is R1*
shows $(P \ ;; \ Q)$ *is R3c*
by (*metis Healthy-def' R3c-semir-form assms*)

lemma *R3c-R3-left-seq-closure*:
assumes *P is R3 Q is R3c*
shows $(P \ ;; \ Q)$ *is R3c*
proof $-$
have $(P \ ;; \ Q) = ((P \ ;; \ Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \ ;; \ Q))$
by (*metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens*)
also have ... $= (((II \triangleleft \$wait \triangleright P) \ ;; \ Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \ ;; \ Q))$
by (*metis Healthy-def' R3-def assms*(*1*))
also have ... $= ((II[\![true/\$wait]\!] \ ;; \ Q) \triangleleft \$wait \triangleright (P \ ;; \ Q))$
by (*subst-tac*)
also have ... $= ((II \wedge \$wait' \ ;; \ Q) \triangleleft \$wait \triangleright (P \ ;; \ Q))$

by (*metis* (*no-types*, *lifting*) *cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred.inf-left-idem wait-vwb-lens*)
also have ... = $((II[\![true/\$wait']\!] ;; Q[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... = $((II[\![true/\$wait']\!] ;; (II_r \triangleleft \$wait \triangleright Q)[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis Healthy-def' R3c-def assms*(*2*))
also have ... = $((II[\![true/\$wait']\!] ;; II_r[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P ;; Q))$
by (*subst-tac*)
also have ... = $((II \wedge \$wait' ;; II_r) \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... = $((II ;; II_r) \triangleleft \$wait \triangleright (P ;; Q))$
by (*simp add*: *cond-def seqr-pre-transfer utp-rel.unrest-ouvar*)
also have ... = $(II_r \triangleleft \$wait \triangleright (P ;; Q))$
by *simp*
also have ... = $R3c(P ;; Q)$
by (*simp add*: *R3c-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def'*)
qed

lemma *R3c-cases*: $R3c(P) = ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$
by (*rel-auto*)

lemma *R3c-subst-wait*: $R3c(P) = R3c(P\ _f)$
by (*metis R3c-def cond-var-subst-right wait-vwb-lens*)

lemma *R1-R3-commute*: $R1(R3(P)) = R3(R1(P))$
by *rel-auto*

lemma *R1-R3c-commute*: $R1(R3c(P)) = R3c(R1(P))$
by *rel-auto*

lemma *R2-R3-commute*: $R2(R3(P)) = R3(R2(P))$
apply (*rel-auto*)
using *minus-zero-eq* **apply** *blast+*
done

lemma *R2-R3c-commute*: $R2(R3c(P)) = R3c(R2(P))$
apply (*rel-auto*)
using *minus-zero-eq* **apply** *blast+*
done

lemma *R2c-R3c-commute*: $R2c(R3c(P)) = R3c(R2c(P))$
by (*simp add*: *R3c-def R2c-condr R2c-wait R2c-skip-rea*)

lemma *R1-H1-R3c-commute*:
$R1(H1(R3c(P))) = R3c(R1(H1(P)))$
by *rel-auto*

lemma *R3c-H2-commute*: $R3c(H2(P)) = H2(R3c(P))$
by (*simp add*: *H2-split R3c-def usubst*, *rel-auto*)

lemma *R3c-idem*: $R3c(R3c(P)) = R3c(P)$

by *rel-auto*

lemma *R3c-Idempotent*: *Idempotent R3c*
using *Idempotent-def R3c-idem* **by** *blast*

lemma *R3c-mono*: $P \sqsubseteq Q \Longrightarrow R3c(P) \sqsubseteq R3c(Q)$
by *rel-auto*

lemma *R3c-Monotonic*: *Monotonic R3c*
by (*simp add*: *Monotonic-def R3c-mono*)

lemma *R3c-conj*: $R3c(P \wedge Q) = (R3c(P) \wedge R3c(Q))$
by (*rel-auto*)

lemma *R3c-disj*: $R3c(P \vee Q) = (R3c(P) \vee R3c(Q))$
by *rel-auto*

lemma *R3c-USUP*:
assumes $A \neq \{\}$
shows $R3c(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot R3c(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *R3c-UINF*:
assumes $A \neq \{\}$
shows $R3c(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot R3c(P(i)))$
using *assms* **by** (*rel-auto*)

14.5 RH laws

definition *RH-def* [*upred-defs*]: $RH(P) = R1(R2s(R3c(P)))$

notation *RH* (**R**)

definition *reactive-sup* :: - *set* ⇒ - ($\bigsqcap_r$) **where**
$\bigsqcap_r\ A = (if\ (A = \{\})\ then\ \mathbf{R}(false)\ else\ \bigsqcap\ A)$

definition *reactive-inf* :: - *set* ⇒ - ($\bigsqcup_r$) **where**
$\bigsqcup_r\ A = (if\ (A = \{\})\ then\ \mathbf{R}(true)\ else\ \bigsqcup\ A)$

lemma *RH-alt-def*:
$\mathbf{R}(P) = R1(R2(R3c(P)))$
by (*simp add*: *R1-idem R2-def RH-def*)

lemma *RH-alt-def'*:
$\mathbf{R}(P) = R2(R3c(P))$
by (*simp add*: *R2-def RH-def*)

lemma *RH-alt-def''*:
$\mathbf{R}(P) = R1(R2c(R3c(P)))$
by (*simp add*: *R1-R2s-R2c RH-def*)

lemma *RH-idem*:
$\mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$
by (*metis R2-R3c-commute R2-def R2-idem R3c-idem RH-def*)

lemma *RH-Idempotent*: *Idempotent* **R**

by (*simp add*: *Idempotent-def RH-idem*)

lemma *RH-monotone*:
$P \sqsubseteq Q \Longrightarrow \mathbf{R}(P) \sqsubseteq \mathbf{R}(Q)$
by *rel-auto*

lemma *RH-disj*: $\mathbf{R}(P \vee Q) = (\mathbf{R}(P) \vee \mathbf{R}(Q))$
by (*simp add*: *RH-def R3c-disj R2s-disj R1-disj*)

lemma *RH-USUP*:
assumes $A \neq \{\}$
shows $\mathbf{R}(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot \mathbf{R}(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *RH-UINF*:
assumes $A \neq \{\}$
shows $\mathbf{R}(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot \mathbf{R}(P(i)))$
using *assms* **by** (*rel-auto*)

lemma *RH-intro*:
$[\![\ P\ \textit{is}\ R1;\ P\ \textit{is}\ R2;\ P\ \textit{is}\ R3c\ ]\!] \Longrightarrow P\ \textit{is}\ \mathbf{R}$
by (*simp add*: *Healthy-def′ R2-def RH-def*)

lemma *R1-true-left-zero-R*: $(R1(true)\ ;;\ \mathbf{R}(P)) = R1(true)$
by (*rel-auto*)

lemma *RH-seq-closure*:
assumes $P\ \textit{is}\ \mathbf{R}\ Q\ \textit{is}\ \mathbf{R}$
shows $(P\ ;;\ Q)\ \textit{is}\ \mathbf{R}$
proof (*rule RH-intro*)
show $(P\ ;;\ Q)\ \textit{is}\ R1$
by (*metis Healthy-def′ R1-seqr-closure R2-def RH-alt-def RH-def assms*(*1*) *assms*(*2*))
show $(P\ ;;\ Q)\ \textit{is}\ R2$
by (*metis Healthy-def′ R2-def R2-idem R2-seqr-closure RH-def assms*(*1*) *assms*(*2*))
show $(P\ ;;\ Q)\ \textit{is}\ R3c$
by (*metis Healthy-def′ R2-R3c-commute R2-def R3c-idem R3c-seq-closure RH-alt-def RH-def assms*(*1*) *assms*(*2*))
qed

lemma *RH-R2c-def*: $\mathbf{R}(P) = R1(R2c(R3c(P)))$
by (*rel-auto*)

lemma *RH-absorbs-R2c*: $\mathbf{R}(R2c(P)) = \mathbf{R}(P)$
by (*metis R1-R2-commute R1-R2c-is-R2 R1-R3c-commute R2-R3c-commute R2-idem RH-alt-def RH-alt-def′*)

lemma *RH-subst-wait*: $\mathbf{R}(P\ _f) = \mathbf{R}(P)$
by (*metis R3c-subst-wait RH-alt-def′*)

lemma *RH-false*: $\mathbf{R}(false) = (\$wait \wedge II_r)$
by (*rel-auto*, *metis minus-zero-eq*)

lemma *RH-true*: $\mathbf{R}(true) = (II_r \triangleleft \$wait \triangleright \$tr \leq_u \$tr')$
by (*rel-auto*, *metis minus-zero-eq*)

lemma *RH-false-top*:
  $\mathbf{R}(P) \sqsubseteq \mathbf{R}(\mathit{false})$
  by (*simp add*: *RH-monotone*)

lemma *RH-false-bottom*:
  $\mathbf{R}(\mathit{true}) \sqsubseteq \mathbf{R}(P)$
  by (*simp add*: *RH-monotone*)

14.6 UTP theory

typedef *REA* = *UNIV* :: *unit set* **by** *simp*

abbreviation $\mathit{REA} \equiv \mathit{TYPE}(\mathit{REA} \times ('t{::}\mathit{ordered\text{-}cancel\text{-}monoid\text{-}diff},'\alpha)\ \mathit{alphabet\text{-}rp})$

overloading
  rea-hcond == *utp-hcond* :: $(\mathit{REA} \times ('t{::}\mathit{ordered\text{-}cancel\text{-}monoid\text{-}diff},'\alpha)\ \mathit{alphabet\text{-}rp})\ \mathit{itself} \Rightarrow (('t,'\alpha)\ \mathit{alphabet\text{-}rp} \times ('t,'\alpha)\ \mathit{alphabet\text{-}rp})\ \mathit{Healthiness\text{-}condition}$
begin
  definition *rea-hcond* :: $(\mathit{REA} \times ('t{::}\mathit{ordered\text{-}cancel\text{-}monoid\text{-}diff},'\alpha)\ \mathit{alphabet\text{-}rp})\ \mathit{itself} \Rightarrow (('t,'\alpha)\ \mathit{alphabet\text{-}rp} \times ('t,'\alpha)\ \mathit{alphabet\text{-}rp})\ \mathit{Healthiness\text{-}condition}$ **where**
  [*upred-defs*]: *rea-hcond t* = $\mathbf{R}$
end

interpretation *rea-utp-theory*: *utp-theory* $\mathit{TYPE}(\mathit{REA} \times ('t{::}\mathit{ordered\text{-}cancel\text{-}monoid\text{-}diff},'\alpha)\ \mathit{alphabet\text{-}rp})$
  by (*simp add*: *rea-hcond-def utp-theory-def RH-idem*)

interpretation *rea-utp-theory-mono*: *utp-theory-mono* $\mathit{TYPE}(\mathit{REA} \times ('t{::}\mathit{ordered\text{-}cancel\text{-}monoid\text{-}diff},'\alpha)\ \mathit{alphabet\text{-}rp})$
  by (*unfold-locales*, *simp add*: *Monotonic-def RH-monotone rea-hcond-def*)

lemma *rea-top*: $\top_{REA} = (\$wait \wedge II_r)$
proof −
  have $\top_{REA} = \mathbf{R}(\mathit{false})$
    by (*simp add*: *rea-hcond-def rea-utp-theory-mono.healthy-top*)
  also have ... $= (\$wait \wedge II_r)$
    by (*rel-auto*, *metis minus-zero-eq*)
  finally show *?thesis* **.**
qed

lemma *rea-bottom*: $\bot_{REA} = R1(\$wait \Rightarrow II_r)$
proof −
  have $\bot_{REA} = \mathbf{R}(\mathit{true})$
    by (*simp add*: *rea-hcond-def rea-utp-theory-mono.healthy-bottom*)
  also have ... $= R1(\$wait \Rightarrow II_r)$
    by (*rel-auto*, *metis minus-zero-eq*)
  finally show *?thesis* **.**
qed

14.7 Reactive parallel-by-merge

We show closure of parallel by merge under the reactive healthiness conditions by means of suitable restrictions on the merge predicate. We first define healthiness conditions for R1 and R2 merge predicates.

definition [*upred-defs*]: $R1m(M) = (M \wedge \$tr_< \leq_u \$tr')$

definition [*upred-defs*]: $R1m'(M) = (M \wedge \$tr_< \leq_u \$tr' \wedge \$tr_< \leq_u \$0{-}tr \wedge \$tr_< \leq_u \$1{-}tr)$

A merge predicate can access the history through tr, as usual, but also through $0.tr$ and $1.tr$. Thus we have to remove the latter two histories as well to satisfy R2 for the overall construction.

definition [*upred-defs*]: $R2m(M) = R1m(M[\![0,\$tr' - \$tr_<,\$0{-}tr - \$tr_<,\$1{-}tr - \$tr_</\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])$

definition [*upred-defs*]: $R2m'(M) = R1m'(M[\![0,\$tr' - \$tr_<,\$0{-}tr - \$tr_<,\$1{-}tr - \$tr_</\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])$

lemma *R2m'-form*:
$R2m'(M) =$
$(\exists\ tt,\ tt_0,\ tt_1 \cdot M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!]$
$\wedge \$tr' =_u \$tr_< + \ll tt\gg$
$\wedge \$0{-}tr =_u \$tr_< + \ll tt_0\gg$
$\wedge \$1{-}tr =_u \$tr_< + \ll tt_1\gg)$
by (*rel-auto*, *metis diff-add-cancel-left'*)

lemma *R1-par-by-merge*:
M *is* $R1m \Longrightarrow (P \parallel_M Q)$ *is* $R1$
by (*rel-blast*)

lemma *R2-par-by-merge*:
assumes P *is* $R2$ Q *is* $R2$ M *is* $R2m$
shows $(P \parallel_M Q)$ *is* $R2$
proof −
have $(P \parallel_M Q) = (P \parallel_{R2m(M)} Q)$
by (*metis Healthy-def' assms(3)*)
also have ... $= (R2(P) \parallel_{R2m(M)} R2(Q))$
using *assms* **by** (*simp add*: *Healthy-def'*)
also have ... $= (R2(P) \parallel_{R2m'(M)} R2(Q))$
by (*rel-blast*)
also have ... $= (P \parallel_{R2m'(M)} Q)$
using *assms* **by** (*simp add*: *Healthy-def'*)
also have ... $= ((P \parallel_s Q)\ ;;$
$(\exists\ tt,\ tt_0,\ tt_1 \cdot M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!]$
$\wedge \$tr' =_u \$tr_< + \ll tt\gg$
$\wedge \$0{-}tr =_u \$tr_< + \ll tt_0\gg$
$\wedge \$1{-}tr =_u \$tr_< + \ll tt_1\gg))$
by (*simp add*: *par-by-merge-def R2m'-form*)
also have ... $= (\exists\ tt,\ tt_0,\ tt_1 \cdot ((P \parallel_s Q)\ ;;\ (M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!]$
$\wedge \$tr' =_u \$tr_< + \ll tt\gg$
$\wedge \$0{-}tr =_u \$tr_< + \ll tt_0\gg$
$\wedge \$1{-}tr =_u \$tr_< + \ll tt_1\gg)))$
by (*rel-blast*)
also have ... $= (\exists\ tt,\ tt_0,\ tt_1 \cdot ((P \parallel_s Q) \wedge \$0{-}tr' =_u \$tr_<' + \ll tt_0\gg \wedge \$1{-}tr' =_u \$tr_<' + \ll tt_1\gg\ ;;$
$(M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!] \wedge \$tr' =_u \$tr_< + \ll tt\gg)))$
by (*rel-blast*)
also have ... $= (\exists\ tt,\ tt_0,\ tt_1 \cdot ((P \parallel_s Q) \wedge \$0{-}tr' =_u \$tr_<' + \ll tt_0\gg \wedge \$1{-}tr' =_u \$tr_<' + \ll tt_1\gg\ ;;$
$(M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])) \wedge \$tr' =_u \$tr + \ll tt\gg)$
by (*rel-blast*)
also have ... $= (\exists\ tt,\ tt_0,\ tt_1 \cdot (((P \wedge \$tr' =_u \$tr + \ll tt_0\gg) \parallel_s (Q \wedge \$tr' =_u \$tr + \ll tt_1\gg))\ ;;$
$(M[\![0,\ll tt\gg,\ll tt_0\gg,\ll tt_1\gg/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])) \wedge \$tr' =_u \$tr + \ll tt\gg)$
by (*rel-blast*)

also have ... = $(\exists\ tt, tt_0, tt_1 \cdot (((R2(P) \wedge \$tr' =_u \$tr + «tt_0») \|_s (R2(Q) \wedge \$tr' =_u \$tr + «tt_1»))$;;
$(M[\![0,«tt»,«tt_0»,«tt_1»/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])) \wedge \$tr' =_u \$tr + «tt»)$
using *assms(1−2)* **by** (*simp add*: *Healthy-def'*)
also have ... = $(\exists\ tt, tt_0, tt_1 \cdot ((\ ((\exists\ tt_0' \cdot P[\![0,«tt_0'»/\$tr,\$tr']\!] \wedge \$tr' =_u \$tr + «tt_0'») \wedge \$tr' =_u \$tr + «tt_0»)$
$\|_s ((\exists\ tt_1' \cdot Q[\![0,«tt_1'»/\$tr,\$tr']\!] \wedge \$tr' =_u \$tr + «tt_1'») \wedge \$tr' =_u \$tr + «tt_1»))$;;
$(M[\![0,«tt»,«tt_0»,«tt_1»/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])) \wedge \$tr' =_u \$tr + «tt»)$
by (*simp add*: *R2-form usubst*)
also have ... = $(\exists\ tt, tt_0, tt_1 \cdot ((\ (P[\![0,«tt_0»/\$tr,\$tr']\!]\ \wedge \$tr' =_u \$tr + «tt_0»)$
$\|_s (Q[\![0,«tt_1»/\$tr,\$tr']\!] \wedge \$tr' =_u \$tr + «tt_1»))$;;
$(M[\![0,«tt»,«tt_0»,«tt_1»/\$tr_<,\$tr',\$0{-}tr,\$1{-}tr]\!])) \wedge \$tr' =_u \$tr + «tt»)$
by (*rel-auto*, *metis left-cancel-monoid-class.add-left-imp-eq*, *blast*)
also have ... = $R2(P \|_M Q)$
by (*rel-auto*, *blast*, *metis diff-add-cancel-left'*)
finally show *?thesis*
by (*simp add*: *Healthy-def*)
qed

For R3, we can't easily define an idempotent healthiness function of mege predicates. Thus we define some units and anhilators instead. Each of these defines the behaviour of an indexed parallel system of predicates to be merged.

definition [*upred-defs*]: $skip_m = (\$0{-}\Sigma' =_u \$\Sigma \wedge \$1{-}\Sigma' =_u \$\Sigma \wedge \$\Sigma_<' =_u \$\Sigma)$

$skip_m$ is the system which does nothing to the variables in both predicates. A merge predicate which is R3 must yield II when composed with it.

lemma *R3-par-by-merge*:
assumes
P is R3 Q is R3 $(skip_m$;; $M) = II$
shows $(P \|_M Q)$ *is R3*
proof −
have $(P \|_M Q) = ((P \|_M Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \|_M Q))$
by (*metis cond-L6 cond-var-split in-var-uvar wait-vwb-lens*)
also have ... = $((P[\![true/\$wait]\!] \|_M Q[\![true/\$wait]\!])[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \|_M Q))$
by (*rel-auto*)
also have ... = $((P[\![true/\$wait]\!] \|_M Q[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \|_M Q))$
by (*metis cond-var-subst-left wait-vwb-lens*)
also have ... = $(((II \triangleleft \$wait \triangleright P)[\![true/\$wait]\!] \|_M (II \triangleleft \$wait \triangleright Q)[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \|_M Q))$
by (*metis Healthy-if R3-def assms(1) assms(2)*)
also have ... = $((II[\![true/\$wait]\!] \|_M II[\![true/\$wait]\!]) \triangleleft \$wait \triangleright (P \|_M Q))$
by (*subst-tac*)
also have ... = $((II \|_M II) \triangleleft \$wait \triangleright (P \|_M Q))$
by (*rel-auto*)
also have ... = $((skip_m$;; $M) \triangleleft \$wait \triangleright (P \|_M Q))$
by (*rel-auto*)
also have ... = $(II \triangleleft \$wait \triangleright (P \|_M Q))$
by (*simp add*: *assms(3)*)
also have ... = $R3(P \|_M Q)$
by (*simp add*: *R3-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def'*)
qed

end

15 Reactive designs

theory *utp-rea-designs*
 imports *utp-reactive*
begin

15.1 Commutativity properties

lemma *H2-R1-comm*: $H2(R1(P)) = R1(H2(P))$
 by (*rel-auto*)

lemma *H2-R2s-comm*: $H2(R2s(P)) = R2s(H2(P))$
 by (*rel-auto*)

lemma *H2-R2-comm*: $H2(R2(P)) = R2(H2(P))$
 by (*simp add*: *H2-R1-comm H2-R2s-comm R2-def*)

lemma *H2-R3-comm*: $H2(R3c(P)) = R3c(H2(P))$
 by (*simp add*: *R3c-H2-commute*)

lemma *R3c-via-H1*: $R1(R3c(H1(P))) = R1(H1(R3(P)))$
 by *rel-auto*

lemma *skip-rea-via-H1*: $II_r = R1(H1(R3(II)))$
 by *rel-auto*

lemma *R1-true-left-zero-R*: $(R1(true) \;;\; \mathbf{R}(P)) = R1(true)$
 by (*rel-auto*)

lemma *skip-rea-R1-lemma*: $II_r = R1(\$ok \Rightarrow II)$
 by (*rel-auto*)

lemma *skip-rea-R1-dskip*: $II_r = R1(II_D)$
 by (*rel-auto*)

15.2 Reactive design composition

Pedro's proof for R1 design composition

lemma *R1-design-composition*:
 fixes $P\ Q$ :: $('t{::}ordered\text{-}cancel\text{-}monoid\text{-}diff, '\alpha, '\beta)$ *relation-rp*
 and $R\ S$ :: $('t, '\beta, '\gamma)$ *relation-rp*
 assumes $\$ok' \sharp P\ \$ok' \sharp Q\ \$ok \sharp R\ \$ok \sharp S$
 shows
 $(R1(P \vdash Q) \;;\; R1(R \vdash S)) =$
 $R1((\neg (R1(\neg P) \;;\; R1(true)) \wedge \neg (R1(Q) \;;\; R1(\neg R))) \vdash (R1(Q) \;;\; R1(S)))$
proof −
 have $(R1(P \vdash Q) \;;\; R1(R \vdash S)) = (\exists\ ok_0 \cdot (R1(P \vdash Q))[\![\ll ok_0 \gg/\$ok']\!] \;;\; (R1(R \vdash S))[\![\ll ok_0 \gg/\$ok]\!])$
  using *seqr-middle vwb-lens-ok* **by** *blast*
 also from *assms* **have** ... $= (\exists\ ok_0 \cdot R1((\$ok \wedge P) \Rightarrow (\ll ok_0 \gg \wedge Q)) \;;\; R1((\ll ok_0 \gg\ \wedge R) \Rightarrow (\$ok' \wedge S)))$
  by (*simp add*: *design-def R1-def usubst unrest*)
 also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) \;;\; R1((true \wedge R) \Rightarrow (\$ok' \wedge S)))$
  $\vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) \;;\; R1((false \wedge R) \Rightarrow (\$ok' \wedge S))))$

by (*simp add*: *false-alt-def true-alt-def*)
also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow Q) \;;\; R1(R \Rightarrow (\$ok' \wedge S)))$
$\vee\ (R1(\neg\ (\$ok \wedge P)) \;;\; R1(true)))$
by *simp*
also from *assms* **have** ... $= ((R1(\neg\ \$ok \vee \neg\ P \vee Q) \;;\; R1(\neg\ R \vee (\$ok' \wedge S)))$
$\vee\ (R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(true)))$
by (*simp add*: *impl-alt-def utp-pred.sup.assoc*)
also from *assms* **have** ... $= (((R1(\neg\ \$ok \vee \neg\ P) \vee R1(Q)) \;;\; R1(\neg\ R \vee (\$ok' \wedge S)))$
$\vee\ (R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-disj utp-pred.disj-assoc*)
also from *assms* **have** ... $= ((R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(\neg\ R \vee (\$ok' \wedge S)))$
$\vee\ (R1(Q) \;;\; R1(\neg\ R \vee (\$ok' \wedge S)))$
$\vee\ (R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(true)))$
by (*simp add*: *seqr-or-distl utp-pred.sup.assoc*)
also from *assms* **have** ... $= ((R1(Q) \;;\; R1(\neg\ R \vee (\$ok' \wedge S)))$
$\vee\ (R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(true)))$
by *rel-blast*
also from *assms* **have** ... $= ((R1(Q) \;;\; (R1(\neg\ R) \vee R1(S) \wedge \$ok'))$
$\vee\ (R1(\neg\ \$ok \vee \neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-disj R1-extend-conj utp-pred.inf-commute*)
also have ... $= ((R1(Q) \;;\; (R1(\neg\ R) \vee R1(S) \wedge \$ok'))$
$\vee\ ((R1(\neg\ \$ok) :: ('t,'\alpha,'\beta)\ relation\text{-}rp) \;;\; R1(true))$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-disj seqr-or-distl*)
also have ... $= ((R1(Q) \;;\; (R1(\neg\ R) \vee R1(S) \wedge \$ok'))$
$\vee\ (R1(\neg\ \$ok))$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
proof −
have $((R1(\neg\ \$ok) :: ('t,'\alpha,'\beta)\ relation\text{-}rp) \;;\; R1(true)) =$
$(R1(\neg\ \$ok) :: ('t,'\alpha,'\gamma)\ relation\text{-}rp)$
by (*rel-auto*)
thus *?thesis*
by *simp*
qed
also have ... $= ((R1(Q) \;;\; (R1(\neg\ R) \vee (R1(S \wedge \$ok'))))$
$\vee\ R1(\neg\ \$ok)$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-extend-conj*)
also have ... $= (\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\vee\ (R1(Q) \;;\; (R1(S \wedge \$ok')))$
$\vee\ R1(\neg\ \$ok)$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *seqr-or-distr utp-pred.sup.assoc*)
also have ... $= R1(\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\vee\ (R1(Q) \;;\; (R1(S \wedge \$ok')))$
$\vee\ (\neg\ \$ok)$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
by (*simp add*: *R1-disj R1-seqr*)
also have ... $= R1(\ (R1(Q) \;;\; (R1\ (\neg\ R)))$
$\vee\ ((R1(Q) \;;\; R1(S)) \wedge \$ok')$
$\vee\ (\neg\ \$ok)$
$\vee\ (R1(\neg\ P) \;;\; R1(true)))$
by (*rel-blast*)
also have ... $= R1(\neg(\$ok \wedge \neg\ (R1(\neg\ P) \;;\; R1(true)) \wedge \neg\ (R1(Q) \;;\; (R1\ (\neg\ R))))$
$\vee\ \ ((R1(Q) \;;\; R1(S)) \wedge \$ok'))$

by (*rel-blast*)
also have ... = $R1((\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1 (\neg R))))$
$\Rightarrow (\$ok' \wedge (R1(Q) ;; R1(S))))$
by (*simp add*: *impl-alt-def utp-pred.inf-commute*)
also have ... = $R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$
by (*simp add*: *design-def*)
finally show *?thesis* **.**
qed

definition [*upred-defs*]: $R3c\text{-}pre(P) = (true \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: $R3c\text{-}post(P) = (\lceil II \rceil_D \triangleleft \$wait \triangleright P)$

lemma *R3c-pre-conj*: $R3c\text{-}pre(P \wedge Q) = (R3c\text{-}pre(P) \wedge R3c\text{-}pre(Q))$
by *rel-auto*

lemma *R3c-pre-seq*:
$(true ;; Q) = true \Longrightarrow R3c\text{-}pre(P ;; Q) = (R3c\text{-}pre(P) ;; Q)$
by (*rel-auto*)

lemma *R2s-design*: $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$
by (*simp add*: *R2s-def design-def usubst*)

lemma *R2c-design*: $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$
by (*simp add*: *design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok'*)

lemma *R1-R3c-design*:
$R1(R3c(P \vdash Q)) = R1(R3c\text{-}pre(P) \vdash R3c\text{-}post(Q))$
by (*rel-auto*)

lemma *unrest-ok-R2s* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R2s(P)$
by (*simp add*: *R2s-def unrest*)

lemma *unrest-ok'-R2s* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R2s(P)$
by (*simp add*: *R2s-def unrest*)

lemma *unrest-ok-R2c* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R2c(P)$
by (*simp add*: *R2c-def unrest*)

lemma *unrest-ok'-R2c* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R2c(P)$
by (*simp add*: *R2c-def unrest*)

lemma *unrest-ok-R3c-pre* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3c\text{-}pre(P)$
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok'-R3c-pre* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3c\text{-}pre(P)$
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok-R3c-post* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3c\text{-}post(P)$
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3c-post'* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3c\text{-}post(P)$
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *R3c-R1-design-composition*:

assumes $\$ok\acute{} \sharp P$ $\$ok\acute{} \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(R3c(R1(P \vdash Q))$;; $R3c(R1(R \vdash S))) =$
$R3c(R1((\neg (R1(\neg P)$;; $R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait\acute{})$;; $R1(\neg R)))$
$\vdash (R1(Q)$;; $(\lceil II \rceil_D \triangleleft \$wait \triangleright R1(S))))))$

proof −
have *1*:$(\neg (R1 (\neg R3c\text{-}pre\ P)$;; $R1\ true)) = (R3c\text{-}pre\ (\neg (R1\ (\neg P)$;; $R1\ true)))$
by (*rel-auto*)
have *2*:$(\neg (R1\ (R3c\text{-}post\ Q)$;; $R1\ (\neg R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg (R1\ Q \wedge \neg \$wait\acute{}$;; $R1\ (\neg R)))$
by (*rel-auto*)
have *3*:$(R1\ (R3c\text{-}post\ Q)$;; $R1\ (R3c\text{-}post\ S)) = R3c\text{-}post\ (R1\ Q$;; $(\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ S))$
by (*rel-auto*)
show *?thesis*
apply (*simp add*: *R3c-semir-form R1-R3c-commute*[*THEN sym*] *R1-R3c-design unrest*)
apply (*subst R1-design-composition*)
apply (*simp-all add*: *unrest assms R3c-pre-conj 1 2 3*)
done
qed

lemma *R1-des-lift-skip*: $R1(\lceil II \rceil_D) = \lceil II \rceil_D$
by (*rel-auto*)

lemma *R2s-subst-wait-true* [*usubst*]:
$(R2s(P))[\![true/\$wait]\!] = R2s(P[\![true/\$wait]\!])$
by (*simp add*: *R2s-def usubst unrest*)

lemma *R2s-subst-wait'-true* [*usubst*]:
$(R2s(P))[\![true/\$wait\acute{}]\!] = R2s(P[\![true/\$wait\acute{}]\!])$
by (*simp add*: *R2s-def usubst unrest*)

lemma *R2-subst-wait-true* [*usubst*]:
$(R2(P))[\![true/\$wait]\!] = R2(P[\![true/\$wait]\!])$
by (*simp add*: *R2-def R1-def R2s-def usubst unrest*)

lemma *R2-subst-wait'-true* [*usubst*]:
$(R2(P))[\![true/\$wait\acute{}]\!] = R2(P[\![true/\$wait\acute{}]\!])$
by (*simp add*: *R2-def R1-def R2s-def usubst unrest*)

lemma *R2-subst-wait-false* [*usubst*]:
$(R2(P))[\![false/\$wait]\!] = R2(P[\![false/\$wait]\!])$
by (*simp add*: *R2-def R1-def R2s-def usubst unrest*)

lemma *R2-subst-wait'-false* [*usubst*]:
$(R2(P))[\![false/\$wait\acute{}]\!] = R2(P[\![false/\$wait\acute{}]\!])$
by (*simp add*: *R2-def R1-def R2s-def usubst unrest*)

lemma *R2-des-lift-skip*:
$R2(\lceil II \rceil_D) = \lceil II \rceil_D$
by (*rel-auto*, *metis alpha-rp'.cases-scheme alpha-rp'.select-convs*(*2*) *alpha-rp'.update-convs*(*2*) *minus-zero-eq*)

lemma *R2c-R2s-absorb*: $R2c(R2s(P)) = R2s(P)$
by (*rel-auto*)

lemma *R2-design-composition*:
assumes $\$ok\acute{} \sharp P$ $\$ok\acute{} \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(R2(P \vdash Q)$;; $R2(R \vdash S)) =$

$R2((\neg\ (R1\ (\neg\ R2c\ P)\ ;;\ R1\ true) \wedge \neg\ (R1\ (R2c\ Q)\ ;;\ R1\ (\neg\ R2c\ R))) \vdash (R1\ (R2c\ Q)\ ;;\ R1\ (R2c\ S)))$
apply (*simp add*: *R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj R1-R2c-commute*[*THEN sym*] *R2c-idem R2c-R1-seq*)
apply (*metis* (*no-types*, *lifting*) *R2c-R1-seq R2c-not R2c-true*)
done

lemma *RH-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
shows $(RH(P \vdash Q)\ ;;\ RH(R \vdash S)) =$
$RH((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) \wedge \neg\ (R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q)\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))))$
proof −
have *1*: $R2c\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) = (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true)$
proof −
have *1*:$(R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) = (R1(R2\ (\neg\ P)\ ;;\ R2\ true))$
by (*rel-auto*)
have $R2c(R1(R2\ (\neg\ P)\ ;;\ R2\ true)) = R2c(R1(R2\ (\neg\ P)\ ;;\ R2\ true))$
using *R2c-not* **by** *blast*
also have ... $= R2(R2\ (\neg\ P)\ ;;\ R2\ true)$
by (*metis R1-R2c-commute R1-R2c-is-R2*)
also have ... $= (R2\ (\neg\ P)\ ;;\ R2\ true)$
by (*simp add*: *R2-seqr-distribute*)
also have ... $= (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true)$
by (*simp add*: *R2-def R2s-not R2s-true*)
finally show *?thesis*
by (*simp add*: *1*)
qed

have *2*:$R2c\ (R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R)) = (R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R))$
proof −
have $(R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R)) = R1\ (R2\ (Q \wedge \neg\ \$wait')\ ;;\ R2\ (\neg\ R))$
by (*rel-auto*)
hence $R2c\ (R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R)) = (R2\ (Q \wedge \neg\ \$wait')\ ;;\ R2\ (\neg\ R))$
by (*metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)
also have ... $= (R1\ (R2s\ Q) \wedge \neg\ \$wait'\ ;;\ R1\ (\neg\ R2s\ R))$
by *rel-auto*
finally show *?thesis* **.**
qed

have *3*:$R2c((R1\ (R2s\ Q)\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))) = (R1\ (R2s\ Q)\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])) =$
$R2c(R1\ (R2s\ (Q[\![true/\$wait']\!]))\ ;;\ \lceil II \rceil_D[\![true/\$wait]\!])$
by (*simp add*: *usubst cond-unit-T R1-def R2s-def*)
also have ... $= R2c(R2(Q[\![true/\$wait']\!])\ ;;\ R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
by (*metis R2-def R2-des-lift-skip R2-subst-wait-true*)
also have ... $= (R2(Q[\![true/\$wait']\!])\ ;;\ R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
using *R2c-seq* **by** *blast*
also have ... $= ((R1\ (R2s\ Q))[\![true/\$wait']\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
apply (*simp add*: *usubst R2-des-lift-skip*)

apply (*metis R2-def R2-des-lift-skip R2-subst-wait′-true R2-subst-wait-true*)
done
finally show *?thesis* **.**
qed
moreover have $R2c(((R1\ (R2s\ Q))[\![false/\$wait\acute{}]\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![false/\$wait\acute{}]\!]\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!])$
by (*simp add*: *usubst cond-unit-F*, *metis R2-R1-form R2-subst-wait′-false R2-subst-wait-false R2c-seq*)
ultimately show *?thesis*
by (*smt R2-R1-form R2-condr′ R2-des-lift-skip R2c-seq R2s-wait*)
qed

have $(R1(R2s(R3c(P \vdash Q)))\ ;;\ R1(R2s(R3c(R \vdash S)))) =$
$((R3c(R1(R2s(P) \vdash R2s(Q))))\ ;;\ R3c(R1(R2s(R) \vdash R2s(S))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design*)
also have ... $= R3c\ (R1\ ((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) \wedge \neg\ (R1\ (R2s\ Q) \wedge \neg\ \$wait\acute{}\ ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q)\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S)))))$
by (*simp add*: *R3c-R1-design-composition assms unrest*)
also have ... $= R3c(R1(R2c((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) \wedge \neg\ (R1\ (R2s\ Q) \wedge \neg\ \$wait\acute{}\ ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q)\ ;;\ (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S))))))$
by (*simp add*: *R2c-design R2c-and R2c-not 1 2 3*)
finally show *?thesis*
by (*simp add*: *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-R2c-def*)
qed

lemma *RH-design-export-R1*: $RH(P \vdash Q) = RH(P \vdash R1(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2s*: $RH(P \vdash Q) = RH(P \vdash R2s(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2*: $RH(P \vdash Q) = RH(P \vdash R2(Q))$
by (*metis R2-def RH-design-export-R1 RH-design-export-R2s*)

lemma *RH-design-pre-neg-R1*: $RH((\neg\ R1\ P) \vdash Q) = RH((\neg\ P) \vdash Q)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R3c-commute R1-def R1-disj RH-R2c-def design-def impl-alt-def not-conj-deMorgans utp-pred.double-compl utp-pred.inf.orderE utp-pred.inf-le2*)

lemma *RH-design-pre-R2s*: $RH((R2s\ P) \vdash Q) = RH(P \vdash Q)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R1-R2s-R2c R2-R3c-commute R2s-design R2s-idem RH-alt-def′*)

lemma *RH-design-pre-R2c*: $RH((R2c\ P) \vdash Q) = RH(P \vdash Q)$
by (*metis* (*no-types*, *lifting*) *R2c-design R2c-idem RH-absorbs-R2c*)

lemma *RH-design-pre-neg-R1-R2c*: $RH((\neg\ R1\ (R2c\ P)) \vdash Q) = RH((\neg\ P) \vdash Q)$
by (*simp add*: *RH-design-pre-neg-R1*, *metis R2c-not RH-design-pre-R2c*)

lemma *RH-design-refine-intro*:
assumes ‘$P_1 \Rightarrow P_2$‘ ‘$P_1 \wedge Q_2 \Rightarrow Q_1$‘
shows $RH(P_1 \vdash Q_1) \sqsubseteq RH(P_2 \vdash Q_2)$
by (*simp add*: *RH-monotone assms*(*1*) *assms*(*2*) *design-refine-intro*)

Marcel’s proof for reactive design composition

method *rel-auto′* = ((*simp add*: *upred-defs urel-defs*)?, (*transfer*, (*rule-tac ext*)?, *auto simp add*: *uvar-defs lens-defs urel-defs relcomp-unfold fun-eq-iff prod.case-eq-if*)?)

lemma *reactive-design-composition*:
  assumes $out\alpha \sharp p_1$ p_1 *is R2s* P_2 *is R2s* Q_1 *is R2s* Q_2 *is R2s*
  shows
  $(RH(p_1 \vdash Q_1) \;;;\; RH(P_2 \vdash Q_2)) =$
    $RH((p_1 \wedge \neg ((\$ok' \wedge \neg \$wait' \wedge Q_1) \;;;\; R1\ (\neg P_2)))$
      $\vdash (((\$wait' \wedge Q_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) \;;;\; R1(Q_2)))))$ (**is** *?lhs* = *?rhs*)
proof −
  have $?lhs = RH(?lhs)$
    by (*metis Healthy-def′ RH-idem RH-seq-closure*)
  also have ... $= RH\ ((R2 \circ R1)\ (p_1 \vdash Q_1) \;;;\; RH\ (P_2 \vdash Q_2))$
   by (*metis* (*no-types*, *hide-lams*) *R1-R2-commute R1-idem R2-R3c-commute R2-def R2-seqr-distribute R3c-semir-form RH-alt-def′ calculation comp-apply*)
  also have ... $= RH\ (R1\ ((\neg \$ok \vee R2s\ (\neg p_1)) \vee \$ok' \wedge R2s\ Q_1) \;;;\; RH(P_2 \vdash Q_2))$
    by (*simp add*: *design-def R2-R1-form impl-alt-def R2s-not R2s-ok R2s-disj R2s-conj R2s-ok′*)
  also have ... $= RH(((\neg \$ok \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2))$
    $\vee ((\neg R2s(p_1) \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2))$
    $\vee ((\$ok' \wedge R2s(Q_1) \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2)))$
      by (*smt R1-conj R1-def R1-disj R1-negate-R1 R2-def R2s-not seqr-or-distl utp-pred.conj-assoc utp-pred.inf.commute utp-pred.sup.assoc*)
  also have ... $= RH(((\neg \$ok \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2))$
    $\vee ((\neg p_1 \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2))$
    $\vee ((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2)))$
    by (*metis Healthy-def′ assms*(*2*) *assms*(*4*))

  also have ... $= RH((\neg \$ok \wedge \$tr \le_u \$tr')$
    $\vee (\neg p_1 \wedge \$tr \le_u \$tr')$
    $\vee ((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2)))$
  proof −
    have $((\neg \$ok \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2)) = (\neg \$ok \wedge \$tr \le_u \$tr')$
      by (*rel-auto*)
    moreover have $(((\neg p_1 \;;;\; true) \wedge \$tr \le_u \$tr') \;;;\; RH(P_2 \vdash Q_2)) = ((\neg p_1 \;;;\; true) \wedge \$tr \le_u \$tr')$
      by (*rel-auto*)
    ultimately show *?thesis*
      by (*smt assms*(*1*) *precond-right-unit unrest-not*)
  qed

  also have ... $= RH((\neg \$ok \wedge \$tr \le_u \$tr')$
    $\vee (\neg p_1 \wedge \$tr \le_u \$tr')$
    $\vee ((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; (\$wait \wedge \$ok' \wedge II))$
    $\vee ((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; (\neg \$wait \wedge R1(\neg P_2) \wedge \$tr \le_u \$tr'))$
    $\vee ((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; (\neg \$wait \wedge \$ok' \wedge R2(Q_2) \wedge \$tr \le_u \$tr')))$
  proof −
    have *1*:$RH(P_2 \vdash Q_2) = ((\$wait \wedge \neg \$ok \wedge \$tr \le_u \$tr')$
      $\vee (\$wait \wedge \$ok' \wedge II)$
      $\vee (\neg \$wait \wedge \neg \$ok \wedge \$tr \le_u \$tr')$
      $\vee (\neg \$wait \wedge R2(\neg P_2) \wedge \$tr \le_u \$tr')$
      $\vee (\neg \$wait \wedge \$ok' \wedge R2(Q_2) \wedge \$tr \le_u \$tr'))$
      by (*simp add*: *RH-alt-def′ R2-condr′ R2s-wait R2-skip-rea R3c-def usubst*, *rel-auto*)
    have *2*:$((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; (\$wait \wedge \neg \$ok \wedge \$tr \le_u \$tr')) = false$
      by *rel-auto*
    have *3*:$((\$ok' \wedge Q_1 \wedge \$tr \le_u \$tr') \;;;\; (\neg \$wait \wedge \neg \$ok \wedge \$tr \le_u \$tr')) = false$
      by *rel-auto*

have *4*:$R2(\neg P_2) = R1(\neg P_2)$
by (*metis Healthy-def′ R1-negate-R1 R2-def R2s-not assms*(*3*))
show *?thesis*
by (*simp add*: *1 2 3 4 seqr-or-distr*)
qed

also have ... $= RH((\neg \$ok) \vee (\neg p_1)$
$\vee ((\$ok' \wedge Q_1) ;; (\$wait \wedge \$ok' \wedge II))$
$\vee ((\$ok' \wedge Q_1) ;; (\neg \$wait \wedge R1(\neg P_2)))$
$\vee ((\$ok' \wedge Q_1) ;; (\neg \$wait \wedge \$ok' \wedge R2(Q_2))))$
by (*rel-blast*)

also have ... $= RH((\neg \$ok) \vee (\neg p_1)$
$\vee (\$ok' \wedge \$wait' \wedge Q_1)$
$\vee ((\$ok' \wedge Q_1) ;; (\neg \$wait \wedge R1(\neg P_2)))$
$\vee ((\$ok' \wedge Q_1) ;; (\neg \$wait \wedge \$ok' \wedge R1(Q_2))))$
proof –
have $((\$ok' \wedge Q_1) ;; (\$wait \wedge \$ok' \wedge II)) = (\$ok' \wedge \$wait' \wedge Q_1)$
by (*rel-auto*)
moreover have $R2(Q_2) = R1(Q_2)$
by (*metis Healthy-def′ R2-def assms*(*5*))
ultimately show *?thesis* **by** *simp*
qed

also have ... $= RH((\neg \$ok) \vee (\neg p_1)$
$\vee (\$ok' \wedge \$wait' \wedge Q_1)$
$\vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; (R1(\neg P_2)))$
$\vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; (\$ok' \wedge R1(Q_2))))$
by *rel-auto′*

also have ... $= RH((\neg \$ok) \vee (\neg p_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(\neg P_2))$
$\vee (\$ok' \wedge ((\$wait' \wedge Q_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(Q_2)))))$
by *rel-auto′*

also have ... $= RH(\neg (\$ok \wedge p_1 \wedge \neg ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(\neg P_2)))$
$\vee (\$ok' \wedge ((\$wait' \wedge Q_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(Q_2)))))$
by *rel-auto′*

also have ... = *?rhs*
proof –
have $(\neg (\$ok \wedge p_1 \wedge \neg ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(\neg P_2)))$
$\vee (\$ok' \wedge ((\$wait' \wedge Q_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(Q_2)))))$
$= ((\$ok \wedge (p_1 \wedge \neg ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(\neg P_2)))) \Rightarrow$
$(\$ok' \wedge ((\$wait' \wedge Q_1) \vee ((\$ok' \wedge \neg \$wait' \wedge Q_1) ;; R1(Q_2)))))$
by *pred-auto*
thus *?thesis*
by (*simp add*: *design-def*)
qed

finally show *?thesis* **.**
qed

15.3 Healthiness conditions

definition [*upred-defs*]: $CSP1(P) = (P \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

CSP2 is just H2 since the type system will automatically have J identifying the reactive variables as required.

definition [*upred-defs*]: $CSP2(P) = H2(P)$

abbreviation $CSP(P) \equiv CSP1(CSP2(RH(P)))$

lemma *CSP1-idem*:
$CSP1(CSP1(P)) = CSP1(P)$
by *pred-auto*

lemma *CSP2-idem*:
$CSP2(CSP2(P)) = CSP2(P)$
by (*simp add*: *CSP2-def H2-idem*)

lemma *CSP1-CSP2-commute*:
$CSP1(CSP2(P)) = CSP2(CSP1(P))$
by (*simp add*: *CSP1-def CSP2-def H2-split usubst*, *rel-auto*)

lemma *CSP1-R1-commute*:
$CSP1(R1(P)) = R1(CSP1(P))$
by (*rel-auto*)

lemma *CSP1-R2c-commute*:
$CSP1(R2c(P)) = R2c(CSP1(P))$
by (*rel-auto*)

lemma *CSP1-R3c-commute*:
$CSP1(R3c(P)) = R3c(CSP1(P))$
by (*rel-auto*)

lemma *CSP-idem*: $CSP(CSP(P)) = CSP(P)$
by (*metis* (*no-types*, *hide-lams*) *CSP1-CSP2-commute CSP1-R1-commute CSP1-R2c-commute CSP1-R3c-commute CSP1-idem CSP2-def CSP2-idem R1-H2-commute R2c-H2-commute R3c-H2-commute RH-R2c-def RH-idem*)

lemma *CSP-Idempotent*: *Idempotent CSP*
by (*simp add*: *CSP-idem Idempotent-def*)

lemma *CSP1-via-H1*: $R1(H1(P)) = R1(CSP1(P))$
by *rel-auto*

lemma *CSP1-R3c*: $CSP1(R3(P)) = R3c(CSP1(P))$
by *rel-auto*

lemma *CSP1-R1-H1*:
$CSP1(R1(P)) = R1(H1(P))$
by *rel-auto*

lemma *CSP1-algebraic-intro*:
assumes
P *is* $R1$ $(R1(true_h) ;; P) = R1(true_h)$ $(II_r ;; P) = P$
shows P *is* $CSP1$
proof $-$
have $P = (II_r ;; P)$
by (*simp add*: *assms*(*3*))
also have ... $= (R1(\$ok \Rightarrow II) ;; P)$

by (*simp add*: *skip-rea-R1-lemma*)
also have ... = $(((\neg \$ok \wedge R1(true)) ;; P) \vee P)$
by (*metis* (*no-types*, *lifting*) *R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred.inf-top-left utp-pred.sup-commute*)
also have ... = $(((R1(\neg \$ok) ;; R1(true_h)) ;; P) \vee P)$
by (*rel-auto*, *metis order-trans*)
also have ... = $((R1(\neg \$ok) ;; (R1(true_h) ;; P)) \vee P)$
by (*simp add*: *seqr-assoc*)
also have ... = $((R1(\neg \$ok) ;; R1(true_h)) \vee P)$
by (*simp add*: *assms*(*2*))
also have ... = $(R1(\neg \$ok) \vee P)$
by (*rel-auto*)
also have ... = $CSP1(P)$
by (*rel-auto*)
finally show *?thesis*
by (*simp add*: *Healthy-def*)
qed

theorem *CSP1-left-zero*:
assumes *P is R1 P is CSP1*
shows $(R1(true) ;; P) = R1(true)$
proof −
have $(R1(true) ;; R1(CSP1(P))) = R1(true)$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

theorem *CSP1-left-unit*:
assumes *P is R1 P is CSP1*
shows $(II_r ;; P) = P$
proof −
have $(II_r ;; R1(CSP1(P))) = R1(CSP1(P))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

lemma *CSP1-alt-def*:
assumes *P is R1*
shows $CSP1(P) = (P \triangleleft \$ok \triangleright R1(true))$
using *assms*
proof −
have $CSP1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*)
qed

theorem *CSP1-algebraic*:
assumes *P is R1*
shows *P is CSP1* $\longleftrightarrow (R1(true_h) ;; P) = R1(true_h) \wedge (II_r ;; P) = P$
using *CSP1-algebraic-intro CSP1-left-unit CSP1-left-zero assms* **by** *blast*

lemma *CSP1-reactive-design*: $CSP1(RH(P \vdash Q)) = RH(P \vdash Q)$

by *rel-auto*

lemma *CSP2-reactive-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $CSP2(RH(P \vdash Q)) = RH(P \vdash Q)$
using *assms*
by (*simp add*: *CSP2-def H2-R1-comm H2-R2-comm H2-R3-comm H2-design RH-def H2-R2s-comm*)

lemma *wait-false-design*:
$(P \vdash Q)_f = ((P_f) \vdash (Q_f))$
by (*rel-auto*)

lemma *CSP-RH-design-form*:
$CSP(P) = RH((\neg P^f{}_f) \vdash P^t{}_f)$
proof −
have $CSP(P) = CSP1(CSP2(R1(R2s(R3c(P)))))$
by (*metis Healthy-def' RH-def assms*)
also have ... $= CSP1(H2(R1(R2s(R3c(P)))))$
by (*simp add*: *CSP2-def*)
also have ... $= CSP1(R1(H2(R2s(R3c(P)))))$
by (*simp add*: *R1-H2-commute*)
also have ... $= R1(H1(R1(H2(R2s(R3c(P))))))$
by (*simp add*: *CSP1-R1-commute CSP1-via-H1 R1-idem*)
also have ... $= R1(H1(H2(R2s(R3c(R1(P))))))$
by (*metis* (*no-types*, *hide-lams*) *CSP1-R1-H1 R1-H2-commute R1-R2-commute R1-idem R2-R3c-commute R2-def*)
also have ... $= R1(R2s(H1(H2(R3c(R1(P))))))$
by (*simp add*: *R2s-H1-commute R2s-H2-commute*)
also have ... $= R1(R2s(H1(R3c(H2(R1(P))))))$
by (*simp add*: *R3c-H2-commute*)
also have ... $= R2(R1(H1(R3c(H2(R1(P))))))$
by (*metis R1-R2-commute R1-idem R2-def*)
also have ... $= R2(R3c(R1(H1(H2(R1(P))))))$
by (*simp add*: *R1-H1-R3c-commute*)
also have ... $= RH(H1\text{-}H2(R1(P)))$
by (*metis R1-R2-commute R1-idem R2-R3c-commute R2-def RH-def*)
also have ... $= RH(H1\text{-}H2(P))$
by (*metis* (*no-types*, *hide-lams*) *CSP1-R1-H1 R1-H2-commute R1-R2-commute R1-R3c-commute R1-idem RH-alt-def*)
also have ... $= RH((\neg P^f) \vdash P^t)$
proof −
have 0:$(\neg (H1\text{-}H2(P))^f) = (\$ok \wedge \neg P^f)$
by (*simp add*: *H1-def H2-split*, *pred-auto*)
have 1:$(H1\text{-}H2(P))^t = (\$ok \Rightarrow (P^f \vee P^t))$
by (*simp add*: *H1-def H2-split*, *pred-auto*)
have $(\neg (H1\text{-}H2(P))^f) \vdash (H1\text{-}H2(P))^t = ((\neg P^f) \vdash P^t)$
by (*simp add*: *0 1*, *pred-auto*)
thus *?thesis*
by (*metis H1-H2-commute H1-H2-is-design H1-idem H2-idem Healthy-def'*)
qed
also have ... $= RH((\neg P^f{}_f) \vdash P^t{}_f)$
by (*metis* (*no-types*, *lifting*) *RH-subst-wait subst-not wait-false-design*)
finally show *?thesis* **.**
qed

lemma *CSP-reactive-design*:
  assumes *P is CSP*
  shows $RH((\neg\ P^f{}_f) \vdash P^t{}_f) = P$
  by (*metis CSP-RH-design-form Healthy-def' assms*)

lemma *CSP-RH-design*:
  assumes $\$ok\acute{}\ \sharp\ P$ $\$ok\acute{}\ \sharp\ Q$
  shows $CSP(RH(P \vdash Q)) = RH(P \vdash Q)$
  by (*metis CSP1-reactive-design CSP2-reactive-design RH-idem assms*(*1*) *assms*(*2*))

lemma *RH-design-is-CSP*:
  assumes $\$ok\acute{}\ \sharp\ P$ $\$ok\acute{}\ \sharp\ Q$
  shows $\mathbf{R}(P \vdash Q)$ *is CSP*
  by (*simp add*: *CSP-RH-design Healthy-def' assms*(*1*) *assms*(*2*))

lemma *CSP2-R3c-commute*: $CSP2(R3c(P)) = R3c(CSP2(P))$
  by (*rel-auto*)

lemma *R3c-via-CSP1-R3*:
  $[\![\ P\ is\ CSP1;\ P\ is\ R3\ ]\!] \Longrightarrow P\ is\ R3c$
  by (*metis CSP1-R3c Healthy-def'*)

lemma *R3c-CSP1-form*:
  $P\ is\ R1 \Longrightarrow R3c(CSP1(P)) = (R1(true) \triangleleft \neg\$ok \triangleright (II \triangleleft \$wait \triangleright P))$
  by (*rel-blast*)

lemma *R3c-CSP*: $R3c(CSP(P)) = CSP(P)$
  by (*simp add*: *CSP1-R3c-commute CSP2-R3c-commute R2-R3c-commute R3c-idem RH-alt-def'*)

lemma *CSP-R1-R2s*: $P\ is\ CSP \Longrightarrow R1\ (R2s\ P) = P$
  by (*metis* (*no-types*) *CSP-reactive-design R1-R2c-is-R2 R1-R2s-R2c R2-idem RH-alt-def'*)

lemma *CSP-healths*:
  assumes *P is CSP*
  shows *P is R1 P is R2 P is R3c P is CSP1 P is CSP2*
  apply (*metis* (*mono-tags*) *CSP-R1-R2s Healthy-def' R1-idem assms*(*1*))
  apply (*metis CSP-R1-R2s Healthy-def R2-def assms*)
  apply (*metis Healthy-def R3c-CSP assms*)
  apply (*metis CSP1-idem Healthy-def' assms*)
  apply (*metis CSP1-CSP2-commute CSP2-idem Healthy-def' assms*)
done

lemma *CSP-intro*:
  assumes *P is R1 P is R2 P is R3c P is CSP1 P is CSP2*
  shows *P is CSP*
  by (*metis Healthy-def RH-alt-def' assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*))

15.4 Reactive design triples

definition *wait'-cond* :: - ⇒ - ⇒ - (**infix** ⋄ *65*) **where**
[*upred-defs*]: $P \diamond Q = (P \triangleleft \$wait\acute{} \triangleright Q)$

lemma *wait'-cond-unrest* [*unrest*]:
  $[\![\ out\text{-}var\ wait \bowtie x;\ x\ \sharp\ P;\ x\ \sharp\ Q\ ]\!] \Longrightarrow x\ \sharp\ (P \diamond Q)$
  by (*simp add*: *wait'-cond-def unrest*)

lemma *wait′-cond-subst* [*usubst*]:
$\$wait' \sharp \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$
by (*simp add*: *wait′-cond-def usubst unrest*)

lemma *wait′-cond-left-false*: $false \diamond P = (\neg \$wait' \wedge P)$
by (*rel-auto*)

lemma *wait′-cond-seq*: $((P \diamond Q) \;;\; R) = ((P \;;\; \$wait \wedge R) \vee (Q \;;\; \neg\$wait \wedge R))$
by (*simp add*: *wait′-cond-def cond-def seqr-or-distl*, *rel-blast*)

lemma *wait′-cond-true*: $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$
by (*rel-auto*)

lemma *wait′-cond-false*: $(P \diamond Q \wedge (\neg\$wait')) = (Q \wedge (\neg\$wait'))$
by (*rel-auto*)

lemma *wait′-cond-idem*: $P \diamond P = P$
by (*rel-auto*)

lemma *wait′-cond-conj-exchange*:
$((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$
by *rel-auto*

lemma *subst-wait′-cond-true* [*usubst*]: $(P \diamond Q)[\![true/\$wait']\!] = P[\![true/\$wait']\!]$
by *rel-auto*

lemma *subst-wait′-cond-false* [*usubst*]: $(P \diamond Q)[\![false/\$wait']\!] = Q[\![false/\$wait']\!]$
by *rel-auto*

lemma *subst-wait′-left-subst*: $(P[\![true/\$wait']\!] \diamond Q) = (P \diamond Q)$
by (*metis wait′-cond-def cond-def conj-comm conj-eq-out-var-subst upred-eq-true wait-vwb-lens*)

lemma *subst-wait′-right-subst*: $(P \diamond Q[\![false/\$wait']\!]) = (P \diamond Q)$
by (*metis cond-def conj-eq-out-var-subst upred-eq-false utp-pred.inf.commute wait′-cond-def wait-vwb-lens*)

lemma *wait′-cond-split*: $P[\![true/\$wait']\!] \diamond P[\![false/\$wait']\!] = P$
by (*simp add*: *wait′-cond-def cond-var-split*)

lemma *R1-wait′-cond*: $R1(P \diamond Q) = R1(P) \diamond R1(Q)$
by *rel-auto*

lemma *R2s-wait′-cond*: $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$
by (*simp add*: *wait′-cond-def R2s-def R2s-def usubst*)

lemma *R2-wait′-cond*: $R2(P \diamond Q) = R2(P) \diamond R2(Q)$
by (*simp add*: *R2-def R2s-wait′-cond R1-wait′-cond*)

lemma *RH-design-peri-R1*: $RH(P \vdash R1(Q) \diamond R) = RH(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R1-idem R1-wait′-cond RH-design-export-R1*)

lemma *RH-design-post-R1*: $RH(P \vdash Q \diamond R1(R)) = RH(P \vdash Q \diamond R)$
by (*metis R1-wait′-cond RH-design-export-R1 RH-design-peri-R1*)

lemma *RH-design-peri-R2s*: $RH(P \vdash R2s(Q) \diamond R) = RH(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-post-R2s*: $RH(P \vdash Q \diamond R2s(R)) = RH(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-peri-R2c*: $RH(P \vdash R2c(Q) \diamond R) = RH(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R2-wait′-cond R2c-idem RH-design-export-R2*)

lemma *RH-design-post-R2c*: $RH(P \vdash Q \diamond R2c(R)) = RH(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R2-wait′-cond R2c-idem RH-design-export-R2*)

lemma *RH-design-lemma1*:
$RH(P \vdash (R1(R2c(Q)) \lor R) \diamond S) = RH(P \vdash (Q \lor R) \diamond S)$
by (*simp add*: *design-def impl-alt-def wait′-cond-def RH-R2c-def R2c-R3c-commute R1-R3c-commute R1-disj R2c-disj R2c-and R1-cond R2c-condr R1-R2c-commute R2c-idem R1-extend-conj′ R1-idem*)

lemma *RH-tri-design-composition*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q_1$ $\$ok' \sharp Q_2$ $\$ok \sharp R$ $\$ok \sharp S_1$ $\$ok \sharp S_2$
$\$wait' \sharp Q_2$ $\$wait \sharp S_1$ $\$wait \sharp S_2$
shows $(RH(P \vdash Q_1 \diamond Q_2) \;;\; RH(R \vdash S_1 \diamond S_2)) =$
$RH((\neg (R1\ (\neg\ R2s\ P) \;;\; R1\ true) \land \neg (R1\ (R2s\ Q_2) \land \neg\ \$wait' \;;\; R1\ (\neg\ R2s\ R))) \vdash$
$((Q_1 \lor (R1\ (R2s\ Q_2) \;;\; R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2) \;;\; R1\ (R2s\ S_2)))))$
proof −
have *1*:$(\neg (R1\ (R2s\ (Q_1 \diamond Q_2)) \land \neg\ \$wait' \;;\; R1\ (\neg\ R2s\ R))) =$
$(\neg (R1\ (R2s\ Q_2) \land \neg\ \$wait' \;;\; R1\ (\neg\ R2s\ R)))$
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait′ wait′-cond-false*)
have *2*: $(R1\ (R2s\ (Q_1 \diamond Q_2)) \;;\; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ (S_1 \diamond S_2)))) =$
$((R1\ (R2s\ Q_1) \lor (R1\ (R2s\ Q_2) \;;\; R1\ (R2s\ S_1))) \diamond (R1\ (R2s\ Q_2) \;;\; R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_1) \;;\; \$wait \land (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
$= (R1\ (R2s\ Q_1) \land \$wait')$
proof −
have $(R1\ (R2s\ Q_1) \;;\; \$wait \land (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
$= (R1\ (R2s\ Q_1) \;;\; \$wait \land \lceil II \rceil_D)$
by (*rel-auto*)
also have ... $= ((R1\ (R2s\ Q_1) \;;\; \lceil II \rceil_D) \land \$wait')$
by (*rel-auto*)
also from *assms*(*2*) **have** ... $= ((R1\ (R2s\ Q_1)) \land \$wait')$
by (*simp add*: *lift-des-skip-dr-unit-unrest unrest*)
finally show *?thesis* **.**
qed

moreover have $(R1\ (R2s\ Q_2) \;;\; \neg\ \$wait \land (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
$= ((R1\ (R2s\ Q_2)) \;;\; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_2) \;;\; \neg\ \$wait \land (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
$= (R1\ (R2s\ Q_2) \;;\; \neg\ \$wait \land (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred.double-compl utp-pred.inf.left-idem utp-pred.sup-assoc utp-pred.sup-inf-absorb*)

also have ... $= ((R1\ (R2s\ Q_2))[\![false/\$wait']\!] \;;\; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))[\![false/\$wait]\!])$
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have ... $= ((R1\ (R2s\ Q_2)) \;;\; (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
by (*simp add*: *wait′-cond-def usubst unrest assms*)

finally show *?thesis* **.**
qed

moreover
have $((R1\ (R2s\ Q_1) \wedge \$wait\acute{}) \vee ((R1\ (R2s\ Q_2))\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \vee (R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_2)))$
by (*simp add*: *wait′-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*
by (*simp add*: *R2s-wait′-cond R1-wait′-cond wait′-cond-seq*)
qed

show *?thesis*
apply (*subst RH-design-composition*)
apply (*simp-all add*: *assms*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *1 2*)
apply (*simp add*: *R1-R2s-R2c RH-design-lemma1*)
done
qed

Syntax for pre-, post-, and periconditions

abbreviation $pre_s \equiv [\$ok \mapsto_s true, \$ok\acute{} \mapsto_s false, \$wait \mapsto_s false]$
abbreviation $cmt_s \equiv [\$ok \mapsto_s true, \$ok\acute{} \mapsto_s true, \$wait \mapsto_s false]$
abbreviation $peri_s \equiv [\$ok \mapsto_s true, \$ok\acute{} \mapsto_s true, \$wait \mapsto_s false, \$wait\acute{} \mapsto_s true]$
abbreviation $post_s \equiv [\$ok \mapsto_s true, \$ok\acute{} \mapsto_s true, \$wait \mapsto_s false, \$wait\acute{} \mapsto_s false]$

abbreviation $npre_R(P) \equiv pre_s \dagger P$

definition [*upred-defs*]: $pre_R(P) = (\neg\ (npre_R(P)))$
definition [*upred-defs*]: $cmt_R(P) = (cmt_s \dagger P)$
definition [*upred-defs*]: $peri_R(P) = (peri_s \dagger P)$
definition [*upred-defs*]: $post_R(P) = (post_s \dagger P)$

lemma *ok-pre-unrest* [*unrest*]: $\$ok \sharp pre_R\ P$
by (*simp add*: pre_R*-def unrest usubst*)

lemma *ok-peri-unrest* [*unrest*]: $\$ok \sharp peri_R\ P$
by (*simp add*: $peri_R$*-def unrest usubst*)

lemma *ok-post-unrest* [*unrest*]: $\$ok \sharp post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

lemma *ok′-pre-unrest* [*unrest*]: $\$ok\acute{} \sharp pre_R\ P$
by (*simp add*: pre_R*-def unrest usubst*)

lemma *ok′-peri-unrest* [*unrest*]: $\$ok\acute{} \sharp peri_R\ P$
by (*simp add*: $peri_R$*-def unrest usubst*)

lemma *ok′-post-unrest* [*unrest*]: $\$ok\acute{} \sharp post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

lemma *wait-pre-unrest* [*unrest*]: $\$wait \sharp pre_R\ P$
by (*simp add*: pre_R*-def unrest usubst*)

lemma *wait-peri-unrest* [*unrest*]: $\$wait \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *wait-post-unrest* [*unrest*]: $\$wait \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

lemma *wait′-peri-unrest* [*unrest*]: $\$wait\acute{} \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *wait′-post-unrest* [*unrest*]: $\$wait\acute{} \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

lemma pre_s-*design*: $pre_s \dagger (P \vdash Q) = (\neg\ pre_s \dagger P)$
by (*simp add*: *design-def* pre_R-*def usubst*)

lemma $peri_s$-*design*: $peri_s \dagger (P \vdash Q \diamond R) = peri_s \dagger (P \Rightarrow Q)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma $post_s$-*design*: $post_s \dagger (P \vdash Q \diamond R) = post_s \dagger (P \Rightarrow R)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma pre_s-*R1* [*usubst*]: $pre_s \dagger R1(P) = R1(pre_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma pre_s-*R2c* [*usubst*]: $pre_s \dagger R2c(P) = R2c(pre_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma $peri_s$-*R1* [*usubst*]: $peri_s \dagger R1(P) = R1(peri_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma $peri_s$-*R2c* [*usubst*]: $peri_s \dagger R2c(P) = R2c(peri_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma $post_s$-*R1* [*usubst*]: $post_s \dagger R1(P) = R1(post_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma $post_s$-*R2c* [*usubst*]: $post_s \dagger R2c(P) = R2c(post_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma *rea-pre-RH-design*: $pre_R(RH(P \vdash Q)) = (\neg\ R1(R2c(pre_s \dagger (\neg\ P))))$
by (*simp add*: *RH-R2c-def usubst R3c-def* pre_R-*def* pre_s-*design*)

lemma *rea-peri-RH-design*: $peri_R(RH(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow Q)))$
by (*simp add*:*RH-R2c-def usubst* $peri_R$-*def R3c-def* $peri_s$-*design*)

lemma *rea-post-RH-design*: $post_R(RH(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow R)))$
by (*simp add*:*RH-R2c-def usubst* $post_R$-*def R3c-def* $post_s$-*design*)

lemma *CSP-reactive-tri-design-lemma*:
assumes *P is CSP*
shows $RH((\neg\ P^f{}_f) \vdash P^t{}_f[\![true/\$wait\acute{}]\!] \diamond P^t{}_f[\![false/\$wait\acute{}]\!]) = P$
by (*simp add*: *CSP-reactive-design assms wait′-cond-split*)

lemma *CSP-reactive-tri-design*:

assumes P *is* CSP
shows $RH(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$
proof −
have $P = RH((\neg P^f{}_f) \vdash P^t{}_f[\![true/\$wait']\!] \diamond P^t{}_f[\![false/\$wait']\!])$
by (*simp add*: *CSP-reactive-tri-design-lemma assms*)
also have ... $= RH(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
apply (*simp add*: *usubst*)
apply (*subst design-subst-ok-ok'*[*THEN sym*])
apply (*simp add*: pre_R*-def* $peri_R$*-def* $post_R$*-def usubst unrest*)
done
finally show *?thesis* **..**
qed

lemma *skip-rea-reactive-design*:
$II_r = RH(true \vdash II)$
proof −
have $RH(true \vdash II) = R1(R2c(R3c(true \vdash II)))$
by (*metis RH-R2c-def*)
also have ... $= R1(R3c(R2c(true \vdash II)))$
by (*metis R2c-R3c-commute RH-R2c-def*)
also have ... $= R1(R3c(true \vdash II))$
by (*simp add*: *design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-skip-r R2c-ok'*)
also have ... $= R1(II_r \triangleleft \$wait \triangleright true \vdash II)$
by (*metis R3c-def*)
also have ... $= II_r$
by (*rel-auto*)
finally show *?thesis* **..**
qed

lemma *skip-rea-reactive-design'*:
$II_r = RH(true \vdash \lceil II \rceil_D)$
by (*metis aext-true rdesign-def skip-d-alt-def skip-d-def skip-rea-reactive-design*)

lemma *RH-design-subst-wait*: $RH(P\ _f \vdash Q\ _f) = RH(P \vdash Q)$
by (*metis RH-subst-wait wait-false-design*)

lemma *RH-design-subst-wait-pre*: $RH(P\ _f \vdash Q) = RH(P \vdash Q)$
by (*subst RH-design-subst-wait*[*THEN sym*], *simp add*: *usubst RH-design-subst-wait*)

lemma *RH-design-subst-wait-post*: $RH(P \vdash Q\ _f) = RH(P \vdash Q)$
by (*subst RH-design-subst-wait*[*THEN sym*], *simp add*: *usubst RH-design-subst-wait*)

lemma *RH-peri-subst-false-wait*: $RH(P \vdash Q\ _f \diamond R) = RH(P \vdash Q \diamond R)$
apply (*subst RH-design-subst-wait-post*[*THEN sym*])
apply (*simp add*: *usubst unrest*)
apply (*metis RH-design-subst-wait RH-design-subst-wait-pre out-in-indep out-var-uvar unrest-false unrest-usubst-id unrest-usubst-upd vwb-lens.axioms*(*2*) *wait'-cond-subst wait-vwb-lens*)
done

lemma *RH-post-subst-false-wait*: $RH(P \vdash Q \diamond R\ _f) = RH(P \vdash Q \diamond R)$
apply (*subst RH-design-subst-wait-post*[*THEN sym*])
apply (*simp add*: *usubst unrest*)
apply (*metis RH-design-subst-wait RH-design-subst-wait-pre out-in-indep out-var-uvar unrest-false unrest-usubst-id unrest-usubst-upd vwb-lens.axioms*(*2*) *wait'-cond-subst wait-vwb-lens*)
done

lemma *skip-rea-reactive-tri-design*:
$II_r = RH(true \vdash false \diamond \lceil II \rceil_D)$ (**is** *?lhs* = *?rhs*)
proof −
have $?rhs = RH\ (true \vdash (\neg\ \$wait' \wedge \lceil II \rceil_D))$
by (*simp add*: *wait′-cond-def cond-def*)
have ... $= RH\ (true \vdash (\neg\ \$wait \wedge \lceil II \rceil_D))$ (**is** $RH\ (true \vdash ?Q1) = RH\ (true \vdash ?Q2)$)
proof −
have $?Q1 = ?Q2$
by (*rel-auto*)
thus *?thesis* **by** *simp*
qed
also have ... $= RH\ (true \vdash \lceil II \rceil_D)$
by (*rel-auto*)
finally show *?thesis*
by (*simp add*: *skip-rea-reactive-design′ wait′-cond-def cond-def*)
qed

lemma *skip-d-lift-rea*:
$\lceil II \rceil_D = (\$wait' =_u \$wait \wedge \$tr' =_u \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R)$
by (*rel-auto*)

lemma *skip-rea-reactive-tri-design′*:
$II_r = RH(true \vdash false \diamond (\$tr' =_u \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R))$ (**is** *?lhs* = *?rhs*)
proof −
have $?rhs = RH\ (true \vdash (\neg\ \$wait' \wedge \$tr' =_u \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R))$
by (*simp add*: *wait′-cond-def cond-def*)
also have ... $= RH\ (true \vdash (\$wait' =_u \$wait \wedge \$tr' =_u \$tr \wedge \$\Sigma_R' =_u \$\Sigma_R))$ (**is** $RH\ (true \vdash ?Q1)$ $= RH\ (true \vdash ?Q2)$)
proof −
have $?Q1\ _f = ?Q2\ _f$
by (*rel-auto*)
thus *?thesis*
by (*metis RH-design-subst-wait*)
qed
also have ... $= RH\ (true \vdash \lceil II \rceil_D)$
by (*metis skip-d-lift-rea*)
finally show *?thesis*
by (*simp add*: *skip-rea-reactive-design′*)
qed

lemma *R1-neg-pre*: $R1\ (\neg\ pre_R\ P) = (\neg\ pre_R\ (R1(P)))$
by (*simp add*: pre_R*-def R1-def usubst*)

lemma *R1-peri*: $R1\ (peri_R\ P) = peri_R\ (R1(P))$
by (*simp add*: $peri_R$*-def R1-def usubst*)

lemma *R1-post*: $R1\ (post_R\ P) = post_R\ (R1(P))$
by (*simp add*: $post_R$*-def R1-def usubst*)

lemma *R2s-pre*:
$R2s\ (pre_R\ P) = pre_R\ (R2s\ P)$
by (*simp add*: pre_R*-def R2s-def usubst*)

lemma *R2s-peri*: $R2s\ (peri_R\ P) = peri_R\ (R2s\ P)$

by (*simp add*: *peri*$_R$-*def R2s-def usubst*)

lemma *R2s-post*: $R2s\ (post_R\ P) = post_R\ (R2s\ P)$
by (*simp add*: *post*$_R$-*def R2s-def usubst*)

lemma *RH-pre-RH-design*:
$\$ok' \sharp P \Longrightarrow RH(pre_R(RH(P \vdash Q)) \vdash R) = RH(P \vdash R)$
apply (*simp add*: *rea-pre-RH-design RH-design-pre-neg-R1-R2c usubst*)
apply (*subst subst-to-singleton*)
apply (*simp add*: *unrest*)
apply (*simp add*: *RH-design-subst-wait-pre*)
apply (*simp add*: *usubst*)
apply (*metis conj-pos-var-subst design-def vwb-lens-ok*)
done

lemma *RH-postcondition*: $(RH(P \vdash Q))^t{}_f = R1(R2s(\$ok \wedge P^t{}_f \Rightarrow Q^t{}_f))$
by (*simp add*: *RH-def R1-def R3c-def usubst R2s-def design-def*)

lemma *RH-postcondition-RH*: $RH(P \vdash (RH(P \vdash Q))^t{}_f) = RH(P \vdash Q)$
proof −
have $RH(P \vdash (RH(P \vdash Q))^t{}_f) = RH\ (P \vdash (\$ok \wedge P^t{}_f \Rightarrow Q^t{}_f))$
by (*simp add*: *RH-postcondition RH-design-export-R1*[*THEN sym*] *RH-design-export-R2s*[*THEN sym*])
also have ... $= RH\ (P \vdash (\$ok \wedge P^t \Rightarrow Q^t))$
by (*subst RH-design-subst-wait-post*[*THEN sym*, *of* - $(\$ok \wedge P^t \Rightarrow Q^t)$], *simp add*: *usubst*)
also have ... $= RH\ (P \vdash (P^t \Rightarrow Q^t))$
by (*rel-auto*)
also have ... $= RH\ (P \vdash (P \Rightarrow Q))$
by (*subst design-subst-ok'*[*THEN sym*, *of* - $P \Rightarrow Q$], *simp add*: *usubst*)
also have ... $= RH\ (P \vdash Q)$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *peri*$_R$-*alt-def*: $peri_R(P) = (P^t{}_f)[\![true/\$ok]\!][\![true/\$wait']\!]$
by (*simp add*: *peri*$_R$-*def usubst*)

lemma *post*$_R$-*alt-def*: $post_R(P) = (P^t{}_f)[\![true/\$ok]\!][\![false/\$wait']\!]$
by (*simp add*: *post*$_R$-*def usubst*)

lemma *design-export-ok-true*: $P \vdash Q[\![true/\$ok]\!] = P \vdash Q$
by (*metis conj-pos-var-subst design-export-ok vwb-lens-ok*)

lemma *design-export-peri-ok-true*: $P \vdash Q[\![true/\$ok]\!] \diamond R = P \vdash Q \diamond R$
apply (*subst design-export-ok-true*[*THEN sym*])
apply (*simp add*: *usubst unrest*)
apply (*metis design-export-ok-true out-in-indep out-var-uvar unrest-true unrest-usubst-id unrest-usubst-upd vwb-lens-mwb wait'-cond-subst wait-vwb-lens*)
done

lemma *design-export-post-ok-true*: $P \vdash Q \diamond R[\![true/\$ok]\!] = P \vdash Q \diamond R$
apply (*subst design-export-ok-true*[*THEN sym*])
apply (*simp add*: *usubst unrest*)
apply (*metis design-export-ok-true out-in-indep out-var-uvar unrest-true unrest-usubst-id unrest-usubst-upd vwb-lens-mwb wait'-cond-subst wait-vwb-lens*)

done

lemma *RH-peri-RH-design*:
$RH(P \vdash peri_R(RH(P \vdash Q \diamond R)) \diamond S) = RH(P \vdash Q \diamond S)$
apply (*simp add*: *peri$_R$-alt-def subst-wait′-left-subst design-export-peri-ok-true RH-postcondition*)
apply (*simp add*: *rea-peri-RH-design RH-design-peri-R1 RH-design-peri-R2s*)
oops

lemma *R1-R2s-tr-diff-conj*: $(R1\ (R2s\ (\$tr\acute{} =_u \$tr \wedge P))) = (\$tr\acute{} =_u \$tr \wedge R2s(P))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *R2s-state′-eq-state*: $R2s\ (\$\Sigma_R\acute{} =_u \$\Sigma_R) = (\$\Sigma_R\acute{} =_u \$\Sigma_R)$
by (*simp add*: *R2s-def usubst*)

lemma *skip-r-rea*: $II = (\$ok\acute{} =_u \$ok \wedge \$wait\acute{} =_u \$wait \wedge \$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R)$
by (*rel-auto*)

lemma *wait-pre-lemma*:
assumes $\$wait\acute{} \sharp P$
shows $(P \wedge \neg \$wait\acute{} ;; \neg pre_R\ Q) = (P ;; \neg pre_R\ Q)$
proof −
have $(P \wedge \neg \$wait\acute{} ;; \neg pre_R\ Q) = (P \wedge \$wait\acute{} =_u false ;; \neg pre_R\ Q)$
by (*rel-auto*)
also have ... $= (P[\![false/\$wait\acute{}]\!] ;; (\neg pre_R\ Q)[\![false/\$wait]\!])$
by (*metis false-alt-def seqr-left-one-point wait-vwb-lens*)
also have ... $= (P ;; \neg pre_R\ Q)$
by (*simp add*: *usubst unrest assms*)
finally show *?thesis* **.**
qed

lemma *rea-left-unit-lemma*:
assumes $\$ok \sharp P$ $\$wait \sharp P$
shows $((\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R) ;; P) = P$
proof −
have $P = (II ;; P)$
by *simp*
also have ... $= ((\$ok\acute{} =_u \$ok \wedge \$wait\acute{} =_u \$wait \wedge \$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R) ;; P)$
by (*metis skip-r-rea*)
also from *assms* **have** ... $= ((\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R) ;; P)$
by (*simp add*: *seqr-insert-ident-left assms unrest*)
finally show *?thesis* **..**
qed

lemma *rea-right-unit-lemma*:
assumes $\$ok\acute{} \sharp P$ $\$wait\acute{} \sharp P$
shows $(P ;; (\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R)) = P$
proof −
have $P = (P ;; II)$
by *simp*
also have ... $= (P ;; (\$ok\acute{} =_u \$ok \wedge \$wait\acute{} =_u \$wait \wedge \$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R))$
by (*metis skip-r-rea*)
also from *assms* **have** ... $= (P ;; (\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R))$
by (*simp add*: *seqr-insert-ident-right assms unrest*)
finally show *?thesis* **..**
qed

lemma *skip-rea-left-unit*:
assumes P *is CSP*
shows $(II_r \;;\; P) = P$
proof −
have $(II_r \;;\; P) = (II_r \;;\; RH\ (pre_R\ P \vdash peri_R\ P \diamond post_R\ P))$
by (*metis CSP-reactive-tri-design assms*)
also have ... $= (RH(true \vdash false \diamond (\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R)) \;;\; RH\ (pre_R\ P \vdash peri_R\ P \diamond post_R\ P))$
by (*metis skip-rea-reactive-tri-design′*)
also have ... $= RH\ (pre_R\ P \vdash peri_R\ P \diamond post_R\ P)$
apply (*subst RH-tri-design-composition*)
apply (*simp-all add*: *unrest R2s-true R1-false R1-neg-pre R1-peri R1-post R2s-pre R2s-peri R2s-post CSP-R1-R2s R1-R2s-tr-diff-conj assms*)
apply (*simp add*: *R2s-conj R2s-state′-eq-state wait-pre-lemma rea-left-unit-lemma unrest*)
done
also have ... $= P$
by (*metis CSP-reactive-tri-design assms*)
finally show *?thesis* **.**
qed

lemma *skip-rea-left-semi-unit*:
assumes P *is CSP*
shows $(P \;;\; II_r) = RH\ ((\neg\ (\neg\ pre_R\ P \;;\; R1\ true)) \vdash peri_R\ P \diamond post_R\ P)$
proof −
have $(P \;;\; II_r) = (RH\ (pre_R\ P \vdash peri_R\ P \diamond post_R\ P) \;;\; II_r)$
by (*metis CSP-reactive-tri-design assms*)
also have ... $= (RH\ (pre_R\ P \vdash peri_R\ P \diamond post_R\ P) \;;\; RH(true \vdash false \diamond (\$tr\acute{} =_u \$tr \wedge \$\Sigma_R\acute{} =_u \$\Sigma_R)))$
by (*metis skip-rea-reactive-tri-design′*)
also have ... $= RH\ ((\neg\ (\neg\ pre_R\ P \;;\; R1\ true)) \vdash peri_R\ P \diamond post_R\ P)$
apply (*subst RH-tri-design-composition*)
apply (*simp-all add*: *unrest R2s-true R1-false R2s-false R1-neg-pre R1-peri R1-post R2s-pre R2s-peri R2s-post CSP-R1-R2s R1-R2s-tr-diff-conj assms*)
apply (*simp add*: *R2s-conj R2s-state′-eq-state wait-pre-lemma rea-right-unit-lemma unrest*)
done
finally show *?thesis* **.**
qed

lemma *HR-design-wait-false*: $RH(P_{\,f} \vdash Q_{\,f}) = RH(P \vdash Q)$
by (*metis R3c-subst-wait RH-R2c-def wait-false-design*)

lemma *RH-design-R1-neg-precond*: $RH((\neg\ R1(\neg\ P)) \vdash Q) = RH(P \vdash Q)$
by (*rel-auto*)

lemma *RH-design-pre-neg-conj-R1*: $RH((\neg\ R1\ P \wedge \neg\ R1\ Q) \vdash R) = RH((\neg\ P \wedge \neg\ Q) \vdash R)$
by (*rel-auto*)

15.5 Signature

definition [*urel-defs*]: $Miracle = RH(true \vdash false \diamond false)$

definition [*urel-defs*]: $Chaos = RH(false \vdash true \diamond true)$

definition [*urel-defs*]: $Term = RH(true \vdash true \diamond true)$

definition *assigns-rea* :: $'\alpha$ *usubst* $\Rightarrow$ ($'t$::*ordered-cancel-monoid-diff*, $'\alpha$) *hrelation-rp* ($\langle$-$\rangle_R$) **where**
assigns-rea $\sigma = RH(true \vdash false \diamond (\$tr' =_u \$tr \wedge \lceil\langle\sigma\rangle_a\rceil_R))$

definition *rea-design-sup* :: - *set* $\Rightarrow$ - ($\bigsqcap_R$) **where**
$\bigsqcap_R A = (if\ (A = \{\})\ then\ Miracle\ else\ \bigsqcap\ A)$

definition *rea-design-inf* :: - *set* $\Rightarrow$ - ($\bigsqcup_R$) **where**
$\bigsqcup_R A = (if\ (A = \{\})\ then\ Chaos\ else\ \bigsqcup\ A)$

definition *rea-design-par* :: - $\Rightarrow$ - $\Rightarrow$ - (**infixr** $\|_R$ *85*) **where**
$P \|_R Q = RH((pre_R(P)\ \wedge pre_R(Q)) \vdash (P^t{}_f \wedge Q^t{}_f))$

lemma *Miracle-greatest*:
  assumes *P is CSP*
  shows $P \sqsubseteq Miracle$
proof –
  have $P = RH\ (pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
    by (*metis CSP-reactive-tri-design assms*)
  also have ... $\sqsubseteq RH(true \vdash false)$
    by (*rule RH-monotone*, *rel-auto*)
  also have $RH(true \vdash false) = RH(true \vdash false \diamond false)$
    by (*simp add*: *wait'-cond-def cond-def*)
  finally show *?thesis*
    by (*simp add*: *Miracle-def*)
qed

lemma *Chaos-least*:
  assumes *P is CSP*
  shows $Chaos \sqsubseteq P$
proof –
  have $Chaos = RH(true)$
    by (*simp add*: *Chaos-def design-def*)
  also have ... $\sqsubseteq RH(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
    by (*simp add*: *RH-monotone*)
  also have $RH(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$
    by (*metis CSP-reactive-tri-design assms*)
  finally show *?thesis* **.**
qed

lemma *Miracle-left-zero*:
  assumes *P is CSP*
  shows $(Miracle\ ;;\ P) = Miracle$
proof –
  have $(Miracle\ ;;\ P) = (RH(true \vdash false \diamond false)\ ;;\ RH\ (pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
    by (*metis CSP-reactive-tri-design Miracle-def assms*)
  also have ... $= RH(true \vdash false \diamond false)$
    by (*simp add*: *RH-tri-design-composition R1-false R2s-true R2s-false R2c-true R1-true-comp unrest usubst*)
  also have ... $= Miracle$
    by (*simp add*: *Miracle-def*)
  finally show *?thesis* **.**
qed

lemma *Chaos-def'*: $Chaos = RH(false \vdash true)$
  by (*simp add*: *Chaos-def design-false-pre*)

lemma *Miracle-CSP-false*: $Miracle = CSP(false)$
by (*rel-auto*)

lemma *Chaos-CSP-true*: $Chaos = CSP(true)$
by (*rel-auto*)

lemma *Chaos-left-zero*:
assumes *P is CSP*
shows $(Chaos\ ;;\ P) = Chaos$
proof –
have $(Chaos\ ;;\ P) = (RH(false \vdash true \diamond true)\ ;;\ RH\ (pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*metis CSP-reactive-tri-design Chaos-def assms*)
also have ... $= RH\ ((\neg\ R1\ true \wedge \neg\ (R1\ true \wedge \neg\ \$wait\'\ ;;\ R1\ (\neg\ R2c\ (pre_R\ P)))) \vdash$
$(true \vee (R1\ true\ ;;\ R1\ (R2c\ (peri_R\ P)))) \diamond (R1\ true\ ;;\ R1\ (R2c\ (post_R\ P))))$
by (*simp add*: *RH-tri-design-composition R2s-true R1-true-comp R2s-false unrest*, *metis* (*no-types*) *R1-R2s-R2c R1-negate-R1*)
also have ... $= RH\ ((\neg\ \$ok \vee R1\ true \vee (R1\ true \wedge \neg\ \$wait\'\ ;;\ R1\ (\neg\ R2c\ (pre_R\ P)))) \vee$
$\$ok\' \wedge (true \vee (R1\ true\ ;;\ R1\ (R2c\ (peri_R\ P)))) \diamond (R1\ true\ ;;\ R1\ (R2c\ (post_R\ P))))$
by (*simp add*: *design-def impl-alt-def*)
also have ... $= RH(R1((\neg\ \$ok \vee R1\ true \vee (R1\ true \wedge \neg\ \$wait\'\ ;;\ R1\ (\neg\ R2c\ (pre_R\ P)))) \vee$
$\$ok\' \wedge (true \vee (R1\ true\ ;;\ R1\ (R2c\ (peri_R\ P)))) \diamond (R1\ true\ ;;\ R1\ (R2c\ (post_R\ P)))))$
by (*simp add*: *R1-R2c-commute R1-R3c-commute R1-idem RH-R2c-def*)
also have ... $= RH(R1((\neg\ \$ok \vee true \vee (R1\ true \wedge \neg\ \$wait\'\ ;;\ R1\ (\neg\ R2c\ (pre_R\ P)))) \vee$
$\$ok\' \wedge (true \vee (R1\ true\ ;;\ R1\ (R2c\ (peri_R\ P)))) \diamond (R1\ true\ ;;\ R1\ (R2c\ (post_R\ P)))))$
by (*metis* (*no-types*, *hide-lams*) *R1-disj R1-idem*)
also have ... $= RH(true)$
by (*simp add*: *R1-R2c-commute R1-R3c-commute R1-idem RH-R2c-def*)
also have ... $= Chaos$
by (*simp add*: *Chaos-def design-def*)
finally show *?thesis* **.**
qed

lemma *RH-design-choice*:
$(RH(P \vdash Q_1 \diamond Q_2) \sqcap RH(R \vdash S_1 \diamond S_2)) = RH((P \wedge R) \vdash ((Q_1 \vee S_1) \diamond (Q_2 \vee S_2)))$
proof –
have $(RH(P \vdash Q_1 \diamond Q_2) \sqcap RH(R \vdash S_1 \diamond S_2)) = RH((P \vdash Q_1 \diamond Q_2) \sqcap (R \vdash S_1 \diamond S_2))$
by (*simp add*: *disj-upred-def*[*THEN sym*] *RH-disj*[*THEN sym*])
also have ... $= RH\ ((P \wedge R) \vdash (Q_1 \diamond Q_2 \vee S_1 \diamond S_2))$
by (*simp add*: *design-choice*)
also have ... $= RH\ ((P \wedge R) \vdash ((Q_1 \vee S_1) \diamond (Q_2 \vee S_2)))$
proof –
have $(Q_1 \diamond Q_2 \vee S_1 \diamond S_2) = ((Q_1 \vee S_1) \diamond (Q_2 \vee S_2))$
by (*rel-auto*)
thus *?thesis* **by** *simp*
qed
finally show *?thesis* **.**
qed

lemma *USUP-CSP-closed*:
assumes $A \neq \{\}\ \forall\ P \in A.\ P$ *is CSP*
shows $(\bigsqcap\ A)$ *is CSP*
proof –
from *assms* **have** *A*: $A = CSP\ `\ A$
by (*auto simp add*: *Healthy-def rev-image-eqI*)

also have $(\bigsqcap \ldots) = (\bigsqcap\ P \in A.\ CSP(P))$
  by *auto*
also have $\ldots = (\bigsqcap\ P \in A \cdot CSP(P))$
  by (*simp add*: *USUP-as-Sup-collect*)
also have $\ldots = (\bigsqcap\ P \in A \cdot RH((\neg\ P^f{}_f) \vdash P^t{}_f))$
  by (*metis* (*no-types*) *CSP-RH-design-form*)
also have $\ldots = RH(\bigsqcap\ P \in A \cdot (\neg\ P^f{}_f) \vdash P^t{}_f)$
  by (*simp add*: *RH-USUP assms*(*1*))
also have $\ldots = RH((\bigsqcup\ P \in A \cdot \neg\ P^f{}_f) \vdash (\bigsqcap\ P \in A \cdot P^t{}_f))$
  by (*simp add*: *design-USUP assms*)
also have $\ldots = CSP(\ldots)$
  by (*simp add*: *CSP-RH-design unrest*)
finally show *?thesis*
  by (*simp add*: *Healthy-def CSP-idem*)
qed

lemma *UINF-CSP-closed*:
assumes $A \neq \{\}\ \forall\ P \in A.\ P$ *is CSP*
shows $(\bigsqcup\ A)$ *is CSP*
proof −
from *assms* **have** A: $A = CSP$ ‘ A
  by (*auto simp add*: *Healthy-def rev-image-eqI*)
also have $(\bigsqcup \ldots) = (\bigsqcup\ P \in A.\ CSP(P))$
  by *auto*
also have $\ldots = (\bigsqcup\ P \in A \cdot CSP(P))$
  by (*simp add*: *UINF-as-Inf-collect*)
also have $\ldots = (\bigsqcup\ P \in A \cdot RH((\neg\ P^f{}_f) \vdash P^t{}_f))$
  by (*simp add*: *CSP-RH-design-form*)
also have $\ldots = RH(\bigsqcup\ P \in A \cdot (\neg\ P^f{}_f) \vdash P^t{}_f)$
  by (*simp add*: *RH-UINF assms*(*1*))
also have $\ldots = RH\ ((\bigsqcap\ P \in A \cdot \neg\ P^f{}_f) \vdash (\bigsqcup\ P \in A \cdot \neg\ P^f{}_f \Rightarrow P^t{}_f))$
  by (*simp add*: *design-UINF*)
also have $\ldots = CSP(\ldots)$
  by (*simp add*: *CSP-RH-design unrest*)
finally show *?thesis*
  by (*simp add*: *Healthy-def CSP-idem*)
qed

lemma *CSP-sup-closed*:
assumes $\forall\ P \in A.\ P$ *is CSP*
shows $(\bigsqcap_R\ A)$ *is CSP*
proof (*cases* $A = \{\}$)
case *True*
moreover have *Miracle is CSP*
  by (*simp add*: *Miracle-def Healthy-def CSP-RH-design unrest*)
ultimately show *?thesis*
  by (*simp add*: *rea-design-sup-def*)
next
case *False*
with *USUP-CSP-closed assms* **show** *?thesis*
  by (*auto simp add*: *rea-design-sup-def*)
qed

lemma *CSP-sup-below*:
assumes $\forall\ Q \in A.\ Q$ *is CSP* $P \in A$

shows $\bigsqcap_R A \sqsubseteq P$
using *assms*
by (*auto simp add*: *rea-design-sup-def Sup-upper*)

lemma *CSP-sup-upper-bound*:
assumes $\forall\ Q \in A.\ Q$ *is CSP* $\forall\ Q \in A.\ P \sqsubseteq Q$ P *is CSP*
shows $P \sqsubseteq \bigsqcap_R A$
proof (*cases* $A = \{\}$)
case *True*
thus *?thesis*
by (*simp add*: *rea-design-sup-def Miracle-greatest assms*)
next
case *False*
thus *?thesis*
by (*simp add*: *rea-design-sup-def cSup-least assms*)
qed

lemma *CSP-inf-closed*:
assumes $\forall\ P \in A.\ P$ *is CSP*
shows $(\bigsqcup_R A)$ *is CSP*
proof (*cases* $A = \{\}$)
case *True*
moreover have *Chaos is CSP*
by (*simp add*: *Chaos-def Healthy-def CSP-RH-design unrest*)
ultimately show *?thesis*
by (*simp add*: *rea-design-inf-def*)
next
case *False*
with *UINF-CSP-closed assms* **show** *?thesis*
by (*auto simp add*: *rea-design-inf-def*)
qed

lemma *CSP-inf-above*:
assumes $\forall\ Q \in A.\ Q$ *is CSP* $P \in A$
shows $P \sqsubseteq \bigsqcup_R A$
using *assms*
by (*auto simp add*: *rea-design-inf-def Inf-lower*)

lemma *CSP-inf-lower-bound*:
assumes $\forall\ P \in A.\ P$ *is CSP* $\forall\ P \in A.\ P \sqsubseteq Q$ Q *is CSP*
shows $\bigsqcup_R A \sqsubseteq Q$
proof (*cases* $A = \{\}$)
case *True*
thus *?thesis*
by (*simp add*: *rea-design-inf-def Chaos-least assms*)
next
case *False*
thus *?thesis*
by (*simp add*: *rea-design-inf-def cInf-greatest assms*)
qed

lemma *assigns-lift-rea-unfold*:
$(\$wait' =_u \$wait \wedge \$tr' =_u \$tr \wedge \lceil\langle\sigma\rangle_a\rceil_R) = \lceil\langle\sigma \oplus_s \Sigma_r\rangle_a\rceil_D$
by (*rel-auto*)

lemma *assigns-lift-des-unfold*:
  $(\$ok' =_u \$ok \wedge \lceil \langle\sigma\rangle_a \rceil_D) = \langle \sigma \oplus_s \Sigma_D \rangle_a$
  by (*rel-auto*)

lemma *assigns-rea-comp-lemma*:
  assumes $\$ok \sharp P$ $\$wait \sharp P$
  shows $((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \;;\; P) = (\lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger P)$
proof −
  have $((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \;;\; P) =$
        $((\$ok' =_u \$ok \wedge \$wait' =_u \$wait \wedge \$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \;;\; P)$
    by (*simp add*: *seqr-insert-ident-left unrest assms*)
  also have ... $= (\langle \sigma \oplus_s \Sigma_R \rangle_a \;;\; P)$
    by (*simp add*: *assigns-lift-rea-unfold assigns-lift-des-unfold*, *rel-auto*)
  also have ... $= (\lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger P)$
    by (*simp add*: *assigns-r-comp*)
  finally show *?thesis* **.**
qed

lemma *R1-R2s-frame*:
  $R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil P \rceil_R)) = (\$tr' =_u \$tr \wedge \lceil P \rceil_R)$
    apply (*rel-auto*)
    using *minus-zero-eq* **apply** *blast*
done

lemma *assigns-rea-comp*:
  assumes $\$ok \sharp P$ $\$ok \sharp Q_1$ $\$ok \sharp Q_2$ $\$wait \sharp P$ $\$wait \sharp Q_1$ $\$wait \sharp Q_2$
          Q_1 *is R1* Q_2 *is R1* P *is R2s* Q_1 *is R2s* Q_2 *is R2s*
  shows $(\langle\sigma\rangle_R \;;\; RH(P \vdash Q_1 \diamond Q_2)) = RH(\lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger P \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
proof −
  have $(\langle\sigma\rangle_R \;;\; RH(P \vdash Q_1 \diamond Q_2)) =$
        $(RH\ (true \vdash false \diamond (\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R)) \;;\; RH\ (P \vdash Q_1 \diamond Q_2))$
    by (*simp add*: *assigns-rea-def*)
  also have ... $= RH\ ((\neg\ ((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \wedge \neg\ \$wait' \;;$
                    $R1\ (\neg\ P))) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *RH-tri-design-composition unrest assms R2s-true R1-false R1-R2s-frame Healthy-if assigns-rea-comp-lemma*)
  also have ... $= RH\ ((\neg\ ((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \wedge \$wait' =_u \ll False \gg \;;$
                    $R1\ (\neg\ P))) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *false-alt-def*[*THEN sym*])
  also have ... $= RH\ ((\neg\ ((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R)[\![false/\$wait']\!] \;;$
                    $(R1\ (\neg\ P))[\![false/\$wait]\!])) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *seqr-left-one-point false-alt-def*)
  also have ... $= RH\ ((\neg\ ((\$tr' =_u \$tr \wedge \lceil \langle\sigma\rangle_a \rceil_R) \;;\; (R1\ (\neg\ P)))) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *R1-def usubst unrest assms*)
  also have ... $= RH\ ((\neg\ \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger R1\ (\neg\ P)) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *assigns-rea-comp-lemma assms unrest*)
  also have ... $= RH\ ((\neg\ R1\ (\neg\ \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger P)) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *R1-def usubst unrest*)
  also have ... $= RH\ ((\lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger P) \vdash \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_1 \diamond \lceil \sigma \oplus_s \Sigma_R \rceil_s \dagger Q_2)$
    by (*simp add*: *RH-design-R1-neg-precond*)
  finally show *?thesis* **.**
qed

lemma *RH-design-par*:

assumes
$\$ok' \sharp P_1$ $\$wait \sharp P_1$ $\$ok' \sharp P_2$ $\$wait \sharp P_2$
$\$ok' \sharp Q_1$ $\$wait \sharp Q_1$ $\$ok' \sharp Q_2$ $\$wait \sharp Q_2$
shows $RH(P_1 \vdash Q_1) \parallel_R RH(P_2 \vdash Q_2) = RH((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
proof −
have $RH(P_1 \vdash Q_1) \parallel_R RH(P_2 \vdash Q_2) =$
$RH\ ((\neg\ R1\ (R2c\ (\neg\ P_1[\![true/\$ok]\!])) \wedge \neg\ R1\ (R2c\ (\neg\ P_2[\![true/\$ok]\!]))) \vdash$
$(R1\ (R2s\ (\$ok \wedge P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (\$ok \wedge P_2 \Rightarrow Q_2))))$
by (*simp add*: *rea-design-par-def rea-pre-RH-design RH-postcondition*, *simp add*: *usubst assms*)
also have ... =
$RH\ ((P_1[\![true/\$ok]\!] \wedge P_2[\![true/\$ok]\!]) \vdash$
$(R1\ (R2s\ (\$ok \wedge P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (\$ok \wedge P_2 \Rightarrow Q_2))))$
by (*metis* (*no-types*, *hide-lams*) *R2c-and R2c-not RH-design-pre-R2c RH-design-pre-neg-conj-R1 double-negation*)
also have ... = $RH\ ((P_1 \wedge P_2) \vdash (R1\ (R2s\ (\$ok \wedge P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (\$ok \wedge P_2 \Rightarrow Q_2))))$
by (*metis conj-pos-var-subst design-def subst-conj vwb-lens-ok*)
also have ... = $RH\ ((P_1 \wedge P_2) \vdash (R1\ (R2s\ ((\$ok \wedge P_1 \Rightarrow Q_1) \wedge (\$ok \wedge P_2 \Rightarrow Q_2)))))$
by (*simp add*: *R1-conj R2s-conj*)
also have ... = $RH\ ((P_1 \wedge P_2) \vdash ((\$ok \wedge P_1 \Rightarrow Q_1) \wedge (\$ok \wedge P_2 \Rightarrow Q_2)))$
by (*metis* (*mono-tags*, *lifting*) *RH-design-export-R1 RH-design-export-R2s*)
also have ... = $RH\ ((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RH-tri-design-par*:
assumes
$\$ok' \sharp P_1$ $\$wait \sharp P_1$ $\$ok' \sharp P_2$ $\$wait \sharp P_2$
$\$ok' \sharp Q_1$ $\$wait \sharp Q_1$ $\$ok' \sharp Q_2$ $\$wait \sharp Q_2$
$\$ok' \sharp R_1$ $\$wait \sharp R_1$ $\$ok' \sharp R_2$ $\$wait \sharp R_2$
shows $RH(P_1 \vdash Q_1 \diamond R_1) \parallel_R RH(P_2 \vdash Q_2 \diamond R_2) = RH((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
by (*simp add*: *RH-design-par assms unrest wait′-cond-conj-exchange*)

lemma *RH-design-par-comm*:
$P \parallel_R Q = Q \parallel_R P$
by (*simp add*: *rea-design-par-def utp-pred.inf-commute*)

lemma *RH-design-par-zero*:
assumes *P is CSP*
shows $Chaos \parallel_R P = Chaos$
proof −
have $Chaos \parallel_R P = RH\ (false \vdash true \diamond true) \parallel_R RH\ (pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
by (*simp add*: *Chaos-def CSP-reactive-tri-design assms*)
also have ... = $RH\ (false \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *RH-tri-design-par unrest*)
also have ... = *Chaos*
by (*simp add*: *Chaos-def design-false-pre*)
finally show *?thesis* **.**
qed

lemma *RH-design-par-unit*:
assumes *P is CSP*
shows $Term \parallel_R P = P$
proof −
have $Term \parallel_R P = RH\ (true \vdash true \diamond true) \parallel_R RH\ (pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

  by (*simp add*: *Term-def CSP-reactive-tri-design assms*)
 also have ... = *RH* (pre_R *P* ⊢ $peri_R$ *P* ⋄ $post_R$ *P*)
  by (*simp add*: *RH-tri-design-par unrest*)
 also have ... = *P*
  by (*simp add*: *CSP-reactive-tri-design assms*)
 finally show *?thesis* **.**
qed

15.6 Complete lattice

typedef *RDES* = *UNIV* :: *unit set* **..**
typedef *R1DES* = *UNIV* :: *unit set* **..**

abbreviation *R1DES* ≡ *TYPE*(*R1DES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*)

overloading
 r1des-hcond == *utp-hcond* :: (*R1DES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*) *itself* ⇒ ((*'t*,*'α*) *alphabet-rp* × (*'t*,*'α*) *alphabet-rp*) *Healthiness-condition*
begin
 definition *r1des-hcond* :: (*R1DES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*) *itself* ⇒ ((*'t*,*'α*) *alphabet-rp* × (*'t*,*'α*) *alphabet-rp*) *Healthiness-condition* **where**
 [*upred-defs*]: *r1des-hcond T* = *R1* ∘ **H**
end

interpretation *r1des-theory*: *utp-theory TYPE*(*R1DES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*)
 by (*unfold-locales*, *simp-all add*: *r1des-hcond-def*, *metis CSP1-R1-H1 H1-H2-idempotent H2-R1-comm R1-idem*)

abbreviation *RDES* ≡ *TYPE*(*RDES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*)

overloading
 rdes-hcond == *utp-hcond* :: (*RDES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*) *itself* ⇒ ((*'t*,*'α*) *alphabet-rp* × (*'t*,*'α*) *alphabet-rp*) *Healthiness-condition*
begin
 definition *rdes-hcond* :: (*RDES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*) *itself* ⇒ ((*'t*,*'α*) *alphabet-rp* × (*'t*,*'α*) *alphabet-rp*) *Healthiness-condition* **where**
 [*upred-defs*]: *rdes-hcond T* = *CSP*
end

interpretation *rdes-theory*: *utp-theory TYPE*(*RDES* × (*'t*::*ordered-cancel-monoid-diff*,*'α*) *alphabet-rp*)
 by (*unfold-locales*, *simp-all add*: *rdes-hcond-def CSP-idem*)

lemma *Miracle-is-top*: $\top_{utp\text{-}order\ RDES}$ = *Miracle*
 apply (*auto intro*!:*some-equality simp add*: *atop-def some-equality greatest-def utp-order-def rdes-hcond-def*)
 apply (*metis CSP-sup-closed emptyE rea-design-sup-def*)
 using *Miracle-greatest* **apply** *blast*
 apply (*metis CSP-sup-closed dual-order.antisym equals0D rea-design-sup-def Miracle-greatest*)
done

lemma *Chaos-is-bot*: $\bot_{utp\text{-}order\ RDES}$ = *Chaos*
 apply (*auto intro*!:*some-equality simp add*: *abottom-def some-equality least-def utp-order-def rdes-hcond-def*)
 apply (*metis CSP-inf-closed emptyE rea-design-inf-def*)
 using *Chaos-least* **apply** *blast*
 apply (*metis Chaos-least CSP-inf-closed dual-order.antisym equals0D rea-design-inf-def*)
done

interpretation *hrd-lattice*: *utp-theory-lattice TYPE*(*RDES* × ('*t*::*ordered-cancel-monoid-diff*,'α) *alphabet-rp*)
rewrites *carrier* (*utp-order RDES*) = $[\![CSP]\!]_H$
and $\top_{utp\text{-}order\ RDES}$ = *Miracle*
and $\bot_{utp\text{-}order\ RDES}$ = *Chaos*
apply (*unfold-locales*)
apply (*simp-all add*: *Miracle-is-top Chaos-is-bot*)
apply (*simp-all add*: *utp-order-def rdes-hcond-def*)
apply (*rename-tac A*)
apply (*rule-tac x*=$\bigsqcup_R$ *A* **in** *exI*, *auto intro*:*CSP-inf-above CSP-inf-lower-bound CSP-inf-closed simp add*: *least-def Upper-def CSP-inf-above*)
apply (*rename-tac A*)
apply (*rule-tac x*=$\bigsqcap_R$ *A* **in** *exI*, *auto intro*:*CSP-sup-below CSP-sup-upper-bound CSP-sup-closed simp add*: *greatest-def Lower-def CSP-inf-above*)
done

abbreviation *rdes-lfp* :: - ⇒ - (μ_R) **where**
$\mu_R\ F \equiv \mu_{utp\text{-}order\ RDES}\ F$

abbreviation *rdes-gfp* :: - ⇒ - (ν_R) **where**
$\nu_R\ F \equiv \nu_{utp\text{-}order\ RDES}\ F$

lemma *rdes-lfp-copy*: $[\![$ *mono F*; $F \in [\![CSP]\!]_H \to [\![CSP]\!]_H$ $]\!] \Longrightarrow \mu_R\ F = F\ (\mu_R\ F)$
by (*metis hrd-lattice.LFP-unfold mono-Monotone-utp-order*)

lemma *rdes-gfp-copy*: $[\![$ *mono F*; $F \in [\![CSP]\!]_H \to [\![CSP]\!]_H$ $]\!] \Longrightarrow \nu_R\ F = F\ (\nu_R\ F)$
by (*metis hrd-lattice.GFP-unfold mono-Monotone-utp-order*)

lemma *RH-H1-H2-eq-CSP*: **R** (**H** *P*) = *CSP P*
by (*metis* (*no-types*, *lifting*) *CSP1-R1-H1 CSP1-R2c-commute CSP1-R3c-commute CSP2-def R1-H2-commute R1-R2c-commute R1-R2c-is-R2 R2-R3c-commute R2c-H2-commute R3c-H2-commute RH-alt-def''*)

lemma *Des-Rea-galois-lemma-1*: *R1*(**H**(*R1*(*P*))) ⊑ *R1*(*P*)
by (*rel-auto*)

lemma **R**(*CSP*(*P*)) = *CSP*(*P*)
by (*rel-auto*)

lemma *Des-Rea-galois-lemma-2*: *CSP*(*P*) ⊑ **H**(**R**(*CSP*(*P*)))
apply (*rel-auto*)
oops

lemma *R2c-H1-H2-commute*: *R2c*(**H**(*P*)) = **H**(*R2c*(*P*))
by (*rel-auto*)

lemma *funcset-into-Idempotent*: *Idempotent H* ⟹ $H \in X \to [\![H]\!]_H$
by (*simp add*: *Healthy-def' Idempotent-def*)

interpretation *galois-connection R1DES* ←⟨*id*,*R2c* ∘ *R3c*⟩→ *RDES*
proof (*simp add*: *mk-conn-def*, *rule galois-connectionI'*, *simp-all add*: *utp-partial-order r1des-hcond-def rdes-hcond-def r1des-hcond-def*)
show *R2c* ∘ *R3c* ∈ $[\![R1 \circ \mathbf{H}]\!]_H \to [\![CSP]\!]_H$
by (*simp add*: *Pi-iff Healthy-def'*, *metis R1-R2c-commute R1-R3c-commute R3c-idem RH-H1-H2-eq-CSP RH-absorbs-R2c RH-alt-def''*)
show *id* ∈ $[\![CSP]\!]_H \to [\![R1 \circ \mathbf{H}]\!]_H$

by (*simp add*: *Pi-iff Healthy-def'*, *metis CSP1-via-H1 CSP2-def RH-H1-H2-eq-CSP RH-alt-def RH-alt-def' RH-idem*)
show *isotone* (*utp-order R1DES*) (*utp-order RDES*) (*R2c* ∘ *R3c*)
by (*auto intro*: *isotone-utp-orderI Monotonic-comp R2c-Monotonic R3c-Monotonic*)
show *isotone* (*utp-order RDES*) (*utp-order R1DES*) *id*
by (*auto intro*: *isotone-utp-orderI Monotonic-comp Monotonic-id*)
show $\forall P.\ P\ is\ CSP \longrightarrow R2c\ (R3c\ P) \sqsubseteq P$
by (*metis* (*no-types*, *lifting*) *CSP-R1-R2s CSP-healths*(*3*) *Healthy-def' R1-R2c-commute R2c-R2s-absorb eq-refl*)
show $\forall P.\ P\ is\ R1 \circ \mathbf{H} \longrightarrow P \sqsubseteq R2c\ (R3c\ P)$
oops

interpretation *Des-Rea-galois*: *galois-connection DES* $\leftarrow\langle\mathbf{H},\mathbf{R}\rangle\rightarrow$ *RDES*
proof (*simp add*: *mk-conn-def*, *rule galois-connectionI'*, *simp-all add*: *utp-partial-order rdes-hcond-def des-hcond-def*)
show $\mathbf{R} \in [\![\mathbf{H}]\!]_H \to [\![CSP]\!]_H$
by (*metis* (*no-types*, *lifting*) *CSP-idem Healthy-def' Pi-I' RH-H1-H2-eq-CSP mem-Collect-eq*)
show $\mathbf{H} \in [\![CSP]\!]_H \to [\![\mathbf{H}]\!]_H$
by (*rule funcset-into-Idempotent*, *rule H1-H2-Idempotent*)
show *isotone* (*utp-order DES*) (*utp-order RDES*) **R**
by (*rule isotone-utp-orderI*, *metis rea-hcond-def rea-utp-theory-mono.HCond-Mono*)
show *isotone* (*utp-order RDES*) (*utp-order DES*) **H**
by (*rule isotone-utp-orderI*, *simp add*: *H1-H2-monotonic*)
show $\forall X.\ X\ is\ CSP \longrightarrow \mathbf{R}\ (\mathbf{H}\ X) \sqsubseteq X$
by (*simp add*: *CSP-RH-design-form CSP-reactive-design RH-H1-H2-eq-CSP*)
show $\forall X.\ X\ is\ \mathbf{H} \longrightarrow X \sqsubseteq \mathbf{H}\ (\mathbf{R}\ X)$
proof (*auto*)
fix P :: (*'t*::*ordered-cancel-monoid-diff*,*'α*) *hrelation-rp*
assume $P\ is\ \mathbf{H}$
hence $(P \sqsubseteq \mathbf{H}\ (\mathbf{R}\ P)) \longleftrightarrow (\mathbf{H}(P) \sqsubseteq \mathbf{H}(\mathbf{R}(\mathbf{H}(P))))$
by (*simp add*: *Healthy-def'*)
also have ... $\longleftrightarrow (\mathbf{H}(P) \sqsubseteq \mathbf{H}(R1(\mathbf{H}(P))))$
oops

15.7 Reactive design parallel-by-merge

definition [*upred-defs*]: $nil_{rm} = (nil_m \triangleleft \$0{-}ok \wedge \$1{-}ok \triangleright \$tr_{<} \leq_u \$tr')$

nil_{rm} is the parallel system which does nothing if the parallel predicates have both terminated ($0.ok \wedge 1.ok$), and otherwise guarantees only the merged trace gets longer.

definition [*upred-defs*]: $div_m = (\$tr \leq_u \$0{-}tr' \wedge \$tr \leq_u \$1{-}tr' \wedge \$\Sigma_{<}' =_u \$\Sigma)$

div_m is the parallel system where both sides traces get longer than the initial trace and identifies the before values of the variables.

definition [*upred-defs*]: $wait_m = skip_m[\![true,true/\$ok,\$wait]\!]$

$wait_m$ is the parallel system where ok and wait are both true and all other variables are identified
.

R3c implicitly depends on CSP1, and therefore it requires that both sides be CSP1. We also require that both sides are R3c, and that $wait_m$ is a quasi-unit, and div_m yields divergence.

lemma *R3c-par-by-merge*:
assumes
P is R1 Q is R1 P is CSP1 Q is CSP1 P is R3c Q is R3c
$(wait_m\ ;;\ M) = II[\![true,true/\$ok,\$wait]\!]$ $(div_m\ ;;\ M) = R1(true)$

shows $(P \parallel_M Q)$ *is R3c*
proof −
have $(P \parallel_M Q) = (((P \parallel_M Q)[\![true/\$ok]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!])[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*metis cond-idem cond-var-subst-left cond-var-subst-right vwb-lens-ok wait-vwb-lens*)
also have ... $= (((P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!])[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= (((P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!]) \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*metis cond-var-subst-left wait-vwb-lens*)
also have ... $= ((II[\![true,true/\$ok,\$wait]\!] \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q))$
proof −
have $(P \parallel_M Q)[\![false/\$ok]\!] = R1(true)$
proof −
have $(P \parallel_M Q)[\![false/\$ok]\!] = ((P \triangleleft \$ok \triangleright R1(true)) \parallel_M (Q \triangleleft \$ok \triangleright R1(true)))[\![false/\$ok]\!]$
by (*metis CSP1-alt-def Healthy-if assms*)
also have ... $= (R1(true) \parallel_{M[\![false/\$ok_<]\!]} R1(true))$
by (*rel-auto*, *metis*, *metis*)
also have ... $= (div_m \;;\; M)[\![false/\$ok]\!]$
by (*rel-auto*, *metis*, *metis*)
also have ... $= (R1(true))[\![false/\$ok]\!]$
by (*simp add*: *assms*(*8*))
also have ... $= (R1(true))$
by *rel-auto*
finally show *?thesis*
by *simp*
qed
moreover have $(P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] = II[\![true,true/\$ok,\$wait]\!]$
proof −
have $(P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] = (P[\![true,true/\$ok,\$wait]\!] \parallel_M Q[\![true,true/\$ok,\$wait]\!])[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
also have ... $= (((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)[\![true,true/\$ok,\$wait]\!] \parallel_M ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright Q)[\![true,true/\$ok,\$wait]\!])[\![true,true/\$ok,\$wait]\!]$
by (*metis Healthy-def′ R3c-cases assms*(*5*) *assms*(*6*))
also have ... $= (II[\![true,true/\$ok,\$wait]\!] \parallel_M II[\![true,true/\$ok,\$wait]\!])[\![true,true/\$ok,\$wait]\!]$
by (*subst-tac*)
also have ... $= (wait_m \;;\; M)[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
also have ... $= II[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *assms usubst*)
finally show *?thesis* **.**
qed
ultimately show *?thesis* **by** *simp*
qed
also have ... $= ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= R3c(P \parallel_M Q)$
by (*simp add*: *R3c-cases*)
finally show *?thesis*
by (*simp add*: *Healthy-def′*)
qed

lemma *CSP1-par-by-merge*:
assumes *P is R1 Q is R1 P is CSP1 Q is CSP1 M is R1m* $(div_m \;;\; M) = R1(true)$

shows $(P \parallel_M Q)$ *is CSP1*
proof –
have $(P \parallel_M Q) = ((P \parallel_M Q) \triangleleft \$ok \triangleright (P \parallel_M Q)[\![false/\$ok]\!])$
by (*metis cond-idem cond-var-subst-right vwb-lens-ok*)
also have ... $= ((P \parallel_M Q) \triangleleft \$ok \triangleright R1(true))$
proof –
have $(P \parallel_M Q)[\![false/\$ok]\!] = ((P \triangleleft \$ok \triangleright R1(true)) \parallel_M (Q \triangleleft \$ok \triangleright R1(true)))[\![false/\$ok]\!]$
by (*metis CSP1-alt-def Healthy-if assms*)
also have ... $= (R1(true) \parallel_{M[\![false/\$ok_<]\!]} R1(true))$
by (*rel-auto*, *metis*, *metis*)
also have ... $= (div_m \;;;\; M)[\![false/\$ok]\!]$
by (*rel-auto*, *metis*, *metis*)
also have ... $= (R1(true))[\![false/\$ok]\!]$
by (*simp add*: *assms*(*6*))
also have ... $= (R1(true))$
by *rel-auto*
finally show *?thesis*
by *simp*
qed
finally show *?thesis*
by (*metis CSP1-alt-def Healthy-def R1-par-by-merge assms*(*5*))
qed

lemma *CSP2-par-by-merge*:
assumes *M is CSP2*
shows $(P \parallel_M Q)$ *is CSP2*
proof –
have $(P \parallel_M Q) = ((P \parallel_s Q) \;;;\; M)$
by (*simp add*: *par-by-merge-def*)
also from *assms* **have** ... $= ((P \parallel_s Q) \;;;\; (M \;;;\; J))$
by (*simp add*: *Healthy-def′ CSP2-def H2-def*)
also from *assms* **have** ... $= (((P \parallel_s Q) \;;;\; M) \;;;\; J)$
by (*meson seqr-assoc*)
also from *assms* **have** ... $= CSP2(P \parallel_M Q)$
by (*simp add*: *CSP2-def H2-def par-by-merge-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def′*)
qed

end

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