

Reactive Designs

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1 Introduction

This document contains a mechanisation in Isabelle/UTP [2] of our theory of reactive designs. Reactive designs form an important semantic foundation for reactive modelling languages such as Circus [3]. For more details of this work, please see our recent paper [1].

2 Reactive Designs Healthiness Conditions

```
theory utp-rdes-healths
  imports UTP-Reactive.utp-reactive
begin
```

2.1 Preliminaries

```
named-theorems rdes and rdes-def and RD-elim
```

```
type-synonym ('s,'t) rdes = ('s,'t,unit) hrel-rsp
```

```
translations
```

```
(type) ('s,'t) rdes <= (type) ('s, 't, unit) hrel-rsp
```

```
lemma R2-st-ex: R2 ( $\exists$  $st  $\cdot$  P) = ( $\exists$  $st  $\cdot$  R2(P))
  by (rel-auto)
```

```
lemma R2s-st'-eq-st:
  R2s($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R2c-st'-eq-st:
  R2c($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R1-des-lift-skip: R1( $\lceil$  II  $\rceil_D$ ) =  $\lceil$  II  $\rceil_D$ 
  by (rel-auto)
```

```
lemma R2-des-lift-skip:
  R2( $\lceil$  II  $\rceil_D$ ) =  $\lceil$  II  $\rceil_D$ 
  apply (rel-auto) using minus-zero-eq by blast
```

```
lemma R1-R2c-ex-st: R1 (R2c ( $\exists$  $st'  $\cdot$  Q1)) = ( $\exists$  $st'  $\cdot$  R1 (R2c Q1))
  by (rel-auto)
```

2.2 Identities

We define two identities fro reactive designs, which correspond to the regular and state-sensitive versions of reactive designs, respectively. The former is the one used in the UTP book and related publications for CSP.

```
definition skip-rea :: ('t::trace, 'α) hrel-rp (IIc) where
skip-rea-def [urel-defs]: IIc = (II  $\vee$  ( $\neg$  $ok  $\wedge$  $tr  $\leq_u$  $tr'))
```

```
definition skip-srea :: ('s, 't::trace, 'α) hrel-rsp (IIR) where
skip-srea-def [urel-defs]: IIR = (( $\exists$  $st  $\cdot$  IIc)  $\triangleleft$  $wait  $\triangleright$  IIc)
```

```
lemma skip-rea-R1-lemma: IIc = R1($ok  $\Rightarrow$  II)
  by (rel-auto)
```

```
lemma skip-rea-form: IIc = (II  $\triangleleft$  $ok  $\triangleright$  R1(true))
  by (rel-auto)
```

```
lemma skip-srea-form: IIR = (( $\exists$  $st  $\cdot$  II)  $\triangleleft$  $wait  $\triangleright$  II)  $\triangleleft$  $ok  $\triangleright$  R1(true)
  by (rel-auto)
```

lemma *R1-skip-rea*: $R1(H_c) = H_c$
by (*rel-auto*)

lemma *R2c-skip-rea*: $R2c H_c = H_c$
by (*simp add: skip-rea-def R2c-and R2c-disj R2c-skip-r R2c-not R2c-ok R2c-tr'-ge-tr*)

lemma *R2-skip-rea*: $R2(H_c) = H_c$
by (*metis R1-R2c-is-R2 R1-skip-rea R2c-skip-rea*)

lemma *R2c-skip-srea*: $R2c(H_R) = H_R$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *skip-srea-R1 [closure]*: H_R is *R1*
by (*rel-auto*)

lemma *skip-srea-R2c [closure]*: H_R is *R2c*
by (*simp add: Healthy-def R2c-skip-srea*)

lemma *skip-srea-R2 [closure]*: H_R is *R2*
by (*metis Healthy-def' R1-R2c-is-R2 R2c-skip-srea skip-srea-R1*)

2.3 RD1: Divergence yields arbitrary traces

definition *RD1* :: $(t::trace, 'α, 'β)$ *rel-rp* $\Rightarrow (t, 'α, 'β)$ *rel-rp* **where**
 $[upred-defs]: RD1(P) = (P \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

RD1 is essentially *H1* from the theory of designs, but viewed through the prism of reactive processes.

lemma *RD1-idem*: $RD1(RD1(P)) = RD1(P)$
by (*rel-auto*)

lemma *RD1-Idempotent*: *Idempotent RD1*
by (*simp add: Idempotent-def RD1-idem*)

lemma *RD1-mono*: $P \sqsubseteq Q \implies RD1(P) \sqsubseteq RD1(Q)$
by (*rel-auto*)

lemma *RD1-Monotonic*: *Monotonic RD1*
using *mono-def RD1-mono* **by** *blast*

lemma *RD1-Continuous*: *Continuous RD1*
by (*rel-auto*)

lemma *R1-true-RD1-closed [closure]*: *R1(true)* is *RD1*
by (*rel-auto*)

lemma *RD1-wait-false [closure]*: P is *RD1* $\implies P[[false/\$wait]]$ is *RD1*
by (*rel-auto*)

lemma *RD1-wait'-false [closure]*: P is *RD1* $\implies P[[false/\$wait']]$ is *RD1*
by (*rel-auto*)

lemma *RD1-seq*: $RD1(RD1(P) ;; RD1(Q)) = RD1(P) ;; RD1(Q)$
by (*rel-auto*)

lemma *RD1-seq-closure* [closure]: $\llbracket P \text{ is } RD1; Q \text{ is } RD1 \rrbracket \implies P ;; Q \text{ is } RD1$
by (*metis Healthy-def' RD1-seq*)

lemma *RD1-R1-commute*: $RD1(R1(P)) = R1(RD1(P))$
by (*rel-auto*)

lemma *RD1-R2c-commute*: $RD1(R2c(P)) = R2c(RD1(P))$
by (*rel-auto*)

lemma *RD1-via-R1*: $R1(H1(P)) = RD1(R1(P))$
by (*rel-auto*)

lemma *RD1-R1-cases*: $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$
by (*rel-auto*)

lemma *skip-rea-RD1-skip*: $II_c = RD1(II)$
by (*rel-auto*)

lemma *skip-srea-RD1* [closure]: $II_R \text{ is } RD1$
by (*rel-auto*)

lemma *RD1-algebraic-intro*:

assumes

$P \text{ is } R1 \ (R1(true_h) ;; P) = R1(true_h) \ (II_c ;; P) = P$

shows $P \text{ is } RD1$

proof –

have $P = (II_c ;; P)$

by (*simp add: assms(3)*)

also have $\dots = (R1(\$ok \Rightarrow II) ;; P)$

by (*simp add: skip-rea-R1-lemma*)

also have $\dots = (((\neg \$ok \wedge R1(true)) ;; P) \vee P)$

by (*metis (no-types, lifting) R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred-laws.inf-top-left utp-pred-laws.sup-commute*)

also have $\dots = ((R1(\neg \$ok) ;; (R1(true_h) ;; P)) \vee P)$

using *dual-order.trans* **by** (*rel-blast*)

also have $\dots = ((R1(\neg \$ok) ;; R1(true_h)) \vee P)$

by (*simp add: assms(2)*)

also have $\dots = (R1(\neg \$ok) \vee P)$

by (*rel-auto*)

also have $\dots = RD1(P)$

by (*rel-auto*)

finally show *?thesis*

by (*simp add: Healthy-def*)

qed

theorem *RD1-left-zero*:

assumes $P \text{ is } R1 \ P \text{ is } RD1$

shows $(R1(true) ;; P) = R1(true)$

proof –

have $(R1(true) ;; R1(RD1(P))) = R1(true)$

by (*rel-auto*)

thus *?thesis*

by (*simp add: Healthy-if assms(1) assms(2)*)

qed

theorem *RD1-left-unit*:
assumes P is $R1$ P is $RD1$
shows $(II_c ;; P) = P$
proof –
have $(II_c ;; R1(RD1(P))) = R1(RD1(P))$
by (*rel-auto*)
thus ?thesis
by (*simp add: Healthy-if assms(1) assms(2)*)
qed

lemma *RD1-alt-def*:
assumes P is $R1$
shows $RD1(P) = (P \triangleleft \$ok \triangleright R1(true))$
proof –
have $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$
by (*rel-auto*)
thus ?thesis
by (*simp add: Healthy-if assms*)
qed

theorem *RD1-algebraic*:
assumes P is $R1$
shows P is $RD1 \iff (R1(true_h) ;; P) = R1(true_h) \wedge (II_c ;; P) = P$
using *RD1-algebraic-intro RD1-left-unit RD1-left-zero assms* **by** *blast*

2.4 R3c and R3h: Reactive design versions of R3

definition $R3c :: ('t::trace, 'α) hrel-rp \Rightarrow ('t, 'α) hrel-rp$ **where**
 $[upred-defs]: R3c(P) = (II_c \triangleleft \$wait \triangleright P)$

definition $R3h :: ('s, 't::trace, 'α) hrel-rsp \Rightarrow ('s, 't, 'α) hrel-rsp$ **where**
 $R3h-def [upred-defs]: R3h(P) = ((\exists \$st \cdot II_c) \triangleleft \$wait \triangleright P)$

lemma *R3c-idem*: $R3c(R3c(P)) = R3c(P)$
by (*rel-auto*)

lemma *R3c-Idempotent*: *Idempotent R3c*
by (*simp add: Idempotent-def R3c-idem*)

lemma *R3c-mono*: $P \sqsubseteq Q \implies R3c(P) \sqsubseteq R3c(Q)$
by (*rel-auto*)

lemma *R3c-Monotonic*: *Monotonic R3c*
by (*simp add: mono-def R3c-mono*)

lemma *R3c-Continuous*: *Continuous R3c*
by (*rel-auto*)

lemma *R3h-idem*: $R3h(R3h(P)) = R3h(P)$
by (*rel-auto*)

lemma *R3h-Idempotent*: *Idempotent R3h*
by (*simp add: Idempotent-def R3h-idem*)

lemma *R3h-mono*: $P \sqsubseteq Q \implies R3h(P) \sqsubseteq R3h(Q)$

by (rel-auto)

lemma *R3h-Monotonic: Monotonic R3h*
 by (simp add: mono-def R3h-mono)

lemma *R3h-Continuous: Continuous R3h*
 by (rel-auto)

lemma *R3h-inf: $R3h(P \sqcap Q) = R3h(P) \sqcap R3h(Q)$*
 by (rel-auto)

lemma *R3h-UNF:*
 $A \neq \{\} \implies R3h(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot R3h(P(i)))$
 by (rel-auto)

lemma *R3h-cond: $R3h(P \triangleleft b \triangleright Q) = (R3h(P) \triangleleft b \triangleright R3h(Q))$*
 by (rel-auto)

lemma *R3c-via-RD1-R3: $RD1(R3(P)) = R3c(RD1(P))$*
 by (rel-auto)

lemma *R3c-RD1-def: P is $RD1 \implies R3c(P) = RD1(R3(P))$*
 by (simp add: Healthy-if R3c-via-RD1-R3)

lemma *RD1-R3c-commute: $RD1(R3c(P)) = R3c(RD1(P))$*
 by (rel-auto)

lemma *R1-R3c-commute: $R1(R3c(P)) = R3c(R1(P))$*
 by (rel-auto)

lemma *R2c-R3c-commute: $R2c(R3c(P)) = R3c(R2c(P))$*
 apply (rel-auto) using minus-zero-eq by blast+

lemma *R1-R3h-commute: $R1(R3h(P)) = R3h(R1(P))$*
 by (rel-auto)

lemma *R2c-R3h-commute: $R2c(R3h(P)) = R3h(R2c(P))$*
 apply (rel-auto) using minus-zero-eq by blast+

lemma *RD1-R3h-commute: $RD1(R3h(P)) = R3h(RD1(P))$*
 by (rel-auto)

lemma *R3c-cancels-R3: $R3c(R3(P)) = R3c(P)$*
 by (rel-auto)

lemma *R3-cancels-R3c: $R3(R3c(P)) = R3(P)$*
 by (rel-auto)

lemma *R3h-cancels-R3c: $R3h(R3c(P)) = R3h(P)$*
 by (rel-auto)

lemma *R3c-semir-form:*
 $(R3c(P) ;; R3c(R1(Q))) = R3c(P ;; R3c(R1(Q)))$
 by (rel-simp, safe, auto intro: order-trans)

lemma *R3h-semir-form*:

$(R3h(P) ;; R3h(R1(Q))) = R3h(P ;; R3h(R1(Q)))$
 by (rel-simp, safe, auto intro: order-trans, blast+)

lemma *R3c-seq-closure*:

assumes P is *R3c* Q is *R3c* Q is *R1*
 shows $(P ;; Q)$ is *R3c*
 by (metis Healthy-def' R3c-semir-form assms)

lemma *R3h-seq-closure* [closure]:

assumes P is *R3h* Q is *R3h* Q is *R1*
 shows $(P ;; Q)$ is *R3h*
 by (metis Healthy-def' R3h-semir-form assms)

lemma *R3c-R3-left-seq-closure*:

assumes P is *R3* Q is *R3c*
 shows $(P ;; Q)$ is *R3c*

proof –

have $(P ;; Q) = ((P ;; Q) \llbracket \text{true}/\$wait \rrbracket \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens)
 also have $\dots = (((II \triangleleft \$wait \triangleright P) ;; Q) \llbracket \text{true}/\$wait \rrbracket \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis Healthy-def' R3-def assms(1))
 also have $\dots = ((II \llbracket \text{true}/\$wait \rrbracket ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$
 by (subst-tac)
 also have $\dots = (((II \wedge \$wait') ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis (no-types, lifting) cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred-laws.inf-left-idem wait-vwb-lens)
 also have $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; Q \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens)
 also have $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; (II_c \triangleleft \$wait \triangleright Q) \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis Healthy-def' R3c-def assms(2))
 also have $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; II_c \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$
 by (subst-tac)
 also have $\dots = (((II \wedge \$wait') ;; II_c) \triangleleft \$wait \triangleright (P ;; Q))$
 by (metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens)
 also have $\dots = ((II ;; II_c) \triangleleft \$wait \triangleright (P ;; Q))$
 by (simp add: cond-def seqr-pre-transfer utp-rel.unrest-ouvar)
 also have $\dots = (II_c \triangleleft \$wait \triangleright (P ;; Q))$
 by simp
 also have $\dots = R3c(P ;; Q)$
 by (simp add: R3c-def)
 finally show ?thesis
 by (simp add: Healthy-def')

qed

lemma *R3c-cases*: $R3c(P) = ((II \triangleleft \$ok \triangleright R1(\text{true})) \triangleleft \$wait \triangleright P)$

by (rel-auto)

lemma *R3h-cases*: $R3h(P) = (((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(\text{true})) \triangleleft \$wait \triangleright P)$

by (rel-auto)

lemma *R3h-form*: $R3h(P) = II_R \triangleleft \$wait \triangleright P$

by (rel-auto)

lemma *R3c-subst-wait*: $R3c(P) = R3c(P_f)$
 by (*simp add: R3c-def cond-var-subst-right*)

lemma *R3h-subst-wait*: $R3h(P) = R3h(P_f)$
 by (*simp add: R3h-cases cond-var-subst-right*)

lemma *skip-srea-R3h [closure]*: II_R is *R3h*
 by (*rel-auto*)

lemma *R3h-wait-true*:
 assumes P is *R3h*
 shows $P_t = II_R t$
proof –
 have $P_t = (II_R \triangleleft \$wait \triangleright P)_t$
 by (*metis Healthy-if R3h-form assms*)
 also have $\dots = II_R t$
 by (*simp add: usubst*)
 finally show ?thesis .
qed

2.5 RD2: A reactive specification cannot require non-termination

definition *RD2 where*
[upred-defs]: $RD2(P) = H2(P)$

RD2 is just *H2* since the type system will automatically have *J* identifying the reactive variables as required.

lemma *RD2-idem*: $RD2(RD2(P)) = RD2(P)$
 by (*simp add: H2-idem RD2-def*)

lemma *RD2-Idempotent*: *Idempotent RD2*
 by (*simp add: Idempotent-def RD2-idem*)

lemma *RD2-mono*: $P \sqsubseteq Q \implies RD2(P) \sqsubseteq RD2(Q)$
 by (*simp add: H2-def RD2-def seqr-mono*)

lemma *RD2-Monotonic*: *Monotonic RD2*
 using *mono-def RD2-mono* by *blast*

lemma *RD2-Continuous*: *Continuous RD2*
 by (*rel-auto*)

lemma *RD1-RD2-commute*: $RD1(RD2(P)) = RD2(RD1(P))$
 by (*rel-auto*)

lemma *RD2-R3c-commute*: $RD2(R3c(P)) = R3c(RD2(P))$
 by (*rel-auto*)

lemma *RD2-R3h-commute*: $RD2(R3h(P)) = R3h(RD2(P))$
 by (*rel-auto*)

2.6 Major healthiness conditions

definition $RH :: ('t::trace, 'α) hrel-rp \Rightarrow ('t, 'α) hrel-rp$ (**R**)
 where *[upred-defs]*: $RH(P) = R1(R2c(R3c(P)))$

definition $RHS :: ('s, 't :: trace, 'α) hrel-rsp \Rightarrow ('s, 't, 'α) hrel-rsp \text{ (}\mathbf{R}_s\text{)}$
where $[upred-defs]: RHS(P) = R1(R2c(R3h(P)))$

definition $RD :: ('t :: trace, 'α) hrel-rp \Rightarrow ('t, 'α) hrel-rp$
where $[upred-defs]: RD(P) = RD1(RD2(RP(P)))$

definition $SRD :: ('s, 't :: trace, 'α) hrel-rsp \Rightarrow ('s, 't, 'α) hrel-rsp$
where $[upred-defs]: SRD(P) = RD1(RD2(RHS(P)))$

lemma $RH\text{-comp}: RH = R1 \circ R2c \circ R3c$
by $(auto \text{ simp add: } RH\text{-def})$

lemma $RHS\text{-comp}: RHS = R1 \circ R2c \circ R3h$
by $(auto \text{ simp add: } RHS\text{-def})$

lemma $RD\text{-comp}: RD = RD1 \circ RD2 \circ RP$
by $(auto \text{ simp add: } RD\text{-def})$

lemma $SRD\text{-comp}: SRD = RD1 \circ RD2 \circ RHS$
by $(auto \text{ simp add: } SRD\text{-def})$

lemma $RH\text{-idem}: \mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$
by $(simp \text{ add: } R1\text{-}R2c\text{-commute } R1\text{-}R3c\text{-commute } R1\text{-idem } R2c\text{-}R3c\text{-commute } R2c\text{-idem } R3c\text{-idem } RH\text{-def})$

lemma $RH\text{-Idempotent}: \text{Idempotent } \mathbf{R}$
by $(simp \text{ add: } Idempotent\text{-def } RH\text{-idem})$

lemma $RH\text{-Monotonic}: \text{Monotonic } \mathbf{R}$
by $(metis \text{ (no-types, lifting) } R1\text{-Monotonic } R2c\text{-Monotonic } R3c\text{-mono } RH\text{-def mono-def})$

lemma $RH\text{-Continuous}: \text{Continuous } \mathbf{R}$
by $(simp \text{ add: } Continuous\text{-comp } R1\text{-Continuous } R2c\text{-Continuous } R3c\text{-Continuous } RH\text{-comp})$

lemma $RHS\text{-idem}: \mathbf{R}_s(\mathbf{R}_s(P)) = \mathbf{R}_s(P)$
by $(simp \text{ add: } R1\text{-}R2c\text{-is-}R2 \text{ } R1\text{-}R3h\text{-commute } R2\text{-idem } R2c\text{-}R3h\text{-commute } R3h\text{-idem } RHS\text{-def})$

lemma $RHS\text{-Idempotent [closure]}: \text{Idempotent } \mathbf{R}_s$
by $(simp \text{ add: } Idempotent\text{-def } RHS\text{-idem})$

lemma $RHS\text{-Monotonic}: \text{Monotonic } \mathbf{R}_s$
by $(simp \text{ add: } mono\text{-def } R1\text{-}R2c\text{-is-}R2 \text{ } R2\text{-mono } R3h\text{-mono } RHS\text{-def})$

lemma $RHS\text{-mono}: P \sqsubseteq Q \implies \mathbf{R}_s(P) \sqsubseteq \mathbf{R}_s(Q)$
using $mono\text{-def } RHS\text{-Monotonic by blast}$

lemma $RHS\text{-Continuous [closure]}: \text{Continuous } \mathbf{R}_s$
by $(simp \text{ add: } Continuous\text{-comp } R1\text{-Continuous } R2c\text{-Continuous } R3h\text{-Continuous } RHS\text{-comp})$

lemma $RHS\text{-inf}: \mathbf{R}_s(P \sqcap Q) = \mathbf{R}_s(P) \sqcap \mathbf{R}_s(Q)$
using $Continuous\text{-Disjunctuous } Disjunctuous\text{-def } RHS\text{-Continuous by auto}$

lemma $RHS\text{-INF}: A \neq \{\} \implies \mathbf{R}_s(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot \mathbf{R}_s(P(i)))$

by (simp add: RHS-def R3h-UINF R2c-USUP R1-USUP)

lemma *RHS-sup*: $\mathbf{R}_s(P \sqcup Q) = \mathbf{R}_s(P) \sqcup \mathbf{R}_s(Q)$
 by (rel-auto)

lemma *RHS-SUP*:
 $A \neq \{\} \implies \mathbf{R}_s(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot \mathbf{R}_s(P(i)))$
 by (rel-auto)

lemma *RHS-cond*: $\mathbf{R}_s(P \triangleleft b \triangleright Q) = (\mathbf{R}_s(P) \triangleleft R2c\ b \triangleright \mathbf{R}_s(Q))$
 by (simp add: RHS-def R3h-cond R2c-condr R1-cond)

lemma *RD-alt-def*: $RD(P) = RD1(RD2(\mathbf{R}(P)))$
 by (simp add: R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute
 RH-def RD-def RP-def)

lemma *RD1-RH-commute*: $RD1(\mathbf{R}(P)) = \mathbf{R}(RD1(P))$
 by (simp add: RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RH-def)

lemma *RD2-RH-commute*: $RD2(\mathbf{R}(P)) = \mathbf{R}(RD2(P))$
 by (metis R1-H2-commute R2c-H2-commute RD2-R3c-commute RD2-def RH-def)

lemma *RD-idem*: $RD(RD(P)) = RD(P)$
 by (simp add: RD-alt-def RD1-RH-commute RD2-RH-commute RD1-RD2-commute RD2-idem RD1-idem
 RH-idem)

lemma *RD-Monotonic*: *Monotonic RD*
 by (simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic RD-comp RP-Monotonic)

lemma *RD-Continuous*: *Continuous RD*
 by (simp add: Continuous-comp RD1-Continuous RD2-Continuous RD-comp RP-Continuous)

lemma *R3-RD-RP*: $R3(RD(P)) = RP(RD1(RD2(P)))$
 by (metis (no-types, lifting) R1-R2c-is-R2 R2-R3-commute R3-cancels-R3c RD1-RH-commute RD2-RH-commute
 RD-alt-def RH-def RP-def)

lemma *RD1-RHS-commute*: $RD1(\mathbf{R}_s(P)) = \mathbf{R}_s(RD1(P))$
 by (simp add: RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RHS-def)

lemma *RD2-RHS-commute*: $RD2(\mathbf{R}_s(P)) = \mathbf{R}_s(RD2(P))$
 by (metis R1-H2-commute R2c-H2-commute RD2-R3h-commute RD2-def RHS-def)

lemma *SRD-idem*: $SRD(SRD(P)) = SRD(P)$
 by (simp add: RD1-RD2-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RHS-idem
 SRD-def)

lemma *SRD-Idempotent [closure]*: *Idempotent SRD*
 by (simp add: Idempotent-def SRD-idem)

lemma *SRD-Monotonic*: *Monotonic SRD*
 by (simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic RHS-Monotonic SRD-comp)

lemma *SRD-Continuous [closure]*: *Continuous SRD*
 by (simp add: Continuous-comp RD1-Continuous RD2-Continuous RHS-Continuous SRD-comp)

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*rel-auto*)

lemma *SRD-healths [closure]*:

assumes P is *SRD*
shows P is *R1* P is *R2* P is *R3h* P is *RD1* P is *RD2*
apply (*metis Healthy-def R1-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def R1-R2c-is-R2 R2-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def R1-R3h-commute R2c-R3h-commute R3h-idem RD1-R3h-commute RD2-R3h-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def' RD1-idem SRD-def assms*)
apply (*metis Healthy-def' RD1-RD2-commute RD2-idem SRD-def assms*)
done

lemma *SRD-intro*:

assumes P is *R1* P is *R2* P is *R3h* P is *RD1* P is *RD2*
shows P is *SRD*
by (*metis Healthy-def R1-R2c-is-R2 RHS-def SRD-def assms(2) assms(3) assms(4) assms(5)*)

lemma *SRD-ok-false [usubst]*: P is *SRD* $\implies P \llbracket \text{false}/\$ok \rrbracket = R1(\text{true})$

by (*metis (no-types, hide-lams) H1-H2-eq-design Healthy-def R1-ok-false RD1-R1-commute RD1-via-R1 RD2-def SRD-def SRD-healths(1) design-ok-false*)

lemma *SRD-ok-true-wait-true [usubst]*:

assumes P is *SRD*
shows $P \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$
proof –
have $P = (\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P$
by (*metis Healthy-def R3h-cases SRD-healths(3) assms*)
moreover have $((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$
by (*simp add: usubst*)
ultimately show *?thesis*
by (*simp*)
qed

lemma *SRD-left-zero-1*: P is *SRD* $\implies R1(\text{true}) ;; P = R1(\text{true})$

by (*simp add: RD1-left-zero SRD-healths(1) SRD-healths(4)*)

lemma *SRD-left-zero-2*:

assumes P is *SRD*
shows $(\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket ;; P = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$
proof –
have $(\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket ;; R3h(P) = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$
by (*rel-auto*)
thus *?thesis*
by (*simp add: Healthy-if SRD-healths(3) assms*)
qed

2.7 UTP theories

We create two theory objects: one for reactive designs and one for stateful reactive designs.

typedecl *RDES*

typedecl *SRDES*

abbreviation $RDES \equiv UTHY(RDES, ('t::trace, 'α) rp)$
abbreviation $SRDES \equiv UTHY(SRDES, ('s, 't::trace, 'α) rsp)$

overloading

$rdes-hcond == utp-hcond :: (RDES, ('t::trace, 'α) rp) uthy \Rightarrow (('t, 'α) rp \times ('t, 'α) rp) health$
 $srdes-hcond == utp-hcond :: (SRDES, ('s, 't::trace, 'α) rsp) uthy \Rightarrow (('s, 't, 'α) rsp \times ('s, 't, 'α) rsp) health$

begin

definition $rdes-hcond :: (RDES, ('t::trace, 'α) rp) uthy \Rightarrow (('t, 'α) rp \times ('t, 'α) rp) health$ **where**
 $[upred-defs]: rdes-hcond T = RD$

definition $srdes-hcond :: (SRDES, ('s, 't::trace, 'α) rsp) uthy \Rightarrow (('s, 't, 'α) rsp \times ('s, 't, 'α) rsp) health$
where

$[upred-defs]: srdes-hcond T = SRD$

end

interpretation $rdes-theory: utp-theory UTHY(RDES, ('t::trace, 'α) rp)$
by $(unfold-locales, simp-all add: rdes-hcond-def RD-idem)$

interpretation $rdes-theory-continuous: utp-theory-continuous UTHY(RDES, ('t::trace, 'α) rp)$

rewrites $\bigwedge P. P \in carrier (uthy-order RDES) \longleftrightarrow P \text{ is } RD$

and $carrier (uthy-order RDES) \rightarrow carrier (uthy-order RDES) \equiv \llbracket RD \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

and $le (uthy-order RDES) = op \sqsubseteq$

and $eq (uthy-order RDES) = op =$

by $(unfold-locales, simp-all add: rdes-hcond-def RD-Continuous)$

interpretation $rdes-rea-galois:$

$galois-connection (RDES \leftarrow \langle RD1 \circ RD2, R3 \rangle \rightarrow REA)$

proof $(simp add: mk-conn-def, rule galois-connectionI', simp-all add: utp-partial-order rdes-hcond-def rea-hcond-def)$

show $R3 \in \llbracket RD \rrbracket_H \rightarrow \llbracket RP \rrbracket_H$

by $(metis (no-types, lifting) Healthy-def' Pi-I R3-RD-RP RP-idem mem-Collect-eq)$

show $RD1 \circ RD2 \in \llbracket RP \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

by $(simp add: Pi-iff Healthy-def, metis RD-def RD-idem)$

show $isotone (utp-order RD) (utp-order RP) R3$

by $(simp add: R3-Monotonic isotone-utp-orderI)$

show $isotone (utp-order RP) (utp-order RD) (RD1 \circ RD2)$

by $(simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic isotone-utp-orderI)$

fix $P :: ('a, 'b) hrel-rp$

assume $P \text{ is } RD$

thus $P \sqsubseteq RD1 (RD2 (R3 P))$

by $(metis Healthy-if R3-RD-RP RD-def RP-idem eq-iff)$

next

fix $P :: ('a, 'b) hrel-rp$

assume $a: P \text{ is } RP$

thus $R3 (RD1 (RD2 P)) \sqsubseteq P$

proof –

have $R3 (RD1 (RD2 P)) = RP (RD1 (RD2(P)))$

by $(metis Healthy-if R3-RD-RP RD-def a)$

moreover have $RD1(RD2(P)) \sqsubseteq P$

by $(rel-auto)$

ultimately show $?thesis$

by $(metis Healthy-if RP-mono a)$

qed

qed

interpretation *rdes-rea-retract*:

retract ($RDES \leftarrow \langle RD1 \circ RD2, R3 \rangle \rightarrow REA$)

by (*unfold-locales*, *simp-all* *add*: *mk-conn-def* *utp-partial-order* *rdes-hcond-def* *rea-hcond-def*)
(metis Healthy-if R3-RD-RP RD-def RP-idem eq-refl)

interpretation *srdes-theory*: *utp-theory* *UTHY*(*SRDES*, (*'s, 't::trace, 'α*) *rsp*)

by (*unfold-locales*, *simp-all* *add*: *srdes-hcond-def* *SRD-idem*)

interpretation *srdes-theory-continuous*: *utp-theory-continuous* *UTHY*(*SRDES*, (*'s, 't::trace, 'α*) *rsp*)

rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } SRDES) \longleftrightarrow P \text{ is } SRD$

and $P \text{ is } \mathcal{H}_{SRDES} \longleftrightarrow P \text{ is } SRD$

and $(\mu X \cdot F (\mathcal{H}_{SRDES} X)) = (\mu X \cdot F (SRD X))$

and $\text{carrier } (\text{uthy-order } SRDES) \rightarrow \text{carrier } (\text{uthy-order } SRDES) \equiv \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$

and $\llbracket \mathcal{H}_{SRDES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{SRDES} \rrbracket_H \equiv \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$

and $le (\text{uthy-order } SRDES) = op \sqsubseteq$

and $eq (\text{uthy-order } SRDES) = op =$

by (*unfold-locales*, *simp-all* *add*: *srdes-hcond-def* *SRD-Continuous*)

declare *srdes-theory-continuous.top-healthy* [*simp del*]

declare *srdes-theory-continuous.bottom-healthy* [*simp del*]

abbreviation *Chaos* :: (*'s, 't::trace, 'α*) *hrel-rsp* **where**

Chaos $\equiv \perp_{SRDES}$

abbreviation *Miracle* :: (*'s, 't::trace, 'α*) *hrel-rsp* **where**

Miracle $\equiv \top_{SRDES}$

thm *srdes-theory-continuous.weak.bottom-lower*

thm *srdes-theory-continuous.weak.top-higher*

thm *srdes-theory-continuous.meet-bottom*

thm *srdes-theory-continuous.meet-top*

abbreviation *srd-lfp* (μ_R) **where** $\mu_R F \equiv \mu_{SRDES} F$

abbreviation *srd-gfp* (ν_R) **where** $\nu_R F \equiv \nu_{SRDES} F$

syntax

-srd-mu :: *pttrn* \Rightarrow *logic* \Rightarrow *logic* ($\mu_R \cdot \cdot \cdot [0, 10] 10$)

-srd-nu :: *pttrn* \Rightarrow *logic* \Rightarrow *logic* ($\nu_R \cdot \cdot \cdot [0, 10] 10$)

translations

$\mu_R X \cdot P == \mu_R (\lambda X. P)$

$\nu_R X \cdot P == \mu_R (\lambda X. P)$

The reactive design weakest fixed-point can be defined in terms of relational calculus one.

lemma *srd-mu-equiv*:

assumes *Monotonic* $F F \in \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$

shows $(\mu_R X \cdot F(X)) = (\mu X \cdot F(SRD(X)))$

by (*metis* *assms* *srdes-hcond-def* *srdes-theory-continuous.utp-lfp-def*)

end

3 Reactive Design Specifications

```
theory utp-rdes-designs
  imports utp-rdes-healths
begin
```

3.1 Reactive design forms

```
lemma srdes-skip-def:  $II_R = \mathbf{R}_s(\text{true} \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil II \rceil_R))$ 
  apply (rel-auto) using minus-zero-eq by blast+
```

```
lemma Chaos-def:  $\text{Chaos} = \mathbf{R}_s(\text{false} \vdash \text{true})$ 
```

```
proof -
```

```
  have  $\text{Chaos} = \text{SRD}(\text{true})$ 
    by (metis srdes-hcond-def srdes-theory-continuous.healthy-bottom)
  also have  $\dots = \mathbf{R}_s(\mathbf{H}(\text{true}))$ 
    by (simp add: SRD-RHS-H1-H2)
  also have  $\dots = \mathbf{R}_s(\text{false} \vdash \text{true})$ 
    by (metis H1-design H2-true design-false-pre)
  finally show ?thesis .
```

```
qed
```

```
lemma Miracle-def:  $\text{Miracle} = \mathbf{R}_s(\text{true} \vdash \text{false})$ 
```

```
proof -
```

```
  have  $\text{Miracle} = \text{SRD}(\text{false})$ 
    by (metis srdes-hcond-def srdes-theory-continuous.healthy-top)
  also have  $\dots = \mathbf{R}_s(\mathbf{H}(\text{false}))$ 
    by (simp add: SRD-RHS-H1-H2)
  also have  $\dots = \mathbf{R}_s(\text{true} \vdash \text{false})$ 
    by (metis (no-types, lifting) H1-H2-eq-design p-imp-p subst-impl subst-not utp-pred-laws.compl-bot-eq
      utp-pred-laws.compl-top-eq)
  finally show ?thesis .
```

```
qed
```

```
lemma RD1-reactive-design:  $\text{RD1}(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$ 
  by (rel-auto)
```

```
lemma RD2-reactive-design:
```

```
  assumes  $\$ok' \# P \ \$ok' \# Q$ 
  shows  $\text{RD2}(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$ 
  using assms
  by (metis H2-design RD2-RH-commute RD2-def)
```

```
lemma RD1-st-reactive-design:  $\text{RD1}(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$ 
  by (rel-auto)
```

```
lemma RD2-st-reactive-design:
```

```
  assumes  $\$ok' \# P \ \$ok' \# Q$ 
  shows  $\text{RD2}(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$ 
  using assms
  by (metis H2-design RD2-RHS-commute RD2-def)
```

```
lemma wait-false-design:
```

```
   $(P \vdash Q)_f = ((P_f) \vdash (Q_f))$ 
  by (rel-auto)
```

lemma *RD-RH-design-form*:

$RD(P) = \mathbf{R}((\neg P^f_f) \vdash P^t_f)$

proof –

have $RD(P) = RD1(RD2(R1(R2c(R3c(P))))))$
 by (*simp add: RD-alt-def RH-def*)
 also have $\dots = RD1(H2(R1(R2s(R3c(P))))))$
 by (*simp add: R1-R2s-R2c RD2-def*)
 also have $\dots = RD1(R1(H2(R2s(R3c(P))))))$
 by (*simp add: R1-H2-commute*)
 also have $\dots = R1(H1(R1(H2(R2s(R3c(P))))))$
 by (*simp add: R1-idem RD1-via-R1*)
 also have $\dots = R1(H1(H2(R2s(R3c(R1(P))))))$
 by (*simp add: R1-H2-commute R1-R2c-commute R1-R2s-R2c R1-R3c-commute RD1-via-R1*)
 also have $\dots = R1(R2s(H1(H2(R3c(R1(P))))))$
 by (*simp add: R2s-H1-commute R2s-H2-commute*)
 also have $\dots = R1(R2s(H1(R3c(H2(R1(P))))))$
 by (*metis RD2-R3c-commute RD2-def*)
 also have $\dots = R2(R1(H1(R3c(H2(R1(P))))))$
 by (*metis R1-R2-commute R1-idem R2-def*)
 also have $\dots = R2(R3c(R1(\mathbf{H}(R1(P)))))$
 by (*simp add: R1-R3c-commute RD1-R3c-commute RD1-via-R1*)
 also have $\dots = RH(\mathbf{H}(R1(P)))$
 by (*metis R1-R2s-R2c R1-R3c-commute R2-R1-form RH-def*)
 also have $\dots = RH(\mathbf{H}(P))$
 by (*simp add: R1-H2-commute R1-R2c-commute R1-R3c-commute R1-idem RD1-via-R1 RH-def*)
 also have $\dots = RH((\neg P^f_f) \vdash P^t_f)$
 by (*simp add: H1-H2-eq-design*)
 also have $\dots = \mathbf{R}((\neg P^f_f) \vdash P^t_f)$
 by (*metis (no-types, lifting) R3c-subst-wait RH-def subst-not wait-false-design*)
 finally show *?thesis* .

qed

lemma *RD-reactive-design*:

assumes *P is RD*

shows $\mathbf{R}((\neg P^f_f) \vdash P^t_f) = P$

by (*metis RD-RH-design-form Healthy-def' assms*)

lemma *RD-RH-design*:

assumes $\$ok' \# P \ \$ok' \# Q$

shows $RD(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$

by (*simp add: RD1-reactive-design RD2-reactive-design RD-alt-def RH-idem assms(1) assms(2)*)

lemma *RH-design-is-RD*:

assumes $\$ok' \# P \ \$ok' \# Q$

shows $\mathbf{R}(P \vdash Q)$ is RD

by (*simp add: RD-RH-design Healthy-def' assms(1) assms(2)*)

lemma *SRD-RH-design-form*:

$SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

proof –

have $SRD(P) = R1(R2c(R3h(RD1(RD2(R1(P))))))$
 by (*metis (no-types, lifting) R1-H2-commute R1-R2c-commute R1-R3h-commute R1-idem R2c-H2-commute RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RD2-R3h-commute RD2-def RHS-def SRD-def*)
 also have $\dots = R1(R2s(R3h(\mathbf{H}(P))))$
 by (*metis (no-types, lifting) R1-H2-commute R1-R2c-is-R2 R1-R3h-commute R2-R1-form RD1-via-R1*)

$RD2\text{-def}$
also have $\dots = \mathbf{R}_s(\mathbf{H}(P))$
by (*simp add: R1-R2s-R2c RHS-def*)
also have $\dots = \mathbf{R}_s((\neg P^f) \vdash P^t)$
by (*simp add: H1-H2-eq-design*)
also have $\dots = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$
by (*metis (no-types, lifting) R3h-subst-wait RHS-def subst-not wait-false-design*)
finally show *?thesis* .
qed

lemma *SRD-reactive-design*:
assumes P is *SRD*
shows $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = P$
by (*metis SRD-RH-design-form Healthy-def' assms*)

lemma *SRD-RH-design*:
assumes $\$ok' \# P \ \$ok' \# Q$
shows $SRD(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
by (*simp add: RD1-st-reactive-design RD2-st-reactive-design RHS-idem SRD-def assms(1) assms(2)*)

lemma *RHS-design-is-SRD*:
assumes $\$ok' \# P \ \$ok' \# Q$
shows $\mathbf{R}_s(P \vdash Q)$ is *SRD*
by (*simp add: Healthy-def' SRD-RH-design assms(1) assms(2)*)

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*metis (no-types, lifting) H1-H2-eq-design R3h-subst-wait RHS-def SRD-RH-design-form subst-not wait-false-design*)

3.2 Auxiliary healthiness conditions

definition [*upred-defs*]: $R3c\text{-pre}(P) = (true \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: $R3c\text{-post}(P) = ([II]_D \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: $R3h\text{-post}(P) = ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright P)$

lemma *R3c-pre-conj*: $R3c\text{-pre}(P \wedge Q) = (R3c\text{-pre}(P) \wedge R3c\text{-pre}(Q))$
by (*rel-auto*)

lemma *R3c-pre-seq*:
 $(true ;; Q) = true \implies R3c\text{-pre}(P ;; Q) = (R3c\text{-pre}(P) ;; Q)$
by (*rel-auto*)

lemma *unrest-ok-R3c-pre* [*unrest*]: $\$ok \# P \implies \$ok \# R3c\text{-pre}(P)$
by (*simp add: R3c-pre-def cond-def unrest*)

lemma *unrest-ok'-R3c-pre* [*unrest*]: $\$ok' \# P \implies \$ok' \# R3c\text{-pre}(P)$
by (*simp add: R3c-pre-def cond-def unrest*)

lemma *unrest-ok-R3c-post* [*unrest*]: $\$ok \# P \implies \$ok \# R3c\text{-post}(P)$
by (*simp add: R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3c-post'* [*unrest*]: $\$ok' \# P \implies \$ok' \# R3c\text{-post}(P)$
by (*simp add: R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post* [*unrest*]: $\$ok \# P \implies \$ok \# R3h\text{-post}(P)$
 by (*simp add: R3h-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post'* [*unrest*]: $\$ok' \# P \implies \$ok' \# R3h\text{-post}(P)$
 by (*simp add: R3h-post-def cond-def unrest*)

3.3 Composition laws

theorem *R1-design-composition*:

fixes $P\ Q :: ('t::\text{trace}, 'a, 'b) \text{ rel-rp}$

and $R\ S :: ('t, 'b, 'c) \text{ rel-rp}$

assumes $\$ok' \# P\ \$ok' \# Q\ \$ok \# R\ \$ok \# S$

shows

$(R1(P \vdash Q) ;; R1(R \vdash S)) =$

$R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$

proof –

have $(R1(P \vdash Q) ;; R1(R \vdash S)) = (\exists\ ok_0 \cdot (R1(P \vdash Q)) \llbracket \llcorner ok_0 \rceil / \$ok' \rrbracket ;; (R1(R \vdash S)) \llbracket \llcorner ok_0 \rceil / \$ok \rrbracket)$

using *segr-middle ok-vwb-lens* **by** *blast*

also from *assms* **have** $\dots = (\exists\ ok_0 \cdot R1((\$ok \wedge P) \Rightarrow (\llcorner ok_0 \rceil \wedge Q)) ;; R1((\llcorner ok_0 \rceil \wedge R) \Rightarrow (\$ok' \wedge S)))$

by (*simp add: design-def R1-def usubst unrest*)

also from *assms* **have** $\dots = ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) ;; R1((true \wedge R) \Rightarrow (\$ok' \wedge S)))$
 $\vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) ;; R1((false \wedge R) \Rightarrow (\$ok' \wedge S))))$

by (*simp add: false-alt-def true-alt-def*)

also from *assms* **have** $\dots = ((R1((\$ok \wedge P) \Rightarrow Q) ;; R1(R \Rightarrow (\$ok' \wedge S)))$
 $\vee (R1(\neg (\$ok \wedge P)) ;; R1(true)))$

by *simp*

also from *assms* **have** $\dots = ((R1(\neg \$ok \vee \neg P \vee Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
 $\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*simp add: impl-alt-def utp-pred-laws.sup.assoc*)

also from *assms* **have** $\dots = (((R1(\neg \$ok \vee \neg P) \vee R1(Q)) ;; R1(\neg R \vee (\$ok' \wedge S)))$
 $\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*simp add: R1-disj utp-pred-laws.disj-assoc*)

also from *assms* **have** $\dots = ((R1(\neg \$ok \vee \neg P) ;; R1(\neg R \vee (\$ok' \wedge S)))$
 $\vee (R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
 $\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*simp add: segr-or-distl utp-pred-laws.sup.assoc*)

also from *assms* **have** $\dots = ((R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$
 $\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*rel-blast*)

also from *assms* **have** $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
 $\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*simp add: R1-disj R1-extend-conj utp-pred-laws.inf-commute*)

also have $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
 $\vee ((R1(\neg \$ok) :: ('t, 'a, 'b) \text{ rel-rp}) ;; R1(true)))$
 $\vee (R1(\neg P) ;; R1(true)))$

by (*simp add: R1-disj segr-or-distl*)

also have $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$
 $\vee (R1(\neg \$ok))$
 $\vee (R1(\neg P) ;; R1(true)))$

proof –

have $((R1(\neg \$ok) :: ('t, 'a, 'b) \text{ rel-rp}) ;; R1(true)) =$
 $(R1(\neg \$ok) :: ('t, 'a, 'c) \text{ rel-rp})$

by (*rel-auto*)

thus *?thesis*

by *simp*

qed
also have ... = (($R1(Q) ;; (R1(\neg R) \vee (R1(S \wedge \$ok')))$)
 $\vee R1(\neg \$ok)$
 $\vee (R1(\neg P) ;; R1(true))$)
by (simp add: R1-extend-conj)
also have ... = ($R1(Q) ;; (R1(\neg R))$)
 $\vee (R1(Q) ;; (R1(S \wedge \$ok')))$
 $\vee R1(\neg \$ok)$
 $\vee (R1(\neg P) ;; R1(true))$
by (simp add: segr-or-distr utp-pred-laws.sup.assoc)
also have ... = $R1(R1(Q) ;; (R1(\neg R)))$
 $\vee (R1(Q) ;; (R1(S \wedge \$ok')))$
 $\vee (\neg \$ok)$
 $\vee (R1(\neg P) ;; R1(true))$
by (simp add: R1-disj R1-segr)
also have ... = $R1(R1(Q) ;; (R1(\neg R)))$
 $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$
 $\vee (\neg \$ok)$
 $\vee (R1(\neg P) ;; R1(true))$
by (rel-blast)
also have ... = $R1(\neg(\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$
 $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$
by (rel-blast)
also have ... = $R1(\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$
 $\Rightarrow (\$ok' \wedge (R1(Q) ;; R1(S)))$
by (simp add: impl-alt-def utp-pred-laws.inf-commute)
also have ... = $R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$
by (simp add: design-def)
finally show ?thesis .
qed

theorem R1-design-composition-RR:
assumes P is RR Q is RR R is RR S is RR
shows
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$
apply (subst R1-design-composition)
apply (simp-all add: assms unrest wp-rea-def Healthy-if closure)
apply (rel-auto)
done

theorem R1-design-composition-RC:
assumes P is RC Q is RR R is RR S is RR
shows
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$
by (simp add: R1-design-composition-RR assms unrest Healthy-if closure wp)

lemma R2s-design: $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$
by (simp add: R2s-def design-def usubst)

lemma R2c-design: $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$
by (simp add: design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok')

lemma R1-R3c-design:
 $R1(R3c(P \vdash Q)) = R1(R3c\text{-pre}(P) \vdash R3c\text{-post}(Q))$
by (rel-auto)

lemma *R1-R3h-design:*

$R1(R3h(P \vdash Q)) = R1(R3c-pre(P) \vdash R3h-post(Q))$
 by (rel-auto)

lemma *R3c-R1-design-composition:*

assumes $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$
 shows $(R3c(R1(P \vdash Q)) ;; R3c(R1(R \vdash S))) =$
 $R3c(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$
 $\vdash (R1(Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1(S))))$

proof –

have 1: $(\neg (R1(\neg R3c-pre P) ;; R1 true)) = (R3c-pre(\neg (R1(\neg P) ;; R1 true)))$
 by (rel-auto)
 have 2: $(\neg (R1(R3c-post Q) ;; R1(\neg R3c-pre R))) = R3c-pre(\neg ((R1 Q \wedge \neg \$wait') ;; R1(\neg R)))$
 by (rel-auto, blast+)
 have 3: $(R1(R3c-post Q) ;; R1(R3c-post S)) = R3c-post(R1 Q ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 S))$
 by (rel-auto)
 show ?thesis
 apply (simp add: R3c-semir-form R1-R3c-commute[THEN sym] R1-R3c-design unrest)
 apply (subst R1-design-composition)
 apply (simp-all add: unrest assms R3c-pre-conj 1 2 3)
 done
qed

lemma *R3h-R1-design-composition:*

assumes $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$
 shows $(R3h(R1(P \vdash Q)) ;; R3h(R1(R \vdash S))) =$
 $R3h(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$
 $\vdash (R1(Q) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(S))))$

proof –

have 1: $(\neg (R1(\neg R3c-pre P) ;; R1 true)) = (R3c-pre(\neg (R1(\neg P) ;; R1 true)))$
 by (rel-auto)
 have 2: $(\neg (R1(R3h-post Q) ;; R1(\neg R3c-pre R))) = R3c-pre(\neg ((R1 Q \wedge \neg \$wait') ;; R1(\neg R)))$
 by (rel-auto, blast+)
 have 3: $(R1(R3h-post Q) ;; R1(R3h-post S)) = R3h-post(R1 Q ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 S))$
 by (rel-auto, blast+)
 show ?thesis
 apply (simp add: R3h-semir-form R1-R3h-commute[THEN sym] R1-R3h-design unrest)
 apply (subst R1-design-composition)
 apply (simp-all add: unrest assms R3c-pre-conj 1 2 3)
 done
qed

lemma *R2-design-composition:*

assumes $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$
 shows $(R2(P \vdash Q) ;; R2(R \vdash S)) =$
 $R2((\neg (R1(\neg R2c P) ;; R1 true) \wedge \neg (R1(R2c Q) ;; R1(\neg R2c R))) \vdash (R1(R2c Q) ;; R1(R2c S)))$
 apply (simp add: R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj
 R1-R2c-commute[THEN sym] R2c-idem R2c-R1-seq)
 apply (metis (no-types, lifting) R2c-R1-seq R2c-not R2c-true)
 done

lemma *RH-design-composition:*

assumes $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$

shows $(RH(P \vdash Q) ;; RH(R \vdash S)) =$
 $RH((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge (\neg \$wait')) ;; R1 (\neg R2s R))) \vdash$
 $(R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))))$

proof –

have 1: $R2c(R1 (\neg R2s P) ;; R1 true) = (R1 (\neg R2s P) ;; R1 true)$

proof –

have 1: $(R1 (\neg R2s P) ;; R1 true) = (R1(R2 (\neg P) ;; R2 true))$
by (rel-auto)

have $R2c(R1(R2 (\neg P) ;; R2 true)) = R2c(R1(R2 (\neg P) ;; R2 true))$
using R2c-not by blast

also have $\dots = R2(R2 (\neg P) ;; R2 true)$
by (metis R1-R2c-commute R1-R2c-is-R2)

also have $\dots = (R2 (\neg P) ;; R2 true)$
by (simp add: R2-seqr-distribute)

also have $\dots = (R1 (\neg R2s P) ;; R1 true)$
by (simp add: R2-def R2s-not R2s-true)

finally show ?thesis
by (simp add: 1)

qed

have 2: $R2c((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$

proof –

have $((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = R1 (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$
by (rel-auto)

hence $R2c((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$
by (metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute)

also have $\dots = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$
by (rel-auto)

finally show ?thesis .

qed

have 3: $R2c((R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S)))) = (R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S)))$

proof –

have $R2c(((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket))$
 $= ((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$

proof –

have $R2c(((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)) =$
 $R2c(R1 (R2s (Q\llbracket true/\$wait' \rrbracket)) ;; \lceil II \rceil_D \llbracket true/\$wait \rrbracket)$
by (simp add: usubst cond-unit-T R1-def R2s-def)

also have $\dots = R2c(R2(Q\llbracket true/\$wait' \rrbracket) ;; R2(\lceil II \rceil_D \llbracket true/\$wait \rrbracket))$
by (metis R2-def R2-des-lift-skip R2-subst-wait-true)

also have $\dots = (R2(Q\llbracket true/\$wait' \rrbracket) ;; R2(\lceil II \rceil_D \llbracket true/\$wait \rrbracket))$
using R2c-seq by blast

also have $\dots = ((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$
apply (simp add: usubst R2-des-lift-skip)

apply (metis R2-def R2-des-lift-skip R2-subst-wait'-true R2-subst-wait-true)

done

finally show ?thesis .

qed

moreover have $R2c(((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket))$
 $= ((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket)$
by (simp add: usubst cond-unit-F)

(metis (no-types, hide-lams) R1-wait'-false R1-wait-false R2-def R2-subst-wait'-false R2-subst-wait-false)

R2c-seq)

ultimately show *?thesis*

proof –

have $\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S) = R2 (\lceil II \rceil_D \triangleleft \$wait \triangleright S)$

by (*simp add: R1-R2c-is-R2 R1-R2s-R2c R2-condr' R2-des-lift-skip R2s-wait*)

then show *?thesis*

by (*simp add: R1-R2c-is-R2 R1-R2s-R2c R2c-seq*)

qed

qed

have $(R1(R2s(R3c(P \vdash Q))) ;; R1(R2s(R3c(R \vdash S)))) =$

$((R3c(R1(R2s(P) \vdash R2s(Q)))) ;; R3c(R1(R2s(R) \vdash R2s(S))))$

by (*metis (no-types, hide-lams) R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design*)

also have $\dots = R3c (R1 ((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)))) \vdash$

$(R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))))$

by (*simp add: R3c-R1-design-composition assms unrest*)

also have $\dots = R3c(R1(R2c((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)))) \vdash$

$(R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))))$

by (*simp add: R2c-design R2c-and R2c-not 1 2 3*)

finally show *?thesis*

by (*simp add: R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-def*)

qed

lemma *RHS-design-composition:*

assumes $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$

$\mathbf{R}_s((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge (\neg \$wait')) ;; R1 (\neg R2s R))) \vdash$
 $(R1 (R2s Q) ;; ((\exists \$st \cdot \lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))))$

proof –

have $1: R2c (R1 (\neg R2s P) ;; R1 true) = (R1 (\neg R2s P) ;; R1 true)$

proof –

have $1: (R1 (\neg R2s P) ;; R1 true) = (R1(R2 (\neg P) ;; R2 true))$

by (*rel-auto, blast*)

have $R2c(R1(R2 (\neg P) ;; R2 true)) = R2c(R1(R2 (\neg P) ;; R2 true))$

using *R2c-not* **by** *blast*

also have $\dots = R2(R2 (\neg P) ;; R2 true)$

by (*metis R1-R2c-commute R1-R2c-is-R2*)

also have $\dots = (R2 (\neg P) ;; R2 true)$

by (*simp add: R2-seqr-distribute*)

also have $\dots = (R1 (\neg R2s P) ;; R1 true)$

by (*simp add: R2-def R2s-not R2s-true*)

finally show *?thesis*

by (*simp add: 1*)

qed

have $2: R2c ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$

proof –

have $((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = R1 (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$

by (*rel-auto, blast+*)

hence $R2c ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$

by (*metis (no-types, lifting) R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)

also have $\dots = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$

by (*rel-auto*, *blast+*)
 finally show ?thesis .
 qed

have $3: R2c((R1 (R2s Q)) \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))) =$
 $(R1 (R2s Q)) \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))$
 proof –
 have $R2c(((R1 (R2s Q))\llbracket true/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket))$
 $= ((R1 (R2s Q))\llbracket true/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$
 proof –
 have $R2c(((R1 (R2s Q))\llbracket true/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)) =$
 $R2c(R1 (R2s (Q\llbracket true/\$wait' \rrbracket)) \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$
 by (*simp add: usubst cond-unit-T R1-def R2s-def*)
 also have ... = $R2c(R2(Q\llbracket true/\$wait' \rrbracket) \;; R2((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket))$
 by (*metis (no-types, lifting) R2-def R2-des-lift-skip R2-subst-wait-true R2-st-ex*)
 also have ... = $(R2(Q\llbracket true/\$wait' \rrbracket) \;; R2((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket))$
 using *R2c-seq* by *blast*
 also have ... = $((R1 (R2s Q))\llbracket true/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$
 apply (*simp add: usubst R2-des-lift-skip*)
 apply (*metis (no-types) R2-def R2-des-lift-skip R2-st-ex R2-subst-wait'-true R2-subst-wait-true*)
 done
 finally show ?thesis .
 qed

moreover have $R2c(((R1 (R2s Q))\llbracket false/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket))$
 $= ((R1 (R2s Q))\llbracket false/\$wait' \rrbracket \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket)$
 by (*simp add: usubst*)
 (*metis (no-types, lifting) R1-wait'-false R1-wait-false R2-R1-form R2-subst-wait'-false R2-subst-wait-false*)
R2c-seq
 ultimately show ?thesis
 by (*smt R2-R1-form R2-condr' R2-des-lift-skip R2-st-ex R2c-seq R2s-wait*)
 qed

have $(R1(R2s(R3h(P \vdash Q))) \;; R1(R2s(R3h(R \vdash S)))) =$
 $((R3h(R1(R2s(P) \vdash R2s(Q)))) \;; R3h(R1(R2s(R) \vdash R2s(S))))$
 by (*metis (no-types, hide-lams) R1-R2s-R2c R1-R3h-commute R2c-R3h-commute R2s-design*)
 also have ... = $R3h(R1 ((\neg (R1 (\neg R2s P)) \;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait') \;; R1 (\neg R2s R))) \vdash$
 $(R1 (R2s Q) \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))))$
 by (*simp add: R3h-R1-design-composition assms unrest*)
 also have ... = $R3h(R1(R2c((\neg (R1 (\neg R2s P)) \;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait') \;; R1 (\neg R2s R))) \vdash$
 $(R1 (R2s Q) \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))))$
 by (*simp add: R2c-design R2c-and R2c-not 1 2 3*)
 finally show ?thesis
 by (*simp add: R1-R2s-R2c R1-R3h-commute R2c-R3h-commute RHS-def*)
 qed

lemma *RHS-R2s-design-composition:*
 assumes
 $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$
 $P \text{ is } R2s \ Q \text{ is } R2s \ R \text{ is } R2s \ S \text{ is } R2s$
 shows $(\mathbf{R}_s(P \vdash Q) \;; \mathbf{R}_s(R \vdash S)) =$
 $\mathbf{R}_s((\neg (R1 (\neg P)) \;; R1 true) \wedge \neg ((R1 Q \wedge \neg \$wait') \;; R1 (\neg R))) \vdash$
 $(R1 Q \;; (\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 S))$
 proof –

```

have f1: R2s P = P
  by (meson Healthy-def assms(5))
have f2: R2s Q = Q
  by (meson Healthy-def assms(6))
have f3: R2s R = R
  by (meson Healthy-def assms(7))
have R2s S = S
  by (meson Healthy-def assms(8))
then show ?thesis
  using f3 f2 f1 by (simp add: RHS-design-composition assms(1) assms(2) assms(3) assms(4))
qed

```

```

lemma RH-design-export-R1:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R1(Q))$ 
  by (rel-auto)

```

```

lemma RH-design-export-R2s:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2s(Q))$ 
  by (rel-auto)

```

```

lemma RH-design-export-R2c:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2c(Q))$ 
  by (rel-auto)

```

```

lemma RHS-design-export-R1:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R1(Q))$ 
  by (rel-auto)

```

```

lemma RHS-design-export-R2s:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2s(Q))$ 
  by (rel-auto)

```

```

lemma RHS-design-export-R2c:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2c(Q))$ 
  by (rel-auto)

```

```

lemma RHS-design-export-R2:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2(Q))$ 
  by (rel-auto)

```

```

lemma R1-design-R1-pre:
 $\mathbf{R}_s(R1(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$ 
  by (rel-auto)

```

```

lemma RHS-design-ok-wait:  $\mathbf{R}_s(P \llbracket true, false / \$ok, \$wait \rrbracket \vdash Q \llbracket true, false / \$ok, \$wait \rrbracket) = \mathbf{R}_s(P \vdash Q)$ 
  by (rel-auto)

```

```

lemma RHS-design-neg-R1-pre:
 $\mathbf{R}_s((\neg R1 P) \vdash R) = \mathbf{R}_s((\neg P) \vdash R)$ 
  by (rel-auto)

```

```

lemma RHS-design-conj-neg-R1-pre:
 $\mathbf{R}_s(((\neg R1 P) \wedge Q) \vdash R) = \mathbf{R}_s(((\neg P) \wedge Q) \vdash R)$ 
  by (rel-auto)

```

```

lemma RHS-pre-lemma:  $(\mathbf{R}_s P)^f_f = R1(R2c(P^f_f))$ 
  by (rel-auto)

```

```

lemma RHS-design-R2c-pre:
 $\mathbf{R}_s(R2c(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$ 
  by (rel-auto)

```


3.4 Refinement introduction laws

lemma *R1-design-refine*:

assumes

$P_1 \text{ is } R1 \ P_2 \text{ is } R1 \ Q_1 \text{ is } R1 \ Q_2 \text{ is } R1$
 $\$ok \# P_1 \ \$ok' \# P_1 \ \$ok \# P_2 \ \$ok' \# P_2$
 $\$ok \# Q_1 \ \$ok' \# Q_1 \ \$ok \# Q_2 \ \$ok' \# Q_2$

shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

proof –

have $R1((\exists \$ok; \$ok' \cdot P_1) \vdash (\exists \$ok; \$ok' \cdot P_2)) \sqsubseteq R1((\exists \$ok; \$ok' \cdot Q_1) \vdash (\exists \$ok; \$ok' \cdot Q_2))$
 $\longleftrightarrow 'R1(\exists \$ok; \$ok' \cdot P_1) \Rightarrow R1(\exists \$ok; \$ok' \cdot Q_1)' \wedge 'R1(\exists \$ok; \$ok' \cdot P_1) \wedge R1(\exists \$ok; \$ok'$

$\cdot Q_2) \Rightarrow R1(\exists \$ok; \$ok' \cdot P_2)'$

by (*rel-auto, meson+*)

thus *?thesis*

by (*simp-all add: ex-unrest ex-plus Healthy-if assms*)

qed

lemma *R1-design-refine-RR*:

assumes $P_1 \text{ is } RR \ P_2 \text{ is } RR \ Q_1 \text{ is } RR \ Q_2 \text{ is } RR$

shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

by (*simp add: R1-design-refine assms unrest closure*)

lemma *RHS-design-refine*:

assumes

$P_1 \text{ is } R1 \ P_2 \text{ is } R1 \ Q_1 \text{ is } R1 \ Q_2 \text{ is } R1$
 $P_1 \text{ is } R2c \ P_2 \text{ is } R2c \ Q_1 \text{ is } R2c \ Q_2 \text{ is } R2c$
 $\$ok \# P_1 \ \$ok' \# P_1 \ \$ok \# P_2 \ \$ok' \# P_2$
 $\$ok \# Q_1 \ \$ok' \# Q_1 \ \$ok \# Q_2 \ \$ok' \# Q_2$
 $\$wait \# P_1 \ \$wait \# P_2 \ \$wait \# Q_1 \ \$wait \# Q_2$

shows $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

proof –

have $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow R1(R3h(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3h(R2c(Q_1 \vdash Q_2)))$

by (*simp add: R2c-R3h-commute RHS-def*)

also have $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \sqsubseteq R1(R3h(Q_1 \vdash Q_2))$

by (*simp add: Healthy-if R2c-design assms*)

also have $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \llbracket false/\$wait \rrbracket \sqsubseteq R1(R3h(Q_1 \vdash Q_2)) \llbracket false/\$wait \rrbracket$

by (*rel-auto, metis+*)

also have $\dots \longleftrightarrow R1(P_1 \vdash P_2) \llbracket false/\$wait \rrbracket \sqsubseteq R1(Q_1 \vdash Q_2) \llbracket false/\$wait \rrbracket$

by (*rel-auto*)

also have $\dots \longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$

by (*simp add: usubst assms closure unrest*)

also have $\dots \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

by (*simp add: R1-design-refine assms*)

finally show *?thesis* .

qed

lemma *srdes-refine-intro*:

assumes $'P_1 \Rightarrow P_2' \wedge 'P_1 \wedge Q_2 \Rightarrow Q_1'$

shows $\mathbf{R}_s(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2)$

by (*simp add: RHS-mono assms design-refine-intro*)

3.5 Distribution laws

lemma *RHS-design-choice*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcap \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \vee Q_2))$

by (*metis RHS-inf design-choice*)

lemma *RHS-design-sup*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcup \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \vee P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
by (*metis RHS-sup design-inf*)

lemma *RHS-design-USUP*:

assumes $A \neq \{\}$

shows $(\prod i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\prod i \in A \cdot P(i)) \vdash (\prod i \in A \cdot Q(i)))$

by (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms*)

end

4 Reactive Design Triples

theory *utp-rdes-triples*

imports *utp-rdes-designs*

begin

4.1 Diamond notation

definition *wait'-cond* ::

$(t::trace, 'a, 'b) \text{ rel-rp} \Rightarrow (t, 'a, 'b) \text{ rel-rp} \Rightarrow (t, 'a, 'b) \text{ rel-rp}$ (**infixr** \diamond 65) **where**
[upred-defs]: $P \diamond Q = (P \triangleleft \$wait' \triangleright Q)$

lemma *wait'-cond-unrest* [*unrest*]:

$\llbracket \text{out-var wait} \bowtie x; x \# P; x \# Q \rrbracket \Longrightarrow x \# (P \diamond Q)$

by (*simp add: wait'-cond-def unrest*)

lemma *wait'-cond-subst* [*usubst*]:

$\$wait' \# \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$

by (*simp add: wait'-cond-def usubst unrest*)

lemma *wait'-cond-left-false*: $false \diamond P = (\neg \$wait' \wedge P)$

by (*rel-auto*)

lemma *wait'-cond-seq*: $((P \diamond Q) ;; R) = ((P ;; (\$wait' \wedge R)) \vee (Q ;; (\neg \$wait' \wedge R)))$

by (*simp add: wait'-cond-def cond-def segr-or-distl, rel-blast*)

lemma *wait'-cond-true*: $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$

by (*rel-auto*)

lemma *wait'-cond-false*: $(P \diamond Q \wedge (\neg \$wait')) = (Q \wedge (\neg \$wait'))$

by (*rel-auto*)

lemma *wait'-cond-idem*: $P \diamond P = P$

by (*rel-auto*)

lemma *wait'-cond-conj-exchange*:

$((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$

by (*rel-auto*)

lemma *subst-wait'-cond-true* [*usubst*]: $(P \diamond Q) \llbracket true / \$wait' \rrbracket = P \llbracket true / \$wait' \rrbracket$

by (*rel-auto*)

lemma *subst-wait'-cond-false* [*usubst*]: $(P \diamond Q) \llbracket false / \$wait' \rrbracket = Q \llbracket false / \$wait' \rrbracket$

by (*rel-auto*)

lemma *subst-wait'-left-subst*: $(P \llbracket \text{true} / \$\text{wait}' \rrbracket \diamond Q) = (P \diamond Q)$
by (*rel-auto*)

lemma *subst-wait'-right-subst*: $(P \diamond Q \llbracket \text{false} / \$\text{wait}' \rrbracket) = (P \diamond Q)$
by (*rel-auto*)

lemma *wait'-cond-split*: $P \llbracket \text{true} / \$\text{wait}' \rrbracket \diamond P \llbracket \text{false} / \$\text{wait}' \rrbracket = P$
by (*simp add: wait'-cond-def cond-var-split*)

lemma *wait-cond'-assoc* [*simp*]: $P \diamond Q \diamond R = P \diamond R$
by (*rel-auto*)

lemma *wait-cond'-shadow*: $(P \diamond Q) \diamond R = P \diamond Q \diamond R$
by (*rel-auto*)

lemma *wait-cond'-conj* [*simp*]: $P \diamond (Q \wedge (R \diamond S)) = P \diamond (Q \wedge S)$
by (*rel-auto*)

lemma *R1-wait'-cond*: $R1(P \diamond Q) = R1(P) \diamond R1(Q)$
by (*rel-auto*)

lemma *R2s-wait'-cond*: $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$
by (*simp add: wait'-cond-def R2s-def R2s-def usubst*)

lemma *R2-wait'-cond*: $R2(P \diamond Q) = R2(P) \diamond R2(Q)$
by (*simp add: R2-def R2s-wait'-cond R1-wait'-cond*)

lemma *wait'-cond-R1-closed* [*closure*]:
 $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies P \diamond Q \text{ is } R1$
by (*simp add: Healthy-def R1-wait'-cond*)

lemma *wait'-cond-R2c-closed* [*closure*]: $\llbracket P \text{ is } R2c; Q \text{ is } R2c \rrbracket \implies P \diamond Q \text{ is } R2c$
by (*simp add: R2c-condr wait'-cond-def Healthy-def, rel-auto*)

4.2 Export laws

lemma *RH-design-peri-R1*: $\mathbf{R}(P \vdash R1(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis (no-types, lifting) R1-idem R1-wait'-cond RH-design-export-R1*)

lemma *RH-design-post-R1*: $\mathbf{R}(P \vdash Q \diamond R1(R)) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis R1-wait'-cond RH-design-export-R1 RH-design-peri-R1*)

lemma *RH-design-peri-R2s*: $\mathbf{R}(P \vdash R2s(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s*)

lemma *RH-design-post-R2s*: $\mathbf{R}(P \vdash Q \diamond R2s(R)) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s*)

lemma *RH-design-peri-R2c*: $\mathbf{R}(P \vdash R2c(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis R1-R2s-R2c RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-peri-R1*: $\mathbf{R}_s(P \vdash R1(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis (no-types, lifting) R1-idem R1-wait'-cond RHS-design-export-R1*)

lemma *RHS-design-post-R1*: $\mathbf{R}_s(P \vdash Q \diamond R1(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R1-wait'-cond RHS-design-export-R1 RHS-design-peri-R1*)

lemma *RHS-design-peri-R2s*: $\mathbf{R}_s(P \vdash R2s(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
 by (metis (no-types, lifting) *R2s-idem R2s-wait'-cond RHS-design-export-R2s*)

lemma *RHS-design-post-R2s*: $\mathbf{R}_s(P \vdash Q \diamond R2s(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
 by (metis *R2s-wait'-cond RHS-design-export-R2s RHS-design-peri-R2s*)

lemma *RHS-design-peri-R2c*: $\mathbf{R}_s(P \vdash R2c(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
 by (metis *R1-R2s-R2c RHS-design-peri-R1 RHS-design-peri-R2s*)

lemma *RH-design-lemma1*:
 $RH(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RH(P \vdash (Q \vee R) \diamond S)$
 by (metis (no-types, lifting) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-lemma1*:
 $RHS(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RHS(P \vdash (Q \vee R) \diamond S)$
 by (metis (no-types, lifting) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RHS-design-peri-R1 RHS-design-peri-R2s*)

4.3 Pre-, peri-, and postconditions

4.3.1 Definitions

abbreviation $pre_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s false, \$wait \mapsto_s false]$
abbreviation $cmt_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false]$
abbreviation $peri_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true]$
abbreviation $post_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s false]$

abbreviation $npre_R(P) \equiv pre_s \dagger P$

definition [*upred-defs*]: $pre_R(P) = (\neg_r npre_R(P))$

definition [*upred-defs*]: $cmt_R(P) = R1(cmt_s \dagger P)$

definition [*upred-defs*]: $peri_R(P) = R1(peri_s \dagger P)$

definition [*upred-defs*]: $post_R(P) = R1(post_s \dagger P)$

4.3.2 Unrestriction laws

lemma *ok-pre-unrest* [*unrest*]: $\$ok \# pre_R P$
 by (simp add: *pre_R-def unrest usubst*)

lemma *ok-peri-unrest* [*unrest*]: $\$ok \# peri_R P$
 by (simp add: *peri_R-def unrest usubst*)

lemma *ok-post-unrest* [*unrest*]: $\$ok \# post_R P$
 by (simp add: *post_R-def unrest usubst*)

lemma *ok-cmt-unrest* [*unrest*]: $\$ok \# cmt_R P$
 by (simp add: *cmt_R-def unrest usubst*)

lemma *ok'-pre-unrest* [*unrest*]: $\$ok' \# pre_R P$
 by (simp add: *pre_R-def unrest usubst*)

lemma *ok'-peri-unrest* [*unrest*]: $\$ok' \# peri_R P$
 by (simp add: *peri_R-def unrest usubst*)

lemma *ok'-post-unrest* [*unrest*]: $\$ok' \# post_R P$
 by (*simp add: post_R-def unrest usubst*)

lemma *ok'-cmt-unrest* [*unrest*]: $\$ok' \# cmt_R P$
 by (*simp add: cmt_R-def unrest usubst*)

lemma *wait-pre-unrest* [*unrest*]: $\$wait \# pre_R P$
 by (*simp add: pre_R-def unrest usubst*)

lemma *wait-peri-unrest* [*unrest*]: $\$wait \# peri_R P$
 by (*simp add: peri_R-def unrest usubst*)

lemma *wait-post-unrest* [*unrest*]: $\$wait \# post_R P$
 by (*simp add: post_R-def unrest usubst*)

lemma *wait-cmt-unrest* [*unrest*]: $\$wait \# cmt_R P$
 by (*simp add: cmt_R-def unrest usubst*)

lemma *wait'-peri-unrest* [*unrest*]: $\$wait' \# peri_R P$
 by (*simp add: peri_R-def unrest usubst*)

lemma *wait'-post-unrest* [*unrest*]: $\$wait' \# post_R P$
 by (*simp add: post_R-def unrest usubst*)

4.3.3 Substitution laws

lemma *pre_s-design*: $pre_s \dagger (P \vdash Q) = (\neg pre_s \dagger P)$
 by (*simp add: design-def pre_R-def usubst*)

lemma *peri_s-design*: $peri_s \dagger (P \vdash Q \diamond R) = peri_s \dagger (P \Rightarrow Q)$
 by (*simp add: design-def usubst wait'-cond-def*)

lemma *post_s-design*: $post_s \dagger (P \vdash Q \diamond R) = post_s \dagger (P \Rightarrow R)$
 by (*simp add: design-def usubst wait'-cond-def*)

lemma *cmt_s-design*: $cmt_s \dagger (P \vdash Q) = cmt_s \dagger (P \Rightarrow Q)$
 by (*simp add: design-def usubst wait'-cond-def*)

lemma *pre_s-R1* [*usubst*]: $pre_s \dagger R1(P) = R1(pre_s \dagger P)$
 by (*simp add: R1-def usubst*)

lemma *pre_s-R2c* [*usubst*]: $pre_s \dagger R2c(P) = R2c(pre_s \dagger P)$
 by (*simp add: R2c-def R2s-def usubst*)

lemma *peri_s-R1* [*usubst*]: $peri_s \dagger R1(P) = R1(peri_s \dagger P)$
 by (*simp add: R1-def usubst*)

lemma *peri_s-R2c* [*usubst*]: $peri_s \dagger R2c(P) = R2c(peri_s \dagger P)$
 by (*simp add: R2c-def R2s-def usubst*)

lemma *post_s-R1* [*usubst*]: $post_s \dagger R1(P) = R1(post_s \dagger P)$
 by (*simp add: R1-def usubst*)

lemma *post_s-R2c* [*usubst*]: $post_s \dagger R2c(P) = R2c(post_s \dagger P)$
 by (*simp add: R2c-def R2s-def usubst*)

lemma *cmt_s-R1* [*usubst*]: $cmt_s \dagger R1(P) = R1(cmt_s \dagger P)$
 by (*simp add: R1-def usubst*)

lemma *cmt_s-R2c* [*usubst*]: $cmt_s \dagger R2c(P) = R2c(cmt_s \dagger P)$
 by (*simp add: R2c-def R2s-def usubst*)

lemma *pre-wait-false*:
 $pre_R(P \llbracket false/\$wait \rrbracket) = pre_R(P)$
 by (*rel-auto*)

lemma *cmt-wait-false*:
 $cmt_R(P \llbracket false/\$wait \rrbracket) = cmt_R(P)$
 by (*rel-auto*)

lemma *rea-pre-RHS-design*: $pre_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(pre_s \dagger P))$
 by (*simp add: RHS-def usubst R3h-def pre_R-def pre_s-design R1-negate-R1 R2c-not rea-not-def*)

lemma *rea-cmt-RHS-design*: $cmt_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$
 by (*simp add: RHS-def usubst R3h-def cmt_R-def cmt_s-design R1-idem*)

lemma *rea-peri-RHS-design*: $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$
 by (*simp add: RHS-def usubst peri_R-def R3h-def peri_s-design, rel-auto*)

lemma *rea-post-RHS-design*: $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$
 by (*simp add: RHS-def usubst post_R-def R3h-def post_s-design, rel-auto*)

lemma *peri-cmt-def*: $peri_R(P) = (cmt_R(P)) \llbracket true/\$wait' \rrbracket$
 by (*rel-auto*)

lemma *post-cmt-def*: $post_R(P) = (cmt_R(P)) \llbracket false/\$wait' \rrbracket$
 by (*rel-auto*)

lemma *rdes-export-cmt*: $\mathbf{R}_s(P \vdash cmt_s \dagger Q) = \mathbf{R}_s(P \vdash Q)$
 by (*rel-auto*)

lemma *rdes-export-pre*: $\mathbf{R}_s((P \llbracket true, false/\$ok, \$wait \rrbracket) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
 by (*rel-auto*)

4.3.4 Healthiness laws

lemma *wait'-unrest-pre-SRD* [*unrest*]:
 $\$wait' \# pre_R(P) \Longrightarrow \$wait' \# pre_R(SRD P)$
 apply (*rel-auto*)
 using *least-zero apply blast+*
 done

lemma *R1-R2s-cmt-SRD*:
 assumes *P is SRD*
 shows $R1(R2s(cmt_R(P))) = cmt_R(P)$
 by (*metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-idem R2c-idem SRD-reactive-design*
assms rea-cmt-RHS-design)

lemma *R1-R2s-peri-SRD*:
 assumes *P is SRD*
 shows $R1(R2s(peri_R(P))) = peri_R(P)$
 by (*metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R2-def R2-idem RHS-def SRD-RH-design-form*)

assms R1-idem peri_R-def peri_s-R1 peri_s-R2c)

lemma *R1-peri-SRD*:

assumes *P is SRD*

shows $R1(per_i_R(P)) = per_i_R(P)$

proof –

have $R1(per_i_R(P)) = R1(R1(R2s(per_i_R(P))))$

by (*simp add: R1-R2s-peri-SRD assms*)

also have $\dots = per_i_R(P)$

by (*simp add: R1-idem, simp add: R1-R2s-peri-SRD assms*)

finally show *?thesis* .

qed

lemma *periR-SRD-R1 [closure]*: $P \text{ is SRD} \implies per_i_R(P) \text{ is R1}$

by (*simp add: Healthy-def' R1-peri-SRD*)

lemma *R1-R2c-peri-RHS*:

assumes *P is SRD*

shows $R1(R2c(per_i_R(P))) = per_i_R(P)$

by (*metis R1-R2s-R2c R1-R2s-peri-SRD assms*)

lemma *R1-R2s-post-SRD*:

assumes *P is SRD*

shows $R1(R2s(post_R(P))) = post_R(P)$

by (*metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R1-idem R2-def R2-idem RHS-def SRD-RH-design-form assms post_R-def post_s-R1 post_s-R2c*)

lemma *R2c-peri-SRD*:

assumes *P is SRD*

shows $R2c(per_i_R(P)) = per_i_R(P)$

by (*metis R1-R2c-commute R1-R2c-peri-RHS R1-peri-SRD assms*)

lemma *R1-post-SRD*:

assumes *P is SRD*

shows $R1(post_R(P)) = post_R(P)$

proof –

have $R1(post_R(P)) = R1(R1(R2s(post_R(P))))$

by (*simp add: R1-R2s-post-SRD assms*)

also have $\dots = post_R(P)$

by (*simp add: R1-idem, simp add: R1-R2s-post-SRD assms*)

finally show *?thesis* .

qed

lemma *R2c-post-SRD*:

assumes *P is SRD*

shows $R2c(post_R(P)) = post_R(P)$

by (*metis R1-R2c-commute R1-R2s-R2c R1-R2s-post-SRD R1-post-SRD assms*)

lemma *postR-SRD-R1 [closure]*: $P \text{ is SRD} \implies post_R(P) \text{ is R1}$

by (*simp add: Healthy-def' R1-post-SRD*)

lemma *R1-R2c-post-RHS*:

assumes *P is SRD*

shows $R1(R2c(post_R(P))) = post_R(P)$

by (*metis R1-R2s-R2c R1-R2s-post-SRD assms*)

lemma *R2-cmt-conj-wait'*:

$P \text{ is } SRD \implies R2(cmt_R P \wedge \neg \$wait') = (cmt_R P \wedge \neg \$wait')$

by (*simp add: R2-def R2s-conj R2s-not R2s-wait' R1-extend-conj R1-R2s-cmt-SRD*)

lemma *R2c-preR*:

$P \text{ is } SRD \implies R2c(pre_R(P)) = pre_R(P)$

by (*metis (no-types, lifting) R1-R2c-commute R2c-idem SRD-reactive-design rea-pre-RHS-design*)

lemma *preR-R2c-closed [closure]*: $P \text{ is } SRD \implies pre_R(P) \text{ is } R2c$

by (*simp add: Healthy-def' R2c-preR*)

lemma *R2c-periR*:

$P \text{ is } SRD \implies R2c(per_iR(P)) = per_iR(P)$

by (*metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-R2s-peri-SRD R2c-idem*)

lemma *periR-R2c-closed [closure]*: $P \text{ is } SRD \implies per_iR(P) \text{ is } R2c$

by (*simp add: Healthy-def R2c-peri-SRD*)

lemma *R2c-postR*:

$P \text{ is } SRD \implies R2c(post_R(P)) = post_R(P)$

by (*metis (no-types, hide-lams) R1-R2c-commute R1-R2c-is-R2 R1-R2s-post-SRD R2-def R2s-idem*)

lemma *postR-R2c-closed [closure]*: $P \text{ is } SRD \implies post_R(P) \text{ is } R2c$

by (*simp add: Healthy-def R2c-post-SRD*)

lemma *periR-RR [closure]*: $P \text{ is } SRD \implies per_iR(P) \text{ is } RR$

by (*rule RR-intro, simp-all add: closure unrest*)

lemma *postR-RR [closure]*: $P \text{ is } SRD \implies post_R(P) \text{ is } RR$

by (*rule RR-intro, simp-all add: closure unrest*)

lemma *wpR-trace-ident-pre [wp]*:

$(\$tr' =_u \$tr \wedge \lceil II \rceil_R) \wp_r pre_R P = pre_R P$

by (*rel-auto*)

lemma *R1-preR [closure]*:

$pre_R(P) \text{ is } R1$

by (*rel-auto*)

lemma *trace-ident-left-periR*:

$(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; per_iR(P) = per_iR(P)$

by (*rel-auto*)

lemma *trace-ident-left-postR*:

$(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; post_R(P) = post_R(P)$

by (*rel-auto*)

lemma *trace-ident-right-postR*:

$post_R(P) ;; (\$tr' =_u \$tr \wedge \lceil II \rceil_R) = post_R(P)$

by (*rel-auto*)

lemma *preR-R2-closed [closure]*: $P \text{ is } SRD \implies pre_R(P) \text{ is } R2$

by (*simp add: R2-comp-def Healthy-comp closure*)

lemma *periR-R2-closed* [closure]: P is SRD \implies $\text{peri}_R(P)$ is R2
 by (simp add: Healthy-def' R1-R2c-peri-RHS R2-R2c-def)

lemma *postR-R2-closed* [closure]: P is SRD \implies $\text{post}_R(P)$ is R2
 by (simp add: Healthy-def' R1-R2c-post-RHS R2-R2c-def)

4.3.5 Calculation laws

lemma *wait'-cond-peri-post-cmt* [rdes]:
 $\text{cmt}_R P = \text{peri}_R P \diamond \text{post}_R P$
 by (rel-auto)

lemma *preR-rdes* [rdes]:
 assumes P is RR
 shows $\text{pre}_R(\mathbf{R}_s(P \vdash Q \diamond R)) = P$
 by (simp add: rea-pre-RHS-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1)

lemma *periR-rdes* [rdes]:
 assumes P is RR Q is RR
 shows $\text{peri}_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$
 by (simp add: rea-peri-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure)

lemma *postR-rdes* [rdes]:
 assumes P is RR R is RR
 shows $\text{post}_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$
 by (simp add: rea-post-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure)

lemma *preR-Chaos* [rdes]: $\text{pre}_R(\text{Chaos}) = \text{false}$
 by (simp add: Chaos-def, rel-simp)

lemma *periR-Chaos* [rdes]: $\text{peri}_R(\text{Chaos}) = \text{true}_r$
 by (simp add: Chaos-def, rel-simp)

lemma *postR-Chaos* [rdes]: $\text{post}_R(\text{Chaos}) = \text{true}_r$
 by (simp add: Chaos-def, rel-simp)

lemma *preR-Miracle* [rdes]: $\text{pre}_R(\text{Miracle}) = \text{true}_r$
 by (simp add: Miracle-def, rel-auto)

lemma *periR-Miracle* [rdes]: $\text{peri}_R(\text{Miracle}) = \text{false}$
 by (simp add: Miracle-def, rel-auto)

lemma *postR-Miracle* [rdes]: $\text{post}_R(\text{Miracle}) = \text{false}$
 by (simp add: Miracle-def, rel-auto)

lemma *preR-srdes-skip* [rdes]: $\text{pre}_R(\text{II}_R) = \text{true}_r$
 by (rel-auto)

lemma *periR-srdes-skip* [rdes]: $\text{peri}_R(\text{II}_R) = \text{false}$
 by (rel-auto)

lemma *postR-srdes-skip* [rdes]: $\text{post}_R(\text{II}_R) = (\$tr' =_u \$tr \wedge [\text{II}]_R)$
 by (rel-auto)

lemma *preR-INF* [rdes]: $A \neq \{\}$ \implies $\text{pre}_R(\bigcap A) = (\bigwedge P \in A \cdot \text{pre}_R(P))$
 by (rel-auto)

lemma *periR-INF* [rdes]: $\text{peri}_R(\sqcap A) = (\bigvee P \in A \cdot \text{peri}_R(P))$
by (*rel-auto*)

lemma *postR-INF* [rdes]: $\text{post}_R(\sqcap A) = (\bigvee P \in A \cdot \text{post}_R(P))$
by (*rel-auto*)

lemma *preR-UINF* [rdes]: $\text{pre}_R(\sqcap i \cdot P(i)) = (\bigsqcup i \cdot \text{pre}_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF* [rdes]: $\text{peri}_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot \text{peri}_R(P(i)))$
by (*rel-auto*)

lemma *postR-UINF* [rdes]: $\text{post}_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot \text{post}_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member* [rdes]: $A \neq \{\} \implies \text{pre}_R(\sqcap i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot \text{pre}_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member-2* [rdes]: $A \neq \{\} \implies \text{pre}_R(\sqcap (i,j) \in A \cdot P i j) = (\bigsqcup (i,j) \in A \cdot \text{pre}_R(P i j))$
by (*rel-auto*)

lemma *preR-UINF-member-3* [rdes]: $A \neq \{\} \implies \text{pre}_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\bigsqcup (i,j,k) \in A \cdot \text{pre}_R(P i j k))$
by (*rel-auto*)

lemma *periR-UINF-member* [rdes]: $\text{peri}_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot \text{peri}_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF-member-2* [rdes]: $\text{peri}_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot \text{peri}_R(P i j))$
by (*rel-auto*)

lemma *periR-UINF-member-3* [rdes]: $\text{peri}_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot \text{peri}_R(P i j k))$
by (*rel-auto*)

lemma *postR-UINF-member* [rdes]: $\text{post}_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot \text{post}_R(P(i)))$
by (*rel-auto*)

lemma *postR-UINF-member-2* [rdes]: $\text{post}_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot \text{post}_R(P i j))$
by (*rel-auto*)

lemma *postR-UINF-member-3* [rdes]: $\text{post}_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot \text{post}_R(P i j k))$
by (*rel-auto*)

lemma *preR-inf* [rdes]: $\text{pre}_R(P \sqcap Q) = (\text{pre}_R(P) \wedge \text{pre}_R(Q))$
by (*rel-auto*)

lemma *periR-inf* [rdes]: $\text{peri}_R(P \sqcap Q) = (\text{peri}_R(P) \vee \text{peri}_R(Q))$
by (*rel-auto*)

lemma *postR-inf* [rdes]: $\text{post}_R(P \sqcap Q) = (\text{post}_R(P) \vee \text{post}_R(Q))$
by (*rel-auto*)

lemma *preR-SUP* [rdes]: $\text{pre}_R(\bigsqcup A) = (\bigvee P \in A \cdot \text{pre}_R(P))$
by (*rel-auto*)

lemma *periR-SUP* [rdes]: $A \neq \{\}$ \implies $\text{peri}_R(\bigsqcup A) = (\bigwedge P \in A \cdot \text{peri}_R(P))$
 by (*rel-auto*)

lemma *postR-SUP* [rdes]: $A \neq \{\}$ \implies $\text{post}_R(\bigsqcup A) = (\bigwedge P \in A \cdot \text{post}_R(P))$
 by (*rel-auto*)

4.4 Formation laws

lemma *srdes-skip-tri-design* [rdes-def]: $II_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond II_r)$
 by (*simp add: srdes-skip-def, rel-auto*)

lemma *Chaos-tri-def* [rdes-def]: $\text{Chaos} = \mathbf{R}_s(\text{false} \vdash \text{false} \diamond \text{false})$
 by (*simp add: Chaos-def design-false-pre*)

lemma *Miracle-tri-def* [rdes-def]: $\text{Miracle} = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \text{false})$
 by (*simp add: Miracle-def R1-design-R1-pre wait'-cond-idem*)

lemma *RHS-tri-design-form*:

assumes P_1 is RR P_2 is RR P_3 is RR

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$

proof –

have $\mathbf{R}_s(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3)))))$

apply (*rel-auto*) using *minus-zero-eq* by *blast*

thus *?thesis*

by (*simp add: Healthy-if assms*)

qed

lemma *RHS-design-pre-post-form*:

$\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

proof –

have $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s((\neg P^f_f) \llbracket \text{true}/\$ok \rrbracket \vdash P^t_f \llbracket \text{true}/\$ok \rrbracket)$

by (*simp add: design-subst-ok*)

also have $\dots = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (*simp add: pre_R-def cmt_R-def usubst, rel-auto*)

finally show *?thesis* .

qed

lemma *SRD-as-reactive-design*:

$\text{SRD}(P) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (*simp add: RHS-design-pre-post-form SRD-RH-design-form*)

lemma *SRD-reactive-design-alt*:

assumes P is *SRD*

shows $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) = P$

proof –

have $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

by (*simp add: RHS-design-pre-post-form*)

thus *?thesis*

by (*simp add: SRD-reactive-design assms*)

qed

lemma *SRD-reactive-tri-design-lemma*:

$\text{SRD}(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket \text{true}/\$wait' \rrbracket \diamond P^t_f \llbracket \text{false}/\$wait' \rrbracket)$

by (*simp add: SRD-RH-design-form wait'-cond-split*)

lemma *SRD-as-reactive-tri-design*:

$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

proof –

have $SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket true/\$wait' \rrbracket \diamond P^t_f \llbracket false/\$wait' \rrbracket)$

by (*simp add: SRD-RH-design-form wait'-cond-split*)

also have $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

apply (*simp add: usubst*)

apply (*subst design-subst-ok-ok'[THEN sym]*)

apply (*simp add: pre_R-def peri_R-def post_R-def usubst unrest*)

apply (*rel-auto*)

done

finally show *?thesis* .

qed

lemma *SRD-reactive-tri-design*:

assumes *P is SRD*

shows $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$

by (*metis Healthy-if SRD-as-reactive-tri-design assms*)

lemma *SRD-elim* [*RD-elim*]: $\llbracket P \text{ is } SRD; Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))) \rrbracket \implies Q(P)$

by (*simp add: SRD-reactive-tri-design*)

lemma *RHS-tri-design-is-SRD* [*closure*]:

assumes $\$ok' \# P \ \$ok' \# Q \ \$ok' \# R$

shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*

by (*rule RHS-design-is-SRD, simp-all add: unrest assms*)

lemma *SRD-rdes-intro* [*closure*]:

assumes *P is RR Q is RR R is RR*

shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*

by (*rule RHS-tri-design-is-SRD, simp-all add: unrest closure assms*)

lemma *USUP-R1-R2s-cmt-SRD*:

assumes $A \subseteq \llbracket SRD \rrbracket_H$

shows $(\bigsqcup P \in A \cdot R1 (R2s (cmt_R P))) = (\bigsqcup P \in A \cdot cmt_R P)$

by (*rule USUP-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms*)

lemma *UINF-R1-R2s-cmt-SRD*:

assumes $A \subseteq \llbracket SRD \rrbracket_H$

shows $(\bigsqcap P \in A \cdot R1 (R2s (cmt_R P))) = (\bigsqcap P \in A \cdot cmt_R P)$

by (*rule UINF-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms*)

4.4.1 Order laws

lemma *preR-antitone*: $P \sqsubseteq Q \implies pre_R(Q) \sqsubseteq pre_R(P)$

by (*rel-auto*)

lemma *periR-monotone*: $P \sqsubseteq Q \implies peri_R(P) \sqsubseteq peri_R(Q)$

by (*rel-auto*)

lemma *postR-monotone*: $P \sqsubseteq Q \implies post_R(P) \sqsubseteq post_R(Q)$

by (*rel-auto*)

4.5 Composition laws

theorem *RH-tri-design-composition*:

assumes $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$
 $\$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

shows $(RH(P \vdash Q_1 \diamond Q_2) ;; RH(R \vdash S_1 \diamond S_2)) =$
 $RH((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R))) \vdash$
 $((Q_1 \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

proof –

have $1: (\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$
 $(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)
have $2: (R1 (R2s (Q_1 \diamond Q_2)) ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$
 $((R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2)))$

proof –

have $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (R1 (R2s Q_1) \wedge \$wait')$

proof –

have $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (R1 (R2s Q_1) ;; (\$wait \wedge [II]_D))$

by (*rel-auto*)

also have $\dots = ((R1 (R2s Q_1) ;; [II]_D) \wedge \$wait')$

by (*rel-auto*)

also from *assms*(2) **have** $\dots = ((R1 (R2s Q_1)) \wedge \$wait')$

by (*simp add: lift-des-skip-dr-unit-unrest unrest*)

finally show *?thesis* .

qed

moreover have $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

proof –

have $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (R1 (R2s Q_2) ;; (\neg \$wait \wedge (R1 (R2s S_1) \diamond R1 (R2s S_2))))$

by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem*
utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb)

also have $\dots = ((R1 (R2s Q_2)) \llbracket false/\$wait' \rrbracket ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)) \llbracket false/\$wait \rrbracket)$
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have $\dots = ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$
by (*simp add: wait'-cond-def usubst unrest assms*)

finally show *?thesis* .

qed

moreover

have $((R1 (R2s Q_1) \wedge \$wait') \vee ((R1 (R2s Q_2) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2)))$
by (*simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*

by (*simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq*)

qed

show *?thesis*

apply (*subst RH-design-composition*)

apply (simp-all add: assms)
 apply (simp add: assms wait'-cond-def unrest)
 apply (simp add: assms wait'-cond-def unrest)
 apply (simp add: 1 2)
 apply (simp add: R1-R2s-R2c RH-design-lemma1)
 done
 qed

theorem R1-design-composition-RR:
 assumes P is RR Q is RR R is RR S is RR
 shows
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$
 apply (subst R1-design-composition)
 apply (simp-all add: assms unrest wp-rea-def Healthy-if closure)
 apply (rel-auto)
 done

theorem R1-design-composition-RC:
 assumes P is RC Q is RR R is RR S is RR
 shows
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$
 by (simp add: R1-design-composition-RR assms unrest Healthy-if closure wp)

theorem RHS-tri-design-composition:
 assumes $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$
 $\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$
 shows $(\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)) =$
 $\mathbf{R}_s((\neg (R1 (\neg R2s P) ;; R1 \text{ true}) \wedge \neg (R1(R2s Q_2) ;; R1 (\neg R2s R))) \vdash$
 $((\exists \$st' \cdot Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

proof –
 have 1: $(\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$
 $(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$
 by (metis (no-types, hide-lams) R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false)
 have 2: $(R1 (R2s (Q_1 \diamond Q_2)) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$
 $((\exists \$st' \cdot R1 (R2s Q_1)) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2)))$

proof –
 have $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$

proof –
 have $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= (R1 (R2s Q_1) ;; (\$wait \wedge (\exists \$st \cdot \lceil II \rceil_D)))$
 by (rel-auto, blast+)
 also have ... $= ((R1 (R2s Q_1) ;; (\exists \$st \cdot \lceil II \rceil_D)) \wedge \$wait')$
 by (rel-auto)
 also from assms(2) have ... $= (\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$
 by (rel-auto, blast)
 finally show ?thesis .
 qed

moreover have $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$
 $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

proof –
 have $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$$= (R1 \ (R2s \ Q_2) \ ;\ ;\ (\neg \ \$wait \wedge (R1 \ (R2s \ S_1) \diamond R1 \ (R2s \ S_2))))$$
 by (metis (no-types, lifting) cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb)

also have ... = ((R1 (R2s Q₂)) \llbracket false/\$wait \rrbracket ; (R1 (R2s S₁) \diamond R1 (R2s S₂)) \llbracket false/\$wait \rrbracket)
 by (metis false-alt-def segr-right-one-point upred-eq-false wait-vwb-lens)

also have ... = ((R1 (R2s Q₂)) ; (R1 (R2s S₁) \diamond R1 (R2s S₂)))
 by (simp add: wait'-cond-def usubst unrest assms)

finally show ?thesis .
 qed

moreover
 have ((R1 (R2s Q₁) \wedge \$wait') \vee ((R1 (R2s Q₂)) ; (R1 (R2s S₁) \diamond R1 (R2s S₂))))

$$= (R1 \ (R2s \ Q_1) \ \vee \ (R1 \ (R2s \ Q_2) \ ;\ ;\ R1 \ (R2s \ S_1))) \diamond ((R1 \ (R2s \ Q_2) \ ;\ ;\ R1 \ (R2s \ S_2)))$$
 by (simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest)

ultimately show ?thesis
 by (simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq ex-conj-contr-right unrest)
 (simp add: cond-and-T-integrate cond-seq-right-distr unrest-var wait'-cond-def)

qed

from assms(7,8) have \exists : (R1 (R2s Q₂) \wedge \neg \$wait') ; R1 (\neg R2s R) = R1 (R2s Q₂) ; R1 (\neg R2s R)
 by (rel-auto, blast, meson)

show ?thesis
 apply (subst RHS-design-composition)
 apply (simp-all add: assms)
 apply (simp add: assms wait'-cond-def unrest)
 apply (simp add: assms wait'-cond-def unrest)
 apply (simp add: 1 2 3)
 apply (simp add: R1-R2s-R2c RHS-design-lemma1)
 apply (metis R1-R2c-ex-st RHS-design-lemma1)

done

qed

theorem RHS-tri-design-composition-wp:

assumes \$ok' $\#$ P \$ok' $\#$ Q₁ \$ok' $\#$ Q₂ \$ok $\#$ R \$ok $\#$ S₁ \$ok $\#$ S₂
 $\$wait \# R \ \$wait' \# Q_2 \ \$wait \# S_1 \ \$wait \# S_2$
 $P \text{ is } R2c \ Q_1 \text{ is } R1 \ Q_1 \text{ is } R2c \ Q_2 \text{ is } R1 \ Q_2 \text{ is } R2c$
 $R \text{ is } R2c \ S_1 \text{ is } R1 \ S_1 \text{ is } R2c \ S_2 \text{ is } R1 \ S_2 \text{ is } R2c$

shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
 $\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \ \$st' \cdot Q_1) \sqcap (Q_2 ; S_1)) \diamond (Q_2 ; S_2)))$ (is ?lhs = ?rhs)

proof –

have ?lhs = $\mathbf{R}_s((\neg R1 (\neg P) ; R1 \text{ true} \wedge \neg Q_2 ; R1 (\neg R)) \vdash ((\exists \ \$st' \cdot Q_1) \sqcap Q_2 ; S_1) \diamond Q_2 ; S_2)$

by (simp add: RHS-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def)
 (metis (no-types, hide-lams) R1-negate-R1 R2c-healthy-R2s assms(11,16))

also have ... = ?rhs

by (rel-auto)

finally show ?thesis .

qed

theorem *RHS-tri-design-composition-RR-wp*:

assumes P is RR Q_1 is RR Q_2 is RR

R is RR S_1 is RR S_2 is RR

shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$ (**is** ?lhs = ?rhs)

by (*simp add: RHS-tri-design-composition-wp add: closure assms unrest RR-implies-R2c*)

lemma *RHS-tri-normal-design-composition*:

assumes

$\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

P is R2c Q_1 is R1 Q_1 is R2c Q_2 is R1 Q_2 is R2c

R is R2c S_1 is R1 S_1 is R2c S_2 is R1 S_2 is R2c

$R1 (\neg P) ;; R1(true) = R1(\neg P) \$st' \# Q_1$

shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

proof –

have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((R1 (\neg P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

by (*simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest*)

also have ... $= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

by (*simp add: assms wp-rea-def ex-unrest, rel-auto*)

finally show ?thesis .

qed

lemma *RHS-tri-normal-design-composition' [rdes-def]*:

assumes P is RC Q_1 is RR $\$st' \# Q_1$ Q_2 is RR R is RR S_1 is RR S_2 is RR

shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

proof –

have $R1 (\neg P) ;; R1 true = R1(\neg P)$

using *RC-implies-RC1[OF assms(1)]*

by (*simp add: Healthy-def RC1-def rea-not-def*)

(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)

thus ?thesis

by (*simp add: RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)

qed

lemma *RHS-tri-design-right-unit-lemma*:

assumes $\$ok' \# P \$ok' \# Q \$ok' \# R \$wait' \# R$

shows $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$

proof –

have $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(true \vdash false \diamond (\$tr' =_u \$tr \wedge [II]_R))$

by (*simp add: srdes-skip-tri-design, rel-auto*)

also have ... $= \mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 true) \vdash (\exists \$st' \cdot Q) \diamond (R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))))$

by (*simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)

also have ... $= \mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 true) \vdash (\exists \$st' \cdot Q) \diamond R1 (R2s R))$

proof –

from *assms(3,4)* **have** $(R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 (R2s R)$

by (*rel-auto, metis (no-types, lifting) minus-zero-eq, meson order-refl trace-class.diff-cancel*)

thus ?thesis

by *simp*

qed
 also have ... = $\mathbf{R}_s((\neg (\neg P) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$
 by (metis (no-types, lifting) R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre
 RHS-design-post-R1 RHS-design-post-R2s)
 also have ... = $\mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$
 by (rel-auto)
 finally show ?thesis .
 qed

lemma SRD-composition-wp:

assumes P is SRD Q is SRD

shows $(P ;; Q) = \mathbf{R}_s(((\neg_r pre_R P) wp_r false \wedge post_R P wp_r pre_R Q) \vdash$
 $((\exists \$st' \cdot peri_R P) \vee (post_R P ;; peri_R Q)) \diamond (post_R P ;; post_R Q))$

(is ?lhs = ?rhs)

proof –

have $(P ;; Q) = (\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q)))$

by (simp add: SRD-reactive-tri-design assms(1) assms(2))

also from assms

have ... = ?rhs

by (simp add: RHS-tri-design-composition-wp disj-upred-def unrest assms closure)

finally show ?thesis .

qed

4.6 Refinement introduction laws

lemma RHS-tri-design-refine:

assumes P_1 is RR P_2 is RR P_3 is RR Q_1 is RR Q_2 is RR Q_3 is RR

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2' \wedge 'P_1 \wedge Q_3 \Rightarrow$
 P_3'

(is ?lhs = ?rhs)

proof –

have $?lhs \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3'$

by (simp add: RHS-design-refine assms closure RR-implies-R2c unrest ex-unrest)

also have ... $\longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge '(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3'$

by (rel-auto)

also have ... $\longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge '((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[[true/\$wait]]' \wedge '((P_1 \wedge Q_2) \diamond$
 $(P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[[false/\$wait]]'$

by (rel-auto, metis)

also have ... $\longleftrightarrow ?rhs$

by (simp add: usubst unrest assms)

finally show ?thesis .

qed

lemma sdes-tri-refine-intro:

assumes $'P_1 \Rightarrow P_2' 'P_1 \wedge Q_2 \Rightarrow Q_1' 'P_1 \wedge R_2 \Rightarrow R_1'$

shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$

using assms

by (rule-tac sdes-refine-intro, simp-all, rel-auto)

lemma sdes-tri-eq-intro:

assumes $P_1 = Q_1 P_2 = Q_2 P_3 = Q_3$

shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$

using assms by (simp)

lemma sdes-tri-refine-intro':

assumes $P_2 \sqsubseteq P_1 Q_1 \sqsubseteq (P_1 \wedge Q_2) R_1 \sqsubseteq (P_1 \wedge R_2)$

shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$
 using *assms*
 by (rule-tac *srdes-tri-refine-intro*, simp-all add: *refBy-order*)

lemma *SRD-peri-under-pre*:

assumes *P is SRD \$wait' \# pre_R(P)*
 shows $(pre_R(P) \Rightarrow_r peri_R(P)) = peri_R(P)$

proof –

have $peri_R(P) =$
 $peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
 by (simp add: *SRD-reactive-tri-design assms*)
 also have $\dots = (pre_R P \Rightarrow_r peri_R P)$
 by (simp add: *rea-pre-RHS-design rea-peri-RHS-design assms*
unrest usubst R1-peri-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-periR)
 finally show ?thesis ..

qed

lemma *SRD-post-under-pre*:

assumes *P is SRD \$wait' \# pre_R(P)*
 shows $(pre_R(P) \Rightarrow_r post_R(P)) = post_R(P)$

proof –

have $post_R(P) =$
 $post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
 by (simp add: *SRD-reactive-tri-design assms*)
 also have $\dots = (pre_R P \Rightarrow_r post_R P)$
 by (simp add: *rea-pre-RHS-design rea-post-RHS-design assms*
unrest usubst R1-post-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-postR)
 finally show ?thesis ..

qed

lemma *SRD-refine-intro*:

assumes
P is SRD Q is SRD
 $'pre_R(P) \Rightarrow pre_R(Q)'$ $'pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)'$ $'pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)'$
 shows $P \sqsubseteq Q$
 by (metis *SRD-reactive-tri-design assms(1) assms(2) assms(3) assms(4) assms(5) srdes-tri-refine-intro*)

lemma *SRD-refine-intro'*:

assumes
P is SRD Q is SRD
 $'pre_R(P) \Rightarrow pre_R(Q)'$ $peri_R(P) \sqsubseteq (pre_R(P) \wedge peri_R(Q))$ $post_R(P) \sqsubseteq (pre_R(P) \wedge post_R(Q))$
 shows $P \sqsubseteq Q$
 using *assms* by (rule-tac *SRD-refine-intro*, simp-all add: *refBy-order*)

lemma *SRD-eq-intro*:

assumes
P is SRD Q is SRD $pre_R(P) = pre_R(Q)$ $peri_R(P) = peri_R(Q)$ $post_R(P) = post_R(Q)$
 shows $P = Q$
 by (metis *SRD-reactive-tri-design assms*)

4.7 Closure laws

lemma *SRD-srdes-skip [closure]*: *II_R is SRD*

by (simp add: *srdes-skip-def RHS-design-is-SRD unrest*)

lemma *SRD-seqr-closure [closure]*:

assumes P is SRD Q is SRD
 shows $(P ;; Q)$ is SRD
 proof –
 have $(P ;; Q) = \mathbf{R}_s (((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q) \vdash$
 $((\exists \$st' \cdot \text{peri}_R P) \vee \text{post}_R P ;; \text{peri}_R Q) \diamond \text{post}_R P ;; \text{post}_R Q)$
 by (simp add: SRD-composition-wp assms(1) assms(2))
 also have ... is SRD
 by (rule RHS-design-is-SRD, simp-all add: wp-rea-def unrest)
 finally show ?thesis .
 qed

lemma SRD-power-Suc [closure]: P is SRD $\implies P^\wedge(\text{Suc } n)$ is SRD
 proof (induct n)
 case 0
 then show ?case
 by (simp)
 next
 case (Suc n)
 then show ?case
 using SRD-seqr-closure by (simp add: SRD-seqr-closure upred-semiring.power-Suc)
 qed

lemma SRD-power-comp [closure]: P is SRD $\implies P ;; P^\wedge n$ is SRD
 by (metis SRD-power-Suc upred-semiring.power-Suc)

lemma uplus-SRD-closed [closure]: P is SRD $\implies P^+$ is SRD
 by (simp add: uplus-power-def closure)

lemma SRD-Sup-closure [closure]:
 assumes $A \subseteq \llbracket \text{SRD} \rrbracket_H$ $A \neq \{\}$
 shows $(\bigsqcup A)$ is SRD
 proof –
 have $\text{SRD } (\bigsqcup A) = (\bigsqcup (\text{SRD } `A))$
 by (simp add: ContinuousD SRD-Continuous assms(2))
 also have ... = $(\bigsqcup A)$
 by (simp only: Healthy-carrier-image assms)
 finally show ?thesis by (simp add: Healthy-def)
 qed

4.8 Distribution laws

lemma RHS-tri-design-choice [rdes-def]:
 $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$
 apply (simp add: RHS-design-choice)
 apply (rule cong[of \mathbf{R}_s \mathbf{R}_s])
 apply (simp)
 apply (rel-auto)
 done

lemma RHS-tri-design-sup [rdes-def]:
 $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcup \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$
 by (simp add: RHS-design-sup, rel-auto)

lemma RHS-tri-design-conj [rdes-def]:
 $(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$

$\Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$
 by (*simp add: RHS-tri-design-sup conj-upred-def*)

lemma *SRD-UINF* [*rdes-def*]:

assumes $A \neq \{\}$ $A \subseteq \llbracket SRD \rrbracket_H$

shows $\sqcap A = \mathbf{R}_s((\sqcap P \in A \cdot pre_R(P)) \vdash (\bigvee P \in A \cdot peri_R(P)) \diamond (\bigvee P \in A \cdot post_R(P)))$

proof –

have $\sqcap A = \mathbf{R}_s(pre_R(\sqcap A) \vdash peri_R(\sqcap A) \diamond post_R(\sqcap A))$

by (*metis SRD-as-reactive-tri-design assms srdes-hcond-def*

srdes-theory-continuous.healthy-inf srdes-theory-continuous.healthy-inf-def)

also have $\dots = \mathbf{R}_s((\sqcap P \in A \cdot pre_R(P)) \vdash (\bigvee P \in A \cdot peri_R(P)) \diamond (\bigvee P \in A \cdot post_R(P)))$

by (*simp add: preR-INF periR-INF postR-INF assms*)

finally show *?thesis* .

qed

lemma *RHS-tri-design-USUP* [*rdes-def*]:

assumes $A \neq \{\}$

shows $(\sqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i) \diamond R(i))) = \mathbf{R}_s((\sqcap i \in A \cdot P(i)) \vdash (\sqcap i \in A \cdot Q(i)) \diamond (\sqcap i \in A \cdot R(i)))$

by (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms, rel-auto*)

lemma *SRD-UINF-mem*:

assumes $A \neq \{\}$ $\bigwedge i. P \ i \text{ is } SRD$

shows $(\sqcap i \in A \cdot P \ i) = \mathbf{R}_s((\sqcap i \in A \cdot pre_R(P \ i)) \vdash (\bigvee i \in A \cdot peri_R(P \ i)) \diamond (\bigvee i \in A \cdot post_R(P \ i)))$
 (is *?lhs = ?rhs*)

proof –

have *?lhs* = $(\sqcap (P \ ' \ A))$

by (*rel-auto*)

also have $\dots = \mathbf{R}_s((\sqcap Pa \in P \ ' \ A \cdot pre_R \ Pa) \vdash (\sqcap Pa \in P \ ' \ A \cdot peri_R \ Pa) \diamond (\sqcap Pa \in P \ ' \ A \cdot post_R \ Pa))$

by (*subst rdes-def, simp-all add: assms image-subsetI*)

also have $\dots = ?rhs$

by (*rel-auto*)

finally show *?thesis* .

qed

lemma *RHS-tri-design-UINF-ind* [*rdes-def*]:

$(\sqcap i \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) = \mathbf{R}_s((\sqcap i \cdot P_1 \ i) \vdash (\bigvee i \cdot P_2(i)) \diamond (\bigvee i \cdot P_3(i)))$

by (*rel-auto*)

lemma *cond-srea-form* [*rdes-def*]:

$\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$

proof –

have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft R2c(\lceil b \rceil_{S<}) \triangleright \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

by (*pred-auto*)

also have $\dots = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2 \triangleleft b \triangleright_R R \vdash S_1 \diamond S_2)$

by (*simp add: RHS-cond lift-cond-srea-def*)

also have $\dots = \mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \diamond Q_2 \triangleleft b \triangleright_R S_1 \diamond S_2))$

by (*simp add: design-condr lift-cond-srea-def*)

also have $\dots = \mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$

by (*rule cong[of $\mathbf{R}_s \ \mathbf{R}_s$], simp, rel-auto*)

finally show *?thesis* .

qed

lemma *SRD-cond-srea* [closure]:

assumes *P* is *SRD* *Q* is *SRD*

shows $P \triangleleft b \triangleright_R Q$ is *SRD*

proof –

have $P \triangleleft b \triangleright_R Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \triangleleft b \triangleright_R \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$
 by (simp add: *SRD-reactive-tri-design* *assms*)

also have $\dots = \mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond (\text{post}_R P \triangleleft b \triangleright_R \text{post}_R Q))$

by (simp add: *cond-srea-form*)

also have \dots is *SRD*

by (simp add: *RHS-tri-design-is-SRD* *lift-cond-srea-def* *unrest*)

finally show ?thesis .

qed

4.9 Algebraic laws

lemma *SRD-left-unit*:

assumes *P* is *SRD*

shows $\Pi_R \mathbin{;;} P = P$

by (simp add: *SRD-composition-wp* *closure* *rdes* *wp* *C1* *R1-negate-R1* *R1-false* *rpred* *trace-ident-left-periR* *trace-ident-left-postR* *SRD-reactive-tri-design* *assms*)

lemma *skip-srea-self-unit* [simp]:

$\Pi_R \mathbin{;;} \Pi_R = \Pi_R$

by (simp add: *SRD-left-unit* *closure*)

lemma *SRD-right-unit-tri-lemma*:

assumes *P* is *SRD*

shows $P \mathbin{;;} \Pi_R = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P)$

by (simp add: *SRD-composition-wp* *closure* *rdes* *wp* *rpred* *trace-ident-right-postR* *assms*)

lemma *Miracle-left-zero*:

assumes *P* is *SRD*

shows *Miracle* $\mathbin{;;} P = \text{Miracle}$

proof –

have $\text{Miracle} \mathbin{;;} P = \mathbf{R}_s(\text{true} \vdash \text{false}) \mathbin{;;} \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (simp add: *Miracle-def* *SRD-reactive-design-alt* *assms*)

also have $\dots = \mathbf{R}_s(\text{true} \vdash \text{false})$

by (simp add: *RHS-design-composition* *unrest* *R1-false* *R2s-false* *R2s-true*)

also have $\dots = \text{Miracle}$

by (simp add: *Miracle-def*)

finally show ?thesis .

qed

lemma *Chaos-left-zero*:

assumes *P* is *SRD*

shows (*Chaos* $\mathbin{;;} P$) = *Chaos*

proof –

have $\text{Chaos} \mathbin{;;} P = \mathbf{R}_s(\text{false} \vdash \text{true}) \mathbin{;;} \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (simp add: *Chaos-def* *SRD-reactive-design-alt* *assms*)

also have $\dots = \mathbf{R}_s((\neg R1 \text{ true} \wedge \neg (R1 \text{ true} \wedge \neg \$wait')) \mathbin{;;} R1 (\neg R2s (\text{pre}_R P))) \vdash$
 $R1 \text{ true} \mathbin{;;} ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (\text{cmt}_R P)))$

by (simp add: *RHS-design-composition* *unrest* *R2s-false* *R2s-true* *R1-false*)

also have $\dots = \mathbf{R}_s((\text{false} \wedge \neg (R1 \text{ true} \wedge \neg \$wait')) \mathbin{;;} R1 (\neg R2s (\text{pre}_R P))) \vdash$
 $R1 \text{ true} \mathbin{;;} ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (\text{cmt}_R P)))$

by (simp add: RHS-design-conj-neg-R1-pre)
 also have ... = $\mathbf{R}_s(\text{true})$
 by (simp add: design-false-pre)
 also have ... = $\mathbf{R}_s(\text{false} \vdash \text{true})$
 by (simp add: design-def)
 also have ... = *Chaos*
 by (simp add: Chaos-def)
 finally show ?thesis .
 qed

lemma *SRD-right-Chaos-tri-lemma*:

assumes *P* is *SRD*
 shows $P \;; \text{Chaos} = \mathbf{R}_s(((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{false}) \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{false})$
 by (simp add: SRD-composition-wp closure rdes assms wp, rel-auto)

lemma *SRD-right-Miracle-tri-lemma*:

assumes *P* is *SRD*
 shows $P \;; \text{Miracle} = \mathbf{R}_s(((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{false})$
 by (simp add: SRD-composition-wp closure rdes assms wp, rel-auto)

Stateful reactive designs are left unital

overloading

$\text{srdes-unit} == \text{utp-unit} :: (\text{SRDES}, ('s, 't::\text{trace}, 'a) \text{rsp}) \text{uthy} \Rightarrow ('s, 't, 'a) \text{hrel-rsp}$

begin

definition $\text{srdes-unit} :: (\text{SRDES}, ('s, 't::\text{trace}, 'a) \text{rsp}) \text{uthy} \Rightarrow ('s, 't, 'a) \text{hrel-rsp}$ **where**

$\text{srdes-unit } T = \text{II}_R$

end

interpretation *srdes-left-unital*: *utp-theory-left-unital SRDES*

by (unfold-locales, simp-all add: srdes-hcond-def srdes-unit-def SRD-seqr-closure SRD-srdes-skip SRD-left-unit)

4.10 Recursion laws

lemma *mono-srd-iter*:

assumes $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$
 shows $\text{mono } (\lambda X. \mathbf{R}_s(\text{pre}_R(F X) \vdash \text{peri}_R(F X) \diamond \text{post}_R(F X)))$
 apply (rule monoI)
 apply (rule srdes-tri-refine-intro')
 apply (meson assms(1) monoE preR-antitone utp-pred-laws.le-infI2)
 apply (meson assms(1) monoE periR-monotone utp-pred-laws.le-infI2)
 apply (meson assms(1) monoE postR-monotone utp-pred-laws.le-infI2)
 done

lemma *mu-srd-SRD*:

assumes $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$
 shows $(\mu X \cdot \mathbf{R}_s(\text{pre}_R(F X) \vdash \text{peri}_R(F X) \diamond \text{post}_R(F X))) \text{ is SRD}$
 apply (subst gfp-unfold)
 apply (simp add: mono-srd-iter assms)
 apply (rule RHS-tri-design-is-SRD)
 apply (simp-all add: unrest)
 done

lemma *mu-srd-iter*:

assumes $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$
 shows $(\mu X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X)))) = F(\mu X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$

apply (*subst gfp-unfold*)
apply (*simp add: mono-srd-iter assms*)
apply (*subst SRD-as-reactive-tri-design [THEN sym]*)
using *Healthy-func assms(1) assms(2) mu-srd-SRD* **apply** *blast*
done

lemma *mu-srd-form:*

assumes *mono* $F \in \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$
shows $\mu_R F = (\mu X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$

proof –

have 1: $F (\mu X \cdot \mathbf{R}_s(\text{pre}_R(F X) \vdash \text{peri}_R(F X) \diamond \text{post}_R(F X)))$ *is SRD*
by (*simp add: Healthy-apply-closed assms(1) assms(2) mu-srd-SRD*)
have 2: *Mono_{uthy-order} SRDES* F
by (*simp add: assms(1) mono-Monotone-utp-order*)
hence 3: $\mu_R F = F (\mu_R F)$
by (*simp add: srdes-theory-continuous.LFP-unfold [THEN sym] assms*)
hence $\mathbf{R}_s(\text{pre}_R(F (F (\mu_R F))) \vdash \text{peri}_R(F (F (\mu_R F))) \diamond \text{post}_R(F (F (\mu_R F)))) = \mu_R F$
using *SRD-reactive-tri-design* **by** *force*
hence $(\mu X \cdot \mathbf{R}_s(\text{pre}_R(F X) \vdash \text{peri}_R(F X) \diamond \text{post}_R(F X))) \sqsubseteq F (\mu_R F)$
by (*simp add: 2 srdes-theory-continuous.weak.LFP-lemma3 gfp-upperbound assms*)
thus *?thesis*
using *assms 1 3 srdes-theory-continuous.weak.LFP-lowerbound eq-iff mu-srd-iter*
by (*metis (mono-tags, lifting)*)

qed

lemma *Monotonic-SRD-comp [closure]: Monotonic* (*op* ;; $P \circ SRD$)

by (*simp add: mono-def R1-R2c-is-R2 R2-mono R3h-mono RD1-mono RD2-mono RHS-def SRD-def seqr-mono*)

end

5 Normal Reactive Designs

theory *utp-rdes-normal*

imports

utp-rdes-triples

UTP-KAT.utp-kleene

begin

This additional healthiness condition is analogous to H3

definition *RD3* **where**

[upred-defs]: $RD3(P) = P ;; II_R$

lemma *RD3-idem:* $RD3(RD3(P)) = RD3(P)$

proof –

have $a: II_R ;; II_R = II_R$
by (*simp add: SRD-left-unit SRD-srdes-skip*)
show *?thesis*
by (*simp add: RD3-def seqr-assoc a*)

qed

lemma *RD3-Idempotent [closure]: Idempotent* *RD3*

by (*simp add: Idempotent-def RD3-idem*)

lemma *RD3-continuous:* $RD3(\bigcap A) = (\bigcap P \in A. RD3(P))$

by (simp add: RD3-def seq-Sup-distr)

lemma *RD3-Continuous [closure]: Continuous RD3*

by (simp add: Continuous-def RD3-continuous)

lemma *RD3-right-subsumes-RD2: RD2(RD3(P)) = RD3(P)*

proof –

have $a:II_R \;; J = II_R$

by (rel-auto)

show ?thesis

by (metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc)

qed

lemma *RD3-left-subsumes-RD2: RD3(RD2(P)) = RD3(P)*

proof –

have $a:J \;; II_R = II_R$

by (rel-simp, safe, blast+)

show ?thesis

by (metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc)

qed

lemma *RD3-implies-RD2: P is RD3 \implies P is RD2*

by (metis Healthy-def RD3-right-subsumes-RD2)

lemma *RD3-intro-pre:*

assumes $P \text{ is } SRD \ (\neg_r \text{ pre}_R(P)) \;; \text{true}_r = (\neg_r \text{ pre}_R(P)) \ \$st' \ \# \ \text{peri}_R(P)$

shows $P \text{ is } RD3$

proof –

have $RD3(P) = \mathbf{R}_s ((\neg_r \text{ pre}_R P) \text{ wp}_r \text{ false} \vdash (\exists \ \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P)$

by (simp add: RD3-def SRD-right-unit-tri-lemma assms)

also have $\dots = \mathbf{R}_s ((\neg_r \text{ pre}_R P) \text{ wp}_r \text{ false} \vdash \text{peri}_R P \diamond \text{post}_R P)$

by (simp add: assms(3) ex-unrest)

also have $\dots = \mathbf{R}_s ((\neg_r \text{ pre}_R P) \text{ wp}_r \text{ false} \vdash \text{cmt}_R P)$

by (simp add: wait'-cond-peri-post-cmt)

also have $\dots = \mathbf{R}_s (\text{pre}_R P \vdash \text{cmt}_R P)$

by (simp add: assms(2) rpred wp-rea-def R1-preR)

finally show ?thesis

by (metis Healthy-def SRD-as-reactive-design assms(1))

qed

lemma *RHS-tri-design-right-unit-lemma:*

assumes $\$ok' \ \# \ P \ \$ok' \ \# \ Q \ \$ok' \ \# \ R \ \$wait' \ \# \ R$

shows $\mathbf{R}_s(P \vdash Q \diamond R) \;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) \;; \text{true}_r) \vdash ((\exists \ \$st' \cdot Q) \diamond R))$

proof –

have $\mathbf{R}_s(P \vdash Q \diamond R) \;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) \;; \mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [II]_R))$

by (simp add: srdes-skip-tri-design, rel-auto)

also have $\dots = \mathbf{R}_s ((\neg R1 \ (\neg R2s P) \;; R1 \text{ true}) \vdash (\exists \ \$st' \cdot Q) \diamond (R1 \ (R2s R) \;; R1 \ (R2s (\$tr' =_u \$tr \wedge [II]_R))))$

by (simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false)

also have $\dots = \mathbf{R}_s ((\neg R1 \ (\neg R2s P) \;; R1 \text{ true}) \vdash (\exists \ \$st' \cdot Q) \diamond R1 \ (R2s R))$

proof –

from $\text{assms}(3,4)$ have $(R1 \ (R2s R) \;; R1 \ (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 \ (R2s R)$

by (rel-auto, metis (no-types, lifting) minus-zero-eq, meson order-refl trace-class.diff-cancel)

thus ?thesis

by simp

qed
 also have ... = $\mathbf{R}_s((\neg (\neg P) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$
 by (metis (no-types, lifting) R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre
 RHS-design-post-R1 RHS-design-post-R2s)
 also have ... = $\mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$
 by (rel-auto)
 finally show ?thesis .
 qed

lemma *RHS-tri-design-RD3-intro*:

assumes
 $\$ok' \# P \$ok' \# Q \$ok' \# R \$st' \# Q \$wait' \# R$
 $P \text{ is } R1 (\neg_r P) ;; true_r = (\neg_r P)$
 shows $\mathbf{R}_s(P \vdash Q \diamond R) \text{ is } RD3$
 apply (simp add: Healthy-def RD3-def)
 apply (subst RHS-tri-design-right-unit-lemma)
 apply (simp-all add: assms ex-unrest rpred)
 done

RD3 reactive designs are those whose assumption can be written as a conjunction of a precondition on (undashed) program variables, and a negated statement about the trace. The latter allows us to state that certain events must not occur in the trace – which are effectively safety properties.

lemma *R1-right-unit-lemma*:

$\llbracket out\alpha \# b; out\alpha \# e \rrbracket \implies (\neg_r b \vee \$tr \hat{^}_u e \leq_u \$tr') ;; R1(true) = (\neg_r b \vee \$tr \hat{^}_u e \leq_u \$tr')$
 by (rel-auto, blast, metis (no-types, lifting) dual-order.trans)

lemma *RHS-tri-design-RD3-intro-form*:

assumes
 $out\alpha \# b out\alpha \# e \$ok' \# Q \$st' \# Q \$ok' \# R \$wait' \# R$
 shows $\mathbf{R}_s((b \wedge \neg_r \$tr \hat{^}_u e \leq_u \$tr') \vdash Q \diamond R) \text{ is } RD3$
 apply (rule RHS-tri-design-RD3-intro)
 apply (simp-all add: assms unrest closure rpred)
 apply (subst R1-right-unit-lemma)
 apply (simp-all add: assms unrest)
 done

definition *NSRD* :: $('s, 't :: trace, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$
 where [upred-defs]: $NSRD = RD1 \circ RD3 \circ RHS$

lemma *RD1-RD3-commute*: $RD1(RD3(P)) = RD3(RD1(P))$
 by (rel-auto, blast+)

lemma *NSRD-is-SRD [closure]*: $P \text{ is } NSRD \implies P \text{ is } SRD$

by (simp add: Healthy-def NSRD-def SRD-def, metis Healthy-def RD1-RD3-commute RD2-RHS-commute
 RD3-def RD3-right-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip)

lemma *NSRD-elim [RD-elim]*:

$\llbracket P \text{ is } NSRD; Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))) \rrbracket \implies Q(P)$
 by (simp add: RD-elim closure)

lemma *NSRD-Idempotent [closure]*: *Idempotent NSRD*

by (clarsimp simp add: Idempotent-def NSRD-def, metis (no-types, hide-lams) Healthy-def RD1-RD3-commute
 RD3-def RD3-idem RD3-left-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip)

lemma *NSRD-Continuous* [closure]: *Continuous NSRD*

by (*simp add: Continuous-comp NSRD-def RD1-Continuous RD3-Continuous RHS-Continuous*)

lemma *NSRD-form*:

$NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1\ true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$

proof –

have $NSRD(P) = RD3(SRD(P))$

by (*metis (no-types, lifting) NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-def*)

also have $\dots = RD3(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$

by (*simp add: SRD-as-reactive-tri-design*)

also have $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) ;; II_R$

by (*simp add: RD3-def*)

also have $\dots = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1\ true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$

by (*simp add: RHS-tri-design-right-unit-lemma unrest*)

finally show *?thesis* .

qed

lemma *NSRD-healthy-form*:

assumes P is *NSRD*

shows $\mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1\ true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P))) = P$

by (*metis Healthy-def NSRD-form assms*)

lemma *NSRD-Sup-closure* [closure]:

assumes $A \subseteq \llbracket NSRD \rrbracket_H$ $A \neq \{\}$

shows $\sqcap A$ is *NSRD*

proof –

have $NSRD (\sqcap A) = (\sqcap (NSRD\ 'A))$

by (*simp add: ContinuousD NSRD-Continuous assms(2)*)

also have $\dots = (\sqcap A)$

by (*simp only: Healthy-carrier-image assms*)

finally show *?thesis* **by** (*simp add: Healthy-def*)

qed

lemma *intChoice-NSRD-closed* [closure]:

assumes P is *NSRD* Q is *NSRD*

shows $P \sqcap Q$ is *NSRD*

using *NSRD-Sup-closure*[of $\{P, Q\}$] **by** (*simp add: assms*)

lemma *NRSD-SUP-closure* [closure]:

$\llbracket \bigwedge i. i \in A \implies P(i) \text{ is } NSRD; A \neq \{\} \rrbracket \implies (\sqcap i \in A. P(i)) \text{ is } NSRD$

by (*rule NSRD-Sup-closure, auto*)

lemma *NSRD-neg-pre-unit*:

assumes P is *NSRD*

shows $(\neg_r pre_R(P)) ;; true_r = (\neg_r pre_R(P))$

proof –

have $(\neg_r pre_R(P)) = (\neg_r pre_R(\mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1\ true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))))$

by (*simp add: NSRD-healthy-form assms*)

also have $\dots = R1\ (R2c\ ((\neg_r pre_R\ P) ;; R1\ true))$

by (*simp add: rea-pre-RHS-design R1-negate-R1 R1-idem R1-rea-not' R2c-rea-not usubst rpred unrest closure*)

also have $\dots = (\neg_r pre_R\ P) ;; R1\ true$

by (*simp add: R1-R2c-seqr-distribute closure assms*)

finally show *?thesis*

by (*simp add: rea-not-def*)

qed

lemma *NSRD-neg-pre-left-zero*:

assumes *P is NSRD Q is R1 Q is RD1*

shows $(\neg_r \text{pre}_R(P)) ;; Q = (\neg_r \text{pre}_R(P))$

by (*metis* (*no-types*, *hide-lams*) *NSRD-neg-pre-unit RD1-left-zero assms(1) assms(2) assms(3) seqr-assoc*)

lemma *NSRD-st'-unrest-peri* [*unrest*]:

assumes *P is NSRD*

shows $\$st' \# \text{peri}_R(P)$

proof –

have $\text{peri}_R(P) = \text{peri}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$

by (*simp add: NSRD-healthy-form assms*)

also have $\dots = R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true} \Rightarrow_r (\exists \$st' \cdot \text{peri}_R P)))$

by (*simp add: rea-peri-RHS-design usubst unrest*)

also have $\$st' \# \dots$

by (*simp add: R1-def R2c-def unrest*)

finally show *?thesis* .

qed

lemma *NSRD-wait'-unrest-pre* [*unrest*]:

assumes *P is NSRD*

shows $\$wait' \# \text{pre}_R(P)$

proof –

have $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$

by (*simp add: NSRD-healthy-form assms*)

also have $\dots = (R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true})))$

by (*simp add: rea-pre-RHS-design usubst unrest*)

also have $\$wait' \# \dots$

by (*simp add: R1-def R2c-def unrest*)

finally show *?thesis* .

qed

lemma *NSRD-st'-unrest-pre* [*unrest*]:

assumes *P is NSRD*

shows $\$st' \# \text{pre}_R(P)$

proof –

have $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$

by (*simp add: NSRD-healthy-form assms*)

also have $\dots = R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true}))$

by (*simp add: rea-pre-RHS-design usubst unrest*)

also have $\$st' \# \dots$

by (*simp add: R1-def R2c-def unrest*)

finally show *?thesis* .

qed

lemma *NSRD-alt-def*: $\text{NSRD}(P) = \text{RD3}(\text{SRD}(P))$

by (*metis* *NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-eq-dest-lhs*)

lemma *preR-RR* [*closure*]: $P \text{ is NSRD} \Rightarrow \text{pre}_R(P) \text{ is RR}$

by (*rule* *RR-intro*, *simp-all add: closure unrest*)

lemma *NSRD-neg-pre-RC* [*closure*]:

assumes *P is NSRD*

shows $\text{pre}_R(P) \text{ is RC}$

by (rule *RC-intro*, *simp-all add: closure assms NSRD-neg-pre-unit rpred*)

lemma *NSRD-intro*:

assumes P is *SRD* $(\neg_r \text{pre}_R(P)) \;; \text{true}_r = (\neg_r \text{pre}_R(P)) \ \$st' \ \# \ \text{peri}_R(P)$
 shows P is *NSRD*

proof –

have $\text{NSRD}(P) = \mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) \;; R1 \text{ true}) \vdash ((\exists \ \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$
 by (*simp add: NSRD-form*)
 also have $\dots = \mathbf{R}_s(\text{pre}_R P \vdash \text{peri}_R P \diamond \text{post}_R P)$
 by (*simp add: assms ex-unrest rpred closure*)
 also have $\dots = P$
 by (*simp add: SRD-reactive-tri-design assms(1)*)
 finally show *?thesis*
 using *Healthy-def* by *blast*

qed

lemma *NSRD-intro'*:

assumes P is *R2* P is *R3h* P is *RD1* P is *RD3*
 shows P is *NSRD*
 by (*metis (no-types, hide-lams) Healthy-def NSRD-def R1-R2c-is-R2 RHS-def assms comp-apply*)

lemma *NSRD-RC-intro*:

assumes P is *SRD* $\text{pre}_R(P)$ is *RC* $\$st' \ \# \ \text{peri}_R(P)$
 shows P is *NSRD*
 by (*metis Healthy-def NSRD-form SRD-reactive-tri-design assms(1) assms(2) assms(3) ex-unrest rea-not-false wp-rea-RC-false wp-rea-def*)

lemma *NSRD-rdes-intro* [*closure*]:

assumes P is *RC* Q is *RR* R is *RR* $\$st' \ \# \ Q$
 shows $\mathbf{R}_s(P \vdash Q \diamond R)$ is *NSRD*
 by (*rule NSRD-RC-intro, simp-all add: rdes closure assms unrest*)

lemma *SRD-RD3-implies-NSRD*:

$\llbracket P \text{ is } \text{SRD}; P \text{ is } \text{RD3} \rrbracket \implies P \text{ is } \text{NSRD}$
 by (*metis (no-types, lifting) Healthy-def NSRD-def RHS-idem SRD-healths(4) SRD-reactive-design comp-apply*)

lemma *NSRD-iff*:

$P \text{ is } \text{NSRD} \iff ((P \text{ is } \text{SRD}) \wedge (\neg_r \text{pre}_R(P)) \;; R1(\text{true}) = (\neg_r \text{pre}_R(P)) \wedge (\$st' \ \# \ \text{peri}_R(P)))$
 by (*meson NSRD-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri*)

lemma *NSRD-is-RD3* [*closure*]:

assumes P is *NSRD*
 shows P is *RD3*
 by (*simp add: NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri RD3-intro-pre assms*)

lemma *NSRD-refine-elim*:

assumes
 $P \sqsubseteq Q$ P is *NSRD* Q is *NSRD*
 $\llbracket 'pre_R(P) \Rightarrow pre_R(Q)'; 'pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)'; 'pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)' \rrbracket$
 $\implies R$
 shows R

proof –

have $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$

by (simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) assms(2) assms(3))
 hence 1: 'pre_R P ⇒ pre_R Q' and 2: 'pre_R P ∧ peri_R Q ⇒ peri_R P' and 3: 'pre_R P ∧ post_R Q ⇒
 post_R P'
 by (simp-all add: RHS-tri-design-refine assms closure)
 with assms(4) show ?thesis
 by simp
 qed

lemma NSRD-right-unit: P is NSRD ⇒ P ;; II_R = P
 by (metis Healthy-if NSRD-is-RD3 RD3-def)

lemma NSRD-composition-wp:

assumes P is NSRD Q is SRD

shows P ;; Q =

$\mathbf{R}_s ((pre_R P \wedge post_R P \wp_r pre_R Q) \vdash (peri_R P \vee (post_R P ;; peri_R Q)) \diamond (post_R P ;; post_R Q))$

by (simp add: SRD-composition-wp assms NSRD-is-SRD wp-rea-def NSRD-neg-pre-unit NSRD-st'-unrest-peri R1-negate-R1 R1-preR ex-unrest rpred)

lemma preR-NSRD-seq-lemma:

assumes P is NSRD Q is SRD

shows R1 (R2c (post_R P ;; (¬_r pre_R Q))) = post_R P ;; (¬_r pre_R Q)

proof –

have post_R P ;; (¬_r pre_R Q) = R1 (R2c (post_R P)) ;; R1 (R2c (¬_r pre_R Q))

by (simp add: NSRD-is-SRD R1-R2c-post-RHS R1-rea-not R2c-preR R2c-rea-not assms(1) assms(2))

also have ... = R1 (R2c (post_R P ;; (¬_r pre_R Q)))

by (simp add: R1-seqr R2c-R1-seq calculation)

finally show ?thesis ..

qed

lemma preR-NSRD-seq [rdes]:

assumes P is NSRD Q is SRD

shows pre_R(P ;; Q) = (pre_R P ∧ post_R P wp_r pre_R Q)

by (simp add: NSRD-composition-wp assms rea-pre-RHS-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute[THEN sym] R1-extend-conj' R1-idem R2c-not closure) (metis (no-types, lifting) Healthy-def Healthy-if NSRD-is-SRD R1-R2c-commute R1-R2c-seqr-distribute R1-seqr-closure assms(1) assms(2) postR-R2c-closed postR-SRD-R1 preR-R2c-closed rea-not-R1 rea-not-R2c)

lemma periR-NSRD-seq [rdes]:

assumes P is NSRD Q is NSRD

shows peri_R(P ;; Q) = ((pre_R P ∧ post_R P wp_r pre_R Q) ⇒_r (peri_R P ∨ (post_R P ;; peri_R Q)))

by (simp add: NSRD-composition-wp assms closure rea-peri-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not R1-peri-SRD)

lemma postR-NSRD-seq [rdes]:

assumes P is NSRD Q is NSRD

shows post_R(P ;; Q) = ((pre_R P ∧ post_R P wp_r pre_R Q) ⇒_r (post_R P ;; post_R Q))

by (simp add: NSRD-composition-wp assms closure rea-post-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not)

lemma NSRD-seqr-closure [closure]:

assumes P is NSRD Q is NSRD

shows $(P ;; Q)$ is NSRD
proof –
 have $(\neg_r \text{ post}_R P \text{ wp}_r \text{ pre}_R Q) ;; \text{ true}_r = (\neg_r \text{ post}_R P \text{ wp}_r \text{ pre}_R Q)$
 by (simp add: wp-rea-def rpred assms closure segr-assoc NSRD-neg-pre-unit)
 moreover have $\$st' \# \text{ pre}_R P \wedge \text{ post}_R P \text{ wp}_r \text{ pre}_R Q \Rightarrow_r \text{ peri}_R P \vee \text{ post}_R P ;; \text{ peri}_R Q$
 by (simp add: unrest assms wp-rea-def)
 ultimately show ?thesis
 by (rule-tac NSRD-intro, simp-all add: segr-or-distl NSRD-neg-pre-unit assms closure rdes unrest)
qed

lemma *RHS-tri-normal-design-composition*:

assumes
 $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$
 $\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$
 $P \text{ is } R2c \ Q_1 \text{ is } R1 \ Q_1 \text{ is } R2c \ Q_2 \text{ is } R1 \ Q_2 \text{ is } R2c$
 $R \text{ is } R2c \ S_1 \text{ is } R1 \ S_1 \text{ is } R2c \ S_2 \text{ is } R1 \ S_2 \text{ is } R2c$
 $R1 (\neg P) ;; R1(\text{true}) = R1(\neg P) \$st' \# Q_1$
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
 $= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof –
 have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
 $\mathbf{R}_s((R1 (\neg P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
 by (simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest)
 also have $\dots = \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
 by (simp add: assms wp-rea-def ex-unrest, rel-auto)
 finally show ?thesis .
qed

lemma *RHS-tri-normal-design-composition'* [rdes-def]:

assumes $P \text{ is } RC \ Q_1 \text{ is } RR \ \$st' \# Q_1 \ Q_2 \text{ is } RR \ R \text{ is } RR \ S_1 \text{ is } RR \ S_2 \text{ is } RR$
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
 $= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof –
 have $R1 (\neg P) ;; R1 \text{ true} = R1(\neg P)$
 using *RC-implies-RC1* [OF assms(1)]
 by (simp add: Healthy-def RC1-def rea-not-def)
 (metis *R1-negate-R1 R1-segr utp-pred-laws.double-compl*)
 thus ?thesis
 by (simp add: RHS-tri-normal-design-composition assms closure unrest *RR-implies-R2c*)
qed

If a normal reactive design has postcondition false, then it is a left zero for sequential composition.

lemma *NSRD-seq-post-false*:

assumes $P \text{ is } NSRD \ Q \text{ is } SRD \ \text{post}_R(P) = \text{false}$
shows $P ;; Q = P$
apply (simp add: NSRD-composition-wp assms wp rpred closure)
using *NSRD-is-SRD SRD-reactive-tri-design* assms(1,3) **apply** fastforce
done

lemma *NSRD-srd-skip* [closure]: Π_R is NSRD

by (rule NSRD-intro, simp-all add: rdes closure unrest)

lemma *NSRD-Chaos* [closure]: *Chaos* is NSRD

by (rule NSRD-intro, simp-all add: closure rdes unrest)

lemma *NSRD-Miracle* [closure]: *Miracle is NSRD*
 by (rule *NSRD-intro*, simp-all add: closure rdes unrest)

Post-composing a miracle filters out the non-terminating behaviours

lemma *NSRD-right-Miracle-tri-lemma*:
 assumes *P is NSRD*
 shows $P ;; \text{Miracle} = \mathbf{R}_s (pre_R P \vdash peri_R P \diamond false)$
 by (simp add: *NSRD-composition-wp closure assms rdes wp rpred*)

The set of non-terminating behaviours is a subset

lemma *NSRD-right-Miracle-refines*:
 assumes *P is NSRD*
 shows $P \sqsubseteq P ;; \text{Miracle}$
proof –
 have $\mathbf{R}_s (pre_R P \vdash peri_R P \diamond post_R P) \sqsubseteq \mathbf{R}_s (pre_R P \vdash peri_R P \diamond false)$
 by (rule *sdes-tri-refine-intro*, rel-auto+)
 thus ?thesis
 by (simp add: *NSRD-elim NSRD-right-Miracle-tri-lemma assms*)
qed

lemma *upower-Suc-NSRD-closed* [closure]:
 $P \text{ is NSRD} \implies P \wedge \text{Suc } n \text{ is NSRD}$
proof (induct *n*)
 case 0
 then show ?case
 by (simp)
next
 case (Suc *n*)
 then show ?case
 by (simp add: *NSRD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *NSRD-power-Suc* [closure]:
 $P \text{ is NSRD} \implies P ;; P \wedge n \text{ is NSRD}$
 by (metis *upower-Suc-NSRD-closed upred-semiring.power-Suc*)

lemma *uplus-NSRD-closed* [closure]: $P \text{ is NSRD} \implies P^+ \text{ is NSRD}$
 by (simp add: *uplus-power-def closure*)

lemma *preR-power*:
 assumes *P is NSRD*
 shows $pre_R(P ;; P \wedge n) = (\bigsqcup i \in \{0..n\}. (post_R(P) \wedge i) \text{ wp}_r (pre_R(P)))$
proof (induct *n*)
 case 0
 then show ?case
 by (simp add: wp closure)
next
 case (Suc *n*) **note** *hyp = this*
 have $pre_R(P \wedge (Suc\ n + 1)) = pre_R(P ;; P \wedge (n+1))$
 by (simp add: *upred-semiring.power-Suc*)
 also have $\dots = (pre_R P \wedge post_R P \text{ wp}_r pre_R(P \wedge (Suc\ n)))$
 using *NSRD-iff assms preR-NSRD-seq upower-Suc-NSRD-closed* **by** fastforce
 also have $\dots = (pre_R P \wedge post_R P \text{ wp}_r (\bigsqcup i \in \{0..n\}. post_R P \wedge i \text{ wp}_r pre_R P))$
 by (simp add: *hyp upred-semiring.power-Suc*)

also have ... = $(pre_R P \wedge (\bigsqcup_{i \in \{0..n\}} post_R P wp_r (post_R P \wedge i wp_r pre_R P)))$
by (*simp add: wp*)
also have ... = $(pre_R P \wedge (\bigsqcup_{i \in \{0..n\}} (post_R P \wedge (i+1) wp_r pre_R P)))$
proof –
have $\bigwedge i. R1 (post_R P \wedge i ;; (\neg_r pre_R P)) = (post_R P \wedge i ;; (\neg_r pre_R P))$
by (*induct-tac i, simp-all add: closure Healthy-if assms*)
thus ?thesis
by (*simp add: wp-rea-def upred-semiring.power-Suc seqr-assoc rpred closure assms*)
qed
also have ... = $(post_R P \wedge 0 wp_r pre_R P \wedge (\bigsqcup_{i \in \{0..n\}} (post_R P \wedge (i+1) wp_r pre_R P)))$
by (*simp add: wp assms closure*)
also have ... = $(post_R P \wedge 0 wp_r pre_R P \wedge (\bigsqcup_{i \in \{1..Suc\ n\}} (post_R P \wedge i wp_r pre_R P)))$
proof –
have $(\bigsqcup_{i \in \{0..n\}} (post_R P \wedge (i+1) wp_r pre_R P)) = (\bigsqcup_{i \in \{1..Suc\ n\}} (post_R P \wedge i wp_r pre_R P))$
by (*rule cong[of Inf], simp-all add: fun-eq-iff*)
(metis (no-types, lifting) image-Suc-atLeastAtMost image-cong image-image)
thus ?thesis **by** *simp*
qed
also have ... = $(\bigsqcup_{i \in insert\ 0\ \{1..Suc\ n\}} (post_R P \wedge i wp_r pre_R P))$
by (*simp add: conj-upred-def*)
also have ... = $(\bigsqcup_{i \in \{0..Suc\ n\}} post_R P \wedge i wp_r pre_R P)$
by (*simp add: atLeast0-atMost-Suc-eq-insert-0*)
finally show ?case **by** (*simp add: upred-semiring.power-Suc*)
qed

lemma *preR-power'* [rdes]:
assumes *P is NSRD*
shows $pre_R(P ;; P^n) = (\bigsqcup_{i \in \{0..n\}} (post_R(P) \wedge i) wp_r (pre_R(P)))$
by (*simp add: preR-power assms USUP-as-Inf[THEN sym]*)

lemma *preR-power-Suc* [rdes]:
assumes *P is NSRD*
shows $pre_R(P \wedge (Suc\ n)) = (\bigsqcup_{i \in \{0..n\}} (post_R(P) \wedge i) wp_r (pre_R(P)))$
by (*simp add: upred-semiring.power-Suc rdes assms*)

declare *upred-semiring.power-Suc* [*simp*]

lemma *periR-power*:
assumes *P is NSRD*
shows $peri_R(P ;; P^n) = (pre_R(P \wedge (Suc\ n)) \Rightarrow_r (\bigcap_{i \in \{0..n\}} post_R(P) \wedge i) ;; peri_R(P))$

proof (*induct n*)
case 0
then show ?case
by (*simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-peri-under-pre assms*)
next
case (*Suc n*) **note** *hyp = this*
have $peri_R(P \wedge (Suc\ n + 1)) = peri_R(P ;; P \wedge (n+1))$
by (*simp*)
also have ... = $(pre_R(P \wedge (Suc\ n + 1)) \Rightarrow_r (peri_R P \vee post_R P ;; peri_R(P ;; P \wedge n)))$
by (*simp add: closure assms rdes*)
also have ... = $(pre_R(P \wedge (Suc\ n + 1)) \Rightarrow_r (peri_R P \vee post_R P ;; (pre_R(P \wedge (Suc\ n)) \Rightarrow_r (\bigcap_{i \in \{0..n\}} post_R P \wedge i) ;; peri_R P)))$
by (*simp only: hyp*)
also
have ... = $(pre_R P \Rightarrow_r peri_R P \vee (post_R P wp_r pre_R(P ;; P \wedge n) \Rightarrow_r post_R P ;; (pre_R(P ;; P \wedge n)))$


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 $\Rightarrow_r (\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P))$ 
  by (simp add: rdes closure assms, rel-blast)
  also
  have ... = (pre_R P  $\Rightarrow_r$  peri_R P  $\vee$  (post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  post_R P ;; ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ )))
  proof -
    have ( $\bigwedge i \in \{0..n\}. post_R P \wedge i$ ) is R1
    by (simp add: NSRD-is-SRD R1-Continuous R1-power Sup-Continuous-closed assms postR-SRD-R1)
    hence 1: ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ ) is R1
    by (simp add: closure assms)
    hence (pre_R (P ;; P ^ n)  $\Rightarrow_r$  ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ ) is R1
    by (simp add: closure)
    hence (post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  post_R P ;; (pre_R (P ;; P ^ n)  $\Rightarrow_r$  ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ )))
    = (post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  R1(post_R P) ;; R1(pre_R (P ;; P ^ n)  $\Rightarrow_r$  ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ )))
    by (simp add: Healthy-if R1-post-SRD assms closure)
    thus ?thesis
    by (simp only: wp-rea-impl-lemma, simp add: Healthy-if 1, simp add: R1-post-SRD assms closure)
  qed
  also
  have ... = (pre_R P  $\wedge$  post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  peri_R P  $\vee$  post_R P ;; ( $\bigwedge i \in \{0..n\}. post_R P \wedge i ;; peri_R P$ ))
  by (pred-auto)
  also
  have ... = (pre_R P  $\wedge$  post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  peri_R P  $\vee$  (( $\bigwedge i \in \{0..n\}. post_R P \wedge (Suc i) ;; peri_R P$ )))
  by (simp add: seq-Sup-distl seqr-assoc[THEN sym])
  also
  have ... = (pre_R P  $\wedge$  post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  peri_R P  $\vee$  (( $\bigwedge i \in \{1..Suc n\}. post_R P \wedge i ;; peri_R P$ )))
  proof -
    have ( $\bigwedge i \in \{0..n\}. post_R P \wedge Suc i$ ) = ( $\bigwedge i \in \{1..Suc n\}. post_R P \wedge i$ )
    apply (rule cong[of Sup], auto)
    apply (metis atLeast0AtMost atMost-iff image-Suc-atLeastAtMost rev-image-eqI upred-semiring.power-Suc)
    using Suc-le-D apply fastforce
  done
  thus ?thesis by simp
  qed
  also
  have ... = (pre_R P  $\wedge$  post_R P wp_r pre_R (P ;; P ^ n)  $\Rightarrow_r$  (( $\bigwedge i \in \{0..Suc n\}. post_R P \wedge i$ ) ;; peri_R P)
  by (simp add: SUP-atLeastAtMost-first uinf-or seqr-or-distl seqr-or-distr)
  also
  have ... = (pre_R (P ^ (Suc (Suc n)))  $\Rightarrow_r$  (( $\bigwedge i \in \{0..Suc n\}. post_R P \wedge i$ ) ;; peri_R P)
  by (simp add: rdes closure assms)
  finally show ?case by (simp)
  qed

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lemma periR-power' [rdes]:
  assumes P is NSRD
  shows peri_R(P ;; P ^ n) = (pre_R(P ^ (Suc n))  $\Rightarrow_r$  ( $\bigwedge i \in \{0..n\}. post_R(P) \wedge i ;; peri_R(P)$ ))
  by (simp add: periR-power assms UINF-as-Sup[THEN sym])

```

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lemma periR-power-Suc [rdes]:
  assumes P is NSRD

```

shows $\text{peri}_R(P^\wedge(\text{Suc } n)) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r (\bigwedge i \in \{0..n\} \cdot \text{post}_R(P) \wedge i) ;; \text{peri}_R(P))$
by (*simp add: rdes assms*)

lemma *postR-power [rdes]*:

assumes *P is NSRD*

shows $\text{post}_R(P ;; P^\wedge n) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r \text{post}_R(P) \wedge \text{Suc } n)$

proof (*induct n*)

case 0

then show ?case

by (*simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-post-under-pre assms*)

next

case (*Suc n*) **note** *hyp = this*

have $\text{post}_R(P \wedge (\text{Suc } n + 1)) = \text{post}_R(P ;; P^\wedge (n+1))$

by (*simp*)

also have $\dots = (\text{pre}_R(P \wedge (\text{Suc } n + 1)) \Rightarrow_r (\text{post}_R P ;; \text{post}_R(P \wedge n)))$

by (*simp add: closure assms rdes*)

also have $\dots = (\text{pre}_R(P \wedge (\text{Suc } n + 1)) \Rightarrow_r (\text{post}_R P ;; (\text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } n)))$

by (*simp only: hyp*)

also

have $\dots = (\text{pre}_R P \Rightarrow_r (\text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P ;; (\text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } n)))$

by (*simp add: rdes closure assms, pred-auto*)

also

have $\dots = (\text{pre}_R P \Rightarrow_r (\text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P ;; \text{post}_R P \wedge \text{Suc } n))$

by (*metis (no-types, lifting) Healthy-if NSRD-is-SRD NSRD-power-Suc R1-power assms hyp postR-SRD-R1 upred-semiring.power-Suc wp-rea-impl-lemma*)

also

have $\dots = (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } (\text{Suc } n))$

by (*pred-auto*)

also have $\dots = (\text{pre}_R(P^\wedge(\text{Suc } (\text{Suc } n))) \Rightarrow_r \text{post}_R P \wedge \text{Suc } (\text{Suc } n))$

by (*simp add: rdes closure assms*)

finally show ?case **by** (*simp*)

qed

lemma *postR-power-Suc [rdes]*:

assumes *P is NSRD*

shows $\text{post}_R(P^\wedge(\text{Suc } n)) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r \text{post}_R(P) \wedge \text{Suc } n)$

by (*simp add: rdes assms*)

lemma *power-rdes-def [rdes-def]*:

assumes *P is RC Q is RR R is RR \$st' \# Q*

shows $(\mathbf{R}_s(P \vdash Q \diamond R))^\wedge(\text{Suc } n)$

$= \mathbf{R}_s((\bigwedge i \in \{0..n\} \cdot (R \wedge i) \text{ wp}_r P) \vdash ((\bigwedge i \in \{0..n\} \cdot R \wedge i) ;; Q) \diamond (R \wedge \text{Suc } n))$

proof (*induct n*)

case 0

then show ?case

by (*simp add: wp assms closure*)

next

case (*Suc n*)

have 1: $(P \wedge (\bigwedge i \in \{0..n\} \cdot R \text{ wp}_r (R \wedge i \text{ wp}_r P))) = (\bigwedge i \in \{0..\text{Suc } n\} \cdot R \wedge i \text{ wp}_r P)$
(is ?lhs = ?rhs)

proof –

have ?lhs = $(P \wedge (\bigwedge i \in \{0..n\} \cdot (R \wedge \text{Suc } i \text{ wp}_r P)))$

by (*simp add: wp closure assms*)

```

also have ... = (P ∧ (⋀ i ∈ {0..n}. (R ^ Suc i wpr P)))
  by (simp only: USUP-as-Inf-collect)
also have ... = (P ∧ (⋀ i ∈ {1..Suc n}. (R ^ i wpr P)))
  by (metis (no-types, lifting) INF-cong One-nat-def image-Suc-atLeastAtMost image-image)
also have ... = (⋀ i ∈ insert 0 {1..Suc n}. (R ^ i wpr P))
  by (simp add: wp assms closure conj-upred-def)
also have ... = (⋀ i ∈ {0..Suc n}. (R ^ i wpr P))
  by (simp add: atLeastAtMost-insertL)
finally show ?thesis
  by (simp add: USUP-as-Inf-collect)
qed

have 2: (Q ∨ R ;; (⋀ i ∈ {0..n}. R ^ i) ;; Q) = (⋀ i ∈ {0..Suc n}. R ^ i) ;; Q
  (is ?lhs = ?rhs)
proof -
  have ?lhs = (Q ∨ (⋀ i ∈ {0..n}. R ^ Suc i) ;; Q)
    by (simp add: seqr-assoc[THEN sym] seq-UINF-distl)
  also have ... = (Q ∨ (⋀ i ∈ {0..n}. R ^ Suc i) ;; Q)
    by (simp only: UINF-as-Sup-collect)
  also have ... = (Q ∨ (⋀ i ∈ {1..Suc n}. R ^ i) ;; Q)
    by (metis One-nat-def image-Suc-atLeastAtMost image-image)
  also have ... = ((⋀ i ∈ insert 0 {1..Suc n}. R ^ i) ;; Q)
    by (simp add: disj-upred-def[THEN sym] seqr-or-distl)
  also have ... = ((⋀ i ∈ {0..Suc n}. R ^ i) ;; Q)
    by (simp add: atLeastAtMost-insertL)
  finally show ?thesis
    by (simp add: UINF-as-Sup-collect)
qed

have 3: (⋀ i ∈ {0..n}. R ^ i) ;; Q is RR
proof -
  have (⋀ i ∈ {0..n}. R ^ i) ;; Q = (⋀ i ∈ {0..n}. R ^ i) ;; Q
    by (simp add: UINF-as-Sup-collect)
  also have ... = (⋀ i ∈ insert 0 {1..n}. R ^ i) ;; Q
    by (simp add: atLeastAtMost-insertL)
  also have ... = (Q ∨ (⋀ i ∈ {1..n}. R ^ i) ;; Q)
    by (metis (no-types, lifting) SUP-insert disj-upred-def seqr-left-unit seqr-or-distl upred-semiring.power-0)
  also have ... = (Q ∨ (⋀ i ∈ {0..<n}. R ^ Suc i) ;; Q)
    by (metis One-nat-def atLeastLessThanSuc-atLeastAtMost image-Suc-atLeastLessThan image-image)
  also have ... = (Q ∨ (⋀ i ∈ {0..<n}. R ^ Suc i) ;; Q)
    by (simp add: UINF-as-Sup-collect)
  also have ... is RR
    by (simp-all add: closure assms)
  finally show ?thesis .
qed
from 1 2 3 Suc show ?case
  by (simp add: Suc RHS-tri-normal-design-composition' closure assms wp)
qed

declare upred-semiring.power-Suc [simp del]

theorem uplus-rdes-def [rdes-def]:
  assumes P is RC Q is RR R is RR $st' ≠ Q
  shows (Rs(P ⊢ Q ◊ R))+ = Rs(R*r wpr P ⊢ R*r ;; Q ◊ R+)
proof -

```

```

have 1:( $\prod i \cdot R \hat{=} i$ ) ;;  $Q = R^{*r}$  ;;  $Q$ 
  by (metis (no-types) RA1 assms(2) rea-skip-unit(2) rrel-thy.Star-def ustar-alt-def)
show ?thesis
  by (simp add: uplus-power-def seq-UINF-distr wp closure assms rdes-def)
    (metis 1 seq-UINF-distr')
qed

```

5.1 UTP theory

typeddecl *NSRDES*

abbreviation *NSRDES* \equiv *UTHY*(*NSRDES*, ('s,'t::trace,' α) *rsp*)

overloading

nsrdes-hcond == *utp-hcond* :: (*NSRDES*, ('s,'t::trace,' α) *rsp*) *uthy* \Rightarrow (('s,'t,' α) *rsp* \times ('s,'t,' α) *rsp*) *health*

nsrdes-unit == *utp-unit* :: (*NSRDES*, ('s,'t::trace,' α) *rsp*) *uthy* \Rightarrow ('s, 't, ' α) *hrel-rsp*

begin

definition *nsrdes-hcond* :: (*NSRDES*, ('s,'t::trace,' α) *rsp*) *uthy* \Rightarrow (('s,'t,' α) *rsp* \times ('s,'t,' α) *rsp*) *health* **where**

[*upred-defs*]: *nsrdes-hcond* *T* = *NSRD*

definition *nsrdes-unit* :: (*NSRDES*, ('s,'t::trace,' α) *rsp*) *uthy* \Rightarrow ('s, 't, ' α) *hrel-rsp* **where**

[*upred-defs*]: *nsrdes-unit* *T* = *II_R*

end

interpretation *nsrd-thy*: *utp-theory-kleene* *UTHY*(*NSRDES*, ('s,'t::trace,' α) *rsp*)

rewrites $\bigwedge P. P \in \text{carrier } (\text{uthy-order } \text{NSRDES}) \longleftrightarrow P \text{ is } \text{NSRD}$

and $P \text{ is } \mathcal{H}_{\text{NSRDES}} \longleftrightarrow P \text{ is } \text{NSRD}$

and $(\mu X \cdot F (\mathcal{H}_{\text{NSRDES}} X)) = (\mu X \cdot F (\text{NSRD } X))$

and $\text{carrier } (\text{uthy-order } \text{NSRDES}) \rightarrow \text{carrier } (\text{uthy-order } \text{NSRDES}) \equiv \llbracket \text{NSRD} \rrbracket_H \rightarrow \llbracket \text{NSRD} \rrbracket_H$

and $\llbracket \mathcal{H}_{\text{NSRDES}} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{\text{NSRDES}} \rrbracket_H \equiv \llbracket \text{NSRD} \rrbracket_H \rightarrow \llbracket \text{NSRD} \rrbracket_H$

and $\top_{\text{NSRDES}} = \text{Miracle}$

and $\mathcal{II}_{\text{NSRDES}} = \text{II}_R$

and $\text{le } (\text{uthy-order } \text{NSRDES}) = \text{op } \sqsubseteq$

proof –

interpret *lat*: *utp-theory-continuous* *UTHY*(*NSRDES*, ('s,'t,' α) *rsp*)

by (*unfold-locales*, *simp-all* add: *nsrdes-hcond-def* *nsrdes-unit-def* *closure* *Healthy-if*)

show 1: $\top_{\text{NSRDES}} = (\text{Miracle} :: ('s,'t,' α) \text{hrel-rsp})$

by (*metis* *NSRD-Miracle* *NSRD-is-SRD* *lat.top-healthy* *lat.utp-theory-continuous-axioms* *nsrdes-hcond-def* *srdes-theory-continuous.meet-top* *upred-semiring.add-commute* *utp-theory-continuous.meet-top*)

thus *utp-theory-kleene* *UTHY*(*NSRDES*, ('s,'t,' α) *rsp*)

by (*unfold-locales*, *simp-all* add: *nsrdes-hcond-def* *nsrdes-unit-def* *closure* *Healthy-if* *Miracle-left-zero* *SRD-left-unit* *NSRD-right-unit*)

qed (*simp-all* add: *nsrdes-hcond-def* *nsrdes-unit-def* *closure* *Healthy-if*)

declare *nsrd-thy.top-healthy* [*simp del*]

declare *nsrd-thy.bottom-healthy* [*simp del*]

abbreviation *TestR* (*test_R*) **where**

test_R *P* \equiv *utest* *NSRDES* *P*

abbreviation *StarR* :: ('s, 't::trace, ' α) *hrel-rsp* \Rightarrow ('s, 't, ' α) *hrel-rsp* ($-^{*R}$ [999] 999) **where**

StarR *P* $\equiv P \star \text{NSRDES}$

lemma *StarR-rdes-def* [*rdes-def*]:

assumes P is RC Q is RR R is RR $\$st' \# Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))^{*R} = \mathbf{R}_s((R^{*r} \text{ wp}_r P) \vdash R^{*r} ;; Q \diamond R^{*r})$
by (*simp add: rrel-thy.Star-alt-def nsrd-thy.Star-alt-def assms closure rdes-def unrest rpred disj-upred-def*)
end

6 Syntax for reactive design contracts

theory *utp-rdes-contracts*
imports *utp-rdes-normal*
begin

We give an experimental syntax for reactive design contracts $[P \vdash Q | R]_R$, where P is a precondition on undashed state variables only, Q is a pericondition that can refer to the trace and before state but not the after state, and R is a postcondition. Both Q and R can refer only to the trace contribution through a HOL variable *trace* which is bound to $\&tt$.

definition *mk-RD* :: $'s \text{ upred} \Rightarrow ('t::\text{trace} \Rightarrow 's \text{ upred}) \Rightarrow ('t \Rightarrow 's \text{ hrel}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$ **where**
mk-RD $P \ Q \ R = \mathbf{R}_s([\![P]\!]_{S<} \vdash [\![Q(x)]\!]_{S<} \llbracket x \rightarrow \&tt \rrbracket \diamond [\![R(x)]\!]_S \llbracket x \rightarrow \&tt \rrbracket)$

definition *trace-pred* :: $('t::\text{trace} \Rightarrow 's \text{ upred}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$ **where**
 $[\text{upred-defs}]: \text{trace-pred } P = [\![P \ x]\!]_{S<} \llbracket x \rightarrow \&tt \rrbracket$

syntax

$\text{-trace-var} :: \text{logic}$
 $\text{-mk-RD} \quad :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\[/\vdash \ / \mid \ /]_R)$
 $\text{-trace-pred} \quad :: \text{logic} \Rightarrow \text{logic} \ (\[/\!]_t)$

parse-translation \ll

let
 $\text{fun trace-var-tr } [] = \text{Syntax.free trace}$
 $\quad | \text{ trace-var-tr } - = \text{raise Match};$
 in
 $[(\text{@}\{\text{syntax-const } \text{-trace-var}\}, K \text{ trace-var-tr})]$
 end
 \gg

translations

$[P \vdash Q \mid R]_R \Rightarrow \text{CONST } \text{mk-RD } P \ (\lambda \text{-trace-var. } Q) \ (\lambda \text{-trace-var. } R)$
 $[P \vdash Q \mid R]_R \Leftarrow \text{CONST } \text{mk-RD } P \ (\lambda x. Q) \ (\lambda y. R)$
 $[P]_t \Rightarrow \text{CONST } \text{trace-pred } (\lambda \text{-trace-var. } P)$
 $[P]_t \Leftarrow \text{CONST } \text{trace-pred } (\lambda t. P)$

lemma *SRD-mk-RD* [*closure*]: $[P \vdash Q(\text{trace}) \mid R(\text{trace})]_R$ is *SRD*
by (*simp add: mk-RD-def closure unrest*)

lemma *preR-mk-RD* [*rdes*]: $\text{pre}_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = R1([\![P]\!]_{S<})$
by (*simp add: mk-RD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

lemma *trace-pred-RR-closed* [*closure*]:
 $[P \text{ trace}]_t$ is *RR*
by (*rel-auto*)

lemma *unrest-trace-pred-st'* [*unrest*]:

$\$st' \# [P \text{ trace}]_t$
by (*rel-auto*)

lemma *R2c-msubst-tt*: $R2c \ (msubst \ (\lambda x. \lceil Q \ x \rceil_S) \ \&tt) = (msubst \ (\lambda x. \lceil Q \ x \rceil_S) \ \&tt)$
by (*rel-auto*)

lemma *periR-mk-RD* [*rdes*]: $peri_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil Q(\text{trace}) \rceil_{S<}) \llbracket \text{trace} \rightarrow \&tt \rrbracket))$
by (*simp add: mk-RD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre R2c-disj R2c-msubst-tt R1-disj R2c-rea-impl R1-rea-impl*)

lemma *postR-mk-RD* [*rdes*]: $post_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil R(\text{trace}) \rceil_S) \llbracket \text{trace} \rightarrow \&tt \rrbracket))$
by (*simp add: mk-RD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R2c-rea-impl R1-rea-impl*)

Refinement introduction law for contracts

lemma *RD-contract-refine*:

assumes

$Q \text{ is } SRD \ \lceil P_1 \rceil_{S<} \Rightarrow pre_R \ Q$
 $\lceil P_1 \rceil_{S<} \wedge peri_R \ Q \Rightarrow \lceil P_2 \ x \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket$
 $\lceil P_1 \rceil_{S<} \wedge post_R \ Q \Rightarrow \lceil P_3 \ x \rceil_S \llbracket x \rightarrow \&tt \rrbracket$

shows $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq Q$

proof –

have $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$

using *assms*

by (*simp add: mk-RD-def, rule-tac srdes-tri-refine-intro, simp-all*)

thus *?thesis*

by (*simp add: SRD-reactive-tri-design assms(1)*)

qed

end

7 Reactive design tactics

theory *utp-rdes-tactics*

imports *utp-rdes-triples*

begin

Theorems for normalisation

lemmas *rdes-rel-norms* =

prod.case-eq-if

conj-assoc

disj-assoc

conj-UINF-dist

conj-UINF-ind-dist

seqr-or-distl

seqr-or-distr

seq-UINF-distl

seq-UINF-distl'

seq-UINF-distr

seq-UINF-distr'

The following tactic can be used to simply and evaluate reactive predicates.

method *rpred-simp* = (*ueexpr-simp_simps: rpred usubst closure unrest*)

Tactic to expand out healthy reactive design predicates into the syntactic triple form.

method *rdes-expand* **uses** *cls* = (*insert cls*, (*erule RD-elim*)⁺)

Tactic to simplify the definition of a reactive design

method *rdes-simp* **uses** *cls cong_simps* =
 ((*rdes-expand cls*: *cls*)?, (*simp add*: *rdes-def rdes-rel-norms rdes rpred cls closure alpha usubst unrest*
wp_simps cong: *cong*))

Tactic to split a refinement conjecture into three POs

method *rdes-refine-split* **uses** *cls cong_simps* =
 (*rdes-simp cls*: *cls cong*: *cong_simps*: *simps*; *rule-tac srdes-tri-refine-intro'*)

Tactic to split an equality conjecture into three POs

method *rdes-eq-split* **uses** *cls cong_simps* =
 (*rdes-simp cls*: *cls cong*: *cong_simps*: *simps*; (*rule-tac srdes-tri-eq-intro*))

Tactic to prove a refinement

method *rdes-refine* **uses** *cls cong_simps* =
 (*rdes-refine-split cls*: *cls cong*: *cong_simps*: *simps*; (*insert cls*; *rel-auto*))

Tactics to prove an equality

method *rdes-eq* **uses** *cls cong_simps* =
 (*rdes-eq-split cls*: *cls cong*: *cong_simps*: *simps*; *rel-auto*)

Via antisymmetry

method *rdes-eq-anti* **uses** *cls cong_simps* =
 (*rdes-simp cls*: *cls cong*: *cong_simps*: *simps*; (*rule-tac antisym*; (*rule-tac srdes-tri-refine-intro*; *rel-auto*))))

Tactic to calculate pre/peri/postconditions from reactive designs

method *rdes-calc* = (*simp add*: *rdes rpred closure alpha usubst unrest wp prod.case-eq-if*)

The following tactic attempts to prove a reactive design refinement by calculation of the pre-, peri-, and postconditions and then showing three implications between them using *rel-blast*.

method *rdspl-refine* =
 (*rule-tac SRD-refine-intro*; (*simp add*: *closure rdes unrest usubst* ; *rel-blast?*))

The following tactic combines antisymmetry with the previous tactic to prove an equality.

method *rdspl-eq* =
 (*rule-tac antisym*, *rdes-refine*, *rdes-refine*)

end

8 Reactive design parallel-by-merge

theory *utp-rdes-parallel*

imports

utp-rdes-normal

utp-rdes-tactics

begin

R3h implicitly depends on RD1, and therefore it requires that both sides be RD1. We also require that both sides are R3c, and that $wait_m$ is a quasi-unit, and div_m yields divergence.

lemma *st-U0-alpha*: $\lceil \exists \$st \cdot II \rceil_0 = (\exists \$st \cdot \lceil II \rceil_0)$

by (rel-auto)

lemma *st-U1-alpha*: $\lceil \exists \$st \cdot II \rceil_1 = (\exists \$st \cdot \lceil II \rceil_1)$
 by (rel-auto)

definition *skip-rm* :: $(\text{'s, 't::trace, '}\alpha) \text{ rsp merge } (II_{RM})$ **where**
 $[upred-defs]: II_{RM} = (\exists \$st_{<} \cdot skip_m \vee (\neg \$ok_{<} \wedge \$tr_{<} \leq_u \$tr'))$

definition $[upred-defs]: R3hm(M) = (II_{RM} \triangleleft \$wait_{<} \triangleright M)$

lemma *R3hm-idem*: $R3hm(R3hm(P)) = R3hm(P)$
 by (rel-auto)

lemma *R3h-par-by-merge* [closure]:
 assumes *P is R3h Q is R3h M is R3hm*
 shows $(P \parallel_M Q)$ is R3h

proof –

have $(P \parallel_M Q) = (((P \parallel_M Q) \llbracket true/\$ok \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false/\$ok \rrbracket) \llbracket true/\$wait \rrbracket \triangleleft \$wait \triangleright (P \parallel_M Q))$

by (simp add: cond-var-subst-left cond-var-subst-right)

also have $\dots = (((P \parallel_M Q) \llbracket true, true/\$ok, \$wait \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false, true/\$ok, \$wait \rrbracket) \triangleleft \$wait \triangleright (P \parallel_M Q))$

by (rel-auto)

also have $\dots = (((\exists \$st \cdot II) \llbracket true, true/\$ok, \$wait \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false, true/\$ok, \$wait \rrbracket) \triangleleft \$wait \triangleright (P \parallel_M Q))$

proof –

have $(P \parallel_M Q) \llbracket true, true/\$ok, \$wait \rrbracket = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<}' =_u \$v) ;; R3hm(M)) \llbracket true, true/\$ok, \$wait \rrbracket$

by (simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if)

also have $\dots = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<}' =_u \$v) ;; (\exists \$st_{<} \cdot \$v' =_u \$v_{<})) \llbracket true, true/\$ok, \$wait \rrbracket$

by (rel-blast)

also have $\dots = ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$v_{<}' =_u \$v) ;; (\exists \$st_{<} \cdot \$v' =_u \$v_{<})) \llbracket true, true/\$ok, \$wait \rrbracket$

by (simp add: assms Healthy-if)

also have $\dots = (\exists \$st \cdot II) \llbracket true, true/\$ok, \$wait \rrbracket$

by (rel-auto)

finally show ?thesis by (simp add: closure assms unrest)

qed

also have $\dots = (((\exists \$st \cdot II) \llbracket true, true/\$ok, \$wait \rrbracket \triangleleft \$ok \triangleright (R1(true)) \llbracket false, true/\$ok, \$wait \rrbracket) \triangleleft \$wait \triangleright (P \parallel_M Q))$

proof –

have $(P \parallel_M Q) \llbracket false, true/\$ok, \$wait \rrbracket = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<}' =_u \$v) ;; R3hm(M)) \llbracket false, true/\$ok, \$wait \rrbracket$

by (simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if)

also have $\dots = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<}' =_u \$v) ;; (\$tr_{<} \leq_u \$tr')) \llbracket false, true/\$ok, \$wait \rrbracket$

by (rel-blast)

also have $\dots = ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$v_{<}' =_u \$v) ;; (\$tr_{<} \leq_u \$tr')) \llbracket false, true/\$ok, \$wait \rrbracket$

by (simp add: assms Healthy-if)

also have $\dots = (R1(true)) \llbracket false, true/\$ok, \$wait \rrbracket$

by (rel-blast)

finally show ?thesis by simp

qed

also have $\dots = (((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q))$

by (rel-auto)

also have $\dots = R3h(P \parallel_M Q)$

by (simp add: R3h-cases)

finally show ?thesis

by (simp add: Healthy-def)

qed

definition $[upred-defs]$: $RD1m(M) = (M \vee \neg \$ok_{<} \wedge \$tr_{<} \leq_u \$tr')$

lemma $RD1\text{-}par\text{-}by\text{-}merge$ $[closure]$:

assumes P is $R1$ Q is $R1$ M is $R1m$ P is $RD1$ Q is $RD1$ M is $RD1m$
 shows $(P \parallel_M Q)$ is $RD1$

proof –

have $1: (RD1(R1(P)) \parallel_{RD1m(R1m(M))} RD1(R1(Q))) \llbracket false/\$ok \rrbracket = R1(true)$

by $(rel\text{-}blast)$

have $(P \parallel_M Q) = (P \parallel_M Q) \llbracket true/\$ok \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false/\$ok \rrbracket$

by $(simp\ add: cond\text{-}var\text{-}split)$

also have $\dots = R1(P \parallel_M Q) \triangleleft \$ok \triangleright R1(true)$

by $(metis\ 1\ Healthy\text{-}if\ R1\text{-}par\text{-}by\text{-}merge\ assms\ calculation\ cond\text{-}idem\ cond\text{-}var\text{-}subst\text{-}right\ in\text{-}var\text{-}uvar\ ok\text{-}vwb\text{-}lens)$

also have $\dots = RD1(P \parallel_M Q)$

by $(simp\ add: Healthy\text{-}if\ R1\text{-}par\text{-}by\text{-}merge\ RD1\text{-}alt\text{-}def\ assms(3))$

finally show $?thesis$

by $(simp\ add: Healthy\text{-}def)$

qed

lemma $RD2\text{-}par\text{-}by\text{-}merge$ $[closure]$:

assumes M is $RD2$

shows $(P \parallel_M Q)$ is $RD2$

proof –

have $(P \parallel_M Q) = ((P \parallel_s Q) ;; M)$

by $(simp\ add: par\text{-}by\text{-}merge\text{-}def)$

also from $assms$ have $\dots = ((P \parallel_s Q) ;; (M ;; J))$

by $(simp\ add: Healthy\text{-}def'\ RD2\text{-}def\ H2\text{-}def)$

also from $assms$ have $\dots = (((P \parallel_s Q) ;; M) ;; J)$

by $(simp\ add: seqr\text{-}assoc)$

also from $assms$ have $\dots = RD2(P \parallel_M Q)$

by $(simp\ add: RD2\text{-}def\ H2\text{-}def\ par\text{-}by\text{-}merge\text{-}def)$

finally show $?thesis$

by $(simp\ add: Healthy\text{-}def')$

qed

lemma $SRD\text{-}par\text{-}by\text{-}merge$:

assumes P is SRD Q is SRD M is $R1m$ M is $R2m$ M is $R3hm$ M is $RD1m$ M is $RD2$

shows $(P \parallel_M Q)$ is SRD

by $(rule\ SRD\text{-}intro, simp\text{-}all\ add: assms\ closure\ SRD\text{-}healths)$

definition $nmerge\text{-}rd0$ (N_0) **where**

$[upred-defs]$: $N_0(M) = (\$wait' =_u (\$0\text{-}wait \vee \$1\text{-}wait) \wedge \$tr_{<} \leq_u \$tr' \wedge (\exists \$0\text{-}ok; \$1\text{-}ok; \$ok_{<} \$ok'; \$0\text{-}wait; \$1\text{-}wait; \$wait_{<} \$wait' \cdot M))$

definition $nmerge\text{-}rd1$ (N_1) **where**

$[upred-defs]$: $N_1(M) = (\$ok' =_u (\$0\text{-}ok \wedge \$1\text{-}ok) \wedge N_0(M))$

definition $nmerge\text{-}rd$ (N_R) **where**

$[upred-defs]$: $N_R(M) = ((\exists \$st_{<} \cdot \$v' =_u \$v_{<}) \triangleleft \$wait_{<} \triangleright N_1(M)) \triangleleft \$ok_{<} \triangleright (\$tr_{<} \leq_u \$tr')$

definition $merge\text{-}rd1$ (M_1) **where**

$[upred-defs]$: $M_1(M) = (N_1(M) ;; II_R)$

definition *merge-rd* (M_R) **where**
 $[upred-defs]: M_R(M) = N_R(M) ;; II_R$

abbreviation *rdes-par* ($- \parallel_R - [85, 0, 86] 85$) **where**
 $P \parallel_{RM} Q \equiv P \parallel_{M_R(M)} Q$

Healthiness condition for reactive design merge predicates

definition $[upred-defs]: RDM(M) = R2m(\exists \$0-ok; \$1-ok; \$ok_<; \$ok'; \$0-wait; \$1-wait; \$wait_<; \$wait' \cdot M)$

lemma *nmerge-rd-is-R1m* [closure]:
 $N_R(M)$ is *R1m*
by (*rel-blast*)

lemma *R2m-nmerge-rd*: $R2m(N_R(R2m(M))) = N_R(R2m(M))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *nmerge-rd-is-R2m* [closure]:
 M is *R2m* $\implies N_R(M)$ is *R2m*
by (*metis Healthy-def' R2m-nmerge-rd*)

lemma *nmerge-rd-is-R3hm* [closure]: $N_R(M)$ is *R3hm*
by (*rel-blast*)

lemma *nmerge-rd-is-RD1m* [closure]: $N_R(M)$ is *RD1m*
by (*rel-blast*)

lemma *merge-rd-is-RD3*: $M_R(M)$ is *RD3*
by (*metis Healthy-Idempotent RD3-Idempotent RD3-def merge-rd-def*)

lemma *merge-rd-is-RD2*: $M_R(M)$ is *RD2*
by (*simp add: RD3-implies-RD2 merge-rd-is-RD3*)

lemma *par-rdes-NSRD* [closure]:
assumes P is *SRD* Q is *SRD* M is *RDM*
shows $P \parallel_{RM} Q$ is *NSRD*

proof –

have ($P \parallel_{N_R M} Q ;; II_R$) is *NSRD*
by (*rule NSRD-intro', simp-all add: SRD-healths closure assms*)
(metis (no-types, lifting) Healthy-def R2-par-by-merge R2-seqr-closure R2m-nmerge-rd RDM-def SRD-healths(2) assms skip-srea-R2
,metis Healthy-Idempotent RD3-Idempotent RD3-def)
thus *?thesis*
by (*simp add: merge-rd-def par-by-merge-def seqr-assoc*)
qed

lemma *RDM-intro*:
assumes M is *R2m* $\$0-ok \# M \$1-ok \# M \$ok_< \# M \$ok' \# M$
 $\$0-wait \# M \$1-wait \# M \$wait_< \# M \$wait' \# M$
shows M is *RDM*
using *assms*
by (*simp add: Healthy-def RDM-def ex-unrest unrest*)

lemma *RDM-unrests* [unrest]:
assumes M is *RDM*

shows $\$0-ok \# M \ \$1-ok \# M \ \$ok_{<} \# M \ \$ok' \# M$
 $\$0-wait \# M \ \$1-wait \# M \ \$wait_{<} \# M \ \$wait' \# M$
 by (subst Healthy-if[OF assms, THEN sym], simp-all add: RDM-def unrest, rel-auto)+

lemma RDM-R1m [closure]: M is RDM $\implies M$ is R1m
 by (metis (no-types, hide-lams) Healthy-def R1m-idem R2m-def RDM-def)

lemma RDM-R2m [closure]: M is RDM $\implies M$ is R2m
 by (metis (no-types, hide-lams) Healthy-def R2m-idem RDM-def)

lemma ex-st'-R2m-closed [closure]:

assumes P is R2m
 shows $(\exists \ \$st' \cdot P)$ is R2m

proof –

have $R2m(\exists \ \$st' \cdot R2m \ P) = (\exists \ \$st' \cdot R2m \ P)$
 by (rel-auto)

thus ?thesis

by (metis Healthy-def' assms)

qed

lemma parallel-RR-closed:

assumes P is RR Q is RR M is R2m
 $\$ok_{<} \# M \ \$wait_{<} \# M \ \$ok' \# M \ \$wait' \# M$

shows $P \parallel_M Q$ is RR

by (rule RR-R2-intro, simp-all add: unrest assms RR-implies-R2 closure)

lemma parallel-ok-cases:

$((P \parallel_s Q) ;; M) =$
 $((P^t \parallel_s Q^t) ;; (M \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^f \parallel_s Q^t) ;; (M \llbracket false, true / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^t \parallel_s Q^f) ;; (M \llbracket true, false / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^f \parallel_s Q^f) ;; (M \llbracket false, false / \$0-ok, \$1-ok \rrbracket))$

proof –

have $((P \parallel_s Q) ;; M) = (\exists \ ok_0 \cdot (P \parallel_s Q) \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket ;; M \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket)$
 by (subst segr-middle[of left-uvar ok], simp-all)

also have $\dots = (\exists \ ok_0 \cdot \exists \ ok_1 \cdot ((P \parallel_s Q) \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket \llbracket \llbracket ok_1 \rrbracket / \$1-ok \rrbracket ;; (M \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket \llbracket \llbracket ok_1 \rrbracket / \$1-ok \rrbracket))$
 by (subst segr-middle[of right-uvar ok], simp-all)

also have $\dots = (\exists \ ok_0 \cdot \exists \ ok_1 \cdot (P \llbracket \llbracket ok_0 \rrbracket / \$ok \rrbracket \parallel_s Q \llbracket \llbracket ok_1 \rrbracket / \$ok \rrbracket) ;; (M \llbracket \llbracket ok_0 \rrbracket, \llbracket ok_1 \rrbracket / \$0-ok, \$1-ok \rrbracket))$
 by (rel-auto robust)

also have $\dots =$
 $((P^t \parallel_s Q^t) ;; (M \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^f \parallel_s Q^t) ;; (M \llbracket false, true / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^t \parallel_s Q^f) ;; (M \llbracket true, false / \$0-ok, \$1-ok \rrbracket)) \vee$
 $((P^f \parallel_s Q^f) ;; (M \llbracket false, false / \$0-ok, \$1-ok \rrbracket))$

by (simp add: true-alt-def[THEN sym] false-alt-def[THEN sym] disj-assoc
 utp-pred-laws.sup.left-commute utp-pred-laws.sup-commute usubst)

finally show ?thesis .

qed

lemma skip-srea-ok-f [usubst]:

$\Pi_R^f = R1(\neg \$ok)$

by (rel-auto)

lemma nmerge0-rd-unrest [unrest]:

$\$0-ok \# N_0 \ M \ \$1-ok \# N_0 \ M$

by (*pred-auto*) +

lemma *parallel-assm-lemma*:

assumes P is *RD2*

shows $pre_s \uparrow (P \parallel_{M_R(M)} Q) = ((pre_s \uparrow P) \parallel_{N_0(M)} ;; R1(true) (cmt_s \uparrow Q))$
 $\vee ((cmt_s \uparrow P) \parallel_{N_0(M)} ;; R1(true) (pre_s \uparrow Q))$

proof –

have $pre_s \uparrow (P \parallel_{M_R(M)} Q) = pre_s \uparrow ((P \parallel_s Q) ;; M_R(M))$

by (*simp add: par-by-merge-def*)

also have ... = $((P \parallel_s Q) \llbracket true, false / \$ok, \$wait \rrbracket ;; N_R M ;; R1(\neg \$ok))$

by (*simp add: merge-rd-def usubst, rel-auto*)

also have ... = $((P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_s Q \llbracket true, false / \$ok, \$wait \rrbracket) ;; N_1(M) ;; R1(\neg \$ok))$

by (*rel-auto robust, (metis) +*)

also have ... = $(($
 $((P \llbracket true, false / \$ok, \$wait \rrbracket)^t \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^t) ;; ((N_1 M) \llbracket true, true / \$0-ok, \$1-ok \rrbracket$
 $;; R1(\neg \$ok))) \vee$
 $((P \llbracket true, false / \$ok, \$wait \rrbracket)^f \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^t) ;; ((N_1 M) \llbracket false, true / \$0-ok, \$1-ok \rrbracket$
 $;; R1(\neg \$ok))) \vee$
 $((P \llbracket true, false / \$ok, \$wait \rrbracket)^t \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^f) ;; ((N_1 M) \llbracket true, false / \$0-ok, \$1-ok \rrbracket$
 $;; R1(\neg \$ok))) \vee$
 $((P \llbracket true, false / \$ok, \$wait \rrbracket)^f \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^f) ;; ((N_1 M) \llbracket false, false / \$0-ok, \$1-ok \rrbracket$
 $;; R1(\neg \$ok))))$

(*is - = (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4)*)

by (*subst parallel-ok-cases, subst-tac*)

also have ... = $(?C2 \vee ?C3)$

proof –

have $?C1 = false$

by (*rel-auto*)

moreover have $'?C4 \Rightarrow ?C3'$ (*is* $'(?A ;; ?B) \Rightarrow (?C ;; ?D)'$)

proof –

from *assms* have $'P^f \Rightarrow P^t'$

by (*metis RD2-def H2-equivalence Healthy-def*)

hence $P: 'P^f_f \Rightarrow P^t_f'$

by (*rel-auto*)

have $'?A \Rightarrow ?C'$

using P by (*rel-auto*)

moreover have $'?B \Rightarrow ?D'$

by (*rel-auto*)

ultimately show *?thesis*

by (*simp add: impl-seqr-mono*)

qed

ultimately show *?thesis*

by (*simp add: subsumption2*)

qed

also have ... = $($
 $((pre_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; ((N_0 M) ;; R1(true))) \vee$
 $((cmt_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; ((N_0 M) ;; R1(true)))$
 $by (rel-auto, metis+)$
 $also have ... = ($
 $((pre_s \uparrow P) \parallel_{N_0 M} ;; R1(true) (cmt_s \uparrow Q)) \vee$
 $((cmt_s \uparrow P) \parallel_{N_0 M} ;; R1(true) (pre_s \uparrow Q)))$
 $by (simp add: par-by-merge-def)$
finally show *?thesis* .

qed

lemma *pre_s-SRD*:

assumes *P is SRD*

shows $pre_s \uparrow P = (\neg_r pre_R(P))$

proof –

have $pre_s \uparrow P = pre_s \uparrow \mathbf{R}_s(pre_R P \vdash peri_R P \diamond post_R P)$

by (*simp add: SRD-reactive-tri-design assms*)

also have $\dots = R1(R2c(\neg pre_s \uparrow pre_R P))$

by (*simp add: RHS-def usubst R3h-def pre_s-design*)

also have $\dots = R1(R2c(\neg pre_R P))$

by (*rel-auto*)

also have $\dots = (\neg_r pre_R P)$

by (*simp add: R2c-not R2c-preR assms rea-not-def*)

finally show *?thesis* .

qed

lemma *parallel-assm*:

assumes *P is SRD Q is SRD*

shows $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r ((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q)) \wedge$
 $\neg_r (cmt_R(P) \parallel_{N_0(M)} ;; R1(true) (\neg_r pre_R(Q))))$

(*is ?lhs = ?rhs*)

proof –

have $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r (pre_s \uparrow P) \parallel_{N_0 M} ;; R1 true (cmt_s \uparrow Q) \wedge$

$\neg_r (cmt_s \uparrow P) \parallel_{N_0 M} ;; R1 true (pre_s \uparrow Q))$

by (*simp add: pre_R-def parallel-assm-lemma assms SRD-healths R1-conj rea-not-def [THEN sym]*)

also have $\dots = ?rhs$

by (*simp add: pre_s-SRD assms cmt_R-def Healthy-if closure unrest*)

finally show *?thesis* .

qed

lemma *parallel-assm-unrest-wait' [unrest]*:

$\llbracket P \text{ is SRD}; Q \text{ is SRD} \rrbracket \implies \$wait' \nmid pre_R(P \parallel_{M_R(M)} Q)$

by (*simp add: parallel-assm, simp add: par-by-merge-def unrest*)

lemma *JL1*: $(M_1 M)^t \llbracket false, true / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$

by (*rel-blast*)

lemma *JL2*: $(M_1 M)^t \llbracket true, false / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$

by (*rel-blast*)

lemma *JL3*: $(M_1 M)^t \llbracket false, false / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$

by (*rel-blast*)

lemma *JL4*: $(M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket = (\$ok' \wedge N_0 M) ;; II_R^t$

by (*simp add: merge-rd1-def usubst nmerge-rd1-def unrest*)

lemma *parallel-commitment-lemma-1*:

assumes *P is RD2*

shows $cmt_s \uparrow (P \parallel_{M_R(M)} Q) = ($

$((cmt_s \uparrow P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t (cmt_s \uparrow Q)) \vee$

$((pre_s \uparrow P) \parallel_{N_0(M)} ;; R1(true) (cmt_s \uparrow Q)) \vee$

$((cmt_s \uparrow P) \parallel_{N_0(M)} ;; R1(true) (pre_s \uparrow Q)))$

proof –

have $cmt_s \dagger (P \parallel_{M_R(M)} Q) = (P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_{(M_1(M))^t} Q \llbracket true, false / \$ok, \$wait \rrbracket)$
by (*simp add: usubst, rel-auto*)
also have $\dots = ((P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_s Q \llbracket true, false / \$ok, \$wait \rrbracket) ;; (M_1 M)^t)$
by (*simp add: par-by-merge-def*)
also have $\dots =$ (
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket) \vee$
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket false, true / \$0-ok, \$1-ok \rrbracket) \vee$
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, false / \$0-ok, \$1-ok \rrbracket) \vee$
 $((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((M_1 M)^t \llbracket false, false / \$0-ok, \$1-ok \rrbracket))$
by (*subst parallel-ok-cases, subst-tac*)
also have $\dots =$ (
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket) \vee$
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$
 $((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)))$
 $(\text{is } - = (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4))$
by (*simp add: JL1 JL2 JL3*)
also have $\dots =$ (
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1(M))^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket) \vee$
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)))$
proof –
from *assms* **have** $P^f \Rightarrow P^t$
by (*metis RD2-def H2-equivalence Healthy-def*)
hence $P: P^f_f \Rightarrow P^t_f$
by (*rel-auto*)
have $?C4 \Rightarrow ?C3$ (**is** $(?A ;; ?B) \Rightarrow (?C ;; ?D)$)
proof –
have $?A \Rightarrow ?C$
using P **by** (*rel-auto*)
thus $?thesis$
by (*simp add: impl-seqr-mono*)
qed
thus $?thesis$
by (*simp add: subsumption2*)
qed
finally show $?thesis$
by (*simp add: par-by-merge-def JL4*)
qed

lemma *parallel-commitment-lemma-2:*

assumes P *is* $RD2$

shows $cmt_s \dagger (P \parallel_{M_R(M)} Q) =$

$$(((cmt_s \dagger P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t (cmt_s \dagger Q)) \vee pre_s \dagger (P \parallel_{M_R(M)} Q))$$

by (*simp add: parallel-commitment-lemma-1 assms parallel-assm-lemma*)

lemma *parallel-commitment-lemma-3:*

M *is* $R1m \implies (\$ok' \wedge N_0 M) ;; II_R^t$ *is* $R1m$

by (*rel-simp, safe, metis+*)

lemma *parallel-commitment:*

assumes P *is* SRD Q *is* SRD M *is* RDM

shows $cmt_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t cmt_R(Q))$

by (*simp add: parallel-commitment-lemma-2 parallel-commitment-lemma-3 Healthy-if assms cmt_R-def pre_s-SRD closure rea-impl-def disj-comm unrest*)

theorem *parallel-reactive-design*:

assumes P is SRD Q is SRD M is RDM

shows $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s($

$(\neg_r ((\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q)) \wedge$

$\neg_r (\text{cmt}_R(P) \parallel_{N_0(M)} ;; R1(\text{true}) (\neg_r \text{pre}_R(Q)))) \vdash$

$(\text{cmt}_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t \text{cmt}_R(Q))$ (**is** $?lhs = ?rhs$)

proof –

have $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s(\text{pre}_R(P \parallel_{M_R(M)} Q) \vdash \text{cmt}_R(P \parallel_{M_R(M)} Q))$

by (*metis Healthy-def NSRD-is-SRD SRD-as-reactive-design assms(1) assms(2) assms(3) par-rdes-NSRD*)

also have $\dots = ?rhs$

by (*simp add: parallel-assm parallel-commitment design-export-spec assms, rel-auto*)

finally show $?thesis$.

qed

lemma *parallel-pericondition-lemma1*:

$(\$ok' \wedge P) ;; II_R[\text{true}, \text{true}/\$ok', \$wait'] = (\exists \$st' \cdot P)[\text{true}, \text{true}/\$ok', \$wait']$

(**is** $?lhs = ?rhs$)

proof –

have $?lhs = (\$ok' \wedge P) ;; (\exists \$st' \cdot II)[\text{true}, \text{true}/\$ok', \$wait']$

by (*rel-blast*)

also have $\dots = ?rhs$

by (*rel-auto*)

finally show $?thesis$.

qed

lemma *parallel-pericondition-lemma2*:

assumes M is RDM

shows $(\exists \$st' \cdot N_0(M))[\text{true}, \text{true}/\$ok', \$wait'] = ((\$0\text{--wait} \vee \$1\text{--wait}) \wedge (\exists \$st' \cdot M))$

proof –

have $(\exists \$st' \cdot N_0(M))[\text{true}, \text{true}/\$ok', \$wait'] = (\exists \$st' \cdot (\$0\text{--wait} \vee \$1\text{--wait}) \wedge \$tr' \geq_u \$tr_<$

$\wedge M)$

by (*simp add: usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)

also have $\dots = (\exists \$st' \cdot (\$0\text{--wait} \vee \$1\text{--wait}) \wedge M)$

by (*metis (no-types, hide-lams) Healthy-if R1m-def R1m-idem R2m-def RDM-def assms utp-pred-laws.inf-commute*)

also have $\dots = ((\$0\text{--wait} \vee \$1\text{--wait}) \wedge (\exists \$st' \cdot M))$

by (*rel-auto*)

finally show $?thesis$.

qed

lemma *parallel-pericondition-lemma3*:

$((\$0\text{--wait} \vee \$1\text{--wait}) \wedge (\exists \$st' \cdot M)) = ((\$0\text{--wait} \wedge \$1\text{--wait} \wedge (\exists \$st' \cdot M)) \vee (\neg \$0\text{--wait} \wedge$

$\$1\text{--wait} \wedge (\exists \$st' \cdot M)) \vee (\$0\text{--wait} \wedge \neg \$1\text{--wait} \wedge (\exists \$st' \cdot M)))$

by (*rel-auto*)

lemma *parallel-pericondition [rdes]*:

fixes $M :: ('s, 't :: \text{trace}, 'a) \text{rsp merge}$

assumes P is SRD Q is SRD M is RDM

shows $\text{peri}_R(P \parallel_{M_R(M)} Q) = (\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{peri}_R(P) \parallel_{\exists \$st' \cdot M} \text{peri}_R(Q)$

$\vee \text{post}_R(P) \parallel_{\exists \$st' \cdot M} \text{peri}_R(Q)$

$\vee \text{peri}_R(P) \parallel_{\exists \$st' \cdot M} \text{post}_R(Q))$

proof –

have $\text{peri}_R(P \parallel_{M_R(M)} Q) =$

$(\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{cmt}_R P \parallel_{(\$ok' \wedge N_0 M)} ;; II_R[\text{true}, \text{true}/\$ok', \$wait'] \text{cmt}_R Q)$

by (simp add: parallel-postcondition-lemma1 parallel-postcondition-lemma2 assms,
 simp add: utp-pred-laws.inf-commute utp-pred-laws.inf-left-commute)
 also have ... = ($\text{pre}_R(P \parallel_{M_R} M Q) \Rightarrow_r \text{post}_R P \parallel_M \text{post}_R Q$)
 by (simp add: par-by-merge-alt-def segr-right-one-point-false usubst unrest cmt_R-def post_R-def assms)
 finally show ?thesis .
 qed

lemma parallel-precondition-lemma:

fixes $M :: ('s, 't :: \text{trace}, 'a) \text{rsp merge}$
 assumes P is NSRD Q is NSRD M is RDM
 shows $(\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q) =$
 $((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$
 proof –
 have $((\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q)) =$
 $((\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) (\text{peri}_R(Q) \diamond \text{post}_R(Q)))$
 by (simp add: wait'-cond-peri-post-cmt)
 also have ... = $((\neg_r \text{pre}_R(P))_0 \wedge [\text{peri}_R(Q) \diamond \text{post}_R(Q)]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; N_0(M) ;; R1(\text{true}))$
 by (simp add: par-by-merge-alt-def)
 also have ... = $((\neg_r \text{pre}_R(P))_0 \wedge [\text{peri}_R(Q)]_1 \triangleleft \$1\text{--wait}' \triangleright [\text{post}_R(Q)]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; N_0(M)$
 $;; R1(\text{true}))$
 by (simp add: wait'-cond-def alpha)
 also have ... = $(([\neg_r \text{pre}_R(P)]_0 \wedge [\text{peri}_R(Q)]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) \triangleleft \$1\text{--wait}' \triangleright ([\neg_r \text{pre}_R(P)]_0 \wedge$
 $[\text{post}_R(Q)]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v})) ;; N_0(M) ;; R1(\text{true}))$
 (is $(?P ;; -) = (?Q ;; -)$)
 proof –
 have $?P = ?Q$
 by (rel-auto)
 thus ?thesis by simp
 qed

also have ... = $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v})[\text{true}/\$1\text{--wait}'] ;; (N_0 M ;; R1$
 $\text{true})[\text{true}/\$1\text{--wait}'] \vee$
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v})[\text{false}/\$1\text{--wait}'] ;; (N_0 M ;; R1$
 $\text{true})[\text{false}/\$1\text{--wait}']$)
 by (simp add: cond-inter-var-split)
 also have ... = $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; N_0 M[\text{true}/\$1\text{--wait}'] ;; R1 \text{true} \vee$
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; N_0 M[\text{false}/\$1\text{--wait}'] ;; R1 \text{true})$
 by (simp add: usubst unrest)
 also have ... = $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; (\$wait' \wedge M) ;; R1 \text{true} \vee$
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; (\$wait' =_u \$0\text{--wait} \wedge M) ;; R1 \text{true})$
 proof –
 have $(\$tr' \geq_u \$tr_{<} \wedge M) = M$
 using RDM-R1m[OF assms(3)]
 by (simp add: Healthy-def R1m-def conj-comm)
 thus ?thesis
 by (simp add: nmerge-rd0-def unrest assms closure ex-unrest usubst)
 qed

also have ... = $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; M ;; R1 \text{true} \vee$
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; M ;; R1 \text{true})$
 (is $(?P_1 \vee_p ?P_2) = (?Q_1 \vee ?Q_2)$)
 proof –
 have $?P_1 = ([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; (M \wedge \$wait')$;; $R1 \text{true}$
 by (simp add: conj-comm)
 hence 1: $?P_1 = ?Q_1$
 by (simp add: segr-left-one-point-true segr-left-one-point-false add: unrest usubst closure assms)
 have $?P_2 = ([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; (M \wedge \$wait')$;; $R1 \text{true} \vee$

$(\lceil \neg_r \text{pre}_R P \rceil_0 \wedge \lceil \text{post}_R Q \rceil_1 \wedge \$\mathbf{v}_{<} =_u \$\mathbf{v}) ;; (M \wedge \neg \$\text{wait}') ;; R1 \text{ true})$
 by (subst seqr-bool-split[of left-uvar wait], simp-all add: usubst unrest assms closure conj-comm)
 hence 2: $?P_2 = ?Q_2$
 by (simp add: seqr-left-one-point-true seqr-left-one-point-false unrest usubst closure assms)
 from 1 2 show ?thesis by simp
 qed
 also have ... = $((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$
 by (simp add: par-by-merge-alt-def)
 finally show ?thesis .
 qed

lemma swap-nmerge-rd0:
 $\text{swap}_m ;; N_0(M) = N_0(\text{swap}_m ;; M)$
 by (rel-auto, meson+)

lemma SymMerge-nmerge-rd0 [closure]:
 $M \text{ is SymMerge} \implies N_0(M) \text{ is SymMerge}$
 by (rel-auto, meson+)

lemma swap-merge-rd':
 $\text{swap}_m ;; N_R(M) = N_R(\text{swap}_m ;; M)$
 by (rel-blast)

lemma swap-merge-rd:
 $\text{swap}_m ;; M_R(M) = M_R(\text{swap}_m ;; M)$
 by (simp add: merge-rd-def seqr-assoc[THEN sym] swap-merge-rd')

lemma SymMerge-merge-rd [closure]:
 $M \text{ is SymMerge} \implies M_R(M) \text{ is SymMerge}$
 by (simp add: Healthy-def swap-merge-rd)

lemma nmerge-rd1-merge3:
 assumes $M \text{ is RDM}$
 shows $\mathbf{M3}(N_1(M)) = (\$ok' =_u (\$0-ok \wedge \$1-0-ok \wedge \$1-1-ok) \wedge$
 $\$wait' =_u (\$0-wait \vee \$1-0-wait \vee \$1-1-wait) \wedge$
 $\mathbf{M3}(M))$

proof –

have $\mathbf{M3}(N_1(M)) = \mathbf{M3}(\$ok' =_u (\$0-ok \wedge \$1-ok) \wedge$
 $\$wait' =_u (\$0-wait \vee \$1-wait) \wedge$
 $\$tr_{<} \leq_u \$tr' \wedge$
 $(\exists \{\$0-ok, \$1-ok, \$ok_{<}, \$ok', \$0-wait, \$1-wait, \$wait_{<}, \$wait'\} \cdot RDM(M)))$
 by (simp add: nmerge-rd1-def nmerge-rd0-def assms Healthy-if)
 also have ... = $\mathbf{M3}(\$ok' =_u (\$0-ok \wedge \$1-ok) \wedge \$wait' =_u (\$0-wait \vee \$1-wait) \wedge RDM(M))$
 by (rel-blast)
 also have ... = $(\$ok' =_u (\$0-ok \wedge \$1-0-ok \wedge \$1-1-ok) \wedge \$wait' =_u (\$0-wait \vee \$1-0-wait$
 $\vee \$1-1-wait) \wedge \mathbf{M3}(RDM(M)))$
 by (rel-blast)
 also have ... = $(\$ok' =_u (\$0-ok \wedge \$1-0-ok \wedge \$1-1-ok) \wedge \$wait' =_u (\$0-wait \vee \$1-0-wait$
 $\vee \$1-1-wait) \wedge \mathbf{M3}(M))$
 by (simp add: assms Healthy-if)
 finally show ?thesis .
 qed

lemma nmerge-rd-merge3:
 $\mathbf{M3}(N_R(M)) = (\exists \$st_{<} \cdot \$\mathbf{v}' =_u \$\mathbf{v}_{<} \triangleleft \$wait_{<} \triangleright \mathbf{M3}(N_1 M) \triangleleft \$ok_{<} \triangleright (\$tr_{<} \leq_u \$tr'))$

by (rel-blast)

lemma *swap-merge-RDM-closed* [closure]:

assumes *M is RDM*

shows $swap_m \;; M \text{ is RDM}$

proof –

have $RDM(swap_m \;; RDM(M)) = (swap_m \;; RDM(M))$

by (rel-auto)

thus ?thesis

by (metis Healthy-def' assms)

qed

lemma *parallel-precondition*:

fixes $M :: ('s, 't :: trace, 'α) \text{ rsp merge}$

assumes *P is NSRD Q is NSRD M is RDM*

shows $pre_R(P \parallel_{M_R(M)} Q) =$

$(\neg_r ((\neg_r pre_R P) \parallel_M \;; R1(true) \text{ peri}_R Q) \wedge$

$\neg_r ((\neg_r pre_R P) \parallel_M \;; R1(true) \text{ post}_R Q) \wedge$

$\neg_r ((\neg_r pre_R Q) \parallel_{(swap_m \;; M)} \;; R1(true) \text{ peri}_R P) \wedge$

$\neg_r ((\neg_r pre_R Q) \parallel_{(swap_m \;; M)} \;; R1(true) \text{ post}_R P))$

proof –

have $a: (\neg_r pre_R(P)) \parallel_{N_0(M)} \;; R1(true) \text{ cmt}_R(Q) =$

$((\neg_r pre_R P) \parallel_M \;; R1(true) \text{ peri}_R Q \vee (\neg_r pre_R P) \parallel_M \;; R1(true) \text{ post}_R Q)$

by (simp add: parallel-precondition-lemma assms)

have $b: (\neg_r \text{ cmt}_R P \parallel_{N_0 M} \;; R1 \text{ true } (\neg_r pre_R Q)) =$

$(\neg_r (\neg_r pre_R(Q)) \parallel_{N_0(swap_m \;; M)} \;; R1(true) \text{ cmt}_R(P))$

by (simp add: swap-nmerge-rd0[THEN sym] seqr-assoc[THEN sym] par-by-merge-def par-sep-swap)

have $c: (\neg_r pre_R(Q)) \parallel_{N_0(swap_m \;; M)} \;; R1(true) \text{ cmt}_R(P) =$

$((\neg_r pre_R Q) \parallel_{(swap_m \;; M)} \;; R1(true) \text{ peri}_R P \vee (\neg_r pre_R Q) \parallel_{(swap_m \;; M)} \;; R1(true) \text{ post}_R$

$P)$

by (simp add: parallel-precondition-lemma closure assms)

show ?thesis

by (simp add: parallel-assm closure assms a b c, rel-auto)

qed

Weakest Parallel Precondition

definition *wrR* ::

$('t :: trace, 'α) \text{ hrel-rp} \Rightarrow$

$('t :: trace, 'α) \text{ rp merge} \Rightarrow$

$('t, 'α) \text{ hrel-rp} \Rightarrow$

$('t, 'α) \text{ hrel-rp } (- \text{ wr}_R'(-) - [60, 0, 61] \ 61)$

where [upred-defs]: $Q \text{ wr}_R(M) P = (\neg_r ((\neg_r P) \parallel_M \;; R1(true) Q))$

lemma *wrR-R1* [closure]:

$M \text{ is R1m} \Longrightarrow Q \text{ wr}_R(M) P \text{ is R1}$

by (simp add: wrR-def closure)

lemma *R2-rea-not*: $R2(\neg_r P) = (\neg_r R2(P))$

by (rel-auto)

lemma *wrR-R2-lemma*:

assumes *P is R2 Q is R2 M is R2m*

shows $((\neg_r P) \parallel_M Q) ;; R1(true_h) \text{ is } R2$

proof –

have $(\neg_r P) \parallel_M Q \text{ is } R2$
 by (simp add: closure assms)
 thus ?thesis
 by (simp add: closure)

qed

lemma wrR-R2 [closure]:

assumes $P \text{ is } R2 \ Q \text{ is } R2 \ M \text{ is } R2m$
 shows $Q \text{ wr}_R(M) \ P \text{ is } R2$

proof –

have $((\neg_r P) \parallel_M Q) ;; R1(true_h) \text{ is } R2$
 by (simp add: wrR-R2-lemma assms)
 thus ?thesis
 by (simp add: wrR-def wrR-R2-lemma par-by-merge-seq-add closure)

qed

lemma wrR-RR [closure]:

assumes $P \text{ is } RR \ Q \text{ is } RR \ M \text{ is } RDM$
 shows $Q \text{ wr}_R(M) \ P \text{ is } RR$
 apply (rule RR-intro)
 apply (simp-all add: unrest assms closure wrR-def rpred)
 apply (metis (no-types, lifting) Healthy-def' R1-R2c-commute R1-R2c-is-R2 R1-rea-not RDM-R2m
 RR-implies-R2 assms(1) assms(2) assms(3) par-by-merge-seq-add rea-not-R2-closed
 wrR-R2-lemma)

done

lemma wrR-RC [closure]:

assumes $P \text{ is } RR \ Q \text{ is } RR \ M \text{ is } RDM$
 shows $(Q \text{ wr}_R(M) \ P) \text{ is } RC$
 apply (rule RC-intro)
 apply (simp add: closure assms)
 apply (simp add: wrR-def rpred closure assms)
 apply (simp add: par-by-merge-def seqr-assoc)

done

lemma wppR-choice [wp]: $(P \vee Q) \text{ wr}_R(M) \ R = (P \text{ wr}_R(M) \ R \wedge Q \text{ wr}_R(M) \ R)$

proof –

have $(P \vee Q) \text{ wr}_R(M) \ R =$
 $(\neg_r ((\neg_r R) ;; U0 \wedge (P ;; U1 \vee Q ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r)$
 by (simp add: wrR-def par-by-merge-def seqr-or-distl)
 also have $\dots = (\neg_r ((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v} \vee (\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r)$
 by (simp add: conj-disj-distr utp-pred-laws.inf-sup-distrib2)
 also have $\dots = (\neg_r (((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r \vee$
 $((\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r))$
 by (simp add: seqr-or-distl)
 also have $\dots = (P \text{ wr}_R(M) \ R \wedge Q \text{ wr}_R(M) \ R)$
 by (simp add: wrR-def par-by-merge-def)
 finally show ?thesis .

qed

lemma wppR-miracle [wp]: $false \text{ wr}_R(M) \ P = true_r$

by (simp add: wrR-def)

lemma *wppR-true* [wp]: $P \text{ wr}_R(M) \text{ true}_r = \text{true}_r$
 by (simp add: wrR-def)

lemma *parallel-precondition-wr* [rdes]:
 assumes P is NSRD Q is NSRD M is RDM
 shows $\text{pre}_R(P \parallel_{M_R(M)} Q) = (\text{peri}_R(Q) \text{ wr}_R(M) \text{ pre}_R(P) \wedge \text{post}_R(Q) \text{ wr}_R(M) \text{ pre}_R(P) \wedge$
 $\text{peri}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{ pre}_R(Q) \wedge \text{post}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{ pre}_R(Q))$
 by (simp add: assms parallel-precondition wrR-def)

lemma *parallel-rdes-def* [rdes-def]:
 assumes P_1 is RC P_2 is RR P_3 is RR Q_1 is RC Q_2 is RR Q_3 is RR
 $\$st' \# P_2 \$st' \# Q_2$
 M is RDM
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \parallel_{M_R(M)} \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge (Q_1 \Rightarrow_r Q_3) \text{ wr}_R(M) P_1 \wedge$
 $(P_1 \Rightarrow_r P_2) \text{ wr}_R(\text{swap}_m ;; M) Q_1 \wedge (P_1 \Rightarrow_r P_3) \text{ wr}_R(\text{swap}_m ;; M) Q_1) \vdash$
 $((P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee$
 $(P_1 \Rightarrow_r P_3) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee (P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_3)) \diamond$
 $((P_1 \Rightarrow_r P_3) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_3)))$ (is ?lhs = ?rhs)

proof –
 have ?lhs = $\mathbf{R}_s(\text{pre}_R ?lhs \vdash \text{peri}_R ?lhs \diamond \text{post}_R ?lhs)$
 by (simp add: SRD-reactive-tri-design assms closure)
 also have ... = ?rhs
 by (simp add: rdes closure unrest assms, rel-auto)
 finally show ?thesis .
qed

lemma *Miracle-parallel-left-zero*:
 assumes P is SRD M is RDM
 shows $\text{Miracle} \parallel_{RM} P = \text{Miracle}$

proof –
 have $\text{pre}_R(\text{Miracle} \parallel_{RM} P) = \text{true}_r$
 by (simp add: parallel-assm wait'-cond-idem rdes closure assms)
 moreover hence $\text{cmt}_R(\text{Miracle} \parallel_{RM} P) = \text{false}$
 by (simp add: rdes closure wait'-cond-idem SRD-healths assms)
 ultimately have $\text{Miracle} \parallel_{RM} P = \mathbf{R}_s(\text{true}_r \vdash \text{false})$
 by (metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed)
 thus ?thesis
 by (simp add: Miracle-def R1-design-R1-pre)
qed

lemma *Miracle-parallel-right-zero*:
 assumes P is SRD M is RDM
 shows $P \parallel_{RM} \text{Miracle} = \text{Miracle}$

proof –
 have $\text{pre}_R(P \parallel_{RM} \text{Miracle}) = \text{true}_r$
 by (simp add: wait'-cond-idem parallel-assm rdes closure assms)
 moreover hence $\text{cmt}_R(P \parallel_{RM} \text{Miracle}) = \text{false}$
 by (simp add: wait'-cond-idem rdes closure SRD-healths assms)
 ultimately have $P \parallel_{RM} \text{Miracle} = \mathbf{R}_s(\text{true}_r \vdash \text{false})$
 by (metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed)
 thus ?thesis
 by (simp add: Miracle-def R1-design-R1-pre)
qed

8.1 Example basic merge

definition *BasicMerge* :: (('s, 't::trace, unit) rsp) merge (N_B) **where**
 $[upred-defs]: BasicMerge = (\$tr_{<} \leq_u \$tr' \wedge \$tr' - \$tr_{<} =_u \$0 - tr - \$tr_{<} \wedge \$tr' - \$tr_{<} =_u \$1 - tr - \$tr_{<} \wedge \$st' =_u \$st_{<})$

abbreviation *rbasic-par* ($- \parallel_B - [85, 86]$ 85) **where**
 $P \parallel_B Q \equiv P \parallel_{M_R(N_B)} Q$

lemma *BasicMerge-RDM* [closure]: N_B is RDM
by (rule RDM-intro, (rel-auto)+)

lemma *BasicMerge-SymMerge* [closure]:
 N_B is SymMerge
by (rel-auto)

lemma *BasicMerge'-calc*:
assumes $\$ok' \# P \$wait' \# P \$ok' \# Q \$wait' \# Q$ P is R2 Q is R2
shows $P \parallel_{N_B} Q = ((\exists \$st' \cdot P) \wedge (\exists \$st' \cdot Q) \wedge \$st' =_u \$st)$
using *assms*

proof –
have $P: (\exists \{ \$ok', \$wait' \} \cdot R2(P)) = P$ (**is** ? $P' = -$)
by (simp add: ex-unrest ex-plus Healthy-if assms)
have $Q: (\exists \{ \$ok', \$wait' \} \cdot R2(Q)) = Q$ (**is** ? $Q' = -$)
by (simp add: ex-unrest ex-plus Healthy-if assms)
have $?P' \parallel_{N_B} ?Q' = ((\exists \$st' \cdot ?P') \wedge (\exists \$st' \cdot ?Q') \wedge \$st' =_u \$st)$
by (simp add: par-by-merge-alt-def, rel-auto, blast+)
thus ?thesis
by (simp add: P Q)
qed

8.2 Simple parallel composition

definition *rea-design-par* ::
 $('s, 't::trace, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp$ (**infixr** \parallel_R 85)
where $[upred-defs]: P \parallel_R Q = \mathbf{R}_s((pre_R(P) \wedge pre_R(Q)) \vdash (cmt_R(P) \wedge cmt_R(Q)))$

lemma *RHS-design-par*:
assumes
 $\$ok' \# P_1 \$ok' \# P_2$
shows $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$

proof –
have $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) =$
 $\mathbf{R}_s(P_1 \llbracket true, false / \$ok, \$wait \rrbracket \vdash Q_1 \llbracket true, false / \$ok, \$wait \rrbracket) \parallel_R \mathbf{R}_s(P_2 \llbracket true, false / \$ok, \$wait \rrbracket \vdash$
 $Q_2 \llbracket true, false / \$ok, \$wait \rrbracket)$
by (simp add: RHS-design-ok-wait)

also from *assms*

have ... =
 $\mathbf{R}_s((R1 (R2c (P_1)) \wedge R1 (R2c (P_2))) \llbracket true, false / \$ok, \$wait \rrbracket \vdash$
 $(R1 (R2c (P_1 \Rightarrow Q_1)) \wedge R1 (R2c (P_2 \Rightarrow Q_2))) \llbracket true, false / \$ok, \$wait \rrbracket)$
apply (simp add: rea-design-par-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest assms)
apply (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp)
using *assms* **apply** (rel-auto)

done

also have ... =

$\mathbf{R}_s((R2c(P_1) \wedge R2c(P_2)) \vdash$
 $(R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$
 by (*metis (no-types, hide-lams) R1-R2s-R2c R1-conj R1-design-R1-pre RHS-design-ok-wait*)
 also have ... =
 $\mathbf{R}_s((P_1 \wedge P_2) \vdash (R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$
 by (*simp add: R2c-R3h-commute R2c-and R2c-design R2c-idem R2c-not RHS-def*)
 also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
 by (*metis (no-types, lifting) R1-conj R2s-conj RHS-design-export-R1 RHS-design-export-R2s*)
 also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
 by (*rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto*)
 finally show ?thesis .
 qed

lemma *RHS-tri-design-par*:

assumes $\$ok' \# P_1 \$ok' \# P_2$
 shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
 by (*simp add: RHS-design-par assms unrest wait'-cond-conj-exchange*)

lemma *RHS-tri-design-par-RR [rdes-def]*:

assumes P_1 is *RR* P_2 is *RR*
 shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
 by (*simp add: RHS-tri-design-par unrest assms*)

lemma *RHS-comp-assoc*:

assumes P is *NSRD* Q is *NSRD* R is *NSRD*
 shows $(P \parallel_R Q) \parallel_R R = P \parallel_R Q \parallel_R R$
 by (*rdes-eq cls: assms*)

end

9 Productive Reactive Designs

theory *utp-rdes-productive*

imports *utp-rdes-parallel*

begin

9.1 Healthiness condition

A reactive design is productive if it strictly increases the trace, whenever it terminates. If it does not terminate, it is also classed as productive.

definition *Productive* :: $(\text{'s}, \text{'t}::\text{trace}, \text{'}\alpha) \text{ hrel-rsp} \Rightarrow (\text{'s}, \text{'t}, \text{'}\alpha) \text{ hrel-rsp}$ **where**
 $[\text{upred-defs}]: \text{Productive}(P) = P \parallel_R \mathbf{R}_s(\text{true} \vdash \text{true} \diamond (\$tr <_u \$tr'))$

lemma *Productive-RHS-design-form*:

assumes $\$ok' \# P \$ok' \# Q \$ok' \# R$
 shows $\text{Productive}(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash Q \diamond (R \wedge \$tr <_u \$tr'))$
 using *assms* by (*simp add: Productive-def RHS-tri-design-par unrest*)

lemma *Productive-form*:

$\text{Productive}(\text{SRD}(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$

proof –

have $\text{Productive}(\text{SRD}(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \parallel_R \mathbf{R}_s(\text{true} \vdash \text{true} \diamond (\$tr <_u \$tr'))$
 by (*simp add: Productive-def SRD-as-reactive-tri-design*)
 also have ... = $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$

by (simp add: RHS-tri-design-par unrest)
 finally show ?thesis .
 qed

A reactive design is productive provided that the postcondition, under the precondition, strictly increases the trace.

lemma *Productive-intro*:

assumes P is SRD $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P)) \$wait' \# pre_R(P)$
 shows P is Productive
proof –
 have $P : \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) = P$
proof –
 have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge post_R(P)))$
 by (metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem)
 also have $\dots = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge (post_R(P) \wedge \$tr <_u \$tr')))$
 by (metis assms(2) utp-pred-laws.inf.absorb1 utp-pred-laws.inf.assoc)
 also have $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$
 by (metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem)
 finally show ?thesis
 by (simp add: SRD-reactive-tri-design assms(1))
 qed
 thus ?thesis
 by (metis Healthy-def RHS-tri-design-par Productive-def ok'-pre-unrest unrest-true utp-pred-laws.inf-right-idem
 utp-pred-laws.inf-top-right)
 qed

lemma *Productive-post-refines-tr-increase*:

assumes P is SRD P is Productive $\$wait' \# pre_R(P)$
 shows $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$
proof –
 have $post_R(P) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
 by (metis Healthy-def Productive-form assms(1) assms(2))
 also have $\dots = R1(R2c(pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
 by (simp add: rea-post-RHS-design unrest usubst assms, rel-auto)
 also have $\dots = R1((pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
 by (simp add: R2c-impl R2c-preR R2c-postR R2c-and R2c-tr-less-tr' assms)
 also have $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge \dots)$
 by (rel-auto)
 finally show ?thesis .
 qed

lemma *Continuous-Productive [closure]: Continuous Productive*

by (simp add: Continuous-def Productive-def, rel-auto)

9.2 Reactive design calculations

lemma *preR-Productive [rdes]*:

assumes P is SRD
 shows $pre_R(Productive(P)) = pre_R(P)$
proof –
 have $pre_R(Productive(P)) = pre_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
 by (metis Healthy-def Productive-form assms)
 thus ?thesis
 by (simp add: rea-pre-RHS-design usubst unrest R2c-not R2c-preR R1-preR Healthy-if assms)

qed

lemma *periR-Productive* [rdes]:

assumes *P* is NSRD

shows $\text{peri}_R(\text{Productive}(P)) = \text{peri}_R(P)$

proof –

have $\text{peri}_R(\text{Productive}(P)) = \text{peri}_R(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')))$

by (metis *Healthy-def NSRD-is-SRD Productive-form assms*)

also have $\dots = R1 (R2c (\text{pre}_R P \Rightarrow_r \text{peri}_R P))$

by (simp add: *rea-peri-RHS-design usubst unrest R2c-not assms closure*)

also have $\dots = (\text{pre}_R P \Rightarrow_r \text{peri}_R P)$

by (simp add: *R1-rea-impl R2c-rea-impl R2c-preR R2c-peri-SRD
R1-peri-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)

finally show ?thesis

by (simp add: *SRD-peri-under-pre assms unrest closure*)

qed

lemma *postR-Productive* [rdes]:

assumes *P* is NSRD

shows $\text{post}_R(\text{Productive}(P)) = (\text{pre}_R(P) \Rightarrow_r \text{post}_R(P) \wedge \$tr <_u \$tr')$

proof –

have $\text{post}_R(\text{Productive}(P)) = \text{post}_R(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')))$

by (metis *Healthy-def NSRD-is-SRD Productive-form assms*)

also have $\dots = R1 (R2c (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u \$tr))$

by (simp add: *rea-post-RHS-design usubst unrest assms closure*)

also have $\dots = (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u \$tr)$

by (simp add: *R1-rea-impl R2c-rea-impl R2c-preR R2c-and R1-extend-conj' R2c-post-SRD
R1-post-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)

finally show ?thesis .

qed

lemma *preR-frame-seq-export*:

assumes *P* is NSRD *P* is Productive *Q* is NSRD

shows $(\text{pre}_R P \wedge (\text{pre}_R P \wedge \text{post}_R P) ;; Q) = (\text{pre}_R P \wedge (\text{post}_R P ;; Q))$

proof –

have $(\text{pre}_R P \wedge (\text{post}_R P ;; Q)) = (\text{pre}_R P \wedge ((\text{pre}_R P \Rightarrow_r \text{post}_R P) ;; Q))$

by (simp add: *SRD-post-under-pre assms closure unrest*)

also have $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) ;; Q \vee (\text{pre}_R P \Rightarrow_r R1(\text{post}_R P)) ;; Q)))$

by (simp add: *NSRD-is-SRD R1-post-SRD assms(1) rea-impl-def seqr-or-distl R1-preR Healthy-if*)

also have $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) ;; Q \vee (\text{pre}_R P \wedge \text{post}_R P) ;; Q)))$

proof –

have $(\text{pre}_R P \vee \neg_r \text{pre}_R P) = R1 \text{ true}$

by (simp add: *R1-preR rea-not-or*)

then show ?thesis

by (metis (no-types, lifting) *R1-def conj-comm disj-comm disj-conj-distr rea-impl-def seqr-or-distl*

utp-pred-laws.inf-top-left utp-pred-laws.sup.left-idem)

qed

also have $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) \vee (\text{pre}_R P \wedge \text{post}_R P) ;; Q)))$

by (simp add: *NSRD-neg-pre-left-zero assms closure SRD-healths*)

also have $\dots = (\text{pre}_R P \wedge (\text{pre}_R P \wedge \text{post}_R P) ;; Q)$

by (rel-blast)

finally show ?thesis ..

qed

9.3 Closure laws

lemma *Productive-rdes-intro*:

assumes $(\$tr <_u \$tr') \sqsubseteq R \$ok' \# P \$ok' \# Q \$ok' \# R \$wait \# P \$wait' \# P$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ is *Productive*

proof (rule *Productive-intro*)

show $\mathbf{R}_s(P \vdash Q \diamond R)$ is *SRD*

by (simp add: *RHS-tri-design-is-SRD* assms)

from *assms*(1) **show** $(\$tr' >_u \$tr) \sqsubseteq (pre_R(\mathbf{R}_s(P \vdash Q \diamond R)) \wedge post_R(\mathbf{R}_s(P \vdash Q \diamond R)))$

apply (simp add: *rea-pre-RHS-design* *rea-post-RHS-design* *usubst* *assms* *unrest*)

using *assms*(1) **apply** (*rel-auto*)

apply *fastforce*

done

show $\$wait' \# pre_R(\mathbf{R}_s(P \vdash Q \diamond R))$

by (simp add: *rea-pre-RHS-design* *rea-post-RHS-design* *usubst* *R1-def* *R2c-def* *R2s-def* *assms* *unrest*)

qed

We use the *R4* healthiness condition to characterise that the postcondition must extend the trace for a reactive design to be productive.

lemma *Productive-rdes-RR-intro*:

assumes *P is RR Q is RR R is RR R is R4*

shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ is *Productive*

using *assms* **by** (simp add: *Productive-rdes-intro* *R4-iff-refine* *unrest*)

lemma *Productive-Miracle* [closure]: *Miracle is Productive*

unfolding *Miracle-tri-def* *Healthy-def*

by (subst *Productive-RHS-design-form*, *simp-all* add: *unrest*)

lemma *Productive-Chaos* [closure]: *Chaos is Productive*

unfolding *Chaos-tri-def* *Healthy-def*

by (subst *Productive-RHS-design-form*, *simp-all* add: *unrest*)

lemma *Productive-intChoice* [closure]:

assumes *P is SRD P is Productive Q is SRD Q is Productive*

shows $P \sqcap Q$ is *Productive*

proof –

have $P \sqcap Q =$

$\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$

by (*metis* *Healthy-if* *Productive-form* *assms*)

also have $\dots = \mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond ((post_R P \wedge \$tr' >_u \$tr) \vee (post_R Q \wedge \$tr' >_u \$tr)))$

by (simp add: *RHS-tri-design-choice*)

also have $\dots = \mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond (((post_R P) \vee (post_R Q)) \wedge \$tr' >_u \$tr))$

by (rule *cong*[of \mathbf{R}_s \mathbf{R}_s], *simp*, *rel-auto*)

also have \dots is *Productive*

by (simp add: *Healthy-def* *Productive-RHS-design-form* *unrest*)

finally show *?thesis* .

qed

lemma *Productive-cond-rea* [closure]:

assumes *P is SRD P is Productive Q is SRD Q is Productive*

shows $P \triangleleft b \triangleright_R Q$ is *Productive*

proof –

have $P \triangleleft b \triangleright_R Q =$
 $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) \triangleleft b \triangleright_R \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond (\text{post}_R(Q) \wedge \$tr <_u \$tr'))$
 by (metis Healthy-if Productive-form assms)
 also have $\dots = \mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond ((\text{post}_R P \wedge \$tr' >_u \$tr) \triangleleft b \triangleright_R (\text{post}_R Q \wedge \$tr' >_u \$tr)))$
 by (simp add: cond-srea-form)
 also have $\dots = \mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond (((\text{post}_R P) \triangleleft b \triangleright_R (\text{post}_R Q)) \wedge \$tr' >_u \$tr))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have \dots is Productive
 by (simp add: Healthy-def Productive-RHS-design-form unrest)
 finally show ?thesis .
qed

lemma Productive-seq-1 [closure]:

assumes P is NSRD P is Productive Q is NSRD
 shows $P ;; Q$ is Productive

proof –

have $P ;; Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) ;; \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond (\text{post}_R(Q)))$
 by (metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms(1) assms(2) assms(3))
 also have $\dots = \mathbf{R}_s((\text{pre}_R P \wedge (\text{post}_R P \wedge \$tr' >_u \$tr) \text{wp}_r \text{pre}_R Q) \vdash (\text{peri}_R P \vee ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{peri}_R Q)) \diamond ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{post}_R Q))$
 by (simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def)
 also have $\dots = \mathbf{R}_s((\text{pre}_R P \wedge (\text{post}_R P \wedge \$tr' >_u \$tr) \text{wp}_r \text{pre}_R Q) \vdash (\text{peri}_R P \vee ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{peri}_R Q)) \diamond ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{post}_R Q \wedge \$tr' >_u \$tr))$
proof –
 have $((\text{post}_R P \wedge \$tr' >_u \$tr) ;; R1(\text{post}_R Q)) = ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; R1(\text{post}_R Q) \wedge \$tr' >_u \$tr)$
 by (rel-auto)
 thus ?thesis
 by (simp add: NSRD-is-SRD R1-post-SRD assms)
qed
 also have \dots is Productive
 by (rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def)
 finally show ?thesis .
qed

lemma Productive-seq-2 [closure]:

assumes P is NSRD Q is NSRD Q is Productive
 shows $P ;; Q$ is Productive

proof –

have $P ;; Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P))) ;; \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond (\text{post}_R(Q) \wedge \$tr <_u \$tr'))$
 by (metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms)
 also have $\dots = \mathbf{R}_s((\text{pre}_R P \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q) \vdash (\text{peri}_R P \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; (\text{post}_R Q \wedge \$tr' >_u \$tr)))$
 by (simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def)

```

also have ... =  $\mathbf{R}_s$  (( $pre_R P \wedge post_R P \wp_r pre_R Q$ )  $\vdash$  ( $peri_R P \vee (post_R P ;; peri_R Q)$ )  $\diamond$  ( $post_R P ;; (post_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr$ ))
proof –
  have ( $R1(post_R P) ;; (post_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr$ ) = ( $R1(post_R P) ;; (post_R Q \wedge \$tr' >_u \$tr)$ )
  by (rel-auto)
  thus ?thesis
  by (simp add: NSRD-is-SRD R1-post-SRD assms)
qed
also have ... is Productive
  by (rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def)
finally show ?thesis .
qed
end

```

10 Guarded Recursion

```

theory utp-rdes-guarded
imports utp-rdes-productive
begin

```

10.1 Traces with a size measure

Guarded recursion relies on our ability to measure the trace's size, in order to see if it is decreasing on each iteration. Thus, we here equip the trace algebra with the *ucard* function that provides this.

```

class size-trace = trace + size +
assumes

```

```

  size-zero:  $size\ 0 = 0$  and
  size-nzero:  $s > 0 \implies size(s) > 0$  and
  size-plus:  $size\ (s + t) = size(s) + size(t)$ 

```

— These axioms may be stronger than necessary. In particular, $0 < ?s \implies 0 < \#_u(?s)$ requires that a non-empty trace have a positive size. But this may not be the case with all trace models and is possibly more restrictive than necessary. In future we will explore weakening.

```

begin

```

```

lemma size-mono:  $s \leq t \implies size(s) \leq size(t)$ 
by (metis le-add1 local.diff-add-cancel-left' local.size-plus)

```

```

lemma size-strict-mono:  $s < t \implies size(s) < size(t)$ 
by (metis cancel-ab-semigroup-add-class.add-diff-cancel-left' local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero local.size-plus zero-less-diff)

```

```

lemma trace-strict-prefixE:  $xs < ys \implies (\bigwedge zs. \llbracket ys = xs + zs; size(zs) > 0 \rrbracket \implies thesis) \implies thesis$ 
by (metis local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero)

```

```

lemma size-minus-trace:  $y \leq x \implies size(x - y) = size(x) - size(y)$ 
by (metis diff-add-inverse local.diff-add-cancel-left' local.size-plus)

```

```

end

```

Both natural numbers and lists are measurable trace algebras.

```

instance nat :: size-trace

```

```

by (intro-classes, simp-all)

instance list :: (type) size-trace
  by (intro-classes, simp-all add: zero-list-def less-list-def' plus-list-def prefix-length-less)

syntax
  -usize      :: logic  $\Rightarrow$  logic (sizeu'(-))

translations
  sizeu(t) == CONST uop CONST size t

```

10.2 Guardedness

definition *gvert* :: ((*t*::size-trace, α) *rp* \times (*t*, α) *rp*) *chain* **where**
 [upred-defs]: *gvert*(*n*) \equiv ($\$tr \leq_u \$tr' \wedge size_u(\&tt) <_u \ll n \gg$)

lemma *gvert-chain*: *chain gvert*
 apply (simp add: chain-def, safe)
 apply (rel-simp)
 apply (rel-simp)+
 done

lemma *gvert-limit*: \sqcap (range *gvert*) = ($\$tr \leq_u \tr')
 by (rel-auto)

definition *Guarded* :: ((*t*::size-trace, α) *hrel-rp* \Rightarrow (*t*, α) *hrel-rp*) \Rightarrow bool **where**
 [upred-defs]: *Guarded*(*F*) = ($\forall X n. (F(X) \wedge gvert(n+1)) = (F(X \wedge gvert(n)) \wedge gvert(n+1))$)

lemma *GuardedI*: $\ll \bigwedge X n. (F(X) \wedge gvert(n+1)) = (F(X \wedge gvert(n)) \wedge gvert(n+1)) \gg \implies \text{Guarded } F$
 by (simp add: Guarded-def)

Guarded reactive designs yield unique fixed-points.

theorem *guarded-fp-uniq*:
 assumes *mono* *F* $F \in \ll id \gg_H \rightarrow \ll SRD \gg_H \text{ Guarded } F$
 shows $\mu F = \nu F$
proof –
 have *constr* *F* *gvert*
 using *assms*
 by (auto simp add: constr-def gvert-chain Guarded-def tcontr-alt-def')
 hence ($\$tr \leq_u \$tr' \wedge \mu F$) = ($\$tr \leq_u \$tr' \wedge \nu F$)
 apply (rule constr-fp-uniq)
 apply (simp add: assms)
 using *gvert-limit* apply blast
 done
 moreover have ($\$tr \leq_u \$tr' \wedge \mu F$) = μF
proof –
 have μF is *R1*
 by (rule SRD-healths(1), rule Healthy-mu, simp-all add: assms)
 thus ?thesis
 by (metis Healthy-def R1-def conj-comm)
 qed
 moreover have ($\$tr \leq_u \$tr' \wedge \nu F$) = νF
proof –
 have νF is *R1*
 by (rule SRD-healths(1), rule Healthy-nu, simp-all add: assms)
 thus ?thesis

by (metis Healthy-def R1-def conj-comm)
 qed
 ultimately show ?thesis
 by (simp)
 qed

lemma Guarded-const [closure]: Guarded ($\lambda X. P$)
 by (simp add: Guarded-def)

lemma UINF-Guarded [closure]:

assumes $\bigwedge P. P \in A \implies \text{Guarded } P$
 shows Guarded ($\lambda X. \bigcap P \in A \cdot P(X)$)

proof (rule GuardedI)

fix $X n$

have $\bigwedge Y. ((\bigcap P \in A \cdot P Y) \wedge \text{gvert}(n+1)) = ((\bigcap P \in A \cdot (P Y \wedge \text{gvert}(n+1))) \wedge \text{gvert}(n+1))$

proof –

fix Y

let $?lhs = ((\bigcap P \in A \cdot P Y) \wedge \text{gvert}(n+1))$ and $?rhs = ((\bigcap P \in A \cdot (P Y \wedge \text{gvert}(n+1))) \wedge \text{gvert}(n+1))$

have $a: ?lhs \llbracket \text{false} / \$ok \rrbracket = ?rhs \llbracket \text{false} / \$ok \rrbracket$

by (rel-auto)

have $b: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket$

by (rel-auto)

have $c: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket$

by (rel-auto)

show $?lhs = ?rhs$

using $a b c$

by (rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all)

qed

moreover have $((\bigcap P \in A \cdot (P X \wedge \text{gvert}(n+1))) \wedge \text{gvert}(n+1)) = ((\bigcap P \in A \cdot (P (X \wedge \text{gvert}(n))) \wedge \text{gvert}(n+1))) \wedge \text{gvert}(n+1))$

proof –

have $(\bigcap P \in A \cdot (P X \wedge \text{gvert}(n+1))) = (\bigcap P \in A \cdot (P (X \wedge \text{gvert}(n)) \wedge \text{gvert}(n+1)))$

proof (rule UINF-cong)

fix P assume $P \in A$

thus $(P X \wedge \text{gvert}(n+1)) = (P (X \wedge \text{gvert}(n)) \wedge \text{gvert}(n+1))$

using Guarded-def assms by blast

qed

thus ?thesis by simp

qed

ultimately show $((\bigcap P \in A \cdot P X) \wedge \text{gvert}(n+1)) = ((\bigcap P \in A \cdot (P (X \wedge \text{gvert}(n)))) \wedge \text{gvert}(n+1))$

by simp

qed

lemma intChoice-Guarded [closure]:

assumes Guarded P Guarded Q

shows Guarded ($\lambda X. P(X) \sqcap Q(X)$)

proof –

have Guarded ($\lambda X. \bigcap F \in \{P, Q\} \cdot F(X)$)

by (rule UINF-Guarded, auto simp add: assms)

thus ?thesis

by (simp)

qed

lemma cond-srea-Guarded [closure]:

assumes Guarded P Guarded Q

shows *Guarded* $(\lambda X. P(X) \triangleleft b \triangleright_R Q(X))$
using *assms* **by** (*rel-auto*)

A tail recursive reactive design with a productive body is guarded.

lemma *Guarded-if-Productive* [*closure*]:

fixes $P :: ('s, 't :: \text{size-trace}, 'a) \text{ hrel-rsp}$

assumes $P \text{ is NSRD } P \text{ is Productive}$

shows *Guarded* $(\lambda X. P ;; \text{SRD}(X))$

proof (*clarsimp simp add: Guarded-def*)

— We split the proof into three cases corresponding to valuations for ok, wait, and wait' respectively.

fix $X \ n$

have $a: (P ;; \text{SRD}(X) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket =$

$(P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket$

by (*simp add: usubst closure SRD-left-zero-1 assms*)

have $b: ((P ;; \text{SRD}(X) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket =$

$((P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket$

by (*simp add: usubst closure SRD-left-zero-2 assms*)

have $c: (P ;; \text{SRD}(X) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket \llbracket \text{false}/\$wait \rrbracket =$

$((P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{false}/\$wait \rrbracket$

proof —

have $1: (P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$

$(P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvert } n)) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

by (*metis (no-types, lifting) Healthy-def R3h-wait-true SRD-healths(3) SRD-idem*)

have $2: (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$

$(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvert } n)) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

proof —

have $\text{exp}: \bigwedge Y :: ('s, 't, 'a) \text{ hrel-rsp}. (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

=

$((((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD } Y) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket)) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

proof —

fix $Y :: ('s, 't, 'a) \text{ hrel-rsp}$

have $(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$

$((\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

by (*metis (no-types) Healthy-def Productive-form assms(1) assms(2) NSRD-is-SRD*)

also have ... =

$((R1(R2c(\text{pre}_R(P) \Rightarrow (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')))) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def RD1-def RD2-def usubst unrest assms closure design-def*)

also have ... =

$((((\neg_r \text{pre}_R(P) \vee (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

by (*simp add: impl-alt-def R2c-disj R1-disj R2c-not assms closure R2c-and*

$R2c\text{-preR } \text{rea-not-def } R1\text{-extend-conj}' R2c\text{-ok}' R2c\text{-post-SRD } R1\text{-tr-less-tr}' R2c\text{-tr-less-tr}'$)

also have ... =

$((((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\$ok' \wedge \text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket)) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

by (*simp add: usubst unrest assms closure seqr-or-distl NSRD-neg-pre-left-zero SRD-healths*)

also have ... =

$((((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD } Y) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket)) \wedge \text{gvert } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

```

proof –
  have ( $\$ok' \wedge post_R P \wedge \$tr' >_u \$tr$ ) ;; ( $SRD\ Y$ ) $\llbracket false/\$wait \rrbracket =$ 
    ( $(post_R P \wedge \$tr' >_u \$tr) \wedge \$ok' =_u true$ ) ;; ( $SRD\ Y$ ) $\llbracket false/\$wait \rrbracket$ 
  by (rel-blast)
  also have ... = ( $post_R P \wedge \$tr' >_u \$tr$ ) $\llbracket true/\$ok' \rrbracket$  ;; ( $SRD\ Y$ ) $\llbracket false/\$wait \rrbracket$  $\llbracket true/\$ok \rrbracket$ 
  using seqr-left-one-point[of ok (post_R P \wedge \$tr' >_u \$tr) True (SRD Y)\llbracket false/\$wait \rrbracket]
  by (simp add: true-alt-def[THEN sym])
  finally show ?thesis by (simp add: usubst unrest)
qed
finally
show ( $P\llbracket false/\$wait' \rrbracket$  ;; ( $SRD\ Y$ ) $\llbracket false/\$wait \rrbracket \wedge gvt (Suc\ n)\llbracket true,false/\$ok,\$wait \rrbracket =$ 
  ( $((\neg_r\ pre_R\ P) ;; (SRD(Y))\llbracket false/\$wait \rrbracket \vee (post_R P \wedge \$tr' >_u \$tr) ;; (SRD$ 
 $Y)\llbracket true,false/\$ok,\$wait \rrbracket)$ 
   $\wedge gvt (Suc\ n)\llbracket true,false/\$ok,\$wait \rrbracket$  .
qed

have 1: ( $(post_R P \wedge \$tr' >_u \$tr) ;; (SRD\ X)\llbracket true,false/\$ok,\$wait \rrbracket \wedge gvt (Suc\ n) =$ 
  ( $(post_R P \wedge \$tr' >_u \$tr) ;; (SRD\ (X \wedge gvt\ n))\llbracket true,false/\$ok,\$wait \rrbracket \wedge gvt (Suc\ n)$ )
apply (rel-auto)
apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok')
apply (rule-tac x=tr0 in exI, rule-tac x=st0 in exI, rule-tac x=more0 in exI)
apply (simp)
apply (erule trace-strict-prefixE)
apply (rename-tac tr st ref ok wait tr' st' ref' tr0 st0 ref0 ok' zs)
apply (rule-tac x=False in exI)
apply (simp add: size-minus-trace)
apply (subgoal-tac size(tr) < size(tr0))
apply (simp add: less-diff-conv2 size-mono)
using size-strict-mono apply blast
apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok')
apply (rule-tac x=tr0 in exI, rule-tac x=st0 in exI, rule-tac x=more0 in exI)
apply (simp)
apply (erule trace-strict-prefixE)
apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok' zs)
apply (auto simp add: size-minus-trace)
apply (subgoal-tac size(tr) < size(tr0))
apply (simp add: less-diff-conv2 size-mono)
using size-strict-mono apply blast
done
have 2: ( $\neg_r\ pre_R\ P$ ) ;; ( $SRD\ X$ ) $\llbracket false/\$wait \rrbracket = (\neg_r\ pre_R\ P) ;; (SRD(X \wedge gvt\ n))\llbracket false/\$wait \rrbracket$ 
by (simp add: NSRD-neg-pre-left-zero closure assms SRD-healths)
show ?thesis
by (simp add: exp 1 2 utp-pred-laws.inf-sup-distrib2)
qed

show ?thesis
proof –
  have ( $P ;; (SRD\ X) \wedge gvt (n+1)\llbracket true,false/\$ok,\$wait \rrbracket =$ 
    ( $(P\llbracket true/\$wait' \rrbracket ;; (SRD\ X)\llbracket true/\$wait \rrbracket \wedge gvt (n+1)\llbracket true,false/\$ok,\$wait \rrbracket \vee$ 
    ( $P\llbracket false/\$wait' \rrbracket ;; (SRD\ X)\llbracket false/\$wait \rrbracket \wedge gvt (n+1)\llbracket true,false/\$ok,\$wait \rrbracket)$ 
    by (subst seqr-bool-split[of wait], simp-all add: usubst utp-pred-laws.distrib(4))

  also
  have ... = ( $(P\llbracket true/\$wait' \rrbracket ;; (SRD\ (X \wedge gvt\ n))\llbracket true/\$wait \rrbracket \wedge gvt (n+1)\llbracket true,false/\$ok,\$wait \rrbracket$ 

```

∇


```

    (P[[false/$wait']] ;; (SRD (X ∧ gvirt n))[[false/$wait]] ∧ gvirt (n+1))[[true,false/$ok,$wait]])
  by (simp add: 1 2)

  also
  have ... = ((P[[true/$wait']] ;; (SRD (X ∧ gvirt n))[[true/$wait]] ∨
    P[[false/$wait']] ;; (SRD (X ∧ gvirt n))[[false/$wait]] ∧ gvirt (n+1))[[true,false/$ok,$wait]])
  by (simp add: usubst utp-pred-laws.distrib(4))

  also have ... = (P ;; (SRD (X ∧ gvirt n)) ∧ gvirt (n+1))[[true,false/$ok,$wait]]
  by (subst seqr-bool-split[of wait], simp-all add: usubst)
  finally show ?thesis by (simp add: usubst)
qed

qed
show (P ;; SRD(X) ∧ gvirt (Suc n)) = (P ;; SRD(X ∧ gvirt n) ∧ gvirt (Suc n))
  apply (rule-tac bool-eq-splitI[of in-var ok])
  apply (simp-all add: a)
  apply (rule-tac bool-eq-splitI[of in-var wait])
  apply (simp-all add: b c)
done
qed



### 10.3 Tail recursive fixed-point calculations



declare upred-semiring.power-Suc [simp]

lemma mu-csp-form-1 [rdes]:
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows (μ X · P ;; SRD(X)) = (⊓ i · P ^ (i+1)) ;; Miracle
proof -
  have 1: Continuous (λX. P ;; SRD X)
    using SRD-Continuous
    by (clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac x=A in spec, simp)
  have 2: (λX. P ;; SRD X) ∈ [[id]]H → [[SRD]]H
    by (blast intro: funcsetI closure assms)
  with 1 2 have (μ X · P ;; SRD(X)) = (ν X · P ;; SRD(X))
    by (simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure)
  also have ... = (⊓ i. ((λX. P ;; SRD X) ^^ i) false)
    by (simp add: sup-continuous-lfp 1 sup-continuous-Continuous false-upred-def)
  also have ... = ((λX. P ;; SRD X) ^^ 0) false ⊓ (⊓ i. ((λX. P ;; SRD X) ^^ (i+1)) false)
    by (subst Sup-power-expand, simp)
  also have ... = (⊓ i. ((λX. P ;; SRD X) ^^ (i+1)) false)
    by (simp)
  also have ... = (⊓ i. P ^ (i+1)) ;; Miracle
proof (rule SUP-cong, simp-all)
  fix i
  show P ;; SRD (((λX. P ;; SRD X) ^^ i) false) = (P ;; P ^ i) ;; Miracle
proof (induct i)
  case 0
  then show ?case
    by (simp, metis srdes-hcond-def srdes-theory-continuous.healthy-top)
next
  case (Suc i)
  then show ?case
    by (simp add: Healthy-if NSRD-is-SRD SRD-power-comp SRD-seqr-closure assms(1) seqr-assoc[THEN

```

```

sym] srdes-theory-continuous.weak.top-closed)
  qed
qed
also have ... = ( $\prod i. P \wedge (i+1)$ ) ;; Miracle
  by (simp add: seq-Sup-distr)
finally show ?thesis
  by (simp add: UINF-as-Sup[THEN sym])
qed

lemma mu-csp-form-NSRD [closure]:
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows ( $\mu X \cdot P$  ;; SRD(X)) is NSRD
  by (simp add: mu-csp-form-1 assms closure)

lemma mu-csp-form-1':
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows ( $\mu X \cdot P$  ;; SRD(X)) = (P ;; P*) ;; Miracle
proof -
  have ( $\mu X \cdot P$  ;; SRD(X)) = ( $\prod i \in UNIV \cdot P$  ;;  $P \wedge i$ ) ;; Miracle
    by (simp add: mu-csp-form-1 assms closure ustar-def)
  also have ... = (P ;; P*) ;; Miracle
    by (simp only: seq-UINF-distl[THEN sym], simp add: ustar-def)
  finally show ?thesis .
qed

declare upred-semiring.power-Suc [simp del]

end

```

11 Reactive Design Programs

```

theory utp-rdes-prog
  imports
    utp-rdes-normal
    utp-rdes-tactics
    utp-rdes-parallel
    utp-rdes-guarded
    UTP-KAT.utp-kleene
begin

```

11.1 State substitution

```

lemma srd-subst-RHS-tri-design [usubst]:
  
$$[\sigma]_{S\sigma} \dagger \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s([\sigma]_{S\sigma} \dagger P) \vdash ([\sigma]_{S\sigma} \dagger Q) \diamond ([\sigma]_{S\sigma} \dagger R)$$

  by (rel-auto)

lemma srd-subst-SRD-closed [closure]:
  assumes P is SRD
  shows  $[\sigma]_{S\sigma} \dagger P$  is SRD
proof -
  have  $SRD([\sigma]_{S\sigma} \dagger (SRD P)) = [\sigma]_{S\sigma} \dagger (SRD P)$ 
    by (rel-auto)
  thus ?thesis

```

by (metis Healthy-def assms)
qed

lemma *preR-srd-subst* [rdes]:
 $pre_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger pre_R(P)$
 by (rel-auto)

lemma *periR-srd-subst* [rdes]:
 $peri_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger peri_R(P)$
 by (rel-auto)

lemma *postR-srd-subst* [rdes]:
 $post_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger post_R(P)$
 by (rel-auto)

lemma *srd-subst-NSRD-closed* [closure]:
 assumes *P is NSRD*
 shows $\lceil \sigma \rceil_{S\sigma} \dagger P$ is NSRD
 by (rule NSRD-RC-intro, simp-all add: closure rdes assms unrest)

11.2 Assignment

definition *assigns-srd* :: 's usubst \Rightarrow ('s, 't::trace, 'α) hrel-rsp ($\langle \cdot \rangle_R$) **where**
 $[upred-defs]: assigns-srd \sigma = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil \langle \sigma \rangle_a \rceil_S \wedge \$\Sigma_S' =_u \$\Sigma_S))$

syntax

-assign-srd :: svids \Rightarrow uexprs \Rightarrow logic ('(-) :=_R '(-))
 -assign-srd :: svids \Rightarrow uexprs \Rightarrow logic (**infixr** :=_R 90)

translations

-assign-srd *xs vs* \Rightarrow *CONST assigns-srd (-mk-usubst (CONST id) xs vs)*
 -assign-srd *x v* \leq *CONST assigns-srd (CONST subst-upd (CONST id) x v)*
 -assign-srd *x v* \leq -assign-srd (-spvar *x*) *v*
 $x, y :=_R u, v \leq$ *CONST assigns-srd (CONST subst-upd (CONST subst-upd (CONST id) (CONST svar x) u) (CONST svar y) v)*

lemma *assigns-srd-RHS-tri-des* [rdes-def]:
 $\langle \sigma \rangle_R = \mathbf{R}_s(true_r \vdash false \diamond \langle \sigma \rangle_r)$
 by (rel-auto)

lemma *assigns-srd-NSRD-closed* [closure]: $\langle \sigma \rangle_R$ is NSRD
 by (simp add: rdes-def closure unrest)

lemma *preR-assigs-srd* [rdes]: $pre_R(\langle \sigma \rangle_R) = true_r$
 by (simp add: rdes-def rdes closure)

lemma *periR-assigs-srd* [rdes]: $peri_R(\langle \sigma \rangle_R) = false$
 by (simp add: rdes-def rdes closure)

lemma *postR-assigs-srd* [rdes]: $post_R(\langle \sigma \rangle_R) = \langle \sigma \rangle_r$
 by (simp add: rdes-def rdes closure rpred)

11.3 Conditional

lemma *preR-cond-srea* [rdes]:
 $pre_R(P \triangleleft b \triangleright_R Q) = (\lceil b \rceil_{S<} \wedge pre_R(P) \vee [\neg b]_{S<} \wedge pre_R(Q))$

by (rel-auto)

lemma *periR-cond-srea* [rdes]:
 assumes *P is SRD Q is SRD*
 shows $\text{peri}_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge \text{peri}_R(P) \vee [\neg b]_{S<} \wedge \text{peri}_R(Q))$
proof –
 have $\text{peri}_R(P \triangleleft b \triangleright_R Q) = \text{peri}_R(R1(P) \triangleleft b \triangleright_R R1(Q))$
 by (simp add: Healthy-if SRD-healths assms)
 thus ?thesis
 by (rel-auto)
qed

lemma *postR-cond-srea* [rdes]:
 assumes *P is SRD Q is SRD*
 shows $\text{post}_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge \text{post}_R(P) \vee [\neg b]_{S<} \wedge \text{post}_R(Q))$
proof –
 have $\text{post}_R(P \triangleleft b \triangleright_R Q) = \text{post}_R(R1(P) \triangleleft b \triangleright_R R1(Q))$
 by (simp add: Healthy-if SRD-healths assms)
 thus ?thesis
 by (rel-auto)
qed

lemma *NSRD-cond-srea* [closure]:
 assumes *P is NSRD Q is NSRD*
 shows *P \triangleleft b \triangleright_R Q is NSRD*
proof (rule NSRD-RC-intro)
 show *P \triangleleft b \triangleright_R Q is SRD*
 by (simp add: closure assms)
 show *pre_R (P \triangleleft b \triangleright_R Q) is RC*
proof –
 have 1: $([\neg b]_{S<} \vee \neg_r \text{pre}_R P) ;; R1(\text{true}) = ([\neg b]_{S<} \vee \neg_r \text{pre}_R P)$
 by (metis (no-types, lifting) NSRD-neg-pre-unit aext-not assms(1) seqr-or-distl st-lift-R1-true-right)
 have 2: $([b]_{S<} \vee \neg_r \text{pre}_R Q) ;; R1(\text{true}) = ([b]_{S<} \vee \neg_r \text{pre}_R Q)$
 by (simp add: NSRD-neg-pre-unit assms seqr-or-distl st-lift-R1-true-right)
 show ?thesis
 by (simp add: rdes closure assms)
qed
 show $\$st' \# \text{peri}_R (P \triangleleft b \triangleright_R Q)$
 by (simp add: rdes assms closure unrest)
qed

11.4 Assumptions

definition *AssumeR* :: '*s cond* \Rightarrow ('*s*, '*t::trace*, '*α*) *hrel-rsp* ($[-]^\top_R$) **where**
[upred-defs]: *AssumeR b = II_R \triangleleft b \triangleright_R Miracle*

lemma *AssumeR-rdes-def* [rdes-def]:
 $[b]^\top_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond [b]^\top_r)$
unfolding *AssumeR-def* **by** (rdes-eq)

lemma *AssumeR-NSRD* [closure]: $[b]^\top_R$ *is NSRD*
by (simp add: AssumeR-def closure)

lemma *AssumeR-false*: $[false]^\top_R = \text{Miracle}$
by (rel-auto)

lemma *AssumeR-true*: $[true]^\top_R = II_R$
by (*rel-auto*)

lemma *AssumeR-comp*: $[b]^\top_R ;; [c]^\top_R = [b \wedge c]^\top_R$
by (*rdes-simp*)

lemma *AssumeR-choice*: $[b]^\top_R \sqcap [c]^\top_R = [b \vee c]^\top_R$
by (*rdes-eq*)

lemma *AssumeR-refine-skip*: $II_R \sqsubseteq [b]^\top_R$
by (*rdes-refine*)

lemma *AssumeR-test* [*closure*]: $test_R [b]^\top_R$
by (*simp add: AssumeR-refine-skip nsrd-thy.utest-intro*)

lemma *Star-AssumeR*: $[b]^\top_R^{\star R} = II_R$
by (*simp add: AssumeR-NSRD AssumeR-test nsrd-thy.Star-test*)

lemma *AssumeR-choice-skip*: $II_R \sqcap [b]^\top_R = II_R$
by (*rdes-eq*)

lemma *cond-srea-AssumeR-form*:
assumes P is NSRD Q is NSRD
shows $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P \sqcap [\neg b]^\top_R ;; Q)$
by (*rdes-eq cls: assms*)

lemma *cond-srea-insert-assume*:
assumes P is NSRD Q is NSRD
shows $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P \triangleleft b \triangleright_R [\neg b]^\top_R ;; Q)$
by (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure RA1 assms cond-srea-AssumeR-form*)

lemma *AssumeR-cond-left*:
assumes P is NSRD Q is NSRD
shows $[b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([b]^\top_R ;; P)$
by (*rdes-eq cls: assms*)

lemma *AssumeR-cond-right*:
assumes P is NSRD Q is NSRD
shows $[\neg b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([\neg b]^\top_R ;; Q)$
by (*rdes-eq cls: assms*)

11.5 Guarded commands

definition *GuardedCommR* :: $'s \text{ cond} \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} (- \rightarrow_R - [85, 86] \ 85)$ **where**
gcmd-def[*rdes-def*]: $\text{GuardedCommR } g \ A = A \triangleleft g \triangleright_R \text{Miracle}$

lemma *gcmd-false*[*simp*]: $(false \rightarrow_R A) = \text{Miracle}$
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-true*[*simp*]: $(true \rightarrow_R A) = A$
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-SRD*:
assumes A is SRD
shows $(g \rightarrow_R A)$ is SRD

by (simp add: gcmd-def SRD-cond-srea assms srdes-theory-continuous.weak.top-closed)

lemma gcmd-NSRD [closure]:

assumes A is NSRD

shows $(g \rightarrow_R A)$ is NSRD

by (simp add: gcmd-def NSRD-cond-srea assms NSRD-Miracle)

lemma gcmd-Productive [closure]:

assumes A is NSRD A is Productive

shows $(g \rightarrow_R A)$ is Productive

by (simp add: gcmd-def closure assms)

lemma gcmd-seq-distr:

assumes B is NSRD

shows $(g \rightarrow_R A) ;; B = (g \rightarrow_R (A ;; B))$

by (simp add: Miracle-left-zero NSRD-is-SRD assms cond-st-distr gcmd-def)

lemma gcmd-nondet-distr:

assumes A is NSRD B is NSRD

shows $(g \rightarrow_R (A \sqcap B)) = (g \rightarrow_R A) \sqcap (g \rightarrow_R B)$

by (rdes-eq cls: assms)

lemma AssumeR-as-gcmd:

$[b]^\top_R = b \rightarrow_R II_R$

by (rdes-eq)

12 Generalised Alternation

definition AlternateR

$:: 'a \text{ set} \Rightarrow ('a \Rightarrow 's \text{ upred}) \Rightarrow ('a \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$ **where**

[upred-defs, rdes-def]: $\text{AlternateR } I \ g \ A \ B = (\bigcap i \in I \cdot ((g \ i) \rightarrow_R (A \ i))) \sqcap ((\neg (\bigvee i \in I \cdot g \ i)) \rightarrow_R B)$

definition AlternateR-list

$:: ('s \text{ upred} \times ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \text{ list} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$ **where**

[upred-defs, ndes-simp]:

$\text{AlternateR-list } xs \ P = \text{AlternateR } \{0..<\text{length } xs\} \ (\lambda i. \text{map fst } xs \ ! \ i) \ (\lambda i. \text{map snd } xs \ ! \ i) \ P$

syntax

-altindR-els $:: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \in \ - \rightarrow \ - \text{ else } - \text{ fi})$

-altindR $:: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \in \ - \rightarrow \ - \text{ fi})$

-altgcommR-els $:: \text{gcomms} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \text{ else } - \text{ fi})$

-altgcommR $:: \text{gcomms} \Rightarrow \text{logic} \ (if_R \ - \text{ fi})$

translations

$if_R \ i \in I \cdot g \rightarrow A \text{ else } B \text{ fi} \rightarrow \text{CONST AlternateR } I \ (\lambda i. g) \ (\lambda i. A) \ B$

$if_R \ i \in I \cdot g \rightarrow A \text{ fi} \rightarrow \text{CONST AlternateR } I \ (\lambda i. g) \ (\lambda i. A) \ (\text{CONST Chaos})$

$if_R \ i \in I \cdot (g \ i) \rightarrow A \text{ else } B \text{ fi} \leftarrow \text{CONST AlternateR } I \ g \ (\lambda i. A) \ B$

-altgcommR $cs \rightarrow \text{CONST AlternateR-list } cs \ (\text{CONST Chaos})$

-altgcommR $(\text{-gcomm-show } cs) \leftarrow \text{CONST AlternateR-list } cs \ (\text{CONST Chaos})$

-altgcommR-els $cs \ P \rightarrow \text{CONST AlternateR-list } cs \ P$

-altgcommR-els $(\text{-gcomm-show } cs) \ P \leftarrow \text{CONST AlternateR-list } cs \ P$

lemma *AlternateR-NSRD-closed* [closure]:
assumes $\bigwedge i. i \in I \implies A \text{ } i \text{ is NSRD } B \text{ is NSRD}$
shows $(\text{if}_R i \in I \cdot g \text{ } i \rightarrow A \text{ } i \text{ else } B \text{ fi}) \text{ is NSRD}$
proof (cases $I = \{\}$)
 case *True*
 then show ?thesis **by** (simp add: AlternateR-def assms)
next
 case *False*
 then show ?thesis **by** (simp add: AlternateR-def closure assms)
qed

lemma *AlternateR-empty* [simp]:
 $(\text{if}_R i \in \{\} \cdot g \text{ } i \rightarrow A \text{ } i \text{ else } B \text{ fi}) = B$
by (rdes-simp)

lemma *AlternateR-Productive* [closure]:
assumes
 $\bigwedge i. i \in I \implies A \text{ } i \text{ is NSRD } B \text{ is NSRD}$
 $\bigwedge i. i \in I \implies A \text{ } i \text{ is Productive } B \text{ is Productive}$
shows $(\text{if}_R i \in I \cdot g \text{ } i \rightarrow A \text{ } i \text{ else } B \text{ fi}) \text{ is Productive}$
proof (cases $I = \{\}$)
 case *True*
 then show ?thesis
 by (simp add: assms(4))
next
 case *False*
 then show ?thesis
 by (simp add: AlternateR-def closure assms)
qed

lemma *AlternateR-singleton*:
assumes $A \text{ } k \text{ is NSRD } B \text{ is NSRD}$
shows $(\text{if}_R i \in \{k\} \cdot g \text{ } i \rightarrow A \text{ } i \text{ else } B \text{ fi}) = (A(k) \triangleleft g(k) \triangleright_R B)$
by (simp add: AlternateR-def, rdes-eq cls: assms)

Convert an alternation over disjoint guards into a cascading if-then-else

lemma *AlternateR-insert-cascade*:
assumes
 $\bigwedge i. i \in I \implies A \text{ } i \text{ is NSRD}$
 $A \text{ } k \text{ is NSRD } B \text{ is NSRD}$
 $(g(k) \wedge (\bigvee i \in I \cdot g(i))) = \text{false}$
shows $(\text{if}_R i \in \text{insert } k \text{ } I \cdot g \text{ } i \rightarrow A \text{ } i \text{ else } B \text{ fi}) = (A(k) \triangleleft g(k) \triangleright_R (\text{if}_R i \in I \cdot g(i) \rightarrow A(i) \text{ else } B \text{ fi}))$
proof (cases $I = \{\}$)
 case *True*
 then show ?thesis **by** (simp add: AlternateR-singleton assms)
next
 case *False*
 have 1: $(\bigcap i \in I \cdot g \text{ } i \rightarrow_R A \text{ } i) = (\bigcap i \in I \cdot g \text{ } i \rightarrow_R \mathbf{R}_s(\text{pre}_R(A \text{ } i) \vdash \text{peri}_R(A \text{ } i) \diamond \text{post}_R(A \text{ } i)))$
 by (simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) cong: UINF-cong)
 from assms(4) **show** ?thesis
 by (simp add: AlternateR-def 1 False)
 (rdes-eq cls: assms(1-3) False cong: UINF-cong)
qed

12.1 Choose

definition *choose-srd* :: (*'s, 't::trace, 'α*) *hrel-rsp* (*choose_R*) **where**
[upred-defs, rdes-def]: choose_R = R_s(true_r ⊢ false ∠ true_r)

lemma *preR-choose* [*rdes*]: *pre_R(choose_R) = true_r*
by (*rel-auto*)

lemma *periR-choose* [*rdes*]: *peri_R(choose_R) = false*
by (*rel-auto*)

lemma *postR-choose* [*rdes*]: *post_R(choose_R) = true_r*
by (*rel-auto*)

lemma *choose-srd-SRD* [*closure*]: *choose_R is SRD*
by (*simp add: choose-srd-def closure unrest*)

lemma *NSRD-choose-srd* [*closure*]: *choose_R is NSRD*
by (*rule NSRD-intro, simp-all add: closure unrest rdes*)

12.2 State Abstraction

definition *state-srea* ::
's itself ⇒ ('s, 't::trace, 'α, 'β) rel-rsp ⇒ (unit, 't, 'α, 'β) rel-rsp **where**
[upred-defs]: state-srea t P = ⟨∃ { \$st, \$st' } • P⟩_S

syntax
-state-srea :: type ⇒ logic ⇒ logic (state - - [0,200] 200)

translations
state 'a • P == CONST state-srea TYPE('a) P

lemma *R1-state-srea*: *R1(state 'a • P) = (state 'a • R1(P))*
by (*rel-auto*)

lemma *R2c-state-srea*: *R2c(state 'a • P) = (state 'a • R2c(P))*
by (*rel-auto*)

lemma *R3h-state-srea*: *R3h(state 'a • P) = (state 'a • R3h(P))*
by (*rel-auto*)

lemma *RD1-state-srea*: *RD1(state 'a • P) = (state 'a • RD1(P))*
by (*rel-auto*)

lemma *RD2-state-srea*: *RD2(state 'a • P) = (state 'a • RD2(P))*
by (*rel-auto*)

lemma *RD3-state-srea*: *RD3(state 'a • P) = (state 'a • RD3(P))*
by (*rel-auto, blast+*)

lemma *SRD-state-srea* [*closure*]: *P is SRD ⇒ state 'a • P is SRD*
by (*simp add: Healthy-def R1-state-srea R2c-state-srea R3h-state-srea RD1-state-srea RD2-state-srea RHS-def SRD-def*)

lemma *NSRD-state-srea* [*closure*]: *P is NSRD ⇒ state 'a • P is NSRD*
by (*metis Healthy-def NSRD-is-RD3 NSRD-is-SRD RD3-state-srea SRD-RD3-implies-NSRD SRD-state-srea*)

lemma *preR-state-srea* [rdes]: $pre_R(state\ 'a \cdot P) = \langle \forall\ \{\$st, \$st'\} \cdot pre_R(P) \rangle_S$
 by (*simp add: state-srea-def, rel-auto*)

lemma *periR-state-srea* [rdes]: $peri_R(state\ 'a \cdot P) = state\ 'a \cdot peri_R(P)$
 by (*rel-auto*)

lemma *postR-state-srea* [rdes]: $post_R(state\ 'a \cdot P) = state\ 'a \cdot post_R(P)$
 by (*rel-auto*)

12.3 While Loop

definition *WhileR* :: $'s\ upred \Rightarrow ('s, 't::size\ trace, 'a)\ hrel\ rsp \Rightarrow ('s, 't, 'a)\ hrel\ rsp$ (*while_R - do - od*)
 where

WhileR $b\ P = (\mu_R\ X \cdot (P\ ;;\ X) \triangleleft b \triangleright_R\ II_R)$

lemma *Sup-power-false*:

fixes $F :: 'a\ upred \Rightarrow 'a\ upred$

shows $(\bigcap i. (F\ \hat{\wedge}\ i)\ false) = (\bigcap i. (F\ \hat{\wedge}\ (i+1))\ false)$

proof –

have $(\bigcap i. (F\ \hat{\wedge}\ i)\ false) = (F\ \hat{\wedge}\ 0)\ false \sqcap (\bigcap i. (F\ \hat{\wedge}\ (i+1))\ false)$

by (*subst Sup-power-expand, simp*)

also have $\dots = (\bigcap i. (F\ \hat{\wedge}\ (i+1))\ false)$

by (*simp*)

finally show *?thesis* .

qed

theorem *WhileR-iter-expand*:

assumes P is NSRD P is Productive

shows $while_R\ b\ do\ P\ od = (\bigcap i. (P \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R\ II_R))\ (is\ ?lhs = ?rhs)$

proof –

have $1:Continuous\ (\lambda X. P\ ;;\ SRD\ X)$

using *SRD-Continuous*

by (*clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac $x=A$ in spec, simp*)

have $2: Continuous\ (\lambda X. P\ ;;\ SRD\ X \triangleleft b \triangleright_R\ II_R)$

by (*simp add: 1 closure assms*)

have $?lhs = (\mu_R\ X \cdot P\ ;;\ X \triangleleft b \triangleright_R\ II_R)$

by (*simp add: WhileR-def*)

also have $\dots = (\mu\ X \cdot P\ ;;\ SRD(X) \triangleleft b \triangleright_R\ II_R)$

by (*auto simp add: srd-mu-equiv closure assms*)

also have $\dots = (\nu\ X \cdot P\ ;;\ SRD(X) \triangleleft b \triangleright_R\ II_R)$

by (*auto simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure assms*)

also have $\dots = (\bigcap i. ((\lambda X. P\ ;;\ SRD\ X \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ i)\ false)$

by (*simp add: sup-continuous-lfp 2 sup-continuous-Continuous false-upred-def*)

also have $\dots = (\bigcap i. ((\lambda X. P\ ;;\ SRD\ X \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ (i+1))\ false)$

by (*simp add: Sup-power-false*)

also have $\dots = (\bigcap i. (P \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R\ II_R))$

proof (*rule SUP-cong, simp*)

fix i

show $((\lambda X. P\ ;;\ SRD\ X \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ (i+1))\ false = (P \triangleleft b \triangleright_R\ II_R) \hat{\wedge}\ i\ ;;\ (P\ ;;\ Miracle \triangleleft b \triangleright_R\ II_R)$

proof (*induct i*)

case 0

thm *if-eq-cancel*

then show *?case*

by (*simp, metis srdes-hcond-def srdes-theory-continuous.healthy-top*)

```

next
  case (Suc i)
  show ?case
  proof -
    have (( $\lambda X. P ;; SRD X \triangleleft b \triangleright_R II_R$ )  $\wedge$  ( $Suc i + 1$ )) false =
       $P ;; SRD ((\lambda X. P ;; SRD X \triangleleft b \triangleright_R II_R) \wedge (i + 1)) false \triangleleft b \triangleright_R II_R$ 
    by simp
    also have ... =  $P ;; SRD ((P \triangleleft b \triangleright_R II_R) \wedge i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$ 
    using Suc.hyps by auto
    also have ... =  $P ;; ((P \triangleleft b \triangleright_R II_R) \wedge i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$ 
    by (metis (no-types, lifting) Healthy-if NSRD-cond-srea NSRD-is-SRD NSRD-power-Suc
      NSRD-srd-skip SRD-cond-srea SRD-seqr-closure assms(1) power.power-eq-if seqr-left-unit srdes-theory-continuous.top-closed)
    also have ... =  $(P \triangleleft b \triangleright_R II_R) \wedge Suc i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
    proof (induct i)
      case 0
      then show ?case
      by (simp add: NSRD-is-SRD SRD-cond-srea SRD-left-unit SRD-seqr-closure SRD-srdes-skip
        assms(1) cond-L6 cond-st-distr srdes-theory-continuous.top-closed)
    next
      case (Suc i)
      have 1:  $II_R ;; ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge i) = ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge i)$ 
      by (simp add: NSRD-is-SRD RA1 SRD-cond-srea SRD-left-unit SRD-srdes-skip assms(1))
      then show ?case
      proof -
        have  $\bigwedge u. (u ;; (P \triangleleft b \triangleright_R II_R) \wedge Suc i) ;; (P ;; (Miracle \triangleleft b \triangleright_R (II_R))) \triangleleft b \triangleright_R (II_R) =$ 
           $((u \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge Suc i) ;; (P ;; (Miracle \triangleleft b \triangleright_R (II_R)))$ 
        by (metis (no-types) Suc.hyps 1 cond-L6 cond-st-distr power.power.power-Suc)
        then show ?thesis
        by (simp add: RA1 upred-semiring.power-Suc)
      qed
    qed
    qed
    finally show ?thesis .
  qed
  qed
  qed
  qed
  also have ... =  $(\bigcap i \cdot (P \triangleleft b \triangleright_R II_R) \wedge i) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: UINF-as-Sup-collect')
  finally show ?thesis .
qed

```

theorem *WhileR-star-expand:*

```

  assumes P is NSRD P is Productive
  shows  $while_R b \text{ do } P \text{ od} = (P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$  (is ?lhs = ?rhs)
proof -
  have ?lhs =  $(\bigcap i \cdot (P \triangleleft b \triangleright_R II_R) \wedge i) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: WhileR-iter-expand seq-UINF-distr' assms)
  also have ... =  $(P \triangleleft b \triangleright_R II_R)^* ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: ustar-def)
  also have ... =  $((P \triangleleft b \triangleright_R II_R)^* ;; II_R) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: seqr-assoc SRD-left-unit closure assms)
  also have ... =  $(P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: nsrd-thy.Star-def)
  finally show ?thesis .
qed

```

lemma *WhileR-NSRD-closed* [closure]:

assumes *P is NSRD P is Productive*

shows *while_R b do P od is NSRD*

by (*simp add: WhileR-star-expand assms closure*)

theorem *WhileR-iter-form-lemma*:

assumes *P is NSRD*

shows $(P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R) = ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R$

proof –

have $(P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R) = ([b]^\top_R ;; P \sqcap [\neg b]^\top_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$

by (*simp add: AssumeR-NSRD NSRD-right-unit NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$

by (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-denest assms(1)*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$

by (*metis (no-types, hide-lams) RD3-def RD3-idem Star-AssumeR nsrd-thy.Star-def*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$

by (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-invol assms(1)*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; ([b]^\top_R ;; P ;; Miracle \sqcap [\neg b]^\top_R)$

by (*simp add: AssumeR-NSRD NSRD-Miracle NSRD-right-unit NSRD-seqr-closure NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; Miracle \sqcap ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R$

by (*simp add: upred-semiring.distrib-left*)

also have $\dots = ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R$

proof –

have $([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R = (II_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P) ;; [\neg b]^\top_R$

by (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-unfoldr-eq assms(1)*)

also have $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R$

by (*metis (no-types, lifting) AssumeR-NSRD AssumeR-as-gcmd NSRD-srd-skip Star-AssumeR nsrd-thy.Star-slide gcmd-seq-distr skip-srea-self-unit urel-dioid.distrib-right'*)

also have $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; [b \vee \neg b]^\top_R ;; [\neg b]^\top_R$

by (*simp add: AssumeR-true NSRD-right-unit assms(1)*)

also have $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; [b]^\top_R ;; [\neg b]^\top_R$
 $\sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R ;; [\neg b]^\top_R$

by (*metis (no-types, hide-lams) AssumeR-choice upred-semiring.add-assoc upred-semiring.distrib-left upred-semiring.distrib-right*)

also have $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; ([b]^\top_R ;; [\neg b]^\top_R)$
 $\sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; ([\neg b]^\top_R ;; [\neg b]^\top_R)$

by (*simp add: RA1*)

also have $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; Miracle)$
 $\sqcap ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R$

by (*simp add: AssumeR-comp AssumeR-false*)

finally have $([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R \sqsubseteq ([b]^\top_R ;; P)^{*R} ;; [b]^\top_R ;; P ;; Miracle$

by (*simp add: semilattice-sup-class.le-supI1*)

thus *?thesis*

by (*simp add: semilattice-sup-class.le-iff-sup*)

qed

finally show *?thesis* .

qed

theorem *WhileR-iter-form*:

assumes *P is NSRD P is Productive*

shows *while_R b do P od = ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R*

by (*simp add: WhileR-iter-form-lemma WhileR-star-expand assms*)

theorem *WhileR-false:*

assumes *P is NSRD*

shows *while_R false do P od = II_R*

by (*simp add: WhileR-def rpred closure sdes-theory-continuous.LFP-const*)

theorem *WhileR-true:*

assumes *P is NSRD P is Productive*

shows *while_R true do P od = P^{*R} ;; Miracle*

by (*simp add: WhileR-iter-form AssumeR-true AssumeR-false SRD-left-unit assms closure*)

lemma *WhileR-insert-assume:*

assumes *P is NSRD P is Productive*

shows *while_R b do ([b][⊤]_R ;; P) od = while_R b do P od*

by (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure Productive-seq-2 RA1 WhileR-iter-form assms*)

theorem *WhileR-rdes-def [rdes-def]:*

assumes *P is RC Q is RR R is RR \$st' # Q R is R4*

shows *while_R b do R_s(P ⊢ Q ◊ R) od =*

*R_s ([b][⊤]_r ;; R)^{*r} wp_r ([b]_{S<} ⇒_r P) ⊢ ([b][⊤]_r ;; R)^{*r} ;; [b][⊤]_r ;; Q ◊ ([b][⊤]_r ;; R)^{*r} ;; [¬ b][⊤]_r)*

(is ?lhs = ?rhs)

proof –

have *?lhs = ([b][⊤]_R ;; R_s (P ⊢ Q ◊ R))^{*R} ;; [¬ b][⊤]_R*

by (*simp add: WhileR-iter-form Productive-rdes-RR-intro assms closure*)

also have *... = ?rhs*

by (*simp add: rdes-def assms closure unrest rpred wp del: rea-star-wp*)

finally show *?thesis .*

qed

12.4 Iteration Construction

definition *IterateR*

:: 'a set ⇒ ('a ⇒ 's upred) ⇒ ('a ⇒ ('s, 't::size-trace, 'α) hrel-rsp) ⇒ ('s, 't, 'α) hrel-rsp

where *IterateR A g P = while_R (⋁ i∈A • g(i)) do (if_R i∈A • g(i) → P(i) fi) od*

syntax

-iter-srd :: ptnr ⇒ logic ⇒ logic ⇒ logic ⇒ logic (do_R -∈- • - → - fi)

translations

-iter-srd x A g P => CONST IterateR A (λ x. g) (λ x. P)

-iter-srd x A g P <= CONST IterateR A (λ x. g) (λ x'. P)

lemma *IterateR-NSRD-closed [closure]:*

assumes

⋀ i. i ∈ I ⇒ P(i) is NSRD

⋀ i. i ∈ I ⇒ P(i) is Productive

shows *do_R i∈I • g(i) → P(i) fi is NSRD*

by (*simp add: IterateR-def closure assms*)

lemma *IterateR-empty:*

do_R i∈{} • g(i) → P(i) fi = II_R

by (*simp add: IterateR-def srd-mu-equiv closure rpred gfp-const WhileR-false*)

lemma *IterateR-singleton:*

assumes *P k is NSRD P k is Productive*

shows *do_R i∈{k} • g(i) → P(i) fi = while_R g(k) do P(k) od (is ?lhs = ?rhs)*

proof –

have $?lhs = \text{while}_R g k \text{ do } P k \triangleleft g k \triangleright_R \text{Chaos} \text{ od}$
 by (*simp add: IterateR-def AlternateR-singleton assms closure*)
 also have $\dots = \text{while}_R g k \text{ do } [g k]^\top_R ;; (P k \triangleleft g k \triangleright_R \text{Chaos}) \text{ od}$
 by (*simp add: WhileR-insert-assume closure assms*)
 also have $\dots = \text{while}_R g k \text{ do } P k \text{ od}$
 by (*simp add: AssumeR-cond-left NSRD-Chaos WhileR-insert-assume assms*)
 finally show $?thesis$.

qed

12.5 Substitution Laws

lemma *srd-subst-Chaos* [*usubst*]:

$\sigma \dagger_S \text{Chaos} = \text{Chaos}$

by (*rdes-simp*)

lemma *srd-subst-Miracle* [*usubst*]:

$\sigma \dagger_S \text{Miracle} = \text{Miracle}$

by (*rdes-simp*)

lemma *srd-subst-skip* [*usubst*]:

$\sigma \dagger_S II_R = \langle \sigma \rangle_R$

by (*rdes-eq*)

lemma *srd-subst-assigns* [*usubst*]:

$\sigma \dagger_S \langle \varrho \rangle_R = \langle \varrho \circ \sigma \rangle_R$

by (*rdes-eq*)

12.6 Algebraic Laws

theorem *assigns-srd-id*: $\langle id \rangle_R = II_R$

by (*rdes-eq*)

theorem *assigns-srd-comp*: $\langle \sigma \rangle_R ;; \langle \varrho \rangle_R = \langle \varrho \circ \sigma \rangle_R$

by (*rdes-eq*)

theorem *assigns-srd-Miracle*: $\langle \sigma \rangle_R ;; \text{Miracle} = \text{Miracle}$

by (*rdes-eq*)

theorem *assigns-srd-Chaos*: $\langle \sigma \rangle_R ;; \text{Chaos} = \text{Chaos}$

by (*rdes-eq*)

theorem *assigns-srd-cond* : $\langle \sigma \rangle_R \triangleleft b \triangleright_R \langle \varrho \rangle_R = \langle \sigma \triangleleft b \triangleright_s \varrho \rangle_R$

by (*rdes-eq*)

theorem *assigns-srd-left-seq*:

assumes P is NSRD

shows $\langle \sigma \rangle_R ;; P = \sigma \dagger_S P$

by (*rdes-simp cls: assms*)

lemma *AlternateR-seq-distr*:

assumes $\bigwedge i. A i$ is NSRD B is NSRD C is NSRD

shows $(\text{if}_R i \in I \cdot g i \rightarrow A i \text{ else } B \text{ fi}) ;; C = (\text{if}_R i \in I \cdot g i \rightarrow A i ;; C \text{ else } B ;; C \text{ fi})$

proof (*cases* $I = \{\}$)

case *True*

then show $?thesis$ by (*simp*)

next
 case *False*
 then show *?thesis*
 by (*simp add: AlternateR-def upred-semiring.distrib-right seq-UINF-distr gcmd-seq-distr assms(3)*)
 qed

lemma *AlternateR-is-cond-srea*:
 assumes *A is NSRD B is NSRD*
 shows $(if_R i \in \{a\} \cdot g \rightarrow A \text{ else } B \text{ fi}) = (A \triangleleft g \triangleright_R B)$
 by (*rdes-eq cls: assms*)

lemma *AlternateR-Chaos*:
 $if_R i \in A \cdot g(i) \rightarrow Chaos \text{ fi} = Chaos$
 by (*cases A = \{\}, simp, rdes-eq*)

lemma *choose-srd-par*:
 $choose_R \parallel_R choose_R = choose_R$
 by (*rdes-eq*)

12.7 Lifting designs to reactive designs

definition *des-rea-lift* :: $'s \text{ hrel-des} \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp } (\mathbf{R}_D) \text{ where}$
 $[upred-defs]: \mathbf{R}_D(P) = \mathbf{R}_s(\lceil pre_D(P) \rceil_S \vdash (false \diamond (\$tr' =_u \$tr \wedge \lceil post_D(P) \rceil_S)))$

definition *des-rea-drop* :: $('s, 't::\text{trace}, 'a) \text{ hrel-rsp} \Rightarrow 's \text{ hrel-des } (\mathbf{D}_R) \text{ where}$
 $[upred-defs]: \mathbf{D}_R(P) = \lfloor (pre_R(P)) \llbracket \$tr/\$tr' \rrbracket \vdash_v \$st \rfloor_{S<} \vdash_n \lfloor (post_R(P)) \llbracket \$tr/\$tr' \rrbracket \vdash_v \{\$st, \$st'\} \rfloor_S$

lemma *ndesign-rea-lift-inverse*: $\mathbf{D}_R(\mathbf{R}_D(p \vdash_n Q)) = p \vdash_n Q$
 apply (*simp add: des-rea-lift-def des-rea-drop-def rea-pre-RHS-design rea-post-RHS-design*)
 apply (*simp add: R1-def R2c-def R2s-def usubst unrest*)
 apply (*rel-auto*)
 done

lemma *ndesign-rea-lift-injective*:
 assumes *P is N Q is N R_D P = R_D Q (is ?RP(P) = ?RQ(Q))*
 shows *P = Q*

proof –
 have $?RP(\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P)) = ?RQ(\lfloor pre_D(Q) \rfloor_{<} \vdash_n post_D(Q))$
 by (*simp add: ndesign-form assms*)
 hence $\lfloor pre_D(P) \rfloor_{<} \vdash_n post_D(P) = \lfloor pre_D(Q) \rfloor_{<} \vdash_n post_D(Q)$
 by (*metis ndesign-rea-lift-inverse*)
 thus *?thesis*
 by (*simp add: ndesign-form assms*)
 qed

lemma *des-rea-lift-closure [closure]*: $\mathbf{R}_D(P) \text{ is SRD}$
 by (*simp add: des-rea-lift-def RHS-design-is-SRD unrest*)

lemma *preR-des-rea-lift [rdes]*:
 $pre_R(\mathbf{R}_D(P)) = R1(\lceil pre_D(P) \rceil_S)$
 by (*rel-auto*)

lemma *periR-des-rea-lift [rdes]*:
 $peri_R(\mathbf{R}_D(P)) = (false \triangleleft \lceil pre_D(P) \rceil_S \triangleright (\$tr \leq_u \$tr'))$
 by (*rel-auto*)

lemma *postR-des-rea-lift* [rdes]:
 $post_R(\mathbf{R}_D(P)) = ((true \triangleleft [pre_D(P)]_S \triangleright (\neg \$tr \leq_u \$tr')) \Rightarrow (\$tr' =_u \$tr \wedge [post_D(P)]_S))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *ndes-rea-lift-closure* [closure]:

assumes *P is N*

shows $\mathbf{R}_D(P)$ *is NSRD*

proof –

obtain *p Q* **where** *P: P = (p ⊢_n Q)*

by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def assms*)

show *?thesis*

apply (*rule NSRD-intro*)

apply (*simp-all add: closure rdes unrest P*)

apply (*rel-auto*)

done

qed

lemma *R-D-mono*:

assumes *P is H Q is H P ⊆ Q*

shows $\mathbf{R}_D(P) \subseteq \mathbf{R}_D(Q)$

apply (*simp add: des-rea-lift-def*)

apply (*rule srdes-tri-refine-intro'*)

apply (*auto intro: H1-H2-refines assms aext-mono*)

apply (*rel-auto*)

apply (*metis (no-types, hide-lams) aext-mono assms(3) design-post-choice*

semilattice-sup-class.sup.orderE utp-pred-laws.inf.coboundedI1 utp-pred-laws.inf commute utp-pred-laws.sup.order-iff)

done

Homomorphism laws

lemma *R-D-Miracle*:

$\mathbf{R}_D(\top_D) = \text{Miracle}$

by (*simp add: Miracle-def, rel-auto*)

lemma *R-D-Chaos*:

$\mathbf{R}_D(\perp_D) = \text{Chaos}$

proof –

have $\mathbf{R}_D(\perp_D) = \mathbf{R}_D(false \vdash_r true)$

by (*rel-auto*)

also have $\dots = \mathbf{R}_s(false \vdash false \diamond (\$tr' =_u \$tr))$

by (*simp add: Chaos-def des-rea-lift-def alpha*)

also have $\dots = \mathbf{R}_s(true)$

by (*rel-auto*)

also have $\dots = \text{Chaos}$

by (*simp add: Chaos-def design-false-pre*)

finally show *?thesis* .

qed

lemma *R-D-inf*:

$\mathbf{R}_D(P \sqcap Q) = \mathbf{R}_D(P) \sqcap \mathbf{R}_D(Q)$

by (*rule antisym, rel-auto+*)

lemma *R-D-cond*:

$\mathbf{R}_D(P \triangleleft [b]_{D<} \triangleright Q) = \mathbf{R}_D(P) \triangleleft b \triangleright_R \mathbf{R}_D(Q)$

by (*rule antisym, rel-auto+*)

```

lemma R-D-seq-ndesign:
   $\mathbf{R}_D(p_1 \vdash_n Q_1) ;; \mathbf{R}_D(p_2 \vdash_n Q_2) = \mathbf{R}_D((p_1 \vdash_n Q_1) ;; (p_2 \vdash_n Q_2))$ 
  apply (rule antisym)
  apply (rule SRD-refine-intro)
    apply (simp-all add: closure rdes ndesign-composition-wp)
  using dual-order.trans apply (rel-blast)
  using dual-order.trans apply (rel-blast)
  apply (rel-auto)
  apply (rule SRD-refine-intro)
    apply (simp-all add: closure rdes ndesign-composition-wp)
    apply (rel-auto)
    apply (rel-auto)
  apply (rel-auto)
done

```

```

lemma R-D-seq:
  assumes P is N Q is N
  shows  $\mathbf{R}_D(P) ;; \mathbf{R}_D(Q) = \mathbf{R}_D(P ;; Q)$ 
  by (metis R-D-seq-ndesign assms ndesign-form)

```

These laws are applicable only when there is no further alphabet extension

```

lemma R-D-skip:
   $\mathbf{R}_D(\Pi_D) = (\Pi_R :: ('s, 't :: \text{trace}, \text{unit}) \text{ hrel-rsp})$ 
  apply (rel-auto) using minus-zero-eq by blast+

```

```

lemma R-D-assigns:
   $\mathbf{R}_D(\langle \sigma \rangle_D) = (\langle \sigma \rangle_R :: ('s, 't :: \text{trace}, \text{unit}) \text{ hrel-rsp})$ 
  by (simp add: assigns-d-def des-rea-lift-def alpha assigns-srd-RHS-tri-des, rel-auto)

```

end

13 Instantaneous Reactive Designs

```

theory utp-rdes-instant
  imports utp-rdes-prog
begin

```

```

definition ISR D1 ::  $('s, 't :: \text{trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  where
  [upred-defs]:  $\text{ISR D1}(P) = P \parallel_R \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond (\$tr' =_u \$tr))$ 

```

```

definition ISR D ::  $('s, 't :: \text{trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  where
  [upred-defs]:  $\text{ISR D} = \text{ISR D1} \circ \text{NSRD}$ 

```

```

lemma ISR D1-idem:  $\text{ISR D1}(\text{ISR D1}(P)) = \text{ISR D1}(P)$ 
  by (rel-auto)

```

```

lemma ISR D1-monotonic:  $P \sqsubseteq Q \Longrightarrow \text{ISR D1}(P) \sqsubseteq \text{ISR D1}(Q)$ 
  by (rel-auto)

```

```

lemma ISR D1-RHS-design-form:
  assumes  $\$ok' \# P \$ok' \# Q \$ok' \# R$ 
  shows  $\text{ISR D1}(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \text{false} \diamond (R \wedge \$tr' =_u \$tr))$ 
  using assms by (simp add: ISR D1-def choose-srd-def RHS-tri-design-par unrest, rel-auto)

```


lemma *ISRD1-form*:

$ISRD1(SRD(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr))$
 by (*simp add: ISRD1-RHS-design-form SRD-as-reactive-tri-design unrest*)

lemma *ISRD1-rdes-def* [*rdes-def*]:

$\llbracket P \text{ is } RR; R \text{ is } RR \rrbracket \implies ISRD1(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \text{false} \diamond (R \wedge \$tr' =_u \$tr))$
 by (*simp add: ISRD1-def rdes-def closure rpred*)

lemma *ISRD-intro*:

assumes $P \text{ is } NSRD \text{ } \text{peri}_R(P) = (\neg_r \text{pre}_R(P)) (\$tr' =_u \$tr) \sqsubseteq \text{post}_R(P)$
shows $P \text{ is } ISRD$

proof –

have $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \text{ is } ISRD1$
apply (*simp add: Healthy-def rdes-def closure assms(1-2)*)
using *assms(3) least-zero* **apply** (*rel-blast*)
done
hence $P \text{ is } ISRD1$
by (*simp add: SRD-reactive-tri-design closure assms(1)*)
thus *?thesis*
by (*simp add: ISRD-def Healthy-comp assms(1)*)

qed

lemma *ISRD1-rdes-intro*:

assumes $P \text{ is } RR \ Q \text{ is } RR \ (\$tr' =_u \$tr) \sqsubseteq Q$
shows $\mathbf{R}_s(P \vdash \text{false} \diamond Q) \text{ is } ISRD1$
unfolding *Healthy-def*
by (*simp add: ISRD1-rdes-def assms closure unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-rdes-intro* [*closure*]:

assumes $P \text{ is } RC \ Q \text{ is } RR \ (\$tr' =_u \$tr) \sqsubseteq Q$
shows $\mathbf{R}_s(P \vdash \text{false} \diamond Q) \text{ is } ISRD$
unfolding *Healthy-def*
by (*simp add: ISRD-def closure Healthy-if ISRD1-rdes-def assms unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-implies-ISRD1*:

assumes $P \text{ is } ISRD$
shows $P \text{ is } ISRD1$

proof –

have $ISRD(P) \text{ is } ISRD1$
by (*simp add: ISRD-def Healthy-def ISRD1-idem*)
thus *?thesis*
by (*simp add: assms Healthy-if*)

qed

lemma *ISRD-implies-SRD*:

assumes $P \text{ is } ISRD$
shows $P \text{ is } SRD$

proof –

have $1: ISRD(P) = \mathbf{R}_s((\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true} \wedge R1 \text{ true}) \vdash \text{false} \diamond (\text{post}_R P \wedge \$tr' =_u \$tr))$
by (*simp add: NSRD-form ISRD1-def ISRD-def RHS-tri-design-par rdes-def unrest closure*)
moreover have $\dots \text{ is } SRD$
by (*simp add: closure unrest*)
ultimately have $ISRD(P) \text{ is } SRD$
by (*simp*)
with *assms* **show** *?thesis*

by (simp add: Healthy-def)
qed

lemma *ISRD-implies-NSRD* [closure]:

assumes P is ISRD
shows P is NSRD

proof –

have $1: \text{ISRD}(P) = \text{ISRD1}(\text{RD3}(\text{SRD}(P)))$

by (simp add: ISRD-def NSRD-def SRD-def, metis RD1-RD3-commute RD3-left-subsumes-RD2)

also have $\dots = \text{ISRD1}(\text{RD3}(P))$

by (simp add: asms ISRD-implies-SRD Healthy-if)

also have $\dots = \text{ISRD1}(\mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false}_h \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P))$

by (simp add: RD3-def, subst SRD-right-unit-tri-lemma, simp-all add: asms ISRD-implies-SRD)

also have $\dots = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false}_h \vdash \text{false} \diamond (\text{post}_R P \wedge \$tr' =_u \$tr))$

by (simp add: RHS-tri-design-par ISRD1-def unrest choose-srd-def rpred closure ISRD-implies-SRD asms)

also have $\dots = (\dots ;; II_R)$

by (rdes-simp, simp add: RHS-tri-normal-design-composition' closure asms unrest ISRD-implies-SRD wp rpred wp-rea-false-RC)

also have \dots is RD3

by (simp add: Healthy-def RD3-def segr-assoc)

finally show ?thesis

by (simp add: SRD-RD3-implies-NSRD Healthy-if asms ISRD-implies-SRD)

qed

lemma *ISRD-form*:

assumes P is ISRD

shows $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr)) = P$

proof –

have $P = \text{ISRD1}(P)$

by (simp add: ISRD-implies-ISRD1 asms Healthy-if)

also have $\dots = \text{ISRD1}(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)))$

by (simp add: SRD-reactive-tri-design ISRD-implies-SRD asms)

also have $\dots = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr))$

by (simp add: ISRD1-rdes-def closure asms)

finally show ?thesis ..

qed

lemma *ISRD-elim* [RD-elim]:

$\llbracket P \text{ is ISRD}; Q(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr))) \rrbracket \implies Q(P)$

by (simp add: ISRD-form)

lemma *skip-srd-ISRD* [closure]: II_R is ISRD

by (rule ISRD-intro, simp-all add: rdes closure)

lemma *assigns-srd-ISRD* [closure]: $\langle \sigma \rangle_R$ is ISRD

by (rule ISRD-intro, simp-all add: rdes closure, rel-auto)

lemma *seq-ISRD-closed*:

assumes P is ISRD Q is ISRD

shows $P ;; Q$ is ISRD

apply (insert asms)

apply (erule ISRD-elim)+

apply (simp add: rdes-def closure asms unrest)

apply (rule ISRD-rdes-intro)

```

  apply (simp-all add: rdes-def closure assms unrest)
apply (rel-auto)
done

```

```

lemma ISRD-Miracle-right-zero:
  assumes  $P$  is ISRD  $\text{pre}_R(P) = \text{true}_r$ 
  shows  $P \;; \text{Miracle} = \text{Miracle}$ 
  by (rdes-simp cls: assms)

```

A recursion whose body does not extend the trace results in divergence

```

lemma ISRD-recurse-Chaos:
  assumes  $P$  is ISRD  $\text{post}_R P \;; \text{true}_r = \text{true}_r$ 
  shows  $(\mu_R X \cdot P \;; X) = \text{Chaos}$ 
proof -
  have 1:  $(\mu_R X \cdot P \;; X) = (\mu X \cdot P \;; \text{SRD}(X))$ 
  by (auto simp add: srdes-theory-continuous.utp-lfp-def closure assms)
  have  $(\mu X \cdot P \;; \text{SRD}(X)) \sqsubseteq \text{Chaos}$ 
proof (rule gfp-upperbound)
  have  $P \;; \text{Chaos} \sqsubseteq \text{Chaos}$ 
  apply (rdes-refine-split cls: assms)
  using assms(2) apply (rel-auto, metis (no-types, lifting) dual-order.antisym order-refl)
  apply (rel-auto)+
  done
  thus  $P \;; \text{SRD } \text{Chaos} \sqsubseteq \text{Chaos}$ 
  by (simp add: Healthy-if srdes-theory-continuous.bottom-closed)
qed
thus ?thesis
  by (metis 1 dual-order.antisym srdes-theory-continuous.LFP-closed srdes-theory-continuous.bottom-lower)
qed

```

```

lemma recursive-assign-Chaos:
   $(\mu_R X \cdot \langle \sigma \rangle_R \;; X) = \text{Chaos}$ 
  by (rule ISRD-recurse-Chaos, simp-all add: closure rdes, rel-auto)

```

end

14 Meta-theory for Reactive Designs

```

theory utp-rea-designs
  imports
    utp-rdes-healths
    utp-rdes-designs
    utp-rdes-triples
    utp-rdes-normal
    utp-rdes-contracts
    utp-rdes-tactics
    utp-rdes-parallel
    utp-rdes-prog
    utp-rdes-instant
    utp-rdes-guarded
begin end

```

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