

# Stateful-Failure Reactive Designs in Isabelle/UTP

Simon Foster      James Baxter      Ana Cavalcanti      Jim Woodcock

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## Abstract

Stateful-Failure Reactive Designs specialise reactive design contracts with failures traces, as present in languages like CSP and Circus. A failure trace consists of a sequence of events and a refusal set. It intuitively represents a quiescent observation, where certain events have previously occurred, and others are currently being accepted. Following the UTP book, we add an observational variable to represent refusal sets, and healthiness conditions that ensure their well-formedness. Using these, we also specialise our theory of reactive relations with operators to characterise both completed and quiescent interactions, and an accompanying equational theory. We use these to define the core operators — including assignment, event occurrence, and external choice — and specialise our proof strategy to support these. We also demonstrate a link with the CSP failures-divergences semantic model.

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## 1 Introduction

This document contains a mechanisation in Isabelle/UTP [1] of an specialisation of stateful reactive designs with refusal information, as present in languages like Circus [2].

## 2 Stateful-Failure Core Types

```
theory utp-sfrd-core
  imports UTP-Reactive-Designs.utp-rea-designs
begin
```

## 2.1 SFRD Alphabet

**alphabet**  $\text{'}\varphi \text{ csp-vars} = \text{'}\sigma \text{ rsp-vars} +$   
 $\text{ref} :: \text{'}\varphi \text{ set}$

**declare**  $\text{csp-vars.defs} [\text{lens-defs}]$   
**declare**  $\text{csp-vars.splits} [\text{alpha-splits}]$

The following two locale interpretations are a technicality to improve the behaviour of the automatic tactics. They enable (re)interpretation of state spaces in order to remove any occurrences of lens types, replacing them by tuple types after the tactics *pred-simp* and *rel-simp* are applied. Eventually, it would be desirable to automate preform these interpretations automatically as part of the **alphabet** command.

**interpretation** *alphabet-csp-prd*:  
 $\text{lens-interp } \lambda(\text{ok}, \text{wait}, \text{tr}, \text{m}). (\text{ok}, \text{wait}, \text{tr}, \text{ref}_v \text{ m}, \text{more m})$   
**apply** (*unfold-locales*)  
**apply** (*rule injI*)  
**apply** (*clarsimp*)  
**done**

**interpretation** *alphabet-csp-rel*:  
 $\text{lens-interp } \lambda(\text{ok}, \text{ok}', \text{wait}, \text{wait}', \text{tr}, \text{tr}', \text{m}, \text{m}').$   
 $(\text{ok}, \text{ok}', \text{wait}, \text{wait}', \text{tr}, \text{tr}', \text{ref}_v \text{ m}, \text{ref}_v \text{ m}', \text{more m}, \text{more m}')$   
**apply** (*unfold-locales*)  
**apply** (*rule injI*)  
**apply** (*clarsimp*)  
**done**

**lemma** *circus-var-ords* [*usubst*]:  
 $\text{\$ref} \prec_v \text{\$ref'}$   
 $\text{\$ok} \prec_v \text{\$ref} \text{\$ok'} \prec_v \text{\$ref'} \text{\$ok} \prec_v \text{\$ref'} \text{\$ok'} \prec_v \text{\$ref}$   
 $\text{\$ref} \prec_v \text{\$wait} \text{\$ref'} \prec_v \text{\$wait'} \text{\$ref} \prec_v \text{\$wait'} \text{\$ref'} \prec_v \text{\$wait}$   
 $\text{\$ref} \prec_v \text{\$st} \text{\$ref'} \prec_v \text{\$st'} \text{\$ref} \prec_v \text{\$st'} \text{\$ref'} \prec_v \text{\$st}$   
 $\text{\$ref} \prec_v \text{\$tr} \text{\$ref'} \prec_v \text{\$tr'} \text{\$ref} \prec_v \text{\$tr'} \text{\$ref'} \prec_v \text{\$tr}$   
**by** (*simp-all add: var-name-ord-def*)

**type-synonym**  $(\text{'}\sigma, \text{'}\varphi) \text{ st-csp} = (\text{'}\sigma, \text{'}\varphi \text{ list}, (\text{'}\varphi, \text{unit}) \text{ csp-vars-scheme}) \text{ rsp}$   
**type-synonym**  $(\text{'}\sigma, \text{'}\varphi) \text{ action} = (\text{'}\sigma, \text{'}\varphi) \text{ st-csp hrel}$   
**type-synonym**  $\text{'}\varphi \text{ csp} = (\text{unit}, \text{'}\varphi) \text{ st-csp}$   
**type-synonym**  $\text{'}\varphi \text{ rel-csp} = \text{'}\varphi \text{ csp hrel}$

There is some slight imprecision with the translations, in that we don't bother to check if the trace event type and refusal set event types are the same. Essentially this is because its very difficult to construct processes where this would be the case. However, it may be better to add a proper ML print translation in the future.

**translations**  
 $(\text{type}) (\text{'}\sigma, \text{'}\varphi) \text{ st-csp} \leq (\text{type}) (\text{'}\sigma, \text{'}\varphi \text{ list}, \text{'}\varphi 1 \text{ csp-vars}) \text{ rsp}$   
 $(\text{type}) (\text{'}\sigma, \text{'}\varphi) \text{ action} \leq (\text{type}) (\text{'}\sigma, \text{'}\varphi) \text{ st-csp hrel}$

**notation**  $\text{csp-vars-child-lens}_a (\Sigma_c)$   
**notation**  $\text{csp-vars-child-lens} (\Sigma_C)$

## 2.2 Basic laws

**lemma** *R2c-tr-ext*:  $R2c (\text{\$tr'} =_u \text{\$tr} \hat{^}_u \langle \lceil a \rceil_{S<} \rangle) = (\text{\$tr'} =_u \text{\$tr} \hat{^}_u \langle \lceil a \rceil_{S<} \rangle)$

by (rel-auto)

**lemma** *circus-alpha-bij-lens*:

*bij-lens* ( $\{\$ok, \$ok', \$wait, \$wait', \$tr, \$tr', \$st, \$st', \$ref, \$ref'\}_\alpha :: - \implies ('s, 'e) \text{ st-csp} \times ('s, 'e) \text{ st-csp}$ )  
by (unfold-locales, lens-simp+)

## 2.3 Unrestriction laws

**lemma** *pre-unrest-ref* [unrest]:  $\$ref \# P \implies \$ref \# pre_R(P)$   
by (simp add: pre\_R-def unrest)

**lemma** *peri-unrest-ref* [unrest]:  $\$ref \# P \implies \$ref \# peri_R(P)$   
by (simp add: peri\_R-def unrest)

**lemma** *post-unrest-ref* [unrest]:  $\$ref \# P \implies \$ref \# post_R(P)$   
by (simp add: post\_R-def unrest)

**lemma** *cmt-unrest-ref* [unrest]:  $\$ref \# P \implies \$ref \# cmt_R(P)$   
by (simp add: cmt\_R-def unrest)

**lemma** *st-lift-unrest-ref'* [unrest]:  $\$ref' \# \lceil b \rceil_{S<} \implies$   
by (rel-auto)

**lemma** *RHS-design-ref-unrest* [unrest]:

$\llbracket \$ref \# P; \$ref \# Q \rrbracket \implies \$ref \# (\mathbf{R}_s(P \vdash Q)) \llbracket false / \$wait \rrbracket$   
by (simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)

**lemma** *R1-ref-unrest* [unrest]:  $\$ref \# P \implies \$ref \# R1(P)$   
by (simp add: R1-def unrest)

**lemma** *R2c-ref-unrest* [unrest]:  $\$ref \# P \implies \$ref \# R2c(P)$   
by (simp add: R2c-def unrest)

**lemma** *R1-ref'-unrest* [unrest]:  $\$ref' \# P \implies \$ref' \# R1(P)$   
by (simp add: R1-def unrest)

**lemma** *R2c-ref'-unrest* [unrest]:  $\$ref' \# P \implies \$ref' \# R2c(P)$   
by (simp add: R2c-def unrest)

**lemma** *R2s-notin-ref'*:  $R2s(\lceil \ll x \gg \rceil_{S<} \notin_u \$ref') = (\lceil \ll x \gg \rceil_{S<} \notin_u \$ref')$   
by (pred-auto)

**lemma** *unrest-circus-alpha*:

fixes  $P :: ('e, 't) \text{ action}$

assumes

$\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$tr \# P$   
 $\$tr' \# P \ \$st \# P \ \$st' \# P \ \$ref \# P \ \$ref' \# P$

shows  $\Sigma \# P$

by (rule bij-lens-unrest-all[OF circus-alpha-bij-lens], simp add: unrest assms)

**lemma** *unrest-all-circus-vars*:

fixes  $P :: ('s, 'e) \text{ action}$

assumes  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \Sigma \# r' \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$   
shows  $\Sigma \# [\$ref' \mapsto_s r', \$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \vdash P$

using assms

by (simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens)

(simp add: unrest usubst closure)

**lemma** *unrest-all-circus-vars-st-st'*:

**fixes**  $P :: ('s, 'e) \text{ action}$

**assumes**  $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \Sigma \# s \Sigma \# s' \Sigma \# t \Sigma \# t'$

**shows**  $\Sigma \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

**using** *assms*

**by** (simp add: bij-lens-unrest-all-eq[*OF circus-alpha-bij-lens*] *unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

**lemma** *unrest-all-circus-vars-st*:

**fixes**  $P :: ('s, 'e) \text{ action}$

**assumes**  $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$

**shows**  $\Sigma \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

**using** *assms*

**by** (simp add: bij-lens-unrest-all-eq[*OF circus-alpha-bij-lens*] *unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

**lemma** *unrest-any-circus-var*:

**fixes**  $P :: ('s, 'e) \text{ action}$

**assumes**  $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \Sigma \# s \Sigma \# s' \Sigma \# t \Sigma \# t'$

**shows**  $x \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

**by** (simp add: unrest-all-var unrest-all-circus-vars-st-st' *assms*)

**lemma** *unrest-any-circus-var-st*:

**fixes**  $P :: ('s, 'e) \text{ action}$

**assumes**  $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$

**shows**  $x \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \uparrow P$

**by** (simp add: unrest-all-var unrest-all-circus-vars-st *assms*)

**end**

### 3 Stateful-Failure Reactive Relations

**theory** *utp-sfrd-rel*

**imports** *utp-sfrd-core*

**begin**

#### 3.1 Healthiness Conditions

CSP Reactive Relations

**definition**  $CRR :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$  **where**

[*upred-defs*]:  $CRR(P) = (\exists \$ref \cdot RR(P))$

**lemma** *CRR-idem*:  $CRR(CRR(P)) = CRR(P)$

**by** (*rel-auto*)

**lemma** *Idempotent-CRR* [*closure*]: *Idempotent CRR*

**by** (simp add: *CRR-idem Idempotent-def*)

**lemma** *CRR-intro*:

**assumes**  $\$ref \# P$  *P is RR*

**shows** *P is CRR*

**by** (simp add: *CRR-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

## CSP Reactive Conditions

**definition**  $CRC :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$  **where**  
 $[upred-defs]: CRC(P) = (\exists \$ref \cdot RC(P))$

**lemma**  $CRC\text{-}intro$ :  
**assumes**  $\$ref \# P$   $P$  *is*  $RC$   
**shows**  $P$  *is*  $CRC$   
**by** (*simp add: CRC-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

**lemma**  $ref\text{-}unrest\text{-}RR$  [*unrest*]:  $\$ref \# P \Longrightarrow \$ref \# RR\ P$   
**by** (*rel-auto, blast+*)

**lemma**  $ref\text{-}unrest\text{-}RC1$  [*unrest*]:  $\$ref \# P \Longrightarrow \$ref \# RC1\ P$   
**by** (*rel-auto, blast+*)

**lemma**  $ref\text{-}unrest\text{-}RC$  [*unrest*]:  $\$ref \# P \Longrightarrow \$ref \# RC\ P$   
**by** (*simp add: RC-R2-def ref-unrest-RC1 ref-unrest-RR*)

**lemma**  $RR\text{-}ex\text{-}ref$ :  $RR (\exists \$ref \cdot RR\ P) = (\exists \$ref \cdot RR\ P)$   
**by** (*rel-auto*)

**lemma**  $RC1\text{-}ex\text{-}ref$ :  $RC1 (\exists \$ref \cdot RC1\ P) = (\exists \$ref \cdot RC1\ P)$   
**by** (*rel-auto, meson dual-order.trans*)

**lemma**  $ex\text{-}ref'\text{-}RR\text{-}closed$  [*closure*]:  
**assumes**  $P$  *is*  $RR$   
**shows**  $(\exists \$ref' \cdot P)$  *is*  $RR$   
**proof** –  
**have**  $RR (\exists \$ref' \cdot RR(P)) = (\exists \$ref' \cdot RR(P))$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*metis Healthy-def assms*)  
**qed**

**lemma**  $CRC\text{-}idem$ :  $CRC(CRC(P)) = CRC(P)$   
**apply** (*simp add: CRC-def ex-unrest unrest*)  
**apply** (*simp add: RC-def RR-ex-ref*)  
**apply** (*metis (no-types, hide-lams) Healthy-def RC1-RR-closed RC1-ex-ref RR-ex-ref RR-idem*)  
**done**

**lemma**  $Idempotent\text{-}CRC$  [*closure*]: *Idempotent*  $CRC$   
**by** (*simp add: CRC-idem Idempotent-def*)

## 3.2 Closure Properties

**lemma**  $CRR\text{-}implies\text{-}RR$  [*closure*]:  
**assumes**  $P$  *is*  $CRR$   
**shows**  $P$  *is*  $RR$   
**proof** –  
**have**  $RR(CRR(P)) = CRR(P)$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*metis Healthy-def' assms*)  
**qed**

**lemma** *CRC-implies-RR* [closure]:  
 assumes *P* is CRC  
 shows *P* is RR  
**proof** –  
 have  $RR(CRC(P)) = CRC(P)$   
 by (*rel-auto*)  
 (*metis* (*no-types*, *lifting*) *Prefix-Order.prefixE Prefix-Order.prefixI append.assoc append-minus*) +  
 thus ?thesis  
 by (*metis Healthy-def assms*)  
**qed**

**lemma** *CRC-implies-RC* [closure]:  
 assumes *P* is CRC  
 shows *P* is RC  
**proof** –  
 have  $RC1(CRC(P)) = CRC(P)$   
 by (*rel-auto*, *meson dual-order.trans*)  
 thus ?thesis  
 by (*simp add: CRC-implies-RR Healthy-if RC1-def RC-intro assms*)  
**qed**

**lemma** *CRR-unrest-ref* [*unrest*]:  $P$  is CRR  $\implies \$ref \# P$   
 by (*metis CRR-def CRR-implies-RR Healthy-def in-var-uvar ref-vwb-lens unrest-as-exists*)

**lemma** *CRC-implies-CRR* [closure]:  
 assumes *P* is CRC  
 shows *P* is CRR  
 apply (*rule CRR-intro*)  
 apply (*simp-all add: unrest assms closure*)  
 apply (*metis CRC-def CRC-implies-RC Healthy-def assms in-var-uvar ref-vwb-lens unrest-as-exists*)  
 done

**lemma** *unrest-ref'-neg-RC* [*unrest*]:  
 assumes *P* is RR *P* is RC  
 shows  $\$ref' \# P$   
**proof** –  
 have  $P = (\neg_r \neg_r P)$   
 by (*simp add: closure rpred assms*)  
 also have  $\dots = (\neg_r (\neg_r P) ;; true_r)$   
 by (*metis Healthy-if RC1-def RC-implies-RC1 assms(2) calculation*)  
 also have  $\$ref' \# \dots$   
 by (*rel-auto*)  
 finally show ?thesis .  
**qed**

**lemma** *rea-true-CRR* [closure]:  $true_r$  is CRR  
 by (*rel-auto*)

**lemma** *rea-true-CRC* [closure]:  $true_r$  is CRC  
 by (*rel-auto*)

**lemma** *false-CRR* [closure]:  $false$  is CRR  
 by (*rel-auto*)

**lemma** *false-CRC* [closure]:  $false$  is CRC

by (rel-auto)

**lemma** *st-pred-CRR* [closure]:  $[P]_{S<} \text{ is CRR}$   
 by (rel-auto)

**lemma** *st-cond-CRC* [closure]:  $[P]_{S<} \text{ is CRC}$   
 by (rel-auto)

**lemma** *conj-CRC-closed* [closure]:  
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \wedge Q) \text{ is CRC}$   
 by (rule CRC-intro, simp-all add: unrest closure)

**lemma** *disj-CRC-closed* [closure]:  
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \vee Q) \text{ is CRC}$   
 by (rule CRC-intro, simp-all add: unrest closure)

**lemma** *shEx-CRR-closed* [closure]:  
 assumes  $\bigwedge x. P\ x \text{ is CRR}$   
 shows  $(\exists x \cdot P(x)) \text{ is CRR}$   
**proof** –  
 have  $\text{CRR}(\exists x \cdot \text{CRR}(P(x))) = (\exists x \cdot \text{CRR}(P(x)))$   
 by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def assms shEx-cong)  
**qed**

**lemma** *USUP-ind-CRR-closed* [closure]:  
 assumes  $\bigwedge i. P\ i \text{ is CRR}$   
 shows  $(\bigsqcup i \cdot P(i)) \text{ is CRR}$   
 by (rule CRR-intro, simp-all add: assms unrest closure)

**lemma** *UINF-ind-CRR-closed* [closure]:  
 assumes  $\bigwedge i. P\ i \text{ is CRR}$   
 shows  $(\bigsqcap i \cdot P(i)) \text{ is CRR}$   
 by (rule CRR-intro, simp-all add: assms unrest closure)

**lemma** *cond-tt-CRR-closed* [closure]:  
 assumes  $P \text{ is CRR } Q \text{ is CRR}$   
 shows  $P \triangleleft \$tr' =_u \$tr \triangleright Q \text{ is CRR}$   
 by (rule CRR-intro, simp-all add: unrest assms closure)

**lemma** *rea-implies-CRR-closed* [closure]:  
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \Rightarrow_r Q) \text{ is CRR}$   
 by (simp-all add: CRR-intro closure unrest)

**lemma** *conj-CRR-closed* [closure]:  
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \wedge Q) \text{ is CRR}$   
 by (simp-all add: CRR-intro closure unrest)

**lemma** *disj-CRR-closed* [closure]:  
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \vee Q) \text{ is CRR}$   
 by (rule CRR-intro, simp-all add: unrest closure)

**lemma** *rea-not-CRR-closed* [closure]:  
 $P \text{ is CRR} \implies (\neg_r P) \text{ is CRR}$



**using** *false-CRR rea-implies-CRR-closed* **by** *fastforce*

**lemma** *disj-R1-closed* [closure]:  $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \vee Q) \text{ is } R1$   
**by** (*rel-blast*)

**lemma** *st-cond-R1-closed* [closure]:  $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } R1$   
**by** (*rel-blast*)

**lemma** *cond-st-RR-closed* [closure]:  
**assumes**  $P \text{ is } RR \ Q \text{ is } RR$   
**shows**  $(P \triangleleft b \triangleright_R Q) \text{ is } RR$   
**apply** (*rule RR-intro, simp-all add: unrest closure assms, simp add: Healthy-def R2c-condr*)  
**apply** (*simp add: Healthy-if assms RR-implies-R2c*)  
**apply** (*rel-auto*)  
**done**

**lemma** *cond-st-CRR-closed* [closure]:  
 $\llbracket P \text{ is } CRR; Q \text{ is } CRR \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } CRR$   
**by** (*simp-all add: CRR-intro closure unrest*)

**lemma** *seq-CRR-closed* [closure]:  
**assumes**  $P \text{ is } CRR \ Q \text{ is } RR$   
**shows**  $(P ;; Q) \text{ is } CRR$   
**by** (*rule CRR-intro, simp-all add: unrest assms closure*)

**lemma** *tr-extend-seqr-lit* [rdes]:  
**fixes**  $P :: ('s, 'e) \text{ action}$   
**assumes**  $\$ok \# P \ \$wait \# P \ \$ref \# P$   
**shows**  $(\$tr' =_u \$tr \hat{\ }_u \langle \ll a \gg \rangle \wedge \$st' =_u \$st) ;; P = P[\$tr \hat{\ }_u \langle \ll a \gg \rangle / \$tr]$   
**using** *assms* **by** (*rel-auto, meson*)

**lemma** *tr-assign-comp* [rdes]:  
**fixes**  $P :: ('s, 'e) \text{ action}$   
**assumes**  $\$ok \# P \ \$wait \# P \ \$ref \# P$   
**shows**  $(\$tr' =_u \$tr \wedge \lceil \langle \sigma \rangle_a \rceil_s) ;; P = \lceil \sigma \rceil_{s\sigma} \dagger P$   
**using** *assms* **by** (*rel-auto, meson*)

**lemma** *RR-msubst-tt*:  $RR((P \ t) \llbracket t \rightarrow \&tt \rrbracket) = (RR \ (P \ t)) \llbracket t \rightarrow \&tt \rrbracket$   
**by** (*rel-auto*)

**lemma** *RR-msubst-ref'*:  $RR((P \ r) \llbracket r \rightarrow \$ref' \rrbracket) = (RR \ (P \ r)) \llbracket r \rightarrow \$ref' \rrbracket$   
**by** (*rel-auto*)

**lemma** *msubst-tt-RR* [closure]:  $\llbracket \bigwedge t. P \ t \text{ is } RR \rrbracket \implies (P \ t) \llbracket t \rightarrow \&tt \rrbracket \text{ is } RR$   
**by** (*simp add: Healthy-def RR-msubst-tt*)

**lemma** *msubst-ref'-RR* [closure]:  $\llbracket \bigwedge r. P \ r \text{ is } RR \rrbracket \implies (P \ r) \llbracket r \rightarrow \$ref' \rrbracket \text{ is } RR$   
**by** (*simp add: Healthy-def RR-msubst-ref'*)

**lemma** *conj-less-tr-RR-closed* [closure]:  
**assumes**  $P \text{ is } CRR$   
**shows**  $(P \wedge \$tr <_u \$tr') \text{ is } CRR$   
**proof** –  
**have**  $CRR(CRR(P) \wedge \$tr <_u \$tr') = (CRR(P) \wedge \$tr <_u \$tr')$   
**apply** (*rel-auto, blast+*)

using *less-le* apply *fastforce+*  
 done  
 thus ?thesis  
 by (metis *Healthy-def* *assms*)  
 qed

lemma *conj-eq-tr-RR-closed* [closure]:

assumes *P* is *CRR*  
 shows  $(P \wedge \$tr' =_u \$tr)$  is *CRR*  
 proof –  
 have  $CRR(CRR(P) \wedge \$tr' =_u \$tr) = (CRR(P) \wedge \$tr' =_u \$tr)$   
 by (rel-auto, blast+)  
 thus ?thesis  
 by (metis *Healthy-def* *assms*)  
 qed

### 3.3 Introduction laws

Extensionality principles for introducing refinement and equality of Circus reactive relations. It is necessary only to consider a subset of the variables that are present.

lemma *CRR-refine-ext*:

assumes  
 $P$  is *CRR*  $Q$  is *CRR*  
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] \sqsubseteq Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 shows  $P \sqsubseteq Q$   
 proof –  
 have  $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 $\sqsubseteq (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 by (simp add: *assms* *Healthy-if*)  
 hence  $CRR P \sqsubseteq CRR Q$   
 by (rel-auto)  
 thus ?thesis  
 by (metis *Healthy-if* *assms*(1) *assms*(2))  
 qed

lemma *CRR-eq-ext*:

assumes  
 $P$  is *CRR*  $Q$  is *CRR*  
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] = Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 shows  $P = Q$   
 proof –  
 have  $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 $= (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 by (simp add: *assms* *Healthy-if*)  
 hence  $CRR P = CRR Q$   
 by (rel-auto)  
 thus ?thesis  
 by (metis *Healthy-if* *assms*(1) *assms*(2))  
 qed

lemma *CRR-refine-impl-prop*:

assumes  $P$  is *CRR*  $Q$  is *CRR*  
 $\bigwedge t s s' r'. 'Q[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']' \implies 'P[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']'$   
 shows  $P \sqsubseteq Q$   
 by (rule *CRR-refine-ext*, simp-all add: *assms* *closure* *unrest* *usubst*)

(rule refine-prop-intro, simp-all add: unrest unrest-all-circus-vars closure assms)

### 3.4 Weakest Precondition

**lemma** *nil-least* [simp]:

$\langle \rangle \leq_u x = \text{true}$  **by** *rel-auto*

**lemma** *minus-nil* [simp]:

$xs - \langle \rangle = xs$  **by** *rel-auto*

**lemma** *wp-rea-circus-lemma-1*:

**assumes**  $P$  is CRR  $\$ref' \# P$

**shows**  $out\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

**proof** –

**have**  $out\alpha \# (CRR (\exists \$ref' \cdot P))[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms(1) assms(2) ex-unrest*)

**qed**

**lemma** *wp-rea-circus-lemma-2*:

**assumes**  $P$  is CRR

**shows**  $in\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

**proof** –

**have**  $in\alpha \# (CRR P)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms ex-unrest*)

**qed**

The meaning of reactive weakest precondition for Circus.  $P \text{ wp}_r Q$  means that, whenever  $P$  terminates in a state  $s_0$  having done the interaction trace  $t_0$ , which is a prefix of the overall trace, then  $Q$  must be satisfied. This in particular means that the remainder of the trace after  $t_0$  must not be a divergent behaviour of  $Q$ .

**lemma** *wp-rea-circus-form*:

**assumes**  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

**shows**  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \ll t_0 \gg \leq_u \$tr' \wedge P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \Rightarrow_r Q[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr])$

**proof** –

**have**  $(P \text{ wp}_r Q) = (\neg_r (\exists t_0 \cdot P[\ll t_0 \gg / \$tr']) ; (\neg_r Q)[\ll t_0 \gg / \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** (*simp-all add: wp-rea-def R2-tr-middle closure assms*)

**also have**  $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) ; (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** (*rel-blast*)

**also have**  $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \wedge (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** (*simp add: seqr-to-conj add: wp-rea-circus-lemma-1 wp-rea-circus-lemma-2 assms closure conj-assoc*)

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** (*rel-auto*)

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** (*simp add: Healthy-if assms closure*)

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee (RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** (*rel-auto*)

also have ... =  $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$   
 by (rel-auto)  
 also have ... =  $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r Q[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$   
 by (simp add: Healthy-if assms closure)  
 finally show ?thesis .  
 qed

**lemma** wp-rea-circus-form-alt:

assumes  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

shows  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

**proof** –

have  $(P \text{ wp}_r Q) = R2(P \text{ wp}_r Q)$

by (simp add: CRC-implies-RR CRR-implies-RR Healthy-if RR-implies-R2 assms wp-rea-R2-closed)

also have ... =  $R2(\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st, \$tr]$

by (simp add: wp-rea-circus-form assms closure Healthy-if)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle, \langle tt_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (simp add: R2-form, rel-auto)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \langle tt_0 - tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto, (metis list-concat-minus-list-concat)+)

also have ... =  $(\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1((RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr'])$

by (rel-auto, blast+)

also have ... =  $(\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

by (simp add: Healthy-if assms closure)

finally show ?thesis .

qed

**lemma** wp-rea-circus-form-alt:

assumes  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

shows  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

oops

### 3.5 Trace Substitution

**definition** trace-subst  $(-[\![\ ]\!]_t [999, 0] 999)$

where [upred-defs]:  $P[\![v]\!]_t = (P[\![\&tt - [v]_{S<} / \&tt]\!] \wedge \$tr + [v]_{S<} \leq_u \$tr')$

**lemma** unrest-trace-subst [unrest]:

$\llbracket \text{mwb-lens } x; x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \# P \rrbracket \implies x \# P[\![v]\!]_t$

by (simp add: trace-subst-def lens-indep-sym unrest)

**lemma** trace-subst-RR-closed [closure]:

assumes  $P$  is RR

shows  $P[\![v]\!]_t$  is RR

**proof** –

```

have (RR P)⟦v⟧t is RR
  apply (rel-auto)
  apply (metis diff-add-cancel-left' trace-class.add-left-mono)
  apply (metis le-add minus-cancel-le trace-class.add-diff-cancel-left)
  using le-add order-trans apply blast
done
thus ?thesis
  by (simp add: Healthy-if assms)
qed

```

```

lemma trace-subst-CRR-closed [closure]:
  assumes P is CRR
  shows P⟦v⟧t is CRR
  by (rule CRR-intro, simp-all add: closure assms unrest)

```

```

lemma tsubst-nil [usubst]:
  assumes P is CRR
  shows P⟦⟨⟩⟧t = P
proof -
  have (CRR P)⟦⟨⟩⟧t = CRR P
  by (rel-auto)
  thus ?thesis
  by (simp add: Healthy-if assms)
qed

```

```

lemma tsubst-false [usubst]: false⟦y⟧t = false
  by rel-auto

```

```

lemma cond-rea-tt-subst [usubst]:
  (P ◁ b ▷R Q)⟦v⟧t = (P⟦v⟧t ◁ b ▷R Q⟦v⟧t)
  by (rel-auto)

```

```

lemma tsubst-conj [usubst]: (P ∧ Q)⟦v⟧t = (P⟦v⟧t ∧ Q⟦v⟧t)
  by (rel-auto)

```

```

lemma tsubst-disj [usubst]: (P ∨ Q)⟦v⟧t = (P⟦v⟧t ∨ Q⟦v⟧t)
  by (rel-auto)

```

```

lemma rea-subst-R1-closed [closure]: P⟦v⟧t is R1
  apply (rel-auto) using le-add order.trans by blast

```

```

lemma tsubst-UINF-ind [usubst]: (⋀ i • P(i))⟦v⟧t = (⋀ i • (P(i))⟦v⟧t)
  by (rel-auto)

```

### 3.6 Initial Interaction

**definition** *rea-init* :: 's upred ⇒ ('t::trace, 's) uexpr ⇒ ('s, 't, 'α, 'β) rel-rsp (*I*'(-,-')) **where**  
*[upred-defs]*: *I*(s,t) = ([s]<sub>S<</sub> ∧ \$tr + [t]<sub>S<</sub> ≤<sub>u</sub> \$tr')

*I*(s,t) is a predicate stating that, if the initial state satisfies state predicate s, then the trace t is an initial trace.

```

lemma unrest-rea-init [unrest]:
  ⟦ x ⋈ ($tr)v; x ⋈ ($tr')v; x ⋈ ($st)v ⟧ ⇒ x # I(s,t)
  by (simp add: rea-init-def unrest lens-indep-sym)

```

**lemma** *rea-init-R1* [closure]:  $\mathcal{I}(s, t)$  is *R1*  
**apply** (*rel-auto*) **using** *dual-order.trans* *le-add* **by** *blast*

**lemma** *rea-init-R2c* [closure]:  $\mathcal{I}(s, t)$  is *R2c*  
**apply** (*rel-auto*)  
**apply** (*metis diff-add-cancel-left' trace-class.add-left-mono*)  
**apply** (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)  
**done**

**lemma** *rea-init-R2* [closure]:  $\mathcal{I}(s, t)$  is *R2*  
**by** (*metis Healthy-def R1-R2c-is-R2 rea-init-R1 rea-init-R2c*)

**lemma** *csp-init-RR* [closure]:  $\mathcal{I}(s, t)$  is *RR*  
**apply** (*rel-auto*)  
**apply** (*metis diff-add-cancel-left' trace-class.add-left-mono*)  
**apply** (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)  
**apply** (*metis le-add less-le less-le-trans*)  
**done**

**lemma** *csp-init-CRR* [closure]:  $\mathcal{I}(s, t)$  is *CRR*  
**by** (*rule CRR-intro, simp-all add: unrest closure*)

**lemma** *rea-init-impl-st* [closure]:  $(\mathcal{I}(b, t) \Rightarrow_r [c]_{S<})$  is *RC*  
**apply** (*rule RC-intro*)  
**apply** (*simp add: closure*)  
**apply** (*rel-auto*)  
**using** *order-trans* **by** *auto*

**lemma** *rea-init-RC1*:  
 $\neg_r \mathcal{I}(P, t)$  is *RC1*  
**apply** (*rel-auto*) **using** *dual-order.trans* **by** *blast*

**lemma** *init-acts-empty* [*rpred*]:  $\mathcal{I}(\text{true}, \langle \rangle) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *rea-not-init* [*rpred*]:  
 $(\neg_r \mathcal{I}(P, \langle \rangle)) = \mathcal{I}(\neg P, \langle \rangle)$   
**by** (*rel-auto*)

**lemma** *rea-init-conj* [*rpred*]:  
 $(\mathcal{I}(P, t) \wedge \mathcal{I}(Q, t)) = \mathcal{I}(P \wedge Q, t)$   
**by** (*rel-auto*)

**lemma** *rea-init-empty-trace* [*rpred*]:  $\mathcal{I}(s, \langle \rangle) = [s]_{S<}$   
**by** (*rel-auto*)

**lemma** *rea-init-disj-same* [*rpred*]:  $(\mathcal{I}(s_1, t) \vee \mathcal{I}(s_2, t)) = \mathcal{I}(s_1 \vee s_2, t)$   
**by** (*rel-auto*)

**lemma** *rea-init-impl-same* [*rpred*]:  $(\mathcal{I}(s_1, t) \Rightarrow_r \mathcal{I}(s_2, t)) = (\mathcal{I}(s_1, t) \Rightarrow_r [s_2]_{S<})$   
**apply** (*rel-auto*) **using** *dual-order.trans* *le-add* **by** *blast+*

**lemma** *tsubst-st-cond* [*usubst*]:  $[P]_{S<} \llbracket t \rrbracket_t = \mathcal{I}(P, t)$   
**by** (*rel-auto*)

**lemma** *tsubst-rea-init* [*usubst*]:  $(\mathcal{I}(s, x))\llbracket y \rrbracket_t = \mathcal{I}(s, y+x)$   
**apply** (*rel-auto*)  
**apply** (*metis add.assoc diff-add-cancel-left' trace-class.add-le-imp-le-left trace-class.add-left-mono*)  
**apply** (*metis add.assoc diff-add-cancel-left' le-add trace-class.add-le-imp-le-left trace-class.add-left-mono*) +  
**done**

**lemma** *tsubst-rea-not* [*usubst*]:  $(\neg_r P)\llbracket v \rrbracket_t = ((\neg_r P)\llbracket v \rrbracket_t) \wedge \mathcal{I}(\text{true}, v)$   
**apply** (*rel-auto*)  
**using** *le-add order-trans* **by** *blast*

**lemma** *tsubst-true* [*usubst*]:  $\text{true}_r\llbracket v \rrbracket_t = \mathcal{I}(\text{true}, v)$   
**by** (*rel-auto*)

**lemma** *R4-csp-init* [*rpred*]:  $R4(\mathcal{I}(s, \text{bop Cons } x \text{ } xs)) = \mathcal{I}(s, \text{bop Cons } x \text{ } xs)$   
**using** *less-list-def* **by** (*rel-blast*)

**lemma** *R5-csp-init* [*rpred*]:  $R5(\mathcal{I}(s, \text{bop Cons } x \text{ } xs)) = \text{false}$   
**by** (*rel-auto*)

**lemma** *R4-trace-subst* [*rpred*]:  
 $R4(P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t) = P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t$   
**using** *le-imp-less-or-eq* **by** (*rel-blast*)

**lemma** *R5-trace-subst* [*rpred*]:  
 $R5(P\llbracket \text{bop Cons } x \text{ } xs \rrbracket_t) = \text{false}$   
**by** (*rel-auto*)

### 3.7 Enabled Events

**definition** *csp-enable* ::  $'s \text{ upred} \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('e \text{ set}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\mathcal{E}'(-, -, -))$   
**where**  
*[upred-defs]*:  $\mathcal{E}(s, t, E) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\_u} [t]_{S<} \wedge (\forall e \in [E]_{S<} \cdot \llbracket e \rrbracket \notin_u \$ref'))$

Predicate  $\mathcal{E}(s, t, E)$  states that, if the initial state satisfies predicate  $s$ , then  $t$  is a possible (failure) trace, such that the events in the set  $E$  are enabled after the given interaction.

**lemma** *csp-enable-R1-closed* [*closure*]:  $\mathcal{E}(s, t, E)$  is *R1*  
**by** (*rel-auto*)

**lemma** *csp-enable-R2-closed* [*closure*]:  $\mathcal{E}(s, t, E)$  is *R2c*  
**by** (*rel-auto*)

**lemma** *csp-enable-RR* [*closure*]:  $\mathcal{E}(s, t, E)$  is *CRR*  
**by** (*rel-auto*)

**lemma** *tsubst-csp-enable* [*usubst*]:  $\mathcal{E}(s, t_2, e)\llbracket t_1 \rrbracket_t = \mathcal{E}(s, t_1 \hat{\_u} t_2, e)$   
**apply** (*rel-auto*)  
**apply** (*metis append.assoc less-eq-list-def prefix-concat-minus*)  
**apply** (*simp add: list-concat-minus-list-concat*)  
**done**

**lemma** *csp-enable-unrests* [*unrest*]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$ref')_v \rrbracket \Longrightarrow x \# \mathcal{E}(s, t, e)$   
**by** (*simp add: csp-enable-def R1-def lens-indep-sym unrest*)

**lemma** *csp-enable-tr'-eq-tr* [*rpred*]:

$\mathcal{E}(s, \langle \rangle, r) \triangleleft \$tr' =_u \$tr \triangleright false = \mathcal{E}(s, \langle \rangle, r)$   
**by** (*rel-auto*)

**lemma** *csp-enable-st-pred* [*rpred*]:  
 $([s_1]_{S<} \wedge \mathcal{E}(s_2, t, E)) = \mathcal{E}(s_1 \wedge s_2, t, E)$   
**by** (*rel-auto*)

**lemma** *csp-enable-conj* [*rpred*]:  
 $(\mathcal{E}(s, t, E_1) \wedge \mathcal{E}(s, t, E_2)) = \mathcal{E}(s, t, E_1 \cup_u E_2)$   
**by** (*rel-auto*)

**lemma** *csp-enable-cond* [*rpred*]:  
 $\mathcal{E}(s_1, t_1, E_1) \triangleleft b \triangleright_R \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_1 \triangleleft b \triangleright s_2, t_1 \triangleleft b \triangleright t_2, E_1 \triangleleft b \triangleright E_2)$   
**by** (*rel-auto*)

**lemma** *csp-enable-rea-assm* [*rpred*]:  
 $[b]^\top_r ;; \mathcal{E}(s, t, E) = \mathcal{E}(b \wedge s, t, E)$   
**by** (*rel-auto*)

**lemma** *csp-enable-tr-empty*:  $\mathcal{E}(true, \langle \rangle, \{v\}_u) = (\$tr' =_u \$tr \wedge \lceil v \rceil_{S<} \notin_u \$ref')$   
**by** (*rel-auto*)

**lemma** *csp-enable-nothing*:  $\mathcal{E}(true, \langle \rangle, \{\}_u) = (\$tr' =_u \$tr)$   
**by** (*rel-auto*)

**lemma** *msubst-nil-csp-enable* [*usubst*]:  
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \langle \rangle \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow \langle \rangle \rrbracket, t(x) \llbracket x \rightarrow \langle \rangle \rrbracket, E(x) \llbracket x \rightarrow \langle \rangle \rrbracket)$   
**by** (*pred-auto*)

**lemma** *msubst-csp-enable* [*usubst*]:  
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \lceil v \rceil_{S\leftarrow} \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow v \rrbracket, t(x) \llbracket x \rightarrow v \rrbracket, E(x) \llbracket x \rightarrow v \rrbracket)$   
**by** (*rel-auto*)

**lemma** *csp-enable-false* [*rpred*]:  $\mathcal{E}(false, t, E) = false$   
**by** (*rel-auto*)

**lemma** *conj-csp-enable* [*rpred*]:  $(\mathcal{E}(b_1, t, E_1) \wedge \mathcal{E}(b_2, t, E_2)) = \mathcal{E}(b_1 \wedge b_2, t, E_1 \cup_u E_2)$   
**by** (*rel-auto*)

**lemma** *USUP-csp-enable* [*rpred*]:  
 $(\bigsqcup x \cdot \mathcal{E}(s, t, A(x))) = \mathcal{E}(s, t, (\bigvee x \cdot A(x)))$   
**by** (*rel-auto*)

**lemma** *R4-csp-enable-nil* [*rpred*]:  
 $R4(\mathcal{E}(s, \langle \rangle, E)) = false$   
**by** (*rel-auto*)

**lemma** *R5-csp-enable-nil* [*rpred*]:  
 $R5(\mathcal{E}(s, \langle \rangle, E)) = \mathcal{E}(s, \langle \rangle, E)$   
**by** (*rel-auto*)

**lemma** *R4-csp-enable-Cons* [*rpred*]:  
 $R4(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = \mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)$   
**by** (*rel-auto*, *simp add: Prefix-Order.strict-prefixI'*)



**lemma** *R5-csp-enable-Cons* [rpred]:  
 $R5(\mathcal{E}(s, \text{bop } \text{Cons } x \text{ } xs, E)) = \text{false}$   
**by** (rel-auto)

### 3.8 Completed Trace Interaction

**definition** *csp-do* ::  $'s \text{ upred} \Rightarrow ('s \Rightarrow 's) \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\Phi'(-, -))$  **where**  
*[upred-defs]*:  $\Phi(s, \sigma, t) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\_}_u [t]_{S<} \wedge [\langle \sigma \rangle_a]_S)$

Predicate  $\Phi(s, \sigma, t)$  states that if the initial state satisfies  $s$ , and the trace  $t$  is performed, then afterwards the state update  $\sigma$  is executed.

**lemma** *unrest-csp-do* [unrest]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$st')_v \rrbracket \Longrightarrow x \# \Phi(s, \sigma, t)$   
**by** (simp-all add: csp-do-def alpha-in-var alpha-out-var prod-as-plus unrest lens-indep-sym)

**lemma** *csp-do-CRR* [closure]:  $\Phi(s, \sigma, t)$  is CRR  
**by** (rel-auto)

**lemma** *csp-do-R4-closed* [closure]:  
 $\Phi(b, \sigma, \text{bop } \text{Cons } x \text{ } xs)$  is R4  
**by** (rel-auto, simp add: Prefix-Order.strict-prefixI')

**lemma** *st-pred-conj-csp-do* [rpred]:  
 $([b]_{S<} \wedge \Phi(s, \sigma, t)) = \Phi(b \wedge s, \sigma, t)$   
**by** (rel-auto)

**lemma** *trea-subst-csp-do* [usubst]:  
 $(\Phi(s, \sigma, t_2)) \llbracket t_1 \rrbracket_t = \Phi(s, \sigma, t_1 \hat{\_}_u t_2)$   
**apply** (rel-auto)  
**apply** (metis append.assoc less-eq-list-def prefix-concat-minus)  
**apply** (simp add: list-concat-minus-list-concat)  
**done**

**lemma** *st-subst-csp-do* [usubst]:  
 $[\sigma]_{S\sigma} \dagger \Phi(s, \varrho, t) = \Phi(\sigma \dagger s, \varrho \circ \sigma, \sigma \dagger t)$   
**by** (rel-auto)

**lemma** *csp-init-do* [rpred]:  $(\mathcal{I}(s1, t) \wedge \Phi(s2, \sigma, t)) = \Phi(s1 \wedge s2, \sigma, t)$   
**by** (rel-auto)

**lemma** *csp-do-false* [rpred]:  $\Phi(\text{false}, s, t) = \text{false}$   
**by** (rel-auto)

**lemma** *csp-do-assign* [rpred]:  
**assumes**  $P$  is CRR  
**shows**  $\Phi(s, \sigma, t) ;; P = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger P) \llbracket t \rrbracket_t)$

**proof** –

**have**  $\Phi(s, \sigma, t) ;; CRR(P) = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger CRR(P)) \llbracket t \rrbracket_t)$   
**by** (rel-blast)  
**thus** ?thesis  
**by** (simp add: Healthy-if assms)

**qed**

**lemma** *subst-state-csp-enable* [usubst]:  
 $[\sigma]_{S\sigma} \dagger \mathcal{E}(s, t_2, e) = \mathcal{E}(\sigma \dagger s, \sigma \dagger t_2, \sigma \dagger e)$

by (rel-auto)

**lemma** *csp-do-assign-enable* [rpred]:  
 $\Phi(s_1, \sigma, t_1) ;; \mathcal{E}(s_2, t_2, e) = \mathcal{E}(s_1 \wedge \sigma \dagger s_2, t_1 \hat{^u} (\sigma \dagger t_2), (\sigma \dagger e))$   
 by (simp add: rpred closure usubst)

**lemma** *csp-do-assign-do* [rpred]:  
 $\Phi(s_1, \sigma, t_1) ;; \Phi(s_2, \varrho, t_2) = \Phi(s_1 \wedge (\sigma \dagger s_2), \varrho \circ \sigma, t_1 \hat{^u} (\sigma \dagger t_2))$   
 by (rel-auto)

**lemma** *csp-do-cond* [rpred]:  
 $\Phi(s_1, \sigma, t_1) \triangleleft b \triangleright_R \Phi(s_2, \varrho, t_2) = \Phi(s_1 \triangleleft b \triangleright s_2, \sigma \triangleleft b \triangleright_s \varrho, t_1 \triangleleft b \triangleright t_2)$   
 by (rel-auto)

**lemma** *rea-assm-csp-do* [rpred]:  
 $[b]^\top_r ;; \Phi(s, \sigma, t) = \Phi(b \wedge s, \sigma, t)$   
 by (rel-auto)

**lemma** *csp-do-skip* [rpred]:  
 assumes *P* is CRR  
 shows  $\Phi(\text{true}, \text{id}, t) ;; P = P \llbracket t \rrbracket_t$   
**proof** –  
 have  $\Phi(\text{true}, \text{id}, t) ;; \text{CRR}(P) = (\text{CRR } P) \llbracket t \rrbracket_t$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: Healthy-if assms)  
**qed**

**lemma** *wp-rea-csp-do-lemma*:  
 fixes *P* :: ('σ, 'φ) action  
 assumes  $\$ok \# P \ \$wait \# P \ \$ref \# P$   
 shows  $(\llbracket \langle \sigma \rangle_a \rrbracket_S \wedge \$tr' =_u \$tr \hat{^u} \llbracket t \rrbracket_{S<}) ;; P = (\llbracket \sigma \rrbracket_{S\sigma} \dagger P) \llbracket \$tr \hat{^u} \llbracket t \rrbracket_{S<} / \$tr \rrbracket$   
 using assms by (rel-auto, meson)

**lemma** *wp-rea-csp-do* [wp]:  
 fixes *P* :: ('σ, 'φ) action  
 assumes *P* is CRR  
 shows  $\Phi(s, \sigma, t) \text{wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r (\llbracket \sigma \rrbracket_{S\sigma} \dagger P) \llbracket t \rrbracket_t)$   
**proof** –  
 have  $\Phi(s, \sigma, t) \text{wp}_r \text{CRR}(P) = (\mathcal{I}(s, t) \Rightarrow_r (\llbracket \sigma \rrbracket_{S\sigma} \dagger \text{CRR}(P)) \llbracket t \rrbracket_t)$   
 by (rel-blast)  
 thus ?thesis  
 by (simp add: assms Healthy-if)  
**qed**

**lemma** *csp-do-power-Suc* [rpred]:  
 $\Phi(\text{true}, \text{id}, t) \hat{^} (\text{Suc } i) = \Phi(\text{true}, \text{id}, \text{iter}[\text{Suc } i](t))$   
 by (induct i, (rel-auto)+)

**lemma** *csp-power-do-comp* [rpred]:  
 assumes *P* is CRR  
 shows  $\Phi(\text{true}, \text{id}, t) \hat{^} i ;; P = \Phi(\text{true}, \text{id}, \text{iter}[i](t)) ;; P$   
 apply (cases i)  
 apply (simp-all add: rpred usubst assms closure)  
 done

**lemma** *wp-rea-csp-do-skip* [*wp*]:  
**fixes**  $Q :: ('σ, 'φ)$  *action*  
**assumes**  $P$  *is CRR*  
**shows**  $\Phi(s, id, t) \text{ wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r P \llbracket t \rrbracket_t)$   
**proof** –  
**have**  $\Phi(s, id, t) \text{ wp}_r P = \Phi(s, id, t) \text{ wp}_r P$   
**by** (*simp add: skip-r-def*)  
**thus** ?thesis **by** (*simp add: wp assms usubst alpha*)  
**qed**

**lemma** *msubst-csp-do* [*usubst*]:  
 $\Phi(s(x), \sigma, t(x)) \llbracket x \rightarrow \lceil v \rceil_{S \leftarrow} \rrbracket = \Phi(s(x) \llbracket x \rightarrow v \rrbracket, \sigma, t(x) \llbracket x \rightarrow v \rrbracket)$   
**by** (*rel-auto*)

**end**

## 4 Stateful-Failure Healthiness Conditions

**theory** *utp-sfrd-healths*  
**imports** *utp-sfrd-rel*  
**begin**

## 5 Definitions

We here define extra healthiness conditions for stateful-failure reactive designs.

**abbreviation**  $CSP1 :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$   
**where**  $CSP1(P) \equiv RD1(P)$

**abbreviation**  $CSP2 :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$   
**where**  $CSP2(P) \equiv RD2(P)$

**abbreviation**  $CSP :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$   
**where**  $CSP(P) \equiv SRD(P)$

**definition**  $STOP :: 'φ \text{ rel-csp}$  **where**  
 $[upred-defs]: STOP = CSP1(\$ok' \wedge R3c(\$tr' =_u \$tr \wedge \$wait'))$

**definition**  $SKIP :: 'φ \text{ rel-csp}$  **where**  
 $[upred-defs]: SKIP = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$

**definition**  $Stop :: ('σ, 'φ) \text{ action}$  **where**  
 $[upred-defs]: Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \$wait'))$

**definition**  $Skip :: ('σ, 'φ) \text{ action}$  **where**  
 $[upred-defs]: Skip = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st))$

**definition**  $CSP3 :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$  **where**  
 $[upred-defs]: CSP3(P) = (Skip ;; P)$

**definition**  $CSP4 :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$  **where**  
 $[upred-defs]: CSP4(P) = (P ;; Skip)$

**definition**  $NCSP :: (('σ, 'φ) \text{ st-csp} \times ('σ, 'φ) \text{ st-csp}) \text{ health}$  **where**

[upred-defs]:  $NCSP = CSP3 \circ CSP4 \circ CSP$

Productive and normal processes

**abbreviation**  $PCSP \equiv Productive \circ NCSP$

Instantaneous and normal processes

**abbreviation**  $ICSP \equiv ISRD1 \circ NCSP$

## 5.1 Healthiness condition properties

$SKIP$  is the same as  $Skip$ , and  $STOP$  is the same as  $Stop$ , when we consider stateless CSP processes. This is because any reference to the  $st$  variable degenerates when the alphabet type coerces its type to be empty. We therefore need not consider  $SKIP$  and  $STOP$  actions.

**theorem** *SKIP-is-Skip*:  $SKIP = Skip$   
by (*rel-auto*)

**theorem** *STOP-is-Stop*:  $STOP = Stop$   
by (*rel-auto*)

**theorem** *Skip-UTP-form*:  $Skip = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$   
by (*rel-auto*)

**lemma** *Skip-is-CSP* [closure]:  
*Skip is CSP*  
by (*simp add: Skip-def RHS-design-is-SRD unrest*)

**lemma** *Skip-RHS-tri-design*:  
 $Skip = \mathbf{R}_s(true \vdash (false \diamond (\$tr' =_u \$tr \wedge \$st' =_u \$st)))$   
by (*rel-auto*)

**lemma** *Skip-RHS-tri-design'* [rdes-def]:  
 $Skip = \mathbf{R}_s(true_r \vdash (false \diamond \Phi(true, id, \langle \rangle)))$   
by (*rel-auto*)

**lemma** *Stop-is-CSP* [closure]:  
*Stop is CSP*  
by (*simp add: Stop-def RHS-design-is-SRD unrest*)

**lemma** *Stop-RHS-tri-design*:  $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr) \diamond false)$   
by (*rel-auto*)

**lemma** *Stop-RHS-rdes-def* [rdes-def]:  $Stop = \mathbf{R}_s(true_r \vdash \mathcal{E}(true, \langle \rangle, \{\}_u) \diamond false)$   
by (*rel-auto*)

**lemma** *preR-Skip* [rdes]:  $pre_R(Skip) = true_r$   
by (*rel-auto*)

**lemma** *periR-Skip* [rdes]:  $peri_R(Skip) = false$   
by (*rel-auto*)

**lemma** *postR-Skip* [rdes]:  $post_R(Skip) = \Phi(true, id, \langle \rangle)$   
by (*rel-auto*)

**lemma** *Productive-Stop* [closure]:

*Stop is Productive*

by (simp add: Stop-RHS-tri-design Healthy-def Productive-RHS-design-form unrest)

**lemma** *Skip-left-lemma:*

assumes *P is CSP*

shows  $Skip \;; P = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$

**proof** –

have  $Skip \;; P =$

$\mathbf{R}_s ((\$tr' =_u \$tr \wedge \$st' =_u \$st) \wp_r pre_R P \vdash$   
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; peri_R P \diamond$   
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; post_R P)$

by (simp add: SRD-composition-wp alpha rdes closure wp assms rpred C1, rel-auto)

also have  $\dots = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash$

$(\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st) \;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright cmt_R P))$

by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)

also have  $\dots = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$

by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)

finally show ?thesis .

**qed**

**lemma** *Skip-left-unit-ref-unrest:*

assumes *P is CSP*  $\$ref \# P \llbracket false/\$wait \rrbracket$

shows  $Skip \;; P = P$

using *assms*

by (simp add: Skip-left-lemma)

(metis SRD-reactive-design-alt all-unrest cmt-unrest-ref cmt-wait-false ex-unrest pre-unrest-ref pre-wait-false)

**lemma** *CSP3-intro:*

$\llbracket P \text{ is CSP}; \$ref \# P \llbracket false/\$wait \rrbracket \rrbracket \implies P \text{ is CSP3}$

by (simp add: CSP3-def Healthy-def' Skip-left-unit-ref-unrest)

**lemma** *ref-unrest-RHS-design:*

assumes  $\$ref \# P \ \$ref \# Q_1 \ \$ref \# Q_2$

shows  $\$ref \# (\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)) \ f$

by (simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst assms)

**lemma** *CSP3-SRD-intro:*

assumes *P is CSP*  $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

shows *P is CSP3*

**proof** –

have  $P: \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$

by (simp add: SRD-reactive-design-alt assms(1) wait'-cond-peri-post-cmt[THEN sym])

have  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$  is *CSP3*

by (rule CSP3-intro, simp add: assms P, simp add: ref-unrest-RHS-design assms)

thus ?thesis

by (simp add: P)

**qed**

**lemma** *Skip-unrest-ref [unrest]:*  $\$ref \# Skip \llbracket false/\$wait \rrbracket$

by (simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)

**lemma** *Skip-unrest-ref' [unrest]:*  $\$ref' \# Skip \llbracket false/\$wait \rrbracket$

by (simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)

**lemma** *CSP3-iff:*

**assumes**  $P$  is CSP  
**shows**  $P$  is CSP3  $\longleftrightarrow$  ( $\$ref \# P \llbracket false/\$wait \rrbracket$ )  
**proof**  
**assume** 1:  $P$  is CSP3  
**have**  $\$ref \# (Skip ;; P) \llbracket false/\$wait \rrbracket$   
**by** (simp add: usubst unrest)  
**with** 1 **show**  $\$ref \# P \llbracket false/\$wait \rrbracket$   
**by** (metis CSP3-def Healthy-def)  
**next**  
**assume** 1:  $\$ref \# P \llbracket false/\$wait \rrbracket$   
**show**  $P$  is CSP3  
**by** (simp add: 1 CSP3-intro assms)  
**qed**

**lemma** CSP3-unrest-ref [unrest]:  
**assumes**  $P$  is CSP  $P$  is CSP3  
**shows**  $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$   
**proof** –  
**have**  $a: (\$ref \# P \llbracket false/\$wait \rrbracket)$   
**using** CSP3-iff assms **by** blast  
**from**  $a$  **show**  $\$ref \# pre_R(P)$   
**by** (rel-blast)  
**from**  $a$  **show**  $\$ref \# peri_R(P)$   
**by** (rel-blast)  
**from**  $a$  **show**  $\$ref \# post_R(P)$   
**by** (rel-blast)  
**qed**

**lemma** CSP3-rdes:  
**assumes**  $P$  is RR  $Q$  is RR  $R$  is RR  
**shows**  $CSP3(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\forall \$ref \cdot P) \vdash (\exists \$ref \cdot Q) \diamond (\exists \$ref \cdot R))$   
**by** (simp add: CSP3-def Skip-left-lemma closure assms rdes, rel-auto)

**lemma** CSP3-form:  
**assumes**  $P$  is CSP  
**shows**  $CSP3(P) = \mathbf{R}_s((\forall \$ref \cdot pre_R(P)) \vdash (\exists \$ref \cdot peri_R(P)) \diamond (\exists \$ref \cdot post_R(P)))$   
**by** (simp add: CSP3-def Skip-left-lemma assms, rel-auto)

**lemma** CSP3-Skip [closure]:  
 $Skip$  is CSP3  
**by** (rule CSP3-intro, simp add: Skip-is-CSP, simp add: Skip-def unrest)

**lemma** CSP3-Stop [closure]:  
 $Stop$  is CSP3  
**by** (rule CSP3-intro, simp add: Stop-is-CSP, simp add: Stop-def unrest)

**lemma** CSP3-Idempotent [closure]: Idempotent CSP3  
**by** (metis (no-types, lifting) CSP3-Skip CSP3-def Healthy-if Idempotent-def seqr-assoc)

**lemma** CSP3-Continuous: Continuous CSP3  
**by** (simp add: Continuous-def CSP3-def seq-Sup-distl)

**lemma** Skip-right-lemma:  
**assumes**  $P$  is CSP  
**shows**  $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$

**proof** –

have  $P \;; \text{Skip} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P \;; (\$tr' =_u \$tr \wedge \$st' =_u \$st))$   
 by (*simp add: SRD-composition-wp closure assms wp rdes rpred, rel-auto*)  
 also have  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\text{cmt}_R P \;; (\exists \$st \cdot [II]_D)) \triangleleft \$wait' \triangleright (\text{cmt}_R P \;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 also have  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\text{cmt}_R P \;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 also have  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 finally show *?thesis* .

**qed**

**lemma** *Skip-right-tri-lemma:*

assumes  $P$  is CSP

shows  $P \;; \text{Skip} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$

**proof** –

have  $((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)) = ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P))$

by (*rel-auto*)

thus *?thesis* by (*simp add: Skip-right-lemma[OF assms]*)

**qed**

**lemma** *CSP4-intro:*

assumes  $P$  is CSP  $(\neg_r \text{pre}_R(P)) \;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$

$\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$

shows  $P$  is CSP4

**proof** –

have  $\text{CSP4}(P) = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$

by (*simp add: CSP4-def Skip-right-lemma assms(1)*)

also have  $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot \text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$

by (*simp add: wp-rea-def assms(2) rpred closure cond-var-subst-left cond-var-subst-right*)

also have  $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket)))$

by (*simp add: usubst unrest*)

also have  $\dots = \mathbf{R}_s (\text{pre}_R P \vdash ((\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$

by (*simp add: ex-unrest assms*)

also have  $\dots = \mathbf{R}_s (\text{pre}_R P \vdash \text{cmt}_R P)$

by (*simp add: cond-var-split*)

also have  $\dots = P$

by (*simp add: SRD-reactive-design-alt assms(1)*)

finally show *?thesis*

by (*simp add: Healthy-def'*)

**qed**

**lemma** *CSP4-RC-intro:*

assumes  $P$  is CSP  $\text{pre}_R(P)$  is RC

$\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$

shows  $P$  is CSP4

**proof** –

have  $(\neg_r \text{pre}_R(P)) \;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$

**by** (*metis* (*no-types*, *lifting*) *R1-seqr-closure* *assms*(2) *rea-not-R1* *rea-not-false* *rea-not-not* *wp-rea-RC-false* *wp-rea-def*)  
**thus** *?thesis*  
**by** (*simp* *add*: *CSP4-intro* *assms*)  
**qed**

**lemma** *CSP4-rdes*:

**assumes** *P* is *RR* *Q* is *RR* *R* is *RR*  
**shows**  $CSP_4(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\neg_r P) \text{ wp}_r \text{ false} \vdash ((\exists \$st' \cdot Q) \diamond (\exists \$ref' \cdot R)))$   
**by** (*simp* *add*: *CSP4-def* *Skip-right-lemma* *closure* *assms* *rdes*, *rel-auto*, *blast*+)

**lemma** *CSP4-form*:

**assumes** *P* is *CSP*  
**shows**  $CSP_4(P) = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{ wp}_r \text{ false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$   
**by** (*simp* *add*: *CSP4-def* *Skip-right-tri-lemma* *assms*)

**lemma** *Skip-srdes-right-unit*:

*(Skip :: ('σ, 'φ) action) ;; II<sub>R</sub> = Skip*  
**by** (*rdes-simp*)

**lemma** *Skip-srdes-left-unit*:

$II_R ;; (Skip :: ('σ, 'φ) \text{ action}) = Skip$   
**by** (*rdes-eq*)

**lemma** *CSP4-right-subsumes-RD3*:  $RD3(CSP_4(P)) = CSP_4(P)$

**by** (*metis* (*no-types*, *hide-lams*) *CSP4-def* *RD3-def* *Skip-srdes-right-unit* *seqr-assoc*)

**lemma** *CSP4-implies-RD3*: *P* is *CSP4*  $\implies$  *P* is *RD3*

**by** (*metis* *CSP4-right-subsumes-RD3* *Healthy-def*)

**lemma** *CSP4-tri-intro*:

**assumes** *P* is *CSP*  $(\neg_r \text{pre}_R(P)) ;; R1(\text{true}) = (\neg_r \text{pre}_R(P)) \$st' \# \text{peri}_R(P) \$ref' \# \text{post}_R(P)$   
**shows** *P* is *CSP4*  
**using** *assms*  
**by** (*rule-tac* *CSP4-intro*, *simp-all* *add*: *pre<sub>R</sub>-def* *peri<sub>R</sub>-def* *post<sub>R</sub>-def* *usubst* *cmt<sub>R</sub>-def*)

**lemma** *CSP4-NSRD-intro*:

**assumes** *P* is *NSRD*  $\$ref' \# \text{post}_R(P)$   
**shows** *P* is *CSP4*  
**by** (*simp* *add*: *CSP4-tri-intro* *NSRD-is-SRD* *NSRD-neg-pre-unit* *NSRD-st'-unrest-peri* *assms*)

**lemma** *CSP3-commutes-CSP4*:  $CSP3(CSP_4(P)) = CSP_4(CSP3(P))$

**by** (*simp* *add*: *CSP3-def* *CSP4-def* *seqr-assoc*)

**lemma** *NCSP-implies-CSP* [*closure*]: *P* is *NCSP*  $\implies$  *P* is *CSP*

**by** (*metis* (*no-types*, *hide-lams*) *CSP3-def* *CSP4-def* *Healthy-def* *NCSP-def* *SRD-idem* *SRD-seqr-closure* *Skip-is-CSP* *comp-apply*)

**lemma** *NCSP-elim* [*RD-elim*]:

$\llbracket X \text{ is } NCSP; P(\mathbf{R}_s(\text{pre}_R(X) \vdash \text{peri}_R(X) \diamond \text{post}_R(X))) \rrbracket \implies P(X)$   
**by** (*simp* *add*: *SRD-reactive-tri-design* *closure*)

**lemma** *NCSP-implies-CSP3* [*closure*]:

$P \text{ is } NCSP \implies P \text{ is } CSP3$   
**by** (*metis* (*no-types*, *lifting*) *CSP3-def* *Healthy-def'* *NCSP-def* *Skip-is-CSP* *Skip-left-unit-ref-unrest*)



*Skip-unrest-ref comp-apply seqr-assoc)*

**lemma** *NCSP-implies-CSP4* [closure]:

*P is NCSP  $\implies$  P is CSP4*

**by** (*metis* (*no-types*, *hide-lams*) *CSP3-commutes-CSP4 Healthy-def NCSP-def NCSP-implies-CSP NCSP-implies-CSP3 comp-apply*)

**lemma** *NCSP-implies-RD3* [closure]: *P is NCSP  $\implies$  P is RD3*

**by** (*metis* *CSP3-commutes-CSP4 CSP4-right-subsumes-RD3 Healthy-def NCSP-def comp-apply*)

**lemma** *NCSP-implies-NSRD* [closure]: *P is NCSP  $\implies$  P is NSRD*

**by** (*simp add: NCSP-implies-CSP NCSP-implies-RD3 SRD-RD3-implies-NSRD*)

**lemma** *NCSP-subset-implies-CSP* [closure]:

*A  $\subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{CSP} \rrbracket_H$*

**using** *NCSP-implies-CSP* **by** *blast*

**lemma** *NCSP-subset-implies-NSRD* [closure]:

*A  $\subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{NSRD} \rrbracket_H$*

**using** *NCSP-implies-NSRD* **by** *blast*

**lemma** *CSP-Healthy-subset-member*:  *$\llbracket P \in A; A \subseteq \llbracket \text{CSP} \rrbracket_H \rrbracket \implies P \text{ is CSP}$*

**by** (*simp add: is-Healthy-subset-member*)

**lemma** *CSP3-Healthy-subset-member*:  *$\llbracket P \in A; A \subseteq \llbracket \text{CSP3} \rrbracket_H \rrbracket \implies P \text{ is CSP3}$*

**by** (*simp add: is-Healthy-subset-member*)

**lemma** *CSP4-Healthy-subset-member*:  *$\llbracket P \in A; A \subseteq \llbracket \text{CSP4} \rrbracket_H \rrbracket \implies P \text{ is CSP4}$*

**by** (*simp add: is-Healthy-subset-member*)

**lemma** *NCSP-Healthy-subset-member*:  *$\llbracket P \in A; A \subseteq \llbracket \text{NCSP} \rrbracket_H \rrbracket \implies P \text{ is NCSP}$*

**by** (*simp add: is-Healthy-subset-member*)

**lemma** *NCSP-intro*:

**assumes** *P is CSP P is CSP3 P is CSP4*

**shows** *P is NCSP*

**by** (*metis* *Healthy-def NCSP-def assms comp-eq-dest-lhs*)

**lemma** *Skip-left-unit*: *P is NCSP  $\implies$  Skip ;; P = P*

**by** (*metis* (*full-types*) *CSP3-def Healthy-if NCSP-implies-CSP3*)

**lemma** *Skip-right-unit*: *P is NCSP  $\implies$  P ;; Skip = P*

**by** (*metis* (*full-types*) *CSP4-def Healthy-if NCSP-implies-CSP4*)

**lemma** *NCSP-NSRD-intro*:

**assumes** *P is NSRD \$ref \# pre\_R(P) \$ref \# peri\_R(P) \$ref \# post\_R(P) \$ref' \# post\_R(P)*

**shows** *P is NCSP*

**by** (*simp add: CSP3-SRD-intro CSP4-NSRD-intro NCSP-intro NSRD-is-SRD assms*)

**lemma** *CSP4-neg-pre-unit*:

**assumes** *P is CSP P is CSP4*

**shows** *( $\neg_r pre_R(P)$ ) ;; R1(true) = ( $\neg_r pre_R(P)$ )*

**by** (*simp add: CSP4-implies-RD3 NSRD-neg-pre-unit SRD-RD3-implies-NSRD assms(1) assms(2)*)

**lemma** *NSRD-CSP4-intro*:

assumes  $P$  is CSP  $P$  is CSP<sub>4</sub>  
 shows  $P$  is NSRD  
 by (simp add: CSP<sub>4</sub>-implies-RD3 SRD-RD3-implies-NSRD assms(1) assms(2))

lemma NCSP-form:

NCSP  $P = \mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R(P)) wp_r false) \vdash ((\exists \$ref \cdot \exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref \cdot \exists \$ref' \cdot post_R(P))))$

proof –

have NCSP  $P = CSP3 (CSP4 (NSRD P))$

by (metis (no-types, hide-lams) CSP<sub>4</sub>-def NCSP-def NSRD-alt-def RA1 RD3-def Skip-srdes-left-unit o-apply)

also

have ... =  $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NSRD P)) wp_r false) \vdash ((\exists \$ref \cdot \exists \$st' \cdot peri_R (NSRD P)) \diamond (\exists \$ref \cdot \exists \$ref' \cdot post_R (NSRD P))))$

by (simp add: CSP<sub>3</sub>-form CSP<sub>4</sub>-form closure unrest rdes, rel-auto)

also have ... =  $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R(P)) wp_r false) \vdash ((\exists \$ref \cdot \exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref \cdot \exists \$ref' \cdot post_R(P))))$

by (simp add: NSRD-form rdes closure, rel-blast)

finally show ?thesis .

qed

lemma CSP<sub>4</sub>-st'-unrest-peri [unrest]:

assumes  $P$  is CSP  $P$  is CSP<sub>4</sub>

shows  $\$st' \not\# peri_R(P)$

by (simp add: NSRD-CSP<sub>4</sub>-intro NSRD-st'-unrest-peri assms)

lemma CSP<sub>4</sub>-healthy-form:

assumes  $P$  is CSP  $P$  is CSP<sub>4</sub>

shows  $P = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$

proof –

have  $P = \mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$

by (metis CSP<sub>4</sub>-def Healthy-def Skip-right-lemma assms(1) assms(2))

also have ... =  $\mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \llbracket true/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P) \llbracket false/\$wait' \rrbracket)))$

by (metis (no-types, hide-lams) subst-wait'-left-subst subst-wait'-right-subst wait'-cond-def)

also have ... =  $\mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$

by (simp add: wait'-cond-def usubst peri\_R-def post\_R-def cmt\_R-def unrest)

finally show ?thesis .

qed

lemma CSP<sub>4</sub>-ref'-unrest-pre [unrest]:

assumes  $P$  is CSP  $P$  is CSP<sub>4</sub>

shows  $\$ref' \not\# pre_R(P)$

proof –

have  $pre_R(P) = pre_R(\mathbf{R}_s ((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$

using CSP<sub>4</sub>-healthy-form assms(1) assms(2) by fastforce

also have ... =  $(\neg_r pre_R P) wp_r false$

by (simp add: rea-pre-RHS-design wp-rea-def usubst unrest)

CSP<sub>4</sub>-neg-pre-unit R1-rea-not R2c-preR R2c-rea-not assms)

also have  $\$ref' \not\# \dots$

by (simp add: wp-rea-def unrest)

finally show ?thesis .

qed

**lemma** *NCSP-set-unrest-pre-wait'*:

**assumes**  $A \subseteq \llbracket \text{NCSP} \rrbracket_H$

**shows**  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$

**proof** –

**fix**  $P$

**assume**  $P \in A$

**hence**  $P$  is NSRD

**using** *NCSP-implies-NSRD* *assms* **by** *auto*

**thus**  $\$wait' \# pre_R(P)$

**using** *NSRD-wait'-unrest-pre* **by** *blast*

**qed**

**lemma** *CSP4-set-unrest-pre-st'*:

**assumes**  $A \subseteq \llbracket \text{CSP} \rrbracket_H$   $A \subseteq \llbracket \text{CSP}_4 \rrbracket_H$

**shows**  $\bigwedge P. P \in A \implies \$st' \# pre_R(P)$

**proof** –

**fix**  $P$

**assume**  $P \in A$

**hence**  $P$  is NSRD

**using** *NSRD-CSP4-intro* *assms*(1) *assms*(2) **by** *blast*

**thus**  $\$st' \# pre_R(P)$

**using** *NSRD-st'-unrest-pre* **by** *blast*

**qed**

**lemma** *CSP4-ref'-unrest-post* [*unrest*]:

**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>

**shows**  $\$ref' \# post_R(P)$

**proof** –

**have**  $post_R(P) = post_R(\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P)))))$

**using** *CSP4-healthy-form* *assms*(1) *assms*(2) **by** *fastforce*

**also have**  $\dots = R1 (R2c ((\neg_r pre_R P) wp_r false \Rightarrow_r (\exists \$ref' \cdot post_R P)))$

**by** (*simp add: rea-post-RHS-design* *usubst unrest wp-rea-def*)

**also have**  $\$ref' \# \dots$

**by** (*simp add: R1-def R2c-def wp-rea-def unrest*)

**finally show** *?thesis* .

**qed**

**lemma** *CSP3-Chaos* [*closure*]: *Chaos* is CSP<sub>3</sub>

**by** (*simp add: Chaos-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

**lemma** *CSP4-Chaos* [*closure*]: *Chaos* is CSP<sub>4</sub>

**by** (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

**lemma** *NCSP-Chaos* [*closure*]: *Chaos* is NCSP

**by** (*simp add: NCSP-intro closure*)

**lemma** *CSP3-Miracle* [*closure*]: *Miracle* is CSP<sub>3</sub>

**by** (*simp add: Miracle-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

**lemma** *CSP4-Miracle* [*closure*]: *Miracle* is CSP<sub>4</sub>

**by** (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

**lemma** *NCSP-Miracle* [*closure*]: *Miracle* is NCSP

**by** (*simp add: NCSP-intro closure*)

**lemma** *NCSP-seqr-closure* [closure]:  
**assumes** *P is NCSP Q is NCSP*  
**shows** *P ;; Q is NCSP*  
**by** (metis (no-types, lifting) *CSP3-def CSP4-def Healthy-def' NCSP-implies-CSP NCSP-implies-CSP3*  
*NCSP-implies-CSP4 NCSP-intro SRD-seqr-closure assms(1) assms(2) seqr-assoc*)

**lemma** *CSP4-Skip* [closure]: *Skip is CSP4*  
**apply** (rule *CSP4-intro, simp-all add: Skip-is-CSP*)  
**apply** (simp-all add: *Skip-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)  
**done**

**lemma** *NCSP-Skip* [closure]: *Skip is NCSP*  
**by** (metis *CSP3-Skip CSP4-Skip Healthy-def NCSP-def Skip-is-CSP comp-apply*)

**lemma** *CSP4-Stop* [closure]: *Stop is CSP4*  
**apply** (rule *CSP4-intro, simp-all add: Stop-is-CSP*)  
**apply** (simp-all add: *Stop-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)  
**done**

**lemma** *NCSP-Stop* [closure]: *Stop is NCSP*  
**by** (metis *CSP3-Stop CSP4-Stop Healthy-def NCSP-def Stop-is-CSP comp-apply*)

**lemma** *CSP4-Idempotent*: *Idempotent CSP4*  
**by** (metis (no-types, lifting) *CSP3-Skip CSP3-def CSP4-def Healthy-if Idempotent-def seqr-assoc*)

**lemma** *CSP4-Continuous*: *Continuous CSP4*  
**by** (simp add: *Continuous-def CSP4-def seq-Sup-distr*)

**lemma** *preR-Stop* [rdes]: *pre<sub>R</sub>(Stop) = true<sub>r</sub>*  
**by** (simp add: *Stop-def Stop-is-CSP rea-pre-RHS-design unrest usubst R2c-true*)

**lemma** *periR-Stop* [rdes]: *peri<sub>R</sub>(Stop) =  $\mathcal{E}(\text{true}, \langle \rangle, \{\}_u)$*   
**by** (rel-auto)

**lemma** *postR-Stop* [rdes]: *post<sub>R</sub>(Stop) = false*  
**by** (rel-auto)

**lemma** *cmtR-Stop* [rdes]: *cmt<sub>R</sub>(Stop) = ( $\$tr' =_u \$tr \wedge \$wait'$ )*  
**by** (rel-auto)

**lemma** *NCSP-Idempotent* [closure]: *Idempotent NCSP*  
**by** (clarsimp simp add: *NCSP-def Idempotent-def*)  
(metis (no-types, hide-lams) *CSP3-Idempotent CSP3-def CSP4-Idempotent CSP4-def Healthy-def*  
*Idempotent-def SRD-idem SRD-seqr-closure Skip-is-CSP seqr-assoc*)

**lemma** *NCSP-Continuous* [closure]: *Continuous NCSP*  
**by** (simp add: *CSP3-Continuous CSP4-Continuous Continuous-comp NCSP-def SRD-Continuous*)

**lemma** *preR-CRR* [closure]: *P is NCSP  $\implies$  pre<sub>R</sub>(P) is CRR*  
**by** (rule *CRR-intro, simp-all add: closure unrest*)

**lemma** *periR-CRR* [closure]: *P is NCSP  $\implies$  peri<sub>R</sub>(P) is CRR*  
**by** (rule *CRR-intro, simp-all add: closure unrest*)

**lemma** *postR-CRR* [closure]: *P is NCSP  $\implies$  post<sub>R</sub>(P) is CRR*

by (rule CRR-intro, simp-all add: closure unrest)

**lemma** NCSP-rdes-intro [closure]:

assumes  $P$  is CRC  $Q$  is CRR  $R$  is CRR

$\$st' \# Q \$ref' \# R$

shows  $\mathbf{R}_s(P \vdash Q \diamond R)$  is NCSP

apply (rule NCSP-intro)

apply (simp-all add: closure assms)

apply (rule CSP3-SRD-intro)

apply (simp-all add: rdes closure assms unrest)

apply (rule CSP4-tri-intro)

apply (simp-all add: rdes closure assms unrest)

apply (metis (no-types, lifting) CRC-implies-RC R1-seqr-closure assms(1) rea-not-R1 rea-not-false  
rea-not-not wp-rea-RC-false wp-rea-def)

done

**lemma** NCSP-preR-CRC [closure]:

assumes  $P$  is NCSP

shows  $\text{pre}_R(P)$  is CRC

by (rule CRC-intro, simp-all add: closure assms unrest)

**lemma** CSP3-Sup-closure [closure]:

$A \subseteq \llbracket \text{CSP3} \rrbracket_H \implies (\bigcap A) \text{ is CSP3}$

apply (auto simp add: CSP3-def Healthy-def seq-Sup-distl)

apply (rule cong[of Sup])

apply (simp)

using image-iff apply force

done

**lemma** CSP4-Sup-closure [closure]:

$A \subseteq \llbracket \text{CSP4} \rrbracket_H \implies (\bigcap A) \text{ is CSP4}$

apply (auto simp add: CSP4-def Healthy-def seq-Sup-distr)

apply (rule cong[of Sup])

apply (simp)

using image-iff apply force

done

**lemma** NCSP-Sup-closure [closure]:  $\llbracket A \subseteq \llbracket \text{NCSP} \rrbracket_H; A \neq \{\} \rrbracket \implies (\bigcap A) \text{ is NCSP}$

apply (rule NCSP-intro, simp-all add: closure)

apply (metis (no-types, lifting) Ball-Collect CSP3-Sup-closure NCSP-implies-CSP3)

apply (metis (no-types, lifting) Ball-Collect CSP4-Sup-closure NCSP-implies-CSP4)

done

**lemma** NCSP-SUP-closure [closure]:  $\llbracket \bigwedge i. P(i) \text{ is NCSP}; A \neq \{\} \rrbracket \implies (\bigcap i \in A. P(i)) \text{ is NCSP}$

by (metis (mono-tags, lifting) Ball-Collect NCSP-Sup-closure image-iff image-is-empty)

**lemma** PCSP-implies-NCSP [closure]:

assumes  $P$  is PCSP

shows  $P$  is NCSP

**proof** –

have  $P = \text{Productive}(\text{NCSP}(\text{NCSP } P))$

by (metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply)

also have  $\dots = \mathbf{R}_s((\forall \$ref \cdot (\neg_r \text{pre}_R(\text{NCSP } P)) \text{ wp}_r \text{ false}) \vdash$   
 $(\exists \$ref \cdot \exists \$st' \cdot \text{peri}_R(\text{NCSP } P)) \diamond$

$((\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P)) \wedge \$tr <_u \$tr')$   
 by (simp add: NCSP-form Productive-RHS-design-form unrest closure)  
 also have ... is NCSP  
 apply (rule NCSP-rdes-intro)  
 apply (rule CRC-intro)  
 apply (simp-all add: unrest ex-unrest all-unrest closure)  
 done  
 finally show ?thesis .  
 qed

**lemma** *PCSP-elim* [RD-elim]:  
 assumes  $X$  is PCSP  $P$  ( $\mathbf{R}_s ((pre_R X) \vdash peri_R X \diamond (R_4(post_R X)))$ )  
 shows  $P X$   
 by (metis *R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms comp-apply*)

**lemma** *ICSP-implies-NCSP* [closure]:  
 assumes  $P$  is ICSP  
 shows  $P$  is NCSP  
**proof** –  
 have  $P = ISRD1(NCSP(NCSP P))$   
 by (metis (no-types, hide-lams) *Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)  
 also have ... =  $ISRD1 (\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$   
 $(\exists \$ref \cdot \exists \$st' \cdot peri_R (NCSP P)) \diamond$   
 $(\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P))))$   
 by (simp add: NCSP-form)  
 also have ... =  $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$   
 $false \diamond$   
 $((\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P)) \wedge \$tr' =_u \$tr))$   
 by (simp-all add: *ISRD1-RHS-design-form closure rdes unrest*)  
 also have ... is NCSP  
 apply (rule NCSP-rdes-intro)  
 apply (rule CRC-intro)  
 apply (simp-all add: unrest ex-unrest all-unrest closure)  
 done  
 finally show ?thesis .  
 qed

**lemma** *ICSP-implies-ISRD* [closure]:  
 assumes  $P$  is ICSP  
 shows  $P$  is ISRD  
 by (metis (no-types, hide-lams) *Healthy-def ICSP-implies-NCSP ISRD-def NCSP-implies-NSRD assms comp-apply*)

**lemma** *ICSP-elim* [RD-elim]:  
 assumes  $X$  is ICSP  $P$  ( $\mathbf{R}_s ((pre_R X) \vdash false \diamond (post_R X \wedge \$tr' =_u \$tr))$ )  
 shows  $P X$   
 by (metis *Healthy-if NCSP-implies-CSP ICSP-implies-NCSP ISRD1-form assms comp-apply*)

**lemma** *ICSP-Stop-right-zero-lemma*:  
 $(P \wedge (\$tr' =_u \$tr)) ;; true_r = true_r \implies (P \wedge (\$tr' =_u \$tr)) ;; (\$tr' =_u \$tr) = (\$tr' =_u \$tr)$   
 by (rel-blast)

**lemma** *ICSP-Stop-right-zero*:  
 assumes  $P$  is ICSP  $pre_R(P) = true_r post_R(P) ;; true_r = true_r$   
 shows  $P ;; Stop = Stop$

**proof** –

**from** *assms*(3) **have**  $1:(post_R P \wedge \$tr' =_u \$tr) ;; true_r = true_r$   
**by** (*rel-auto*, *metis* (*full-types*, *hide-lams*) *dual-order.antisym order-refl*)  
**show** *?thesis*  
**by** (*rdes-simp cls: assms*(1), *simp add: csp-enable-nothing assms*(2) *ICSP-Stop-right-zero-lemma*[*OF*  
1])  
**qed**

**lemma** *ICSP-intro*:  $\llbracket P \text{ is NCSP}; P \text{ is ISRD1} \rrbracket \implies P \text{ is ICSP}$   
**using** *Healthy-comp* **by** *blast*

**lemma** *seq-ICSP-closed* [*closure*]:  
**assumes**  $P \text{ is ICSP } Q \text{ is ICSP}$   
**shows**  $P ;; Q \text{ is ICSP}$   
**by** (*meson ICSP-implies-ISRD ICSP-implies-NCSP ICSP-intro ISRD-implies-ISRD1 NCSP-seqr-closure*  
*assms seq-ISRD-closed*)

**lemma** *Miracle-ICSP* [*closure*]: *Miracle is ICSP*  
**by** (*rule ICSP-intro*, *simp add: closure*, *simp add: ISRD1-rdes-intro rdes-def closure*)

## 5.2 CSP theories

**typeddecl** *TCSP*

**abbreviation**  $TCSP \equiv UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**overloading**

*tcsp-hcond* == *utp-hcond* ::  $(TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (('\sigma, '\varphi) \text{ st-csp} \times (''\sigma, '\varphi) \text{ st-csp}) \text{ health}$   
*tcsp-unit* == *utp-unit* ::  $(TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (''\sigma, '\varphi) \text{ action}$

**begin**

**definition** *tcsp-hcond* ::  $(TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (('\sigma, '\varphi) \text{ st-csp} \times (''\sigma, '\varphi) \text{ st-csp}) \text{ health}$  **where**  
[*upred-defs*]: *tcsp-hcond*  $T = NCSP$

**definition** *tcsp-unit* ::  $(TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (''\sigma, '\varphi) \text{ action}$  **where**  
[*upred-defs*]: *tcsp-unit*  $T = Skip$

**end**

**interpretation** *csp-theory*: *utp-theory-kleene*  $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**rewrites**  $\bigwedge P. P \in \text{carrier } (\text{uthy-order } TCSP) \longleftrightarrow P \text{ is NCSP}$

**and**  $P \text{ is } \mathcal{H}_{TCSP} \longleftrightarrow P \text{ is NCSP}$

**and**  $\mathcal{IL}_{TCSP} = Skip$

**and**  $\top_{TCSP} = Miracle$

**and**  $\text{carrier } (\text{uthy-order } TCSP) \rightarrow \text{carrier } (\text{uthy-order } TCSP) \equiv \llbracket NCSP \rrbracket_H \rightarrow \llbracket NCSP \rrbracket_H$

**and**  $A \subseteq \text{carrier } (\text{uthy-order } TCSP) \longleftrightarrow A \subseteq \llbracket NCSP \rrbracket_H$

**and**  $le (\text{uthy-order } TCSP) = op \sqsubseteq$

**proof** –

**interpret** *lat*: *utp-theory-continuous*  $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**by** (*unfold-locales*, *simp-all add: tcsp-hcond-def closure Healthy-if*)

**show**  $1: \top_{TCSP} = (Miracle :: ('\sigma, '\varphi) \text{ action})$

**by** (*metis NCSP-Miracle NCSP-implies-CSP lat.top-healthy lat.utp-theory-continuous-axioms srdes-theory-continuous.*  
*tcsp-hcond-def upred-semiring.add-commute utp-theory-continuous.meet-top*)

**thus** *utp-theory-kleene*  $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**by** (*unfold-locales*, *simp-all add: tcsp-hcond-def tcsp-unit-def Skip-left-unit Skip-right-unit closure*  
*Healthy-if Miracle-left-zero* )

**qed** (*simp-all add: tcsp-hcond-def tcsp-unit-def closure Healthy-if*)

**declare** *csp-theory.top-healthy* [*simp del*]  
**declare** *csp-theory.bottom-healthy* [*simp del*]

**abbreviation** *TestC* (*test<sub>C</sub>*) **where**  
*test<sub>C</sub>* *P*  $\equiv$  *utest TCSP P*

**abbreviation** *StarC* :: (*'σ*, *'φ*) *action*  $\Rightarrow$  (*'σ*, *'φ*) *action* (*-<sup>\*C</sup>* [999] 999) **where**  
*StarC P*  $\equiv$  *P★<sub>TCSP</sub>*

**lemma** *csp-bottom-Chaos*:  $\perp_{TCSP} = \text{Chaos}$   
**using** *NCSP-Chaos NCSP-implies-CSP* **by** *auto*

**lemma** *csp-top-Miracle*:  $\top_{TCSP} = \text{Miracle}$   
**by** (*simp add: csp-theory.healthy-top csp-theory.utp-theory-mono-axioms utp-theory-mono.healthy-top*)

### 5.3 Algebraic laws

**lemma** *Stop-left-zero*:  
**assumes** *P* *is CSP*  
**shows** *Stop* ;; *P* = *Stop*  
**by** (*simp add: NSRD-seq-post-false assms NCSP-implies-NSRD NCSP-Stop postR-Stop*)

**end**

## 6 Stateful-Failure Reactive Contracts

**theory** *utp-sfrd-contracts*  
**imports** *utp-sfrd-healths*  
**begin**

**definition** *mk-CRD* :: *'s upred*  $\Rightarrow$  (*'e list*  $\Rightarrow$  *'e set*  $\Rightarrow$  *'s upred*)  $\Rightarrow$  (*'e list*  $\Rightarrow$  *'s hrel*)  $\Rightarrow$  (*'s*, *'e*) *action*  
**where**  
 $[rdes-def]: mk-CRD\ P\ Q\ R = \mathbf{R}_s([P]_{S<} \vdash [Q\ x\ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket [r \rightarrow \$ref\ ']] \diamond [R(x)]_S' \llbracket x \rightarrow \&tt \rrbracket$

**syntax**  
*-ref-var* :: *logic*  
*-mk-CRD* :: *logic*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* (*[-/*  $\vdash$  *-/*  $|$  *-]*<sub>*C*</sub>)

**parse-translation**  $\ll$   
*let*  
*fun* *ref-var-tr* [] = *Syntax.free refs*  
*|* *ref-var-tr* - = *raise Match*;  
*in*  
 $[(\@ \{syntax-const\ -ref-var\}, K\ ref-var-tr)]$   
*end*  
 $\gg$

**translations**  
 $[P \vdash Q \mid R]_C \Rightarrow CONST\ mk-CRD\ P\ (\lambda\ -trace-var\ -ref-var.\ Q)\ (\lambda\ -trace-var.\ R)$   
 $[P \vdash Q \mid R]_C \Leftarrow CONST\ mk-CRD\ P\ (\lambda\ x\ r.\ Q)\ (\lambda\ y.\ R)$

**lemma** *CSP-mk-CRD [closure]*:  $[P \vdash Q\ trace\ refs \mid R(trace)]_C$  *is CSP*  
**by** (*simp add: mk-CRD-def closure unrest*)

**lemma** *preR-mk-CRD [rdes]*:  $pre_R([P \vdash Q\ trace\ refs \mid R(trace)]_C) = [P]_{S<}$



**by** (*simp add: mk-CRD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre rea-st-cond-def, rel-auto*)

**lemma** *periR-mk-CRD* [rdes]:  $\text{peri}_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([Q \text{ trace refs}]_{S<}))[(\text{trace}, \text{refs}) \rightarrow (\&tt, \$r)]$   
**by** (*simp add: mk-CRD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto*)

**lemma** *postR-mk-CRD* [rdes]:  $\text{post}_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([R(\text{trace})]_S'))[\text{trace} \rightarrow \&tt]$   
**by** (*simp add: mk-CRD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto*)

Refinement introduction law for contracts

**lemma** *CRD-contract-refine*:

**assumes**  
 $Q \text{ is CSP } '[P_1]_{S<} \Rightarrow \text{pre}_R Q'$   
 $'[P_1]_{S<} \wedge \text{peri}_R Q \Rightarrow [P_2 \ t \ r]_{S<}[\![t \rightarrow \&tt]\!][r \rightarrow \$ref']'$   
 $'[P_1]_{S<} \wedge \text{post}_R Q \Rightarrow [P_3 \ x]_S[\![x \rightarrow \&tt]\!]'$   
**shows**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$   
**proof** –  
**have**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$   
**using** *assms* **by** (*simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+*)  
**thus** *?thesis*  
**by** (*simp add: SRD-reactive-tri-design assms(1)*)  
**qed**

**lemma** *CRD-contract-refine'*:

**assumes**  
 $Q \text{ is CSP } '[P_1]_{S<} \Rightarrow \text{pre}_R Q'$   
 $[P_2 \ t \ r]_{S<}[\![t \rightarrow \&tt]\!][r \rightarrow \$ref'] \sqsubseteq ([P_1]_{S<} \wedge \text{peri}_R Q)$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge \text{post}_R Q)$   
**shows**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$   
**using** *assms* **by** (*rule-tac CRD-contract-refine, simp-all add: refBy-order*)

**lemma** *CRD-refine-CRD*:

**assumes**  
 $'[P_1]_{S<} \Rightarrow ([Q_1]_{S<} :: ('e, 's) \text{ action})'$   
 $([P_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!][r \rightarrow \$ref']) \sqsubseteq ([P_1]_{S<} \wedge [Q_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!][r \rightarrow \$ref']) :: ('e, 's) \text{ action}$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge [Q_3 \ x]_S[\![x \rightarrow \&tt]\!] :: ('e, 's) \text{ action})$   
**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq [Q_1 \vdash Q_2 \text{ trace refs} \mid Q_3 \text{ trace}]_C$   
**using** *assms*  
**by** (*simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+*)

**lemma** *CRD-refine-rdes*:

**assumes**  
 $'[P_1]_{S<} \Rightarrow Q_1'$   
 $([P_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!][r \rightarrow \$ref']) \sqsubseteq ([P_1]_{S<} \wedge Q_2)$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge Q_3)$   
**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
**using** *assms*  
**by** (*simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+*)

**lemma** *CRD-refine-rdes'*:

**assumes**  
 $Q_2 \text{ is RR}$

$Q_3$  is  $RR$   
 $\text{'}[P_1]_{S<} \Rightarrow Q_1 \text{'}$   
 $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2 \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
 $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge Q_3 \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq$   
 $\mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
**proof** (*simp add: mk-CRD-def, rule srdes-tri-refine-intro*)  
**show**  $\text{'}[P_1]_{S<} \Rightarrow Q_1 \text{'}$  **by** (*fact assms(3)*)  
  
**have**  $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_2) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $\text{'}[P_1]_{S<} \wedge RR(Q_2) \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket \text{'}$   
**by** (*rel-simp; meson*)  
**thus**  $\text{'}[P_1]_{S<} \wedge Q_2 \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket \text{'}$   
**by** (*simp add: Healthy-if assms*)  
  
**have**  $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_3) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $\text{'}[P_1]_{S<} \wedge (RR \ Q_3) \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket \text{'}$   
**by** (*rel-simp; meson*)  
**thus**  $\text{'}[P_1]_{S<} \wedge Q_3 \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket \text{'}$   
**by** (*simp add: Healthy-if assms*)  
**qed**  
**end**

## 7 External Choice

**theory** *utp-sfrd-extchoice*  
**imports**  
*utp-sfrd-healths*  
*utp-sfrd-rel*  
**begin**

### 7.1 Definitions and syntax

**definition** *ExtChoice* ::

$(\sigma, \varphi) \text{ action set} \Rightarrow (\sigma, \varphi) \text{ action}$  **where**  
 $[upred-defs]: \text{ExtChoice } A = \mathbf{R}_s((\bigsqcup P \in A \cdot pre_R(P)) \vdash ((\bigsqcup P \in A \cdot cmt_R(P)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigsqcup P \in A \cdot cmt_R(P))))$

**syntax**

$-ExtChoice :: pttrn \Rightarrow 'a \text{ set} \Rightarrow 'b \Rightarrow 'b \ ((\exists \square \ - \in \ - \cdot / \ -) [0, 0, 10] \ 10)$   
 $-ExtChoice-simp :: pttrn \Rightarrow 'b \Rightarrow 'b \ ((\exists \square \ - \cdot / \ -) [0, 10] \ 10)$

**translations**

$\square P \in A \cdot B \quad \equiv \text{CONST } ExtChoice \ ((\lambda P. B) \text{' } A)$   
 $\square P \cdot B \quad \equiv \text{CONST } ExtChoice \ (\text{CONST range } (\lambda P. B))$

**definition** *extChoice* ::

$(\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action}$  (**infixl**  $\square$  65) **where**  
 $[upred-defs]: P \square Q \equiv ExtChoice \ \{P, Q\}$

Small external choice as an indexed big external choice.

**lemma** *extChoice-alt-def*:

$P \sqcap Q = (\sqcap i :: \text{nat} \in \{0,1\} \cdot P \triangleleft \ll i = 0 \gg \triangleright Q)$   
 by (simp add: extChoice-def ExtChoice-def, unliteralise, simp)

## 7.2 Basic laws

## 7.3 Algebraic laws

**lemma** *ExtChoice-empty*:  $\text{ExtChoice } \{\} = \text{Stop}$   
 by (simp add: ExtChoice-def cond-def Stop-def)

**lemma** *ExtChoice-single*:  
 $P \text{ is CSP} \implies \text{ExtChoice } \{P\} = P$   
 by (simp add: ExtChoice-def usup-and uinf-or SRD-reactive-design-alt)

## 7.4 Reactive design calculations

**lemma** *ExtChoice-rdes*:  
 assumes  $\bigwedge i. \$ok' \nmid P(i) \ A \neq \{\}$   
 shows  $(\sqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\sqcap i \in A \cdot P(i)) \vdash ((\sqcap i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot Q(i))))$   
**proof** –  
 have  $(\sqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) =$   
 $\mathbf{R}_s((\sqcap i \in A \cdot \text{pre}_R(\mathbf{R}_s(P(i) \vdash Q(i)))) \vdash$   
 $((\sqcap i \in A \cdot \text{cmt}_R(\mathbf{R}_s(P(i) \vdash Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\sqcap i \in A \cdot \text{cmt}_R(\mathbf{R}_s(P(i) \vdash Q(i))))))$   
 by (simp add: ExtChoice-def)  
 also have ... =  
 $\mathbf{R}_s((\sqcap i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $((\sqcap i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\sqcap i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i)))))))$   
 by (simp add: rea-pre-RHS-design rea-cmt-RHS-design)  
 also have ... =  
 $\mathbf{R}_s((\sqcap i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $R1(R2c$   
 $((\sqcap i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\sqcap i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i)))))))$   
 by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)  
 also have ... =  
 $\mathbf{R}_s((\sqcap i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $R1(R2c$   
 $((\sqcap i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\sqcap i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$   
 by (simp add: R2c-UINF R2c-cond R1-cond R1-idem R1-R2c-commute R2c-idem R1-UINF asms R1-USUP R2c-USUP)  
 also have ... =  
 $\mathbf{R}_s((\sqcap i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $\text{cmt}_s \dagger$   
 $((\sqcap i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\sqcap i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$   
 by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c rdes-export-cmt)  
 also have ... =

$\mathbf{R}_s ((\sqcup i \in A \cdot R1 (R2c (pre_s \dagger P(i)))) \vdash$   
 $\quad cmt_s \dagger$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i)))$   
 $\quad \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $\quad (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (simp add: usubst)  
 also have ... =  
 $\mathbf{R}_s ((\sqcup i \in A \cdot R1 (R2c (pre_s \dagger P(i)))) \vdash$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (simp add: rdes-export-cmt)  
 also have ... =  
 $\mathbf{R}_s ((R1(R2c(\sqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (simp add: not-UIINF R1-UIINF R2c-UIINF assms)  
 also have ... =  
 $\mathbf{R}_s ((R2c(\sqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (simp add: R1-design-R1-pre)  
 also have ... =  
 $\mathbf{R}_s (((\sqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (metis (no-types, lifting) RHS-design-R2c-pre)  
 also have ... =  
 $\mathbf{R}_s ([\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger (\sqcup i \in A \cdot P(i))) \vdash$   
 $\quad ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 proof –  
 from assms have  $\bigwedge i. pre_s \dagger P(i) = [\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger P(i)$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: usubst)  
 qed  
 also have ... =  
 $\mathbf{R}_s ((\sqcup i \in A \cdot P(i)) \vdash ((\sqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot (P(i) \Rightarrow Q(i))))$   
 by (simp add: rdes-export-pre not-UIINF)  
 also have ... =  $\mathbf{R}_s ((\sqcup i \in A \cdot P(i)) \vdash ((\sqcup i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot Q(i))))$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto, blast+)  
  
 finally show ?thesis .  
 qed

**lemma** *ExtChoice-tri-rdes*:

assumes  $\bigwedge i. \$ok' \nmid P_1(i) \ A \neq \{\}$   
 shows  $(\sqcap i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\sqcap i \in A \cdot P_1(i)) \vdash (((\sqcap i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i))) \diamond (\sqcap i \in A \cdot P_3(i))))$   
 proof –  
 have  $(\sqcap i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\sqcap i \in A \cdot P_1(i)) \vdash ((\sqcap i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))))$   
 by (simp add: ExtChoice-rdes assms)  
 also  
 have ... =  
 $\mathbf{R}_s ((\sqcap i \in A \cdot P_1(i)) \vdash ((\sqcap i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))))$

by (simp add: conj-comm)  
 also  
 have ... =  
 $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))) \diamond (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))))$   
 by (simp add: cond-conj wait'-cond-def)  
 also  
 have ... =  $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i))) \diamond (\sqcap i \in A \cdot P_3(i))))$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 finally show ?thesis .  
 qed

**lemma** ExtChoice-tri-rdes' [rdes-def]:  
 assumes  $\bigwedge i . \$ok' \# P_1(i) \ A \neq \{\}$   
 shows  $(\sqcap i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot R5(P_2(i))) \vee (\sqcap i \in A \cdot R4(P_2(i)))) \diamond (\sqcap i \in A \cdot P_3(i))))$   
 by (simp add: ExtChoice-tri-rdes assms, rel-auto, simp-all add: less-le assms)

**lemma** ExtChoice-tri-rdes-def [rdes-def]:  
 assumes  $A \subseteq \llbracket CSP \rrbracket_H$   
 shows  $ExtChoice\ A = \mathbf{R}_s ((\sqcup P \in A \cdot pre_R\ P) \vdash (((\sqcup P \in A \cdot peri_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot peri_R\ P)) \diamond (\sqcap P \in A \cdot post_R\ P)))$   
**proof** –  
 have  $((\sqcup P \in A \cdot cmt_R\ P) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap P \in A \cdot cmt_R\ P)) =$   
 $((\sqcup P \in A \cdot cmt_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot cmt_R\ P)) \diamond (\sqcap P \in A \cdot cmt_R\ P)$   
 by (rel-auto)  
 also have ... =  $((\sqcup P \in A \cdot peri_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot peri_R\ P)) \diamond (\sqcap P \in A \cdot post_R\ P)$   
 by (rel-auto)  
 finally show ?thesis  
 by (simp add: ExtChoice-def)  
 qed

**lemma** extChoice-rdes:  
 assumes  $\$ok' \# P_1 \ \$ok' \# Q_1$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2) = \mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$   
**proof** –  
 have  $(\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s (P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright \mathbf{R}_s (Q_1 \vdash Q_2)) = (\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright (Q_1 \vdash Q_2)))$   
 by (simp only: RHS-cond R2c-lit)  
 also have ... =  $(\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \triangleleft \ll i = 0 \gg \triangleright Q_1) \vdash (P_2 \triangleleft \ll i = 0 \gg \triangleright Q_2)))$   
 by (simp add: design-condr)  
 also have ... =  $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$   
 apply (subst ExtChoice-rdes, simp-all add: assms unrest)  
 apply unliteralise  
 apply (simp add: uinf-or usup-and)  
 done  
 finally show ?thesis by (simp add: extChoice-alt-def)  
 qed

**lemma** extChoice-tri-rdes:  
 assumes  $\$ok' \# P_1 \ \$ok' \# Q_1$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$

**proof** –

**have**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$   
**by** (*simp add: extChoice-rdes assms*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$   
**by** (*simp add: conj-comm*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash$   
 $((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)) \diamond (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$   
**by** (*simp add: cond-conj wait'-cond-def*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**finally show** *?thesis* .

**qed**

**lemma** *extChoice-rdes-def*:

**assumes**  $P_1$  is *RR*  $Q_1$  is *RR*  
**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*subst extChoice-tri-rdes, simp-all add: assms unrest*)

**lemma** *extChoice-rdes-def' [rdes-def]*:

**assumes**  $P_1$  is *RR*  $Q_1$  is *RR*  
**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((R5(P_2 \wedge Q_2) \vee R4(P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*simp add: extChoice-rdes-def assms, rel-auto, simp-all add: less-le*)

**lemma** *CSP-ExtChoice [closure]*:

$ExtChoice A$  is *CSP*  
**by** (*simp add: ExtChoice-def RHS-design-is-SRD unrest*)

**lemma** *CSP-extChoice [closure]*:

$P \square Q$  is *CSP*  
**by** (*simp add: CSP-ExtChoice extChoice-def*)

**lemma** *preR-ExtChoice [rdes]*:

**assumes**  $A \neq \{\}$   $A \subseteq \llbracket CSP \rrbracket_H$   
**shows**  $pre_R(ExtChoice A) = (\bigsqcup P \in A \cdot pre_R(P))$

**proof** –

**have**  $pre_R(ExtChoice A) = (R1 (R2c ((\bigsqcup P \in A \cdot pre_R P))))$   
**by** (*simp add: ExtChoice-def rea-pre-RHS-design usubst unrest*)  
**also from** *assms* **have**  $\dots = (R1 (R2c (\bigsqcup P \in A \cdot (pre_R(CSP(P)))))$   
**by** (*metis USUP-healthy*)  
**also from** *assms* **have**  $\dots = (\bigsqcup P \in A \cdot (pre_R(CSP(P))))$   
**by** (*rel-auto*)  
**also from** *assms* **have**  $\dots = (\bigsqcup P \in A \cdot (pre_R(P)))$   
**by** (*metis USUP-healthy*)  
**finally show** *?thesis* .

**qed**

**lemma** *preR-ExtChoice-ind [rdes]*:

**assumes**  $A \neq \{\} \wedge P. P \in A \implies F(P)$  is *CSP*  
**shows**  $pre_R(\bigsqcup P \in A \cdot F(P)) = (\bigsqcup P \in A \cdot pre_R(F(P)))$

using *assms* by (*subst preR-ExtChoice*, *auto*)

**lemma** *periR-ExtChoice* [*rdes*]:

assumes  $A \subseteq \llbracket NCSP \rrbracket_H$   $A \neq \{\}$

shows  $peri_R(ExtChoice\ A) = ((\bigsqcup P \in A \cdot pre_R(P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R\ P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup P \in A \cdot peri_R\ P)$

**proof** –

have  $peri_R(ExtChoice\ A) = peri_R(\mathbf{R}_s((\bigsqcup P \in A \cdot pre_R\ P)) \vdash ((\bigsqcup P \in A \cdot peri_R\ P) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup P \in A \cdot peri_R\ P)) \diamond (\bigsqcup P \in A \cdot post_R\ P)))$

by (*simp add: ExtChoice-tri-rdes-def assms closure*)

also have  $\dots = peri_R(\mathbf{R}_s((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \vdash ((\bigsqcup P \in A \cdot peri_R(NCSP\ P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup P \in A \cdot peri_R(NCSP\ P)) \diamond (\bigsqcup P \in A \cdot post_R\ P))))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have  $\dots = R1\ (R2c\ ((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup P \in A \cdot peri_R(NCSP\ P))))$

**proof** –

have  $(\bigsqcup P \in A \cdot [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true] \dagger pre_R(NCSP\ P)) = (\bigsqcup P \in A \cdot pre_R(NCSP\ P))$

by (*rule USUP-cong, simp add: closure usubst unrest assms*)

thus ?thesis

by (*simp add: rea-peri-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)

**qed**

also have  $\dots = R1\ ((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup P \in A \cdot peri_R(NCSP\ P)))$

by (*simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-periR R2c-tr'-minus-tr R2c-USUP closure*)

also have  $\dots = (((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P))))$

by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure, rel-auto*)

also have  $\dots = (((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r peri_R(NCSP\ P))))$

by (*simp add: UINF-rea-impl[THEN sym]*)

also have  $\dots = (((\bigsqcup P \in A \cdot pre_R(NCSP\ P)) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R(NCSP\ P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcup P \in A \cdot peri_R(NCSP\ P))))$

by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have  $\dots = (((\bigsqcup P \in A \cdot pre_R\ P) \Rightarrow_r (\bigsqcup P \in A \cdot peri_R\ P)) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcup P \in A \cdot peri_R\ P))))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

finally show ?thesis .

**qed**

**lemma** *periR-ExtChoice'*:

assumes  $A \subseteq \llbracket NCSP \rrbracket_H$   $A \neq \{\}$

**shows**  $\text{peri}_R(\text{ExtChoice } A) = (R5((\bigsqcup P \in A \cdot \text{pre}_R(P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R P)) \vee (\prod P \in A \cdot R4(\text{peri}_R P)))$   
**using**  $\text{assms}(2)$   
**by** ( $\text{simp add: periR-ExtChoice assms}(1), \text{rel-auto}$ )

**lemma**  $\text{periR-ExtChoice-ind}$  [rdes]:

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{peri}_R(\bigsqcup P \in A \cdot F(P)) = ((\bigsqcup P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R (F P))) \triangleleft \$tr' =_u \$tr$   
 $\triangleright (\prod P \in A \cdot \text{peri}_R (F P))$   
**using**  $\text{assms}$  **by** ( $\text{subst periR-ExtChoice}, \text{auto simp add: closure unrest}$ )

**lemma**  $\text{periR-ExtChoice-ind'}$ :

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{peri}_R(\bigsqcup P \in A \cdot F(P)) = (R5((\bigsqcup P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R (F P))) \vee (\prod P \in A \cdot R4(\text{peri}_R (F P))))$   
**using**  $\text{assms}$  **by** ( $\text{subst periR-ExtChoice'}, \text{auto simp add: closure unrest}$ )

**lemma**  $\text{postR-ExtChoice}$  [rdes]:

**assumes**  $A \subseteq \llbracket \text{NCSP} \rrbracket_H A \neq \{\}$   
**shows**  $\text{post}_R(\text{ExtChoice } A) = (\prod P \in A \cdot \text{post}_R P)$

**proof** –

**have**  $\text{post}_R(\text{ExtChoice } A) = \text{post}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R P) \vdash$   
 $((\bigsqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot \text{peri}_R P)) \diamond$   
 $(\prod P \in A \cdot \text{post}_R P)))$   
**by** ( $\text{simp add: ExtChoice-tri-rdes-def closure assms}$ )

**also have**  $\dots = \text{post}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \vdash$   
 $((\bigsqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot \text{peri}_R P)) \diamond$   
 $(\prod P \in A \cdot \text{post}_R(\text{NCSP } P))))$   
**by** ( $\text{simp add: UINF-healthy[OF assms}(1), \text{THEN sym}] \text{USUP-healthy[OF assms}(1), \text{THEN sym}]$ )

**also have**  $\dots = R1(R2c((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P))))$

**proof** –

**have**  $(\bigsqcup P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{false}] \dagger \text{pre}_R(\text{NCSP } P))$   
 $= (\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P))$   
**by** ( $\text{rule USUP-cong}, \text{simp add: usubst closure unrest assms}$ )  
**thus**  $?thesis$   
**by** ( $\text{simp add: rea-post-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms}$ )

**qed**

**also have**  $\dots = R1((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P)))$

**by** ( $\text{simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-postR}$   
 $R2c-tr'-minus-tr R2c-USUP \text{ closure}$ )

**also from**  $\text{assms}(2)$  **have**  $\dots = ((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P)))$

**by** ( $\text{simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure}$ )

**also have**  $\dots = (\prod P \in A \cdot \text{pre}_R(\text{NCSP } P) \Rightarrow_r \text{post}_R(\text{NCSP } P))$

**by** ( $\text{simp add: UINF-rea-impl}$ )

**also have**  $\dots = (\prod P \in A \cdot \text{post}_R(\text{NCSP } P))$

**by** ( $\text{simp add: SRD-post-under-pre closure assms unrest}$ )

**finally show**  $?thesis$

**by** ( $\text{simp add: UINF-healthy[OF assms}(1), \text{THEN sym}] \text{USUP-healthy[OF assms}(1), \text{THEN sym}]$ )

**qed**

**lemma**  $\text{postR-ExtChoice-ind}$  [rdes]:

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{post}_R(\bigsqcup P \in A \cdot F(P)) = (\prod P \in A \cdot \text{post}_R(F(P)))$



**using** *assms* **by** (*subst postR-ExtChoice*, *auto simp add: closure unrest*)

**lemma** *preR-extChoice*:  
**assumes**  $P$  is CSP  $Q$  is CSP  $\$wait' \# pre_R(P) \$wait' \# pre_R(Q)$   
**shows**  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*simp add: extChoice-def preR-ExtChoice assms usup-and*)

**lemma** *preR-extChoice' [rdes]*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  
**shows**  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*simp add: preR-extChoice closure assms unrest*)

**lemma** *periR-extChoice [rdes]*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  
**shows**  $peri_R(P \sqcap Q) = ((pre_R(P) \wedge pre_R(Q) \Rightarrow_r peri_R(P) \wedge peri_R(Q)) \triangleleft \$tr' =_u \$tr \triangleright (peri_R(P) \vee peri_R(Q)))$   
**using** *assms*  
**by** (*simp add: extChoice-def, subst periR-ExtChoice, auto simp add: usup-and uinf-or*)

**lemma** *postR-extChoice [rdes]*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  
**shows**  $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$   
**using** *assms*  
**by** (*simp add: extChoice-def, subst postR-ExtChoice, auto simp add: usup-and uinf-or*)

**lemma** *ExtChoice-cong*:  
**assumes**  $\bigwedge P. P \in A \implies F(P) = G(P)$   
**shows**  $(\sqcap P \in A \cdot F(P)) = (\sqcap P \in A \cdot G(P))$   
**using** *assms image-cong* **by** *force*

**lemma** *ref-unrest-ExtChoice*:  
**assumes**  
 $\bigwedge P. P \in A \implies \$ref \# pre_R(P)$   
 $\bigwedge P. P \in A \implies \$ref \# cmt_R(P)$   
**shows**  $\$ref \# (ExtChoice A) \llbracket false / \$wait \rrbracket$   
**proof** –  
**have**  $\bigwedge P. P \in A \implies \$ref \# pre_R(P \llbracket 0 / \$tr \rrbracket)$   
**using** *assms* **by** (*rel-blast*)  
**with** *assms* **show** *?thesis*  
**by** (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)  
**qed**

**lemma** *CSP4-ExtChoice*:  
**assumes**  $A \subseteq \llbracket NCSP \rrbracket_H$   
**shows** *ExtChoice A* is CSP4  
**proof** (*cases A = {}*)  
**case** *True* **thus** *?thesis*  
**by** (*simp add: ExtChoice-empty Healthy-def CSP4-def, simp add: Skip-is-CSP Stop-left-zero*)  
**next**  
**case** *False*  
**have**  $1 : (\neg_r (\neg_r pre_R (ExtChoice A)) ;;_h R1 \text{ true}) = pre_R (ExtChoice A)$   
**proof** –  
**have**  $\bigwedge P. P \in A \implies (\neg_r pre_R(P)) ;; R1 \text{ true} = (\neg_r pre_R(P))$   
**by** (*simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-neg-pre-unit assms*)  
**thus** *?thesis*

```

apply (simp add: False preR-ExtChoice closure NCSP-set-unrest-pre-wait' assms not-UINF seq-UINF-distr
not-USUP)
apply (rule USUP-cong)
apply (simp add: rpred assms closure)
done
qed
have 2:  $\$st' \# \text{peri}_R (\text{ExtChoice } A)$ 
proof -
  have a:  $\bigwedge P. P \in A \implies \$st' \# \text{pre}_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-pre assms)
  have b:  $\bigwedge P. P \in A \implies \$st' \# \text{peri}_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-peri assms)
  from a b show ?thesis
  apply (subst periR-ExtChoice)
  apply (simp-all add: assms closure unrest CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
False)
  done
qed
have 3:  $\$ref' \# \text{post}_R (\text{ExtChoice } A)$ 
proof -
  have a:  $\bigwedge P. P \in A \implies \$ref' \# \text{pre}_R(P)$ 
    by (simp add: CSP4-ref'-unrest-pre CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  have b:  $\bigwedge P. P \in A \implies \$ref' \# \text{post}_R(P)$ 
    by (simp add: CSP4-ref'-unrest-post CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  from a b show ?thesis
  by (subst postR-ExtChoice, simp-all add: assms CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
unrest False)
qed
show ?thesis
  by (rule CSP4-tri-intro, simp-all add: 1 2 3 assms closure)
  (metis 1 R1-seqr-closure rea-not-R1 rea-not-not rea-true-R1)
qed

```

**lemma** *CSP4-extChoice* [closure]:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  
**shows**  $P \sqcap Q$  is CSP4  
**by** (simp add: extChoice-def, rule CSP4-ExtChoice, simp-all add: assms)

**lemma** *NCSP-ExtChoice* [closure]:  
**assumes**  $A \subseteq \llbracket \text{NCSP} \rrbracket_H$   
**shows** *ExtChoice*  $A$  is NCSP  
**proof** (cases  $A = \{\}$ )  
**case** True  
**then show** ?thesis **by** (simp add: ExtChoice-empty closure)  
**next**  
**case** False  
**show** ?thesis  
**proof** (rule NCSP-intro)  
**from** assms **have** cls:  $A \subseteq \llbracket \text{CSP} \rrbracket_H$   $A \subseteq \llbracket \text{CSP3} \rrbracket_H$   $A \subseteq \llbracket \text{CSP4} \rrbracket_H$   
**using** NCSP-implies-CSP NCSP-implies-CSP3 NCSP-implies-CSP4 **by** blast+  
**have** wu:  $\bigwedge P. P \in A \implies \$wait' \# \text{pre}_R(P)$   
**using** NCSP-implies-NSRD NSRD-wait'-unrest-pre assms **by** force  
**show** 1: *ExtChoice*  $A$  is CSP

by (metis (mono-tags) Ball-Collect CSP-ExtChoice NCSP-implies-CSP assms)  
 from cls show ExtChoice A is CSP3  
 by (rule-tac CSP3-SRD-intro, simp-all add: CSP-Healthy-subset-member CSP3-Healthy-subset-member  
 closure rdes unrest wu assms 1 False)  
 from cls show ExtChoice A is CSP4  
 by (simp add: CSP4-ExtChoice assms)  
 qed  
 qed

**lemma** ExtChoice-NCSP-closed [closure]:  
 assumes  $\bigwedge i. i \in I \implies P(i)$  is NCSP  
 shows  $(\bigwedge i \in I. P(i))$  is NCSP  
 by (simp add: NCSP-ExtChoice assms image-subset-iff)

**lemma** NCSP-extChoice [closure]:  
 assumes  $P$  is NCSP  $Q$  is NCSP  
 shows  $P \sqcap Q$  is NCSP  
 by (simp add: NCSP-ExtChoice assms extChoice-def)

## 7.5 Productivity and Guardedness

**lemma** Productive-ExtChoice [closure]:  
 assumes  $A \neq \{\}$   $A \subseteq \llbracket \text{NCSP} \rrbracket_H$   $A \subseteq \llbracket \text{Productive} \rrbracket_H$   
 shows ExtChoice A is Productive

**proof** –

have 1:  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$   
 using NCSP-implies-NSRD NSRD-wait'-unrest-pre assms(2) by blast  
 show ?thesis

**proof** (rule Productive-intro, simp-all add: assms closure rdes 1 unrest)

have  $((\bigwedge P \in A. pre_R P) \wedge (\bigwedge P \in A. post_R P)) =$   
 $((\bigwedge P \in A. pre_R P) \wedge (\bigwedge P \in A. (pre_R P \wedge post_R P)))$   
 by (rel-auto)

moreover have  $(\bigwedge P \in A. (pre_R P \wedge post_R P)) = (\bigwedge P \in A. ((pre_R P \wedge post_R P) \wedge \$tr <_u$   
 $\$tr'))$

by (rule UINF-cong, metis (no-types, lifting) 1 Ball-Collect NCSP-implies-CSP Productive-post-refines-tr-increase  
 assms utp-pred-laws.inf.absorb1)

ultimately show  $(\$tr' >_u \$tr) \sqsubseteq ((\bigwedge P \in A. pre_R P) \wedge (\bigwedge P \in A. post_R P))$   
 by (rel-auto)

qed  
 qed

**lemma** Productive-extChoice [closure]:  
 assumes  $P$  is NCSP  $Q$  is NCSP  $P$  is Productive  $Q$  is Productive  
 shows  $P \sqcap Q$  is Productive  
 by (simp add: extChoice-def Productive-ExtChoice assms)

**lemma** ExtChoice-Guarded [closure]:  
 assumes  $\bigwedge P. P \in A \implies \text{Guarded } P$   
 shows Guarded  $(\lambda X. \bigwedge P \in A. P(X))$

**proof** (rule GuardedI)

fix X n

have  $\bigwedge Y. ((\bigwedge P \in A. P Y) \wedge gvirt(n+1)) = ((\bigwedge P \in A. (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

**proof** –

fix Y

let ?lhs =  $((\bigwedge P \in A. P Y) \wedge gvirt(n+1))$  and ?rhs =  $((\bigwedge P \in A. (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

```

have a: ?lhs[false/$ok] = ?rhs[false/$ok]
  by (rel-auto)
have b: ?lhs[true/$ok][true/$wait] = ?rhs[true/$ok][true/$wait]
  by (rel-auto)
have c: ?lhs[true/$ok][false/$wait] = ?rhs[true/$ok][false/$wait]
  by (simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest,
rel-blast)
show ?lhs = ?rhs
  using a b c
  by (rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all)
qed
moreover have (( $\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))$ )  $\wedge \ gvirt(n+1)$ ) = (( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n))$ )  $\wedge \ gvirt(n+1)$ )  $\wedge \ gvirt(n+1)$ )
proof -
  have ( $\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))$ ) = ( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)) \ \wedge \ gvirt(n+1))$ )
  proof (rule ExtChoice-cong)
    fix P assume P  $\in A$ 
    thus ( $P \ X \ \wedge \ gvirt(n+1)$ ) = ( $P \ (X \ \wedge \ gvirt(n)) \ \wedge \ gvirt(n+1)$ )
      using Guarded-def assms by blast
  qed
  thus ?thesis by simp
qed
ultimately show (( $\Box P \in A \cdot P \ X$ )  $\wedge \ gvirt(n+1)$ ) = (( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)))$ )  $\wedge \ gvirt(n+1)$ )
  by simp
qed

```

```

lemma extChoice-Guarded [closure]:
  assumes Guarded P Guarded Q
  shows Guarded ( $\lambda X. P(X) \Box Q(X)$ )
proof -
  have Guarded ( $\lambda X. \Box F \in \{P, Q\} \cdot F(X)$ )
    by (rule ExtChoice-Guarded, auto simp add: assms)
  thus ?thesis
    by (simp add: extChoice-def)
qed

```

## 7.6 Algebraic laws

```

lemma extChoice-comm:
  P  $\Box$  Q = Q  $\Box$  P
  by (unfold extChoice-def, simp add: insert-commute)

```

```

lemma extChoice-idem:
  P is CSP  $\implies P \Box P = P$ 
  by (unfold extChoice-def, simp add: ExtChoice-single)

```

```

lemma extChoice-assoc:
  assumes P is CSP Q is CSP R is CSP
  shows P  $\Box$  Q  $\Box$  R = P  $\Box$  (Q  $\Box$  R)
proof -
  have P  $\Box$  Q  $\Box$  R =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \Box \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)) \Box \mathbf{R}_s(\text{pre}_R(R) \vdash \text{cmt}_R(R))$ 
    by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))
  also have ... =
     $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \wedge \text{cmt}_R R)$ 
       $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$ 

```

$((cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q) \vee cmt_R R))$   
 by (simp add: extChoice-rdes unrest)  
 also have ... =  
 $\mathbf{R}_s (((pre_R P \wedge pre_R Q) \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge cmt_R Q) \wedge cmt_R R)$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $((cmt_R P \vee cmt_R Q) \vee cmt_R R)))$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 also have ... =  
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(cmt_R P \vee (cmt_R Q \vee cmt_R R))))$   
 by (simp add: conj-assoc disj-assoc)  
 also have ... =  
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(cmt_R P \vee (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))))$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 also have ... =  $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap (\mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)) \sqcap \mathbf{R}_s(pre_R(R) \vdash cmt_R(R)))$   
 by (simp add: extChoice-rdes unrest)  
 also have ... =  $P \sqcap (Q \sqcap R)$   
 by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))  
 finally show ?thesis .  
 qed

lemma extChoice-Stop:

assumes  $Q$  is CSP  
 shows  $Stop \sqcap Q = Q$   
 using assms

proof –

have  $Stop \sqcap Q = \mathbf{R}_s (true \vdash (\$tr' =_u \$tr \wedge \$wait')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$   
 by (simp add: Stop-def SRD-reactive-design-alt assms)  
 also have ... =  $\mathbf{R}_s (pre_R Q \vdash (((\$tr' =_u \$tr \wedge \$wait') \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\$tr' =_u \$tr \wedge \$wait' \vee cmt_R Q)))$   
 by (simp add: extChoice-rdes unrest)  
 also have ... =  $\mathbf{R}_s (pre_R Q \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright cmt_R Q))$   
 by (metis (no-types, lifting) cond-def eq-upred-sym neg-conj-cancel1 utp-pred-laws.inf.left-idem)  
 also have ... =  $\mathbf{R}_s (pre_R Q \vdash cmt_R Q)$   
 by (simp add: cond-idem)  
 also have ... =  $Q$   
 by (simp add: SRD-reactive-design-alt assms)  
 finally show ?thesis .  
 qed

lemma extChoice-Chaos:

assumes  $Q$  is CSP  
 shows  $Chaos \sqcap Q = Chaos$

proof –

have  $Chaos \sqcap Q = \mathbf{R}_s (false \vdash true) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$   
 by (simp add: Chaos-def SRD-reactive-design-alt assms)  
 also have ... =  $\mathbf{R}_s (false \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright true))$   
 by (simp add: extChoice-rdes unrest)  
 also have ... =  $\mathbf{R}_s (false \vdash true)$

by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
also have ... = Chaos  
by (simp add: Chaos-def)  
finally show ?thesis .  
qed

lemma extChoice-Dist:

assumes  $P$  is CSP  $S \subseteq \llbracket \text{CSP} \rrbracket_H S \neq \{\}$   
shows  $P \sqcap (\bigsqcup S) = (\bigsqcup_{Q \in S} P \sqcap Q)$

proof –

let ?S1 =  $\text{pre}_R \text{ ` } S$  and ?S2 =  $\text{cmt}_R \text{ ` } S$   
have  $P \sqcap (\bigsqcup S) = P \sqcap (\bigsqcup_{Q \in S} \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)))$   
by (simp add: SRD-as-reactive-design[THEN sym] Healthy-SUPREMUM UINF-as-Sup-collect assms)  
also have ... =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \sqcap \mathbf{R}_s((\bigsqcup_{Q \in S} \text{pre}_R(Q)) \vdash (\bigsqcup_{Q \in S} \text{cmt}_R(Q)))$   
by (simp add: RHS-design-USUP SRD-reactive-design-alt assms)  
also have ... =  $\mathbf{R}_s((\text{pre}_R(P) \wedge (\bigsqcup_{Q \in S} \text{pre}_R(Q))) \vdash$   
 $((\text{cmt}_R(P) \wedge (\bigsqcup_{Q \in S} \text{cmt}_R(Q)))$   
 $\triangleleft \text{str}' =_u \text{str} \wedge \text{wait}' \triangleright$   
 $(\text{cmt}_R(P) \vee (\bigsqcup_{Q \in S} \text{cmt}_R(Q))))$   
by (simp add: extChoice-rdes unrest)  
also have ... =  $\mathbf{R}_s((\bigsqcup_{Q \in S} \text{pre}_R P \wedge \text{pre}_R Q) \vdash$   
 $(\bigsqcup_{Q \in S} (\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \text{str}' =_u \text{str} \wedge \text{wait}' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q)))$   
by (simp add: conj-USUP-dist conj-UINF-dist disj-UINF-dist cond-UINF-dist assms)  
also have ... =  $(\bigsqcup_{Q \in S} \mathbf{R}_s((\text{pre}_R P \wedge \text{pre}_R Q) \vdash$   
 $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \text{str}' =_u \text{str} \wedge \text{wait}' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q))))$   
by (simp add: assms RHS-design-USUP)  
also have ... =  $(\bigsqcup_{Q \in S} \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \sqcap \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)))$   
by (simp add: extChoice-rdes unrest)  
also have ... =  $(\bigsqcup_{Q \in S} P \sqcap \text{CSP}(Q))$   
by (simp add: UINF-as-Sup-collect, metis (no-types, lifting) Healthy-if SRD-as-reactive-design  
assms(1))  
also have ... =  $(\bigsqcup_{Q \in S} P \sqcap Q)$   
by (rule SUP-cong, simp-all add: Healthy-subset-member[OF assms(2)])  
finally show ?thesis .  
qed

lemma extChoice-dist:

assumes  $P$  is CSP  $Q$  is CSP  $R$  is CSP  
shows  $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap (P \sqcap R)$   
using assms extChoice-Dist[of  $P \{Q, R\}$ ] by simp

lemma ExtChoice-seq-distr:

assumes  $\bigwedge i. i \in A \implies P i$  is PCSP  $Q$  is NCSP  
shows  $(\bigsqcup_{i \in A} P i) ;; Q = (\bigsqcup_{i \in A} P i) ;; Q$

proof (cases  $A = \{\}$ )

case True

then show ?thesis

by (simp add: ExtChoice-empty NCSP-implies-CSP Stop-left-zero assms(2))

next

case False

show ?thesis

proof –

have  $1: (\bigsqcup_{i \in A} P i) = (\bigsqcup_{i \in A} (\mathbf{R}_s((\text{pre}_R(P i)) \vdash \text{peri}_R(P i) \diamond (R4(\text{post}_R(P i)))))$   
(is ?X = ?Y)

by (rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP

*Productive-form assms(1) comp-apply*  
**have**  $2: (\Box i \in A \cdot P \ i \ ; \ Q) = (\Box i \in A \cdot (\mathbf{R}_s ((pre_R (P \ i)) \vdash peri_R (P \ i) \diamond (R4(post_R (P \ i))))) \ ; \ Q)$   
**(is ?X = ?Y)**  
**by** (rule *ExtChoice-cong*, metis (no-types, hide-lams) *R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP*)  
*Productive-form assms(1) comp-apply*  
**show** ?thesis  
**by** (simp add: 1 2, rdes-eq cls: assms False cong: *ExtChoice-cong USUP-cong*)  
**qed**  
**qed**

**lemma** *extChoice-seq-distr*:  
**assumes** *P is PCSP Q is PCSP R is NCSP*  
**shows**  $(P \Box Q) \ ; \ R = (P \ ; \ R \Box Q \ ; \ R)$   
**by** (rdes-eq cls: assms)

**lemma** *extChoice-seq-distl*:  
**assumes** *P is ICSP Q is ICSP R is NCSP*  
**shows**  $P \ ; \ (Q \Box R) = (P \ ; \ Q \Box P \ ; \ R)$   
**by** (rdes-eq cls: assms)

**end**

## 8 Stateful-Failure Programs

**theory** *utp-sfrd-prog*  
**imports**  
*utp-sfrd-extchoice*  
**begin**

### 8.1 Conditionals

**lemma** *NCSP-cond-srea [closure]*:  
**assumes** *P is NCSP Q is NCSP*  
**shows**  $P \triangleleft b \triangleright_R Q \text{ is NCSP}$   
**by** (rule *NCSP-NSRD-intro*, simp-all add: closure rdes assms unrest)

### 8.2 Guarded commands

**lemma** *GuardedCommR-NCSP-closed [closure]*:  
**assumes** *P is NCSP*  
**shows**  $g \rightarrow_R P \text{ is NCSP}$   
**by** (simp add: gcmd-def closure assms)

### 8.3 Alternation

**lemma** *AlternateR-NCSP-closed [closure]*:  
**assumes**  $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$   
**shows**  $(if_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) \text{ is NCSP}$   
**proof** (cases  $A = \{\}$ )  
**case** *True*  
**then show** ?thesis  
**by** (simp add: assms)  
**next**  
**case** *False*  
**then show** ?thesis

by (simp add: AlternateR-def closure assms)  
qed

## 8.4 While Loops

**lemma** *NSRD-coerce-NCSP*:

*P is NSRD  $\implies$  Skip ;; P ;; Skip is NCSP*

by (metis (no-types, hide-lams) CSP3-Skip CSP3-def CSP4-def Healthy-def NCSP-Skip NCSP-implies-CSP  
NCSP-intro NSRD-is-SRD RA1 SRD-seqr-closure)

**definition** *WhileC* :: 's upred  $\Rightarrow$  ('s, 'e) action  $\Rightarrow$  ('s, 'e) action (while<sub>C</sub> - do - od) **where**  
while<sub>C</sub> b do P od = Skip ;; while<sub>R</sub> b do P od ;; Skip

**lemma** *WhileC-NCSP-closed* [closure]:

**assumes** *P is NCSP P is Productive*

**shows** while<sub>C</sub> b do P od is NCSP

by (simp add: WhileC-def NSRD-coerce-NCSP assms closure)

## 8.5 Assignment

**definition** *AssignsCSP* :: 'σ usubst  $\Rightarrow$  ('σ, 'φ) action ( $\langle \cdot \rangle_C$ ) **where**  
[upred-defs]: *AssignsCSP* σ = **R**<sub>s</sub>(true  $\vdash$  false  $\diamond$  (\$tr' =<sub>u</sub> \$tr  $\wedge$  [ $\langle \sigma \rangle_a$ ]<sub>S</sub>))

**syntax**

-assigns-csp :: svids  $\Rightarrow$  uexprs  $\Rightarrow$  logic ('(-) :=<sub>C</sub> '(-))

-assigns-csp :: svids  $\Rightarrow$  uexprs  $\Rightarrow$  logic (**infixr** :=<sub>C</sub> 90)

**translations**

-assigns-csp xs vs => **CONST** *AssignsCSP* (-mk-usubst (CONST id) xs vs)

-assigns-csp x v <= **CONST** *AssignsCSP* (CONST subst-upd (CONST id) x v)

-assigns-csp x v <= -assigns-csp (-spvar x) v

x, y :=<sub>C</sub> u, v <= **CONST** *AssignsCSP* (CONST subst-upd (CONST subst-upd (CONST id) (CONST svar x) u) (CONST svar y) v)

**lemma** *preR-AssignsCSP* [rdes]: pre<sub>R</sub>( $\langle \sigma \rangle_C$ ) = true<sub>r</sub>  
by (rel-auto)

**lemma** *periR-AssignsCSP* [rdes]: peri<sub>R</sub>( $\langle \sigma \rangle_C$ ) = false  
by (rel-auto)

**lemma** *postR-AssignsCSP* [rdes]: post<sub>R</sub>( $\langle \sigma \rangle_C$ ) = Φ(true, σ,  $\langle \cdot \rangle$ )  
by (rel-auto)

**lemma** *AssignsCSP-rdes-def* [rdes-def] :  $\langle \sigma \rangle_C$  = **R**<sub>s</sub>(true<sub>r</sub>  $\vdash$  false  $\diamond$  Φ(true, σ,  $\langle \cdot \rangle$ ))  
by (rel-auto)

**lemma** *AssignsCSP-CSP* [closure]:  $\langle \sigma \rangle_C$  is CSP  
by (simp add: AssignsCSP-def RHS-tri-design-is-SRD unrest)

**lemma** *AssignsCSP-CSP3* [closure]:  $\langle \sigma \rangle_C$  is CSP3  
by (rule CSP3-intro, simp add: closure, rel-auto)

**lemma** *AssignsCSP-CSP4* [closure]:  $\langle \sigma \rangle_C$  is CSP4  
by (rule CSP4-intro, simp add: closure, rel-auto+)

**lemma** *AssignsCSP-NCSP* [closure]:  $\langle \sigma \rangle_C$  is NCSP



by (simp add: AssignsCSP-CSP AssignsCSP-CSP3 AssignsCSP-CSP4 NCSP-intro)

**lemma** *AssignsCSP-ICSP* [closure]:  $\langle \sigma \rangle_C$  is ICSP  
 apply (rule ICSP-intro, simp add: closure, simp add: rdes-def)  
 apply (rule ISRD1-rdes-intro)  
 apply (simp-all add: closure)  
 apply (rel-auto)  
 done

## 8.6 Assignment with update

There are different collections that we would like to assign to, but they all have different types and perhaps more importantly different conditions on the update being well defined. For example, for a list well-definedness equates to the index being less than the length etc. Thus we here set up a polymorphic constant for CSP assignment updates that can be specialised to different types.

**definition** *AssignCSP-update* ::

$(f \Rightarrow k \text{ set}) \Rightarrow (f \Rightarrow k \Rightarrow v \Rightarrow f) \Rightarrow (f \Rightarrow \sigma) \Rightarrow$   
 $(k, \sigma) \text{ uexpr} \Rightarrow (v, \sigma) \text{ uexpr} \Rightarrow (\sigma, \varphi) \text{ action} \textbf{ where}$

[upred-defs, rdes-def]: *AssignCSP-update* domf updatef  $x \ k \ v =$

$\mathbf{R}_s([k \in_u \text{uop domf } (\&x)]_{S<} \vdash \text{false} \diamond \Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) \ k \ v], \langle \rangle))$

All different assignment updates have the same syntax; the type resolves which implementation to use.

**syntax**

-csp-assign-upd ::  $\text{svid} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (-[-] :=_C - [0, 0, 72] \ 72)$

**translations**

$x[k] :=_C v == \text{CONST } \text{AssignCSP-update } (\text{CONST } \text{uom}) (\text{CONST } \text{uupd}) \ x \ k \ v$

**lemma** *AssignCSP-update-CSP* [closure]:

*AssignCSP-update* domf updatef  $x \ k \ v$  is CSP

by (simp add: AssignCSP-update-def RHS-tri-design-is-SRD unrest)

**lemma** *preR-AssignCSP-update* [rdes]:

$\text{pre}_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \in_u \text{uop domf } (\&x)]_{S<}$

by (rel-auto)

**lemma** *periR-AssignCSP-update* [rdes]:

$\text{peri}_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \notin_u \text{uop domf } (\&x)]_{S<}$

by (rel-simp)

**lemma** *post-AssignCSP-update* [rdes]:

$\text{post}_R(\text{AssignCSP-update domf updatef } x \ k \ v) =$

$(\Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) \ k \ v], \langle \rangle) \triangleleft k \in_u \text{uop domf } (\&x) \triangleright_R R1(\text{true}))$

by (rel-auto)

**lemma** *AssignCSP-update-NCSP* [closure]:

$(\text{AssignCSP-update domf updatef } x \ k \ v)$  is NCSP

**proof** (rule NCSP-intro)

show  $(\text{AssignCSP-update domf updatef } x \ k \ v)$  is CSP

by (simp add: closure)

show  $(\text{AssignCSP-update domf updatef } x \ k \ v)$  is CSP3

by (rule CSP3-SRD-intro, simp-all add: csp-do-def closure rdes unrest)

**show** (*AssignCSP-update domf updatef x k v*) *is CSP4*  
**by** (*rule CSP4-tri-intro, simp-all add: csp-do-def closure rdes unrest, rel-auto*)  
**qed**

## 8.7 State abstraction

**lemma** *ref-unrest-abs-st* [*unrest*]:  
 $\$ref \# P \implies \$ref \# \langle P \rangle_S$   
 $\$ref' \# P \implies \$ref' \# \langle P \rangle_S$   
**by** (*rel-simp*)<sup>+</sup>

**lemma** *NCSP-state-srea* [*closure*]: *P is NCSP  $\implies$  state 'a  $\cdot$  P is NCSP*  
**apply** (*rule NCSP-NSRD-intro*)  
**apply** (*simp-all add: closure rdes*)  
**apply** (*simp-all add: state-srea-def unrest closure*)  
**done**

## 8.8 Assumptions

**definition** *AssumeCircus* ( $\{-\}_C$ ) **where**  
 $[rdes-def]: \{b\}_C = \mathbf{R}_s(\mathcal{I}(b, \langle \rangle) \vdash (false \diamond \Phi(true, id, \langle \rangle)))$

## 8.9 Guards

**definition** *GuardCSP* ::

$'\sigma \text{ cond} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action (infixr } \&_u \ 70) \text{ where}$   
 $[upred-defs]: g \&_u A = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R(A)) \vdash ((\lceil g \rceil_{S<} \wedge cmt_R(A)) \vee (\lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$

**lemma** *Guard-tri-design*:

$g \&_u P = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R P) \vdash (peri_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge post_R(P)))$   
**proof** –  
**have**  $(\lceil g \rceil_{S<} \wedge cmt_R P \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait') = (peri_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge post_R(P))$   
**by** (*rel-auto*)  
**thus** *?thesis* **by** (*simp add: GuardCSP-def*)  
**qed**

**lemma** *csp-do-cond-conj*:

**assumes** *P is CRR*  
**shows**  $(\lceil b \rceil_{S<} \wedge P) = \Phi(b, id, \langle \rangle) ;; P$   
**proof** –  
**have**  $(\lceil b \rceil_{S<} \wedge CRR(P)) = \Phi(b, id, \langle \rangle) ;; CRR(P)$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

**lemma** *Guard-rdes-def* [*rdes-def*]:

**assumes** *P is RR Q is CRR R is CRR*  
**shows**  $g \&_u \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{\}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$   
**(is ?lhs = ?rhs)**  
**proof** –

have ?lhs =  $\mathbf{R}_s (([g]_{S<} \Rightarrow_r P) \vdash ((P \Rightarrow_r Q) \triangleleft [g]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ([g]_{S<} \wedge (P \Rightarrow_r R)))$   
 by (simp add: Guard-tri-design rdes assms closure)  
 also have ... =  $\mathbf{R}_s ((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash (([g]_{S<} \wedge Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{\}_u)) \diamond ([g]_{S<} \wedge R))$   
 by (rel-auto)  
 also have ... =  $\mathbf{R}_s ((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{\}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$   
 by (simp add: assms(2) assms(3) csp-do-cond-conj)  
 finally show ?thesis .  
 qed

lemma Guard-rdes-def':

assumes  $\$ok' \# P$   
 shows  $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge Q \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 proof -  
 have  $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s([g]_{S<} \Rightarrow_r pre_R (\mathbf{R}_s(P \vdash Q))) \vdash ([g]_{S<} \wedge cmt_R (\mathbf{R}_s(P \vdash Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: GuardCSP-def)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q))) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: rea-pre-RHS-design rea-cmt-RHS-design)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c([g]_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q)))) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
 by (simp add: R1-R2c-commute R1-disj R1-extend-conj' R1-idem R2c-and R2c-disj R2c-idem)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger ([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: rdes-export-cmt)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: usubst)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: rdes-export-cmt)  
 also from assms have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r (pre_s \dagger P)) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (rel-auto)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r pre_s \dagger P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: rdes-export-pre)  
 also from assms have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (rel-auto)  
 also from assms have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (simp add: rdes-export-pre)  
 also have ... =  $\mathbf{R}_s([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge Q \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait')$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 finally show ?thesis .  
 qed

lemma CSP-Guard [closure]:  $b \&_u P$  is CSP

by (simp add: GuardCSP-def, rule RHS-design-is-SRD, simp-all add: unrest)

**lemma** *preR-Guard* [rdes]:  $P$  is CSP  $\implies \text{pre}_R(b \&_u P) = ([b]_{S<} \Rightarrow_r \text{pre}_R P)$   
 by (simp add: Guard-tri-design rea-pre-RHS-design usubst unrest R2c-preR R2c-lift-state-pre  
 R2c-rea-impl R1-rea-impl R1-preR Healthy-if, rel-auto)

**lemma** *periR-Guard* [rdes]:

assumes  $P$  is NCSP

shows  $\text{peri}_R(b \&_u P) = (\text{peri}_R P \triangleleft b \triangleright_R \mathcal{E}(\text{true}, \langle \rangle, \{\}_u))$

**proof** –

have  $\text{peri}_R(b \&_u P) = (([b]_{S<} \Rightarrow_r \text{pre}_R P) \Rightarrow_r (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)))$

by (simp add: assms Guard-tri-design rea-peri-RHS-design usubst unrest R1-rea-impl R2c-rea-not  
 R2c-rea-impl R2c-preR R2c-periR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr closure  
 Healthy-if R1-cond R1-tr'-eq-tr)

also have  $\dots = ((\text{pre}_R P \Rightarrow_r \text{peri}_R P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$

by (rel-auto)

also have  $\dots = (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$

by (simp add: SRD-peri-under-pre add: unrest closure assms)

**finally show** ?thesis

by rel-auto

**qed**

**lemma** *postR-Guard* [rdes]:

assumes  $P$  is NCSP

shows  $\text{post}_R(b \&_u P) = ([b]_{S<} \wedge \text{post}_R P)$

**proof** –

have  $\text{post}_R(b \&_u P) = (([b]_{S<} \Rightarrow_r \text{pre}_R P) \Rightarrow_r ([b]_{S<} \wedge \text{post}_R P))$

by (simp add: Guard-tri-design rea-post-RHS-design usubst unrest R2c-rea-not R2c-and R2c-rea-impl  
 R2c-preR R2c-postR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr R1-rea-impl R1-extend-conj'  
 R1-post-SRD closure assms)

also have  $\dots = ([b]_{S<} \wedge (\text{pre}_R P \Rightarrow_r \text{post}_R P))$

by (rel-auto)

also have  $\dots = ([b]_{S<} \wedge \text{post}_R P)$

by (simp add: SRD-post-under-pre add: unrest closure assms)

also have  $\dots = ([b]_{S<} \wedge \text{post}_R P)$

by (metis CSP-Guard R1-extend-conj R1-post-SRD calculation rea-st-cond-def)

**finally show** ?thesis .

**qed**

**lemma** *CSP3-Guard* [closure]:

assumes  $P$  is CSP  $P$  is CSP3

shows  $b \&_u P$  is CSP3

**proof** –

from assms have  $1:\$ref \# P \llbracket \text{false}/\$wait \rrbracket$

by (simp add: CSP-Guard CSP3-iff)

hence  $\$ref \# \text{pre}_R (P \llbracket 0/\$tr \rrbracket) \ \$ref \# \text{pre}_R P \ \$ref \# \text{cmt}_R P$

by (pred-blast)+

hence  $\$ref \# (b \&_u P) \llbracket \text{false}/\$wait \rrbracket$

by (simp add: CSP3-iff GuardCSP-def RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest  
 usubst)

**thus** ?thesis

by (metis CSP3-intro CSP-Guard)

**qed**

**lemma** *CSP4-Guard* [closure]:

**assumes**  $P$  is NCSP  
**shows**  $b \&_u P$  is CSP4  
**proof** (*rule CSP4-tri-intro*[*OF CSP-Guard*])  
**show**  $(\neg_r \text{pre}_R (b \&_u P)) \;; R1 \text{ true} = (\neg_r \text{pre}_R (b \&_u P))$   
**proof** –  
**have**  $a: (\neg_r \text{pre}_R P) \;; R1 \text{ true} = (\neg_r \text{pre}_R P)$   
**by** (*simp add: CSP4-neg-pre-unit assms closure*)  
**have**  $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \;; R1 \text{ true} = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P))$   
**proof** –  
**have**  $1: (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) = ([b]_{S<} \wedge (\neg_r \text{pre}_R P))$   
**by** (*rel-auto*)  
**also have**  $2: \dots = ([b]_{S<} \wedge ((\neg_r \text{pre}_R P) \;; R1 \text{ true}))$   
**by** (*simp add: a*)  
**also have**  $3: \dots = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \;; R1 \text{ true}$   
**by** (*rel-auto*)  
**finally show** *?thesis* ..  
**qed**  
**thus** *?thesis*  
**by** (*simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest*)  
**qed**  
**show**  $\$st' \# \text{peri}_R (b \&_u P)$   
**by** (*simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest*)  
**show**  $\$ref' \# \text{post}_R (b \&_u P)$   
**by** (*simp add: preR-Guard postR-Guard NSRD-CSP4-intro closure assms unrest*)  
**qed**

**lemma** *NCSP-Guard* [*closure*]:

**assumes**  $P$  is NCSP  
**shows**  $b \&_u P$  is NCSP  
**proof** –  
**have**  $P$  is CSP  
**using** *NCSP-implies-CSP assms* **by** *blast*  
**then show** *?thesis*  
**by** (*metis* (*no-types*) *CSP3-Guard CSP3-commutes-CSP4 CSP4-Guard CSP4-Idempotent CSP-Guard Healthy-Idempotent Healthy-def NCSP-def assms comp-apply*)  
**qed**

**lemma** *Productive-Guard* [*closure*]:

**assumes**  $P$  is CSP  $P$  is Productive  $\$wait' \# \text{pre}_R(P)$   
**shows**  $b \&_u P$  is Productive  
**proof** –  
**have**  $b \&_u P = b \&_u \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$   
**by** (*metis Healthy-def Productive-form assms(1) assms(2)*)  
**also have**  $\dots =$   
 $\mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash$   
 $((\text{pre}_R P \Rightarrow_r \text{peri}_R P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ([b]_{S<} \wedge (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u$   
 $\$tr)))$   
**by** (*simp add: Guard-tri-design rea-pre-RHS-design rea-peri-RHS-design rea-post-RHS-design unrest assms*)  
 $\text{usubst } R1\text{-preR } \text{Healthy-if } R1\text{-rea-impl } R1\text{-peri-SRD } R1\text{-extend-conj'} R2c\text{-preR } R2c\text{-not } R2c\text{-rea-impl}$   
 $R2c\text{-periR } R2c\text{-postR } R2c\text{-and } R2c\text{-tr-less-tr'} R1\text{-tr-less-tr'}$   
**also have**  $\dots = \mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (([b]_{S<} \wedge \text{post}_R P)$   
 $\wedge \$tr' >_u \$tr))$   
**by** (*rel-auto*)

also have ... = *Productive*(*b* &<sub>u</sub> *P*)  
 by (*simp add: Productive-def Guard-tri-design RHS-tri-design-par unrest*)  
 finally show ?thesis  
 by (*simp add: Healthy-def*)  
 qed

## 8.10 Basic events

**definition** *do<sub>u</sub>* ::

(*'φ, 'σ*) *uexpr*  $\Rightarrow$  (*'σ, 'φ*) *action* **where**  
 [*upred-defs*]: *do<sub>u</sub>* *e* = ((*\$tr'* =<sub>u</sub> *\$tr*  $\wedge$  [*e*]<sub>S<</sub>  $\notin_u$  *\$ref'*)  $\triangleleft$  *\$wait'*  $\triangleright$  (*\$tr'* =<sub>u</sub> *\$tr*  $\hat{\wedge}_u$  [*e*]<sub>S<</sub>)  $\wedge$  *\$st'* =<sub>u</sub> *\$st*)

**definition** *DoCSP* :: (*'φ, 'σ*) *uexpr*  $\Rightarrow$  (*'σ, 'φ*) *action* (*do<sub>C</sub>*) **where**

[*upred-defs*]: *DoCSP* *a* = **R<sub>s</sub>**(*true*  $\vdash$  *do<sub>u</sub>* *a*)

**lemma** *R1-DoAct*: *R1*(*do<sub>u</sub>*(*a*)) = *do<sub>u</sub>*(*a*)

by (*rel-auto*)

**lemma** *R2c-DoAct*: *R2c*(*do<sub>u</sub>*(*a*)) = *do<sub>u</sub>*(*a*)

by (*rel-auto*)

**lemma** *DoCSP-alt-def*: *do<sub>C</sub>*(*a*) = *R3h*(*CSP1*(*\$ok'*  $\wedge$  *do<sub>u</sub>*(*a*)))

**apply** (*simp add: DoCSP-def RHS-def design-def impl-alt-def R1-R3h-commute R2c-R3h-commute R2c-disj*)

*R2c-not R2c-ok R2c-ok' R2c-and R2c-DoAct R1-disj R1-extend-conj' R1-DoAct*)

**apply** (*rel-auto*)

done

**lemma** *DoAct-unrests* [*unrest*]:

*\$ok*  $\nmid$  *do<sub>u</sub>*(*a*) *\$wait*  $\nmid$  *do<sub>u</sub>*(*a*)

by (*pred-auto*)<sup>+</sup>

**lemma** *DoCSP-RHS-tri* [*rdes-def*]:

*do<sub>C</sub>*(*a*) = **R<sub>s</sub>**(*true<sub>r</sub>*  $\vdash$  ( $\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \diamond \Phi(\text{true}, \text{id}, \langle a \rangle)$ ))

by (*simp add: DoCSP-def do<sub>u</sub>-def wait'-cond-def, rel-auto*)

**lemma** *CSP-DoCSP* [*closure*]: *do<sub>C</sub>*(*a*) is *CSP*

by (*simp add: DoCSP-def do<sub>u</sub>-def RHS-design-is-SRD unrest*)

**lemma** *preR-DoCSP* [*rdes*]: *pre<sub>R</sub>*(*do<sub>C</sub>*(*a*)) = *true<sub>r</sub>*

by (*simp add: DoCSP-def rea-pre-RHS-design unrest usubst R2c-true*)

**lemma** *periR-DoCSP* [*rdes*]: *peri<sub>R</sub>*(*do<sub>C</sub>*(*a*)) =  $\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u)$

by (*rel-auto*)

**lemma** *postR-DoCSP* [*rdes*]: *post<sub>R</sub>*(*do<sub>C</sub>*(*a*)) =  $\Phi(\text{true}, \text{id}, \langle a \rangle)$

by (*rel-auto*)

**lemma** *CSP3-DoCSP* [*closure*]: *do<sub>C</sub>*(*a*) is *CSP3*

by (*rule CSP3-intro[OF CSP-DoCSP]*)

(*simp add: DoCSP-def do<sub>u</sub>-def RHS-def design-def R1-def R2c-def R2s-def R3h-def unrest usubst*)

**lemma** *CSP4-DoCSP* [*closure*]: *do<sub>C</sub>*(*a*) is *CSP4*

by (*rule CSP4-tri-intro[OF CSP-DoCSP], simp-all add: preR-DoCSP periR-DoCSP postR-DoCSP unrest*)

**lemma** *NCSP-DoCSP* [closure]:  $do_C(a)$  is NCSP  
 by (metis *CSP3-DoCSP CSP4-DoCSP CSP-DoCSP Healthy-def NCSP-def comp-apply*)

**lemma** *Productive-DoCSP* [closure]:  
 $(do_C a :: ('σ, 'ψ) \text{ action})$  is Productive

**proof** –

have  $((\Phi(true, id, \langle a \rangle) \wedge \$tr' >_u \$tr) :: ('σ, 'ψ) \text{ action})$   
 $= (\Phi(true, id, \langle a \rangle))$   
 by (rel-auto, simp add: *Prefix-Order.strict-prefixI'*)  
 hence  $Productive(do_C a) = do_C a$   
 by (simp add: *Productive-RHS-design-form DoCSP-RHS-tri unrest*)  
 thus ?thesis  
 by (simp add: *Healthy-def*)

qed

**lemma** *PCSP-DoCSP* [closure]:  
 $(do_C a :: ('σ, 'ψ) \text{ action})$  is PCSP  
 by (simp add: *Healthy-comp NCSP-DoCSP Productive-DoCSP*)

**lemma** *wp-rea-DoCSP-lemma*:

fixes  $P :: ('σ, 'φ) \text{ action}$   
 assumes  $\$ok \# P \$wait \# P$   
 shows  $(\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) ;; P = (\exists \$ref \cdot P[\$tr \hat{^}_u \langle [a]_{S<} \rangle / \$tr])$   
 using *assms*  
 by (rel-auto, meson)

**lemma** *wp-rea-DoCSP*:

assumes  $P$  is NCSP  
 shows  $(\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) \text{ wp}_r \text{ pre}_R P =$   
 $(\neg_r (\neg_r \text{pre}_R P)[\$tr \hat{^}_u \langle [a]_{S<} \rangle / \$tr])$   
 by (simp add: *wp-rea-def wp-rea-DoCSP-lemma unrest usubst ex-unrest assms closure*)

**lemma** *wp-rea-DoCSP-alt*:

assumes  $P$  is NCSP  
 shows  $(\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) \text{ wp}_r \text{ pre}_R P =$   
 $(\$tr' \geq_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \Rightarrow_r (\text{pre}_R P)[\$tr \hat{^}_u \langle [a]_{S<} \rangle / \$tr])$   
 by (simp add: *wp-rea-DoCSP assms rea-not-def R1-def usubst unrest, rel-auto*)

## 8.11 Event prefix

**definition** *PrefixCSP* ::

$('\varphi, '\sigma) \text{ uexpr} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action} (- \rightarrow_C - [81, 80] 80) \text{ where}$   
 $[upred-defs]: \text{PrefixCSP } a \text{ } P = (do_C(a) ;; \text{CSP}(P))$

**abbreviation** *OutputCSP*  $c \text{ } v \text{ } P \equiv \text{PrefixCSP } (c.v)_u \text{ } P$

**lemma** *CSP-PrefixCSP* [closure]: *PrefixCSP*  $a \text{ } P$  is CSP

by (simp add: *PrefixCSP-def closure*)

**lemma** *CSP3-PrefixCSP* [closure]:

*PrefixCSP*  $a \text{ } P$  is CSP3  
 by (metis (no-types, hide-lams) *CSP3-DoCSP CSP3-def Healthy-def PrefixCSP-def seqr-assoc*)

**lemma** *CSP4-PrefixCSP* [closure]:  
**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>  
**shows** PrefixCSP  $a$   $P$  is CSP<sub>4</sub>  
**by** (metis (no-types, hide-lams) CSP4-def Healthy-def PrefixCSP-def assms(1) assms(2) seqr-assoc)

**lemma** *NCSP-PrefixCSP* [closure]:  
**assumes**  $P$  is NCSP  
**shows** PrefixCSP  $a$   $P$  is NCSP  
**by** (metis (no-types, hide-lams) CSP3-PrefixCSP CSP3-commutes-CSP<sub>4</sub> CSP4-Idempotent CSP4-PrefixCSP  
CSP-PrefixCSP Healthy-Idempotent Healthy-def NCSP-def NCSP-implies-CSP assms comp-apply)

**lemma** *Productive-PrefixCSP* [closure]:  $P$  is NCSP  $\implies$  PrefixCSP  $a$   $P$  is Productive  
**by** (simp add: Healthy-if NCSP-DoCSP NCSP-implies-NSRD NSRD-is-SRD PrefixCSP-def Productive-DoCSP  
Productive-seq-1)

**lemma** *PCSP-PrefixCSP* [closure]:  $P$  is NCSP  $\implies$  PrefixCSP  $a$   $P$  is PCSP  
**by** (simp add: Healthy-comp NCSP-PrefixCSP Productive-PrefixCSP)

**lemma** *PrefixCSP-Guarded* [closure]: Guarded (PrefixCSP  $a$ )  
**proof** –  
**have** PrefixCSP  $a = (\lambda X. do_C(a) ;; CSP(X))$   
**by** (simp add: fun-eq-iff PrefixCSP-def)  
**thus** ?thesis  
**using** Guarded-if-Productive NCSP-DoCSP NCSP-implies-NSRD Productive-DoCSP **by** auto  
**qed**

**lemma** *PrefixCSP-type* [closure]: PrefixCSP  $a \in \llbracket H \rrbracket_H \rightarrow \llbracket CSP \rrbracket_H$   
**using** CSP-PrefixCSP **by** blast

**lemma** *PrefixCSP-Continuous* [closure]: Continuous (PrefixCSP  $a$ )  
**by** (simp add: Continuous-def PrefixCSP-def ContinuousD[OF SRD-Continuous] seq-Sup-distl)

**lemma** *PrefixCSP-RHS-tri-lemma1*:  
 $R1 \ (R2s \ (\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge [II]_R)) = (\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge [II]_R)$   
**by** (rel-auto)

**lemma** *PrefixCSP-RHS-tri-lemma2*:  
**fixes**  $P :: ('\sigma, '\varphi)$  action  
**assumes**  $\$ok \# P$   $\$wait \# P$   
**shows**  $((\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) \wedge \neg \$wait') ;; P = (\exists \$ref \cdot P[\$tr \hat{^}_u \langle [a]_{S<} \rangle / \$tr])$   
**using** assms  
**by** (rel-auto, meson, fastforce)

**lemma** *tr-extend-seqr*:  
**fixes**  $P :: ('\sigma, '\varphi)$  action  
**assumes**  $\$ok \# P$   $\$wait \# P$   $\$ref \# P$   
**shows**  $(\$tr' =_u \$tr \hat{^}_u \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) ;; P = P[\$tr \hat{^}_u \langle [a]_{S<} \rangle / \$tr]$   
**using** assms **by** (simp add: wp-rea-DoCSP-lemma assms unrest ex-unrest)

**lemma** *trace-ext-R1-closed* [closure]:  $P$  is R1  $\implies P[\$tr \hat{^}_u e / \$tr]$  is R1  
**by** (rel-blast)

**lemma** *preR-PrefixCSP-NCSP* [rdes]:  
**assumes**  $P$  is NCSP  
**shows**  $pre_R(PrefixCSP \ a \ P) = (\mathcal{I}(true, \langle a \rangle) \Rightarrow_r (pre_R \ P) \llbracket \langle a \rangle \rrbracket_t)$



by (simp add: PrefixCSP-def assms closure rdes rpred Healthy-if wp usubst unrest)

**lemma** *periR-PrefixCSP* [rdes]:

assumes *P* is NCSP

shows  $\text{peri}_R(\text{PrefixCSP } a \ P) = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee (\text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**proof** –

have  $\text{peri}_R(\text{PrefixCSP } a \ P) = \text{peri}_R(\text{do}_C \ a \ ; \ P)$

by (simp add: PrefixCSP-def closure assms Healthy-if)

also have  $\dots = ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r \$tr' =_u \$tr \wedge \lceil a \rceil_{S<} \notin_u \$ref' \vee \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t)$

by (simp add: assms NSRD-CSP<sub>4</sub>-intro csp-enable-tr-empty closure rdes unrest ex-unrest usubst rpred wp)

also have  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t))$

by (rel-auto)

also have  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee ((\text{pre}_R(P) \Rightarrow_r \text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t))$

by (rel-auto)

also have  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee (\text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

by (simp add: SRD-peri-under-pre assms closure unrest)

finally show ?thesis .

qed

**lemma** *postR-PrefixCSP* [rdes]:

assumes *P* is NCSP

shows  $\text{post}_R(\text{PrefixCSP } a \ P) = (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t$

**proof** –

have  $\text{post}_R(\text{PrefixCSP } a \ P) = ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r (\text{pre}_R \ P) \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

by (simp add: PrefixCSP-def assms Healthy-if)

(simp add: assms Healthy-if wp closure rdes rpred wp-rea-DoCSP-lemma unrest ex-unrest usubst)

also have  $\dots = (\mathcal{I}(\text{true}, \langle a \rangle) \wedge (\text{pre}_R \ P \Rightarrow_r \text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

by (rel-auto)

also have  $\dots = (\mathcal{I}(\text{true}, \langle a \rangle) \wedge (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

by (simp add: SRD-post-under-pre assms closure unrest)

also have  $\dots = (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t$

by (rel-auto)

finally show ?thesis .

qed

**lemma** *PrefixCSP-RHS-tri*:

assumes *P* is NCSP

shows  $\text{PrefixCSP } a \ P = \mathbf{R}_s((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t) \diamond \text{post}_R \ P \llbracket \langle a \rangle \rrbracket_t)$

by (simp add: PrefixCSP-def Healthy-if unrest assms closure NSRD-composition-wp rdes rpred usubst wp)

For prefix, we can chose whether to propagate the assumptions or not, hence there are two laws.

**lemma** *PrefixCSP-rdes-def-1* [rdes-def]:

assumes *P* is CRC *Q* is CRR *R* is CRR

$\$st' \# Q \ \$ref' \# R$

shows  $\text{PrefixCSP } a \ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee Q \llbracket \langle a \rangle \rrbracket_t) \diamond R \llbracket \langle a \rangle \rrbracket_t)$

apply (subst PrefixCSP-RHS-tri)

apply (rule NCSP-rdes-intro)

apply (simp-all add: assms rdes closure)

apply (rel-auto)

done

**lemma** *PrefixCSP-rdes-def-2*:

**assumes**  $P$  is CRC  $Q$  is CRR  $R$  is CRR

$\$st' \# Q \$ref' \# R$

**shows**  $PrefixCSP\ a\ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(\mathcal{I}(true, \langle a \rangle) \Rightarrow_r P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee (P \wedge Q) \llbracket \langle a \rangle \rrbracket_t) \diamond (P \wedge R) \llbracket \langle a \rangle \rrbracket_t$

**apply** (*subst PrefixCSP-RHS-tri*)

**apply** (*rule NCSP-rdes-intro*)

**apply** (*simp-all add: assms rdes closure*)

**apply** (*rel-auto*)

**done**

## 8.12 Guarded external choice

**abbreviation** *GuardedChoiceCSP* ::  $'\vartheta$  set  $\Rightarrow (' \vartheta \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow (' \sigma, ' \vartheta)$  action **where**  
*GuardedChoiceCSP*  $A\ P \equiv (\Box x \in A \cdot PrefixCSP \llbracket x \rrbracket (P(x)))$

**syntax**

*-GuardedChoiceCSP* ::  $logic \Rightarrow logic \Rightarrow logic \Rightarrow logic\ (\Box - \in - \rightarrow - [0,0,85] 86)$

**translations**

$\Box x \in A \rightarrow P == CONST\ GuardedChoiceCSP\ A\ (\lambda x. P)$

**lemma** *GuardedChoiceCSP [rdes-def]*:

**assumes**  $\bigwedge x. P(x)$  is NCSP  $A \neq \{\}$

**shows**  $(\Box x \in A \rightarrow P(x)) =$

$\mathbf{R}_s((\bigcup x \in A \cdot \mathcal{I}(true, \langle \llbracket x \rrbracket \rangle)) \Rightarrow_r pre_R(P\ x) \llbracket \langle \llbracket x \rrbracket \rangle \rrbracket_t) \vdash$   
 $((\bigcup x \in A \cdot \mathcal{E}(true, \langle \rangle, \{\llbracket x \rrbracket\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\bigcap x \in A \cdot peri_R(P\ x) \llbracket \langle \llbracket x \rrbracket \rangle \rrbracket_t)) \diamond$   
 $(\bigcap x \in A \cdot post_R(P\ x) \llbracket \langle \llbracket x \rrbracket \rangle \rrbracket_t))$

**by** (*simp add: PrefixCSP-RHS-tri assms ExtChoice-tri-rdes closure unrest, rel-auto*)

## 8.13 Input prefix

**definition** *InputCSP* ::

$('a, ' \vartheta)$  chan  $\Rightarrow ('a \Rightarrow ' \sigma\ upred) \Rightarrow ('a \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow (' \sigma, ' \vartheta)$  action **where**

[*upred-defs*]: *InputCSP*  $c\ A\ P = (\Box x \in UNIV \cdot A(x) \ \&_u\ PrefixCSP\ (c \cdot \llbracket x \rrbracket)_u\ (P\ x))$

**definition** *InputVarCSP* ::  $('a, ' \vartheta)$  chan  $\Rightarrow ('a \Longrightarrow ' \sigma) \Rightarrow ('a \Rightarrow ' \sigma\ upred) \Rightarrow (' \sigma, ' \vartheta)$  action  $\Rightarrow (' \sigma, ' \vartheta)$  action **where**

[*upred-defs, rdes-def*]: *InputVarCSP*  $c\ x\ A\ P = InputCSP\ c\ A\ (\lambda v. \langle [x \mapsto_s \llbracket v \rrbracket] \rangle_C) ;; P$

**definition** *do<sub>I</sub>* ::

$('a, ' \vartheta)$  chan  $\Rightarrow$

$('a \Longrightarrow (' \sigma, ' \vartheta)$  st-csp)  $\Rightarrow$

$('a \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow$

$(' \sigma, ' \vartheta)$  action **where**

*do<sub>I</sub>*  $c\ x\ P =$  ( $\$tr' =_u \$tr \wedge \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot (c \cdot \llbracket e \rrbracket)_u\}_u \cap_u \$ref' =_u \{\}_u$

$\triangleleft \$wait' \triangleright$

$((\$tr' - \$tr) \in_u \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot \langle (c \cdot \llbracket e \rrbracket)_u \rangle_u \wedge (c \cdot \$x')_u =_u last_u(\$tr'))$ )

**lemma** *InputCSP-CSP [closure]*: *InputCSP*  $c\ A\ P$  is CSP

**by** (*simp add: CSP-ExtChoice InputCSP-def*)

**lemma** *InputCSP-NCSP [closure]*:  $\llbracket \bigwedge v. P(v) \text{ is NCSP} \rrbracket \Longrightarrow InputCSP\ c\ A\ P \text{ is NCSP}$

**apply** (*simp add: InputCSP-def*)

**apply** (*rule NCSP-ExtChoice*)

**apply** (*simp add: NCSP-Guard NCSP-PrefixCSP image-Collect-subsetI top-set-def*)  
**done**

**lemma** *Productive-InputCSP* [closure]:

$\llbracket \bigwedge v. P(v) \text{ is NCSP} \rrbracket \implies \text{InputCSP } x \ A \ P \text{ is Productive}$

**by** (*auto simp add: InputCSP-def unrest closure intro: Productive-ExtChoice*)

**lemma** *preR-InputCSP* [rdes]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{pre}_R(\text{InputCSP } a \ A \ P) = (\bigsqcup v \cdot [A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r (\text{pre}_R(P(v))) \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t)$

**by** (*simp add: InputCSP-def rdes closure assms alpha usubst unrest*)

**lemma** *periR-InputCSP* [rdes]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{peri}_R(\text{InputCSP } a \ A \ P) =$

$(\bigsqcup x \cdot [A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u))$   
 $\triangleleft \$tr' =_u \$tr \triangleright$

$(\bigsqcap x \cdot [A(x)]_{S<} \wedge (\text{peri}_R(P \ x)) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

**by** (*simp add: InputCSP-def rdes closure assms, rel-auto*)

**lemma** *postR-InputCSP* [rdes]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{post}_R(\text{InputCSP } a \ A \ P) =$

$(\bigsqcap x \cdot [A \ x]_{S<} \wedge \text{post}_R(P \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

**using** *assms* **by** (*simp add: InputCSP-def rdes closure assms usubst unrest*)

**lemma** *InputCSP-rdes-def* [rdes-def]:

**assumes**  $\bigwedge v. P(v) \text{ is CRC} \ \bigwedge v. Q(v) \text{ is CRR} \ \bigwedge v. R(v) \text{ is CRR}$

$\bigwedge v. \$st' \nmid Q(v) \ \bigwedge v. \$ref' \nmid R(v)$

**shows**  $\text{InputCSP } a \ A \ (\lambda v. \mathbf{R}_s(P(v) \vdash Q(v) \diamond R(v))) =$

$\mathbf{R}_s(\bigsqcup v \cdot ([A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r (P \ v) \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t))$   
 $\vdash (((\bigsqcup x \cdot R5([A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)))$

$\vee$

$(\bigsqcap x \cdot [A(x)]_{S<} \wedge (P \ x \wedge Q \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t))$

$\diamond (\bigsqcap x \cdot [A \ x]_{S<} \wedge (P \ x \wedge R \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t))$  (**is** *?lhs* = *?rhs*)

**proof** –

**have** *1:pre<sub>R</sub>(?lhs)* =  $(\bigsqcup v \cdot [A \ v]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r P \ v \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t)$  (**is** - = *?A*)

**by** (*simp add: rdes NCSP-rdes-intro assms conj-comm closure*)

**have** *2:peri<sub>R</sub>(?lhs)* =  $(\bigsqcup x \cdot [A \ x]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap x \cdot [A \ x]_{S<}$

$\wedge (P \ x \Rightarrow_r Q \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

(**is** - = *?B*)

**by** (*simp add: rdes NCSP-rdes-intro assms closure*)

**have** *3:post<sub>R</sub>(?lhs)* =  $(\bigsqcap x \cdot [A \ x]_{S<} \wedge (P \ x \Rightarrow_r R \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

(**is** - = *?C*)

**by** (*simp add: rdes NCSP-rdes-intro assms closure*)

**have** *?lhs* =  $\mathbf{R}_s(?A \vdash ?B \diamond ?C)$

**by** (*subst SRD-reactive-tri-design[THEN sym], simp-all add: closure 1 2 3*)

**also have** ... = *?rhs*

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

## 8.14 Algebraic laws

**lemma** *AssignCSP-conditional*:

**assumes** *vwb-lens* *x*

**shows**  $x :=_C e \triangleleft b \triangleright_R x :=_C f = x :=_C (e \triangleleft b \triangleright f)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssignsCSP-id*:  $\langle id \rangle_C = Skip$   
**by** (*rel-auto*)

**lemma** *Guard-comp*:  
 $g \&_u h \&_u P = (g \wedge h) \&_u P$   
**by** (*rule antisym, rel-blast, rel-blast*)

**lemma** *Guard-false* [*simp*]:  $false \&_u P = Stop$   
**by** (*simp add: GuardCSP-def Stop-def rpred closure alpha R1-design-R1-pre*)

**lemma** *Guard-true* [*simp*]:  
 $P \text{ is CSP} \implies true \&_u P = P$   
**by** (*simp add: GuardCSP-def alpha SRD-reactive-design-alt closure rpred*)

**lemma** *Guard-conditional*:  
**assumes**  $P \text{ is NCSP}$   
**shows**  $b \&_u P = P \triangleleft b \triangleright_R Stop$   
**by** (*rdes-eq cls: assms*)

**lemma** *Guard-expansion*:  
 $(g_1 \vee g_2) \&_u P = (g_1 \&_u P) \sqcap (g_2 \&_u P)$   
**by** (*rel-auto*)

**lemma** *Conditional-as-Guard*:  
**assumes**  $P \text{ is NCSP } Q \text{ is NCSP}$   
**shows**  $P \triangleleft b \triangleright_R Q = b \&_u P \sqcap (\neg b) \&_u Q$   
**by** (*rdes-eq cls: assms; simp add: le-less*)

**lemma** *PrefixCSP-dist*:  
 $PrefixCSP\ a\ (P \sqcap Q) = (PrefixCSP\ a\ P) \sqcap (PrefixCSP\ a\ Q)$   
**using** *Continuous-Disjunctous Disjunctuous-def PrefixCSP-Continuous* **by** *auto*

**lemma** *DoCSP-is-Prefix*:  
 $do_C(a) = PrefixCSP\ a\ Skip$   
**by** (*simp add: PrefixCSP-def Healthy-if closure, metis CSP4-DoCSP CSP4-def Healthy-def*)

**lemma** *PrefixCSP-seq*:  
**assumes**  $P \text{ is CSP } Q \text{ is CSP}$   
**shows**  $(PrefixCSP\ a\ P) ;; Q = (PrefixCSP\ a\ (P ;; Q))$   
**by** (*simp add: PrefixCSP-def seqr-assoc Healthy-if assms closure*)

**lemma** *PrefixCSP-extChoice-dist*:  
**assumes**  $P \text{ is NCSP } Q \text{ is NCSP } R \text{ is NCSP}$   
**shows**  $((a \rightarrow_C P) \sqcap (b \rightarrow_C Q)) ;; R = (a \rightarrow_C P ;; R) \sqcap (b \rightarrow_C Q ;; R)$   
**by** (*simp add: PCSP-PrefixCSP assms(1) assms(2) assms(3) extChoice-seq-distr*)

**lemma** *GuardedChoiceCSP-dist*:  
**assumes**  $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$   
**shows**  $\sqcap x \in A \rightarrow P(x) ;; Q = \sqcap x \in A \rightarrow (P(x) ;; Q)$   
**by** (*simp add: ExtChoice-seq-distr PrefixCSP-seq closure assms cong: ExtChoice-cong*)

Alternation can be re-expressed as an external choice when the guards are disjoint

```

declare ExtChoice-tri-rdes [rdes-def]
declare ExtChoice-tri-rdes' [rdes-def del]

declare extChoice-rdes-def [rdes-def]
declare extChoice-rdes-def' [rdes-def del]

lemma AlternateR-as-ExtChoice:
  assumes
     $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$ 
     $\bigwedge i j. \llbracket i \in A; j \in A; i \neq j \rrbracket \implies (g i \wedge g j) = \text{false}$ 
  shows  $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) =$ 
     $(\Box i \in A \cdot g(i) \ \&_u \ P(i)) \Box (\bigwedge i \in A \cdot \neg g(i) \ \&_u \ Q)$ 
proof (cases  $A = \{\}$ )
  case True
    then show ?thesis by (simp add: ExtChoice-empty extChoice-Stop closure assms)
  next
    case False
    show ?thesis

  proof –
    have  $1: (\Box i \in A \cdot g i \rightarrow_R P i) = (\Box i \in A \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$ 
      by (rule UINF-cong, simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1))
    have  $2: (\Box i \in A \cdot g(i) \ \&_u \ P(i)) = (\Box i \in A \cdot g(i) \ \&_u \ \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$ 
      by (rule ExtChoice-cong, simp add: NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-tri-design
assms(1))
    from assms(3) show ?thesis
    by (simp add: AlternateR-def 1 2)
    (rdes-eq cls: assms(1-2)_simps: False cong: UINF-cong ExtChoice-cong)
  qed
qed

declare ExtChoice-tri-rdes [rdes-def del]
declare ExtChoice-tri-rdes' [rdes-def]

declare extChoice-rdes-def [rdes-def del]
declare extChoice-rdes-def' [rdes-def]

end

```

## 9 Recursion in Stateful-Failures

```

theory utp-sfrd-recursion
  imports utp-sfrd-contracts utp-sfrd-prog
begin

```

### 9.1 Fixed-points

The CSP weakest fixed-point is obtained simply by precomposing the body with the CSP healthiness condition.

**abbreviation** *mu-CSP* ::  $((\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action}) \Rightarrow (\sigma, \varphi) \text{ action } (\mu_C)$  **where**  
 $\mu_C F \equiv \mu (F \circ \text{CSP})$

**syntax**

$\text{-mu-CSP} :: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\mu_C \cdot - \cdot - [0, 10] \ 10)$

**translations**

$\mu_C \ X \cdot P == \text{CONST mu-CSP} \ (\lambda \ X. P)$

**lemma** *mu-CSP-equiv*:

**assumes** *Monotonic F*  $F \in \llbracket \text{CSP} \rrbracket_H \rightarrow \llbracket \text{CSP} \rrbracket_H$

**shows**  $(\mu_R F) = (\mu_C F)$

**by** (*simp add: srd-mu-equiv assms comp-def*)

**lemma** *mu-CSP-unfold*:

$P \text{ is CSP} \implies (\mu_C \ X \cdot P ;; X) = P ;; (\mu_C \ X \cdot P ;; X)$

**apply** (*subst gfp-unfold*)

**apply** (*simp-all add: closure Healthy-if*)

**done**

**lemma** *mu-csp-expand* [*rdes*]:  $(\mu_C \ (op ;; Q)) = (\mu \ X \cdot Q ;; \text{CSP } X)$

**by** (*simp add: comp-def*)

**lemma** *mu-csp-basic-refine*:

**assumes**

$P \text{ is CSP } Q \text{ is NCSP } Q \text{ is Productive } \text{pre}_R(P) = \text{true}_r \ \text{pre}_R(Q) = \text{true}_r$

$\text{peri}_R P \sqsubseteq \text{peri}_R Q$

$\text{peri}_R P \sqsubseteq \text{post}_R Q ;; \text{peri}_R P$

**shows**  $P \sqsubseteq (\mu_C \ X \cdot Q ;; X)$

**proof** (*rule SRD-refine-intro', simp-all add: closure usubst alpha rpred rdes unrest wp seq-UINF-distr assms*)

**show**  $\text{peri}_R P \sqsubseteq (\bigcap i \cdot \text{post}_R Q \wedge i ;; \text{peri}_R Q)$

**proof** (*rule UINF-refines'*)

**fix**  $i$

**show**  $\text{peri}_R P \sqsubseteq \text{post}_R Q \wedge i ;; \text{peri}_R Q$

**proof** (*induct i*)

**case** 0

**then show** ?case **by** (*simp add: assms*)

**next**

**case** (*Suc i*)

**then show** ?case

**by** (*meson assms(6) assms(7) semilattice-sup-class.le-sup-iff upower-inductl*)

**qed**

**qed**

**qed**

**lemma** *CRD-mu-basic-refine*:

**fixes**  $P :: 'e \text{ list} \Rightarrow 'e \text{ set} \Rightarrow 's \text{ upred}$

**assumes**

$Q \text{ is NCSP } Q \text{ is Productive } \text{pre}_R(Q) = \text{true}_r$

$[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq \text{peri}_R Q$

$[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq \text{post}_R Q ;;_h [P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket$

**shows**  $[\text{true} \vdash P \text{ trace refs} \mid R]_C \sqsubseteq (\mu_C \ X \cdot Q ;; X)$

**proof** (*rule mu-csp-basic-refine, simp-all add: msubst-pair assms closure alpha rdes rpred Healthy-if R1-false*)

**show**  $[P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket \sqsubseteq \text{peri}_R Q$

**using** *assms* **by** (*simp add: msubst-pair*)

**show**  $[P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket \sqsubseteq \text{post}_R Q ;; [P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket$

**using** *assms* **by** (*simp add: msubst-pair*)

qed

## 9.2 Example action expansion

**lemma** *mu-example1*:  $(\mu X \cdot \ll a \gg \rightarrow_C X) = (\bigcap i \cdot do_C(\ll a \gg) \wedge (i+1)) \;; \text{Miracle}$   
**by** (*simp add: PrefixCSP-def mu-csp-form-1 closure*)

**lemma** *preR-mu-example1* [*rdes*]:  $pre_R(\mu X \cdot \ll a \gg \rightarrow_C X) = true_r$   
**by** (*simp add: mu-example1 rdes closure unrest wp*)

**lemma** *periR-mu-example1* [*rdes*]:  
 $peri_R(\mu X \cdot \ll a \gg \rightarrow_C X) = (\bigcap i \cdot \mathcal{E}(true, iter[i](\ll a \gg), \{\ll a \gg\}_u))$   
**by** (*simp add: mu-example1 rdes rpred closure unrest wp seq-UNF-distr alpha usubst*)

**lemma** *postR-mu-example1* [*rdes*]:  
 $post_R(\mu X \cdot \ll a \gg \rightarrow_C X) = false$   
**by** (*simp add: mu-example1 rdes closure unrest wp*)

end

## 10 Linking to the Failures-Divergences Model

**theory** *utp-sfrd-fdsem*  
**imports** *utp-sfrd-recursion*  
**begin**

### 10.1 Failures-Divergences Semantics

The following functions play a similar role to those in Roscoe's CSP semantics, and are calculated from the Circus reactive design semantics. A major difference is that these three functions account for state. Each divergence, trace, and failure is subject to an initial state. Moreover, the traces are terminating traces, and therefore also provide a final state following the given interaction. A more subtle difference from the Roscoe semantics is that the set of traces do not include the divergences. The same semantic information is present, but we construct a direct analogy with the pre-, peri- and postconditions of our reactive designs.

**definition** *divergences* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow 'φ \text{ list set } (dv[-] - [0,100] \ 100)$  **where**  
*[upred-defs]*:  $divergences \ P \ s = \{t \mid t. '(\neg_r \ pre_R(P)) \ll s \gg, \langle \rangle, \ll t \gg / \$st, \$tr, \$tr' \}$

**definition** *traces* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow ('φ \text{ list} \times 'σ) \text{ set } (tr[-] - [0,100] \ 100)$  **where**  
*[upred-defs]*:  $traces \ P \ s = \{(t, s') \mid t \ s'. '(pre_R(P) \wedge post_R(P)) \ll s \gg, \ll s' \gg, \langle \rangle, \ll t \gg / \$st, \$st', \$tr, \$tr' \}$

**definition** *failures* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow ('φ \text{ list} \times 'φ \text{ set}) \text{ set } (fl[-] - [0,100] \ 100)$  **where**  
*[upred-defs]*:  $failures \ P \ s = \{(t, r) \mid t \ r. '(pre_R(P) \wedge peri_R(P)) \ll r \gg, \ll s \gg, \langle \rangle, \ll t \gg / \$ref', \$st, \$tr, \$tr' \}$

**lemma** *trace-divergence-disj*:  
**assumes** *P is NCSP*  $(t, s') \in tr[P] \ s \ t \in dv[P] \ s$   
**shows** *False*  
**using** *assms(2,3)*  
**by** (*simp add: traces-def divergences-def, rdes-simp cls:assms, rel-auto*)

**lemma** *preR-refine-divergences*:  
**assumes** *P is NCSP* *Q is NCSP*  $\bigwedge s. dv[P] \ s \subseteq dv[Q] \ s$   
**shows**  $pre_R(P) \sqsubseteq pre_R(Q)$   
**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure usubst unrest*)

**fix**  $t\ s$   
**assume**  $a$ :  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R Q$   
**with**  $a$  **show**  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P$   
**proof** (*rule-tac ccontr*)  
**from**  $assms(\mathcal{J})[of\ s]$  **have**  $b$ :  $t \in dv[P]s \implies t \in dv[Q]s$   
**by** (*auto*)  
**assume**  $\neg [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P$   
**hence**  $\neg [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger CRC(pre_R P)$   
**by** (*simp add: assms closure Healthy-if*)  
**hence**  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (\neg_r CRC(pre_R P))$   
**by** (*rel-auto*)  
**hence**  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (\neg_r pre_R P)$   
**by** (*simp add: assms closure Healthy-if*)  
**with**  $a\ b$  **show** *False*  
**by** (*rel-auto*)  
**qed**  
**qed**

**lemma** *preR-eq-divergences*:

**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s. dv[P]s = dv[Q]s$   
**shows**  $pre_R(P) = pre_R(Q)$   
**by** (*metis assms dual-order.antisym order-refl preR-refine-divergences*)

**lemma** *periR-refine-failures*:

**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s. fl[Q]s \subseteq fl[P]s$   
**shows**  $(pre_R(P) \wedge peri_R(P)) \sqsubseteq (pre_R(Q) \wedge peri_R(Q))$

**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-3*)

**fix**  $t\ s\ r'$   
**assume**  $a$ :  $[\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R Q \wedge peri_R Q)$   
**from**  $assms(\mathcal{J})[of\ s]$  **have**  $b$ :  $(t, r') \in fl[Q]s \implies (t, r') \in fl[P]s$   
**by** (*auto*)  
**with**  $a$  **show**  $[\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R P \wedge peri_R P)$   
**by** (*simp add: failures-def*)

**qed**

**lemma** *periR-eq-failures*:

**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s. fl[P]s = fl[Q]s$   
**shows**  $(pre_R(P) \wedge peri_R(P)) = (pre_R(Q) \wedge peri_R(Q))$   
**by** (*metis (full-types) assms dual-order.antisym order-refl periR-refine-failures*)

**lemma** *postR-refine-traces*:

**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s. tr[Q]s \subseteq tr[P]s$   
**shows**  $(pre_R(P) \wedge post_R(P)) \sqsubseteq (pre_R(Q) \wedge post_R(Q))$

**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-5*)

**fix**  $t\ s\ s'$   
**assume**  $a$ :  $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R Q \wedge post_R Q)$   
**from**  $assms(\mathcal{J})[of\ s]$  **have**  $b$ :  $(t, s') \in tr[Q]s \implies (t, s') \in tr[P]s$   
**by** (*auto*)  
**with**  $a$  **show**  $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R P \wedge post_R P)$   
**by** (*simp add: traces-def*)

**qed**

**lemma** *postR-eq-traces*:

**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s. tr[P]s = tr[Q]s$   
**shows**  $(pre_R(P) \wedge post_R(P)) = (pre_R(Q) \wedge post_R(Q))$



by (metis assms dual-order.antisym order-refl postR-refine-traces)

**lemma** *circus-fd-refine-intro*:

assumes  $P$  is NCSP  $Q$  is NCSP  $\wedge s. dv\llbracket Q \rrbracket s \subseteq dv\llbracket P \rrbracket s \wedge s. fl\llbracket Q \rrbracket s \subseteq fl\llbracket P \rrbracket s \wedge s. tr\llbracket Q \rrbracket s \subseteq tr\llbracket P \rrbracket s$   
 shows  $P \sqsubseteq Q$

**proof** (rule *SRD-refine-intro'*, simp-all add: closure assms)

show  $a: 'pre_R P \Rightarrow pre_R Q'$

using assms(1) assms(2) assms(3) *preR-refine-divergences refBy-order* by blast

show  $peri_R P \sqsubseteq (pre_R P \wedge peri_R Q)$

**proof** –

have  $peri_R P \sqsubseteq (pre_R Q \wedge peri_R Q)$

by (metis (no-types) assms(1) assms(2) assms(4) *periR-refine-failures utp-pred-laws.le-inf-iff*)

then show ?thesis

by (metis a *refBy-order utp-pred-laws.inf.order-iff utp-pred-laws.inf-assoc*)

qed

show  $post_R P \sqsubseteq (pre_R P \wedge post_R Q)$

**proof** –

have  $post_R P \sqsubseteq (pre_R Q \wedge post_R Q)$

by (meson assms(1) assms(2) assms(5) *postR-refine-traces utp-pred-laws.le-inf-iff*)

then show ?thesis

by (metis a *refBy-order utp-pred-laws.inf.absorb-iff1 utp-pred-laws.inf-assoc*)

qed

qed

## 10.2 Circus Operators

**lemma** *traces-Skip*:

$tr\llbracket Skip \rrbracket s = \{(\llbracket, s\rrbracket)\}$

by (simp add: *traces-def rdes alpha closure, rel-simp*)

**lemma** *failures-Skip*:

$fl\llbracket Skip \rrbracket s = \{\}$

by (simp add: *failures-def, rdes-calc*)

**lemma** *divergences-Skip*:

$dv\llbracket Skip \rrbracket s = \{\}$

by (simp add: *divergences-def, rdes-calc*)

**lemma** *traces-Stop*:

$tr\llbracket Stop \rrbracket s = \{\}$

by (simp add: *traces-def, rdes-calc*)

**lemma** *failures-Stop*:

$fl\llbracket Stop \rrbracket s = \{(\llbracket, E) \mid E. True\}$

by (simp add: *failures-def, rdes-calc, rel-auto*)

**lemma** *divergences-Stop*:

$dv\llbracket Stop \rrbracket s = \{\}$

by (simp add: *divergences-def, rdes-calc*)

**lemma** *traces-AssignsCSP*:

$tr\llbracket \langle \sigma \rangle_C \rrbracket s = \{(\llbracket, \sigma(s))\}$

by (simp add: *traces-def rdes closure usubst alpha, rel-auto*)

**lemma** *failures-AssignsCSP*:

$fl\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$

by (simp add: failures-def, rdes-calc)

**lemma** *divergences-AssignsCSP*:  
 $dv\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$   
 by (simp add: divergences-def, rdes-calc)

**lemma** *failures-Miracle*:  $fl\llbracket Miracle \rrbracket s = \{\}$   
 by (simp add: failures-def rdes closure usubst)

**lemma** *divergences-Miracle*:  $dv\llbracket Miracle \rrbracket s = \{\}$   
 by (simp add: divergences-def rdes closure usubst)

**lemma** *failures-Chaos*:  $fl\llbracket Chaos \rrbracket s = \{\}$   
 by (simp add: failures-def rdes, rel-auto)

**lemma** *divergences-Chaos*:  $dv\llbracket Chaos \rrbracket s = UNIV$   
 by (simp add: divergences-def rdes, rel-auto)

**lemma** *traces-Chaos*:  $tr\llbracket Chaos \rrbracket s = \{\}$   
 by (simp add: traces-def rdes closure usubst)

**lemma** *divergences-cond*:  
 assumes  $P$  is NCSP  $Q$  is NCSP  
 shows  $dv\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if\ (\llbracket b \rrbracket_e s) \text{ then } dv\llbracket P \rrbracket s \text{ else } dv\llbracket Q \rrbracket s)$   
 by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *traces-cond*:  
 assumes  $P$  is NCSP  $Q$  is NCSP  
 shows  $tr\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if\ (\llbracket b \rrbracket_e s) \text{ then } tr\llbracket P \rrbracket s \text{ else } tr\llbracket Q \rrbracket s)$   
 by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *failures-cond*:  
 assumes  $P$  is NCSP  $Q$  is NCSP  
 shows  $fl\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if\ (\llbracket b \rrbracket_e s) \text{ then } fl\llbracket P \rrbracket s \text{ else } fl\llbracket Q \rrbracket s)$   
 by (rdes-simp cls: assms, simp add: divergences-def failures-def rdes closure rpred assms, rel-auto)

**lemma** *divergences-guard*:  
 assumes  $P$  is NCSP  
 shows  $dv\llbracket g \&_u P \rrbracket s = (if\ (\llbracket g \rrbracket_e s) \text{ then } dv\llbracket g \&_u P \rrbracket s \text{ else } \{\})$   
 by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *traces-do*:  $tr\llbracket do_C(e) \rrbracket s = \{(\llbracket e \rrbracket_e s, s)\}$   
 by (rdes-simp, simp add: traces-def rdes closure rpred, rel-auto)

**lemma** *failures-do*:  $fl\llbracket do_C(e) \rrbracket s = \{([], E) \mid E. \llbracket e \rrbracket_e s \notin E\}$   
 by (rdes-simp, simp add: failures-def rdes closure rpred usubst, rel-auto)

**lemma** *divergences-do*:  $dv\llbracket do_C(e) \rrbracket s = \{\}$   
 by (rel-auto)

**lemma** *divergences-seq*:  
 fixes  $P :: ('s, 'e)$  action  
 assumes  $P$  is NCSP  $Q$  is NCSP  
 shows  $dv\llbracket P ;; Q \rrbracket s = dv\llbracket P \rrbracket s \cup \{t_1 @ t_2 \mid t_1 \ t_2 \ s_0. (t_1, s_0) \in tr\llbracket P \rrbracket s \wedge t_2 \in dv\llbracket Q \rrbracket s_0\}$   
 (is ?lhs = ?rhs)

oops

lemma *traces-seq*:

fixes  $P :: ('s, 'e) \text{ action}$

assumes  $P \text{ is NCSP } Q \text{ is NCSP}$

shows  $tr[P ;; Q]_s =$

$$\begin{aligned} & \{ (t_1 @ t_2, s') \mid t_1 \ t_2 \ s_0 \ s'. (t_1, s_0) \in tr[P]_s \wedge (t_2, s') \in tr[Q]_{s_0} \\ & \quad \wedge (t_1 @ t_2) \notin dv[P]_s \\ & \quad \wedge (\forall (t, s_1) \in tr[P]_s. t \leq t_1 @ t_2 \longrightarrow (t_1 @ t_2) - t \notin dv[Q]_{s_1}) \} \end{aligned}$$

(is ?lhs = ?rhs)

proof

show ?lhs  $\subseteq$  ?rhs

proof (rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto)

fix  $t :: 'e \text{ list}$  and  $s' :: 's$

let  $\sigma = [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg]$

assume

a1:  $\sigma \vdash (post_R P ;; post_R Q)'$  and

a2:  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \vdash pre_R P'$  and

a3:  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \vdash (post_R P \text{ wr } pre_R Q)'$

from a1 have  $\sigma \vdash (\exists tr_0. ((post_R P) \ll tr_0 \gg / \$tr') ;; (post_R Q) \ll tr_0 \gg / \$tr) \wedge \ll tr_0 \gg \leq_u \$tr'$   
by (simp add: R2-tr-middle assms closure)

then obtain  $tr_0$  where  $p1: \sigma \vdash ((post_R P) \ll tr_0 \gg / \$tr') ;; (post_R Q) \ll tr_0 \gg / \$tr$  and  $tr0: tr_0 \leq t$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

from p1 have  $\sigma \vdash (\exists st_0. (post_R P) \ll tr_0 \gg / \$tr' \ll st_0 \gg / \$st' ;; (post_R Q) \ll tr_0 \gg / \$tr \ll st_0 \gg / \$st)'$   
by (simp add: seqr-middle[of st, THEN sym])

then obtain  $s_0$  where  $\sigma \vdash ((post_R P) \ll s_0 \gg, \ll tr_0 \gg / \$st', \$tr') ;; (post_R Q) \ll s_0 \gg, \ll tr_0 \gg / \$st, \$tr)$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

hence  $(([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P) ;; ([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q))'$

by (rel-auto)

hence  $(([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P) \wedge ([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q))'$

by (simp add: seqr-to-conj unrest-any-circus-var assms closure unrest)

hence  $postP: ([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P)'$  and

$postQ': ([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q)'$

by (rel-auto)+

from  $postQ'$  have  $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \vdash [\$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll tr_0 \gg + (\ll t \gg - \ll tr_0 \gg)] \vdash post_R Q'$

using  $tr0$  by (rel-auto)

hence  $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \vdash [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t \gg - \ll tr_0 \gg] \vdash post_R Q'$

by (simp add: R2-subst-tr closure assms)

hence  $postQ: [\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - tr_0 \gg] \vdash post_R Q'$

by (rel-auto)

have  $preP: [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash pre_R P'$

proof –

**have**  $(pre_R P)[[0, \ll tr_0 \gg / \$tr, \$tr']] \sqsubseteq (pre_R P)[[0, \ll t \gg / \$tr, \$tr']]$   
**by**  $(simp \text{ add: } RC\text{-prefix-refine closure assms } tr0)$   
**hence**  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P$   
**by**  $(rel\text{-auto})$   
**thus**  $?thesis$   
**by**  $(simp \text{ add: } taut\text{-refine-impl } a2)$   
**qed**

**have**  $preQ: '[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - tr_0 \gg] \dagger pre_R Q'$

**proof** –

**from**  $postP \ a3$  **have**  $'[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \dagger pre_R Q'$   
**apply**  $(simp \text{ add: } wp\text{-rea-def})$   
**apply**  $(rel\text{-auto})$   
**using**  $tr0$  **apply**  $blast+$   
**done**

**hence**  $'[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll tr_0 \gg + (\ll t \gg - \ll tr_0 \gg)] \dagger pre_R Q'$   
**by**  $(rel\text{-auto})$

**hence**  $'[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t \gg - \ll tr_0 \gg] \dagger pre_R Q'$   
**by**  $(simp \text{ add: } R2\text{-subst-tr closure assms})$   
**thus**  $?thesis$   
**by**  $(rel\text{-auto})$   
**qed**

**from**  $a2$  **have**  $ndiv: \neg '[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (\neg_r pre_R P)'$   
**by**  $(rel\text{-auto})$

**have**  $t\text{-minus-}tr0: tr_0 @ (t - tr_0) = t$   
**using**  $append\text{-minus } tr0$  **by**  $blast$

**from**  $a3$

**have**  $wpr: \bigwedge t_0 \ s_1.$

$'[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P' \implies$   
 $'[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P' \implies$   
 $t_0 \leq t \implies '[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - t_0 \gg] \dagger (\neg_r pre_R Q)' \implies False$

**proof** –

**fix**  $t_0 \ s_1$

**assume**  $b:$

$'[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P'$   
 $'[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P'$   
 $t_0 \leq t$   
 $'[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - t_0 \gg] \dagger (\neg_r pre_R Q)'$

**from**  $a3$  **have**  $c: \forall (s_0, t_0) \cdot \ll t_0 \gg \leq_u \ll t \gg$   
 $\wedge [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P$   
 $\implies [\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg - \ll t_0 \gg] \dagger pre_R Q'$   
**by**  $(simp \text{ add: } wp\text{-rea-circus-form-alt[of } post_R P \ pre_R Q] \text{ closure assms unrest usubst})$   
 $(rel\text{-simp})$

**from**  $c \ b(2-4)$  **show**  $False$   
**by**  $(rel\text{-auto})$   
**qed**

**show**  $\exists t_1 \ t_2.$

$t = t_1 @ t_2 \wedge$   
 $(\exists s_0. \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg\text{' } \dagger \text{ } pre_R P \wedge$   
 $\text{'[}\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg\text{' } \dagger \text{ } post_R P' \wedge$   
 $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } pre_R Q \wedge$   
 $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } post_R Q' \wedge$   
 $\neg \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg\text{' } \dagger (\neg_r pre_R P)' \wedge$   
 $(\forall t_0 s_1. \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg\text{' } \dagger \text{ } pre_R P \wedge$   
 $\text{'[}\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg\text{' } \dagger \text{ } post_R P' \longrightarrow$   
 $t_0 \leq t_1 @ t_2 \longrightarrow \neg \text{'[}\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg\text{' } \dagger (\neg_r$   
 $pre_R Q)'$ ))  
**apply** (*rule-tac*  $x=tr_0$  **in**  $exI$ )  
**apply** (*rule-tac*  $x=(t - tr_0)$  **in**  $exI$ )  
**apply** (*auto*)  
**using**  $tr0$  **apply**  $auto[1]$   
**apply** (*rule-tac*  $x=s_0$  **in**  $exI$ )  
**apply** (*auto intro:wpr simp add: taut-conj preP preQ postP postQ ndiv wpr t-minus-tr0*)  
**done**  
**qed**

**show**  $?rhs \subseteq ?lhs$

**proof** (*rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto*)

**fix**  $t_1 t_2 :: 'e \text{ list}$  **and**  $s_0 s' :: 's$

**assume**

$a1: \neg \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg\text{' } \dagger (\neg_r pre_R P)'$  **and**  
 $a2: \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg\text{' } \dagger \text{ } pre_R P'$  **and**  
 $a3: \text{'[}\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg\text{' } \dagger \text{ } post_R P'$  **and**  
 $a4: \text{'[}\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } pre_R Q'$  **and**  
 $a5: \text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } post_R Q'$  **and**  
 $a6: \forall t s_1. \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg\text{' } \dagger \text{ } pre_R P \wedge$   
 $\text{'[}\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg\text{' } \dagger \text{ } post_R P' \longrightarrow$   
 $t \leq t_1 @ t_2 \longrightarrow \neg \text{'[}\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg\text{' } \dagger (\neg_r pre_R Q)'$

**from**  $a1$  **have**  $preP: \text{'[}\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg\text{' } \dagger (pre_R P)'$

**by** (*simp add: taut-not unrest-all-circus-vars-st assms closure unrest, rel-auto*)

**have**  $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg\text{' } \dagger \text{ } post_R Q'$

**proof** –

**have**  $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } post_R Q =$

$\text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg\text{' } \dagger \text{'[}\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg\text{' } \dagger \text{ } post_R Q$

**by** *rel-auto*

**also have**  $\dots = \text{'[}\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg\text{' } \dagger \text{'[}\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg\text{' } \dagger \text{ } post_R Q$

**by** (*simp add: R2-subst-tr assms closure, rel-auto*)

**finally show**  $?thesis$  **using**  $a5$

**by** (*rel-auto*)

**qed**

**with**  $a3$

**have**  $postPQ: \text{'[}\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg\text{' } \dagger (post_R P ;; post_R Q)'$

**by** (*rel-auto, meson Prefix-Order.prefixI*)

**have**  $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg\text{' } \dagger \text{ } pre_R Q'$

**proof** –

**have**  $\text{'[}\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg\text{' } \dagger \text{ } pre_R Q =$

$\text{'[}\$st \mapsto_s \ll s_0 \gg\text{' } \dagger \text{'[}\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg\text{' } \dagger \text{ } pre_R Q$

```

    by rel-auto
  also have ... = [ $\$st \mapsto_s \ll s_0 \gg$ ]  $\dagger$  [ $\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t_2 \gg$ ]  $\dagger$   $pre_R Q$ 
    by (simp add: R2-subst-tr assms closure)
  finally show ?thesis using a4
    by (rel-auto)
qed

from a6
have a6':  $\bigwedge t s_1. \llbracket t \leq t_1 @ t_2; '[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P'; '[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger post_R P' \rrbracket \implies$ 
   $'[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg] \dagger pre_R Q'$ 
  apply (subst (asm) taut-not)
  apply (simp add: unrest-all-circus-vars-st assms closure unrest)
  apply (rel-auto)
  done

have wpR:  $'[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)'$ 
proof -
  have  $\bigwedge s_1 t_0. \llbracket t_0 \leq t_1 @ t_2; '[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P' \rrbracket$ 
     $\implies '[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R Q'$ 
  proof -
    fix s1 t0
    assume c:  $t_0 \leq t_1 @ t_2$   $'[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P'$ 

    have preP':  $'[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P'$ 
    proof -
      have  $(pre_R P) \llbracket 0, \ll t_0 \gg / \$tr, \$tr' \rrbracket \sqsubseteq (pre_R P) \llbracket 0, \ll t_1 @ t_2 \gg / \$tr, \$tr' \rrbracket$ 
        by (simp add: RC-prefix-refine closure assms c)
      hence  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P$ 
        by (rel-auto)
      thus ?thesis
        by (simp add: taut-refine-impl preP)
    qed

    with c a3 preP a6' [of t0 s1] show  $'[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R Q'$ 
      by (simp)
    qed

  thus ?thesis
    apply (simp-all add: wp-rea-circus-form-alt assms closure unrest usubst rea-impl-alt-def)
    apply (simp add: R1-def usubst tcontr-alt-def)
    apply (auto intro!: taut-shAll-intro-2)
    apply (rule taut-impl-intro)
    apply (simp add: unrest-all-circus-vars-st-st' unrest closure assms)
    apply (rel-simp)
  done
qed
show  $'([\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P \wedge$ 
 $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)) \wedge$ 
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P ;; post_R Q)'$ 
  by (auto simp add: taut-conj preP postPQ wpR)

```

qed  
qed

**lemma** *Cons-minus* [*simp*]:  $(a \# t) - [a] = t$   
by (*metis append-Cons append-Nil append-minus*)

**lemma** *traces-prefix*:  
assumes *P* is NCSP  
shows  $tr\llbracket\langle a \rangle\rrbracket \rightarrow_C P\llbracket s \rrbracket = \{(a \# t, s') \mid t \ s'. (t, s') \in tr\llbracket P \rrbracket s\}$   
apply (*auto simp add: PrefixCSP-def traces-seq traces-do divergences-do lit.rep-eq assms closure Healthy-if trace-divergence-disj*)  
apply (*meson assms trace-divergence-disj*)  
done

### 10.3 Deadlock Freedom

The following is a specification for deadlock free actions. In any intermediate observation, there must be at least one enabled event.

**definition** *CDF* ::  $(s, e)$  action **where**  
[*rdes-def*]:  $CDF = \mathbf{R}_s(true_r \vdash (\bigwedge (s, t, E, e) \cdot \mathcal{E}(\langle s \rangle, \langle t \rangle, \langle insert\ e\ E \rangle)) \diamond true_r)$

**lemma** *CDF-NCSP* [*closure*]: *CDF* is NCSP  
apply (*simp add: CDF-def*)  
apply (*rule NCSP-rdes-intro*)  
apply (*simp-all add: closure unrest*)  
apply (*rel-auto*)  
done

**lemma** *Skip-deadlock-free*:  $CDF \sqsubseteq Skip$   
by (*rdes-refine*)

end

## 11 Meta-theory for Stateful-Failure Reactive Designs

**theory** *utp-sf-rdes*  
**imports**  
  *utp-sfrd-core*  
  *utp-sfrd-rel*  
  *utp-sfrd-healths*  
  *utp-sfrd-contracts*  
  *utp-sfrd-extchoice*  
  *utp-sfrd-prog*  
  *utp-sfrd-recursion*  
  *utp-sfrd-fdsem*  
**begin end**

## References

- [1] S. Foster, F. Zeyda, and J. Woodcock. Unifying heterogeneous state-spaces with lenses. In *ICTAC*, LNCS 9965. Springer, 2016.

- [2] M. V. M. Oliveira. *Formal Derivation of State-Rich Reactive Programs using Circus*. PhD thesis, Department of Computer Science - University of York, UK, 2006. YCST-2006-02.