

Stateful-Failure Reactive Designs in Isabelle/UTP

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Abstract

Stateful-Failure Reactive Designs specialise reactive design contracts with failures traces, as present in languages like CSP and Circus. A failure trace consists of a sequence of events and a refusal set. It intuitively represents a quiescent observation, where certain events have previously occurred, and others are currently being accepted. Following the UTP book, we add an observational variable to represent refusal sets, and healthiness conditions that ensure their well-formedness. Using these, we also specialise our theory of reactive relations with operators to characterise both completed and quiescent interactions, and an accompanying equational theory. We use these to define the core operators — including assignment, event occurrence, and external choice — and specialise our proof strategy to support these. We also demonstrate a link with the CSP failures-divergences semantic model.

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1 Introduction

This document contains a mechanisation in Isabelle/UTP [1] of an specialisation of stateful reactive designs with refusal information, as present in languages like Circus [2].

2 Stateful-Failure Core Types

```
theory utp-sfrd-core
  imports UTP-Reactive-Designs.utp-rea-designs
begin
```

2.1 SFRD Alphabet

```
alphabet ('σ, 'φ) sfrd-vars = ('φ list, 'σ) rsp-vars +
  ref :: 'φ set
```

The following two locale interpretations are a technicality to improve the behaviour of the automatic tactics. They enable (re)interpretation of state spaces in order to remove any occurrences of lens types, replacing them by tuple types after the tactics *pred-simp* and *rel-simp* are applied. Eventually, it would be desirable to automate preform these interpretations automatically as part of the **alphabet** command.

```
type-synonym ('σ, 'φ) sfrd = ('σ, 'φ) sfrd-vars
type-synonym ('σ, 'φ) action = ('σ, 'φ) sfrd hrel
type-synonym 'φ csp = (unit, 'φ) sfrd
type-synonym 'φ process = 'φ csp hrel
```

There is some slight imprecision with the translations, in that we don't bother to check if the trace event type and refusal set event types are the same. Essentially this is because its very difficult to construct processes where this would be the case. However, it may be better to add a proper ML print translation in the future.

translations

```
(type) ('σ, 'φ) sfrd <= (type) ('σ, 'φ) sfrd-vars
(type) ('σ, 'φ) action <= (type) ('σ, 'φ) sfrd hrel
(type) 'φ process <= (type) (unit, 'φ) action
```

notation *sfrd-vars.more_L* (Σ_C)

```
declare des-vars.splits [alpha-splits del]
declare rp-vars.splits [alpha-splits del]
declare des-vars.splits [alpha-splits del]
declare rsp-vars.splits [alpha-splits del]
declare rsp-vars.splits [alpha-splits]
declare rp-vars.splits [alpha-splits]
declare des-vars.splits [alpha-splits]
```

2.2 Basic laws

```
lemma R2c-tr-ext: R2c ($tr' =_u $tr ^_u <[a]_{S<}>) = ($tr' =_u $tr ^_u <[a]_{S<}>)
  by (rel-auto)
```

lemma *circus-alpha-bij-lens*:

```
bij-lens ({ $ok, $ok', $wait, $wait', $tr, $tr', $st, $st', $ref, $ref' }_α :: - ==> ('s, 'e) sfrd × ('s, 'e) sfrd)
  by (unfold-locales, lens-simp+)
```

2.3 Unrestriction laws

```
lemma pre-unrest-ref [unrest]: $ref # P ==> $ref # pre_R(P)
  by (simp add: pre_R-def unrest)
```

lemma *peri-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# peri_R(P)$
 by (*simp add: peri_R-def unrest*)

lemma *post-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# post_R(P)$
 by (*simp add: post_R-def unrest*)

lemma *cmt-unrest-ref* [*unrest*]: $\$ref \# P \implies \$ref \# cmt_R(P)$
 by (*simp add: cmt_R-def unrest*)

lemma *st-lift-unrest-ref'* [*unrest*]: $\$ref' \# [b]_{S<} \implies$
 by (*rel-auto*)

lemma *RHS-design-ref-unrest* [*unrest*]:
 $\llbracket \$ref \# P; \$ref \# Q \rrbracket \implies \$ref \# (\mathbf{R}_s(P \vdash Q)) \llbracket false/\$wait \rrbracket$
 by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

lemma *R1-ref-unrest* [*unrest*]: $\$ref \# P \implies \$ref \# R1(P)$
 by (*simp add: R1-def unrest*)

lemma *R2c-ref-unrest* [*unrest*]: $\$ref \# P \implies \$ref \# R2c(P)$
 by (*simp add: R2c-def unrest*)

lemma *R1-ref'-unrest* [*unrest*]: $\$ref' \# P \implies \$ref' \# R1(P)$
 by (*simp add: R1-def unrest*)

lemma *R2c-ref'-unrest* [*unrest*]: $\$ref' \# P \implies \$ref' \# R2c(P)$
 by (*simp add: R2c-def unrest*)

lemma *R2s-notin-ref'*: $R2s([x]_{S<} \notin_u \$ref') = ([x]_{S<} \notin_u \$ref')$
 by (*pred-auto*)

lemma *unrest-circus-alpha*:
 fixes $P :: ('e, 't) \text{ action}$
 assumes
 $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$tr \# P$
 $\$tr' \# P \ \$st \# P \ \$st' \# P \ \$ref \# P \ \$ref' \# P$
 shows $\Sigma \# P$
 by (*rule bij-lens-unrest-all[OF circus-alpha-bij-lens], simp add: unrest asms*)

lemma *unrest-all-circus-vars*:
 fixes $P :: ('s, 'e) \text{ action}$
 assumes $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \Sigma \# r' \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$
 shows $\Sigma \# [\$ref' \mapsto_s r', \$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$
 using *asms*
 by (*simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens*)
 (*simp add: unrest usubst closure*)

lemma *unrest-all-circus-vars-st-st'*:
 fixes $P :: ('s, 'e) \text{ action}$
 assumes $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \$ref' \# P \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$
 shows $\Sigma \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$
 using *asms*
 by (*simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens*)
 (*simp add: unrest usubst closure*)

lemma *unrest-all-circus-vars-st*:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$
shows $\Sigma \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$
using *assms*
by (*simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens*)
(simp add: unrest usubst closure)

lemma *unrest-any-circus-var*:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \Sigma \# s \Sigma \# s' \Sigma \# t \Sigma \# t'$
shows $x \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$
by (*simp add: unrest-all-var unrest-all-circus-vars-st-st' assms*)

lemma *unrest-any-circus-var-st*:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \$ok' \# P \$wait \# P \$wait' \# P \$ref \# P \$ref' \# P \$st' \# P \Sigma \# s \Sigma \# t \Sigma \# t'$
shows $x \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$
by (*simp add: unrest-all-var unrest-all-circus-vars-st assms*)

end

3 Stateful-Failure Reactive Relations

theory *utp-sfrd-rel*
imports *utp-sfrd-core*
begin

3.1 Healthiness Conditions

CSP Reactive Relations

definition $CRR :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ **where**
 $[upred-defs]: CRR(P) = (\exists \$ref \cdot RR(P))$

lemma *CRR-idem*: $CRR(CRR(P)) = CRR(P)$
by (*rel-auto*)

lemma *Idempotent-CRR* [*closure*]: *Idempotent CRR*
by (*simp add: CRR-idem Idempotent-def*)

lemma *Continuous-CRR* [*closure*]: *Continuous CRR*
by (*rel-blast*)

lemma *CRR-intro*:
assumes $\$ref \# P \text{ is } RR$
shows $P \text{ is } CRR$
by (*simp add: CRR-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

lemma *CRR-form*: $CRR(P) = (\exists \{\$ok, \$ok', \$wait, \$wait', \$ref\} \cdot (\exists tt_0 \cdot P[\langle \rangle / \$tr][\ll tt_0 \gg / \$tr'] \wedge \$tr' =_u \$tr \hat{^}_u \ll tt_0 \gg))$
by (*rel-auto; fastforce*)

lemma *CRR-seqr-form*:

$$CRR(P) ;; CRR(Q) =$$

$$(\exists tt_1 \cdot \exists tt_2 \cdot ((\exists \{ \$ok, \$ok', \$wait, \$wait', \$ref \} \cdot P) [\langle \rangle / \$tr] [\ll tt_1 \gg / \$tr'] ;;$$

$$(\exists \{ \$ok, \$ok', \$wait, \$wait', \$ref \} \cdot Q) [\langle \rangle / \$tr] [\ll tt_2 \gg / \$tr'] \wedge \$tr' =_u \$tr \wedge_u$$

$$\ll tt_1 \gg \wedge_u \ll tt_2 \gg))$$
 by (simp add: CRR-form, rel-auto; fastforce)

CSP Reactive Finalisers

definition $CRF :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ **where**
 $[upred-defs]: CRF(P) = (\exists \$ref' \cdot CRR(P))$

lemma $CRF-idem: CRF(CRF(P)) = CRF(P)$
 by (rel-auto)

lemma $Idempotent-CRF$ [closure]: $Idempotent\ CRF$
 by (simp add: CRF-idem Idempotent-def)

lemma $Continuous-CRF$ [closure]: $Continuous\ CRF$
 by (rel-blast)

lemma $CRF-intro$:
 assumes $\$ref \# P \ \$ref' \# P$ P is RR
 shows P is CRF
 by (simp add: CRF-def CRR-def Healthy-def, simp add: Healthy-if assms ex-unrest)

lemma $CRF-implies-CRR$ [closure]:
 assumes P is CRF shows P is CRR

proof –
 have $CRR(CRF(P)) = CRF(P)$
 by (rel-auto)
 thus ?thesis
 by (metis Healthy-def assms)

qed

definition $crel-skip :: ('s, 'e) \text{ action} (II_c)$ **where**
 $[upred-defs]: crel-skip = (\$tr' =_u \$tr \wedge \$st' =_u \$st)$

lemma $crel-skip-CRR$ [closure]: II_c is CRF
 by (rel-auto)

lemma $crel-skip-via-rrel: II_c = CRR(II_r)$
 by (rel-auto)

lemma $crel-skip-left-unit$ [rpred]:
 assumes P is CRR
 shows $II_c ;; P = P$

proof –
 have $II_c ;; CRR(P) = CRR(P)$ by (rel-auto)
 thus ?thesis by (simp add: Healthy-if assms)

qed

lemma $crel-skip-right-unit$ [rpred]:
 assumes P is CRF
 shows $P ;; II_c = P$

proof –
 have $CRF(P) ;; II_c = CRF(P)$ by (rel-auto)

thus *?thesis* **by** (*simp add: Healthy-if assms*)
qed

CSP Reactive Conditions

definition *CRC* :: (s, e) *action* $\Rightarrow (s, e)$ *action* **where**
[upred-defs]: $CRC(P) = (\exists \$ref \cdot RC(P))$

lemma *CRC-intro*:
assumes $\$ref \# P$ *P is RC*
shows *P is CRC*
by (*simp add: CRC-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

lemma *CRC-intro'*:
assumes *P is CRR* *P is RC*
shows *P is CRC*
by (*metis CRC-def CRR-def Healthy-def RC-implies-RR assms*)

lemma *ref-unrest-RR* [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RR\ P$
by (*rel-auto, blast+*)

lemma *ref-unrest-RC1* [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RC1\ P$
by (*rel-auto, blast+*)

lemma *ref-unrest-RC* [*unrest*]: $\$ref \# P \Longrightarrow \$ref \# RC\ P$
by (*simp add: RC-R2-def ref-unrest-RC1 ref-unrest-RR*)

lemma *RR-ex-ref*: $RR (\exists \$ref \cdot RR\ P) = (\exists \$ref \cdot RR\ P)$
by (*rel-auto*)

lemma *RC1-ex-ref*: $RC1 (\exists \$ref \cdot RC1\ P) = (\exists \$ref \cdot RC1\ P)$
by (*rel-auto, meson dual-order.trans*)

lemma *ex-ref'-RR-closed* [*closure*]:
assumes *P is RR*
shows $(\exists \$ref' \cdot P)$ *is RR*

proof –
have $RR (\exists \$ref' \cdot RR(P)) = (\exists \$ref' \cdot RR(P))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma *CRC-idem*: $CRC(CRC(P)) = CRC(P)$
apply (*simp add: CRC-def ex-unrest unrest*)
apply (*simp add: RC-def RR-ex-ref*)
apply (*metis (no-types, hide-lams) Healthy-def RC1-RR-closed RC1-ex-ref RR-ex-ref RR-idem*)
done

lemma *Idempotent-CRC* [*closure*]: *Idempotent CRC*
by (*simp add: CRC-idem Idempotent-def*)

3.2 Closure Properties

lemma *CRR-implies-RR* [*closure*]:
assumes *P is CRR*
shows *P is RR*

```

proof –
  have  $RR(CRR(P)) = CRR(P)$ 
    by (rel-auto)
  thus ?thesis
    by (metis Healthy-def' assms)
qed

lemma CRC-intro'':
  assumes  $P$  is CRR  $P$  is RC1
  shows  $P$  is CRC
  by (simp add: CRC-intro' CRR-implies-RR RC-intro' assms)

lemma CRC-implies-RR [closure]:
  assumes  $P$  is CRC
  shows  $P$  is RR
proof –
  have  $RR(CRC(P)) = CRC(P)$ 
    by (rel-auto)
    (metis (no-types, lifting) Prefix-Order.prefixE Prefix-Order.prefixI append.assoc append-minus)+
  thus ?thesis
    by (metis Healthy-def assms)
qed

lemma CRC-implies-RC [closure]:
  assumes  $P$  is CRC
  shows  $P$  is RC
proof –
  have  $RC1(CRC(P)) = CRC(P)$ 
    by (rel-auto, meson dual-order.trans)
  thus ?thesis
    by (simp add: CRC-implies-RR Healthy-if RC1-def RC-intro assms)
qed

lemma CRR-unrest-ref [unrest]:  $P$  is CRR  $\implies$   $\$ref \# P$ 
  by (metis CRR-def CRR-implies-RR Healthy-def in-var-uvar ref-vwb-lens unrest-as-exists)

lemma CRF-unrest-ref' [unrest]:
  assumes  $P$  is CRF
  shows  $\$ref' \# P$ 
proof –
  have  $\$ref' \# CRF(P)$  by (simp add: CRF-def unrest)
  thus ?thesis by (simp add: assms Healthy-if)
qed

lemma CRC-implies-CRR [closure]:
  assumes  $P$  is CRC
  shows  $P$  is CRR
  apply (rule CRR-intro)
  apply (simp-all add: unrest assms closure)
  apply (metis CRC-def CRC-implies-RC Healthy-def assms in-var-uvar ref-vwb-lens unrest-as-exists)
  done

lemma unrest-ref'-neg-RC [unrest]:
  assumes  $P$  is RR  $P$  is RC
  shows  $\$ref' \# P$ 

```


proof –
 have $P = (\neg_r \neg_r P)$
 by (*simp add: closure rpred assms*)
 also have $\dots = (\neg_r (\neg_r P) ;; true_r)$
 by (*metis Healthy-if RC1-def RC-implies-RC1 assms(2) calculation*)
 also have $\$ref' \# \dots$
 by (*rel-auto*)
 finally show *?thesis* .
qed

lemma *rea-true-CRR* [closure]: *true_r is CRR*
 by (*rel-auto*)

lemma *rea-true-CRC* [closure]: *true_r is CRC*
 by (*rel-auto*)

lemma *false-CRR* [closure]: *false is CRR*
 by (*rel-auto*)

lemma *false-CRC* [closure]: *false is CRC*
 by (*rel-auto*)

lemma *st-pred-CRR* [closure]: *[P]_{S<} is CRR*
 by (*rel-auto*)

lemma *st-post-unrest-ref'* [unrest]: $\$ref' \# [b]_{S>}$
 by (*rel-auto*)

lemma *st-post-CRR* [closure]: *[b]_{S>} is CRR*
 by (*rel-auto*)

lemma *st-cond-CRC* [closure]: *[P]_{S<} is CRC*
 by (*rel-auto*)

lemma *st-cond-CRF* [closure]: *[b]_{S<} is CRF*
 by (*rel-auto*)

lemma *rea-rename-CRR-closed* [closure]:
 assumes *P is CRR*
 shows *P(f)_r is CRR*

proof –
 have $\$ref \# (CRR\ P)(f)_r$
 by (*rel-auto*)
 thus *?thesis*
 by (*rule-tac CRR-intro, simp-all add: closure Healthy-if assms*)
qed

lemma *st-subst-CRR-closed* [closure]:
 assumes *P is CRR*
 shows $(\sigma \uparrow_S P)$ *is CRR*
 by (*rule CRR-intro, simp-all add: unrest closure assms*)

lemma *st-subst-CRC-closed* [closure]:
 assumes *P is CRC*
 shows $(\sigma \uparrow_S P)$ *is CRC*

by (rule CRC-intro, simp-all add: closure assms unrest)

lemma conj-CRC-closed [closure]:

$\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \wedge Q) \text{ is CRC}$

by (rule CRC-intro, simp-all add: unrest closure)

lemma conj-CRF-closed [closure]: $\llbracket P \text{ is CRF}; Q \text{ is CRF} \rrbracket \implies (P \wedge Q) \text{ is CRF}$

by (rule CRF-intro, simp-all add: unrest closure)

lemma disj-CRC-closed [closure]:

$\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \vee Q) \text{ is CRC}$

by (rule CRC-intro, simp-all add: unrest closure)

lemma st-cond-left-impl-CRC-closed [closure]:

$P \text{ is CRC} \implies ([b]_{S<} \Rightarrow_r P) \text{ is CRC}$

by (rule CRC-intro, simp-all add: unrest closure)

lemma unrest-ref-map-st [unrest]: $\$ref \# P \implies \$ref \# P \oplus_r \text{map-st}_L[a]$

by (rel-auto)

lemma unrest-ref'-map-st [unrest]: $\$ref' \# P \implies \$ref' \# P \oplus_r \text{map-st}_L[a]$

by (rel-auto)

lemma unrest-ref-rdes-frame-ext [unrest]:

$\$ref \# P \implies \$ref \# a:[P]_r^+$

by (rel-blast)

lemma unrest-ref'-rdes-frame-ext [unrest]:

$\$ref' \# P \implies \$ref' \# a:[P]_r^+$

by (rel-blast)

lemma map-st-ext-CRR-closed [closure]:

assumes $P \text{ is CRR}$

shows $P \oplus_r \text{map-st}_L[a] \text{ is CRR}$

by (rule CRR-intro, simp-all add: closure unrest assms)

lemma map-st-ext-CRC-closed [closure]:

assumes $P \text{ is CRC}$

shows $P \oplus_r \text{map-st}_L[a] \text{ is CRC}$

by (rule CRC-intro, simp-all add: closure unrest assms)

lemma rdes-frame-ext-CRR-closed [closure]:

assumes $P \text{ is CRR}$

shows $a:[P]_r^+ \text{ is CRR}$

by (rule CRR-intro, simp-all add: closure unrest assms)

lemma USUP-CRC-closed [closure]: $\llbracket A \neq \{\}; \bigwedge i. i \in A \implies P i \text{ is CRC} \rrbracket \implies (\bigsqcup i \in A \cdot P i) \text{ is CRC}$

by (rule CRC-intro, simp-all add: unrest closure)

lemma UINF-CRR-closed [closure]: $\llbracket \bigwedge i. i \in A \implies P i \text{ is CRR} \rrbracket \implies (\bigsqcap i \in A \cdot P i) \text{ is CRR}$

by (rule CRR-intro, simp-all add: unrest closure)

lemma cond-CRC-closed [closure]:

assumes $P \text{ is CRC } Q \text{ is CRC}$

shows $P \triangleleft b \triangleright_R Q$ *is CRC*
by (rule *CRC-intro*, *simp-all add: closure assms unrest*)

lemma *shEx-CRR-closed* [closure]:
assumes $\bigwedge x. P\ x$ *is CRR*
shows $(\exists x. P(x))$ *is CRR*
proof –
have $CRR(\exists x. CRR(P(x))) = (\exists x. CRR(P(x)))$
by (*rel-auto*)
thus ?thesis
by (*metis Healthy-def assms shEx-cong*)
qed

lemma *USUP-ind-CRR-closed* [closure]:
assumes $\bigwedge i. P\ i$ *is CRR*
shows $(\bigsqcup i. P(i))$ *is CRR*
by (rule *CRR-intro*, *simp-all add: assms unrest closure*)

lemma *UINF-ind-CRR-closed* [closure]:
assumes $\bigwedge i. P\ i$ *is CRR*
shows $(\bigcap i. P(i))$ *is CRR*
by (rule *CRR-intro*, *simp-all add: assms unrest closure*)

lemma *cond-tt-CRR-closed* [closure]:
assumes P *is CRR* Q *is CRR*
shows $P \triangleleft \$tr' =_u \$tr \triangleright Q$ *is CRR*
by (rule *CRR-intro*, *simp-all add: unrest assms closure*)

lemma *rea-implies-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \Rightarrow_r Q)$ *is CRR*
by (*simp-all add: CRR-intro closure unrest*)

lemma *conj-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \wedge Q)$ *is CRR*
by (*simp-all add: CRR-intro closure unrest*)

lemma *disj-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \vee Q)$ *is CRR*
by (rule *CRR-intro*, *simp-all add: unrest closure*)

lemma *rea-not-CRR-closed* [closure]:
 $P \text{ is CRR} \implies (\neg_r P)$ *is CRR*
using *false-CRR rea-implies-CRR-closed* **by** *fastforce*

lemma *disj-R1-closed* [closure]: $\llbracket P \text{ is R1}; Q \text{ is R1} \rrbracket \implies (P \vee Q)$ *is R1*
by (*rel-blast*)

lemma *st-cond-R1-closed* [closure]: $\llbracket P \text{ is R1}; Q \text{ is R1} \rrbracket \implies (P \triangleleft b \triangleright_R Q)$ *is R1*
by (*rel-blast*)

lemma *cond-st-RR-closed* [closure]:
assumes P *is RR* Q *is RR*
shows $(P \triangleleft b \triangleright_R Q)$ *is RR*
apply (rule *RR-intro*, *simp-all add: unrest closure assms*, *simp add: Healthy-def R2c-condr*)
apply (*simp add: Healthy-if assms RR-implies-R2c*)

apply (*rel-auto*)
done

lemma *cond-st-CRR-closed* [closure]:
 $\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is CRR}$
by (*simp-all add: CRR-intro closure unrest*)

lemma *seq-CRR-closed* [closure]:
assumes *P is CRR Q is RR*
shows $(P ;; Q) \text{ is CRR}$
by (*rule CRR-intro, simp-all add: unrest assms closure*)

lemma *seq-CRF-closed* [closure]:
assumes *P is CRF Q is CRF*
shows $(P ;; Q) \text{ is CRF}$
by (*rule CRF-intro, simp-all add: unrest assms closure*)

lemma *rea-st-cond-CRF* [closure]: $[b]_{S<} \text{ is CRF}$
by (*rel-auto*)

lemma *conj-CRF* [closure]: $\llbracket P \text{ is CRF}; Q \text{ is CRF} \rrbracket \implies (P \wedge Q) \text{ is CRF}$
by (*simp add: CRF-implies-CRR CRF-intro CRF-unrest-ref' CRR-implies-RR CRR-unrest-ref conj-RR unrest-conj*)

lemma *wp-rea-CRC* [closure]: $\llbracket P \text{ is CRR}; Q \text{ is RC} \rrbracket \implies P \text{ wp}_r Q \text{ is CRC}$
by (*rule CRC-intro, simp-all add: unrest closure*)

lemma *USUP-ind-CRC-closed* [closure]:
 $\llbracket \bigwedge i. P i \text{ is CRC} \rrbracket \implies (\bigsqcup i. P i) \text{ is CRC}$
by (*metis CRC-implies-CRR CRC-implies-RC USUP-ind-CRR-closed USUP-ind-RC-closed false-CRC rea-not-CRR-closed wp-rea-CRC wp-rea-RC-false*)

lemma *tr-extend-seqr-lit* [rdes]:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \ \$wait \# P \ \$ref \# P$
shows $(\$tr' =_u \$tr \hat{\ }_u \langle \ll a \gg) \wedge \$st' =_u \$st) ;; P = P[\$tr \hat{\ }_u \langle \ll a \gg] / \$tr]$
using *assms by (rel-auto, meson)*

lemma *tr-assign-comp* [rdes]:
fixes $P :: ('s, 'e) \text{ action}$
assumes $\$ok \# P \ \$wait \# P \ \$ref \# P$
shows $(\$tr' =_u \$tr \wedge \lceil \langle \sigma \rangle_a \rceil_S) ;; P = \lceil \sigma \rceil_{S\sigma} \dagger P$
using *assms by (rel-auto, meson)*

lemma *RR-msubst-tt*: $RR((P \ t) \llbracket t \rightarrow \&tt \rrbracket) = (RR \ (P \ t)) \llbracket t \rightarrow \&tt \rrbracket$
by (*rel-auto*)

lemma *RR-msubst-ref'*: $RR((P \ r) \llbracket r \rightarrow \$ref' \rrbracket) = (RR \ (P \ r)) \llbracket r \rightarrow \$ref' \rrbracket$
by (*rel-auto*)

lemma *msubst-tt-RR* [closure]: $\llbracket \bigwedge t. P \ t \text{ is RR} \rrbracket \implies (P \ t) \llbracket t \rightarrow \&tt \rrbracket \text{ is RR}$
by (*simp add: Healthy-def RR-msubst-tt*)

lemma *msubst-ref'-RR* [closure]: $\llbracket \bigwedge r. P \ r \text{ is RR} \rrbracket \implies (P \ r) \llbracket r \rightarrow \$ref' \rrbracket \text{ is RR}$
by (*simp add: Healthy-def RR-msubst-ref'*)

lemma *conj-less-tr-RR-closed* [closure]:

assumes P is CRR

shows $(P \wedge \$tr <_u \$tr')$ is CRR

proof –

have $CRR(CRR(P) \wedge \$tr <_u \$tr') = (CRR(P) \wedge \$tr <_u \$tr')$

apply (rel-auto, blast+)

using less-le apply fastforce+

done

thus ?thesis

by (metis Healthy-def assms)

qed

lemma *R4-CRR-closed* [closure]: P is CRR $\implies R_4(P)$ is CRR

by (simp add: R4-def conj-less-tr-RR-closed)

lemma *R5-CRR-closed* [closure]:

assumes P is CRR

shows $R_5(P)$ is CRR

proof –

have $R_5(CRR(P))$ is CRR

by (rel-auto; blast)

thus ?thesis

by (simp add: assms Healthy-if)

qed

lemma *conj-eq-tr-RR-closed* [closure]:

assumes P is CRR

shows $(P \wedge \$tr' =_u \$tr)$ is CRR

proof –

have $CRR(CRR(P) \wedge \$tr' =_u \$tr) = (CRR(P) \wedge \$tr' =_u \$tr)$

by (rel-auto, blast+)

thus ?thesis

by (metis Healthy-def assms)

qed

lemma *all-ref-CRC-closed* [closure]:

P is CRC $\implies (\forall \$ref \cdot P)$ is CRC

by (simp add: CRC-implies-CRR CRR-unrest-ref all-unrest)

lemma *ex-ref-CRR-closed* [closure]:

P is CRR $\implies (\exists \$ref \cdot P)$ is CRR

by (simp add: CRR-unrest-ref ex-unrest)

lemma *ex-st'-CRR-closed* [closure]:

P is CRR $\implies (\exists \$st' \cdot P)$ is CRR

by (rule CRR-intro, simp-all add: closure unrest)

lemma *ex-ref'-CRR-closed* [closure]:

P is CRR $\implies (\exists \$ref' \cdot P)$ is CRR

using CRR-implies-RR CRR-intro CRR-unrest-ref ex-ref'-RR-closed out-in-indep unrest-ex-diff by blast

3.3 Introduction laws

Extensionality principles for introducing refinement and equality of Circus reactive relations. It is necessary only to consider a subset of the variables that are present.

lemma *CRR-refine-ext*:

assumes

P is CRR Q is CRR

$\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] \sqsubseteq Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
shows $P \sqsubseteq Q$

proof –

have $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 $\sqsubseteq (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$

using *assms* **by** (*simp add: Healthy-if*)

hence $CRR P \sqsubseteq CRR Q$

by (*rel-auto*)

thus *?thesis*

by (*metis Healthy-if assms(1) assms(2)*)

qed

lemma *CRR-eq-ext*:

assumes

P is CRR Q is CRR

$\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] = Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
shows $P = Q$

proof –

have $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$
 $= (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$

using *assms* **by** (*simp add: Healthy-if*)

hence $CRR P = CRR Q$

by (*rel-auto*)

thus *?thesis*

by (*metis Healthy-if assms(1) assms(2)*)

qed

lemma *CRR-refine-impl-prop*:

assumes P is CRR Q is CRR

$\bigwedge t s s' r'. 'Q[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']' \implies 'P[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']'$
shows $P \sqsubseteq Q$

by (*rule CRR-refine-ext, simp-all add: assms closure unrest usubst*)

(*rule refine-prop-intro, simp-all add: unrest unrest-all-circus-vars closure assms*)

3.4 UTP Theory

interpretation *crf-theory*: *utp-theory-kleene CRF* II_c

rewrites $P \in \text{carrier } \text{crf-theory.thy-order} \longleftrightarrow P \text{ is CRF}$

and *le* *rrel-theory.thy-order* = (\sqsubseteq)

and *eq* *rrel-theory.thy-order* = (=)

and *crf-top*: *crf-theory.utp-top* = false

and *crf-bottom*: *crf-theory.utp-bottom* = *true_r*

proof –

interpret *utp-theory-continuous CRF*

by (*unfold-locales, simp-all add: add: CRF-idem Continuous-CRF*)

show *top:utp-top* = false

by (*simp add: healthy-top, rel-auto*)

show *bot:utp-bottom* = *true_r*

```

  by (simp add: healthy-bottom, rel-auto)
show utp-theory-kleene CRF  $II_c$ 
  by (unfold-locales, simp-all add: closure rpred top)
qed (simp-all)

```

abbreviation $crf\text{-}star :: - \Rightarrow -$ ($-^{*c}$ [999] 999) **where**
 $P^{*c} \equiv crf\text{-}theory.utp\text{-}star P$

lemma $crf\text{-}star\text{-}as\text{-}rea\text{-}star$:
 $P \text{ is } CRF \implies P^{*c} = P^{*r} ;; II_c$
by (simp add: crf-theory.Star-alt-def rrel-theory.Star-alt-def closure rpred unrest)

lemma $crf\text{-}star\text{-}inductl$: $R \text{ is } CRR \implies Q \sqsubseteq (P ;; Q) \sqcap R \implies Q \sqsubseteq P^{*c} ;; R$
by (simp add: crel-skip-left-unit crf-theory.utp-star-def upred-semiring.mult-assoc urel-ka.star-inductl)

3.5 Weakest Precondition

lemma $nil\text{-}least$ [simp]:
 $\langle \rangle \leq_u x = true$ **by** rel-auto

lemma $minus\text{-}nil$ [simp]:
 $xs - \langle \rangle = xs$ **by** rel-auto

lemma $wp\text{-}rea\text{-}circus\text{-}lemma\text{-}1$:
assumes $P \text{ is } CRR$ $\$ref' \# P$
shows $out\alpha \# P[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr' \rrbracket]$
proof –
have $out\alpha \# (CRR (\exists \$ref' \cdot P))[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr' \rrbracket]$
by (rel-auto)
thus ?thesis
by (simp add: Healthy-if assms(1) assms(2) ex-unrest)
qed

lemma $wp\text{-}rea\text{-}circus\text{-}lemma\text{-}2$:
assumes $P \text{ is } CRR$
shows $in\alpha \# P[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr \rrbracket]$
proof –
have $in\alpha \# (CRR P)[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr \rrbracket]$
by (rel-auto)
thus ?thesis
by (simp add: Healthy-if assms ex-unrest)
qed

The meaning of reactive weakest precondition for Circus. $P \text{ } wp_r \text{ } Q$ means that, whenever P terminates in a state s_0 having done the interaction trace t_0 , which is a prefix of the overall trace, then Q must be satisfied. This in particular means that the remainder of the trace after t_0 must not be a divergent behaviour of Q .

lemma $wp\text{-}rea\text{-}circus\text{-}form$:
assumes $P \text{ is } CRR$ $\$ref' \# P$ $Q \text{ is } CRC$
shows $(P \text{ } wp_r \text{ } Q) = (\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr' \rrbracket] \Rightarrow_r Q[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr \rrbracket])$
proof –
have $(P \text{ } wp_r \text{ } Q) = (\neg_r (\exists t_0 \cdot P[\llbracket \langle t_0 \rangle / \$tr' \rrbracket] ;; (\neg_r Q)[\llbracket \langle t_0 \rangle / \$tr \rrbracket] \wedge \langle t_0 \rangle \leq_u \$tr'))$
by (simp-all add: wp-rea-def R2-tr-middle closure assms)
also have $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr' \rrbracket] ;; (\neg_r Q)[\llbracket \langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr \rrbracket] \wedge \langle t_0 \rangle \leq_u \$tr'))$

by (*rel-blast*)
 also have ... = $(\neg_r (\exists (s_0, t_0) \cdot P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \wedge (\neg_r Q)[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr] \wedge \llbracket t_0 \rrbracket \leq_u \$tr'))$
 by (*simp add: segr-to-conj add: wp-rea-circus-lemma-1 wp-rea-circus-lemma-2 assms closure conj-assoc*)
 also have ... = $(\forall (s_0, t_0) \cdot \neg_r P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \vee \neg_r (\neg_r Q)[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr] \vee \neg_r \llbracket t_0 \rrbracket \leq_u \$tr')$
 by (*rel-auto*)
 also have ... = $(\forall (s_0, t_0) \cdot \neg_r P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \vee \neg_r (\neg_r RR Q)[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr] \vee \neg_r \llbracket t_0 \rrbracket \leq_u \$tr')$
 by (*simp add: Healthy-if assms closure*)
 also have ... = $(\forall (s_0, t_0) \cdot \neg_r P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \vee (RR Q)[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr] \vee \neg_r \llbracket t_0 \rrbracket \leq_u \$tr')$
 by (*rel-auto*)
 also have ... = $(\forall (s_0, t_0) \cdot \llbracket t_0 \rrbracket \leq_u \$tr' \wedge P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \Rightarrow_r (RR Q)[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr])$
 by (*rel-auto*)
 also have ... = $(\forall (s_0, t_0) \cdot \llbracket t_0 \rrbracket \leq_u \$tr' \wedge P[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st', \$tr'] \Rightarrow_r Q[\llbracket s_0 \rrbracket, \llbracket t_0 \rrbracket / \$st, \$tr])$
 by (*simp add: Healthy-if assms closure*)
 finally show ?thesis .
 qed

lemma *wp-rea-circus-form-alt*:

assumes *P* is CRR $\$ref' \# P$ *Q* is CRC

shows $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=}_u \llbracket t_0 \rrbracket \leq_u \$tr' \wedge P[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket t_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r R1(Q[\llbracket s_0 \rrbracket, \langle \rangle, (\&tt - \llbracket t_0 \rrbracket) / \$st, \$tr, \$tr']))$

proof –

have $(P \text{ wp}_r Q) = R2(P \text{ wp}_r Q)$

by (*simp add: CRC-implies-RR CRR-implies-RR Healthy-if RR-implies-R2 assms wp-rea-R2-closed*)

also have ... = $R2(\forall (s_0, tr_0) \cdot \llbracket tr_0 \rrbracket \leq_u \$tr' \wedge (RR P)[\llbracket s_0 \rrbracket, \llbracket tr_0 \rrbracket / \$st', \$tr'] \Rightarrow_r (RR Q)[\llbracket s_0 \rrbracket, \llbracket tr_0 \rrbracket / \$st, \$tr])$

by (*simp add: wp-rea-circus-form assms closure Healthy-if*)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \llbracket tr_0 \rrbracket \leq_u \llbracket tt_0 \rrbracket \wedge (RR P)[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket tr_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r (RR Q)[\llbracket s_0 \rrbracket, \llbracket tr_0 \rrbracket, \llbracket tt_0 \rrbracket / \$st, \$tr, \$tr'] \wedge \$tr' =_u \$tr \hat{=}_u \llbracket tt_0 \rrbracket))$

by (*simp add: R2-form, rel-auto*)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \llbracket tr_0 \rrbracket \leq_u \llbracket tt_0 \rrbracket \wedge (RR P)[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket tr_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r (RR Q)[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket tt_0 - tr_0 \rrbracket / \$st, \$tr, \$tr'] \wedge \$tr' =_u \$tr \hat{=}_u \llbracket tt_0 \rrbracket))$

by (*rel-auto*)

also have ... = $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \$tr \hat{=}_u \llbracket tr_0 \rrbracket \leq_u \$tr' \wedge (RR P)[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket tr_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r (RR Q)[\llbracket s_0 \rrbracket, \langle \rangle, (\&tt - \llbracket tr_0 \rrbracket) / \$st, \$tr, \$tr'] \wedge \$tr' =_u \$tr \hat{=}_u \llbracket tt_0 \rrbracket))$

by (*rel-auto, (metis list-concat-minus-list-concat)+*)

also have ... = $(\forall (s_0, tr_0) \cdot \$tr \hat{=}_u \llbracket tr_0 \rrbracket \leq_u \$tr' \wedge (RR P)[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket tr_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r R1((RR Q)[\llbracket s_0 \rrbracket, \langle \rangle, (\&tt - \llbracket tr_0 \rrbracket) / \$st, \$tr, \$tr']))$

by (*rel-auto, blast+*)

also have ... = $(\forall (s_0, t_0) \cdot \$tr \hat{=}_u \llbracket t_0 \rrbracket \leq_u \$tr' \wedge P[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket t_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r R1(Q[\llbracket s_0 \rrbracket, \langle \rangle, (\&tt - \llbracket t_0 \rrbracket) / \$st, \$tr, \$tr']))$

by (*simp add: Healthy-if assms closure*)

finally show ?thesis .

qed

lemma *wp-rea-circus-form-alt*:

assumes *P* is CRR $\$ref' \# P$ *Q* is CRC

shows $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=}_u \llbracket t_0 \rrbracket \leq_u \$tr' \wedge P[\llbracket s_0 \rrbracket, \langle \rangle, \llbracket t_0 \rrbracket / \$st', \$tr, \$tr'] \Rightarrow_r R1(Q[\llbracket s_0 \rrbracket, \langle \rangle, (\&tt - \llbracket t_0 \rrbracket) / \$st, \$tr, \$tr']))$

oops

3.6 Trace Substitution

definition *trace-subst* $(-\llbracket - \rrbracket_t [999, 0] 999)$
where $[upred-defs]: P\llbracket v \rrbracket_t = (P\llbracket (\&tt - \lceil v \rceil_{S<})/\&tt \rrbracket \wedge \$tr + \lceil v \rceil_{S<} \leq_u \$tr')$

lemma *unrest-trace-subst* $[unrest]$:
 $\llbracket mwb-lens\ x; x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \# P \rrbracket \implies x \# P\llbracket v \rrbracket_t$
by (*simp add: trace-subst-def lens-indep-sym unrest*)

lemma *trace-subst-RR-closed* $[closure]$:
assumes P *is* RR
shows $P\llbracket v \rrbracket_t$ *is* RR
proof –
have $(RR\ P)\llbracket v \rrbracket_t$ *is* RR
apply (*rel-auto*)
apply (*metis diff-add-cancel-left' trace-class.add-left-mono*)
apply (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)
using *le-add order-trans* **apply** *blast*
done
thus *?thesis*
by (*simp add: Healthy-if assms*)
qed

lemma *trace-subst-CRR-closed* $[closure]$:
assumes P *is* CRR
shows $P\llbracket v \rrbracket_t$ *is* CRR
by (*rule CRR-intro, simp-all add: closure assms unrest*)

lemma *tsubst-nil* $[usubst]$:
assumes P *is* CRR
shows $P\llbracket \langle \rangle \rrbracket_t = P$
proof –
have $(CRR\ P)\llbracket \langle \rangle \rrbracket_t = CRR\ P$
by (*rel-auto*)
thus *?thesis*
by (*simp add: Healthy-if assms*)
qed

lemma *tsubst-false* $[usubst]$: $false\llbracket y \rrbracket_t = false$
by *rel-auto*

lemma *cond-rea-tt-subst* $[usubst]$:
 $(P \triangleleft b \triangleright_R Q)\llbracket v \rrbracket_t = (P\llbracket v \rrbracket_t \triangleleft b \triangleright_R Q\llbracket v \rrbracket_t)$
by (*rel-auto*)

lemma *tsubst-conj* $[usubst]$: $(P \wedge Q)\llbracket v \rrbracket_t = (P\llbracket v \rrbracket_t \wedge Q\llbracket v \rrbracket_t)$
by (*rel-auto*)

lemma *tsubst-disj* $[usubst]$: $(P \vee Q)\llbracket v \rrbracket_t = (P\llbracket v \rrbracket_t \vee Q\llbracket v \rrbracket_t)$
by (*rel-auto*)

lemma *rea-subst-R1-closed* $[closure]$: $P\llbracket v \rrbracket_t$ *is* $R1$
apply (*rel-auto*) **using** *le-add order.trans* **by** *blast*

lemma *tsubst-UINF-ind* $[usubst]$: $(\bigcap i \cdot P(i))\llbracket v \rrbracket_t = (\bigcap i \cdot (P(i))\llbracket v \rrbracket_t)$
by (*rel-auto*)

3.7 Initial Interaction

definition $rea-init :: 's \text{ upred} \Rightarrow ('t::trace, 's) \text{ uexpr} \Rightarrow ('s, 't, 'a, 'b) \text{ rel-rsp } (\mathcal{I}'(-,-))$ **where**
 $[upred-defs]: \mathcal{I}(s,t) = ([s]_{S<} \Rightarrow_r \neg_r \$tr + [t]_{S<} \leq_u \$tr')$

lemma $usubst-rea-init' [usubst]:$

$\sigma \dagger_S \mathcal{I}(s,t) = \mathcal{I}(\sigma \dagger_S s, \sigma \dagger_S t)$

by $(rel-auto)$

lemma $unrest-rea-init [unrest]:$

$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v \rrbracket \Longrightarrow x \# \mathcal{I}(s,t)$

by $(simp \text{ add: } rea-init-def \text{ unrest } lens-indep-sym)$

lemma $rea-init-R1 [closure]: \mathcal{I}(s,t) \text{ is } R1$

by $(rel-auto)$

lemma $rea-init-R2c [closure]: \mathcal{I}(s,t) \text{ is } R2c$

apply $(rel-auto)$

apply $(metis \text{ le-add minus-cancel-le } trace-class.add-diff-cancel-left)$

apply $(metis \text{ diff-add-cancel-left' } trace-class.add-left-mono)$

done

lemma $rea-init-R2 [closure]: \mathcal{I}(s,t) \text{ is } R2$

by $(metis \text{ Healthy-def } R1-R2c-is-R2 \text{ rea-init-R1 } rea-init-R2c)$

lemma $csp-init-RR [closure]: \mathcal{I}(s,t) \text{ is } RR$

apply $(rel-auto)$

apply $(metis \text{ le-add minus-cancel-le } trace-class.add-diff-cancel-left)$

apply $(metis \text{ diff-add-cancel-left' } trace-class.add-left-mono)$

done

lemma $csp-init-CRR [closure]: \mathcal{I}(s,t) \text{ is } CRR$

by $(rule \text{ CRR-intro, simp-all add: unrest closure})$

lemma $rea-init-RC [closure]: \mathcal{I}(s,t) \text{ is } CRC$

apply $(rel-auto)$ **by** $fastforce$

lemma $rea-init-false [rpred]: \mathcal{I}(false, t) = true_r$

by $(rel-auto)$

lemma $rea-init-nil [rpred]: \mathcal{I}(s, \langle \rangle) = [\neg s]_{S<}$

by $(rel-auto)$

lemma $rea-not-init [rpred]: (\neg_r \mathcal{I}(P, \langle \rangle)) = \mathcal{I}(\neg P, \langle \rangle)$

by $(rel-auto)$

lemma $rea-init-conj [rpred]:$

$(\mathcal{I}(s_1, t) \wedge \mathcal{I}(s_2, t)) = \mathcal{I}(s_1 \vee s_2, t)$

by $(rel-auto)$

lemma $rea-init-disj-same [rpred]: (\mathcal{I}(s_1, t) \vee \mathcal{I}(s_2, t)) = \mathcal{I}(s_1 \wedge s_2, t)$

by $(rel-auto)$

3.8 Enabled Events

definition $csp\text{-}enable :: 's \text{ upred} \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('e \text{ set}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\mathcal{E}'(-, -, -))$
where

$[upred\text{-}defs]: \mathcal{E}(s, t, E) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\ }_u [t]_{S<} \wedge (\forall e \in [E]_{S<} \cdot \ll e \gg \notin_u \$ref'))$

Predicate $\mathcal{E}(s, t, E)$ states that, if the initial state satisfies predicate s , then t is a possible (failure) trace, such that the events in the set E are enabled after the given interaction.

lemma $csp\text{-}enable\text{-}R1\text{-}closed$ $[closure]: \mathcal{E}(s, t, E)$ is $R1$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}R2\text{-}closed$ $[closure]: \mathcal{E}(s, t, E)$ is $R2c$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}RR$ $[closure]: \mathcal{E}(s, t, E)$ is CRR
by $(rel\text{-}auto)$

lemma $tsubst\text{-}csp\text{-}enable$ $[usubst]: \mathcal{E}(s, t_2, e) \llbracket t_1 \rrbracket_t = \mathcal{E}(s, t_1 \hat{\ }_u t_2, e)$
apply $(rel\text{-}auto)$
apply $(metis \text{ append.assoc less-eq-list-def prefix-concat-minus})$
apply $(simp \text{ add: list-concat-minus-list-concat})$
done

lemma $csp\text{-}enable\text{-}unrests$ $[unrest]:$
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$ref')_v \rrbracket \Longrightarrow x \# \mathcal{E}(s, t, e)$
by $(simp \text{ add: csp-enable-def } R1\text{-def lens-indep-sym } unrest)$

lemma $st\text{-}unrest\text{-}csp\text{-}enable$ $[unrest]: \llbracket \&\mathbf{v} \# s; \&\mathbf{v} \# t; \&\mathbf{v} \# E \rrbracket \Longrightarrow \$st \# \mathcal{E}(s, t, E)$
by $(simp \text{ add: csp-enable-def } unrest)$

lemma $csp\text{-}enable\text{-}tr'\text{-}eq\text{-}tr$ $[rpred]:$
 $\mathcal{E}(s, \langle \rangle, r) \triangleleft \$tr' =_u \$tr \triangleright false = \mathcal{E}(s, \langle \rangle, r)$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}st\text{-}pred$ $[rpred]:$
 $([s_1]_{S<} \wedge \mathcal{E}(s_2, t, E)) = \mathcal{E}(s_1 \wedge s_2, t, E)$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}conj$ $[rpred]:$
 $(\mathcal{E}(s, t, E_1) \wedge \mathcal{E}(s, t, E_2)) = \mathcal{E}(s, t, E_1 \cup_u E_2)$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}cond$ $[rpred]:$
 $\mathcal{E}(s_1, t_1, E_1) \triangleleft b \triangleright_R \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_1 \triangleleft b \triangleright s_2, t_1 \triangleleft b \triangleright t_2, E_1 \triangleleft b \triangleright E_2)$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}rea\text{-}assm$ $[rpred]:$
 $[b]^\top_r ;; \mathcal{E}(s, t, E) = \mathcal{E}(b \wedge s, t, E)$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}tr\text{-}empty: \mathcal{E}(true, \langle \rangle, \{v\}_u) = (\$tr' =_u \$tr \wedge [v]_{S<} \notin_u \$ref')$
by $(rel\text{-}auto)$

lemma $csp\text{-}enable\text{-}nothing: \mathcal{E}(true, \langle \rangle, \{\}_u) = (\$tr' =_u \$tr)$
by $(rel\text{-}auto)$

lemma *msubst-nil-csp-enable* [*usubst*]:

$$\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \langle \rangle \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow \langle \rangle \rrbracket, t(x) \llbracket x \rightarrow \langle \rangle \rrbracket, E(x) \llbracket x \rightarrow \langle \rangle \rrbracket)$$

by (*pred-auto*)

lemma *msubst-csp-enable* [*usubst*]:

$$\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \lceil v \rceil_{S \leftarrow} \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow v \rrbracket, t(x) \llbracket x \rightarrow v \rrbracket, E(x) \llbracket x \rightarrow v \rrbracket)$$

by (*rel-auto*)

lemma *csp-enable-false* [*rpred*]: $\mathcal{E}(\text{false}, t, E) = \text{false}$

by (*rel-auto*)

lemma *conj-csp-enable* [*rpred*]: $(\mathcal{E}(b_1, t, E_1) \wedge \mathcal{E}(b_2, t, E_2)) = \mathcal{E}(b_1 \wedge b_2, t, E_1 \cup_u E_2)$

by (*rel-auto*)

lemma *refine-csp-enable*: $\mathcal{E}(b_1, t, E_1) \sqsubseteq \mathcal{E}(b_2, t, E_2) \longleftrightarrow ('b_2 \Rightarrow b_1 \wedge E_1 \subseteq_u E_2')$

by (*rel-blast*)

lemma *USUP-csp-enable* [*rpred*]:

$$(\bigsqcup x \cdot \mathcal{E}(s, t, A(x))) = \mathcal{E}(s, t, (\bigvee x \cdot A(x)))$$

by (*rel-auto*)

lemma *R4-csp-enable-nil* [*rpred*]:

$$R4(\mathcal{E}(s, \langle \rangle, E)) = \text{false}$$

by (*rel-auto*)

lemma *R5-csp-enable-nil* [*rpred*]:

$$R5(\mathcal{E}(s, \langle \rangle, E)) = \mathcal{E}(s, \langle \rangle, E)$$

by (*rel-auto*)

lemma *R4-csp-enable-Cons* [*rpred*]:

$$R4(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = \mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)$$

by (*rel-auto*, *simp add: Prefix-Order.strict-prefixI'*)

lemma *R5-csp-enable-Cons* [*rpred*]:

$$R5(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = \text{false}$$

by (*rel-auto*)

lemma *rel-aext-csp-enable* [*alpha*]:

$$vwb\text{-}lens \ a \Longrightarrow \mathcal{E}(s, t, E) \oplus_r map\text{-}st_L[a] = \mathcal{E}(s \oplus_p a, t \oplus_p a, E \oplus_p a)$$

by (*rel-auto*)

3.9 Completed Trace Interaction

definition *csp-do* :: $'s \text{ upred} \Rightarrow ('s \Rightarrow 's) \Rightarrow ('e \text{ list}, 's) \text{ uexpr} \Rightarrow ('s, 'e) \text{ action } (\Phi'(-, -, -))$ **where**
[upred-defs]: $\Phi(s, \sigma, t) = (\lceil s \rceil_{S <} \wedge \$tr' =_u \$tr \hat{\ }_u \lceil t \rceil_{S <} \wedge \lceil \langle \sigma \rangle_a \rceil_S)$

lemma *csp-do-eq-intro*:

$$\begin{aligned} &\text{assumes } s_1 = s_2 \ \sigma_1 = \sigma_2 \ t_1 = t_2 \\ &\text{shows } \Phi(s_1, \sigma_1, t_1) = \Phi(s_2, \sigma_2, t_2) \\ &\text{by (simp add: assms)} \end{aligned}$$

Predicate $\Phi(s, \sigma, t)$ states that if the initial state satisfies s , and the trace t is performed, then afterwards the state update σ is executed.

lemma *unrest-csp-do* [*unrest*]:

$$\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$st')_v \rrbracket \Longrightarrow x \# \Phi(s, \sigma, t)$$

by (simp-all add: csp-do-def alpha-in-var alpha-out-var prod-as-plus unrest lens-indep-sym)

lemma csp-do-CRF [closure]: $\Phi(s, \sigma, t)$ is CRF
 by (rel-auto)

lemma csp-do-R4-closed [closure]:
 $\Phi(b, \sigma, \text{bop Cons } x \text{ } xs)$ is R4
 by (rel-auto, simp add: Prefix-Order.strict-prefixI')

lemma st-pred-conj-csp-do [rpred]:
 $([b]_{s <} \wedge \Phi(s, \sigma, t)) = \Phi(b \wedge s, \sigma, t)$
 by (rel-auto)

lemma trea-subst-csp-do [usubst]:
 $(\Phi(s, \sigma, t_2)) \llbracket t_1 \rrbracket_t = \Phi(s, \sigma, t_1 \hat{\ }_u t_2)$
 apply (rel-auto)
 apply (metis append.assoc less-eq-list-def prefix-concat-minus)
 apply (simp add: list-concat-minus-list-concat)
 done

lemma st-subst-csp-do [usubst]:
 $[\sigma]_{s \sigma} \dagger \Phi(s, \varrho, t) = \Phi(\sigma \dagger s, \varrho \circ \sigma, \sigma \dagger t)$
 by (rel-auto)

lemma csp-do-nothing: $\Phi(\text{true}, \text{id}, \langle \rangle) = II_c$
 by (rel-auto)

lemma csp-do-nothing-0: $\Phi(\text{true}, \text{id}, 0) = II_c$
 by (rel-auto)

lemma csp-do-false [rpred]: $\Phi(\text{false}, s, t) = \text{false}$
 by (rel-auto)

lemma subst-state-csp-enable [usubst]:
 $[\sigma]_{s \sigma} \dagger \mathcal{E}(s, t_2, e) = \mathcal{E}(\sigma \dagger s, \sigma \dagger t_2, \sigma \dagger e)$
 by (rel-auto)

lemma csp-do-assign-enable [rpred]:
 $\Phi(s_1, \sigma, t_1) ;; \mathcal{E}(s_2, t_2, e) = \mathcal{E}(s_1 \wedge \sigma \dagger s_2, t_1 \hat{\ }_u (\sigma \dagger t_2), (\sigma \dagger e))$
 by (rel-auto)

lemma csp-do-assign-do [rpred]:
 $\Phi(s_1, \sigma, t_1) ;; \Phi(s_2, \varrho, t_2) = \Phi(s_1 \wedge (\sigma \dagger s_2), \varrho \circ \sigma, t_1 \hat{\ }_u (\sigma \dagger t_2))$
 by (rel-auto)

lemma csp-do-cond [rpred]:
 $\Phi(s_1, \sigma, t_1) \triangleleft b \triangleright_R \Phi(s_2, \varrho, t_2) = \Phi(s_1 \triangleleft b \triangleright s_2, \sigma \triangleleft b \triangleright_s \varrho, t_1 \triangleleft b \triangleright t_2)$
 by (rel-auto)

lemma rea-assm-csp-do [rpred]:
 $[b]^\top_r ;; \Phi(s, \sigma, t) = \Phi(b \wedge s, \sigma, t)$
 by (rel-auto)

lemma csp-do-comp:
 assumes P is CRR

shows $\Phi(s, \sigma, t) ;; P = ([s]_{S<} \wedge (\sigma \dagger_S P)) \llbracket t \rrbracket_t$
proof –
have $\Phi(s, \sigma, t) ;; (CRR P) = ([s]_{S<} \wedge ((\sigma \dagger_S CRR P)) \llbracket t \rrbracket_t)$
by (*rel-auto*; *blast*)
thus *?thesis*
by (*simp add: Healthy-if assms*)
qed

lemma *wp-rea-csp-do-lemma*:
fixes $P :: ('\sigma, '\varphi) \text{ action}$
assumes $\$ok \# P \ \$wait \# P \ \$ref \# P$
shows $(\llbracket \langle \sigma \rangle_a \rrbracket_S \wedge \$tr' =_u \$tr \hat{^}_u \llbracket t \rrbracket_{S<}) ;; P = (\llbracket \sigma \rrbracket_{S\sigma} \dagger P) \llbracket \$tr \hat{^}_u \llbracket t \rrbracket_{S<} / \$tr \rrbracket$
using *assms* **by** (*rel-auto*, *meson*)

This operator sets an upper bound on the permissible traces, when starting from a particular state

lemma *wp-rea-csp-do [wp]*:
 $\Phi(s_1, \sigma, t_1) \text{ wp}_r \mathcal{I}(s_2, t_2) = \mathcal{I}(s_1 \wedge \sigma \dagger s_2, t_1 \hat{^}_u \sigma \dagger t_2)$
by (*rel-auto*)

lemma *wp-rea-csp-do-false' [wp]*:
 $\Phi(s_1, \sigma, t_1) \text{ wp}_r \text{ false} = \mathcal{I}(s_1, t_1)$
by (*rel-auto*)

lemma *st-pred-impl-csp-do-wp [rpred]*:
 $([s_1]_{S<} \Rightarrow_r \Phi(s_2, \sigma, t) \text{ wp}_r P) = \Phi(s_1 \wedge s_2, \sigma, t) \text{ wp}_r P$
by (*rel-auto*)

lemma *csp-do-seq-USUP-distl [rpred]*:
assumes $\bigwedge i. i \in A \implies P(i) \text{ is } CRR A \neq \{\}$
shows $\Phi(s, \sigma, t) ;; (\bigwedge i \in A. P(i)) = (\bigwedge i \in A. \Phi(s, \sigma, t) ;; P(i))$
proof –
from *assms(2)* **have** $\Phi(s, \sigma, t) ;; (\bigsqcup i \in A. CRR(P(i))) = (\bigsqcup i \in A. \Phi(s, \sigma, t) ;; CRR(P(i)))$
by (*rel-blast*)
thus *?thesis*
by (*simp cong: USUP-cong add: assms(1) Healthy-if*)
qed

lemma *csp-do-seq-conj-distl*:
assumes $P \text{ is } CRR Q \text{ is } CRR$
shows $\Phi(s, \sigma, t) ;; (P \wedge Q) = (\Phi(s, \sigma, t) ;; P \wedge \Phi(s, \sigma, t) ;; Q)$
proof –
have $\Phi(s, \sigma, t) ;; (CRR(P) \wedge CRR(Q)) = ((\Phi(s, \sigma, t) ;; (CRR P)) \wedge (\Phi(s, \sigma, t) ;; (CRR Q)))$
by (*rel-blast*)
thus *?thesis*
by (*simp add: assms Healthy-if*)
qed

lemma *csp-do-power-Suc [rpred]*:
 $\Phi(\text{true}, id, t) \hat{^} (Suc i) = \Phi(\text{true}, id, \text{iter}[Suc i](t))$
by (*induct i, (rel-auto)+*)

lemma *csp-power-do-comp [rpred]*:
assumes $P \text{ is } CRR$
shows $\Phi(\text{true}, id, t) \hat{^} i ;; P = \Phi(\text{true}, id, \text{iter}[i](t)) ;; P$

apply (*cases i*)
apply (*simp-all add: csp-do-comp rpred usubst assms closure*)
done

lemma *csp-do-id* [*rpred*]:
 $P \text{ is } CRR \implies \Phi(b, id, \langle \rangle) ;; P = ([b]_{S<} \wedge P)$
by (*simp add: csp-do-comp usubst*)

lemma *csp-do-id-wp* [*wp*]:
 $P \text{ is } CRR \implies \Phi(b, id, \langle \rangle) \text{ wp}_r P = ([b]_{S<} \Rightarrow_r P)$
by (*metis (no-types, lifting) CRR-implies-RR RR-implies-R1 csp-do-id rea-impl-conj rea-impl-false rea-not-CRR-closed rea-not-not wp-rea-def*)

lemma *wp-rea-csp-do-st-pre* [*wp*]: $\Phi(s_1, \sigma, t_1) \text{ wp}_r [s_2]_{S<} = \mathcal{I}(s_1 \wedge \neg \sigma \dagger s_2, t_1)$
by (*rel-auto*)

lemma *wp-rea-csp-do-skip* [*wp*]:
fixes $Q :: (' \sigma, ' \varphi) \text{ action}$
assumes $P \text{ is } CRR$
shows $\Phi(s, \sigma, t) \text{ wp}_r P = (\mathcal{I}(s, t) \wedge (\sigma \dagger_S P) \llbracket t \rrbracket_t)$
apply (*simp add: wp-rea-def*)
apply (*subst csp-do-comp*)
apply (*simp-all add: closure assms usubst*)
oops

lemma *msubst-csp-do* [*usubst*]:
 $\Phi(s(x), \sigma, t(x)) \llbracket x \rightarrow [v]_{S\leftarrow} \rrbracket = \Phi(s(x) \llbracket x \rightarrow v \rrbracket, \sigma, t(x) \llbracket x \rightarrow v \rrbracket)$
by (*rel-auto*)

lemma *rea-frame-ext-csp-do* [*frame*]:
 $vwb\text{-}lens\ a \implies a: [\Phi(s, \sigma, t)]_r^+ = \Phi(s \oplus_p a, \sigma \oplus_s a, t \oplus_p a)$
by (*rel-auto*)

lemma *R5-csp-do-nil* [*rpred*]: $R5(\Phi(s, \sigma, \langle \rangle)) = \Phi(s, \sigma, \langle \rangle)$
by (*rel-auto*)

lemma *R5-csp-do-Cons* [*rpred*]: $R5(\Phi(s, \sigma, x \#_u xs)) = false$
by (*rel-auto*)

Iterated do relations

fun *titr* :: $nat \Rightarrow 's \text{ usubst} \Rightarrow ('a \text{ list}, 's) \text{ uexpr} \Rightarrow ('a \text{ list}, 's) \text{ uexpr}$ **where**
titr 0 σ $t = 0$ |
titr (*Suc n*) σ $t = (\textit{titr n } \sigma \textit{ t}) + (\sigma \hat{\wedge} n) \dagger t$

lemma *titr-as-list-sum*: $\textit{titr n } \sigma \textit{ t} = \textit{list-sum} (\textit{map} (\lambda i. (\sigma \hat{\wedge} i) \dagger t) [0..<n])$
apply (*induct n*)
apply (*auto simp add: usubst fold-plus-sum-list-rev foldr-conv-fold*)
done

lemma *titr-as-foldr*: $\textit{titr n } \sigma \textit{ t} = \textit{foldr} (\lambda i e. (\sigma \hat{\wedge} i) \dagger t + e) [0..<n] 0$
by (*simp add: titr-as-list-sum foldr-map comp-def*)

lemma *list-sum-uexpr-rep-eq*: $\llbracket \textit{list-sum xs} \rrbracket_e s = \textit{list-sum} (\textit{map} (\lambda e. \llbracket e \rrbracket_e s) xs)$
apply (*induct xs*)
apply (*simp-all*)

apply (pred-simp+)
done

lemma titr-rep-eq: $\llbracket \text{titr } n \ \sigma \ t \rrbracket_e s = \text{foldr } (@) \ (\text{map } (\lambda x. \llbracket t \rrbracket_e ((\sigma \hat{\ } x) \ s)) \ [0..<n]) \ []$
by (simp add: titr-as-list-sum list-sum-uexpr-rep-eq comp-def, rel-auto)

update-uexpr-rep-eq-thms

lemma funpow-lemma: $(\lambda x. (f \hat{\ } n) (f x)) = (f \hat{\ } n) \circ f$
by (simp add: fun-eq-iff funpow-swap1)

lemma titr-lemma:
 $t + (\sigma \dagger \text{titr } n \ \sigma \ t) + (\sigma \hat{\ } n \circ \sigma) \dagger t = (\text{titr } n \ \sigma \ t + (\sigma \hat{\ } n) \dagger t) + (\sigma \circ \sigma \hat{\ } n) \dagger t$
by (induct n, simp-all add: usubst funpow-lemma add.assoc funpow-swap1)

lemma csp-do-power [rpred]:
 $\Phi(s, \sigma, t)^\wedge(\text{Suc } n) = \Phi(\bigwedge i \in \{0..n\} \cdot (\sigma \hat{\ } i) \dagger s, \sigma \hat{\ } \text{Suc } n, \text{titr } (\text{Suc } n) \ \sigma \ t)$
apply (induct n)
apply (rel-auto)
apply (simp add: power.power.power-Suc rpred usubst)
apply (thin-tac -)
apply (rule csp-do-eq-intro)
apply (rel-auto)
apply (case-tac x=0)
apply (simp-all add: titr-lemma)
apply (metis Suc-le-mono funpow-simps-right(2) gr0-implies-Suc o-def)
apply force
apply (metis Suc-leI funpow-simps-right(2) less-Suc-eq-le o-apply)
apply (metis comp-apply funpow-swap1)
apply (metis add.assoc plus-list-def plus-uexpr-def titr-lemma)
done

lemma csp-do-rea-star [rpred]:
 $\Phi(s, \sigma, t)^{*r} = II_r \sqcap (\bigcap n \cdot \Phi(\bigwedge i \in \{0..n\} \cdot (\sigma \hat{\ } i) \dagger s, \sigma \hat{\ } \text{Suc } n, \text{titr } (\text{Suc } n) \ \sigma \ t))$
by (simp add: rrel-theory.Star-alt-def closure uplus-power-def rpred)

lemma csp-do-csp-star [rpred]:
 $\Phi(s, \sigma, t)^{*c} = (\bigcap n \cdot \Phi(\bigcup i \in \{0..<n\} \cdot (\sigma \hat{\ } i) \dagger s, \sigma \hat{\ } n, \text{titr } n \ \sigma \ t))$
(is ?lhs = $(\bigcap n \cdot ?G(n))$)

proof -

have ?lhs = $II_c \sqcap (\bigcap n \cdot \Phi(\bigwedge i \in \{0..n\} \cdot (\sigma \hat{\ } i) \dagger s, \sigma \hat{\ } \text{Suc } n, \text{titr } (\text{Suc } n) \ \sigma \ t))$
(is - = $II_c \sqcap (\bigcap n \cdot ?F(n))$)

by (simp add: crf-theory.Star-alt-def closure uplus-power-def rpred)

also have ... = $II_c \sqcap (\bigcap n \in \{1..\} \cdot ?F(n - 1))$

by (simp add: UINF-atLeast-Suc)

also have ... = $II_c \sqcap (\bigcap n \in \{1..\} \cdot \Phi(\bigcup i \in \{0..<n\} \cdot (\sigma \hat{\ } i) \dagger s, \sigma \hat{\ } n, \text{titr } n \ \sigma \ t))$

proof -

have $(\bigcap n \in \{1..\} \cdot ?F(n - 1)) = (\bigcap n \in \{1..\} \cdot ?G(n))$

by (rule UINF-cong, simp, metis Suc-pred atLeastLessThanSuc-atLeastAtMost diff-is-0-eq not0-implies-Suc not-less-eq-eq zero-less-Suc)

thus ?thesis by simp

qed

also have ... = $?G(0) \sqcap (\bigcap n \in \{1..\} \cdot ?G(n))$

by (simp add: usubst csp-do-nothing-0)

also have ... = $(\bigcap n \in \text{insert } 0 \ \{1..\} \cdot ?G(n))$


```

  by (simp)
also have ... = ( $\prod$  n • ?G(n))
proof -
  have insert (0::nat) {1..} = {0..} by auto
  thus ?thesis
    by simp
qed
finally show ?thesis .
qed

```

3.10 Assumptions

abbreviation *crf-assume* :: 's upred \Rightarrow ('s, 'e) action ($[-]_c$) **where**
 $[b]_c \equiv \Phi(b, id, \langle \rangle)$

lemma *crf-assume-true* [*rpred*]: *P* is CRR \Longrightarrow $[true]_c$;; $P = P$
 by (simp add: *crel-skip-left-unit csp-do-nothing*)

3.11 Downward closure of refusals

We define downward closure of the pericondition by the following healthiness condition

definition *CDC* :: ('s, 'e) action \Rightarrow ('s, 'e) action **where**
 $[upred-defs]$: $CDC(P) = (\exists \text{ ref}_0 \cdot P \llbracket \llbracket \text{ref}_0 \rrbracket / \$\text{ref}' \rrbracket \wedge \$\text{ref}' \subseteq_u \llbracket \text{ref}_0 \rrbracket)$

lemma *CDC-idem*: $CDC(CDC(P)) = CDC(P)$
 by (rel-auto)

lemma *CDC-Continuous* [*closure*]: *Continuous CDC*
 by (rel-auto)

lemma *CDC-RR-commute*: $CDC(RR(P)) = RR(CDC(P))$
 by (rel-blast)

lemma *CDC-RR-closed* [*closure*]: *P* is RR \Longrightarrow $CDC(P)$ is RR
 by (metis *CDC-RR-commute Healthy-def*)

lemma *CDC-CRR-commute*: $CDC(CRR P) = CRR(CDC P)$
 by (rel-blast)

lemma *CDC-CRR-closed* [*closure*]:
 assumes *P* is CRR
 shows $CDC(P)$ is CRR
 by (rule *CRR-intro*, simp add: *CDC-def unrest assms closure*, simp add: *unrest assms closure*)

lemma *CDC-unrest* [*unrest*]: $\llbracket vwb\text{-lens } x; (\$ref')_v \bowtie x; x \# P \rrbracket \Longrightarrow x \# CDC(P)$
 by (simp add: *CDC-def unrest usubst lens-indep-sym*)

lemma *CDC-R4-commute*: $CDC(R4(P)) = R4(CDC(P))$
 by (rel-auto)

lemma *R4-CDC-closed* [*closure*]: *P* is CDC \Longrightarrow $R4(P)$ is CDC
 by (simp add: *CDC-R4-commute Healthy-def*)

lemma *CDC-R5-commute*: $CDC(R5(P)) = R5(CDC(P))$
 by (rel-auto)

lemma *R5-CDC-closed* [closure]: P is CDC $\implies R5(P)$ is CDC
 by (simp add: CDC-R5-commute Healthy-def)

lemma *rea-true-CDC* [closure]: $true_r$ is CDC
 by (rel-auto)

lemma *false-CDC* [closure]: $false$ is CDC
 by (rel-auto)

lemma *CDC-UINF-closed* [closure]:
 assumes $\bigwedge i. i \in I \implies P\ i$ is CDC
 shows $(\bigcap i \in I. P\ i)$ is CDC
 using assms by (rel-blast)

lemma *CDC-disj-closed* [closure]:
 assumes P is CDC Q is CDC
 shows $(P \vee Q)$ is CDC
proof –
 have $CDC(P \vee Q) = (CDC(P) \vee CDC(Q))$
 by (rel-auto)
 thus ?thesis
 by (metis Healthy-def assms(1) assms(2))
qed

lemma *CDC-USUP-closed* [closure]:
 assumes $\bigwedge i. i \in I \implies P\ i$ is CDC
 shows $(\bigcup i \in I. P\ i)$ is CDC
 using assms by (rel-blast)

lemma *CDC-conj-closed* [closure]:
 assumes P is CDC Q is CDC
 shows $(P \wedge Q)$ is CDC
 using assms by (rel-auto, blast, meson)

lemma *CDC-rea-impl* [rpred]:
 $\$ref' \# P \implies CDC(P \Rightarrow_r Q) = (P \Rightarrow_r CDC(Q))$
 by (rel-auto)

lemma *rea-impl-CDC-closed* [closure]:
 assumes $\$ref' \# P\ Q$ is CDC
 shows $(P \Rightarrow_r Q)$ is CDC
 using assms by (simp add: CDC-rea-impl Healthy-def)

lemma *seq-CDC-closed* [closure]:
 assumes Q is CDC
 shows $(P ;; Q)$ is CDC
proof –
 have $CDC(P ;; Q) = P ;; CDC(Q)$
 by (rel-blast)
 thus ?thesis
 by (metis Healthy-def assms)
qed

lemma *st-subst-CDC-closed* [closure]:

assumes P is CDC
shows $(\sigma \uparrow_S P)$ is CDC
proof –
have $(\sigma \uparrow_S CDC P)$ is CDC
by (rel-auto)
thus ?thesis
by (simp add: assms Healthy-if)
qed

lemma *rea-st-cond-CDC* [closure]: $[g]_{S<}$ is CDC
by (rel-auto)

lemma *csp-enable-CDC* [closure]: $\mathcal{E}(s, t, E)$ is CDC
by (rel-auto)

lemma *state-srea-CDC-closed* [closure]:
assumes P is CDC
shows $state \ 'a \cdot P$ is CDC
proof –
have $state \ 'a \cdot CDC(P)$ is CDC
by (rel-blast)
thus ?thesis
by (simp add: Healthy-if assms)
qed

3.12 Renaming

abbreviation *pre-image* $f B \equiv \{x. f(x) \in B\}$

definition *csp-rename* :: $(\ 's, \ 'e) \text{ action} \Rightarrow (\ 'e \Rightarrow \ 'f) \Rightarrow (\ 's, \ 'f) \text{ action} ((-) \Downarrow_c [999, 0] 999)$ **where**
 $[upred-defs]: P \Downarrow_c = R2((\$tr' =_u \langle \rangle \wedge \$st' =_u \$st) ;; P ;; (\$tr' =_u map_u \ll f \gg \$tr \wedge \$st' =_u \$st \wedge uop \ (pre\text{-}image \ f) \ \$ref' \subseteq_u \$ref))$

lemma *csp-rename-CRR-closed* [closure]:
assumes P is CRR
shows $P \Downarrow_c$ is CRR
proof –
have $(CRR \ P) \Downarrow_c$ is CRR
by (rel-auto)
thus ?thesis **by** (simp add: assms Healthy-if)
qed

lemma *csp-rename-disj* [rpred]: $(P \vee Q) \Downarrow_c = (P \Downarrow_c \vee Q \Downarrow_c)$
by (rel-blast)

lemma *csp-rename-UINF-ind* [rpred]: $(\bigcap i \cdot P \ i) \Downarrow_c = (\bigcap i \cdot (P \ i) \Downarrow_c)$
by (rel-blast)

lemma *csp-rename-UINF-mem* [rpred]: $(\bigcap i \in A \cdot P \ i) \Downarrow_c = (\bigcap i \in A \cdot (P \ i) \Downarrow_c)$
by (rel-blast)

Renaming distributes through conjunction only when both sides are downward closed

lemma *csp-rename-conj* [rpred]:
assumes $inj \ f$ P is CRR Q is CRR P is CDC Q is CDC
shows $(P \wedge Q) \Downarrow_c = (P \Downarrow_c \wedge Q \Downarrow_c)$
proof –

```

from assms(1) have ((CDC (CRR P))  $\wedge$  (CDC (CRR Q)))( $\llbracket f \rrbracket_c$ ) = ((CDC (CRR P))( $\llbracket f \rrbracket_c$ )  $\wedge$  (CDC
(CRR Q))( $\llbracket f \rrbracket_c$ ))
  apply (rel-auto)
  apply blast
  apply blast
  apply (meson order-refl order-trans)
  done
thus ?thesis
  by (simp add: assms Healthy-if)
qed

```

```

lemma csp-rename-seq [rpred]:
  assumes P is CRR Q is CRR
  shows (P ;; Q)( $\llbracket f \rrbracket_c$ ) = P( $\llbracket f \rrbracket_c$ ) ;; Q( $\llbracket f \rrbracket_c$ )
  oops

```

```

lemma csp-rename-R4 [rpred]:
  (R4(P))( $\llbracket f \rrbracket_c$ ) = R4(P( $\llbracket f \rrbracket_c$ ))
  apply (rel-auto, blast)
  using less-le apply fastforce
  apply (metis (mono-tags, lifting) Prefix-Order.Nil-prefix append-Nil2 diff-add-cancel-left' less-le list.simps(8)
plus-list-def)
  done

```

```

lemma csp-rename-R5 [rpred]:
  (R5(P))( $\llbracket f \rrbracket_c$ ) = R5(P( $\llbracket f \rrbracket_c$ ))
  apply (rel-auto, blast)
  using less-le apply fastforce
  done

```

```

lemma csp-rename-do [rpred]:  $\Phi(s, \sigma, t)(\llbracket f \rrbracket_c) = \Phi(s, \sigma, \text{map}_u \llbracket f \rrbracket_c t)$ 
  by (rel-auto)

```

```

lemma csp-rename-enable [rpred]:  $\mathcal{E}(s, t, E)(\llbracket f \rrbracket_c) = \mathcal{E}(s, \text{map}_u \llbracket f \rrbracket_c t, \text{uop}(\text{image } f) E)$ 
  by (rel-auto)

```

```

lemma st'-unrest-csp-rename [unrest]:  $\$st' \# P \implies \$st' \# P(\llbracket f \rrbracket_c)$ 
  by (rel-blast)

```

```

lemma ref'-unrest-csp-rename [unrest]:  $\$ref' \# P \implies \$ref' \# P(\llbracket f \rrbracket_c)$ 
  by (rel-blast)

```

```

lemma csp-rename-CDC-closed [closure]:
  P is CDC  $\implies P(\llbracket f \rrbracket_c)$  is CDC
  by (rel-blast)

```

```

lemma csp-do-CDC [closure]:  $\Phi(s, \sigma, t)$  is CDC
  by (rel-auto)

```

end

4 Stateful-Failure Healthiness Conditions

```

theory utp-sfrd-healths
imports utp-sfrd-rel

```

begin

5 Definitions

We here define extra healthiness conditions for stateful-failure reactive designs.

abbreviation $CSP1 :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health}$
where $CSP1(P) \equiv RD1(P)$

abbreviation $CSP2 :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health}$
where $CSP2(P) \equiv RD2(P)$

abbreviation $CSP :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health}$
where $CSP(P) \equiv SRD(P)$

definition $STOP :: \varphi \text{ process where}$
 $[upred-defs]: STOP = CSP1(\$ok' \wedge R3c(\$tr' =_u \$tr \wedge \$wait'))$

definition $SKIP :: \varphi \text{ process where}$
 $[upred-defs]: SKIP = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$

definition $Stop :: (\sigma, \varphi) \text{ action where}$
 $[upred-defs]: Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \$wait'))$

definition $Skip :: (\sigma, \varphi) \text{ action where}$
 $[upred-defs]: Skip = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st))$

definition $CSP3 :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health where}$
 $[upred-defs]: CSP3(P) = (Skip ;; P)$

definition $CSP4 :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health where}$
 $[upred-defs]: CSP4(P) = (P ;; Skip)$

definition $NCSP :: ((\sigma, \varphi) \text{ sfrd} \times (\sigma, \varphi) \text{ sfrd}) \text{ health where}$
 $[upred-defs]: NCSP = CSP3 \circ CSP4 \circ CSP$

Productive and normal processes

abbreviation $PCSP \equiv Productive \circ NCSP$

Instantaneous and normal processes

abbreviation $ICSP \equiv ISRD1 \circ NCSP$

5.1 Healthiness condition properties

$SKIP$ is the same as $Skip$, and $STOP$ is the same as $Stop$, when we consider stateless CSP processes. This is because any reference to the st variable degenerates when the alphabet type coerces its type to be empty. We therefore need not consider $SKIP$ and $STOP$ actions.

theorem $SKIP\text{-is-Skip}$ $[simp]: SKIP = Skip$
by $(rel\text{-auto})$

theorem $STOP\text{-is-Stop}$ $[simp]: STOP = Stop$
by $(rel\text{-auto})$

theorem $Skip\text{-UTP-form}$: $Skip = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$

by (rel-auto)

lemma *Skip-is-CSP* [closure]:
Skip is CSP
 by (simp add: Skip-def RHS-design-is-SRD unrest)

lemma *Skip-RHS-tri-design*:
 $Skip = \mathbf{R}_s(true \vdash (false \diamond (\$tr' =_u \$tr \wedge \$st' =_u \$st)))$
 by (rel-auto)

lemma *Skip-RHS-tri-design'* [rdes-def]:
 $Skip = \mathbf{R}_s(true_r \vdash (false \diamond \Phi(true, id, \langle \rangle)))$
 by (rel-auto)

lemma *Skip-frame* [frame]: $vwb\text{-}lens\ a \implies a:[Skip]_R^+ = Skip$
 by (rdes-eq)

lemma *Stop-is-CSP* [closure]:
Stop is CSP
 by (simp add: Stop-def RHS-design-is-SRD unrest)

lemma *Stop-RHS-tri-design*: $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr) \diamond false)$
 by (rel-auto)

lemma *Stop-RHS-rdes-def* [rdes-def]: $Stop = \mathbf{R}_s(true_r \vdash \mathcal{E}(true, \langle \rangle, \{\}_u) \diamond false)$
 by (rel-auto)

lemma *preR-Skip* [rdes]: $pre_R(Skip) = true_r$
 by (rel-auto)

lemma *periR-Skip* [rdes]: $peri_R(Skip) = false$
 by (rel-auto)

lemma *postR-Skip* [rdes]: $post_R(Skip) = \Phi(true, id, \langle \rangle)$
 by (rel-auto)

lemma *Productive-Stop* [closure]:
Stop is Productive
 by (simp add: Stop-RHS-tri-design Healthy-def Productive-RHS-design-form unrest)

lemma *Skip-left-lemma*:
 assumes P is CSP
 shows $Skip \;; P = \mathbf{R}_s((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$
proof –
 have $Skip \;; P =$
 $\mathbf{R}_s((\$tr' =_u \$tr \wedge \$st' =_u \$st) \wp_r pre_R P \vdash$
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; peri_R P \diamond$
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; post_R P)$
 by (simp add: SRD-composition-wp alpha rdes closure wp assms rpred C1, rel-auto)
 also have $\dots = \mathbf{R}_s((\forall \$ref \cdot pre_R P) \vdash$
 $(\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st) \;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright cmt_R P))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have $\dots = \mathbf{R}_s((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 finally show ?thesis .

qed

lemma *Skip-left-unit-ref-unrest:*

assumes P is CSP $\$ref \# P \llbracket false/\$wait \rrbracket$

shows $Skip \;; P = P$

using *assms*

by (*simp add: Skip-left-lemma*)

(*metis SRD-reactive-design-alt all-unrest cmt-unrest-ref cmt-wait-false ex-unrest pre-unrest-ref pre-wait-false*)

lemma *CSP3-intro:*

$\llbracket P \text{ is CSP}; \$ref \# P \llbracket false/\$wait \rrbracket \rrbracket \implies P \text{ is CSP3}$

by (*simp add: CSP3-def Healthy-def' Skip-left-unit-ref-unrest*)

lemma *ref-unrest-RHS-design:*

assumes $\$ref \# P \ \$ref \# Q_1 \ \$ref \# Q_2$

shows $\$ref \# (\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)) \ f$

by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst assms*)

lemma *CSP3-SRD-intro:*

assumes P is CSP $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

shows P is CSP3

proof –

have $P: \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$

by (*simp add: SRD-reactive-design-alt assms(1) wait'-cond-peri-post-cmt[THEN sym]*)

have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$ is CSP3

by (*rule CSP3-intro, simp add: assms P, simp add: ref-unrest-RHS-design assms*)

thus *?thesis*

by (*simp add: P*)

qed

lemma *Skip-unrest-ref [unrest]:* $\$ref \# Skip \llbracket false/\$wait \rrbracket$

by (*simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

lemma *Skip-unrest-ref' [unrest]:* $\$ref' \# Skip \llbracket false/\$wait \rrbracket$

by (*simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

lemma *CSP3-iff:*

assumes P is CSP

shows P is CSP3 $\longleftrightarrow (\$ref \# P \llbracket false/\$wait \rrbracket)$

proof

assume $1: P$ is CSP3

have $\$ref \# (Skip \;; P) \llbracket false/\$wait \rrbracket$

by (*simp add: usubst unrest*)

with 1 **show** $\$ref \# P \llbracket false/\$wait \rrbracket$

by (*metis CSP3-def Healthy-def*)

next

assume $1: \$ref \# P \llbracket false/\$wait \rrbracket$

show P is CSP3

by (*simp add: 1 CSP3-intro assms*)

qed

lemma *CSP3-unrest-ref [unrest]:*

assumes P is CSP P is CSP3

shows $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

proof –

have $a: (\$ref \# P \llbracket false / \$wait \rrbracket)$
using $CSP3\text{-}iff\text{ assms}$ **by** $blast$
from a **show** $\$ref \# pre_R(P)$
by $(rel\text{-}blast)$
from a **show** $\$ref \# peri_R(P)$
by $(rel\text{-}blast)$
from a **show** $\$ref \# post_R(P)$
by $(rel\text{-}blast)$
qed

lemma $CSP3\text{-}rdes$:

assumes P is RR Q is RR R is RR
shows $CSP3(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\forall \$ref \cdot P) \vdash (\exists \$ref \cdot Q) \diamond (\exists \$ref \cdot R))$
by $(simp\ add: CSP3\text{-}def\ Skip\text{-}left\text{-}lemma\ closure\ assms\ rdes, rel\text{-}auto)$

lemma $CSP3\text{-}form$:

assumes P is CSP
shows $CSP3(P) = \mathbf{R}_s((\forall \$ref \cdot pre_R(P)) \vdash (\exists \$ref \cdot peri_R(P)) \diamond (\exists \$ref \cdot post_R(P)))$
by $(simp\ add: CSP3\text{-}def\ Skip\text{-}left\text{-}lemma\ assms, rel\text{-}auto)$

lemma $CSP3\text{-}Skip$ [closure]:

$Skip$ is $CSP3$
by $(rule\ CSP3\text{-}intro, simp\ add: Skip\text{-}is\text{-}CSP, simp\ add: Skip\text{-}def\ unrest)$

lemma $CSP3\text{-}Stop$ [closure]:

$Stop$ is $CSP3$
by $(rule\ CSP3\text{-}intro, simp\ add: Stop\text{-}is\text{-}CSP, simp\ add: Stop\text{-}def\ unrest)$

lemma $CSP3\text{-}Idempotent$ [closure]: $Idempotent\ CSP3$

by $(metis\ (no\text{-}types, lifting)\ CSP3\text{-}Skip\ CSP3\text{-}def\ Healthy\text{-}if\ Idempotent\text{-}def\ seqr\text{-}assoc)$

lemma $CSP3\text{-}Continuous$: $Continuous\ CSP3$

by $(simp\ add: Continuous\text{-}def\ CSP3\text{-}def\ seq\text{-}Sup\text{-}distl)$

lemma $Skip\text{-}right\text{-}lemma$:

assumes P is CSP
shows $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$

proof –

have $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash (\exists \$st' \cdot peri_R P) \diamond post_R P ;; (\$tr' =_u \$tr \wedge \$st' =_u \$st))$

by $(simp\ add: SRD\text{-}composition\text{-}wp\ closure\ assms\ wp\ rdes\ rpred, rel\text{-}auto)$

also have $\dots = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((cmt_R P ;; (\exists \$st \cdot [II]_D)) \triangleleft \$wait' \triangleright (cmt_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$

by $(rule\ cong[of\ \mathbf{R}_s\ \mathbf{R}_s], simp, rel\text{-}auto)$

also have $\dots = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (cmt_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$

by $(rule\ cong[of\ \mathbf{R}_s\ \mathbf{R}_s], simp, rel\text{-}auto)$

also have $\dots = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$

by $(rule\ cong[of\ \mathbf{R}_s\ \mathbf{R}_s], simp, rel\text{-}auto)$

finally show $?thesis$.

qed

lemma $Skip\text{-}right\text{-}tri\text{-}lemma$:

assumes P is CSP

shows $P \;; \text{Skip} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$
proof –
 have $((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)) = ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P))$
 by (*rel-auto*)
 thus ?thesis by (*simp add: Skip-right-lemma[OF assms]*)
qed

lemma CSP4-intro:

assumes $P \text{ is CSP } (\neg_r \text{pre}_R(P)) \;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$
 $\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$
 shows $P \text{ is CSP}_4$
proof –
 have $\text{CSP}_4(P) = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$
 by (*simp add: CSP4-def Skip-right-lemma assms(1)*)
 also have $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot \text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$
 by (*simp add: wp-rea-def assms(2) rpred closure cond-var-subst-left cond-var-subst-right*)
 also have $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket)))$
 by (*simp add: usubst unrest*)
 also have $\dots = \mathbf{R}_s (\text{pre}_R P \vdash ((\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$
 by (*simp add: ex-unrest assms*)
 also have $\dots = \mathbf{R}_s (\text{pre}_R P \vdash \text{cmt}_R P)$
 by (*simp add: cond-var-split*)
 also have $\dots = P$
 by (*simp add: SRD-reactive-design-alt assms(1)*)
 finally show ?thesis
 by (*simp add: Healthy-def'*)
qed

lemma CSP4-RC-intro:

assumes $P \text{ is CSP } \text{pre}_R(P) \text{ is RC}$
 $\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$
 shows $P \text{ is CSP}_4$
proof –
 have $(\neg_r \text{pre}_R(P)) \;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$
 by (*metis (no-types, lifting) R1-seqr-closure assms(2) rea-not-R1 rea-not-false rea-not-not wp-rea-RC-false wp-rea-def*)
 thus ?thesis
 by (*simp add: CSP4-intro assms*)
qed

lemma CSP4-rdes:

assumes $P \text{ is RR } Q \text{ is RR } R \text{ is RR}$
 shows $\text{CSP}_4(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s ((\neg_r P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot Q) \diamond (\exists \$ref' \cdot R)))$
 by (*simp add: CSP4-def Skip-right-lemma closure assms rdes, rel-auto, blast+*)

lemma CSP4-form:

assumes $P \text{ is CSP}$
 shows $\text{CSP}_4(P) = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$
 by (*simp add: CSP4-def Skip-right-tri-lemma assms*)

lemma Skip-srdes-right-unit:

$(\text{Skip} :: ('\sigma, '\varphi) \text{ action}) \;; \text{II}_R = \text{Skip}$

by (*rdes-simp*)

lemma *Skip-srdes-left-unit*:
 $\Pi_R \;; (Skip \:: (' \sigma, ' \varphi) \text{ action}) = Skip$
 by (*rdes-eq*)

lemma *CSP4-right-subsumes-RD3*: $RD3(CSP4(P)) = CSP4(P)$
 by (*metis (no-types, hide-lams) CSP4-def RD3-def Skip-srdes-right-unit seqr-assoc*)

lemma *CSP4-implies-RD3*: $P \text{ is } CSP4 \implies P \text{ is } RD3$
 by (*metis CSP4-right-subsumes-RD3 Healthy-def*)

lemma *CSP4-tri-intro*:
 assumes $P \text{ is } CSP (\neg_r \text{ pre}_R(P)) \;; R1(true) = (\neg_r \text{ pre}_R(P)) \$st' \# \text{peri}_R(P) \$ref' \# \text{post}_R(P)$
 shows $P \text{ is } CSP4$
 using *assms*
 by (*rule-tac CSP4-intro, simp-all add: pre_R-def peri_R-def post_R-def usubst cmt_R-def*)

lemma *CSP4-NSRD-intro*:
 assumes $P \text{ is } NSRD \$ref' \# \text{post}_R(P)$
 shows $P \text{ is } CSP4$
 by (*simp add: CSP4-tri-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri assms*)

lemma *CSP3-commutes-CSP4*: $CSP3(CSP4(P)) = CSP4(CSP3(P))$
 by (*simp add: CSP3-def CSP4-def seqr-assoc*)

lemma *NCSP-implies-CSP [closure]*: $P \text{ is } NCSP \implies P \text{ is } CSP$
 by (*metis (no-types, hide-lams) CSP3-def CSP4-def Healthy-def NCSP-def SRD-idem SRD-seqr-closure Skip-is-CSP comp-apply*)

lemma *NCSP-elim [RD-elim]*:
 $\llbracket X \text{ is } NCSP; P(\mathbf{R}_s(\text{pre}_R(X) \vdash \text{peri}_R(X) \diamond \text{post}_R(X))) \rrbracket \implies P(X)$
 by (*simp add: SRD-reactive-tri-design closure*)

lemma *NCSP-implies-CSP3 [closure]*:
 $P \text{ is } NCSP \implies P \text{ is } CSP3$
 by (*metis (no-types, lifting) CSP3-def Healthy-def' NCSP-def Skip-is-CSP Skip-left-unit-ref-unrest Skip-unrest-ref comp-apply seqr-assoc*)

lemma *NCSP-implies-CSP4 [closure]*:
 $P \text{ is } NCSP \implies P \text{ is } CSP4$
 by (*metis (no-types, hide-lams) CSP3-commutes-CSP4 Healthy-def NCSP-def NCSP-implies-CSP NCSP-implies-CSP3 comp-apply*)

lemma *NCSP-implies-RD3 [closure]*: $P \text{ is } NCSP \implies P \text{ is } RD3$
 by (*metis CSP3-commutes-CSP4 CSP4-right-subsumes-RD3 Healthy-def NCSP-def comp-apply*)

lemma *NCSP-implies-NSRD [closure]*: $P \text{ is } NCSP \implies P \text{ is } NSRD$
 by (*simp add: NCSP-implies-CSP NCSP-implies-RD3 SRD-RD3-implies-NSRD*)

lemma *NCSP-subset-implies-CSP [closure]*:
 $A \subseteq \llbracket NCSP \rrbracket_H \implies A \subseteq \llbracket CSP \rrbracket_H$
 using *NCSP-implies-CSP* by *blast*

lemma *NCSP-subset-implies-NSRD [closure]*:

$A \subseteq \llbracket NCSP \rrbracket_H \implies A \subseteq \llbracket NSRD \rrbracket_H$
using *NCSP-implies-NSRD* **by** *blast*

lemma *CSP-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket CSP \rrbracket_H \rrbracket \implies P \text{ is } CSP$
by (*simp add: is-Healthy-subset-member*)

lemma *CSP3-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket CSP3 \rrbracket_H \rrbracket \implies P \text{ is } CSP3$
by (*simp add: is-Healthy-subset-member*)

lemma *CSP4-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket CSP4 \rrbracket_H \rrbracket \implies P \text{ is } CSP4$
by (*simp add: is-Healthy-subset-member*)

lemma *NCSP-Healthy-subset-member*: $\llbracket P \in A; A \subseteq \llbracket NCSP \rrbracket_H \rrbracket \implies P \text{ is } NCSP$
by (*simp add: is-Healthy-subset-member*)

lemma *NCSP-intro*:
assumes *P is CSP P is CSP3 P is CSP4*
shows *P is NCSP*
by (*metis Healthy-def NCSP-def assms comp-eq-dest-lhs*)

lemma *Skip-left-unit*: $P \text{ is } NCSP \implies \text{Skip} ;; P = P$
by (*metis (full-types) CSP3-def Healthy-if NCSP-implies-CSP3*)

lemma *Skip-right-unit*: $P \text{ is } NCSP \implies P ;; \text{Skip} = P$
by (*metis (full-types) CSP4-def Healthy-if NCSP-implies-CSP4*)

lemma *NCSP-NSRD-intro*:
assumes $P \text{ is } NSRD \ \$ref \ \# \ pre_R(P) \ \$ref \ \# \ peri_R(P) \ \$ref \ \# \ post_R(P) \ \$ref' \ \# \ post_R(P)$
shows *P is NCSP*
by (*simp add: CSP3-SRD-intro CSP4-NSRD-intro NCSP-intro NSRD-is-SRD assms*)

lemma *CSP4-neg-pre-unit*:
assumes *P is CSP P is CSP4*
shows $(\neg_r \ pre_R(P)) ;; R1(true) = (\neg_r \ pre_R(P))$
by (*simp add: CSP4-implies-RD3 NSRD-neg-pre-unit SRD-RD3-implies-NSRD assms(1) assms(2)*)

lemma *NSRD-CSP4-intro*:
assumes *P is CSP P is CSP4*
shows *P is NSRD*
by (*simp add: CSP4-implies-RD3 SRD-RD3-implies-NSRD assms(1) assms(2)*)

lemma *NCSP-form*:
 $NCSP \ P = \mathbf{R}_s ((\forall \ \$ref \cdot (\neg_r \ pre_R(P)) \ wp_r \ false) \vdash ((\exists \ \$ref \cdot \exists \ \$st' \cdot peri_R(P)) \diamond (\exists \ \$ref \cdot \exists \ \$ref' \cdot post_R(P))))$
proof –
have $NCSP \ P = CSP3 \ (CSP4 \ (NSRD \ P))$
by (*metis (no-types, hide-lams) CSP4-def NCSP-def NSRD-alt-def RA1 RD3-def Skip-srdes-left-unit o-apply*)
also
have $\dots = \mathbf{R}_s ((\forall \ \$ref \cdot (\neg_r \ pre_R \ (NSRD \ P)) \ wp_r \ false) \vdash ((\exists \ \$ref \cdot \exists \ \$st' \cdot peri_R \ (NSRD \ P)) \diamond ((\exists \ \$ref \cdot \exists \ \$ref' \cdot post_R \ (NSRD \ P))))$
by (*simp add: CSP3-form CSP4-form closure unrest rdes, rel-auto*)
also have $\dots = \mathbf{R}_s ((\forall \ \$ref \cdot (\neg_r \ pre_R(P)) \ wp_r \ false) \vdash ((\exists \ \$ref \cdot \exists \ \$st' \cdot peri_R(P)) \diamond (\exists \ \$ref \cdot \exists \ \$ref' \cdot post_R(P))))$

by (simp add: NSRD-form rdes closure, rel-blast)
 finally show ?thesis .
 qed

lemma *CSP4-st'-unrest-peri* [unrest]:
 assumes *P is CSP P is CSP4*
 shows $\$st' \# \text{peri}_R(P)$
 by (simp add: NSRD-CSP4-intro NSRD-st'-unrest-peri assms)

lemma *CSP4-healthy-form*:
 assumes *P is CSP P is CSP4*
 shows $P = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond (\exists \$ref' \cdot \text{post}_R(P))))$
 proof –
 have $P = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$
 by (metis *CSP4-def Healthy-def Skip-right-lemma assms(1) assms(2)*)
 also have $\dots = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$
 by (metis (no-types, hide-lams) *subst-wait'-left-subst subst-wait'-right-subst wait'-cond-def*)
 also have $\dots = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond (\exists \$ref' \cdot \text{post}_R(P))))$
 by (simp add: *wait'-cond-def usubst peri_R-def post_R-def cmt_R-def unrest*)
 finally show ?thesis .
 qed

lemma *CSP4-ref'-unrest-pre* [unrest]:
 assumes *P is CSP P is CSP4*
 shows $\$ref' \# \text{pre}_R(P)$
 proof –
 have $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond (\exists \$ref' \cdot \text{post}_R(P))))$
 using *CSP4-healthy-form assms(1) assms(2)* by fastforce
 also have $\dots = (\neg_r \text{pre}_R P) \text{wp}_r \text{false}$
 by (simp add: *rea-pre-RHS-design wp-rea-def usubst unrest CSP4-neg-pre-unit R1-rea-not R2c-preR R2c-rea-not assms*)
 also have $\$ref' \# \dots$
 by (simp add: *wp-rea-def unrest*)
 finally show ?thesis .
 qed

lemma *NCSP-set-unrest-pre-wait'*:
 assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H$
 shows $\bigwedge P. P \in A \implies \$wait' \# \text{pre}_R(P)$
 proof –
 fix *P*
 assume $P \in A$
 hence *P is NSRD*
 using *NCSP-implies-NSRD assms* by auto
 thus $\$wait' \# \text{pre}_R(P)$
 using *NSRD-wait'-unrest-pre* by blast
 qed

lemma *CSP4-set-unrest-pre-st'*:
 assumes $A \subseteq \llbracket \text{CSP} \rrbracket_H \ A \subseteq \llbracket \text{CSP4} \rrbracket_H$
 shows $\bigwedge P. P \in A \implies \$st' \# \text{pre}_R(P)$
 proof –
 fix *P*
 assume $P \in A$

hence P is NSRD
 using NSRD-CSP₄-intro *assms(1) assms(2)* by blast
 thus $\$st' \# pre_R(P)$
 using NSRD- st' -unrest-pre by blast
 qed

lemma CSP₄-ref'-unrest-post [unrest]:
 assumes P is CSP P is CSP₄
 shows $\$ref' \# post_R(P)$
proof –
 have $post_R(P) = post_R(\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))$
 using CSP₄-healthy-form *assms(1) assms(2)* by fastforce
 also have $\dots = R1 (R2c ((\neg_r pre_R P) wp_r false \Rightarrow_r (\exists \$ref' \cdot post_R P)))$
 by (simp add: rea-post-RHS-design usubst unrest wp-rea-def)
 also have $\$ref' \# \dots$
 by (simp add: R1-def R2c-def wp-rea-def unrest)
 finally show ?thesis .
 qed

lemma CSP₃-Chaos [closure]: Chaos is CSP₃
 by (simp add: Chaos-def, rule CSP₃-intro, simp-all add: RHS-design-is-SRD unrest)

lemma CSP₄-Chaos [closure]: Chaos is CSP₄
 by (rule CSP₄-tri-intro, simp-all add: closure rdes unrest)

lemma NCSP-Chaos [closure]: Chaos is NCSP
 by (simp add: NCSP-intro closure)

lemma CSP₃-Miracle [closure]: Miracle is CSP₃
 by (simp add: Miracle-def, rule CSP₃-intro, simp-all add: RHS-design-is-SRD unrest)

lemma CSP₄-Miracle [closure]: Miracle is CSP₄
 by (rule CSP₄-tri-intro, simp-all add: closure rdes unrest)

lemma NCSP-Miracle [closure]: Miracle is NCSP
 by (simp add: NCSP-intro closure)

lemma NCSP-seqr-closure [closure]:
 assumes P is NCSP Q is NCSP
 shows $P ;; Q$ is NCSP
 by (metis (no-types, lifting) CSP₃-def CSP₄-def Healthy-def' NCSP-implies-CSP NCSP-implies-CSP₃
 NCSP-implies-CSP₄ NCSP-intro SRD-seqr-closure *assms(1) assms(2) seqr-assoc*)

lemma CSP₄-Skip [closure]: Skip is CSP₄
 apply (rule CSP₄-intro, simp-all add: Skip-is-CSP)
 apply (simp-all add: Skip-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true)
 done

lemma NCSP-Skip [closure]: Skip is NCSP
 by (metis CSP₃-Skip CSP₄-Skip Healthy-def NCSP-def Skip-is-CSP comp-apply)

lemma CSP₄-Stop [closure]: Stop is CSP₄
 apply (rule CSP₄-intro, simp-all add: Stop-is-CSP)
 apply (simp-all add: Stop-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true)
 done

lemma *NCSP-Stop* [closure]: *Stop* is *NCSP*

by (metis *CSP3-Stop CSP4-Stop Healthy-def NCSP-def Stop-is-CSP comp-apply*)

lemma *CSP4-Idempotent*: *Idempotent CSP4*

by (metis (no-types, lifting) *CSP3-Skip CSP3-def CSP4-def Healthy-if Idempotent-def seqr-assoc*)

lemma *CSP4-Continuous*: *Continuous CSP4*

by (simp add: *Continuous-def CSP4-def seq-Sup-distr*)

lemma *rdes-frame-ext-NCSP-closed* [closure]:

assumes *vwb-lens a P* is *NCSP*

shows $a:[P]_R^+$ is *NCSP*

by (metis (no-types, lifting) *CSP3-def CSP4-def Healthy-intro NCSP-Skip NCSP-implies-NSRD NCSP-intro NSRD-is-SRD Skip-frame Skip-left-unit Skip-right-unit assms(1) assms(2) rdes-frame-ext-NSRD-closed seq-srea-frame*)

lemma *preR-Stop* [rdes]: $\text{pre}_R(\text{Stop}) = \text{true}_r$

by (simp add: *Stop-def Stop-is-CSP rea-pre-RHS-design unrest usubst R2c-true*)

lemma *periR-Stop* [rdes]: $\text{peri}_R(\text{Stop}) = \mathcal{E}(\text{true}, \langle \rangle, \{\}_u)$

by (rel-auto)

lemma *postR-Stop* [rdes]: $\text{post}_R(\text{Stop}) = \text{false}$

by (rel-auto)

lemma *cmtR-Stop* [rdes]: $\text{cmt}_R(\text{Stop}) = (\$tr' =_u \$tr \wedge \$wait')$

by (rel-auto)

lemma *NCSP-Idempotent* [closure]: *Idempotent NCSP*

by (clarsimp simp add: *NCSP-def Idempotent-def*)

(metis (no-types, hide-lams) *CSP3-Idempotent CSP3-def CSP4-Idempotent CSP4-def Healthy-def Idempotent-def SRD-idem SRD-seqr-closure Skip-is-CSP seqr-assoc*)

lemma *NCSP-Continuous* [closure]: *Continuous NCSP*

by (simp add: *CSP3-Continuous CSP4-Continuous Continuous-comp NCSP-def SRD-Continuous*)

lemma *preR-CRR* [closure]: P is *NCSP* $\implies \text{pre}_R(P)$ is *CRR*

by (rule *CRR-intro*, simp-all add: closure unrest)

lemma *periR-CRR* [closure]: P is *NCSP* $\implies \text{peri}_R(P)$ is *CRR*

by (rule *CRR-intro*, simp-all add: closure unrest)

lemma *postR-CRR* [closure]: P is *NCSP* $\implies \text{post}_R(P)$ is *CRR*

by (rule *CRR-intro*, simp-all add: closure unrest)

lemma *NCSP-rdes-intro* [closure]:

assumes P is *CRC* Q is *CRR* R is *CRR*

$\$st' \# Q \ \$ref' \# R$

shows $\mathbf{R}_s(P \vdash Q \diamond R)$ is *NCSP*

apply (rule *NCSP-intro*)

apply (simp-all add: closure assms)

apply (rule *CSP3-SRD-intro*)

apply (simp-all add: rdes closure assms unrest)

apply (rule *CSP4-tri-intro*)

apply (*simp-all add: rdes closure assms unrest*)
apply (*metis (no-types, lifting) CRC-implies-RC R1-seqr-closure assms(1) rea-not-R1 rea-not-false*
rea-not-not wp-rea-RC-false wp-rea-def)
done

lemma *NCSP-preR-CRC [closure]:*
assumes *P is NCSP*
shows *pre_R(P) is CRC*
by (*rule CRC-intro, simp-all add: closure assms unrest*)

lemma *NCSP-postR-CRF [closure]: P is NCSP \implies post_R P is CRF*
by (*rule CRF-intro, simp-all add: unrest closure*)

lemma *CSP3-Sup-closure [closure]:*
 $A \subseteq \llbracket \text{CSP3} \rrbracket_H \implies (\bigwedge A) \text{ is CSP3}$
apply (*auto simp add: CSP3-def Healthy-def seq-Sup-distl*)
apply (*rule cong[of Sup]*)
apply (*simp*)
using *image-iff* **apply** *force*
done

lemma *CSP4-Sup-closure [closure]:*
 $A \subseteq \llbracket \text{CSP4} \rrbracket_H \implies (\bigwedge A) \text{ is CSP4}$
apply (*auto simp add: CSP4-def Healthy-def seq-Sup-distr*)
apply (*rule cong[of Sup]*)
apply (*simp*)
using *image-iff* **apply** *force*
done

lemma *NCSP-Sup-closure [closure]: $\llbracket A \subseteq \llbracket \text{NCSP} \rrbracket_H; A \neq \{\} \rrbracket \implies (\bigwedge A) \text{ is NCSP}$*
apply (*rule NCSP-intro, simp-all add: closure*)
apply (*metis (no-types, lifting) Ball-Collect CSP3-Sup-closure NCSP-implies-CSP3*)
apply (*metis (no-types, lifting) Ball-Collect CSP4-Sup-closure NCSP-implies-CSP4*)
done

lemma *NCSP-SUP-closure [closure]: $\llbracket \bigwedge i. P(i) \text{ is NCSP}; A \neq \{\} \rrbracket \implies (\bigwedge i \in A. P(i)) \text{ is NCSP}$*
by (*metis (mono-tags, lifting) Ball-Collect NCSP-Sup-closure image-iff image-is-empty*)

lemma *PCSP-implies-NCSP [closure]:*
assumes *P is PCSP*
shows *P is NCSP*
proof –
have $P = \text{Productive}(\text{NCSP}(\text{NCSP } P))$
by (*metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)

also have $\dots = \mathbf{R}_s ((\forall \$ref \cdot (\neg_r \text{pre}_R(\text{NCSP } P)) \text{ wp}_r \text{ false}) \vdash$
 $(\exists \$ref \cdot \exists \$st' \cdot \text{peri}_R(\text{NCSP } P)) \diamond$
 $((\exists \$ref \cdot \exists \$ref' \cdot \text{post}_R(\text{NCSP } P)) \wedge \$tr <_u \$tr'))$

by (*simp add: NCSP-form Productive-RHS-design-form unrest closure*)

also have $\dots \text{ is NCSP}$
apply (*rule NCSP-rdes-intro*)
apply (*rule CRC-intro*)
apply (*simp-all add: unrest ex-unrest all-unrest closure*)
done

finally show *?thesis* .

qed

lemma *PCSP-elim* [RD-elim]:

assumes *X is PCSP P* ($\mathbf{R}_s ((pre_R X) \vdash peri_R X \diamond (R4(post_R X))))$)

shows *P X*

by (*metis R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms comp-apply*)

lemma *ICSP-implies-NCSP* [closure]:

assumes *P is ICSP*

shows *P is NCSP*

proof –

have $P = ISRD1(NCSP(NCSP P))$

by (*metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)

also have $\dots = ISRD1(\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$
 $(\exists \$ref \cdot \exists \$st' \cdot peri_R (NCSP P)) \diamond$
 $(\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P))))$

by (*simp add: NCSP-form*)

also have $\dots = \mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$
 $false \diamond$
 $((\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P)) \wedge \$tr' =_u \$tr))$

by (*simp-all add: ISRD1-RHS-design-form closure rdes unrest*)

also have \dots *is NCSP*

apply (*rule NCSP-rdes-intro*)

apply (*rule CRC-intro*)

apply (*simp-all add: unrest ex-unrest all-unrest closure*)

done

finally show *?thesis* .

qed

lemma *ICSP-implies-ISRD* [closure]:

assumes *P is ICSP*

shows *P is ISRD*

by (*metis (no-types, hide-lams) Healthy-def ICSP-implies-NCSP ISRD-def NCSP-implies-NSRD assms comp-apply*)

lemma *ICSP-elim* [RD-elim]:

assumes *X is ICSP P* ($\mathbf{R}_s ((pre_R X) \vdash false \diamond (post_R X \wedge \$tr' =_u \$tr))$)

shows *P X*

by (*metis Healthy-if NCSP-implies-CSP ICSP-implies-NCSP ISRD1-form assms comp-apply*)

lemma *ICSP-Stop-right-zero-lemma*:

$(P \wedge (\$tr' =_u \$tr)) ;; true_r = true_r \implies (P \wedge (\$tr' =_u \$tr)) ;; (\$tr' =_u \$tr) = (\$tr' =_u \$tr)$

by (*rel-blast*)

lemma *ICSP-Stop-right-zero*:

assumes *P is ICSP* $pre_R(P) = true_r post_R(P) ;; true_r = true_r$

shows $P ;; Stop = Stop$

proof –

from *assms(3)* **have** $1:(post_R P \wedge \$tr' =_u \$tr) ;; true_r = true_r$

by (*rel-auto, metis (full-types, hide-lams) dual-order.antisym order-refl*)

show *?thesis*

by (*rdes-simp cls: assms(1), simp add: csp-enable-nothing assms(2) ICSP-Stop-right-zero-lemma[OF 1]*)

qed

lemma *ICSP-intro*: $\llbracket P \text{ is NCSP}; P \text{ is ISRD1} \rrbracket \implies P \text{ is ICSP}$
using *Healthy-comp* **by** *blast*

lemma *seq-ICSP-closed* [*closure*]:
assumes $P \text{ is ICSP } Q \text{ is ICSP}$
shows $P ;; Q \text{ is ICSP}$
by (*meson ICSP-implies-ISRD ICSP-implies-NCSP ICSP-intro ISRD-implies-ISRD1 NCSP-seqr-closure assms seq-ISRD-closed*)

lemma *Miracle-ICSP* [*closure*]: *Miracle is ICSP*
by (*rule ICSP-intro, simp add: closure, simp add: ISRD1-rdes-intro rdes-def closure*)

5.2 CSP theories

lemma *NCSP-false*: $NCSP \text{ false} = \text{Miracle}$
by (*simp add: NCSP-def srdes-theory.healthy-top[THEN sym], simp add: closure Healthy-if*)

lemma *NCSP-true*: $NCSP \text{ true} = \text{Chaos}$
by (*simp add: NCSP-def srdes-theory.healthy-bottom[THEN sym], simp add: closure Healthy-if*)

interpretation *csp-theory*: *utp-theory-kleene NCSP Skip*
rewrites $P \in \text{carrier csp-theory.thy-order} \longleftrightarrow P \text{ is NCSP}$
and $\text{carrier csp-theory.thy-order} \rightarrow \text{carrier csp-theory.thy-order} \equiv \llbracket NCSP \rrbracket_H \rightarrow \llbracket NCSP \rrbracket_H$
and $\text{le csp-theory.thy-order} = (\sqsubseteq)$
and $\text{eq csp-theory.thy-order} = (=)$
and *csp-top*: $\text{csp-theory.utp-top} = \text{Miracle}$
and *csp-bottom*: $\text{csp-theory.utp-bottom} = \text{Chaos}$

proof –

have *utp-theory-continuous NCSP*
by (*unfold-locales, simp-all add: Healthy-Idempotent Healthy-if NCSP-Idempotent NCSP-Continuous*)
then interpret *utp-theory-continuous NCSP*
by *simp*
show $t: \text{utp-top} = \text{Miracle}$ **and** $b: \text{utp-bottom} = \text{Chaos}$
by (*simp-all add: healthy-top healthy-bottom NCSP-false NCSP-true*)
show *utp-theory-kleene NCSP Skip*
by (*unfold-locales, simp-all add: closure Skip-left-unit Skip-right-unit Miracle-left-zero t*)
qed (*simp-all*)

abbreviation *TestC* (*test_C*) **where**
 $\text{test}_C P \equiv \text{csp-theory.utp-test } P$

definition *StarC* :: $(\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action} \rightarrow (-)^C$ [999] 999) **where**
 $\text{StarC } P \equiv \text{csp-theory.utp-star } P$

lemma *StarC-unfold*: $P \text{ is NCSP} \implies P^{*C} = \text{Skip} \sqcap (P ;; P^{*C})$
by (*simp add: StarC-def csp-theory.Star-unfoldl-eq*)

lemma *sfrd-star-as-rdes-star*:
 $P \text{ is NCSP} \implies P^{*R} ;; \text{Skip} = P^{*C}$
by (*simp add: csp-theory.Star-alt-def nsrdes-theory.Star-alt-def StarC-def StarR-def closure unrest Skip-srdes-left-unit csp-theory.Unit-Right*)

lemma *sfrd-star-as-rdes-star'*:
 $P \text{ is NCSP} \implies \text{Skip} ;; P^{*R} = P^{*C}$
by (*simp add: csp-theory.Star-alt-def nsrdes-theory.Star-alt-def StarC-def StarR-def closure unrest Skip-srdes-right-unit csp-theory.Unit-Left upred-semiring.distrib-left*)

theorem *csp-star-rdes-def* [*rdes-def*]:
assumes *P* is CRC *Q* is CRR *R* is CRF $\$st' \# Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))^{*C} = \mathbf{R}_s(R^{*c} \text{ wp}_r P \vdash (R^{*c} ;; Q) \diamond R^{*c})$
apply (*simp add: wp-rea-def sfrd-star-as-rdes-star[THEN sym] crf-star-as-rea-star assms segr-assoc rpred closure unrest StarR-rdes-def*)
apply (*simp add: rdes-def assms closure unrest wp-rea-def[THEN sym]*)
apply (*simp add: wp rpred assms closure*)
apply (*simp add: csp-do-nothing*)
done

5.3 Algebraic laws

lemma *Stop-left-zero*:
assumes *P* is CSP
shows *Stop* ;; *P* = *Stop*
by (*simp add: NSRD-seq-post-false assms NCSP-implies-NSRD NCSP-Stop postR-Stop*)
end

6 Stateful-Failure Reactive Contracts

theory *utp-sfrd-contracts*
imports *utp-sfrd-healths*
begin

definition *mk-CRD* :: '*s* upred \Rightarrow ('*e* list \Rightarrow '*e* set \Rightarrow '*s* upred) \Rightarrow ('*e* list \Rightarrow '*s* hrel) \Rightarrow ('*s*, '*e*) action
where
[*rdes-def*]: *mk-CRD* *P* *Q* *R* = $\mathbf{R}_s([P]_{S<} \vdash [Q \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket \diamond [R(x)]_S \llbracket x \rightarrow \&tt \rrbracket)$

syntax
-ref-var :: *logic*
-mk-CRD :: *uexp* \Rightarrow *uexp* \Rightarrow *logic* \Rightarrow *logic* (*-* / *⊢* *-* / *|* *-*)_{*C*}

parse-translation (
let
fun *ref-var-tr* [] = *Syntax.free refs*
| *ref-var-tr* - = *raise Match*;
in
[(@{*syntax-const -ref-var*}, *K ref-var-tr*)]
end
)

translations
-mk-CRD *P* *Q* *R* => *CONST mk-CRD* *P* (λ *-trace-var -ref-var.* *Q*) (λ *-trace-var.* *R*)
-mk-CRD *P* *Q* *R* <= *CONST mk-CRD* *P* (λ *x r.* *Q*) (λ *y.* *R*)

lemma *CSP-mk-CRD [closure]*: $[P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C$ is CSP
by (*simp add: mk-CRD-def closure unrest*)

lemma *preR-mk-CRD [rdes]*: $\text{pre}_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = [P]_{S<}$
by (*simp add: mk-CRD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre rea-st-cond-def, rel-auto*)

lemma *periR-mk-CRD [rdes]*: $\text{peri}_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([Q \text{ trace refs}]_{S<} \llbracket (\text{trace}, \text{refs}) \rightarrow (\&tt, \n

by (simp add: mk-CRD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre
impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto)

lemma *postR-mk-CRD* [rdes]: $\text{post}_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([R(\text{trace})]_S') \llbracket \text{trace} \rightarrow \& \text{tt} \rrbracket)$
by (simp add: mk-CRD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre
impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto)

Refinement introduction law for contracts

lemma *CRD-contract-refine*:

assumes

$Q \text{ is CSP } '[P_1]_{S<} \Rightarrow \text{pre}_R Q'$
 $'[P_1]_{S<} \wedge \text{peri}_R Q \Rightarrow [P_2 \ t \ r]_{S<} \llbracket t \rightarrow \& \text{tt} \rrbracket \llbracket r \rightarrow \$\text{ref}' \rrbracket'$
 $'[P_1]_{S<} \wedge \text{post}_R Q \Rightarrow [P_3 \ x]_S \llbracket x \rightarrow \& \text{tt} \rrbracket'$

shows $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$

proof –

have $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$

using *assms* **by** (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

thus ?thesis

by (simp add: SRD-reactive-tri-design assms(1))

qed

lemma *CRD-contract-refine'*:

assumes

$Q \text{ is CSP } '[P_1]_{S<} \Rightarrow \text{pre}_R Q'$
 $[P_2 \ t \ r]_{S<} \llbracket t \rightarrow \& \text{tt} \rrbracket \llbracket r \rightarrow \$\text{ref}' \rrbracket \sqsubseteq ([P_1]_{S<} \wedge \text{peri}_R Q)$
 $[P_3 \ x]_S \llbracket x \rightarrow \& \text{tt} \rrbracket \sqsubseteq ([P_1]_{S<} \wedge \text{post}_R Q)$

shows $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$

using *assms* **by** (rule-tac CRD-contract-refine, simp-all add: refBy-order)

lemma *CRD-refine-CRD*:

assumes

$'[P_1]_{S<} \Rightarrow ([Q_1]_{S<} :: ('e, 's) \text{ action})'$
 $([P_2 \ x \ r]_{S<} \llbracket x \rightarrow \& \text{tt} \rrbracket \llbracket r \rightarrow \$\text{ref}' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge [Q_2 \ x \ r]_{S<} \llbracket x \rightarrow \& \text{tt} \rrbracket \llbracket r \rightarrow \$\text{ref}' \rrbracket :: ('e, 's) \text{ action})$
 $[P_3 \ x]_S \llbracket x \rightarrow \& \text{tt} \rrbracket \sqsubseteq ([P_1]_{S<} \wedge [Q_3 \ x]_S \llbracket x \rightarrow \& \text{tt} \rrbracket :: ('e, 's) \text{ action})$

shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq [Q_1 \vdash Q_2 \text{ trace refs} \mid Q_3 \text{ trace}]_C$

using *assms*

by (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

lemma *CRD-refine-rdes*:

assumes

$'[P_1]_{S<} \Rightarrow Q_1'$
 $([P_2 \ x \ r]_{S<} \llbracket x \rightarrow \& \text{tt} \rrbracket \llbracket r \rightarrow \$\text{ref}' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2)$
 $[P_3 \ x]_S \llbracket x \rightarrow \& \text{tt} \rrbracket \sqsubseteq ([P_1]_{S<} \wedge Q_3)$

shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq$

$\mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$

using *assms*

by (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

lemma *CRD-refine-rdes'*:

assumes

$Q_2 \text{ is RR}$
 $Q_3 \text{ is RR}$
 $'[P_1]_{S<} \Rightarrow Q_1'$
 $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$\text{ref}' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2 \llbracket \langle \rangle, \langle t \rangle / \$\text{tr}, \$\text{tr}' \rrbracket)$
 $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge Q_3 \llbracket \langle \rangle, \langle t \rangle / \$\text{tr}, \$\text{tr}' \rrbracket)$

shows $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq$
 $\mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
proof (*simp add: mk-CRD-def, rule srdes-tri-refine-intro*)
show $'[P_1]_{S<} \Rightarrow Q_1'$ **by** (*fact assms(3)*)

have $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_2) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$
by (*simp add: assms Healthy-if*)
hence $'[P_1]_{S<} \wedge RR(Q_2) \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket'$
by (*rel-simp; meson*)
thus $'[P_1]_{S<} \wedge Q_2 \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket'$
by (*simp add: Healthy-if assms*)

have $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_3) \llbracket \langle \rangle, \ll t \gg / \$tr, \$tr' \rrbracket)$
by (*simp add: assms Healthy-if*)
hence $'[P_1]_{S<} \wedge (RR \ Q_3) \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket'$
by (*rel-simp; meson*)
thus $'[P_1]_{S<} \wedge Q_3 \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket'$
by (*simp add: Healthy-if assms*)
qed

end

7 External Choice

theory *utp-sfrd-extchoice*
imports
utp-sfrd-healths
utp-sfrd-rel
begin

7.1 Definitions and syntax

definition *ExtChoice* ::
 $('σ, 'φ) \text{ action set} \Rightarrow ('σ, 'φ) \text{ action where}$
 $[upred-defs]: \text{ExtChoice } A = \mathbf{R}_s((\bigsqcup P \in A \cdot pre_R(P)) \vdash ((\bigsqcup P \in A \cdot cmt_R(P)) \triangleleft \$tr' =_u \$tr \wedge \$wait'$
 $\triangleright (\bigsqcup P \in A \cdot cmt_R(P))))$

syntax
 $-ExtChoice :: ptnr \Rightarrow 'a \text{ set} \Rightarrow 'b \Rightarrow 'b \ ((\exists \square \ - \in \ - \cdot / \ -) [0, 0, 10] \ 10)$
 $-ExtChoice-simp :: ptnr \Rightarrow 'b \Rightarrow 'b \ ((\exists \square \ - \cdot / \ -) [0, 10] \ 10)$

translations
 $\square P \in A \cdot B \quad \Rightarrow \text{CONST } ExtChoice \ ((\lambda P. B) \cdot A)$
 $\square P \cdot B \quad \Rightarrow \text{CONST } ExtChoice \ (\text{CONST range } (\lambda P. B))$

definition *extChoice* ::
 $('σ, 'φ) \text{ action} \Rightarrow ('σ, 'φ) \text{ action} \Rightarrow ('σ, 'φ) \text{ action (infixl } \square \ 59) \text{ where}$
 $[upred-defs]: P \square Q \equiv ExtChoice \ \{P, Q\}$

Small external choice as an indexed big external choice.

lemma *extChoice-alt-def*:
 $P \square Q = (\square i :: nat \in \{0, 1\} \cdot P \triangleleft \ll i = 0 \gg \triangleright Q)$
by (*simp add: extChoice-def ExtChoice-def*)

7.2 Basic laws

7.3 Algebraic laws

lemma *ExtChoice-empty*: $\text{ExtChoice } \{\} = \text{Stop}$
 by (*simp add: ExtChoice-def cond-def Stop-def*)

lemma *ExtChoice-single*:
 $P \text{ is CSP} \implies \text{ExtChoice } \{P\} = P$
 by (*simp add: ExtChoice-def usup-and uinf-or SRD-reactive-design-alt*)

7.4 Reactive design calculations

lemma *ExtChoice-rdes*:
 assumes $\bigwedge i. \$ok' \nmid P(i) \ A \neq \{\}$
 shows $(\bigsqcup_{i \in A} \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\bigsqcup_{i \in A} \cdot P(i)) \vdash ((\bigsqcup_{i \in A} \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigsqcup_{i \in A} \cdot Q(i))))$

proof –

have $(\bigsqcup_{i \in A} \cdot \mathbf{R}_s(P(i) \vdash Q(i))) =$
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot \text{pre}_R(\mathbf{R}_s(P(i) \vdash Q(i))) \vdash$
 $((\bigsqcup_{i \in A} \cdot \text{cmt}_R(\mathbf{R}_s(P(i) \vdash Q(i)))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\bigsqcup_{i \in A} \cdot \text{cmt}_R(\mathbf{R}_s(P(i) \vdash Q(i))))))$
 by (*simp add: ExtChoice-def*)

also have ... =
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$
 $((\bigsqcup_{i \in A} \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\bigsqcup_{i \in A} \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$
 by (*simp add: rea-pre-RHS-design rea-cmt-RHS-design*)

also have ... =
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$
 $R1(R2c$
 $((\bigsqcup_{i \in A} \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\bigsqcup_{i \in A} \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$
 by (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c*)

also have ... =
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$
 $R1(R2c$
 $((\bigsqcup_{i \in A} \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\bigsqcup_{i \in A} \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$
 by (*simp add: R2c-UINF R2c-cond R1-cond R1-idem R1-R2c-commute R2c-idem R1-UINF assms R1-USUP R2c-USUP*)

also have ... =
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$
 $\text{cmt}_s \dagger$
 $((\bigsqcup_{i \in A} \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\bigsqcup_{i \in A} \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$
 by (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c rdes-export-cmt*)

also have ... =
 $\mathbf{R}_s((\bigsqcup_{i \in A} \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$
 $\text{cmt}_s \dagger$
 $((\bigsqcup_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$

$\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
by (*simp add: usubst*)
also have ... =
 $\mathbf{R}_s ((\prod_{i \in A} \cdot R1 (R2c (pre_s \dagger P(i)))) \vdash$
 $((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
by (*simp add: rdes-export-cmt*)
also have ... =
 $\mathbf{R}_s ((R1(R2c(\prod_{i \in A} \cdot (pre_s \dagger P(i)))) \vdash$
 $((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
by (*simp add: not-UINF R1-UINF R2c-UINF assms*)
also have ... =
 $\mathbf{R}_s ((R2c(\prod_{i \in A} \cdot (pre_s \dagger P(i)))) \vdash$
 $((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
by (*simp add: R1-design-R1-pre*)
also have ... =
 $\mathbf{R}_s (((\prod_{i \in A} \cdot (pre_s \dagger P(i)))) \vdash$
 $((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
by (*metis (no-types, lifting) RHS-design-R2c-pre*)
also have ... =
 $\mathbf{R}_s ([\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger (\prod_{i \in A} \cdot P(i))) \vdash$
 $((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))))$
proof –
from *assms* **have** $\bigwedge i. pre_s \dagger P(i) = [\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger P(i)$
by (*rel-auto*)
thus *?thesis*
by (*simp add: usubst*)
qed
also have ... =
 $\mathbf{R}_s ((\prod_{i \in A} \cdot P(i)) \vdash ((\prod_{i \in A} \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot (P(i) \Rightarrow$
 $Q(i))))$
by (*simp add: rdes-export-pre not-UINF*)
also have ... = $\mathbf{R}_s ((\prod_{i \in A} \cdot P(i)) \vdash ((\prod_{i \in A} \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot Q(i))))$
by (*rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto, blast+*)

finally show *?thesis* .
qed

lemma *ExtChoice-tri-rdes*:
assumes $\bigwedge i. \$ok' \nmid P_1(i) \ A \neq \{\}$
shows $(\prod_{i \in A} \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$
 $\mathbf{R}_s ((\prod_{i \in A} \cdot P_1(i)) \vdash (((\prod_{i \in A} \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\prod_{i \in A} \cdot P_2(i))) \diamond (\prod_{i \in A} \cdot$
 $P_3(i))))$
proof –
have $(\prod_{i \in A} \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$
 $\mathbf{R}_s ((\prod_{i \in A} \cdot P_1(i)) \vdash ((\prod_{i \in A} \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod_{i \in A} \cdot P_2(i) \diamond$
 $P_3(i))))$
by (*simp add: ExtChoice-rdes assms*)
also
have ... =
 $\mathbf{R}_s ((\prod_{i \in A} \cdot P_1(i)) \vdash ((\prod_{i \in A} \cdot P_2(i) \diamond P_3(i)) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (\prod_{i \in A} \cdot P_2(i) \diamond$
 $P_3(i))))$
by (*simp add: conj-comm*)
also
have ... =

$\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))) \diamond (\sqcap i \in A \cdot P_2(i) \diamond P_3(i))))$
 by (*simp add: cond-conj wait'-cond-def*)
 also
 have ... = $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap i \in A \cdot P_2(i))) \diamond (\sqcap i \in A \cdot P_3(i))))$
 by (*rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto*)
 finally show ?thesis .
 qed

lemma *ExtChoice-tri-rdes'* [*rdes-def*]:
 assumes $\bigwedge i . \$ok' \# P_1(i) \ A \neq \{\}$
 shows $(\sqcap i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$
 $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot R5(P_2(i))) \vee (\sqcap i \in A \cdot R4(P_2(i)))) \diamond (\sqcap i \in A \cdot P_3(i))))$
 by (*simp add: ExtChoice-tri-rdes assms, rel-auto, simp-all add: less-le assms*)

lemma *ExtChoice-tri-rdes-def* [*rdes-def*]:
 assumes $A \subseteq \llbracket CSP \rrbracket_H$
 shows *ExtChoice* $A = \mathbf{R}_s ((\sqcup P \in A \cdot pre_R P) \vdash (((\sqcup P \in A \cdot peri_R P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot post_R P)) \diamond (\sqcap P \in A \cdot post_R P)))$
proof –
 have $((\sqcup P \in A \cdot cmt_R P) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\sqcap P \in A \cdot cmt_R P)) =$
 $((\sqcup P \in A \cdot cmt_R P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot cmt_R P)) \diamond (\sqcap P \in A \cdot cmt_R P)$
 by (*rel-auto*)
 also have ... = $((\sqcup P \in A \cdot peri_R P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot peri_R P)) \diamond (\sqcap P \in A \cdot post_R P)$
 by (*rel-auto*)
 finally show ?thesis
 by (*simp add: ExtChoice-def*)
 qed

lemma *extChoice-rdes*:
 assumes $\$ok' \# P_1 \ \$ok' \# Q_1$
 shows $\mathbf{R}_s(P_1 \vdash P_2) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2) = \mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$
proof –
 have $(\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s (P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright \mathbf{R}_s (Q_1 \vdash Q_2)) = (\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright (Q_1 \vdash Q_2)))$
 by (*simp only: RHS-cond R2c-lit*)
 also have ... = $(\sqcap i::nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \triangleleft \ll i = 0 \gg \triangleright Q_1) \vdash (P_2 \triangleleft \ll i = 0 \gg \triangleright Q_2)))$
 by (*simp add: design-condr*)
 also have ... = $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$
 by (*subst ExtChoice-rdes, simp-all add: assms unrest uinf-or usup-and*)
 finally show ?thesis by (*simp add: extChoice-alt-def*)
 qed

lemma *extChoice-tri-rdes*:
 assumes $\$ok' \# P_1 \ \$ok' \# Q_1$
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
proof –
 have $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$
 by (*simp add: extChoice-rdes assms*)
 also
 have ... = $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$

by (simp add: conj-comm)
 also
 have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash$
 $((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)) \diamond (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$
 by (simp add: cond-conj wait'-cond-def)
 also
 have ... = $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 finally show ?thesis .
 qed

lemma extChoice-rdes-def:
 assumes P_1 is RR Q_1 is RR
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (subst extChoice-tri-rdes, simp-all add: assms unrest)

lemma extChoice-rdes-def' [rdes-def]:
 assumes P_1 is RR Q_1 is RR
 shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((R5(P_2 \wedge Q_2) \vee R4(P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$
 by (simp add: extChoice-rdes-def assms, rel-auto, simp-all add: less-le)

lemma CSP-ExtChoice [closure]:
 ExtChoice A is CSP
 by (simp add: ExtChoice-def RHS-design-is-SRD unrest)

lemma CSP-extChoice [closure]:
 $P \sqcap Q$ is CSP
 by (simp add: CSP-ExtChoice extChoice-def)

lemma preR-ExtChoice [rdes]:
 assumes $A \neq \{\}$ $A \subseteq \llbracket CSP \rrbracket_H$
 shows $pre_R(ExtChoice A) = (\bigsqcup_{P \in A} P \cdot pre_R(P))$
proof –
 have $pre_R(ExtChoice A) = (R1 (R2c ((\bigsqcup_{P \in A} P \cdot pre_R(P))))$
 by (simp add: ExtChoice-def rea-pre-RHS-design usubst unrest)
 also from assms have ... = $(R1 (R2c (\bigsqcup_{P \in A} P \cdot (pre_R(CSP(P)))))$
 by (metis USUP-healthy)
 also from assms have ... = $(\bigsqcup_{P \in A} P \cdot (pre_R(CSP(P))))$
 by (rel-auto)
 also from assms have ... = $(\bigsqcup_{P \in A} P \cdot (pre_R(P)))$
 by (metis USUP-healthy)
 finally show ?thesis .
 qed

lemma preR-ExtChoice-ind [rdes]:
 assumes $A \neq \{\} \wedge P. P \in A \implies F(P)$ is CSP
 shows $pre_R(\bigsqcup_{P \in A} P \cdot F(P)) = (\bigsqcup_{P \in A} P \cdot pre_R(F(P)))$
 using assms by (subst preR-ExtChoice, auto)

lemma periR-ExtChoice [rdes]:
 assumes $A \subseteq \llbracket NCSP \rrbracket_H$ $A \neq \{\}$
 shows $peri_R(ExtChoice A) = ((\bigsqcup_{P \in A} P \cdot pre_R(P)) \Rightarrow_r (\bigsqcup_{P \in A} P \cdot peri_R(P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcup_{P \in A} P \cdot peri_R(P)))$

proof –

have $\text{peri}_R (\text{ExtChoice } A) = \text{peri}_R (\mathbf{R}_s ((\sqcup P \in A \cdot \text{pre}_R P)) \vdash$
 $((\sqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot \text{peri}_R P)) \diamond$
 $(\sqcap P \in A \cdot \text{post}_R P)))$
by (*simp add: ExtChoice-tri-rdes-def assms closure*)

also have $\dots = \text{peri}_R (\mathbf{R}_s ((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \vdash$
 $((\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P)) \triangleleft \$tr' =_u \$tr \triangleright (\sqcap P \in A \cdot \text{peri}_R (\text{NCSP } P))) \diamond$
 $(\sqcap P \in A \cdot \text{post}_R P)))$
by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have $\dots = R1 (R2c ((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r$
 $(\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\sqcap P \in A \cdot \text{peri}_R (\text{NCSP } P))))$

proof –

have $(\sqcup P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{true}] \dagger \text{pre}_R (\text{NCSP } P))$
 $= (\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P))$

by (*rule USUP-cong, simp add: closure usubst unrest assms*)

thus *?thesis*

by (*simp add: rea-peri-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)

qed

also have $\dots = R1 ((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r$
 $(\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $(\sqcap P \in A \cdot \text{peri}_R (\text{NCSP } P)))$

by (*simp add: R2c-rea-impl R2c-cond R2c-UINF R2c-preR R2c-periR R2c-tr'-minus-tr R2c-USUP closure*)

also have $\dots = (((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\sqcap P \in A \cdot \text{peri}_R (\text{NCSP } P))))$

by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure, rel-auto*)

also have $\dots = (((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\sqcap P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r \text{peri}_R (\text{NCSP } P))))$

by (*simp add: UINF-rea-impl[THEN sym]*)

also have $\dots = (((\sqcup P \in A \cdot \text{pre}_R (\text{NCSP } P)) \Rightarrow_r (\sqcup P \in A \cdot \text{peri}_R (\text{NCSP } P)))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\sqcap P \in A \cdot \text{peri}_R (\text{NCSP } P))))$

by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have $\dots = (((\sqcup P \in A \cdot \text{pre}_R P) \Rightarrow_r (\sqcup P \in A \cdot \text{peri}_R P))$
 $\triangleleft \$tr' =_u \$tr \triangleright$
 $((\sqcap P \in A \cdot \text{peri}_R P))))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

finally show *?thesis* .

qed

lemma *periR-ExtChoice'*:

assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ $A \neq \{\}$

shows $\text{peri}_R (\text{ExtChoice } A) = (R5((\sqcup P \in A \cdot \text{pre}_R(P)) \Rightarrow_r (\sqcup P \in A \cdot \text{peri}_R P)) \vee (\sqcap P \in A \cdot R4(\text{peri}_R P)))$

using *assms(2)*

by (*simp add: periR-ExtChoice assms(1), rel-auto*)

lemma *periR-ExtChoice-ind* [*rdes*]:

assumes $\bigwedge P. P \in A \implies F(P)$ is NCSP $A \neq \{\}$
shows $\text{peri}_R(\Box P \in A \cdot F(P)) = ((\Box P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\Box P \in A \cdot \text{peri}_R(F P))) \triangleleft \$tr' =_u \$tr$
 $\triangleright (\Box P \in A \cdot \text{peri}_R(F P))$
using *assms* **by** (*subst periR-ExtChoice*, *auto simp add: closure unrest*)

lemma *periR-ExtChoice-ind'*:

assumes $\bigwedge P. P \in A \implies F(P)$ is NCSP $A \neq \{\}$
shows $\text{peri}_R(\Box P \in A \cdot F(P)) = (R5((\Box P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\Box P \in A \cdot \text{peri}_R(F P))) \vee (\Box P \in A \cdot R4(\text{peri}_R(F P))))$
using *assms* **by** (*subst periR-ExtChoice'*, *auto simp add: closure unrest*)

lemma *postR-ExtChoice [rdes]*:

assumes $A \subseteq \llbracket \text{NCSP} \rrbracket_H A \neq \{\}$
shows $\text{post}_R(\text{ExtChoice } A) = (\Box P \in A \cdot \text{post}_R P)$

proof –

have $\text{post}_R(\text{ExtChoice } A) = \text{post}_R(\mathbf{R}_s((\Box P \in A \cdot \text{pre}_R P) \vdash ((\Box P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\Box P \in A \cdot \text{peri}_R P)) \diamond (\Box P \in A \cdot \text{post}_R P))))$
by (*simp add: ExtChoice-tri-rdes-def closure assms*)

also have $\dots = \text{post}_R(\mathbf{R}_s((\Box P \in A \cdot \text{pre}_R(\text{NCSP } P)) \vdash ((\Box P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\Box P \in A \cdot \text{peri}_R P)) \diamond (\Box P \in A \cdot \text{post}_R(\text{NCSP } P))))$
by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have $\dots = R1(R2c((\Box P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\Box P \in A \cdot \text{post}_R(\text{NCSP } P))))$

proof –

have $(\Box P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{false}] \dagger \text{pre}_R(\text{NCSP } P)) = (\Box P \in A \cdot \text{pre}_R(\text{NCSP } P))$
by (*rule USUP-cong, simp add: usubst closure unrest assms*)
thus *?thesis*
by (*simp add: rea-post-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)

qed

also have $\dots = R1((\Box P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\Box P \in A \cdot \text{post}_R(\text{NCSP } P)))$

by (*simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-postR R2c-tr'-minus-tr R2c-USUP closure*)

also from *assms(2)* **have** $\dots = ((\Box P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\Box P \in A \cdot \text{post}_R(\text{NCSP } P)))$

by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure*)

also have $\dots = (\Box P \in A \cdot \text{pre}_R(\text{NCSP } P) \Rightarrow_r \text{post}_R(\text{NCSP } P))$

by (*simp add: UINF-rea-impl*)

also have $\dots = (\Box P \in A \cdot \text{post}_R(\text{NCSP } P))$

by (*simp add: SRD-post-under-pre closure assms unrest*)

finally show *?thesis*

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

qed

lemma *postR-ExtChoice-ind [rdes]*:

assumes $\bigwedge P. P \in A \implies F(P)$ is NCSP $A \neq \{\}$
shows $\text{post}_R(\Box P \in A \cdot F(P)) = (\Box P \in A \cdot \text{post}_R(F(P)))$
using *assms* **by** (*subst postR-ExtChoice*, *auto simp add: closure unrest*)

lemma *preR-extChoice*:

assumes P is CSP Q is CSP $\$wait' \nmid \text{pre}_R(P)$ $\$wait' \nmid \text{pre}_R(Q)$
shows $\text{pre}_R(P \Box Q) = (\text{pre}_R(P) \wedge \text{pre}_R(Q))$
by (*simp add: extChoice-def preR-ExtChoice assms usup-and*)

```

lemma preR-extChoice' [rdes]:
  assumes P is NCSP Q is NCSP
  shows  $\text{pre}_R(P \sqcap Q) = (\text{pre}_R(P) \wedge \text{pre}_R(Q))$ 
  by (simp add: preR-extChoice closure assms unrest)

lemma periR-extChoice [rdes]:
  assumes P is NCSP Q is NCSP
  shows  $\text{peri}_R(P \sqcap Q) = ((\text{pre}_R(P) \wedge \text{pre}_R(Q) \Rightarrow_r \text{peri}_R(P) \wedge \text{peri}_R(Q)) \triangleleft \$tr' =_u \$tr \triangleright (\text{peri}_R(P) \vee \text{peri}_R(Q)))$ 
  using assms
  by (simp add: extChoice-def, subst periR-ExtChoice, auto simp add: usup-and uinf-or)

lemma postR-extChoice [rdes]:
  assumes P is NCSP Q is NCSP
  shows  $\text{post}_R(P \sqcap Q) = (\text{post}_R(P) \vee \text{post}_R(Q))$ 
  using assms
  by (simp add: extChoice-def, subst postR-ExtChoice, auto simp add: usup-and uinf-or)

lemma ExtChoice-cong:
  assumes  $\bigwedge P. P \in A \implies F(P) = G(P)$ 
  shows  $(\sqcap P \in A \cdot F(P)) = (\sqcap P \in A \cdot G(P))$ 
  using assms image-cong by force

lemma ref-unrest-ExtChoice:
  assumes
     $\bigwedge P. P \in A \implies \$ref \# \text{pre}_R(P)$ 
     $\bigwedge P. P \in A \implies \$ref \# \text{cmt}_R(P)$ 
  shows  $\$ref \# (\text{ExtChoice } A) \llbracket \text{false} / \$wait \rrbracket$ 
proof –
  have  $\bigwedge P. P \in A \implies \$ref \# \text{pre}_R(P \llbracket 0 / \$tr \rrbracket)$ 
    using assms by (rel-blast)
  with assms show ?thesis
    by (simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest)
qed

lemma CSP4-ExtChoice:
  assumes  $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ 
  shows ExtChoice A is CSP4
proof (cases A = {})
  case True thus ?thesis
    by (simp add: ExtChoice-empty Healthy-def CSP4-def, simp add: Skip-is-CSP Stop-left-zero)
next
  case False
  have  $1: (\neg_r (\neg_r \text{pre}_R (\text{ExtChoice } A)) ;;_h R1 \text{ true}) = \text{pre}_R (\text{ExtChoice } A)$ 
  proof –
  have  $\bigwedge P. P \in A \implies (\neg_r \text{pre}_R(P)) ;; R1 \text{ true} = (\neg_r \text{pre}_R(P))$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-neg-pre-unit assms)
  thus ?thesis
  apply (simp add: False preR-ExtChoice closure NCSP-set-unrest-pre-wait' assms not-UINF seq-UINF-distr not-USUP)
  apply (rule USUP-cong)
  apply (simp add: rpred assms closure)
  done
qed

```

```

have 2: $st' # peri_R (ExtChoice A)
proof -
  have a:  $\bigwedge P. P \in A \implies \$st' \# pre_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-pre assms)
  have b:  $\bigwedge P. P \in A \implies \$st' \# peri_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-peri assms)
  from a b show ?thesis
    apply (subst periR-ExtChoice)
    apply (simp-all add: assms closure unrest CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
False)
  done
qed
have 3: $ref' # post_R (ExtChoice A)
proof -
  have a:  $\bigwedge P. P \in A \implies \$ref' \# pre_R(P)$ 
    by (simp add: CSP4-ref'-unrest-pre CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  have b:  $\bigwedge P. P \in A \implies \$ref' \# post_R(P)$ 
    by (simp add: CSP4-ref'-unrest-post CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  from a b show ?thesis
    by (subst postR-ExtChoice, simp-all add: assms CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
unrest False)
  qed
show ?thesis
  by (rule CSP4-tri-intro, simp-all add: 1 2 3 assms closure)
  (metis 1 R1-seqr-closure rea-not-R1 rea-not-not rea-true-R1)
qed

lemma CSP4-extChoice [closure]:
  assumes P is NCSP Q is NCSP
  shows P  $\square$  Q is CSP4
  by (simp add: extChoice-def, rule CSP4-ExtChoice, simp-all add: assms)

lemma NCSP-ExtChoice [closure]:
  assumes A  $\subseteq \llbracket NCSP \rrbracket_H$ 
  shows ExtChoice A is NCSP
proof (cases A = {})
  case True
  then show ?thesis by (simp add: ExtChoice-empty closure)
next
  case False
  show ?thesis
  proof (rule NCSP-intro)
    from assms have cls: A  $\subseteq \llbracket CSP \rrbracket_H$  A  $\subseteq \llbracket CSP3 \rrbracket_H$  A  $\subseteq \llbracket CSP4 \rrbracket_H$ 
      using NCSP-implies-CSP NCSP-implies-CSP3 NCSP-implies-CSP4 by blast+
    have wu:  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$ 
      using NCSP-implies-NSRD NSRD-wait'-unrest-pre assms by force
    show 1: ExtChoice A is CSP
      by (metis (mono-tags) Ball-Collect CSP-ExtChoice NCSP-implies-CSP assms)
    from cls show ExtChoice A is CSP3
      by (rule-tac CSP3-SRD-intro, simp-all add: CSP-Healthy-subset-member CSP3-Healthy-subset-member
closure rdes unrest wu assms 1 False)
    from cls show ExtChoice A is CSP4
      by (simp add: CSP4-ExtChoice assms)
  qed

```

qed
qed

lemma *ExtChoice-NCSP-closed* [closure]:
assumes $\bigwedge i. i \in I \implies P(i) \text{ is NCSP}$
shows $(\Box i \in I \cdot P(i)) \text{ is NCSP}$
by (*simp add: NCSP-ExtChoice assms image-subset-iff*)

lemma *NCSP-extChoice* [closure]:
assumes $P \text{ is NCSP } Q \text{ is NCSP}$
shows $P \Box Q \text{ is NCSP}$
by (*simp add: NCSP-ExtChoice assms extChoice-def*)

7.5 Productivity and Guardedness

lemma *Productive-ExtChoice* [closure]:
assumes $A \neq \{\}$ $A \subseteq \llbracket \text{NCSP} \rrbracket_H$ $A \subseteq \llbracket \text{Productive} \rrbracket_H$
shows *ExtChoice A is Productive*

proof –

have $1: \bigwedge P. P \in A \implies \$wait' \nmid pre_R(P)$
using *NCSP-implies-NSRD NSRD-wait'-unrest-pre assms(2)* **by** *blast*
show *?thesis*

proof (*rule Productive-intro, simp-all add: assms closure rdes 1 unrest*)

have $((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot post_R P)) =$
 $((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot (pre_R P \wedge post_R P)))$
by (*rel-auto*)

moreover have $(\Box P \in A \cdot (pre_R P \wedge post_R P)) = (\Box P \in A \cdot ((pre_R P \wedge post_R P) \wedge \$tr <_u \$tr'))$

by (*rule UINF-cong, metis (no-types, lifting) 1 Ball-Collect NCSP-implies-CSP Productive-post-refines-tr-increase assms utp-pred-laws.inf.absorb1*)

ultimately show $(\$tr' >_u \$tr) \sqsubseteq ((\Box P \in A \cdot pre_R P) \wedge (\Box P \in A \cdot post_R P))$
by (*rel-auto*)

qed
qed

lemma *Productive-extChoice* [closure]:
assumes $P \text{ is NCSP } Q \text{ is NCSP } P \text{ is Productive } Q \text{ is Productive}$
shows $P \Box Q \text{ is Productive}$
by (*simp add: extChoice-def Productive-ExtChoice assms*)

lemma *ExtChoice-Guarded* [closure]:
assumes $\bigwedge P. P \in A \implies \text{Guarded } P$
shows *Guarded* $(\lambda X. \Box P \in A \cdot P(X))$

proof (*rule GuardedI*)

fix $X n$

have $\bigwedge Y. ((\Box P \in A \cdot P Y) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

proof –

fix Y

let $?lhs = ((\Box P \in A \cdot P Y) \wedge gvirt(n+1))$ **and** $?rhs = ((\Box P \in A \cdot (P Y \wedge gvirt(n+1))) \wedge gvirt(n+1))$

have $a: ?lhs \llbracket false/\$ok \rrbracket = ?rhs \llbracket false/\$ok \rrbracket$

by (*rel-auto*)

have $b: ?lhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket$

by (*rel-auto*)

have $c: ?lhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket$

by (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest,*

```

rel-blast)
  show ?lhs = ?rhs
  using a b c
  by (rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all)
qed
moreover have (( $\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))$ )  $\wedge \ gvirt(n+1)$ ) = (( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)) \ \wedge \ gvirt(n+1))$ )  $\wedge \ gvirt(n+1)$ )
proof -
  have ( $\Box P \in A \cdot (P \ X \ \wedge \ gvirt(n+1))$ ) = ( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)) \ \wedge \ gvirt(n+1))$ )
  proof (rule ExtChoice-cong)
    fix P assume P  $\in$  A
    thus ( $P \ X \ \wedge \ gvirt(n+1)$ ) = ( $P \ (X \ \wedge \ gvirt(n)) \ \wedge \ gvirt(n+1)$ )
    using Guarded-def assms by blast
  qed
  thus ?thesis by simp
qed
ultimately show (( $\Box P \in A \cdot P \ X$ )  $\wedge \ gvirt(n+1)$ ) = (( $\Box P \in A \cdot (P \ (X \ \wedge \ gvirt(n)))$ )  $\wedge \ gvirt(n+1)$ )
  by simp
qed

```

```

lemma extChoice-Guarded [closure]:
  assumes Guarded P Guarded Q
  shows Guarded ( $\lambda X. P(X) \Box Q(X)$ )
proof -
  have Guarded ( $\lambda X. \Box F \in \{P, Q\} \cdot F(X)$ )
  by (rule ExtChoice-Guarded, auto simp add: assms)
  thus ?thesis
  by (simp add: extChoice-def)
qed

```

7.6 Algebraic laws

```

lemma extChoice-comm:
  P  $\Box$  Q = Q  $\Box$  P
  by (unfold extChoice-def, simp add: insert-commute)

```

```

lemma extChoice-idem:
  P is CSP  $\implies P \Box P = P$ 
  by (unfold extChoice-def, simp add: ExtChoice-single)

```

```

lemma extChoice-assoc:
  assumes P is CSP Q is CSP R is CSP
  shows P  $\Box$  Q  $\Box$  R = P  $\Box$  (Q  $\Box$  R)
proof -
  have P  $\Box$  Q  $\Box$  R =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \Box \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)) \Box \mathbf{R}_s(\text{pre}_R(R) \vdash \text{cmt}_R(R))$ 
  by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))
  also have ... =
     $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \wedge \text{cmt}_R R)$ 
       $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \vee \text{cmt}_R R)))$ 
  by (simp add: extChoice-rdes unrest)
  also have ... =
     $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$ 
       $((\text{cmt}_R P \wedge \text{cmt}_R Q) \wedge \text{cmt}_R R)$ 
       $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$ 

```

$((cmt_R P \vee cmt_R Q) \vee cmt_R R))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... =
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R P \vee (cmt_R Q \vee cmt_R R))))$
 by (simp add: conj-assoc disj-assoc)
 also have ... =
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R P \vee (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))))$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap (\mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)) \sqcap \mathbf{R}_s(pre_R(R) \vdash cmt_R(R)))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $P \sqcap (Q \sqcap R)$
 by (simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3))
 finally show ?thesis .
 qed

lemma extChoice-Stop:

assumes Q is CSP
 shows $Stop \sqcap Q = Q$
 using assms

proof –

have $Stop \sqcap Q = \mathbf{R}_s (true \vdash (\$tr' =_u \$tr \wedge \$wait')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$
 by (simp add: Stop-def SRD-reactive-design-alt assms)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash (((\$tr' =_u \$tr \wedge \$wait') \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\$tr' =_u \$tr \wedge \$wait' \vee cmt_R Q)))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright cmt_R Q))$
 by (metis (no-types, lifting) cond-def eq-upred-sym neg-conj-cancel1 utp-pred-laws.inf.left-idem)
 also have ... = $\mathbf{R}_s (pre_R Q \vdash cmt_R Q)$
 by (simp add: cond-idem)
 also have ... = Q
 by (simp add: SRD-reactive-design-alt assms)
 finally show ?thesis .
 qed

lemma extChoice-Chaos:

assumes Q is CSP
 shows $Chaos \sqcap Q = Chaos$

proof –

have $Chaos \sqcap Q = \mathbf{R}_s (false \vdash true) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$
 by (simp add: Chaos-def SRD-reactive-design-alt assms)
 also have ... = $\mathbf{R}_s (false \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright true))$
 by (simp add: extChoice-rdes unrest)
 also have ... = $\mathbf{R}_s (false \vdash true)$
 by (rule cong[of $\mathbf{R}_s \mathbf{R}_s$], simp, rel-auto)
 also have ... = $Chaos$
 by (simp add: Chaos-def)
 finally show ?thesis .
 qed

lemma *extChoice-Dist*:

assumes P is CSP $S \subseteq \llbracket \text{CSP} \rrbracket_H S \neq \{\}$
shows $P \sqcap (\sqcap S) = (\sqcap_{Q \in S} P \sqcap Q)$

proof –

let $?S1 = pre_R \text{ ' } S$ **and** $?S2 = cmt_R \text{ ' } S$

have $P \sqcap (\sqcap S) = P \sqcap (\sqcap_{Q \in S} \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)))$

by (*simp add: SRD-as-reactive-design[THEN sym] Healthy-SUPREMUM UINF-as-Sup-collect assms*)

also have $\dots = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap \mathbf{R}_s(\sqcap_{Q \in S} pre_R(Q) \vdash cmt_R(Q))$

by (*simp add: RHS-design-USUP SRD-reactive-design-alt assms*)

also have $\dots = \mathbf{R}_s((pre_R(P) \wedge (\sqcap_{Q \in S} pre_R(Q))) \vdash$
 $((cmt_R(P) \wedge (\sqcap_{Q \in S} cmt_R(Q)))$
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$
 $(cmt_R(P) \vee (\sqcap_{Q \in S} cmt_R(Q))))$

by (*simp add: extChoice-rdes unrest*)

also have $\dots = \mathbf{R}_s(\sqcap_{Q \in S} pre_R P \wedge pre_R Q \vdash$
 $(\sqcap_{Q \in S} (cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q)))$

by (*simp add: conj-USUP-dist conj-UINF-dist disj-UINF-dist cond-UINF-dist assms*)

also have $\dots = (\sqcap_{Q \in S} \mathbf{R}_s(pre_R P \wedge pre_R Q \vdash$
 $((cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q))))$

by (*simp add: assms RHS-design-USUP*)

also have $\dots = (\sqcap_{Q \in S} \mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \sqcap \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)))$

by (*simp add: extChoice-rdes unrest*)

also have $\dots = (\sqcap_{Q \in S} P \sqcap \text{CSP}(Q))$

by (*simp add: UINF-as-Sup-collect, metis (no-types, lifting) Healthy-if SRD-as-reactive-design assms(1)*)

also have $\dots = (\sqcap_{Q \in S} P \sqcap Q)$

by (*rule SUP-cong, simp-all add: Healthy-subset-member[OF assms(2)]*)

finally show *?thesis* .

qed

lemma *extChoice-dist*:

assumes P is CSP Q is CSP R is CSP

shows $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap (P \sqcap R)$

using *assms extChoice-Dist[of P {Q, R}] by simp*

lemma *ExtChoice-seq-distr*:

assumes $\bigwedge i. i \in A \implies P \ i \text{ is PCSP } Q \text{ is NCSP}$

shows $(\sqcap_{i \in A} P \ i) ;; Q = (\sqcap_{i \in A} P \ i ;; Q)$

proof (*cases A = {}*)

case *True*

then show *?thesis*

by (*simp add: ExtChoice-empty NCSP-implies-CSP Stop-left-zero assms(2)*)

next

case *False*

show *?thesis*

proof –

have $1: (\sqcap_{i \in A} P \ i) = (\sqcap_{i \in A} (\mathbf{R}_s((pre_R(P \ i)) \vdash peri_R(P \ i) \diamond (R4(post_R(P \ i)))))$
 $(\text{is } ?X = ?Y)$

by (*rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms(1) comp-apply*)

have $2: (\sqcap_{i \in A} P \ i ;; Q) = (\sqcap_{i \in A} (\mathbf{R}_s((pre_R(P \ i)) \vdash peri_R(P \ i) \diamond (R4(post_R(P \ i))))) ;; Q)$
 $(\text{is } ?X = ?Y)$

by (*rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms(1) comp-apply*)

show *?thesis*


```

  by (simp add: 1 2, rdes-eq cls: assms False cong: ExtChoice-cong USUP-cong)
qed
qed

```

```

lemma extChoice-seq-distr:
  assumes  $P$  is PCSP  $Q$  is PCSP  $R$  is NCSP
  shows  $(P \sqcap Q) ;; R = (P ;; R \sqcap Q ;; R)$ 
  by (rdes-eq' cls: assms)

```

```

lemma extChoice-seq-distl:
  assumes  $P$  is ICSP  $Q$  is ICSP  $R$  is NCSP
  shows  $P ;; (Q \sqcap R) = (P ;; Q \sqcap P ;; R)$ 
  by (rdes-eq cls: assms)

```

```

lemma extchoice-StateInvR-refine:

```

```

  assumes
     $P$  is NCSP  $Q$  is NCSP
     $\text{inv}_R(b) \sqsubseteq P$   $\text{inv}_R(b) \sqsubseteq Q$ 
  shows  $\text{inv}_R(b) \sqsubseteq P \sqcap Q$ 

```

```

proof -

```

```

  have 1:

```

```

     $\text{pre}_R P \sqsubseteq [b]_{S<} [b]_{S>} \sqsubseteq ([b]_{S<} \wedge \text{post}_R P)$ 

```

```

     $\text{pre}_R Q \sqsubseteq [b]_{S<} [b]_{S>} \sqsubseteq ([b]_{S<} \wedge \text{post}_R Q)$ 

```

```

  by (metis (no-types, lifting) CRR-implies-RR NCSP-implies-CSP RHS-tri-design-refine SRD-reactive-tri-design
    StateInvR-def assms periR-RR postR-RR preR-CRR rea-st-cond-RR rea-true-RR refBy-order st-post-CRR)+

```

```

  show ?thesis

```

```

  by (rdes-refine-split cls: assms(1-2), simp-all add: 1 closure assms truer-bottom-rpred utp-pred-laws.inf-sup-distrib1)

```

```

qed

```

```

end

```

8 Stateful-Failure Programs

```

theory utp-sfrd-prog

```

```

  imports

```

```

    UTP.utp-full

```

```

    utp-sfrd-extchoice

```

```

begin

```

8.1 Conditionals

```

lemma NCSP-cond-srea [closure]:
  assumes  $P$  is NCSP  $Q$  is NCSP
  shows  $P \triangleleft b \triangleright_R Q$  is NCSP
  by (rule NCSP-NSRD-intro, simp-all add: closure rdes assms unrest)

```

8.2 Guarded commands

```

lemma GuardedCommR-NCSP-closed [closure]:
  assumes  $P$  is NCSP
  shows  $g \rightarrow_R P$  is NCSP
  by (simp add: gcmd-def closure assms)

```

8.3 Alternation

```

lemma AlternateR-NCSP-closed [closure]:

```

assumes $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$
shows $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) \text{ is NCSP}$
proof $(\text{cases } A = \{\})$
 case *True*
 then show *?thesis*
 by (simp add: assms)
 next
 case *False*
 then show *?thesis*
 by $(\text{simp add: AlternateR-def closure assms})$
qed

lemma *AlternateR-list-NCSP-closed [closure]:*
assumes $\bigwedge b P. (b, P) \in \text{set } A \implies P \text{ is NCSP } Q \text{ is NCSP}$
shows $(\text{AlternateR-list } A \ Q) \text{ is NCSP}$
apply $(\text{simp add: AlternateR-list-def})$
apply $(\text{rule AlternateR-NCSP-closed})$
apply $(\text{auto simp add: assms})$
apply $(\text{metis assms(1) eq-snd-iff nth-mem})$
done

8.4 Assumptions

definition *AssumeCircus* $([-]_C)$ **where**
 $[b]_C = b \rightarrow_R \text{Skip}$

lemma *AssumeCircus-rdes-def [rdes-def]:* $[b]_C = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond [b]_C)$
unfolding *AssumeCircus-def* **by** *rdes-eq*

lemma *AssumeCircus-NCSP [closure]:* $[b]_C \text{ is NCSP}$
by $(\text{simp add: AssumeCircus-def GuardedCommR-NCSP-closed NCSP-Skip})$

lemma *AssumeCircus-AssumeR:* $\text{Skip} ;; [b]^\top_R = [b]_C [b]^\top_R ;; \text{Skip} = [b]_C$
by $(\text{rdes-eq})+$

lemma *AssumeR-comp-AssumeCircus:* $P \text{ is NCSP} \implies P ;; [b]^\top_R = P ;; [b]_C$
by $(\text{metis (no-types, hide-lams) AssumeCircus-AssumeR(1) RA1 Skip-right-unit})$

lemma *gcmd-AssumeCircus:*
 $P \text{ is NCSP} \implies b \rightarrow_R P = [b]_C ;; P$
by $(\text{simp add: AssumeCircus-def NCSP-implies-NSRD Skip-left-unit gcmd-seq-distr})$

lemma *rdes-assume-pre-refine:*
assumes $P \text{ is NCSP}$
shows $P \sqsubseteq [b]_C ;; P$
by $(\text{rdes-refine cls: assms})$

8.5 While Loops

lemma *NSRD-coerce-NCSP:*
 $P \text{ is NSRD} \implies \text{Skip} ;; P ;; \text{Skip} \text{ is NCSP}$
by $(\text{metis (no-types, hide-lams) CSP3-Skip CSP3-def CSP4-def Healthy-def NCSP-Skip NCSP-implies-CSP NCSP-intro NSRD-is-SRD RA1 SRD-seqr-closure})$

definition *WhileC* $:: 's \text{ upred} \Rightarrow ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$ $(\text{while}_C - \text{do} - \text{od})$ **where**
 $\text{while}_C \ b \ \text{do} \ P \ \text{od} = \text{Skip} ;; \text{while}_R \ b \ \text{do} \ P \ \text{od} ;; \text{Skip}$

lemma *WhileC-NCSP-closed* [closure]:

assumes *P is NCSP P is Productive*

shows *while_C b do P od is NCSP*

by (*simp add: WhileC-def NSRD-coerce-NCSP assms closure*)

theorem *WhileC-iter-form*:

assumes *P is NCSP P is Productive*

shows *while_C b do P od = ([b]_C ;; P)^{*C} ;; [¬ b]_C*

by (*simp add: WhileC-def WhileR-iter-form assms closure*)

(*metis (no-types, lifting) StarC-def AssumeCircus-AssumeR(2) AssumeCircus-NCSP RA1 assms(1) csp-theory.Healthy-Sequence csp-theory.Star-Healthy csp-theory.Unit-Left sfrd-star-as-rdes-star*)

theorem *WhileC-rdes-def* [rdes-def]:

assumes *P is CRC Q is CRR R is CRF \$st' # Q R is R4*

shows *while_C b do R_s(P ⊢ Q ⊔ R) od =*

*R_s ([b]_c ;; R)^{*c} wp_r ([b]_{S<} ⇒_r P) ⊢ ([b]_c ;; R)^{*c} ;; [b]_c ;; Q ⊔ ([b]_c ;; R)^{*c} ;; [¬ b]_c)*
(**is** ?lhs = ?rhs)

proof –

have ?lhs = ([b]_C ;; R_s (P ⊢ Q ⊔ R))^{*C} ;; [¬ b]_C

by (*simp add: WhileC-iter-form assms closure unrest Productive-rdes-RR-intro*)

also have ... = ?rhs

by (*simp add: rdes-def assms closure unrest rpred wp del: rea-star-wp*)

finally show ?thesis .

qed

lemma *WhileC-false*:

P is NCSP ⇒ WhileC false P = Skip

by (*simp add: NCSP-implies-NSRD Skip-srdes-left-unit WhileC-def WhileR-false*)

lemma *WhileC-unfold*:

assumes *P is NCSP P is Productive*

shows *WhileC b P = (P ;; WhileC b P) ◁ b ▷_R Skip*

proof –

have *WhileC b P = (Skip ∨ [b]_C ;; P ;; ([b]_C ;; P)^{*C}) ;; [¬ b]_C*

by (*simp add: WhileC-iter-form assms closure*)

(*metis (no-types, lifting) AssumeCircus-NCSP RA1 StarC-unfold assms(1) csp-theory.Healthy-Sequence disj-upred-def*)

also have ... = ([¬ b]_C ∨ [b]_C ;; P ;; ([b]_C ;; P)^{*C} ;; [¬ b]_C)

by (*metis (no-types, lifting) AssumeCircus-AssumeR(1) RA1 csp-theory.Unit-self seqr-or-distl*)

also have ... = (P ;; WhileC b P) ◁ b ▷_R Skip

by (*metis (no-types, lifting) AssumeCircus-AssumeR(2) NCSP-implies-NSRD RA1 WhileC-NCSP-closed WhileC-iter-form assms(1) assms(2) cond-srea-AssumeR-form csp-theory.Healthy-Sequence csp-theory.Healthy-Unit csp-theory.Unit-Left uinf-or utp-pred-laws.sup-commute*)

finally show ?thesis .

qed

8.6 Iteration Construction

definition *IterateC* :: 'a set ⇒ ('a ⇒ 's upred) ⇒ ('a ⇒ ('s, 'e) action) ⇒ ('s, 'e) action

where [*upred-defs, ndes-simp*]: *IterateC A g P = while_C (⋃ i∈A · g(i)) do (if_R i∈A · g(i) → P(i) fi) od*

lemma *IterateC-IterateR-def*: *IterateC A g P = Skip ;; IterateR A g P ;; Skip*

by (*simp add: IterateC-def IterateR-def WhileC-def*)

definition *IterateC-list* :: ('s upred × ('s, 'e) action) list ⇒ ('s, 'e) action **where**
 [upred-defs, ndes-simp]:
*IterateC-list xs = IterateC {0..*length xs*} (λ i. map fst xs ! i) (λ i. map snd xs ! i)*

syntax

-iter-C :: pttm ⇒ logic ⇒ logic ⇒ logic ⇒ logic (do_C -∈- · - → - od)
 -iter-gcommC :: gcomms ⇒ logic (do_C / - /od)

translations

-iter-C x A g P => CONST IterateC A (λ x. g) (λ x. P)
 -iter-C x A g P <= CONST IterateC A (λ x. g) (λ x'. P)
 -iter-gcommC cs → CONST IterateC-list cs
 -iter-gcommC (-gcomm-show cs) ← CONST IterateC-list cs

lemma *IterateC-NCSP-closed* [closure]:

assumes

⋀ i. i ∈ I ⇒ P(i) is NCSP

⋀ i. i ∈ I ⇒ P(i) is Productive

shows do_C i ∈ I · g(i) → P(i) od is NCSP

by (simp add: IterateC-IterateR-def IterateR-NSRD-closed NCSP-implies-NSRD NSRD-coerce-NCSP
 assms(1) assms(2))

lemma *IterateC-list-NCSP-closed* [closure]:

assumes

⋀ b P. (b, P) ∈ set A ⇒ P is NCSP

⋀ b P. (b, P) ∈ set A ⇒ P is Productive

shows IterateC-list A is NCSP

apply (simp add: IterateC-list-def, rule IterateC-NCSP-closed)

apply (metis assms atLeastLessThan-iff nth-map nth-mem prod.collapse)+

done

lemma *IterateC-list-alt-def*:

IterateC-list xs = while_C (⋀ b ∈ set(map fst xs) · b) do AlternateR-list xs Chaos od

proof –

have (⋀ i ∈ {0..*length(xs)*} · (map fst xs) ! i) = (⋀ b ∈ set(map fst xs) · b)

by (rel-auto, metis nth-mem prod.collapse, metis fst-conv in-set-conv-nth nth-map)

thus ?thesis

by (simp add: IterateC-list-def IterateC-def AlternateR-list-def)

qed

lemma *IterateC-empty*:

do_C i ∈ {} · g(i) → P(i) od = Skip

by (simp add: IterateC-IterateR-def IterateR-empty closure Skip-srdes-left-unit)

lemma *IterateC-singleton*:

assumes P k is NCSP P k is Productive

shows do_C i ∈ {k} · g(i) → P(i) od = while_C g(k) do P(k) od (**is** ?lhs = ?rhs)

by (simp add: IterateC-IterateR-def IterateR-singleton NCSP-implies-NSRD WhileC-def assms)

lemma *IterateC-outer-refine-intro*:

assumes I ≠ {} ⋀ i. i ∈ I ⇒ P i is NCSP ⋀ i. i ∈ I ⇒ P i is Productive

⋀ i. i ∈ I ⇒ S ⊆ (b i →_R P i ;; S) S is NCSP

S ⊆ [¬ (⋀ i ∈ I · b i)][⊤]_R

shows S ⊆ do_C i ∈ I · b(i) → P(i) od

proof –

```

have  $S \sqsubseteq \text{do}_R \ i \in I \cdot b(i) \rightarrow P(i)$  od
  by (simp add: IterateR-outer-refine-intro NCSP-implies-NSRD assms)
thus ?thesis
  unfolding IterateC-IterateR-def
  by (metis (full-types) Skip-left-unit Skip-right-unit assms(5) urel-dioid.mult-isol urel-dioid.mult-isor)
qed

```

lemma *IterateC-outer-refine-init-intro:*

```

assumes
   $\bigwedge i. i \in A \implies P \ i \text{ is NCSP}$ 
   $\bigwedge i. i \in A \implies P \ i \text{ is Productive}$ 
   $S \text{ is NCSP } I \text{ is NCSP}$ 
   $S \sqsubseteq I \ ; \ [\neg (\bigcap i \in A \cdot b \ i)]^\top_R$ 
   $\bigwedge i. i \in A \implies S \sqsubseteq S \ ; \ b \ i \rightarrow_R P \ i$ 
   $\bigwedge i. i \in A \implies S \sqsubseteq I \ ; \ b \ i \rightarrow_R P \ i$ 
shows  $S \sqsubseteq I \ ; \ \text{do}_C \ i \in A \cdot b(i) \rightarrow P(i)$  od
proof (cases  $A = \{\}$ )
case True
  with assms(5) show ?thesis
    by (simp add: IterateC-empty assms closure Skip-right-unit AssumeR-true NSRD-right-unit)
next
case False
  have  $S \sqsubseteq I \ ; \ \text{do}_R \ i \in A \cdot b(i) \rightarrow P(i)$  od
    by (simp add: IterateR-outer-refine-init-intro NCSP-implies-NSRD assms False)
  thus ?thesis
    unfolding IterateC-IterateR-def
    by (metis (no-types, hide-lams) RA1 Skip-right-unit assms(3) assms(4) urel-dioid.mult-isor)
qed

```

lemma *IterateC-list-outer-refine-intro:*

```

assumes
   $A \neq [] \ S \text{ is NCSP}$ 
   $\bigwedge b \ P. (b, P) \in \text{set } A \implies P \text{ is NCSP}$ 
   $\bigwedge b \ P. (b, P) \in \text{set } A \implies P \text{ is Productive}$ 
   $\bigwedge b \ P. (b, P) \in \text{set } A \implies S \sqsubseteq (b \rightarrow_R P \ ; \ S)$ 
   $S \sqsubseteq [\neg (\bigcap (b, P) \in \text{set } A \cdot b)]^\top_R$ 
shows  $S \sqsubseteq \text{IterateC-list } A$ 
proof -
  have  $(\bigcap i \in \{0..<\text{length}(A)\} \cdot (\text{map fst } A) ! i) = (\bigcap (b, P) \in \text{set } A \cdot b)$ 
    by (rel-auto, metis nth-mem prod.exhaust-sel, metis fst-conv in-set-conv-nth nth-map)
  thus ?thesis
    apply (simp add: IterateC-list-def)
    apply (rule IterateC-outer-refine-intro)
    apply (simp-all add: closure assms)
    apply (metis assms(3) nth-mem prod.collapse)
    apply (metis assms(4) nth-mem prod.collapse)
    done
qed

```

lemma *IterateC-list-outer-refine-init-intro:*

```

assumes
   $S \text{ is NCSP } I \text{ is NCSP}$ 
   $\bigwedge b \ P. (b, P) \in \text{set } A \implies P \text{ is NCSP}$ 
   $\bigwedge b \ P. (b, P) \in \text{set } A \implies P \text{ is Productive}$ 

```

$S \sqsubseteq I \;; \; [\neg (\bigcap (b, P) \in \text{set } A \cdot b)]^\top_R$
 $\bigwedge b P. (b, P) \in \text{set } A \implies S \sqsubseteq S \;; \; b \rightarrow_R P$
 $\bigwedge b P. (b, P) \in \text{set } A \implies S \sqsubseteq I \;; \; b \rightarrow_R P$
shows $S \sqsubseteq I \;; \; \text{IterateC-list } A$
proof –
have $(\bigcap i \in \{0..<\text{length}(A)\} \cdot (\text{map fst } A) ! i) = (\bigcap (b, P) \in \text{set } A \cdot b)$
by $(\text{rel-auto}, \text{metis nth-mem prod.exhaust-sel}, \text{metis fst-conv in-set-conv-nth nth-map})$
thus $?thesis$
apply $(\text{simp add: IterateC-list-def})$
apply $(\text{rule IterateC-outer-refine-init-intro})$
apply $(\text{simp-all add: closure assms})$
apply $(\text{metis assms(3) nth-mem prod.collapse})$
apply $(\text{metis assms(4) nth-mem prod.collapse})$
done
qed

8.7 Assignment

definition $\text{AssignsCSP} :: 's \text{ usubst} \Rightarrow ('s, 'v) \text{ action } (\langle \cdot \rangle_C)$ **where**
 $[\text{upred-defs}]: \text{AssignsCSP } \sigma = \mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [\langle \sigma \rangle_a]_S))$

syntax

$\text{-assigns-csp} :: \text{svids} \Rightarrow \text{uexprs} \Rightarrow \text{logic } ('(-) :=_C '(-))$
 $\text{-assigns-csp} :: \text{svids} \Rightarrow \text{uexprs} \Rightarrow \text{logic } (\mathbf{infixr} :=_C 64)$

translations

$\text{-assigns-csp } xs \text{ vs} \Rightarrow \text{CONST AssignsCSP } (\text{-mk-usubst } (\text{CONST id}) \text{ xs vs})$
 $\text{-assigns-csp } x \text{ v} \leq \text{CONST AssignsCSP } (\text{CONST subst-upd } (\text{CONST id}) \text{ x v})$
 $\text{-assigns-csp } x \text{ v} \leq \text{-assigns-csp } (\text{-spvar } x) \text{ v}$
 $x, y :=_C u, v \leq \text{CONST AssignsCSP } (\text{CONST subst-upd } (\text{CONST subst-upd } (\text{CONST id}) (\text{CONST pr-var } x) \text{ u}) (\text{CONST pr-var } y) \text{ v})$

lemma $\text{preR-AssignsCSP } [rdes]: \text{pre}_R(\langle \sigma \rangle_C) = \text{true}_r$
by (rel-auto)

lemma $\text{periR-AssignsCSP } [rdes]: \text{peri}_R(\langle \sigma \rangle_C) = \text{false}$
by (rel-auto)

lemma $\text{postR-AssignsCSP } [rdes]: \text{post}_R(\langle \sigma \rangle_C) = \Phi(\text{true}, \sigma, \langle \rangle)$
by (rel-auto)

lemma $\text{AssignsCSP-rdes-def } [rdes-def]: \langle \sigma \rangle_C = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \Phi(\text{true}, \sigma, \langle \rangle))$
by (rel-auto)

lemma $\text{AssignsCSP-CSP } [\text{closure}]: \langle \sigma \rangle_C \text{ is CSP}$
by $(\text{simp add: AssignsCSP-def RHS-tri-design-is-SRD unrest})$

lemma $\text{AssignsCSP-CSP3 } [\text{closure}]: \langle \sigma \rangle_C \text{ is CSP3}$
by $(\text{rule CSP3-intro}, \text{simp add: closure}, \text{rel-auto})$

lemma $\text{AssignsCSP-CSP4 } [\text{closure}]: \langle \sigma \rangle_C \text{ is CSP4}$
by $(\text{rule CSP4-intro}, \text{simp add: closure}, \text{rel-auto+})$

lemma $\text{AssignsCSP-NCSP } [\text{closure}]: \langle \sigma \rangle_C \text{ is NCSP}$
by $(\text{simp add: AssignsCSP-CSP AssignsCSP-CSP3 AssignsCSP-CSP4 NCSP-intro})$

lemma *AssignsCSP-ICSP* [closure]: $\langle \sigma \rangle_C$ is ICSP

apply (rule ICSP-intro, simp add: closure, simp add: rdes-def)

apply (rule ISRD1-rdes-intro)

apply (simp-all add: closure)

apply (rel-auto)

done

lemma *AssignsCSP-as-AssignsR*: $\langle \sigma \rangle_R ; \text{Skip} = \langle \sigma \rangle_C$

by (rdes-eq)

lemma *AssignC-init-refine-intro*:

assumes

$vwb\text{-}lens\ x\ \$st:x\ \# P_2\ \$st:x\ \# P_3$

P_2 is RR P_3 is RR Q is NCSP

$\mathbf{R}_s([\&x =_u \ll k \gg]_{S<} \vdash P_2 \diamond P_3) \sqsubseteq Q$

shows $\mathbf{R}_s(true_r \vdash P_2 \diamond P_3) \sqsubseteq (x :=_C \ll k \gg) ; Q$

by (simp add: AssignsCSP-as-AssignsR[THEN sym] assms segr-assoc Skip-left-unit AssignR-init-refine-intro closure)

lemma *AssignsCSP-refines-sinv*:

assumes $\sigma \uparrow b$

shows $sinv_R(b) \sqsubseteq \langle \sigma \rangle_C$

apply (rdes-refine-split)

apply (simp-all)

apply (metis rea-st-cond-true st-cond-conj utp-pred-laws.inf.absorb-iff2 utp-pred-laws.inf-top-left)

using assms **apply** (rel-auto)

done

8.8 Assignment with update

There are different collections that we would like to assign to, but they all have different types and perhaps more importantly different conditions on the update being well defined. For example, for a list well-definedness equates to the index being less than the length etc. Thus we here set up a polymorphic constant for CSP assignment updates that can be specialised to different types.

definition *AssignCSP-update* ::

$(f \Rightarrow 'k\ set) \Rightarrow (f \Rightarrow 'k \Rightarrow 'v \Rightarrow 'f) \Rightarrow (f \Rightarrow 'f) \Rightarrow$

$(k, 'f) \Rightarrow ('v, 'f) \Rightarrow ('f, 'f) \Rightarrow ('f, 'f) \Rightarrow ('f, 'f) \Rightarrow$

[upred-defs, rdes-def]: *AssignCSP-update* domf updatef $x\ k\ v =$

$\mathbf{R}_s([k \in_u uop\ domf\ (\&x)]_{S<} \vdash false \diamond \Phi(true, [x \mapsto_s trop\ updatef\ (\&x)\ k\ v], \langle \rangle))$

All different assignment updates have the same syntax; the type resolves which implementation to use.

syntax

$\text{-csp-assign-upd} :: \text{svid} \Rightarrow \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{logic} \ (-[-] :=_C - [61, 0, 62] \ 62)$

translations

$\text{-csp-assign-upd}\ x\ k\ v == \text{CONST AssignCSP-update}\ (\text{CONST udom})\ (\text{CONST uupd})\ x\ k\ v$

lemma *AssignCSP-update-CSP* [closure]:

AssignCSP-update domf updatef $x\ k\ v$ is CSP

by (simp add: AssignCSP-update-def RHS-tri-design-is-SRD unrest)

lemma *preR-AssignCSP-update* [rdes]:

$pre_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \in_u \text{uop domf } (\&x)]_{S<}$
by (*rel-auto*)

lemma *periR-AssignCSP-update* [rdes]:

$peri_R(\text{AssignCSP-update domf updatef } x \ k \ v) = [k \notin_u \text{uop domf } (\&x)]_{S<}$
by (*rel-simp*)

lemma *post-AssignCSP-update* [rdes]:

$post_R(\text{AssignCSP-update domf updatef } x \ k \ v) =$
 $(\Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) \ k \ v], \langle \rangle) \triangleleft (k \in_u \text{uop domf } (\&x)) \triangleright_R R1(\text{true}))$
by (*rel-auto*)

lemma *AssignCSP-update-NCSP* [closure]:

(AssignCSP-update domf updatef } x \ k \ v) is NCSP

proof (*rule NCSP-intro*)

show *(AssignCSP-update domf updatef } x \ k \ v) is CSP*

by (*simp add: closure*)

show *(AssignCSP-update domf updatef } x \ k \ v) is CSP3*

by (*rule CSP3-SRD-intro, simp-all add: csp-do-def closure rdes unrest*)

show *(AssignCSP-update domf updatef } x \ k \ v) is CSP4*

by (*rule CSP4-tri-intro, simp-all add: csp-do-def closure rdes unrest, rel-auto*)

qed

8.9 State abstraction

lemma *ref-unrest-abs-st* [unrest]:

$\$ref \# P \implies \$ref \# \langle P \rangle_S$

$\$ref' \# P \implies \$ref' \# \langle P \rangle_S$

by (*rel-simp*)⁺

lemma *NCSP-state-srea* [closure]: $P \text{ is NCSP} \implies \text{state } 'a \cdot P \text{ is NCSP}$

apply (*rule NCSP-NSRD-intro*)

apply (*simp-all add: closure rdes*)

apply (*simp-all add: state-srea-def unrest closure*)

done

8.10 Guards

definition *GuardCSP* ::

$'\sigma \text{ cond} \Rightarrow$

$(' \sigma, ' \varphi) \text{ action} \Rightarrow$

$(' \sigma, ' \varphi) \text{ action} \text{ where}$

[*upred-defs*]: $\text{GuardCSP } g \ A = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R(A)) \vdash ((\lceil g \rceil_{S<} \wedge cmt_R(A)) \vee (\lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$

syntax

$\text{-GuardCSP} :: \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infixr } \&_C \ 60)$

translations

$\text{-GuardCSP } b \ P == \text{CONST GuardCSP } b \ P$

lemma *Guard-tri-design*:

$g \ \&_C \ P = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_R(P)) \vdash (peri_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge post_R(P)))$

proof –

have $(\lceil g \rceil_{S<} \wedge \text{cmt}_R P \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait') = (\text{peri}_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond$
 $(\lceil g \rceil_{S<} \wedge \text{post}_R(P))$
by *(rel-auto)*
thus *?thesis* **by** *(simp add: GuardCSP-def)*
qed

lemma *csp-do-cond-conj*:

assumes P is CRR
shows $(\lceil b \rceil_{S<} \wedge P) = \Phi(b, id, \langle \rangle) ;; P$

proof –

have $(\lceil b \rceil_{S<} \wedge \text{CRR}(P)) = \Phi(b, id, \langle \rangle) ;; \text{CRR}(P)$
by *(rel-auto)*
thus *?thesis*
by *(simp add: Healthy-if assms)*
qed

lemma *Guard-rdes-def [rdes-def]*:

assumes P is RR Q is CRR R is CRR
shows $g \ \&_C \ \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$
(is ?lhs = ?rhs)

proof –

have $?lhs = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash ((P \Rightarrow_r Q) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge (P \Rightarrow_r R)))$
by *(simp add: Guard-tri-design rdes assms closure)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash ((\lceil g \rceil_{S<} \wedge Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\lceil g \rceil_{S<} \wedge R))$
by *(rel-auto)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{ \}_u)) \diamond (\Phi(g, id, \langle \rangle) ;; R))$
by *(simp add: assms(2) assms(3) csp-do-cond-conj)*
finally show *?thesis* .
qed

lemma *Guard-rdes-def'*:

assumes $\$ok' \nmid P$
shows $g \ \&_C \ (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge Q \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
proof –
have $g \ \&_C \ (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r \text{pre}_R(\mathbf{R}_s(P \vdash Q))) \vdash (\lceil g \rceil_{S<} \wedge \text{cmt}_R(\mathbf{R}_s(P \vdash Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(simp add: GuardCSP-def)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge R1(R2c(\text{cmt}_s \dagger (P \Rightarrow Q))) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(simp add: rea-pre-RHS-design rea-cmt-RHS-design)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash R1(R2c(\lceil g \rceil_{S<} \wedge R1(R2c(\text{cmt}_s \dagger (P \Rightarrow Q)))) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash R1(R2c(\lceil g \rceil_{S<} \wedge (\text{cmt}_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(simp add: R1-R2c-commute R1-disj R1-extend-conj' R1-idem R2c-and R2c-disj R2c-idem)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge (\text{cmt}_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash \text{cmt}_s \dagger (\lceil g \rceil_{S<} \wedge (\text{cmt}_s \dagger (P \Rightarrow Q)) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
by *(simp add: rdes-export-cmt)*
also have $\dots = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(\text{pre}_s \dagger P))) \vdash \text{cmt}_s \dagger (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$

by (simp add: usubst)
 also have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (simp add: rdes-export-cmt)
 also from assms have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r (pre_s \dagger P)) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (rel-auto)
 also have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r pre_s \dagger P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (simp add: rdes-export-pre)
 also from assms have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (rel-auto)
 also from assms have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge (P \Rightarrow Q) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (simp add: rdes-export-pre)
 also have ... = $\mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r P) \vdash (\lceil g \rceil_{S<} \wedge Q \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$
 by (rule cong[of \mathbf{R}_s \mathbf{R}_s], simp, rel-auto)
 finally show ?thesis .
 qed

lemma CSP-Guard [closure]: $b \&_C P$ is CSP

by (simp add: GuardCSP-def, rule RHS-design-is-SRD, simp-all add: unrest)

lemma preR-Guard [rdes]: P is CSP $\implies pre_R(b \&_C P) = (\lceil b \rceil_{S<} \Rightarrow_r pre_R P)$

by (simp add: Guard-tri-design rea-pre-RHS-design usubst unrest R2c-preR R2c-lift-state-pre R2c-rea-impl R1-rea-impl R1-preR Healthy-if, rel-auto)

lemma periR-Guard [rdes]:

assumes P is NCSP

shows $peri_R(b \&_C P) = (peri_R P \triangleleft b \triangleright_R \mathcal{E}(true, \langle \rangle, \{\}_u))$

proof –

have $peri_R(b \&_C P) = ((\lceil b \rceil_{S<} \Rightarrow_r pre_R P) \Rightarrow_r (peri_R P \triangleleft \lceil b \rceil_{S<} \triangleright (\$tr' =_u \$tr)))$

by (simp add: assms Guard-tri-design rea-peri-RHS-design usubst unrest R1-rea-impl R2c-rea-not R2c-rea-impl R2c-preR R2c-periR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr closure Healthy-if R1-cond R1-tr'-eq-tr)

also have ... = $((pre_R P \Rightarrow_r peri_R P) \triangleleft \lceil b \rceil_{S<} \triangleright (\$tr' =_u \$tr))$

by (rel-auto)

also have ... = $(peri_R P \triangleleft \lceil b \rceil_{S<} \triangleright (\$tr' =_u \$tr))$

by (simp add: SRD-peri-under-pre add: unrest closure assms)

finally show ?thesis

by rel-auto

qed

lemma postR-Guard [rdes]:

assumes P is NCSP

shows $post_R(b \&_C P) = (\lceil b \rceil_{S<} \wedge post_R P)$

proof –

have $post_R(b \&_C P) = ((\lceil b \rceil_{S<} \Rightarrow_r pre_R P) \Rightarrow_r (\lceil b \rceil_{S<} \wedge post_R P))$

by (simp add: Guard-tri-design rea-post-RHS-design usubst unrest R2c-rea-not R2c-and R2c-rea-impl R2c-preR R2c-postR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr R1-rea-impl R1-extend-conj' R1-post-SRD closure assms)

also have ... = $(\lceil b \rceil_{S<} \wedge (pre_R P \Rightarrow_r post_R P))$

by (rel-auto)

also have ... = $(\lceil b \rceil_{S<} \wedge post_R P)$

by (simp add: SRD-post-under-pre add: unrest closure assms)
 also have ... = ($[b]_{S<} \wedge \text{post}_R P$)
 by (metis CSP-Guard R1-extend-conj R1-post-SRD calculation rea-st-cond-def)
 finally show ?thesis .
 qed

lemma CSP3-Guard [closure]:
 assumes P is CSP P is CSP3
 shows $b \ \&_C \ P$ is CSP3

proof –

from assms have 1: $\$ref \# P \llbracket \text{false} / \$wait \rrbracket$
 by (simp add: CSP-Guard CSP3-iff)
 hence $\$ref \# \text{pre}_R (P \llbracket 0 / \$tr \rrbracket) \ \$ref \# \text{pre}_R P \ \$ref \# \text{cmt}_R P$
 by (pred-blast)+
 hence $\$ref \# (b \ \&_C \ P) \llbracket \text{false} / \$wait \rrbracket$
 by (simp add: CSP3-iff GuardCSP-def RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest
 usubst)
 thus ?thesis
 by (metis CSP3-intro CSP-Guard)
 qed

lemma CSP4-Guard [closure]:

assumes P is NCSP
 shows $b \ \&_C \ P$ is CSP4
 proof (rule CSP4-tri-intro[OF CSP-Guard])
 show $(\neg_r \text{pre}_R (b \ \&_C \ P)) \ ; \ R1 \ \text{true} = (\neg_r \text{pre}_R (b \ \&_C \ P))$

proof –

have a: $(\neg_r \text{pre}_R P) \ ; \ R1 \ \text{true} = (\neg_r \text{pre}_R P)$
 by (simp add: CSP4-neg-pre-unit assms closure)
 have $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \ ; \ R1 \ \text{true} = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P))$

proof –

have 1: $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) = ([b]_{S<} \wedge (\neg_r \text{pre}_R P))$
 by (rel-auto)
 also have 2: ... = $([b]_{S<} \wedge ((\neg_r \text{pre}_R P) \ ; \ R1 \ \text{true}))$
 by (simp add: a)
 also have 3: ... = $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \ ; \ R1 \ \text{true}$
 by (rel-auto)
 finally show ?thesis ..

qed

thus ?thesis

by (simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest)

qed

show $\$st' \# \text{peri}_R (b \ \&_C \ P)$

by (simp add: preR-Guard periR-Guard NSRD-CSP4-intro closure assms unrest)

show $\$ref' \# \text{post}_R (b \ \&_C \ P)$

by (simp add: preR-Guard postR-Guard NSRD-CSP4-intro closure assms unrest)

qed

lemma NCSP-Guard [closure]:

assumes P is NCSP
 shows $b \ \&_C \ P$ is NCSP

proof –

have P is CSP

using NCSP-implies-CSP assms by blast

then show ?thesis

by (metis (no-types) CSP3-Guard CSP3-commutes-CSP4 CSP4-Guard CSP4-Idempotent CSP-Guard
Healthy-Idempotent Healthy-def NCSP-def assms comp-apply)
qed

lemma *Productive-Guard [closure]*:

assumes P is CSP P is Productive $\$wait' \# pre_R(P)$

shows $b \&_C P$ is Productive

proof –

have $b \&_C P = b \&_C \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

by (metis Healthy-def Productive-form assms(1) assms(2))

also have ... =

$\mathbf{R}_s((\lceil b \rceil_{S<} \Rightarrow_r pre_R P) \vdash$
 $((pre_R P \Rightarrow_r peri_R P) \triangleleft \lceil b \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil b \rceil_{S<} \wedge (pre_R P \Rightarrow_r post_R P \wedge \$tr' >_u$
 $\$tr)))$

by (simp add: Guard-tri-design rea-pre-RHS-design rea-peri-RHS-design rea-post-RHS-design unrest
assms

usubst R1-preR Healthy-if R1-rea-impl R1-peri-SRD R1-extend-conj' R2c-preR R2c-not R2c-rea-impl

$R2c-periR R2c-postR R2c-and R2c-tr-less-tr' R1-tr-less-tr')$

also have ... = $\mathbf{R}_s((\lceil b \rceil_{S<} \Rightarrow_r pre_R P) \vdash (peri_R P \triangleleft \lceil b \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ((\lceil b \rceil_{S<} \wedge post_R P)$
 $\wedge \$tr' >_u \$tr))$

by (rel-auto)

also have ... = $Productive(b \&_C P)$

by (simp add: Productive-def Guard-tri-design RHS-tri-design-par unrest)

finally show ?thesis

by (simp add: Healthy-def')

qed

lemma *Guard-refines-sinv*:

assumes P is NCSP $sinv_R(b) \sqsubseteq P$

shows $sinv_R(b) \sqsubseteq g \&_C P$

proof –

from assms

have $\mathbf{R}_s(\lceil b \rceil_{S<} \vdash R1 \text{ true} \diamond \lceil b \rceil_{S>}) \sqsubseteq \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

by (simp add: rdes-def NCSP-implies-CSP SRD-reactive-tri-design)

thus ?thesis

apply (simp add: RHS-tri-design-refine' closure unrest assms)

apply (safe)

apply (rdes-refine cls: assms(1))

done

qed

8.11 Basic events

definition $do_u ::$

$(\varphi, \sigma) \text{ uexpr} \Rightarrow (\sigma, \varphi) \text{ action}$ **where**

$[upred-defs]: do_u e = ((\$tr' =_u \$tr \wedge \lceil e \rceil_{S<} \notin_u \$ref') \triangleleft \$wait' \triangleright (\$tr' =_u \$tr \wedge_u \lceil e \rceil_{S<} \wedge \$st' =_u$
 $\$st))$

definition $DoCSP :: (\varphi, \sigma) \text{ uexpr} \Rightarrow (\sigma, \varphi) \text{ action}$ (do_C) **where**

$[upred-defs]: DoCSP a = \mathbf{R}_s(true \vdash do_u a)$

lemma $R1-DoAct: R1(do_u(a)) = do_u(a)$

by (rel-auto)

lemma $R2c-DoAct: R2c(do_u(a)) = do_u(a)$

by (rel-auto)

lemma *DoCSP-alt-def*: $do_C(a) = R3h(CSP1(\$ok' \wedge do_u(a)))$
 apply (simp add: DoCSP-def RHS-def design-def impl-alt-def R1-R3h-commute R2c-R3h-commute R2c-disj R2c-not R2c-ok R2c-ok' R2c-and R2c-DoAct R1-disj R1-extend-conj' R1-DoAct)
 apply (rel-auto)
 done

lemma *DoAct-unrests* [unrest]:
 $\$ok \# do_u(a) \ \$wait \# do_u(a)$
 by (pred-auto)+

lemma *DoCSP-RHS-tri* [rdes-def]:
 $do_C(a) = \mathbf{R}_s(true_r \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \diamond \Phi(true, id, \langle a \rangle)))$
 by (simp add: DoCSP-def do_u-def wait'-cond-def, rel-auto)

lemma *CSP-DoCSP* [closure]: $do_C(a)$ is CSP
 by (simp add: DoCSP-def do_u-def RHS-design-is-SRD unrest)

lemma *preR-DoCSP* [rdes]: $pre_R(do_C(a)) = true_r$
 by (simp add: DoCSP-def rea-pre-RHS-design unrest usubst R2c-true)

lemma *periR-DoCSP* [rdes]: $peri_R(do_C(a)) = \mathcal{E}(true, \langle \rangle, \{a\}_u)$
 by (rel-auto)

lemma *postR-DoCSP* [rdes]: $post_R(do_C(a)) = \Phi(true, id, \langle a \rangle)$
 by (rel-auto)

lemma *CSP3-DoCSP* [closure]: $do_C(a)$ is CSP3
 by (rule CSP3-intro[OF CSP-DoCSP])
 (simp add: DoCSP-def do_u-def RHS-def design-def R1-def R2c-def R2s-def R3h-def unrest usubst)

lemma *CSP4-DoCSP* [closure]: $do_C(a)$ is CSP4
 by (rule CSP4-tri-intro[OF CSP-DoCSP], simp-all add: preR-DoCSP periR-DoCSP postR-DoCSP unrest)

lemma *NCSP-DoCSP* [closure]: $do_C(a)$ is NCSP
 by (metis CSP3-DoCSP CSP4-DoCSP CSP-DoCSP Healthy-def NCSP-def comp-apply)

lemma *Productive-DoCSP* [closure]:
 $(do_C a :: ('\sigma, '\psi) \text{ action})$ is Productive
proof –
 have $((\Phi(true, id, \langle a \rangle) \wedge \$tr' >_u \$tr) :: ('\sigma, '\psi) \text{ action})$
 $= (\Phi(true, id, \langle a \rangle))$
 by (rel-auto, simp add: Prefix-Order.strict-prefixI')
 hence $Productive(do_C a) = do_C a$
 by (simp add: Productive-RHS-design-form DoCSP-RHS-tri unrest)
 thus ?thesis
 by (simp add: Healthy-def)
qed

lemma *PCSP-DoCSP* [closure]:
 $(do_C a :: ('\sigma, '\psi) \text{ action})$ is PCSP
 by (simp add: Healthy-comp NCSP-DoCSP Productive-DoCSP)

lemma *wp-rea-DoCSP-lemma*:

fixes $P :: ('σ, 'φ) \text{ action}$

assumes $\$ok \# P \$wait \# P$

shows $(\$tr' =_u \$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle \wedge \$st' =_u \$st) ;; P = (\exists \$ref \cdot P[\$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle / \$tr])$

using *assms*

by (*rel-auto, meson*)

lemma *wp-rea-DoCSP*:

assumes $P \text{ is NCSP}$

shows $(\$tr' =_u \$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle \wedge \$st' =_u \$st) \text{ wp}_r \text{ pre}_R P =$

$(\neg_r (\neg_r \text{pre}_R P)[\$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle / \$tr])$

by (*simp add: wp-rea-def wp-rea-DoCSP-lemma unrest usubst ex-unrest assms closure*)

lemma *wp-rea-DoCSP-alt*:

assumes $P \text{ is NCSP}$

shows $(\$tr' =_u \$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle \wedge \$st' =_u \$st) \text{ wp}_r \text{ pre}_R P =$

$(\$tr' \geq_u \$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle \Rightarrow_r (\text{pre}_R P)[\$tr \hat{^}_u \langle \lceil a \rceil_{S<} \rangle / \$tr])$

by (*simp add: wp-rea-DoCSP assms rea-not-def R1-def usubst unrest, rel-auto*)

lemma *DoCSP-refine-sinv*: $\text{sinv}_R(b) \sqsubseteq \text{do}_C(a)$

by (*rdes-refine*)

8.12 Event prefix

definition *PrefixCSP* ::

$('φ, 'σ) \text{ uexpr} \Rightarrow$

$('σ, 'φ) \text{ action} \Rightarrow$

$('σ, 'φ) \text{ action} (- \rightarrow_C - [81, 80] \ 80) \text{ where}$

[*upred-defs*]: *PrefixCSP* $a \ P = (\text{do}_C(a) ;; \text{CSP}(P))$

abbreviation *OutputCSP* $c \ v \ P \equiv \text{PrefixCSP} \ (c.v)_u \ P$

lemma *CSP-PrefixCSP [closure]*: *PrefixCSP* $a \ P$ is *CSP*

by (*simp add: PrefixCSP-def closure*)

lemma *CSP3-PrefixCSP [closure]*:

PrefixCSP $a \ P$ is *CSP3*

by (*metis (no-types, hide-lams) CSP3-DoCSP CSP3-def Healthy-def PrefixCSP-def seqr-assoc*)

lemma *CSP4-PrefixCSP [closure]*:

assumes $P \text{ is CSP } P \text{ is CSP}_4$

shows *PrefixCSP* $a \ P$ is *CSP4*

by (*metis (no-types, hide-lams) CSP4-def Healthy-def PrefixCSP-def assms(1) assms(2) seqr-assoc*)

lemma *NCSP-PrefixCSP [closure]*:

assumes $P \text{ is NCSP}$

shows *PrefixCSP* $a \ P$ is *NCSP*

by (*metis (no-types, hide-lams) CSP3-PrefixCSP CSP3-commutes-CSP4 CSP4-Idempotent CSP4-PrefixCSP CSP-PrefixCSP Healthy-Idempotent Healthy-def NCSP-def NCSP-implies-CSP assms comp-apply*)

lemma *Productive-PrefixCSP [closure]*: $P \text{ is NCSP} \Longrightarrow \text{PrefixCSP } a \ P \text{ is Productive}$

by (*simp add: Healthy-if NCSP-DoCSP NCSP-implies-NSRD NSRD-is-SRD PrefixCSP-def Productive-DoCSP Productive-seq-1*)

lemma *PCSP-PrefixCSP [closure]*: $P \text{ is NCSP} \Longrightarrow \text{PrefixCSP } a \ P \text{ is PCSP}$

by (simp add: Healthy-comp NCSP-PrefixCSP Productive-PrefixCSP)

lemma *PrefixCSP-Guarded* [closure]: *Guarded* (*PrefixCSP a*)

proof –

have *PrefixCSP a* = ($\lambda X. do_C(a) ;; CSP(X)$)

by (simp add: fun-eq-iff PrefixCSP-def)

thus ?thesis

using *Guarded-if-Productive NCSP-DoCSP NCSP-implies-NSRD Productive-DoCSP* by auto
qed

lemma *PrefixCSP-type* [closure]: *PrefixCSP a* $\in \llbracket H \rrbracket_H \rightarrow \llbracket CSP \rrbracket_H$

using *CSP-PrefixCSP* by blast

lemma *PrefixCSP-Continuous* [closure]: *Continuous* (*PrefixCSP a*)

by (simp add: Continuous-def PrefixCSP-def ContinuousD[OF SRD-Continuous] seq-Sup-distl)

lemma *PrefixCSP-RHS-tri-lemma1*:

R1 (*R2s* ($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle \wedge \lceil II \rceil_R$)) = ($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle \wedge \lceil II \rceil_R$)

by (rel-auto)

lemma *PrefixCSP-RHS-tri-lemma2*:

fixes *P* :: (σ, φ) *action*

assumes $\$ok \nmid P \$wait \nmid P$

shows (($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle \wedge \$st' =_u \$st$) $\wedge \neg \$wait'$) ;; $P = (\exists \$ref \cdot P[\$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle / \$tr])$

using *assms*

by (rel-auto, meson, fastforce)

lemma *tr-extend-seqr*:

fixes *P* :: (σ, φ) *action*

assumes $\$ok \nmid P \$wait \nmid P \$ref \nmid P$

shows ($\$tr' =_u \$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle \wedge \$st' =_u \$st$) ;; $P = P[\$tr \hat{\ }_u \langle \lceil a \rceil_{S<} \rangle / \$tr]$

using *assms* by (simp add: wp-rea-DoCSP-lemma *assms unrest ex-unrest*)

lemma *trace-ext-R1-closed* [closure]: *P* is *R1* $\implies P[\$tr \hat{\ }_u e / \$tr]$ is *R1*

by (rel-blast)

lemma *preR-PrefixCSP-NCSP* [rdes]:

assumes *P* is *NCSP*

shows $pre_R(PrefixCSP a P) = (\Phi(true, id, \langle a \rangle) wp_r pre_R P) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee \Phi(true, id, \langle a \rangle) ;; peri_R P) \diamond \Phi(true, id, \langle a \rangle) ;; post_R P$

by (simp add: PrefixCSP-def *assms closure rdes rpred Healthy-if wp usubst unrest*)

lemma *PrefixCSP-RHS-tri*:

assumes *P* is *NCSP*

shows $PrefixCSP a P = \mathbf{R}_s (\Phi(true, id, \langle a \rangle) wp_r pre_R P \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee \Phi(true, id, \langle a \rangle) ;; peri_R P) \diamond \Phi(true, id, \langle a \rangle) ;; post_R P)$

by (simp add: PrefixCSP-def *Healthy-if unrest assms closure NSRD-composition-wp rdes rpred usubst wp*)

For prefix, we can chose whether to propagate the assumptions or not, hence there are two laws.

lemma *PrefixCSP-rdes-def-1* [rdes-def]:

assumes *P* is *CRC* *Q* is *CRR* *R* is *CRR*

$\$st' \nmid Q \$ref' \nmid R$

shows $PrefixCSP a (\mathbf{R}_s(P \vdash Q \diamond R)) =$

\mathbf{R}_s ($\Phi(\text{true}, \text{id}, \langle a \rangle)$) wp_r $P \vdash (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee \Phi(\text{true}, \text{id}, \langle a \rangle) ;; Q) \diamond \Phi(\text{true}, \text{id}, \langle a \rangle) ;; R$
by (*simp add: PrefixCSP-def Healthy-if assms closure, rdes-simp cls: assms*)

8.13 Guarded external choice

abbreviation *GuardedChoiceCSP* :: $'\vartheta$ set $\Rightarrow (' \vartheta \Rightarrow (' \sigma, ' \vartheta)$ action) $\Rightarrow (' \sigma, ' \vartheta)$ action **where**
GuardedChoiceCSP A $P \equiv (\Box x \in A \cdot \text{PrefixCSP} \ll x \gg (P(x)))$

syntax

-*GuardedChoiceCSP* :: *logic* \Rightarrow *logic* \Rightarrow *logic* \Rightarrow *logic* (\Box - \in - \rightarrow - $[0, 0, 85]$ 86)

translations

$\Box x \in A \rightarrow P == \text{CONST } \text{GuardedChoiceCSP } A (\lambda x. P)$

lemma *GuardedChoiceCSP* [*rdes-def*]:

assumes $\bigwedge x. P(x)$ *is NCSP* $A \neq \{\}$

shows $(\Box x \in A \rightarrow P(x)) =$

$\mathbf{R}_s ((\Box x \in A \cdot \Phi(\text{true}, \text{id}, \langle \ll x \gg \rangle)) \text{wp}_r \text{pre}_R (P x)) \vdash$
 $((\Box x \in A \cdot \mathcal{E}(\text{true}, \langle \rangle, \{\ll x \gg\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\Box x \in A \cdot \Phi(\text{true}, \text{id}, \langle \ll x \gg \rangle) ;; \text{peri}_R$
 $(P x))) \diamond$

$(\Box x \in A \cdot \Phi(\text{true}, \text{id}, \langle \ll x \gg \rangle) ;; \text{post}_R (P x)))$

by (*simp add: PrefixCSP-RHS-tri assms ExtChoice-tri-rdes closure unrest, rel-auto*)

8.14 Input prefix

definition *InputCSP* ::

$('a, ' \vartheta)$ chan $\Rightarrow ('a \Rightarrow (' \sigma \text{ upred}) \Rightarrow ('a \Rightarrow (' \sigma, ' \vartheta)$ action) $\Rightarrow (' \sigma, ' \vartheta)$ action **where**

[*upred-defs*]: *InputCSP* c A $P = (\Box x \in \text{UNIV} \cdot A(x) \ \&_C \ \text{PrefixCSP} (c \cdot \ll x \gg)_u (P x))$

definition *InputVarCSP* :: $('a, ' \vartheta)$ chan $\Rightarrow ('a \Longrightarrow ' \sigma) \Rightarrow ('a \Rightarrow ' \sigma \text{ upred}) \Rightarrow (' \sigma, ' \vartheta)$ action **where**

[*upred-defs, rdes-def*]: *InputVarCSP* c x $A = \text{InputCSP } c$ $A (\lambda v. \langle [x \mapsto_s \ll v \gg] \rangle_C)$

definition *do_I* ::

$('a, ' \vartheta)$ chan \Rightarrow

$('a \Longrightarrow (' \sigma, ' \vartheta)$ sfrd) \Rightarrow

$('a \Rightarrow (' \sigma, ' \vartheta)$ action) \Rightarrow

$(' \sigma, ' \vartheta)$ action **where**

do_I c x $P =$ ($\{$

$(\$tr' =_u \$tr \wedge \{e : \ll \delta_u(c) \gg \mid P(e) \cdot (c \cdot \ll e \gg)_u\}_u \cap_u \$ref' =_u \{\}_u)$

$\triangleleft \$wait' \triangleright$

$((\$tr' - \$tr) \in_u \{e : \ll \delta_u(c) \gg \mid P(e) \cdot \langle (c \cdot \ll e \gg)_u \rangle_u \wedge (c \cdot \$x')_u =_u \text{last}_u(\$tr')\}))$

lemma *InputCSP-CSP* [*closure*]: *InputCSP* c A P *is CSP*

by (*simp add: CSP-ExtChoice InputCSP-def*)

lemma *InputCSP-NCSP* [*closure*]: $\ll \bigwedge v. P(v) \text{ is NCSP} \gg \Longrightarrow \text{InputCSP } c$ A P *is NCSP*

apply (*simp add: InputCSP-def*)

apply (*rule NCSP-ExtChoice*)

apply (*simp add: NCSP-Guard NCSP-PrefixCSP image-Collect-subsetI top-set-def*)

done

lemma *InputVarCSP-NCSP* [*closure*]: *InputVarCSP* c x A *is NCSP*

by (*simp add: AssignsCSP-NCSP InputCSP-NCSP InputVarCSP-def*)

lemma *Productive-InputCSP* [*closure*]:

$\ll \bigwedge v. P(v) \text{ is NCSP} \gg \Longrightarrow \text{InputCSP } x$ A P *is Productive*

by (auto simp add: InputCSP-def unrest closure intro: Productive-ExtChoice)

lemma *Productive-InputVarCSP* [closure]: *InputVarCSP c x A is Productive*
by (simp add: InputVarCSP-def closure)

lemma *R4-st-pred-conj-do* [rpred]:
 $((R4 [s_1]_{S<} \wedge \Phi(s_2, \sigma, t) ;; P) = R4(\Phi(s_1 \wedge s_2, \sigma, t) ;; P)$
by (rel-auto)

lemma *unrest-ref'-R4* [unrest]: $\$ref' \# P \implies \$ref' \# R4(P)$
by (simp add: R4-def unrest)

lemma *st-pred-conj-seq* [rpred]:
 $\llbracket P \text{ is } R4; Q \text{ is } R4 \rrbracket \implies ([s]_{S<} \wedge P ;; Q) = (([s]_{S<} \wedge P) ;; Q)$
by (metis (no-types, lifting) R1-seqr-closure RR-implies-R1 cond-st-distr cond-st-miracle seqr-left-zero)

lemma *InputCSP-rdes-def* [rdes-def]:
 assumes $\bigwedge v. P(v) \text{ is } CRC \bigwedge v. Q(v) \text{ is } CRR \bigwedge v. R(v) \text{ is } CRR$
 $\bigwedge v. \$st' \# Q(v) \bigwedge v. \$ref' \# R(v)$
 shows $InputCSP a A (\lambda v. \mathbf{R}_s(P(v) \vdash Q(v) \diamond R(v))) =$
 $\mathbf{R}_s((\bigsqcup x \cdot \Phi(A x, id, \langle (a \cdot \ll x \gg)_u \rangle) wp_r P x) \vdash$
 $((\bigsqcup x \cdot \mathcal{E}(A x, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u) \vee \mathcal{E}(\neg A x, \langle \rangle, \{\}_u)) \vee (\bigsqcup x \cdot \Phi(A x, id, \langle (a \cdot \ll x \gg)_u \rangle) ;; Q x))$
 \diamond
 $(\bigsqcup x \cdot \Phi(A x, id, \langle (a \cdot \ll x \gg)_u \rangle) ;; R x)$
 by (simp add: InputCSP-def, rdes-simp cls: assms)

8.15 Renaming

definition *RenameCSP* :: $('s, 'e) \text{ action} \Rightarrow ('e \Rightarrow 'f) \Rightarrow ('s, 'f) \text{ action} ((-) \downarrow_C [999, 0] 999)$ **where**
 $[upred-defs]: \text{RenameCSP } P f = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) \downarrow_c ;; true_r) \vdash ((peri_R(P)) \downarrow_c) \diamond ((post_R(P)) \downarrow_c))$

lemma *RenameCSP-rdes-def* [rdes-def]:
 assumes $P \text{ is } CRC \ Q \text{ is } CRR \ R \text{ is } CRR$
 shows $(\mathbf{R}_s(P \vdash Q \diamond R)) \downarrow_c = \mathbf{R}_s((\neg_r (\neg_r P) \downarrow_c ;; true_r) \vdash Q \downarrow_c \diamond R \downarrow_c)$ (is ?lhs = ?rhs)
proof –
 have ?lhs = $\mathbf{R}_s((\neg_r (\neg_r P) \downarrow_c ;; true_r) \vdash (P \Rightarrow_r Q) \downarrow_c \diamond (P \Rightarrow_r R) \downarrow_c)$
 by (simp add: RenameCSP-def rdes closure assms)
 also have ... = $\mathbf{R}_s((\neg_r (\neg_r CRC(P)) \downarrow_c ;; true_r) \vdash (CRC(P) \Rightarrow_r CRR(Q)) \downarrow_c \diamond (CRC(P) \Rightarrow_r CRR(R)) \downarrow_c)$
 by (simp add: Healthy-if assms)
 also have ... = $\mathbf{R}_s((\neg_r (\neg_r CRC(P)) \downarrow_c ;; true_r) \vdash (CRR(Q)) \downarrow_c \diamond (CRR(R)) \downarrow_c)$
 by (rel-auto, (metis order-refl)+)
 also have ... = ?rhs
 by (simp add: Healthy-if assms)
 finally show ?thesis .
qed

lemma *RenameCSP-pre-CRC-closed*:
 assumes $P \text{ is } CRR$
 shows $\neg_r (\neg_r P) \downarrow_c ;; R1 \text{ true is } CRC$
 apply (rule CRC-intro'')
 apply (simp add: unrest closure assms)
 apply (simp add: Healthy-def, simp add: RC1-def rpred closure CRC-idem assms seqr-assoc)
 done

lemma *RenameCSP-NCSP-closed* [closure]:
assumes P is NCSP
shows $P\langle f \rangle_C$ is NCSP
by (simp add: RenameCSP-def RenameCSP-pre-CRC-closed closure assms unrest)

lemma *csp-rename-false* [rpred]:
 $false\langle f \rangle_C = false$
by (rel-auto)

lemma *umap-nil* [simp]: $map_u f \langle \rangle = \langle \rangle$
by (rel-auto)

lemma *rename-Skip*: $Skip\langle f \rangle_C = Skip$
by (rdes-eq)

lemma *rename-Chaos*: $Chaos\langle f \rangle_C = Chaos$
by (rdes-eq-split; rel-simp; force)

lemma *rename-Miracle*: $Miracle\langle f \rangle_C = Miracle$
by (rdes-eq)

lemma *rename-DoCSP*: $(do_C(a))\langle f \rangle_C = do_C(\llbracket f \rrbracket(a)_a)$
by (rdes-eq)

8.16 Algebraic laws

lemma *AssignCSP-conditional*:
assumes vwb -lens x
shows $x :=_C e \triangleleft b \triangleright_R x :=_C f = x :=_C (e \triangleleft b \triangleright f)$
by (rdes-eq cls: assms)

lemma *AssignsCSP-id*: $\langle id \rangle_C = Skip$
by (rel-auto)

lemma *Guard-comp*:
 $g \&_C h \&_C P = (g \wedge h) \&_C P$
by (rule antisym, rel-blast, rel-blast)

lemma *Guard-false* [simp]: $false \&_C P = Stop$
by (simp add: GuardCSP-def Stop-def rpred closure alpha R1-design-R1-pre)

lemma *Guard-true* [simp]:
 $P \text{ is CSP} \implies true \&_C P = P$
by (simp add: GuardCSP-def alpha SRD-reactive-design-alt closure rpred)

lemma *Guard-conditional*:
assumes P is NCSP
shows $b \&_C P = P \triangleleft b \triangleright_R Stop$
by (rdes-eq cls: assms)

lemma *Guard-expansion*:
 $(g_1 \vee g_2) \&_C P = (g_1 \&_C P) \sqcup (g_2 \&_C P)$
by (rel-auto)

lemma *Conditional-as-Guard*:

assumes P is NCSP Q is NCSP
shows $P \triangleleft b \triangleright_R Q = b \&_C P \sqcap (\neg b) \&_C Q$
by (*rdes-eq' cls: assms; simp add: le-less*)

lemma *PrefixCSP-dist*:

$\text{PrefixCSP } a (P \sqcap Q) = (\text{PrefixCSP } a P) \sqcap (\text{PrefixCSP } a Q)$
using *Continuous-Disjunctous Disjunctuous-def PrefixCSP-Continuous* **by** *auto*

lemma *DoCSP-is-Prefix*:

$\text{do}_C(a) = \text{PrefixCSP } a \text{ Skip}$
by (*simp add: PrefixCSP-def Healthy-if closure, metis CSP4-DoCSP CSP4-def Healthy-def*)

lemma *PrefixCSP-seq*:

assumes P is CSP Q is CSP
shows $(\text{PrefixCSP } a P) ;; Q = (\text{PrefixCSP } a (P ;; Q))$
by (*simp add: PrefixCSP-def seqr-assoc Healthy-if assms closure*)

lemma *PrefixCSP-extChoice-dist*:

assumes P is NCSP Q is NCSP R is NCSP
shows $((a \rightarrow_C P) \sqcap (b \rightarrow_C Q)) ;; R = (a \rightarrow_C P ;; R) \sqcap (b \rightarrow_C Q ;; R)$
by (*simp add: PCSP-PrefixCSP assms(1) assms(2) assms(3) extChoice-seq-distr*)

lemma *GuardedChoiceCSP-dist*:

assumes $\bigwedge i. i \in A \implies P(i)$ is NCSP Q is NCSP
shows $\square x \in A \rightarrow P(x) ;; Q = \square x \in A \rightarrow (P(x) ;; Q)$
by (*simp add: ExtChoice-seq-distr PrefixCSP-seq closure assms cong: ExtChoice-cong*)

Alternation can be re-expressed as an external choice when the guards are disjoint

declare *ExtChoice-tri-rdes* [*rdes-def*]
declare *ExtChoice-tri-rdes'* [*rdes-def del*]

declare *extChoice-rdes-def* [*rdes-def*]
declare *extChoice-rdes-def'* [*rdes-def del*]

lemma *AlternateR-as-ExtChoice*:

assumes
 $\bigwedge i. i \in A \implies P(i)$ is NCSP Q is NCSP
 $\bigwedge i j. \llbracket i \in A; j \in A; i \neq j \rrbracket \implies (g i \wedge g j) = \text{false}$
shows $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) =$
 $(\square i \in A \cdot g(i) \&_C P(i)) \sqcap (\bigwedge i \in A \cdot \neg g(i) \&_C Q)$

proof (*cases* $A = \{\}$)

case *True*

then show *?thesis* **by** (*simp add: ExtChoice-empty extChoice-Stop closure assms*)

next

case *False*

show *?thesis*

proof –

have $1: (\square i \in A \cdot g i \rightarrow_R P i) = (\square i \in A \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$
by (*rule UINF-cong, simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1)*)

have $2: (\square i \in A \cdot g(i) \&_C P(i)) = (\square i \in A \cdot g(i) \&_C \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$

by (*rule ExtChoice-cong, simp add: NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-tri-design assms(1)*)

from *assms(3)* **show** *?thesis*

by (*simp add: AlternateR-def 1 2*)

```

      (rdes-eq' cls: assms(1-2)_simps: False cong: UINF-cong ExtChoice-cong)
qed
qed

declare ExtChoice-tri-rdes [rdes-def del]
declare ExtChoice-tri-rdes' [rdes-def]

declare extChoice-rdes-def [rdes-def del]
declare extChoice-rdes-def' [rdes-def]

find-theorems R4

end

```

9 Recursion in Stateful-Failures

```

theory utp-sfrd-recursion
  imports utp-sfrd-contracts utp-sfrd-prog
begin

```

9.1 Fixed-points

The CSP weakest fixed-point is obtained simply by precomposing the body with the CSP healthiness condition.

abbreviation $\mu\text{-CSP} :: ((\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action}) \Rightarrow (\sigma, \varphi) \text{ action} (\mu_C)$ **where**
 $\mu_C F \equiv \mu (F \circ \text{CSP})$

```

syntax
  -mu-CSP :: ptnr => logic => logic (mu_C - - - [0, 10] 10)

```

```

translations
  mu_C X . P == CONST mu-CSP (lambda X. P)

```

lemma $\mu\text{-CSP-equiv}$:
assumes $\text{Monotonic } F \ F \in \llbracket \text{CSP} \rrbracket_H \rightarrow \llbracket \text{CSP} \rrbracket_H$
shows $(\mu_R F) = (\mu_C F)$
by (*simp add: srd-mu-equiv assms comp-def*)

lemma $\mu\text{-CSP-unfold}$:
 $P \text{ is CSP} \implies (\mu_C X \cdot P ;; X) = P ;; (\mu_C X \cdot P ;; X)$
apply (*subst gfp-unfold*)
apply (*simp-all add: closure Healthy-if*)
done

lemma $\mu\text{-csp-expand}$ [rdes]: $(\mu_C ((;;) Q)) = (\mu X \cdot Q ;; \text{CSP } X)$
by (*simp add: comp-def*)

lemma $\mu\text{-csp-basic-refine}$:
assumes
 $P \text{ is CSP } Q \text{ is NCSP } Q \text{ is Productive } \text{pre}_R(P) = \text{true}_r \ \text{pre}_R(Q) = \text{true}_r$
 $\text{peri}_R P \sqsubseteq \text{peri}_R Q$
 $\text{peri}_R P \sqsubseteq \text{post}_R Q ;; \text{peri}_R P$
shows $P \sqsubseteq (\mu_C X \cdot Q ;; X)$

proof (*rule SRD-refine-intro', simp-all add: closure usubst alpha rpred rdes unrest wp seq-UINF-distr*)

```

assms)
show  $\text{peri}_R P \sqsubseteq (\bigcap i \cdot \text{post}_R Q \wedge i ;; \text{peri}_R Q)$ 
proof (rule UINF-refines')
  fix i
  show  $\text{peri}_R P \sqsubseteq \text{post}_R Q \wedge i ;; \text{peri}_R Q$ 
  proof (induct i)
    case 0
    then show ?case by (simp add: assms)
  next
    case (Suc i)
    then show ?case
      by (meson assms(6) assms(7) semilattice-sup-class.le-sup-iff upower-inductl)
  qed
qed
qed
qed

lemma CRD-mu-basic-refine:
  fixes P :: 'e list  $\Rightarrow$  'e set  $\Rightarrow$  's upred
  assumes
    Q is NCSP Q is Productive  $\text{pre}_R(Q) = \text{true}_r$ 
     $[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$\text{ref}')_u \rrbracket \sqsubseteq \text{peri}_R Q$ 
     $[P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$\text{ref}')_u \rrbracket \sqsubseteq \text{post}_R Q ;;_h [P \ t \ r]_{S<} \llbracket (t, r) \rightarrow (\&tt, \$\text{ref}')_u \rrbracket$ 
  shows  $[\text{true} \vdash P \text{ trace refs} \mid R]_C \sqsubseteq (\mu_C X \cdot Q ;; X)$ 
proof (rule mu-csp-basic-refine, simp-all add: msubst-pair assms closure alpha rdes rpred Healthy-if R1-false)
  show  $[P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$\text{ref}' \rrbracket \sqsubseteq \text{peri}_R Q$ 
  using assms by (simp add: msubst-pair)
  show  $[P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$\text{ref}' \rrbracket \sqsubseteq \text{post}_R Q ;; [P \text{ trace refs}]_{S<} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$\text{ref}' \rrbracket$ 
  using assms by (simp add: msubst-pair)
qed

```

9.2 Example action expansion

```

lemma mu-example1:  $(\mu X \cdot \llbracket a \rrbracket \rightarrow_C X) = (\bigcap i \cdot \text{do}_C(\llbracket a \rrbracket) \wedge (i+1)) ;; \text{Miracle}$ 
  by (simp add: PrefixCSP-def mu-csp-form-1 closure)

```

```

lemma preR-mu-example1 [rdes]:  $\text{pre}_R(\mu X \cdot \llbracket a \rrbracket \rightarrow_C X) = \text{true}_r$ 
  by (simp add: mu-example1 rdes closure unrest wp)

```

```

lemma periR-mu-example1 [rdes]:
   $\text{peri}_R(\mu X \cdot \llbracket a \rrbracket \rightarrow_C X) = (\bigcap i \cdot \mathcal{E}(\text{true}, \text{iter}[i](\llbracket a \rrbracket), \{\llbracket a \rrbracket\}_u))$ 
  by (simp add: mu-example1 rdes rpred closure unrest wp seq-UINF-distr alpha usubst)

```

```

lemma postR-mu-example1 [rdes]:
   $\text{post}_R(\mu X \cdot \llbracket a \rrbracket \rightarrow_C X) = \text{false}$ 
  by (simp add: mu-example1 rdes closure unrest wp)

```

end

10 Linking to the Failures-Divergences Model

```

theory utp-sfrd-fdsem
  imports utp-sfrd-recursion
begin

```

10.1 Failures-Divergences Semantics

The following functions play a similar role to those in Roscoe's CSP semantics, and are calculated from the Circus reactive design semantics. A major difference is that these three functions account for state. Each divergence, trace, and failure is subject to an initial state. Moreover, the traces are terminating traces, and therefore also provide a final state following the given interaction. A more subtle difference from the Roscoe semantics is that the set of traces do not include the divergences. The same semantic information is present, but we construct a direct analogy with the pre-, peri- and postconditions of our reactive designs.

definition *divergences* :: ('σ, 'φ) action ⇒ 'σ ⇒ 'φ list set (dv[-] - [0,100] 100) **where**
[upred-defs]: *divergences* P s = {t | t. ' (¬_r pre_R(P)) [[<s>, <>, <t>] / \$st, \$tr, \$tr'] ' }

definition *traces* :: ('σ, 'φ) action ⇒ 'σ ⇒ ('φ list × 'σ) set (tr[-] - [0,100] 100) **where**
[upred-defs]: *traces* P s = {(t, s') | t s'. ' (pre_R(P) ∧ post_R(P)) [[<s>, <s'>, <>, <t>] / \$st, \$st', \$tr, \$tr'] ' }

definition *failures* :: ('σ, 'φ) action ⇒ 'σ ⇒ ('φ list × 'σ) set (fl[-] - [0,100] 100) **where**
[upred-defs]: *failures* P s = {(t, r) | t r. ' (pre_R(P) ∧ peri_R(P)) [[<r>, <s>, <>, <t>] / \$ref', \$st, \$tr, \$tr'] ' }

lemma *trace-divergence-disj*:

assumes P is NCSP (t, s') ∈ tr[P] s t ∈ dv[P] s
shows False
using assms(2,3)
by (simp add: traces-def divergences-def, rdes-simp cls:assms, rel-auto)

lemma *preR-refine-divergences*:

assumes P is NCSP Q is NCSP ∧ s. dv[P] s ⊆ dv[Q] s
shows pre_R(P) ⊆ pre_R(Q)

proof (rule CRR-refine-impl-prop, simp-all add: assms closure usubst unrest)

fix t s

assume a: '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † pre_R Q'

with a show '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † pre_R P'

proof (rule-tac ccontr)

from assms(3)[of s] **have** b: t ∈ dv[P] s ⇒ t ∈ dv[Q] s

by (auto)

assume ¬ '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † pre_R P'

hence ¬ '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † CRC(pre_R P)'

by (simp add: assms closure Healthy-if)

hence '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † (¬_r CRC(pre_R P))'

by (rel-auto)

hence '[\$st ↦_s <s>, \$tr ↦_s <>, \$tr' ↦_s <t>] † (¬_r pre_R P)'

by (simp add: assms closure Healthy-if)

with a b show False

by (rel-auto)

qed

qed

lemma *preR-eq-divergences*:

assumes P is NCSP Q is NCSP ∧ s. dv[P] s = dv[Q] s

shows pre_R(P) = pre_R(Q)

by (metis assms dual-order.antisym order-refl preR-refine-divergences)

lemma *periR-refine-failures*:

assumes P is NCSP Q is NCSP ∧ s. fl[Q] s ⊆ fl[P] s

shows (pre_R(P) ∧ peri_R(P)) ⊆ (pre_R(Q) ∧ peri_R(Q))

proof (rule *CRR-refine-impl-prop*, simp-all add: *assms closure unrest subst-unrest-3*)
 fix $t\ s\ r'$
 assume a : $[\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R\ Q \wedge peri_R\ Q)'$
 from *assms*(3)[of s] **have** b : $(t, r') \in fl[Q]s \implies (t, r') \in fl[P]s$
 by (*auto*)
 with a **show** $[\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R\ P \wedge peri_R\ P)'$
 by (*simp add: failures-def*)
qed

lemma *periR-eq-failures*:

assumes P is NCSP Q is NCSP $\wedge s. fl[P]s = fl[Q]s$
shows $(pre_R(P) \wedge peri_R(P)) = (pre_R(Q) \wedge peri_R(Q))$
by (*metis (full-types) assms dual-order.antisym order-refl periR-refine-failures*)

lemma *postR-refine-traces*:

assumes P is NCSP Q is NCSP $\wedge s. tr[Q]s \subseteq tr[P]s$
shows $(pre_R(P) \wedge post_R(P)) \sqsubseteq (pre_R(Q) \wedge post_R(Q))$

proof (rule *CRR-refine-impl-prop*, simp-all add: *assms closure unrest subst-unrest-5*)

fix $t\ s\ s'$
 assume a : $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R\ Q \wedge post_R\ Q)'$
 from *assms*(3)[of s] **have** b : $(t, s') \in tr[Q]s \implies (t, s') \in tr[P]s$
 by (*auto*)
 with a **show** $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R\ P \wedge post_R\ P)'$
 by (*simp add: traces-def*)
qed

lemma *postR-eq-traces*:

assumes P is NCSP Q is NCSP $\wedge s. tr[P]s = tr[Q]s$
shows $(pre_R(P) \wedge post_R(P)) = (pre_R(Q) \wedge post_R(Q))$
by (*metis assms dual-order.antisym order-refl postR-refine-traces*)

lemma *circus-fd-refine-intro*:

assumes P is NCSP Q is NCSP $\wedge s. dv[Q]s \subseteq dv[P]s \wedge s. fl[Q]s \subseteq fl[P]s \wedge s. tr[Q]s \subseteq tr[P]s$
shows $P \sqsubseteq Q$

proof (rule *SRD-refine-intro'*, simp-all add: *closure assms*)

show a : $pre_R\ P \Rightarrow pre_R\ Q'$
 using *assms*(1) *assms*(2) *assms*(3) *preR-refine-divergences refBy-order* **by** *blast*
show $peri_R\ P \sqsubseteq (pre_R\ P \wedge peri_R\ Q)$
proof –
 have $peri_R\ P \sqsubseteq (pre_R\ Q \wedge peri_R\ Q)$
 by (*metis (no-types) assms*(1) *assms*(2) *assms*(4) *periR-refine-failures utp-pred-laws.le-inf-iff*)
 then **show** *?thesis*
 by (*metis a refBy-order utp-pred-laws.inf.order-iff utp-pred-laws.inf-assoc*)
qed

show $post_R\ P \sqsubseteq (pre_R\ P \wedge post_R\ Q)$

proof –
 have $post_R\ P \sqsubseteq (pre_R\ Q \wedge post_R\ Q)$
 by (*meson assms*(1) *assms*(2) *assms*(5) *postR-refine-traces utp-pred-laws.le-inf-iff*)
 then **show** *?thesis*
 by (*metis a refBy-order utp-pred-laws.inf.absorb-iff1 utp-pred-laws.inf-assoc*)
qed

qed

10.2 Circus Operators

lemma *traces-Skip*:

$tr\llbracket Skip \rrbracket s = \{(\llbracket \cdot \rrbracket, s)\}$
by (*simp add: traces-def rdes alpha closure, rel-simp*)

lemma *failures-Skip*:
 $fl\llbracket Skip \rrbracket s = \{\}$
by (*simp add: failures-def, rdes-calc*)

lemma *divergences-Skip*:
 $dv\llbracket Skip \rrbracket s = \{\}$
by (*simp add: divergences-def, rdes-calc*)

lemma *traces-Stop*:
 $tr\llbracket Stop \rrbracket s = \{\}$
by (*simp add: traces-def, rdes-calc*)

lemma *failures-Stop*:
 $fl\llbracket Stop \rrbracket s = \{(\llbracket \cdot \rrbracket, E) \mid E. True\}$
by (*simp add: failures-def, rdes-calc, rel-auto*)

lemma *divergences-Stop*:
 $dv\llbracket Stop \rrbracket s = \{\}$
by (*simp add: divergences-def, rdes-calc*)

lemma *traces-AssignsCSP*:
 $tr\llbracket \langle \sigma \rangle_C \rrbracket s = \{(\llbracket \cdot \rrbracket, \sigma(s))\}$
by (*simp add: traces-def rdes closure usubst alpha, rel-auto*)

lemma *failures-AssignsCSP*:
 $fl\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$
by (*simp add: failures-def, rdes-calc*)

lemma *divergences-AssignsCSP*:
 $dv\llbracket \langle \sigma \rangle_C \rrbracket s = \{\}$
by (*simp add: divergences-def, rdes-calc*)

lemma *failures-Miracle*: $fl\llbracket Miracle \rrbracket s = \{\}$
by (*simp add: failures-def rdes closure usubst*)

lemma *divergences-Miracle*: $dv\llbracket Miracle \rrbracket s = \{\}$
by (*simp add: divergences-def rdes closure usubst*)

lemma *failures-Chaos*: $fl\llbracket Chaos \rrbracket s = \{\}$
by (*simp add: failures-def rdes, rel-auto*)

lemma *divergences-Chaos*: $dv\llbracket Chaos \rrbracket s = UNIV$
by (*simp add: divergences-def rdes, rel-auto*)

lemma *traces-Chaos*: $tr\llbracket Chaos \rrbracket s = \{\}$
by (*simp add: traces-def rdes closure usubst*)

lemma *divergences-cond*:
assumes P is NCSP Q is NCSP
shows $dv\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then dv\llbracket P \rrbracket s else dv\llbracket Q \rrbracket s)$
by (*rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto*)

lemma *traces-cond*:

assumes P is NCSP Q is NCSP

shows $tr[P \triangleleft b \triangleright_R Q]s = (if (\llbracket b \rrbracket_e s) then tr[P]s else tr[Q]s)$

by (*rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto*)

lemma *failures-cond*:

assumes P is NCSP Q is NCSP

shows $fl[P \triangleleft b \triangleright_R Q]s = (if (\llbracket b \rrbracket_e s) then fl[P]s else fl[Q]s)$

by (*rdes-simp cls: assms, simp add: divergences-def failures-def rdes closure rpred assms, rel-auto*)

lemma *divergences-guard*:

assumes P is NCSP

shows $dv[g \&_C P]s = (if (\llbracket g \rrbracket_e s) then dv[g \&_C P]s else \{\})$

by (*rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto*)

lemma *traces-do*: $tr[do_C(e)]s = \{(\llbracket e \rrbracket_e s, s)\}$

by (*rdes-simp, simp add: traces-def rdes closure rpred, rel-auto*)

lemma *failures-do*: $fl[do_C(e)]s = \{(\llbracket \cdot \rrbracket_e s, E) \mid E. \llbracket e \rrbracket_e s \notin E\}$

by (*rdes-simp, simp add: failures-def rdes closure rpred usubst, rel-auto*)

lemma *divergences-do*: $dv[do_C(e)]s = \{\}$

by (*rel-auto*)

lemma *divergences-seq*:

fixes $P :: ('s, 'e)$ action

assumes P is NCSP Q is NCSP

shows $dv[P ;; Q]s = dv[P]s \cup \{t_1 @ t_2 \mid t_1 \ t_2 \ s_0. (t_1, s_0) \in tr[P]s \wedge t_2 \in dv[Q]s_0\}$

(**is** ?lhs = ?rhs)

oops

lemma *traces-seq*:

fixes $P :: ('s, 'e)$ action

assumes P is NCSP Q is NCSP

shows $tr[P ;; Q]s =$

$$\begin{aligned} & \{(t_1 @ t_2, s') \mid t_1 \ t_2 \ s_0 \ s'. (t_1, s_0) \in tr[P]s \wedge (t_2, s') \in tr[Q]s_0 \\ & \wedge (t_1 @ t_2) \notin dv[P]s \\ & \wedge (\forall (t, s_1) \in tr[P]s. t \leq t_1 @ t_2 \longrightarrow (t_1 @ t_2) - t \notin dv[Q]s_1)\} \end{aligned}$$

(**is** ?lhs = ?rhs)

proof

show ?lhs \subseteq ?rhs

proof (*rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto*)

fix $t :: 'e$ list **and** $s' :: 's$

let $?s = [\$st \mapsto_s \llbracket s \rrbracket_e, \$st' \mapsto_s \llbracket s' \rrbracket_e, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket_e]$

assume

$a1: '?s \dagger (post_R P ;; post_R Q)'$ **and**

$a2: '[\$st \mapsto_s \llbracket s \rrbracket_e, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket_e] \dagger pre_R P'$ **and**

$a3: '[\$st \mapsto_s \llbracket s \rrbracket_e, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \llbracket t \rrbracket_e] \dagger (post_R P \text{ wp}_r pre_R Q)'$

from $a1$ **have** $'?s \dagger (\exists tr_0. ((post_R P) \llbracket \llbracket tr_0 \rrbracket_e / \$tr' \rrbracket ;; (post_R Q) \llbracket \llbracket tr_0 \rrbracket_e / \$tr \rrbracket) \wedge \llbracket tr_0 \rrbracket_e \leq_u \$tr')$

by (*simp add: R2-tr-middle assms closure*)

then obtain tr_0 **where** $p1: '?s \dagger ((post_R P) \llbracket \llbracket tr_0 \rrbracket_e / \$tr' \rrbracket ;; (post_R Q) \llbracket \llbracket tr_0 \rrbracket_e / \$tr \rrbracket)'$ **and** $tr_0: tr_0$

$\leq t$

apply (*simp add: usubst*)

apply (*erule taut-shEx-elim*)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)
 apply (rel-auto)
 done
 from p1 have ' $? \sigma \vdash (\exists st_0 \cdot (post_R P)[\ll tr_0 \gg / \$tr'] [\ll st_0 \gg / \$st'] ;; (post_R Q)[\ll tr_0 \gg / \$tr] [\ll st_0 \gg / \$st])$ '
 by (simp add: segr-middle[of st, THEN sym])
 then obtain s_0 where ' $? \sigma \vdash ((post_R P)[\ll s_0 \gg, \ll tr_0 \gg / \$st', \$tr'] ;; (post_R Q)[\ll s_0 \gg, \ll tr_0 \gg / \$st, \$tr])$ '
 apply (simp add: usubst)
 apply (erule taut-shEx-elim)
 apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)
 apply (rel-auto)
 done
 hence ' $(([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P) ;; ([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q))$ '
 by (rel-auto)
 hence ' $(([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P) \wedge ([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q))$ '
 by (simp add: segr-to-conj unrest-any-circus-var assms closure unrest)
 hence $postP$: ' $([\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash post_R P)$ ' and
 $postQ$ ': ' $([\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash post_R Q)$ '
 by (rel-auto)+
 from $postQ'$ have ' $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \vdash [\$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll tr_0 \gg + (\ll t \gg - \ll tr_0 \gg)] \vdash post_R Q$ '
 using $tr0$ by (rel-auto)
 hence ' $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \vdash [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t \gg - \ll tr_0 \gg] \vdash post_R Q$ '
 by (simp add: R2-subst-tr closure assms)
 hence $postQ$: ' $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - tr_0 \gg] \vdash post_R Q$ '
 by (rel-auto)
 have $preP$: ' $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash pre_R P$ '
 proof -
 have $(pre_R P)[\ll 0, \ll tr_0 \gg / \$tr, \$tr'] \sqsubseteq (pre_R P)[\ll 0, \ll t \gg / \$tr, \$tr']$
 by (simp add: RC-prefix-refine closure assms $tr0$)
 hence $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll tr_0 \gg] \vdash pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \vdash pre_R P$
 by (rel-auto)
 thus ?thesis
 by (simp add: taut-refine-impl a2)
 qed
 have $preQ$: ' $[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t - tr_0 \gg] \vdash pre_R Q$ '
 proof -
 from $postP$ a3 have ' $[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll t \gg] \vdash pre_R Q$ '
 apply (simp add: wp-rea-def)
 apply (rel-auto)
 using $tr0$ apply blast+
 done
 hence ' $[\$st \mapsto_s \ll s_0 \gg] \vdash [\$tr \mapsto_s \ll tr_0 \gg, \$tr' \mapsto_s \ll tr_0 \gg + (\ll t \gg - \ll tr_0 \gg)] \vdash pre_R Q$ '
 by (rel-auto)
 hence ' $[\$st \mapsto_s \ll s_0 \gg] \vdash [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t \gg - \ll tr_0 \gg] \vdash pre_R Q$ '
 by (simp add: R2-subst-tr closure assms)
 thus ?thesis
 by (rel-auto)
 qed
 from a2 have $ndiv$: ' $\neg [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \vdash (\neg_r pre_R P)$ '

```

by (rel-auto)

have t-minus-tr0: tr0 @ (t - tr0) = t
  using append-minus tr0 by blast

from a3
have wpr:  $\bigwedge_{t_0} s_1.$ 
  ' $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P' \implies$ 
  ' $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P' \implies$ 
   $t_0 \leq t \implies '[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - t_0 \rangle] \dagger (\neg_r pre_R Q)' \implies False$ 

proof -
  fix t0 s1
  assume b:
    ' $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P'$ 
    ' $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P'$ 
     $t_0 \leq t$ 
    ' $[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - t_0 \rangle] \dagger (\neg_r pre_R Q)'$ 

  from a3 have c:  $\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \langle t \rangle$ 
     $\wedge [\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P$ 
     $\implies [\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle - \langle t_0 \rangle] \dagger pre_R Q'$ 
  by (simp add: wp-rea-circus-form-alt[of post_R P pre_R Q] closure assms unrest usubst)
    (rel-simp)

  from c b(2-4) show False
  by (rel-auto)

qed

show  $\exists t_1 t_2.$ 
   $t = t_1 @ t_2 \wedge$ 
  ( $\exists s_0. '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \dagger pre_R P \wedge$ 
     $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \dagger post_R P' \wedge$ 
    ' $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \dagger pre_R Q \wedge$ 
     $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \dagger post_R Q' \wedge$ 
     $\neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 @ t_2 \rangle] \dagger (\neg_r pre_R P)' \wedge$ 
    ( $\forall t_0 s_1. '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P \wedge$ 
       $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P' \longrightarrow$ 
       $t_0 \leq t_1 @ t_2 \longrightarrow \neg '[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle (t_1 @ t_2) - t_0 \rangle] \dagger (\neg_r$ 
       $pre_R Q)'$ ))
  apply (rule-tac x=tr0 in exI)
  apply (rule-tac x=(t - tr0) in exI)
  apply (auto)
  using tr0 apply auto[1]
  apply (rule-tac x=s0 in exI)
  apply (auto intro:wpr simp add: taut-conj preP preQ postP postQ ndiv wpr t-minus-tr0)
  done

qed

show ?rhs  $\subseteq$  ?lhs
proof (rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest
  rpred usubst, auto)
  fix t1 t2 :: 'e list and s0 s' :: 's
  assume
    a1:  $\neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 @ t_2 \rangle] \dagger (\neg_r pre_R P)'$  and
    a2: ' $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \dagger pre_R P'$  and

```

$a3$: $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 \rangle] \uparrow post_R P'$ and
 $a4$: $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \uparrow pre_R Q'$ and
 $a5$: $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \uparrow post_R Q'$ and
 $a6$: $\forall t s_1. [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \uparrow pre_R P \wedge$
 $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \uparrow post_R P' \longrightarrow$
 $t \leq t_1 @ t_2 \longrightarrow \neg [\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle (t_1 @ t_2) - t \rangle] \uparrow (\neg_r pre_R Q)'$

from $a1$ **have** $preP$: $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 @ t_2 \rangle] \uparrow (pre_R P)'$
by (*simp add: taut-not unrest-all-circus-vars-st assms closure unrest, rel-auto*)

have $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle t_1 \rangle, \$tr' \mapsto_s \langle t_1 \rangle + \langle t_2 \rangle] \uparrow post_R Q'$

proof –

have $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \uparrow post_R Q =$

$[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \uparrow [\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_2 \rangle] \uparrow post_R Q$

by *rel-auto*

also have $\dots = [\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \uparrow [\$tr \mapsto_s \langle t_1 \rangle, \$tr' \mapsto_s \langle t_1 \rangle + \langle t_2 \rangle] \uparrow post_R Q$

by (*simp add: R2-subst-tr assms closure, rel-auto*)

finally show *?thesis using a5*

by (*rel-auto*)

qed

with $a3$

have $postPQ$: $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 @ t_2 \rangle] \uparrow (post_R P ;; post_R Q)'$

by (*rel-auto, meson Prefix-Order.prefixI*)

have $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle t_1 \rangle, \$tr' \mapsto_s \langle t_1 \rangle + \langle t_2 \rangle] \uparrow pre_R Q'$

proof –

have $[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle t_1 \rangle, \$tr' \mapsto_s \langle t_1 \rangle + \langle t_2 \rangle] \uparrow pre_R Q =$

$[\$st \mapsto_s \langle s_0 \rangle] \uparrow [\$tr \mapsto_s \langle t_1 \rangle, \$tr' \mapsto_s \langle t_1 \rangle + \langle t_2 \rangle] \uparrow pre_R Q$

by *rel-auto*

also have $\dots = [\$st \mapsto_s \langle s_0 \rangle] \uparrow [\$tr \mapsto_s 0, \$tr' \mapsto_s \langle t_2 \rangle] \uparrow pre_R Q$

by (*simp add: R2-subst-tr assms closure*)

finally show *?thesis using a4*

by (*rel-auto*)

qed

from $a6$

have $a6'$: $\bigwedge t s_1. [t \leq t_1 @ t_2; [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \uparrow pre_R P'; [\$st \mapsto_s \langle s \rangle,$

$\$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \uparrow post_R P'] \implies$

$[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle (t_1 @ t_2) - t \rangle] \uparrow pre_R Q'$

apply (*subst (asm) taut-not*)

apply (*simp add: unrest-all-circus-vars-st assms closure unrest*)

apply (*rel-auto*)

done

have wpR : $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_1 @ t_2 \rangle] \uparrow (post_R P wp_r pre_R Q)'$

proof –

have $\bigwedge s_1 t_0. [t_0 \leq t_1 @ t_2; [\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \uparrow post_R P'$

$] \implies$

$[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle (t_1 @ t_2) - t_0 \rangle] \uparrow pre_R Q'$

proof –

fix $s_1 t_0$

assume $c:t_0 \leq t_1 @ t_2$ $[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \uparrow post_R P'$

have $preP'$: $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \uparrow pre_R P'$

proof –
have $(pre_R P) \llbracket 0, \ll t_0 \gg / \$tr, \$tr' \rrbracket \sqsubseteq (pre_R P) \llbracket 0, \ll t_1 @ t_2 \gg / \$tr, \$tr' \rrbracket$
by (*simp add: RC-prefix-refine closure assms c*)
hence $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P$
by (*rel-auto*)
thus *?thesis*
by (*simp add: taut-refine-impl preP*)
qed

with *c a3 preP a6 '[of t0 s1]* **show** $[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R$
Q'.
by (*simp*)
qed

thus *?thesis*
apply (*simp-all add: wp-rea-circus-form-alt assms closure unrest usubst rea-impl-alt-def*)
apply (*simp add: R1-def usubst tcontr-alt-def*)
apply (*auto intro!: taut-shAll-intro-2*)
apply (*rule taut-impl-intro*)
apply (*simp add: unrest-all-circus-vars-st-st' unrest closure assms*)
apply (*rel-simp*)
done
qed
show $([\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P \wedge$
 $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)) \wedge$
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P ;; post_R Q)'$
by (*auto simp add: taut-conj preP postPQ wpR*)
qed
qed

lemma *Cons-minus [simp]:* $(a \# t) - [a] = t$
by (*metis append-Cons append-Nil append-minus*)

lemma *traces-prefix:*
assumes *P is NCSP*
shows $tr \llbracket \ll a \gg \rightarrow_C P \rrbracket s = \{(a \# t, s') \mid t s'. (t, s') \in tr \llbracket P \rrbracket s\}$
apply (*auto simp add: PrefixCSP-def traces-seq traces-do divergences-do lit.rep-eq assms closure Healthy-if trace-divergence-disj*)
apply (*meson assms trace-divergence-disj*)
done

10.3 Deadlock Freedom

The following is a specification for deadlock free actions. In any intermediate observation, there must be at least one enabled event.

definition *CDF :: ('s, 'e) action where*
 $[rdes-def]: CDF = \mathbf{R}_s(true_r \vdash (\bigcap (s, t, E, e) \cdot \mathcal{E}(\ll s \gg, \ll t \gg, \ll insert\ e\ E \gg))) \diamond true_r$

lemma *CDF-NCSP [closure]: CDF is NCSP*
apply (*simp add: CDF-def*)
apply (*rule NCSP-rdes-intro*)
apply (*simp-all add: closure unrest*)
done

```

lemma CDF-seq-idem:  $CDF \;; CDF = CDF$ 
  by (rdes-eq)

lemma CDF-refine-intro:  $CDF \sqsubseteq P \implies CDF \sqsubseteq (CDF \;; P)$ 
  by (metis CDF-seq-idem urel-diod.mult-isol)

lemma Skip-deadlock-free:  $CDF \sqsubseteq \text{Skip}$ 
  by (rdes-refine)

lemma CDF-ext-st [alpha]:  $CDF \oplus_p \text{abs-st}_L = CDF$ 
  by (rdes-eq)

end

```

11 Meta-theory for Stateful-Failure Reactive Designs

```

theory utp-sf-rdes
imports
  utp-sfrd-core
  utp-sfrd-rel
  utp-sfrd-healths
  utp-sfrd-contracts
  utp-sfrd-extchoice
  utp-sfrd-prog
  utp-sfrd-recursion
  utp-sfrd-fdsem
begin end

```

References

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- [2] M. V. M. Oliveira. *Formal Derivation of State-Rich Reactive Programs using Circus*. PhD thesis, Department of Computer Science - University of York, UK, 2006. YCST-2006-02.