SDS 385 Ex 07:
ADMM for LASSO

October 18, 2016

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ADMM for LASSO

Algorithm Details

I used the ADMM algorithm as detailed in Section 6.4 of Boyd, "Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers", 2011.

The setting for this algorithm as applied to lasso is below.

Objective Function:

minimize $\frac{1}{2}||X\beta - y||_2^2 + \lambda||x||_1$
with $\lambda > 0$ as the l1 regularization penalty parameter.

ADMM Form of Problem:

minimize $f(x) + g(x)$
subject to $x - z = 0$
where:
$f(x) = \frac{1}{2}||X\beta - y||_2^2$
$g(x) = \lambda||x||_1$

ADMM Steps:

$\beta^{k+1} = (X^T X + \rho I)^{-1}(X^T y + \rho(z^k - u_k))$
$z^{k+1} = S_{\lambda/\rho}(\beta^{k+1} + u^k)$
$u^{k+1} = u^k + \beta^{k+1} - z^{k+1}$

where $S_{\lambda\rho}(a) = sign(a)(|a| - \lambda/\rho)_+$
and u^k is the augmented lagrangian at the kth iteration.

Implementation Notes:
• Since the step size is fixed, to increase efficiency, I cached the matrix inverse $(X^T X + \rho I)^{-1}$ before beginning iterations.
• Boyd notes that when not using fixed step size, it is best to still cache this inverse, and only update it when the ρ step size is updated.

My Implementation

My ADMM implementation for LASSO converged in 230 iterations. This was considerably faster than my proximal gradient descent (7687 iterations) and my accelerated proximal gradient descent (1853 iterations). I used $\lambda = .01$ for the l1 lasso penalty and $\rho = .01$ for the step size. I used the convergence criteria of absolute change in the lasso objective function $< tol$, where $tol = 1E - 10$.

Note that the lambda and rho were the same values used for exercise 6, for consistency. In practice, I would select an optimal lambda value via cross-validation.
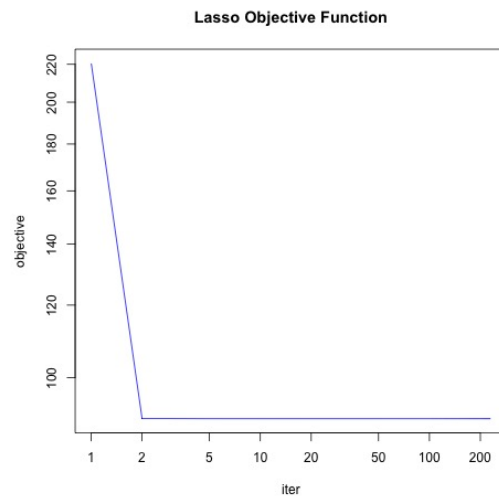
The plot of the lasso objective function is as follows.

Figure 1: LASSO objective using ADMM($\hat{\alpha}$)

The estimated beta coefficients obtained using admm are compared with the coefficients from glmnet below. Most appear to be of reasonably similar sign and magnitude.

```
                 glmnet              admm
   age       0.0084994686    0.03007395
   sex      -0.1274718132   -0.16390462
   bmi       0.3053578438    0.28268400
 5 map       0.1928804642    0.21180397
   tc       -0.0387865853   -0.10040001
   ldl          .               .
   hdl      -0.1635236545   -0.10958174
   tch          .            0.05421951
10 ltg       0.3230766076    0.42695740
   glu       0.0337240035    0.03968565
   age.2     0.0351045119    0.04230375
   bmi.2     0.0238787313    0.02913876
   map.2        .           -0.00496873
15 tc.2         .            3.32992803
```

```
    ldl.2     .            1.78879470
    hdl.2     .            0.86057254
    tch.2     .            0.45016330
    ltg.2     .            0.71081295
20  glu.2    0.0636440798  0.07115055
    age.sex  0.0961729059  0.09425725
    age.bmi   .           -0.00974232
    age.map  0.0145699540  0.01097956
    age.tc    .           -0.10307316
25  age.ldl -0.0389574124 -0.03850246
    age.hdl  0.0258647293  0.13295890
    age.tch   .            0.11473838
    age.ltg  0.0407916229  0.07678043
    age.glu  0.0184222925  0.03852093
30  sex.bmi  0.0270461479  0.04070618
    sex.map  0.0387600107  0.05455798
    sex.tc    .            0.25201571
    sex.ldl -0.0181954985 -0.20580754
    sex.hdl  0.0429324533 -0.06940488
35  sex.tch   .           -0.07673176
    sex.ltg   .           -0.07203117
    sex.glu   .            0.02857881
    bmi.map  0.0837040826  0.09607870
    bmi.tc  -0.0002777938 -0.22702825
40  bmi.ldl   .            0.18330034
    bmi.hdl   .            0.08852731
    bmi.tch   .           -0.01969956
    bmi.ltg   .            0.07900017
    bmi.glu   .            0.01519782
45  map.tc   0.0083855329  0.28434387
    map.ldl   .           -0.19175559
    map.hdl  0.0233788843 -0.11053647
    map.tch   .           -0.03567046
    map.ltg   .           -0.09184804
50  map.glu -0.0362793960 -0.08424349
    tc.ldl    .           -4.61510706
    tc.hdl   0.0072487602 -1.93720146
    tc.tch  -0.0477242186 -1.10984540
    tc.ltg  -0.0098056312 -2.26554521
55  tc.glu    .           -0.08709231
    ldl.hdl   .            1.26150669
    ldl.tch   .            0.52479375
    ldl.ltg  0.0742412609  1.70305929
    ldl.glu  0.0067025050  0.03256134
60  hdl.tch -0.0522309970  0.62815286
    hdl.ltg   .            0.84045023
    hdl.glu   .            0.12679103
    tch.ltg -0.0607125224  0.15900559
    tch.glu  0.0214432611  0.14614305
```

```
65  ltg.glu   .              0.04353096
```

Appendix: R Code

```r
### SDS 385 — Exercises 07 — ADMM for LASSO.
### ADMM = Alternating Direction Method of Multipliers

#Jennifer Starling
#18 October 2016

rm(list=ls())    #Clean workspace.

library(glmnet)
library(Matrix)

#Read in Diabetes.csv data.
X <- read.csv(file='/Users/jennstarling/UTAustin/2016_Fall_SDS 385_Big_Data/
    Exercise 05 R Code/DiabetesX.csv',header=T)
y <- read.csv(file='/Users/jennstarling/UTAustin/2016_Fall_SDS 385_Big_Data/
    Exercise 05 R Code/DiabetesY.csv',header=F)

#Scale X and y.
X = scale(X)
y = scale(y)

#————————————————————————————————————————————————
#LASSO objective function:
#Inputs:
#    X = X matrix (scaled)
#    y = response data (scaled)
#    lambda = a chosen lambda value
#    beta = a vector of beta coefficients.
#Output:
#    Value of the LASSO objective function at specified inputs.
fx <- function(X,y,lambda,beta){
    #f = (1/nrow(X)) * (t(y - X %*% beta) %*% (y - X %*% beta))
    f = (1/2) * (t(y - X %*% beta) %*% (y - X %*% beta))
    g = lambda * sum(abs(beta))
    obj = (f+g)
    return(as.numeric(obj))
}

#————————————————————————————————————————————————
#Proximal L1 Operator function: (soft thresholding operator)
#Inputs:
#    x = vector of values.
#    lambda = the scaling factor of the l1 norm.
#    t = the step size.

#Output:
#    Value of the soft—thresholding proximal operator.
prox_l1 <- function(x,gamma,tau=1) {

    thresh <- gamma*tau
    prox = rep(0,length(x))

    idx.1 = which(x < -thresh)
    idx.2 = which(x > thresh)
    idx.3 = which(abs(x) <= thresh)
```

```r
55      if (length(idx.1) > 0) prox[idx.1] = x[idx.1] + thresh
        if (length(idx.2) > 0) prox[idx.2] = x[idx.2] - thresh
        if (length(idx.3) > 0) prox[idx.3] = 0

        return(prox)
60  }


    #————————————————————————————————————————————————
    #ADMM for Lasso:
    #Inputs:
65  #    X = design matrix
    #    y = response vector
    #    rho = step size
    #    maxiter = maximum iterations
    #    tol = tolerance for convergence
70  #    lambda = l1 norm penalty constant.
    #Output:
    #    List including estimated beta values and objective function.
    admmLasso = function(X,Y,rho=.01,lambda=.1,maxiter=1000,tol=1E-10){

75      i=0                     #Initialize iterator.
        converged <- 0          #Indicator for whether convergence met.

        #1. Initialize matrix to hold beta vector for each iteration.
        betas <- matrix(0,nrow=maxiter,ncol=ncol(X))
80

        #2. Initialize values for objective function.
        obj <- rep(0,maxiter)   #Initialize vector to hold loglikelihood fctn.
        obj[1] <- fx(X,y,lambda,betas[1,])
85
        #3. Cache matrix inverse, since using fixed step size for each iter.
        inv_cache = solve(t(X) %*% X + rho*diag(1,ncol(betas)))

        #4. Initialize values.
90      betas[1,] <- rep(0,ncol(X)) #Initialize beta vector to 0 to start.
        u = rep(0,ncol(betas))  #Initialize the lagrangian to all zeros.
        z = rep(0,ncol(betas))  #Initialize z vector to all zeros.

        #4. ADMM looping.
95      for (i in 2:maxiter){

            #Update betas.
            betas[i,] = inv_cache %*% (t(X) %*% y + rho * (z-u) )

100         #Update z.
            z = prox_l1(betas[i,] + u,lambda/rho)

            #Update u (lagrangian).
            u = u + betas[i,] - z
105
            #Update objective function.
            obj[i] = fx(X,y,lambda=lambda,beta=betas[i,])

            #Convergence check.
110         #Check if convergence met: If yes, exit loop.
            if (abs(obj[i]-obj[i-1])/abs(obj[i-1]+1E-3) < tol ){
                converged=1;
                break;
```

```
          }
115     }


        #Return function values.
        return(list(obj=obj, betas=betas, beta_hat=betas[i,], converged=converged,
            iter=i))
}
120 #————————————————————————————————————————————————

    #Run admm for lasso.
    output <- admmLasso(X,y,rho=.01,lambda=.01,maxiter=1000,tol=1E-14)

125 #Iterations to convergence:
    print(output$iter)
    print(output$converged)

    #Plot objective function.
130 jpeg(file='/Users/jennstarling/UTAustin/2016_Fall_SDS 385_Big_Data/Exercise 07
        LaTeX Files/admm_objective.jpg')
    plot(1:output$iter,output$obj[1:output$iter],type='l',col='blue',log='xy',
        main='Lasso Objective Function',xlab='iter',ylab='objective')
    dev.off()

135 #Compare results to glmnet:
    myLasso <- glmnet(X,y,family='gaussian',alpha=1,lambda=.01) #Fit lasso glmnet
        model.
    beta_glmnet <- myLasso$beta                              #Save glmnet betas.
    cbind(glmnet=beta_glmnet,admm=round(output$beta_hat,8))    #Output comparison
```