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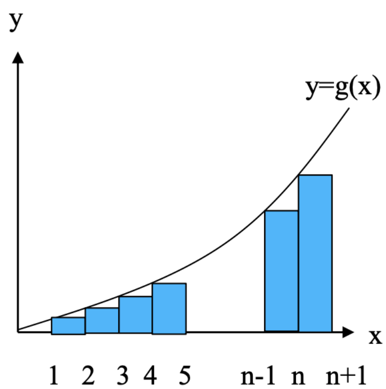
Q5

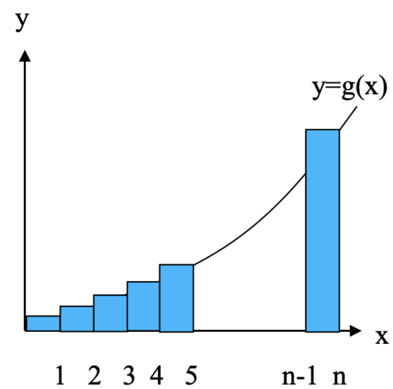
Proof:

Given a non-decreasing function $f(x) : N \to R^+$, we want to show that:

$$\int_0^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq \int_1^{n+1} f(x)\,dx$$

- $f(n) = \sum_{i=1}^{n} g(i)$, where g is a nondecreasing function





Thus:

$\sum_{i=1}^{n} g(i) \leq \int_1^{n+1} g(x)dx$

Thus:

$\int_0^n g(x)dx \leq \sum_{i=1}^{n} g(i)$

Explanation:

1. The left side ($\int_0^n f(x)\,dx$) represents the area under the curve of the function $f(x)$ from $x = 0$ to $x = n$.
2. The middle part ($\sum_{k=1}^{n} f(k)$) represents the sum of function values at discrete points $k = 1, 2, \ldots, n$(blue rectangles).
3. The right side ($\int_1^{n+1} f(x)\,dx$) represents the area under the curve of the function $f(x)$ from $x = 1$ to $x = n + 1$.
   Therefore,we show that the inequality is true.

$$\int_0^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq \int_1^{n+1} f(x)\,dx$$

For $f(x) = x^2$:

1. **Leftmost Inequality:**
$$\int_0^n x^2\,dx = \frac{n^3}{3}$$

2. **Middle Inequality:**
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

3. **Rightmost Inequality:**
$$\int_1^{n+1} x^2\,dx = \frac{(n+1)^3 - 1}{3} = \frac{n^3}{3} + n^2 + n$$

$$\frac{n^3}{3} \leq \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \leq \frac{n^3}{3} + n^2 + n$$

For $f(x) = 2^x$:

1. **Leftmost Inequality:**
$$\int_0^n 2^x\,dx = \frac{2^n - 1}{\ln(2)}$$

2. **Middle Inequality:**
$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2 = (2^n - 1) \cdot 2$$

3. **Rightmost Inequality:**
$$\int_1^{n+1} 2^x\,dx = \frac{2^{n+1} - 2}{\ln(2)} = \frac{2^n - 1}{\ln(2)} \cdot 2$$

Divide sides with $(2^n - 1) henceweget;$
$\frac{1}{\ln(2)} \leq 2 \leq \frac{2}{\ln(2)}$

For $f(x) = 1$:

1. **Leftmost Inequality:**
$$\int_0^n 1\,dx = n$$
2. **Middle Inequality:**
$$\sum_{k=1}^{n} 1 = n$$
3. **Rightmost Inequality:**
$$\int_1^{n+1} 1\,dx = n$$

So, n $\leq n \leq n$