# PS-4

Fourier Series Note: why $a_k = \frac{1}{T_0}\int_0^{T_0} x(t).e^{-jw_0.k.t}.dt$

Assume $x(t) = a_0 . e^{jw_0.0.t} + a_1.e^{jw_0.1.t} + a_{-1}.e^{jw_0(-1).t} + a_2.e^{jw_0.2.t}$

Multiply with $e^{-jw_0 2.t}$ to find $a_2$,

$$x(t) = \underbrace{a_0 . e^{jw_0(0-2).t}}_{\substack{\text{area for one}\\ \text{period}=0}} + a_1.e^{jw_0(-1).t} + a_{-1}.e^{jw_0(-3)t} + \underbrace{a_2 . e^{j\cancel{w_0.0.t}}}_{\substack{\text{area for one}\\ \text{period } a_2.T_0}}\overset{1}{}$$

At the end we multiply by $\frac{1}{T_0}$ to find $a_2$.

4.2

$x(t) = 7.\sin(11\pi t)$. The discrete time signal $x[n]$ is obtained by sampling $x(t)$ at a rate $f_s$, and the resultant $x[n]$ can be written as $x[n] = A.\cos(\hat{w}_0 n + \phi)$.

Determine $A, \phi, \hat{w}_0$. In addition state whether signal is over-sampled or under sampled.

a) $f_s = 10$ samples/sec  b) $f_s = 5$ samples/sec  c) $f_s = 15$ samples/sec

$$\begin{aligned} x(t) &= 7.\sin(11\pi t) \longrightarrow \underset{f_s\uparrow}{\boxed{A/D}} \longrightarrow x[n] = A.\cos(\hat{w}_0 n + \phi) \\ &= 7.\cos(11\pi t - \pi/2) \end{aligned}$$

a) $f_s = 10$ samples/sec

$\hat{w} = w_0 . f_s$

$$\begin{aligned} x(n) &= 7.\cos\left(\frac{11\pi n}{10} - \frac{\pi}{2}\right) \\ &= 7\cos\left(\frac{11\pi n}{10} - 2\pi n - \pi/2\right) \\ &= 7\cos\left(\frac{-9\pi n}{10} - \frac{\pi}{2}\right) = 7.\cos\left(0.9\pi n + \frac{\pi}{2}\right) \end{aligned}$$

$A = 7,\ \hat{w}_0 = 0.9\pi,\ \phi = \pi/2$

b) $f_s = 5$ samples/sec

$$x(t)\Big|_{t=n/f_s} = 7.\cos\left(\frac{11\pi n}{5} - \frac{\pi}{2}\right) = 7\cos\left(\frac{\pi n}{5} - \frac{\pi}{2}\right)$$

$$A=7 \quad \hat{w}_0 = \frac{\pi}{5} \quad \phi = -\frac{\pi}{2}$$

c) $f_s = 15$ samples/sec $\quad x(n) = 7.\cos\left(\frac{11\pi n}{15} - \frac{\pi}{2}\right)$

$$A=7 \quad \hat{w}_0 = \frac{11\pi}{15} \quad \phi = -\frac{\pi}{2}$$

## 4.3

$x[n] = 2.2\cos(0.3n\pi - \pi/3)$, $x[n]$ is obtained by sampling a continious time signal $x(t) = A.\cos(2\pi f_0.t+\phi)$ at a sampling rate $f_s = 6000$ samples/sec. Determine 3 different $x(t)$ that could have produced $x[n]$. All $x(t)$ signals should have a freq. less than 8 kHz.

$$x[n] = 2.2\cos(0.3\pi n - \pi/3)$$

$$x\left(\frac{n}{f_s}\right) = A.\cos\left(2\pi.f_0.\frac{n}{f_s} + \phi\right) \longleftarrow \text{sampling eq.}$$

$$\frac{2\pi.f_0}{f_s} = 0.3\pi + k2\pi \quad \text{where} \quad k \in Z$$

Solve for $k=0$:

$$\frac{2\pi.f_0}{f_s} = 0.3\pi \Rightarrow f_0 = f_s\left(\frac{0.3}{2}\right) = 6000 \times 0,15 = 900 \text{ Hz}$$

$$x(t) = 2.2\cos(1800\pi t - \pi/3)$$

Solve for $k=1$:

$$\frac{2\pi f_0}{f_s} = 2.3\pi \Rightarrow f_0 = f_s\left(\frac{2.3}{2}\right) = 6900 \text{ Hz}$$

$$x(t) = 2.2.\cos(2\pi 6900 t - \pi/3)$$

Solve for $k=-1$:

$$\frac{2\pi f_0}{f_s} = -1.7\pi \Rightarrow f_0 = f_s\left(\frac{-1.7}{2}\right) = -5100 \text{Hz}$$

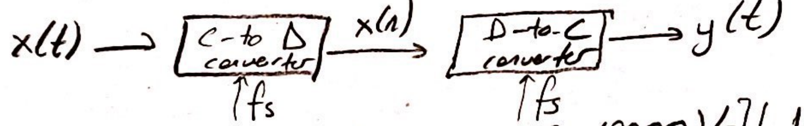
$$x(t) = 2.2\cos(2\pi(-5100)t - \pi/3) = 2.2.\cos(2\pi 5100 t + \pi/3)$$

## 4.4

An amplitude-modulated (AM) cosine wave is represented by the formula:

$x(t)= [10+\cos(2\pi (2000)t)]\cos(2\pi (10^4)t)$

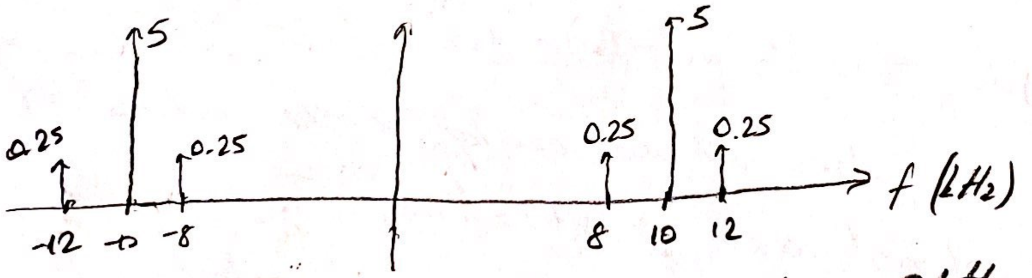
a) Sketch the 2-sided spectrum of this signal.
b) Find the period
c) What relation must fs satisfy so y(t) = x(t) in Fig. 4-26



$x(t)=\left[10+\frac{1}{2}e^{j2\pi(2000)t}+\frac{1}{2}e^{-j2\pi(2000)t}\right]\left(\frac{1}{2}e^{j2\pi 10^4 t}+\frac{1}{2}e^{-j2\pi 10^4 t}\right)$

There are six terms:

$x(t)=5e^{j2\pi 10^4 t}+5e^{-j2\pi 10^4 t}+\frac{1}{4}e^{j2\pi 12000t}+\frac{1}{4}e^{-j2\pi 12000t}$
$+\frac{1}{4}e^{j2\pi 8000t}+\frac{1}{4}e^{-j2\pi 8000t}$



b) Yes, it is periodic. The fundamental freq. 2kHz.

c) $f_s > \underbrace{2f_h} = 2\times 12000 = 24000\text{Hz}$

Greater than twice the highest frequency in x(t)

## 4.5-b

x[n] is given with the equation:

$x[n]=10.\cos(0.2\pi n-\pi/7)$. What signal y(t) will be constructed by an ideal D-to-C converter?

While sampling we replaced t with n/fs. Here we will replace n with tfs.

$$x[n] = 10.\cos(0.2\pi n - \pi/7)$$

$$x(t) = 10.\cos(0.2\pi f_s.t - \pi/7) = 10.\cos(400\pi t - \pi/7)$$

4.6

A nonideal D-C converter takes a seq. of input y(n) and produces y(t) according to the relation:

$$y(t) = \sum_{n=-\infty}^{\infty} y[n].p(t - nT_s) \text{ where } \boxed{T_s = 0.1 \text{ second}}.$$ The input sequence is given by the formula

$$y[n] = \begin{cases} 0.8^n & 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Sketch output y(t) for a and b

a) $$p(t) = \begin{cases} 1 & -0.05 \le t \le 0.05 \\ 0 & \text{otherwise} \end{cases}$$

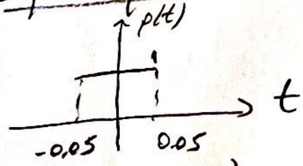
b) $$p(t) = \begin{cases} 1 - 10|t| & -0.1 \le t \le 0.1 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \ldots + y[0]\,p(t) + y[1].p(t - T_s) + y[2].p(t - 2T_s) + \ldots\ldots$$

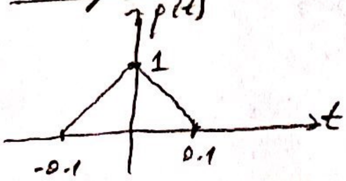
$\boxed{\text{Important}}$: Square pulses will not overlap, so the values of y[n] will be extended over an interval of Ts.

For triangular pulses the neighbouring terms do overlap.

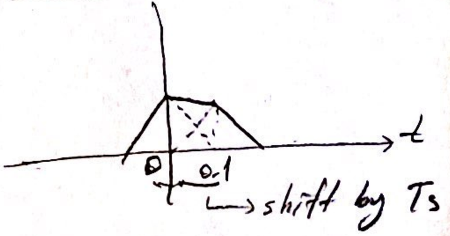
Square pulse:

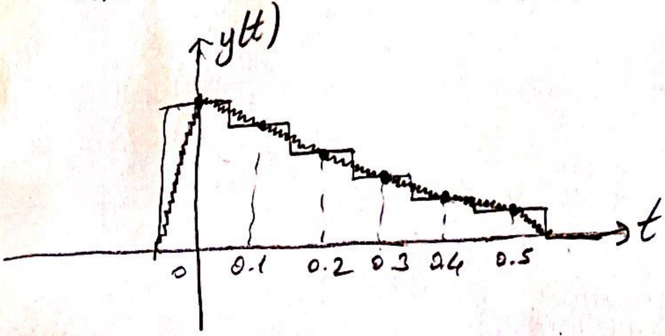


Triangular pulse



Linear interpolation by triangular pulses:





## 4.12

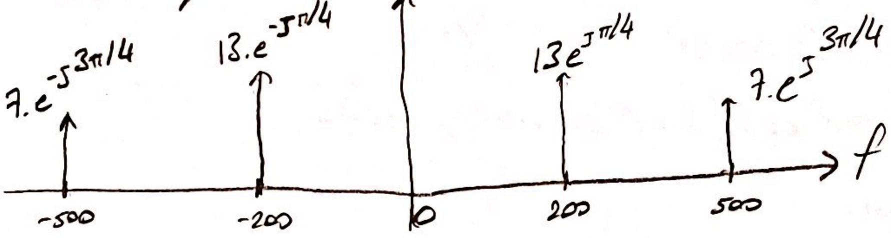
Refer to Fig 4.26 for the system with ideal C-to-D and D-to-C converters.

a) Suppose that the discrete time signal $x[n]$ is.

$x[n] = 10.\cos(0.13\pi n + \pi/13)$

If the $f_s = 1000$ samples/sec. Determine 2 different continuous-time signals $x(t) = x_1(t)$ and $x(t) = x_2(t)$ that could have been inputs to C-to-D system.

b) If the input $x(t)$ is given by the two-sided spectrum. Determine a simple formula for $y(t)$ when $f_s = 700$ samples/sec.



a) $0.13\pi n = 2\pi(0.065)n = 2\pi(65)\frac{.n}{1000}$ $\Rightarrow$ 65 Hz can be fo

Folded case : $1000 - 65$
$= 935$ Hz

$x_1(t) = 10.\cos(2\pi(65)t + \pi/13)$

$x_2(t) = 10.\cos(2\pi(935)t - \pi/13)$ → sign changes for phase for the folded case.

b) $\overset{x(t)}{\overset{\shortparallel}{26}}.\cos(2\pi.200.t + \pi/4) + 14.\cos(2\pi 500t + 3\pi/4)$ ! $f_s = 700 \frac{\text{samples}}{\text{sec.}}$

$x[n] = 26.\cos(2\pi\frac{2}{7}n + \pi/4) + 14.\cos(2\pi\frac{5}{7}n + 3\pi/4)$

$= 26.\cos(2\pi\frac{2}{7}n + \pi/4) + 14.\cos(2\pi(-\frac{2}{7})n + 3\pi/4)$

$y(t) = 26.\cos(2\pi 200t + \pi/4) + 14.\cos(2\pi 200t - 3\pi/4)$

$y(t) = 40.\cos(2\pi 200t + \pi/4$

## 4.13

$x(t) = 2.\cos(2\pi 50t + \pi/2) + \cos(2\pi(150)t)$

a) If the output of the ideal D to C converter is equal to the input $x(t)$, i.e.

$y(t) = 2.\cos(2\pi(50)t + \pi/2) + \cos(2\pi(150)t)$

what can we say about $f_s$? $f_s \geq 300$

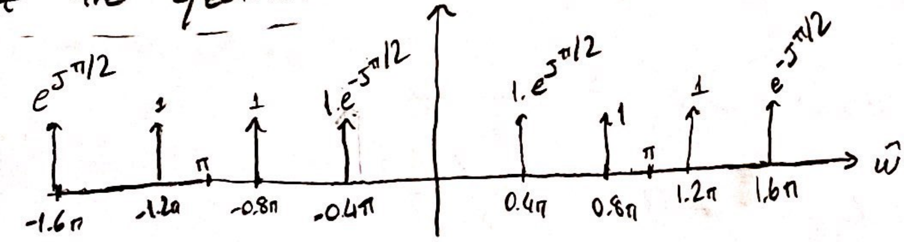
b) If $f_s = 250$ samples/sec, determine $x[n]$ as a sum of cosines. Make sure that all frequencies in your answer are positive and less than $\pi$ radians.

Replace $t$ with $n/250$.

$x[n] = x(n/250) = 2.\cos(2\pi 0.2n + \pi/2) + \cos(2\pi(0.6)n)$

$= 2.\cos(2\pi 0.2n + \pi/2) + \cos(2\pi 0.4n)$

c) Plot the spectrum:



d) If the output of the ideal D-to-C converter is

$y(t) = 2.\cos(2\pi.50t + \pi/2) + 1$

Determine the sampling frequency $f_s$.

150 Hz freq. can not be seen in $y(t)$. That means, it is aliased to 0 Hz. So $f_s = 150$ Hz.

$x[n] = x(n/150) = 2.\cos(2\pi 50.(n/150) + \pi/2) + \cos(2\pi(150)(n/150))$

$= 2.\cos(2\pi n/3 + \pi/2) + \cos(2\pi n)$

$x[n] = 2.\cos(2\pi n/3 + \pi/2) + 1$

$\downarrow$ reconstruction with $f_s = 150$ Hz

$y(t) = 2.\cos(2\pi.50t + \pi/2) + 1$