Q3

List the following functions from the lowest to highest order. Explain your answers in detail, and show the comparisons between the functions clearly. Otherwise, you will not receive any points.

n $\quad$ log(n!) $\quad$ $n^{\cos(n)}$ $\quad$ $n^{\log(n)}$ $\quad$ $\log(n)^n$ $\quad$ $3^{\log(n)}$

Solution

(Disclaimer: I calculated derivates of logarithmic functions as if they are in base e to avoid extra constants that do not affect end results.)

Let's start with initial analysis over all functions.
Before we start let's show log(n) grows slower than n: $\lim_{n\to\infty}\frac{\log(n)}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$ (by L'hopital).

1) n ➔ $n \in \Theta(n)$ This function will be probably one of the lowest orders.

2) log(n!) ➔ We can use Stirling approximation here for n!. $\log(n!) \sim \log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) = \log(\sqrt{2\pi n}) + \log\left(\left(\frac{n}{e}\right)^n\right) = \frac{1}{2}\log(2\pi n) + n\,log(n) - n\,log(e)$
If we take the highest order term in this expression, we find that $\log(n!) \in n\,log(n)$

3) $n^{\cos(n)}$ ➔ Since $-1 \le cosn \le 1$ and $\frac{1}{n} \le n^{\cos(n)} \le n$ and cos(n) oscillates between those boundaries. Since $n^{\cos(n)}$ either equals to n or smaller than it, we can take it as it grows slower than n^1.

4) $n^{\log(n)}$ ➔ Since log(n) grows with n, for any $n_0 \in N, n^{\log(n)} \in \Omega(n^{n_o})$. So, this grows faster than log(n!) since n*log(n) grows slower than n^2.

Since, $\forall n_0 \in N, \exists n$ such that $n_0 < n$; ${n_0}^{\log(n)} < n^{\log(n)}$ so $n^{\log(n)} \in \omega(3^{\log(n)})$. So $n^{\log(n)}$ has the higher order.

5) $\log(n)^n$ ➔ This is possibly the one with highest order so let's compare it with the other competitor $n^{\log(n)}$.
Let's start with taking logarithm of both: $\log(\log(n)^n) = n*\log(\log(n))$
$\log(n^{\log(n)}) = \log(n)*\log(n) = \log(n)^2$

Let's show any constant integer exponent of logarithm grows slower than n:
$\lim_{n\to\infty}\frac{n}{\log(n)^k} = \lim_{n\to\infty}\frac{n}{(k)*\log(n)^{k-1}} = \lim_{n\to\infty}\frac{n}{(k)*(k-1)*\log(n)^{k-2}} = \ldots = \lim_{n\to\infty}\frac{n}{k!} = \infty$

Eventually k is reduced to zero and limit becomes infinity. So we showed any constant exponent of logarithm grows slower than n. Since first expression has n term, log(log(n)) is nondecreasing, and second expression is a constant exponent of logarithm; we can say that first one is higher order than the second.

6) $3^{\log(n)}$ ➔ We know the upper bound, so we need to find a lower bound too. Let's start with the biggest order candidate: log(n!):

Let's take the limit (note $n^{\log(3)} = 3^{\log(n)}$): $\lim_{n\to\infty} \frac{n^{\log(3)}}{n*\log(n)} = \lim_{n\to\infty} \frac{n^{\log(3)-1}}{\log(n)} = \lim_{n\to\infty} \frac{(\log(3)-1)n^{\log(3)-2}}{n^{-1}} = \lim_{n\to\infty} (\log(3)-1)n^{\log(3)-1} = \infty$

Therefore, we found our lower bound too.

So, the following is the ordering of functions from lowest order to highest:

1) $n^{\cos(n)}$
2) n
3) log(n!)
4) $3^{\log(n)}$
5) $n^{\log(n)}$
6) $\log(n)^n$