Q1

Given functions f1 and f2, f1(n) ∈ O(g1(n)) and f2(n) ∈ O(g2(n)). Prove or disprove that f1(n) + f2(n) ∈ O(max{g1(n), g2(n)})

Prove:

Assume $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$

$f_1(n) \in O(g_1(n)) \rightarrow$ there exist positive constants c_1 and n_1 such that $f_1(n) \leq c_1 * g_1(n)$ whenever $n \geq n_1$

$f_2(n) \in O(g_2(n)) \rightarrow$ there exist positive constants c_2 and n_2 such that $f_2(n) \leq c_2 * g_2(n)$ whenever $n \geq n_2$

Add those inequalities together:

$f_1(n) + f_2(n) \leq c_1 * g_1(n) + c_2 * g_2(n)$ whenever $n \geq \max\{n_1, n_2\}$

Want to show $f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$

Lemma: $f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$ if and only if there exist positive constants c_3 and n_3 such that $f_1(n) + f_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$

Want to show c_3 and n_3 exists

Take $c_3 = c_1 + c_2$ and $n_3 = \max\{n1, n2\}$

We need to show $c_1 * g_1(n) + c_2 * g_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$

There are two cases:

<u>First:</u> Assume $g_1(n) \geq g_2(n)$

Then $c_2 * g_2(n) \leq c_2 * g_1(n)$ (since c_2 is positive)

Then $c_1 * g_1(n) + c_2 * g_2(n) \leq c_1 * g_1(n) + c_2 * g_1(n)$

Therefore $c_1 * g_1(n) + c_2 * g_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$ (since $c_3 = c_1 + c_2$)

<u>Second:</u> Assume $g_1(n) < g_2(n)$

Then $c_1 * g_1(n) \leq c_1 * g_2(n)$ (since c_1 is positive)

Therefore $c_1 * g_1(n) + c_2 * g_2(n) \leq c_1 * g_2(n) + c_2 * g_2(n)$

Therefore $c_1 * g_1(n) + c_2 * g_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$ (since $c_3 = c_1 + c_2$)

So $f_1(n) + f_2(n) \leq c_1 * g_1(n) + c_2 * g_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$

Therefore $f_1(n) + f_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$

Therefore positive constants c_3 and n_3 exists such that $f_1(n) + f_2(n) \leq c_3 * O(\max\{g_1(n), g_2(n)\})$ whenever $n \geq n_3$

Therefore $f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$