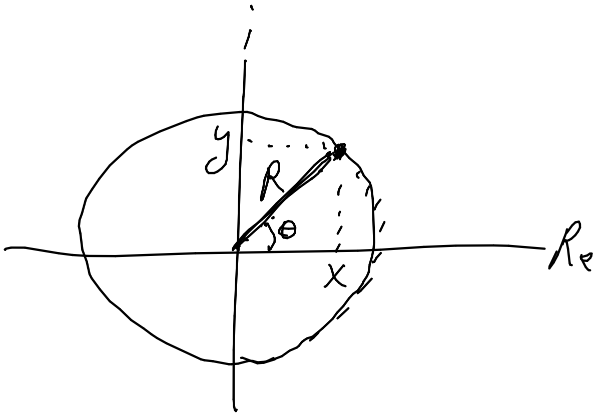
P-2.4 Use the series expansions for e^x, $\cos(\theta)$, and $\sin(\theta)$ given here to verify Euler's formula.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$



Euler's Formula

$$e^{ix} = \cos x + i\sin x$$

$$x + yi = Re^{i\theta}$$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$= 1 + \frac{ix}{1} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$e^{ix} = \cos x + i\sin x$$

P-2.5 Use complex exponentials (i.e., phasors) to show the following trigonometric identities:

(a) $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

(b) $\cos(\theta_1 - \theta_2) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)$

a) $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 + \cos\theta_1\, i\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin(\theta_2)$

b) similar.

P-2.9 Define $x(t)$ as

$$x(t) = 2\sin(\omega_0 t + 45^\circ) + \cos(\omega_0 t)$$

(a) Express $x(t)$ in the form $x(t) = A\cos(\omega_0 t + \phi)$.

(b) Assume that $\omega_0 = 5\pi$. Make a plot of $x(t)$ over the range $-1 \le t \le 2$. How many periods are included in the plot?

(c) Find a complex-valued signal $z(t)$ such that $x(t) = \Re e\{z(t)\}$.

$\text{Mag}(\sqrt{2}+1 - i\sqrt{2}) \approx \underbrace{2.8}_{2R}$

$\text{Angle}(\quad // \quad) = -0.53 \sim \frac{\pi}{6}$

a) and c)

$$x(t) = 2\sin(\underbrace{w_0 t + \tfrac{\pi}{4}}_{-\frac{\pi}{2}}) + \cos(w_0 t)$$

$$= \underbrace{2\cos(w_0 t - \tfrac{\pi}{4})}_{} + \underbrace{\cos(w_0 t)}_{}$$

$$\frac{e^{i(w_0 t - \frac{\pi}{4})} + e^{-i(w_0 t - \frac{\pi}{4})}}{2}$$

$$\frac{e^{iw_0 t} + e^{-iw_0 t}}{2}$$

$\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$ $\quad$ $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$$= \cancel{2}\left[\frac{e^{iw_0 t} e^{-i\frac{\pi}{4}} + e^{-iw_0 t} e^{i\pi/4}}{\cancel{2}}\right] + \frac{e^{iw_0 t} + e^{-iw_0 t}}{2}$$

$$= e^{iw_0 t}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) + e^{-iw_0 t}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + \frac{e^{iw_0 t}}{2} + \frac{e^{-iw_0 t}}{2}$$

$$= e^{iw_0 t}\underbrace{\left(\frac{\sqrt{2}}{2} + \frac{1}{2} - i\frac{\sqrt{2}}{2}\right)}_{Re^{i\varphi}} + e^{-iw_0 t}\underbrace{\left(\frac{\sqrt{2}}{2} + \frac{1}{2} + i\frac{\sqrt{2}}{2}\right)}_{Re^{-i\varphi}}$$

$$= Re^{iw_0 t}e^{i\varphi} + Re^{-iw_0 t}e^{-i\varphi} = 2R\left(\frac{e^{i(w_0 t + \varphi)} + e^{-i(w_0 t + \varphi)}}{2}\right)$$

$$= 2R\cos(w_0 t + \varphi) = 2.8\cos(w_0 t - \underset{-\pi/6}{\overset{}{0.53}})$$

(or)

Easier Approach:

$X(t) = a_1(t) + a_2(t)$ when x_1 & X_2 have same freq.

$a_1(t) = 2\cos(\omega_0 t - \pi/4) \longrightarrow \underbrace{2e^{-\pi/4\, i}} e^{i\omega_0 t}$

$a_2(t) = \cos(\omega_0 t) \longrightarrow \underbrace{1}\, e^{i\omega_0 t}$

add

$2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}\,i}{2}\right) + 1 = \underbrace{(\sqrt{2}+1) - i(\sqrt{2})}$

= 2.8 magnitude

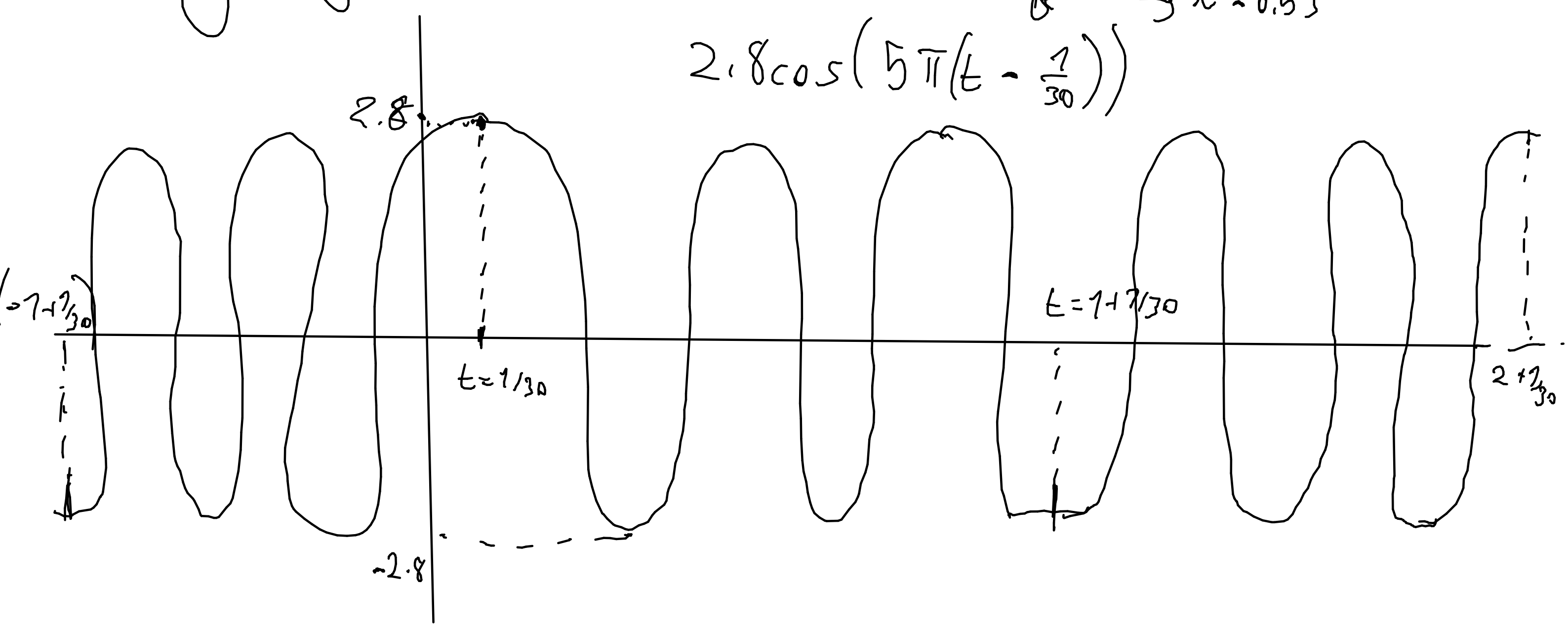
= −0.53 angle in radians

$= 2.8\, e^{-i0.53} \rightarrow$ Complex Phase of $X(t)$

$\Rightarrow 2.8\cos(\omega_0 t - 0.53)$

c) My signal is $2.8\cos\left(2\pi(2.5)t - \frac{\pi}{6}\right) \rightarrow \sim -0.53$

$2.8\cos\left(5\pi\left(t - \frac{1}{30}\right)\right)$



P-3.1 A signal composed of sinusoids is given by the equation

$$x(t) = 10\cos(800\pi t + \pi/4) + 7\cos(1200\pi t - \pi/3) - 3\cos(1600\pi t)$$

(a) Sketch the spectrum of this signal, indicating the complex size of each frequency component. Make separate plots for real/imaginary or magnitude/phase of the complex amplitudes at each frequency.

(b) Is $x(t)$ periodic? If so, what is the period?

(c) Now consider a new signal defined as $y(t) = x(t) + 5\cos(1000\pi t + \pi/2)$. How is the spectrum changed? Is $y(t)$ periodic? If so, what is the period?

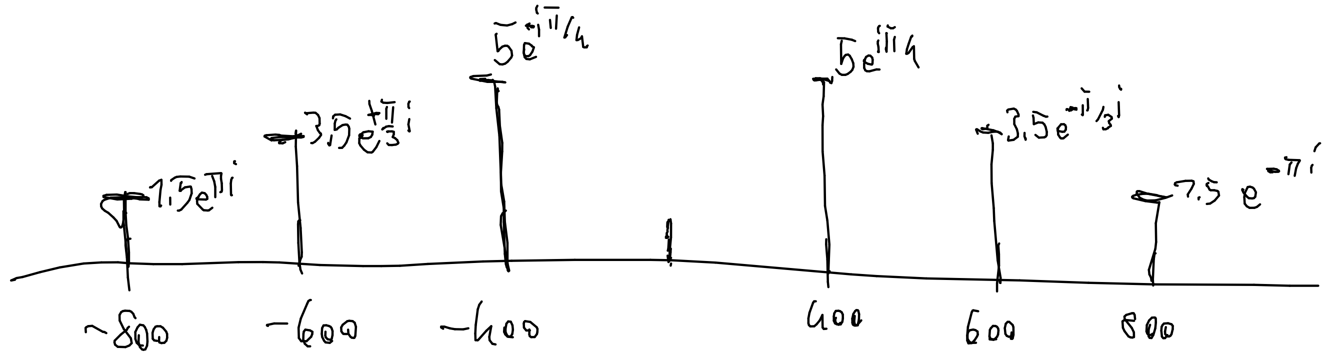
a) $x(t) = 10\cos(800\pi t + \pi/4) + 7\cos(1200\pi t - \pi/3) - 3\cos(1600\pi t)$

$+3\cos(1600\pi t - \pi)$

$= \frac{10}{2}\left(e^{i800\pi t} e^{i\pi/4} + e^{-i800\pi t} e^{-i\pi/4}\right)$

$= \frac{7}{2}\left(e^{i1200\pi t} e^{-\pi/3 i} + e^{-i1200\pi t} e^{\pi/3 i}\right)$

$= \frac{3}{2}\left(e^{i1600\pi t} e^{-\pi i} + e^{-i1600\pi t} e^{\pi i}\right)$ $\left(-1 = e^{\pi i} = e^{-\pi i}\right)$

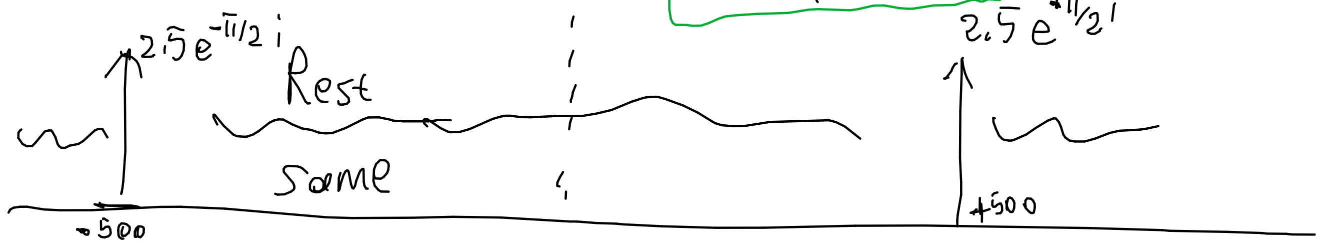


b) Yes, sum of sinusoidals where $\underbrace{f_0}_{\text{fundamental freq.}} = 200$ Hz

$T_0 = \frac{1}{f_0} = \frac{1}{200} \rightarrow$ Fundamental Period

c) $x(t) + 5\cos(1000\pi t + \pi/2)$

$\boxed{\text{New } f_0 = 100 \text{ Hz}}$



P-3.6 An amplitude-modulated (AM) cosine wave is represented by the formula

$$x(t) = [12 + 7\sin(\pi t - \tfrac{1}{3}\pi)]\cos(13\pi t)$$

(a) Use phasors to show that $x(t)$ can be expressed in the form

$$x(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) + A_3\cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find values of the parameters A_1, A_2, A_3, ϕ_1, ϕ_2, ϕ_3, ω_1, ω_2, ω_3.

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the numerical values of A_i, ϕ_i, and ω_i.

$$x(t) = \left[12 + \frac{7}{2j}\left(e^{\pi t j} e^{-\frac{\pi}{3}j} - e^{-\pi t j} e^{\pi/3\, j}\right)\right]\frac{1}{2}\left(e^{13\pi t j} + e^{-13\pi t j}\right)$$

$$\boxed{\frac{1}{j} = e^{-\frac{\pi}{2}j}}$$

$$x(t) = \frac{12}{2}\left(e^{13\pi t j} + e^{-13\pi t j}\right) + \frac{7}{4}\left(e^{\pi t j} e^{-\frac{\pi}{3}j} e^{-\frac{\pi}{2}j} - e^{-\pi t j} e^{\frac{\pi}{3}j} e^{-\frac{\pi}{2}j}\right)\left(e^{13\pi t j} + e^{-13\pi t j}\right)$$

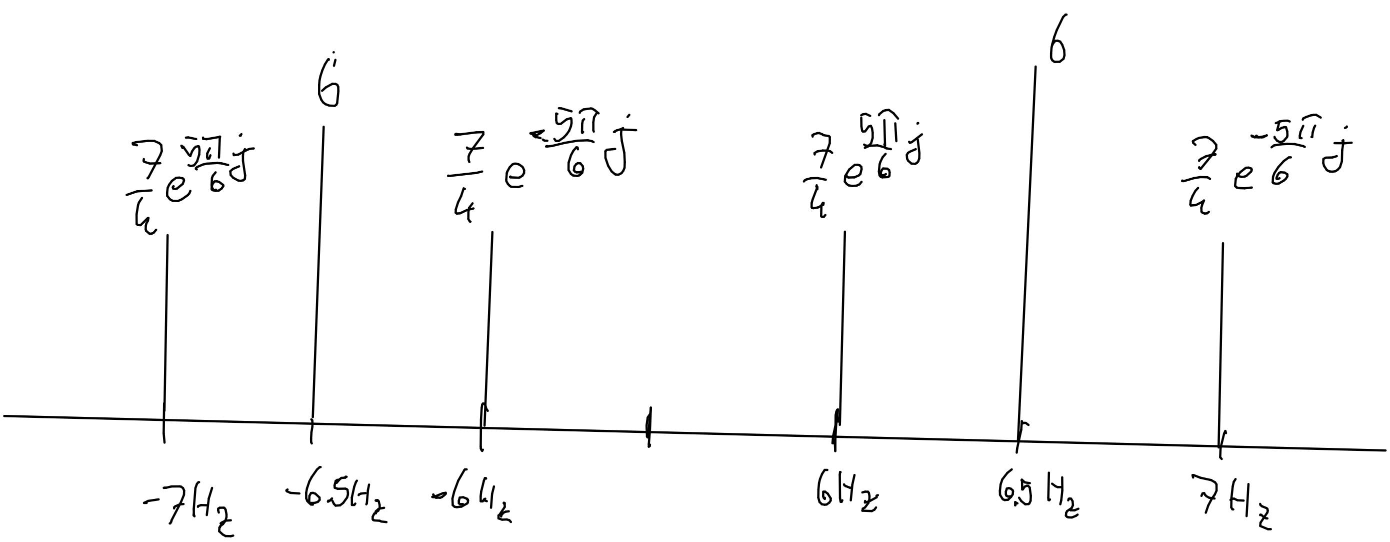
$-5\pi/6$ $\quad (-1 = e^{\pi i})$ $\quad -\pi/6$

$$= 12\cos(13\pi t) + \frac{7}{4}\left(e^{14\pi t j} e^{-\frac{\pi}{3}j} e^{-\frac{\pi}{2}j} - e^{-14\pi t j} e^{\pi/3\, j} e^{-\frac{\pi}{2}j} + e^{-12\pi t j} e^{-\frac{\pi}{3}j} e^{-\frac{\pi}{2}j} - e^{12\pi t j} e^{\pi/3\, j} e^{-\frac{\pi}{2}j}\right)$$

$-5\pi/6$ $\quad (-1 = e^{\pi i})$ $\quad -\pi/6$

$$= 12\cos(13\pi t) + \frac{7}{4}\left(e^{14\pi t j} e^{-5\pi/6\, j} + e^{-14\pi t j} e^{5\pi/6\, j} + e^{12\pi t j} e^{5\pi/6\, j} + e^{-12\pi t j} e^{-5\pi/6\, j}\right)$$

$$= 12\cos(13\pi t) + \frac{7}{2}\cos(12\pi t + 5\pi/6) + \frac{7}{2}\cos(14\pi t - 5\pi/6)$$



P-3.6 An amplitude-modulated (AM) cosine wave is represented by the formula

$$x(t) = [12 + 7\sin(\pi t - \tfrac{1}{3}\pi)]\cos(13\pi t)$$

(a) Use phasors to show that $x(t)$ can be expressed in the form

$$x(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) + A_3\cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find values of the parameters A_1, A_2, A_3, ϕ_1, ϕ_2, ϕ_3, ω_1, ω_2, ω_3.

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the numerical values of A_i, ϕ_i, and ω_i.

$$x(t) = \left[12 + \frac{7}{2i} e^{\pi t i} e^{-\frac{\pi}{3} i} - \frac{7}{2i} e^{-\pi t i} e^{\frac{\pi}{3} i}\right]\left(\frac{1}{2} e^{13\pi t i} + \frac{1}{2} e^{-13\pi t i}\right)$$

$$\frac{1}{i} = e^{-\frac{\pi}{2} i}$$

$$\Rightarrow = \left[12 + \frac{7}{2} e^{\pi t i} e^{-\frac{\pi}{3} i} e^{-\frac{\pi}{2} i} - \frac{7}{2} e^{-\pi t i} e^{\frac{\pi}{3} i} e^{-\frac{\pi}{2} i}\right]\left(\frac{1}{2} e^{13\pi t i} + \frac{1}{2} e^{-13\pi t i}\right)$$

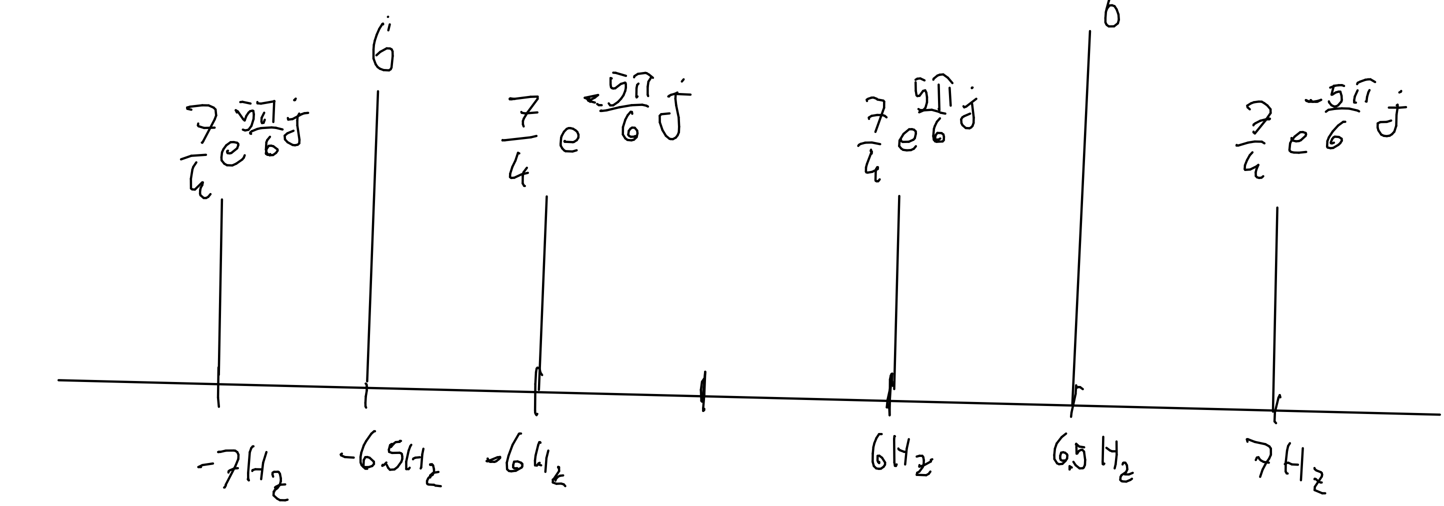
$$= \frac{12}{2}\left(e^{13\pi t i} + e^{-13\pi t i}\right) + \frac{7}{4}\Big(e^{14\pi t i} e^{-\frac{\pi}{3} i} e^{-\frac{\pi}{2} i} + e^{-12\pi t i} e^{-\frac{\pi}{3} i} e^{-\frac{\pi}{2} i}$$

$$\underbrace{-}_{(-1 = e^{\pi i})} e^{12\pi t i} e^{\frac{\pi}{3} i} e^{-\frac{\pi}{2} i} \underbrace{-}_{(-1 = e^{\pi i})} e^{-14\pi t i} e^{\frac{\pi}{3} i} e^{-\frac{\pi}{2} i}\Big)$$

$$= 12\cos(13\pi t) + \frac{7}{4}\left(e^{12\pi t i} e^{\frac{5\pi}{6} i} + e^{-12\pi t i} e^{-\frac{5\pi}{6} i}\right)$$

$$+ \frac{7}{4}\left(e^{14\pi t i} e^{-\frac{5\pi}{6} i} + e^{-14\pi t i} e^{\frac{5\pi}{6} i}\right)$$

$$= 12\cos(13\pi t) + \frac{7}{2}\cos(12\pi t + 5\pi/6) + \frac{7}{2}\cos(12\pi t - 5\pi/6)$$



P-3.10 The periodic waveform in Fig. P-3.10 has the property of *half-wave symmetry*; i.e., the last half of the period is the negative of the first half. More precisely, signals with half-wave symmetry have the property that

$$x(t + T_0/2) = -x(t) \qquad -\infty < t < \infty \qquad (3.47)$$

In this problem we will show that this condition has an interesting effect on the Fourier series coefficients for the signal.

(a) Suppose that $x(t)$ is a periodic signal with half-wave symmetry and is defined over half a period by

$$x(t) = t \qquad \text{for } 0 \le t < T_0/2$$

where T_0 is the period of the signal. Plot this periodic signal for $-T_0 \le t \le T_0$.

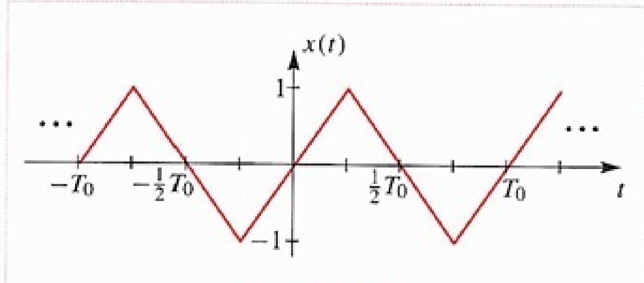
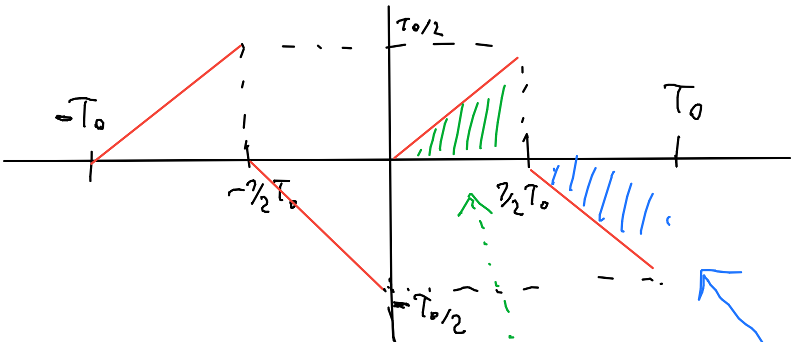


Figure P-3.10

(b) Prove that the DC coefficient a_0 is zero for any periodic signal having half-wave symmetry. *Hint:* Split the integral for a_0 into two parts.

(c) Prove that all the even indexed Fourier series coefficients are zero for a signal with half-wave symmetry; i.e., $a_k = 0$ if k is an even integer.

a)



Recall

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-i(\frac{2\pi}{T_0})kt}\, dt$$

b) $$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j.0}\, dt = \frac{1}{T_0}\int_0^{T_0/2} x(t)\, dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} x(t)\, dt = 0$$

In general, let $u = t - \frac{T_0}{2} \Rightarrow dt = du$

$$a_0 = \frac{1}{T_0}\int_0^{T_0/2} x(t)\, dt + \frac{1}{T_0}\int_0^{T_0/2} \underbrace{x(u+\tfrac{T_0}{2})}_{-x(u)}\, du = 0$$

c) If k even, $k = 2l$ where $l \in \mathbb{N}$

$$a_k = \frac{1}{T_0}\int_0^{T_0/2} x(t)\, e^{-j(\frac{2\pi}{T_0})2lt}\, dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} x(t)\, e^{-j(\frac{2\pi}{T_0})2lt}\, dt$$

let $u = t - T_0/2$

$$a_k = \frac{1}{T_0}\int_0^{T_0/2} x(t)\, e^{-j(\frac{2\pi}{T_0})2lt}\, dt + \frac{1}{T_0}\int_0^{T_0/2} \underbrace{x(u+T_0/2)\, e^{-j(\frac{2\pi}{T_0})2l(u+T_0/2)}}_{= -x(u)\, e^{-j(\frac{2\pi}{T_0})2lu}}\, du$$

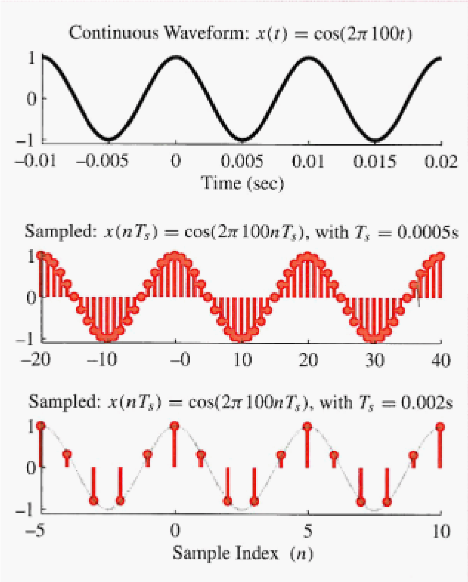


Figure 4-3: A continuous-time 100-Hz sinusoid (top) and two discrete-time sinusoids formed by sampling at $f_s = 2000$ samples/sec (middle) and at $f_s = 500$ samples/sec (bottom).

EXERCISE 4.1: If the sampling rate is $f_s = 1000$ samples/sec. and the continuous-time signal is $x(t) = \cos(\omega t)$, what value of ω will give a sequence of samples identical to the discrete-time signal shown in the bottom panel of Fig. 4-3?

$x(t) = \cos(\omega t)$

$\cos(2\pi 100 n \, 2.10^{-3})$

$= \cos(0.4\pi n)$

$= \cos(\omega n \underbrace{10^{-3}}_{T_s})$

$\Rightarrow 400\pi = \omega$

EXERCISE 4.2: Show that $7\cos(8.4\pi n - 0.2\pi)$ is an alias of $7\cos(0.4\pi n - 0.2\pi)$. In addition, find two more frequencies that are aliases of 0.4π rad.

EXERCISE 4.3: Show that the signal $7\cos(9.6\pi n + 0.2\pi)$ is an alias of the signal $7\cos(0.4\pi n - 0.2\pi)$. It might be instructive to make MATLAB plots of these two signals to verify that the phase must change sign to have identical plots.

exercise 4.2

$$\cos(8.4\pi n - 0.2\pi) = \cos(0.4\pi n - 0.2\pi + 4.(2\pi)n)$$
$$= \cos(0.4\pi n - 0.2\pi)$$

for $l \in \mathbb{Z}$ $\cos(\varphi)$ is an alias

if φ $= 2\pi l n - (0.4\pi n - 0.2\pi)$

or

$= (0.4\pi n - 0.2\pi) + 2\pi l n$

Exercise 4.3 $\cos(9.6\pi n + 0.2\pi) \sim \cos(0.4\pi n - 0.2\pi)$

$=$

$$\cos(10\pi n - 9.6\pi n - 0.2\pi) = \cos(0.4\pi n - 0.2\pi)$$

P-4.1 Consider the cosine wave

$$x(t) = 10\cos(880\pi t + \phi)$$

Suppose that we obtain a sequence of numbers by sampling the waveform at equally spaced time instants nT_s. In this case, the resulting sequence would have values

$$x[n] = x(nT_s) = 10\cos(880\pi nT_s + \phi)$$

for $-\infty < n < \infty$. Suppose that $T_s = 0.0001$ sec.

(a) How many samples will be taken in one period of the cosine wave?

(b) Now consider another waveform $y(t)$ such that

$$y(t) = 10\cos(\omega_0 t + \phi)$$

Find a frequency $\omega_0 > 880\pi$ such that $y(nT_s) = x(nT_s)$ for all integers n.
Hint: Use the fact that $\cos(\theta + 2\pi n) = \cos(\theta)$ if n is an integer.

(c) For the frequency found in (b), what is the total number of samples taken in one period of $x(t)$?

a) $880\pi n T_s \leq 2\pi$

$n \leq \frac{2}{880 \cdot 10^{-4}} = \frac{2000}{88} = \frac{250}{11} \sim 23$

b) we want $x[n] = y[n]$

$880\pi \cancel{n} T_s = w_0 \cancel{n} T_s + 2\pi l \cancel{n}$

$w_0 = 880\pi - \frac{2\pi l}{T_s} \quad l \in \mathbb{Z}$

$w_0 = 20880\pi$

c) $n \leq \frac{2}{2.088}$ only 1.