CENG115 – Discrete Structures
Homework 3

November 15, 2022

Due Date: November 28, 2022

Exercise 1. Unbounded and infinite sets (12 pts)

See the definitions of unbounded (`https://en.wikipedia.org/wiki/Bounded_set#Definition_in_the_real_numbers`) and infinite (`https://en.wikipedia.org/wiki/Infinite_set`) sets. Write the truth values of the following existential propositions. Give an example for true propositions.

. (Example) Some bounded sets are finite.
True for $\{1, 2, 3\}$.

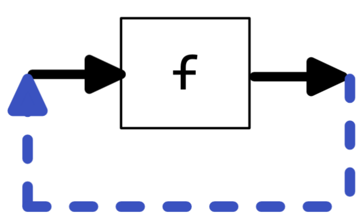
a. (4 pts) Some bounded sets are infinite.

b. (4 pts) Some unbounded sets are finite.

c. (4 pts) Some unbounded sets are infinite.

Exercise 2. Properties of functions (18 pts)

Functions can be thought as machines that consume raw materials and produce fine products. For example $succ(x) = x + 1$ can be seen as a machine that consumes a number and produces another, the successor of the consumed number: when the input is 5, its output must be 6.



A machine f is first given an arbitrary input x to obtain an output y. Then, the same machine is given this output y as the input and this time an output z is obtained. An observer claims that $x = y$. Give truth values to the given propositions: "true" or "false" if it is certain, or "uncertain" if it is uncertain.

Assume that the observer is right:

a. (2 pts) f is the identity function.

b. (2 pts) f is a one-to-one function.

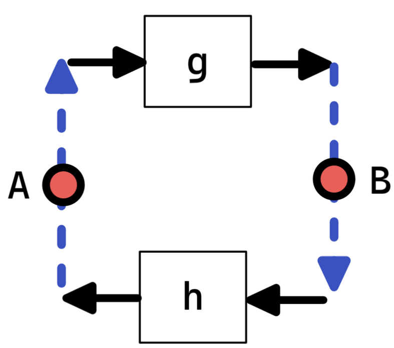
c. (2 pts) $x = z$.

This time assume that the observer is wrong:

d. (2 pts) f is the identity function.

e. (2 pts) x is an element of image of f.

f. (2 pts) $x = z$.



The machines g and h feed each other automatically once the system is initialized. We stand at point A and pass a random object x to initialize the system. Then we walk to point B but machines are fast and we are too late to observe $g(x)$ before it is consumed by h. Let's call this output a. We don't know anything about a. We wait for another tour. Then we observe the output of the second execution of g, which is $g(h(g(x)))$. Let's call this output b. We notice that $b \neq x$. We wait for another tour to observe $g(h(g(h(g(x)))))$. Let's call this output c. We notice that $b = c$. When we tell these to a friend of us (who is a good mathematician), she tells us that this story is true not only for our selection of x but for all possible initial values: it is always the case that $c = b \neq x$.

g. (6 pts) What can be said about the value a and the functions g and h? Think not only about the individual functions but also the relationships between the functions and/or the combined function $h \circ g$ (or $g \circ h$).

Exercise 3. Counting sets by defining one-to-one correspondences (30 pts)

A set can be counted by ordering their elements arbitrarily. The order is mathematically expressed with a one-to-one function from the set to positive integers (a.k.a. natural numbers). The cardinality of the set must be the same as the cardinality of the image of this function as it establishes a one-to-one correspondence between the elements. As an example, we can count $\{\triangle, \square, \bigcirc\}$:

$$f(x) = \left\{ \begin{array}{ll} 4, & \text{for } x = \triangle \\ 7, & \text{for } x = \square \\ 11, & \text{for } x = \bigcirc \end{array} \right\}$$

Image of f is $\{4, 7, 11\}$ and this set has a cardinality of 3. This way of counting may seem unnecessarily complicated for finite sets but it becomes useful when we work with infinite sets. Existence of such functions prove that the set in question is countable. Obviously, all finite sets are countable. Likewise some infinite sets are countable (e.g., rational numbers) but some infinite sets are uncountable (e.g., real numbers. See Cantor's diagonal argument if you are curious.).

Starting from 1 and using the consecutive numbers is a more convenient way of counting. From now on, we will give the examples in this way and you are supposed to follow this convention in your answers as well:

$$f(x) = \left\{ \begin{array}{ll} 1, & \text{for } x = \triangle \\ 2, & \text{for } x = \square \\ 3, & \text{for } x = \bigcirc \end{array} \right\}$$

You can count positive integers using $f(x) = x$, negative integers using $f(x) = -x$, and non-negative integers using $f(x) = x + 1$. Pause here to fully understand the examples. Visualize these functions if you need. Please note that even with the convention, there are multiple ways of counting. For example the following function can be used to count non-negative integers:

$$f(x) = \left\{ \begin{array}{ll} 1, & \text{for } x = 1 \\ 2, & \text{for } x = 0 \\ x + 1, & \textit{otherwise} \end{array} \right\}$$

However $f(x) = x + 1$ is much simpler and more natural. In the following questions you are supposed to find a convenient way of counting the given sets. Simply define the function.

. (Example) Integers that are greater than 100: $\{101, 102, 103, 104, \dots\}$
$f(x) = x - 100$

a. (10 pts) Square numbers: $\{1, 4, 9, 16, \dots\}$
$f(x) = ?$

b. (10 pts) Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
$f(x) = ?$

c. (10 pts) $\mathbb{Z}^+ \times \mathbb{Z}^+ = \{(1,1), (1,2), (1,3), \dots, (2,1), (2,2), (2,3), \dots, \dots\}$
$f(x) = ?$
For this set, please visualize the counting as well.
(Hint: Visualize the set in 2-dimensional space. You cannot afford iterating horizontally or vertically.)

Exercise 4. Infinite seat allocation (20 pts)

Let we have a bus with infinite seats. Seats are numbered using the natural numbers. All seats are already occupied. However, we are greedy and trying to fit more passengers in. (The best thing about the infinite buses is that you can fit more passengers in than you may initially think.)

"Potential passengers" wait outside the bus in a line and everybody knows their number in the line. Find a solution to fit all potential passengers into the bus by defining two functions: *move* which is announced to the passengers inside the bus and *sit* which is announced to potential passengers waiting outside. $move(x) = y$ indicates that the passenger who sits in the seat numbered x must move to the seat numbered y (in order to empty the seat). $sit(x) = y$ indicates that the potential passenger who waits with the line number x must sit in the seat numbered y. After everybody moves and sits according to your function definitions, each seat must be occupied by a single person only.

. (Example) Assume that there is a single person waiting in the line. He/she is number 1. Find a solution to fit him/her.
$move(x) = x + 1 \qquad sit(x) = 1$
(Everybody will shift by 1 and the newcomer will sit at 1.)

a. (10 pts) Assume that there are 100 people waiting in the line. They are numbered $\{1, 2, 3, \ldots, 100\}$. Find a solution to fit all.
$move(x) = ? \qquad sit(x) = ?$

b. (10 pts) Assume that there are infinite people waiting in the line. They are numbered $\{1, 2, 3, \ldots\}$. Find a solution to fit all.
$move(x) = ? \qquad sit(x) = ?$
(Note: This is possible to solve. You can even fit countably infinite lines each of which has countably infinite people! Now, fortunately, you only have a single line of infinite people.)

Exercise 5. Sequences and summations (20 pts)

a. (10 pts) A rabbit runs at 40 km/h. After every hour of running it gets more tired and its speed is decreased by 10%. Given infinite time, can it travel infinite distances? If not, what is the maximum distance it can travel? Prove it.

b. (10 pts) Alice has a single slice of bread on the table. She repeats this algorithm 10 times: chooses the smallest piece on the table, cuts it into three equal pieces, and throws one of them away. When the process is done, how many pieces are there on the table and what is the ratio of the area of the biggest piece to the area of the smallest piece on the table?