Chapter 4

Graph algorithms

4.1 BFS

1. start with an empty (FIFO) list l and a (random) vertex u_1;
2. set $pr(u_1) = \emptyset$;
3. start with a graph T where $V(T) = u_1$ and $E(T) = \emptyset$
4. label vertex u_1 active
5. for every edge $e = (v, u)$ incident with the active vertex v do
   (a) if $u \in V(T)$ skip e and u
   (b) else if $u \notin V(T)$ add u to the list l;
   (c) $V(T) = V(T) \cup u$
   (d) $E(T) = E(T) \cup (v, u)$
   (e) set $pr(u) = v$
6. remove the vertex v from the list and
   (a) if l is not empty label the first vertex active and go to Step 5
   (b) else terminate the algorithm by outputting T

Theorem 4.1.1. *The BFS algorithm terminates by outputting a tree T.*

Proof by induction. Basic step is OK. Hypothesis: for $k \geq 0$ T Is a tree with $k+1$ vertices and k edges. Inductive step. If the algorithm terminates as step $k+1$ we are done. Otherwise, the algorithm adds the same number of edges and vertices, preserves connectivity and does not create cycles. □

Corollary 4.1.1. *If T has less vertices than G then G is disconnected, otherwise T is a spanning tree.*

Definition 4.1.1. *A level of a vertex u in a BFS tree is the integer k such that $pr^k(u) = u_0$.*

Theorem 4.1.2. *Vertices enter a BFS tree in non-decreasing order*

Proof. By induction. The first vertex is at level zero. Assume for the first m vertices the result holds. Consider the next vertex v that joins the tree at stage $m+1$. Then $pr(v) = u$, where u is the active vertex. level$(v) =$ level$(u) + 1$. Consider any other non-root vertex x in the tree at stage $m+1$. Let $pr(x) = y$, hence level$(x) =$ level$(y) + 1$. There are two cases $y = u$ or y was active before u by induction hypothesis level$(y) \leq$level(u). Then

$$\text{level}(v) = \text{level}(u) + 1 \geq \text{level}(y) + 1 = \text{level}(x)$$

hence if x was added before v the level of x is no larger than the level of v. $\square$

Theorem 4.1.3. *In a connected graph G with a breadth-first search tree T each edge $e \in E(G) \setminus E(T)$ connects vertices that are at most one level apart.*

Proof. let $e = (u, v)$ be non-tree edge and without loss of generality let u join the tree before v.

Case 1: v is not in the tree when u is active. Then v and e are added to the growing tree and level$(u) + 1 =$ level(v).

Case 2: v is in the tree when u is active then by Theorem 4.1.2 we have level$(u) \leq$ level(v). Let $pr(v) = w$ and since v is in the tree when u is active then level$(w) \leq$level(u). Thus

$$\text{level}(u) \leq \text{level}(y) = \text{level}(w) + 1 \leq \text{level}(u) + 1$$

$\square$

Theorem 4.1.4. *A connected graph G with BFS tree T has an odd cycle if and only if there is a non-tree edge that joins vertices at the same level*

Proof. Follow the path to the first common ancestor $\square$

Theorem 4.1.5. *The length of a shortest path from u to v in a connected graph G equals the level of v in any BFS tree of G with u as root*

Proof. Homework $\square$