1.4 Connectivity

Definition 1.4.1 (subgraph)**.** *A subgraph G' of a graph G is a graph such that $V(G') \subseteq V(G)$ and the edges of G' are subset of the edges of G such that each edge in G' is incident only with vertices in $V(G')$.*

- clique – a subgraph of a graph that is isomorphic to a complete graph
- independent set (stable set)– a subgraph of a graph that is isomorphic to an empty graph

Definition 1.4.2 (spanning subgraph)**.** *A spanning subgraph of G is a subgraph G' of G such that $V(G') = V(G)$.*

Definition 1.4.3 (connected graph)**.** *A connected graph G is a graph such that there is a path between any two vertices in G.*

Theorem 1.4.1. *Every connected graph has at least two vertices of the same degree.*

Proof. Pigeonhole principle. □

Definition 1.4.4 (maximal connected subgraph)**.** *A* maximal connected subgraph *of G is a subgraph H such that if G' is connected subgraph of G and H is subgraph of G' then $H = G'$.*

Definition 1.4.5 (component)**.** *A maximal connected subgraph of a graph G is called a* component *of G.*

Definition 1.4.6 (decomposition)**.** *A* decomposition *of a graph is a list of subgraphs such that each edge appears in exactly on subgraph in the list.*

Let e be an edge in G with $G - e$ we denote the graph that has vertex set $V(G)$ and edge set $E(G) - e$. Similarly, if $v \in V(G)$ then $G - v$ denotes the graph what has vertex $V(G) - v$ and edge set

$$E(G) - \{e \in E(G) \mid e \text{ is incident with } v\}$$

. The idea can be generalized for a set of edges and set of vertices. For a vertex set T in G we write $G[T]$ for the graph $G - \overline{T}$[2] and call it the subgraph of G *induced* by T.

Theorem 1.4.2. *For every connected graph G the vertices can be enumerated $v_1, v_2, \ldots, v_n$ such that the graph G_i induced by $v_1, \ldots, v_i$ i.e., $G_i = G[v_1, v_2, \ldots, v_i]$ is connected for every i.*

Proof. Pick a vertex at random and denote it as v_1. G_1 is connected and assume by induction that for i, $v_1, \ldots, v_i$ are such that the graph G_i is connected. Let $v \in G - G_i$. Since G is connected there is a path from v_1 to v in G. Define v_{i+1} as the first vertex on the path from v_1 to v that is not in G_i. Then v_{i+1} has a neighbour in $v_1, \ldots, v_i$ and the connectedness of G_i follows by induction. □

Definition 1.4.7. *A* bridge *of a graph G is an edge $e \in E(G)$ such that the number of components of $G - e$ is strictly greater than the number of components of G.*

Theorem 1.4.3. *A graph G is connected if, for some vertex u in $V(G)$, there is a path from u to x for all vertices x in $V(G)$.*

[2]Remove the complement of T, i.e. the graph is restricted to the vertices in T

Proof. Suppose G is connected and let u be any vertex in $V(G)$. Since G is connected there is a path from u to x for any vertex $x \in V(G)$.

Suppose now G has a vertex u such that there is a path from u to any other vertex in $V(G)$. Let x, y be any two vertices in $V(G)$. By assumption

1. there is a path from x to u, i.e., there is a path $p_x = xe_1v_1e_2v_2 \dots e_{n-1}v_{n-1}e_nu$

2. there is a path from y to u, i.e., there is a path $p_y = ue'_1v'_1e'_2v'_2 \dots e'_{k-1}v'_{k-1}e'_ky$

Combining p_x and p_y we obtain the walk

$$p_{xy} = xe_1v_1e_2v_2 \dots e_{n-1}v_{n-1}e_nue'_1v'_1e'_2v'_2 \dots e'_{k-1}v'_{k-1}e'_ky$$

Thus there is a walk from x to y and therefore there is a path from x to y. Since x and y were arbitrary vertices it follows that the graph is connected. □

Theorem 1.4.4. *If $e = (x, y)$ is a bridge of a connected graph G, then $G - e$ has precisely two components; furthermore x and y are in different components.*

Proof. Let $e = (u, v)$. Suppose e is a bridge then $G - e$ has at least two components. Let V_u be the set of vertices in $G - e$ such that there is a path from any vertex $x \in V_u$ to u. Let y be any vertex of $G - e$ such that $y \notin V_u$. Note that there is at least one vertex in not in V_u because $G - e$ has at least two components. Since there is a path from y to u in G and no path from y to u in $G - e$, then any path from y to u in G is of the form

$$ueve_1v_1 \dots v_{n-1}e_ny$$

Then there is a path $ve_1v_1 \dots v_{n-1}e_ny$ from any vertex in $G - e$ to v and therefore every vertex not in V_u is in the same component as v. □

Theorem 1.4.5. *Edge e is a bridge of a graph G if and only if e is not in any cycle of G.*

Proof. First we show that if $e = (ab)$ is an edge in a cycle then e cannot be a bridge. Let $ae_1v_1e_2v_2 \dots v_{n-1}e_nbea$ be a cycle that contains e. Then $ae_1v_1e_2v_2 \dots v_{n-1}e_nb$ is a path from a to b in $G - e$. If e were a bridge then in $G - e$ by Theorem 1.4.4 a and b would be in different components a contradiction. Therefore if e is a bridge then e is not in any cycle of G.

If e is not in any cycle, then there is only one path from a to b which is the path aeb. Indeed if there were to be another path $ae_1v_1 \dots v_{n-1}e_nb$ that do not contain e then $ae_1v_1 \dots v_{n-1}e_nbea$ would be a cycle that contains e. Since e is not in the graph $G - e$ then a and b are in different components of $G - e$. And furthermore $G - e$ has more components than G. Thus e is a bridge. □