Linear Algebra

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Basis, Dimension, Coordinates

Basis

Definition (basis)

Let $\mathbf{V}$ be a vector space, the set of vectors $\mathbf{B} = \{\vec{b}_1, \ldots, \vec{b}_d\}$ is a *basis* for $\mathbf{V}$ if every vector in $\mathbf{V}$ can be represented as a linear combination of the vectors in $\mathbf{B}$ and the vectors in $\mathbf{B}$ are linearly independent.

2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathbb{C}^2$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Different basis

Theorem

Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two distinct basis for a vector space $\mathbf{V}$. Then the number of vectors in $\mathbf{B}_1$ and $\mathbf{B}_2$ is the same.

Dimension

Definition (dimension)

Let $\mathbf{B}$ be a basis for a vector space $\mathbf{V}$, then the size of $\mathbf{B}$ is called the *dimension* of $\mathbf{V}$.

Different basis

Theorem

Let $\vec{e_1}, \ldots, \vec{e_n}$ *be linearly independent. Suppose*

1. $\vec{u} = a_1\vec{e_1} + \cdots + a_n\vec{e_n}$ *and*
2. $\vec{u} = b_1\vec{e_1} + \cdots + b_n\vec{e_n}$.

Then

$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$$

Coordinates

Definition (coordinates)

Let $\vec{u}$ be a vector in a d-dimensional vector space $\mathbf{V}$. Let $\mathbf{B} = \{\vec{b}_1, \ldots, \vec{b}_d\}$ be a basis for $\mathbf{V}$. By definition of basis

$$\vec{u} = \xi_1 \vec{b}_1 + \cdots + \xi_d \vec{b}_d.$$

The values $\xi_1, \ldots, \xi_d$ are the coordinates of $\vec{u}$ with respect to basis $\mathbf{B}$ denoted by

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix}_{\mathbf{B}}$$

$\mathbf{P}_2$

$$p(x) = 3 + 2x + x^2$$

- $\mathbf{B}_0 = \{x^0, x^1, x^2\}$
- $\mathbf{B}_1 = \{x^0, x^0 + x^1, x^0 + x^1 + x^2\}$
- $\mathbf{B}_3 = \{x^0, x^0 + x^1, x^0 + x^2\}$

$$p(x) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}_{\mathbf{B}_0} = \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{\mathbf{B}_1} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}_{\mathbf{B}_2}$$