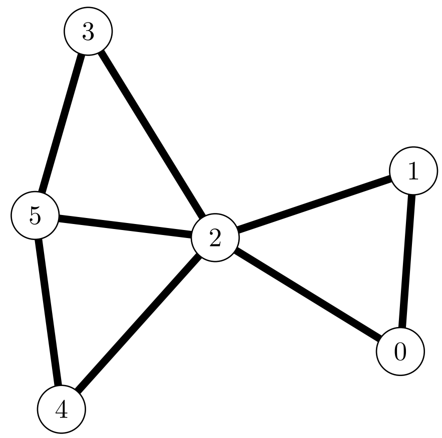

Figure 2.1: The graph GM

2.1 Matchings

2.1.1 Basic definition

Let $G = (V, E)$ be simple undirected finite graph without loops. All other types of graphs that we consider will be explicitly stated.

Problem: In a dormitory each room is a double bed. Each student has a preference for whom the students likes to share the room. Fill as many rooms as possible with pairs of students that accept to share a room. Each student lives in at most one room.

Model. Define a graph where each vertex is a student and there is an edge between two students if and only if they are willing to share a room. Let $M = (x_1y_1, x_2y_2, \ldots, x_ky_k)$ be the filled rooms. It means that all elements in the set $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ are distinct and x_iy_i is an edge in the graph. Our goal is to find an M with maximum number of edges. □

Definition 2.1.1. *A matching in a graph G is a set of edges M such that no two edges in M are incident with the same vertex i.e.,*

$$xy, zw \in M \Rightarrow \{x, y\} \cap \{z, w\} = \emptyset.$$

For a vertex u in the vertex set of G we say that v is *saturated* by a matching M if there is an edge in M incident with u.

A *perfect matching* is a matching in a graph where every vertex is saturated.

Example C_5, K_{2n+1} have no perfect matchings, C_4 has a perfect matching. The graph GM given bellow also has a perfect matching $[(0, 1, \text{None}), (2, 3, \text{None}), (4, 5, \text{None})]$[1]:

Definition 2.1.2. *A* maximal *matching in graph is a matching that cannot be enlarged by adding an edge. A* maximum *matching is a matching with size (number of edges) among all matchings in the graph.*

[1]Here “None” is the name of the edge which we did not define.

For the graph in Figure 2.1 the matching $[(0,1,\mathrm{None}),(2,3,\mathrm{None}),(4,5,\mathrm{None})]$ is maximal and maximum. The matching $[(0,1),(2,5)]$ is maximal but not maximum. The matching $[(0,1)]$ is neither maximal nor maximum.

Definition 2.1.3. *Given a matching M an M-*alternating *path is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are not saturated by M is a M-*augmented path*.*

Definition 2.1.4. *Symmetric difference $G \triangle H$ of graphs G and H is the graph with vertex set $V = V(G) = V(H)$ whose edges are all those edges that appear in exactly one of G or H.*

Theorem 2.1.1. *Every component of the symmetric difference of two matchings is a path or an even cycle.*

Proof. In the symmetric difference a vertex is incident with at most one edge from each matching and therefore each vertex has degree at most two. Thus we either have cycles or paths. If we have a cycle then assuming the cycle is odd it means that it has odd number of edges from the same matching and therefore two edges from the same matching are incident with the same vertex a contradiction thus the cycles are even. □

Theorem 2.1.2. *A matching M in a graph G is a maximum matching if and only if G has no M-augmented path.*

Proof. The only if part holds because we have a maximum matching (switching edges along the augmented path will increase the matching's size which is a contradiction).

Assume that M is a matching that has no M-augmented path. Assume also M' is a matching with larger size. By Theorem 2.1.1 the symmetric difference between M and M' contains either paths or cycles. Since there are no M-augmented paths the number of edges from M in each path in the symmetric difference is at least as much as the edges from M'. Cycles are even and they contain the same number of edges from each matching. Hence the size of M and M' is the same. Thus M and M' have the same number of edges a contradiction with M' having size greater than M. □