45 min lecture

2.9 Rank of a matrix

Theorem 2.9.1. *Let A be a square matrix for which there is a square matrix B such that $AB = I$. Then the columns of A are linearly independent.*

Proof. Let A be $k \times k$ matrix. Since A is invertible then there exists A^{-1} such that $AA^{-1} = I$. In the last multiplication it follows that the columns of I are linear combinations of the columns of A. Denote the columns of A as $\vec{a_1}, \ldots, \vec{a_k}$ and suppose by contradiction that $\vec{a_1}, \ldots, \vec{a_k}$ are linearly dependent. Let $\vec{b_1}, \ldots, \vec{b_m}$, where $m < k$ be a set with largest cardinality such that $\vec{b_1}, \ldots, \vec{b_m}$ are linearly independent. Then the columns of I are linear combinations of $\vec{b_1}, \ldots, \vec{b_m}$. This follows from the fact that $\vec{a_1}, \ldots, \vec{a_k}$ are linear combinations of $\vec{b_1}, \ldots, \vec{b_m}$. Then by Theorem 2.4.1 the columns of I are linearly dependent, which is a contradiction with Theorem 2.3.1. □

Theorem 2.9.2. *If for a matrix A there exists matrices B and C such that $AB = I$ and $AC = 0$ then $C = 0$, where 0 is the zero matrix.*

Proof. Let A be $k \times k$ matrix and assume by contradiction $AB = 0$ and $B \neq 0$. Let the j^{th} row of B contain a non-zero element. Denote the j^{th} row of B by $\vec{b_j}^t$. Then $A\vec{b_j}^t = \vec{0}$, which means that the columns of A are linear dependent contradicting Theorem 2.9.1. □

Theorem 2.9.3. *Let A be $n \times m$ matrix. Then the number of linearly independent rows equals the number of linearly independent columns.*

Proof. Suppose A's columns are spanned by $\vec{b}, \ldots, \vec{b}_r$ then there is an $r \times m$ matrix C such that

$$[\vec{a}_1 \ldots \vec{a}_m] = A = BC = [\vec{b}_1 \ldots \vec{b}_r]C.$$

By properties of matrix multiplication the rows of A are linear combinations of the rows of C and therefore the rows of A contain at most r linearly independent rows. Thus the number of linear independent rows of A do not exceed the number of linear independent columns of A. Applying the same argument for the transpose of A we obtain that the number of linearly independent rows of A equals the number of linearly independent columns of A. □

Theorem 2.9.4. *Suppose the rows (columns) of square A are linearly independent then A can be written as a product of elementary matrices.*

Proof. The columns of A form a basis for the n-dimensional vector space $\mathbb{C}^n$. Indeed if $\vec{e}_i$ is not in the span of the columns of $A = (\vec{c}_1, \ldots, \vec{c}_n)$ then $\{\vec{c}_1, \ldots, \vec{c}_n, \vec{e}_i\}$ are linearly independent (why?), which means in $\mathbb{C}^n$ we have found $n+1$ linearly independent vectors contradiction with the fact that the dimension of $\mathbb{C}^n$ is a n. Therefore the standard basis can be represented as linear combinations of the columns of A. We can solve $A\vec{x} = \vec{e}_i$ so there exists $\vec{c}_i$ such that $A\vec{c}_i = \vec{e}_i$ and by setting $B = (\vec{c}_1, \ldots, \vec{c}_n)$ we obtain that $AB = I$ thus A is invertible. Using the matrix representation of Gaussian operations we can write $A = B^{-1} = E_m \ldots E_1$ where each E_i is an elementary matrix. □

Definition 2.9.1 (rank of a matrix)**.** *The rank of a matrix A is the number of linear independent columns of A denoted by* $\mathbf{rk}(A)$.