Chapter 2

Vector spaces

2.1 Definitions and examples

Definition 2.1.1 (vector space)**.** *A* vector space *over* $\mathbb{R}$ *is a non-empty set* $\mathbf{V}$ *consisting of* vectors *along with two operations:* vector addition *denoted by* $+$ *and scalar vector multiplication, such that the sum*[1] *of two vectors is also in* $\mathbf{V}$ *and for any scalar* $c \in \mathbb{R}$ *and any vector* $\vec{v} \in \mathbf{V}$ *we have* $c\vec{v} \in \mathbf{V}$*. Furthermore, the addition and scalar multiplication satisfy the following properties:*

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. *there is a unique* zero vector $\vec{0} \in \mathbf{V}$ *such that* $\forall \vec{v} \in \mathbf{V} : \vec{0} + \vec{v} = \vec{v}$.
4. *for each vector* $\vec{v} \in \mathbf{V}$ *there exist a unique vector* $\vec{-v}$ *such that* $\vec{v} + (\vec{-v}) = \vec{0}$.
5. $1\vec{v} = \vec{v}$
6. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
7. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{v} + \alpha\vec{u}$
8. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

Examples:

1. vectors from geometry;
2. complex numbers over themselves;
3. matrices of a fixed dimension;
4. planes through the origin in three dimensions with usual addition and scalar multiplication;
5. all functions defined in an interval $[a, b]$;

[1]In general a $\mathbf{V} \times \mathbf{V} \to \mathbf{V}$ binary operation

6. continuous functions defined in an interval $[a, b]$;
7. all sequences;
8. all sequences with finite support;
9. polynomials
10. polynomials of degree at most n

Counterexamples:

1. vectors with integer components;
2. polynomials that evaluate to 1 at 2

More examples and counterexamples:

1. vectors with integer components;
2. polynomials that evaluate to 1 at 2
3. a set with single element z where $\alpha z = z$ and $z + z = z$; here z is the zero vector.
4. $\{a \cos x + b \sin x \mid a, b \in \mathbb{C}\}$
5. $\{(a, b) \mid a, b \in \mathbb{R}\}$ with the following operations:

$$\vec{x} + \vec{y} = (a, b) + (u, v) = (a + u - 2, b + v)$$

$$\alpha\vec{x} = \alpha(a, b) = (\alpha a - 2\alpha + 2, \alpha b)$$

neutral is $(2, 0)$ and inverse is $(-a + 4, -b)$.

Theorem 2.1.1. $\forall \vec{v} \in \mathbf{V}, 0\vec{v} = \vec{0}$ *and* $(-1)\vec{v} = \vec{-v}$.

Proof. Apply condition 8 above with $c_1 = c_2 = 0$ to get

$$0\vec{v} = (0 + 0)\vec{v} \quad = \quad 0\vec{v} + 0\vec{v}$$

Since $0 + 0 = 0$ the right hand side is $0\vec{v}$. Add to both sides the vector equation to $-0\vec{v}$ which exist by 4 to get

$$0\vec{v} + (-0\vec{v}) \quad = \quad 0\vec{v} + 0\vec{v} + (-0\vec{v})$$

The right hand side become $\vec{0}$

$$\vec{0} \quad = \quad 0\vec{v} + \vec{0}$$

By condition 3

$$\vec{0} \quad = \quad 0\vec{v}$$

This shows the first part. For the second part observe that $0 = 1 - 1$ and apply it to the last equation

$$\vec{0} \quad = \quad 0\vec{v} = 1\vec{v} + (-1\vec{v})$$

By condition 5 we get

$$\vec{0} \quad = \quad \vec{v} + (-1\vec{v})$$

Add to both sides $\vec{-v}$

$$\vec{-v} + \vec{0} \quad = \quad \vec{-v} + \vec{v} + (-1\vec{v})$$

which by condition 3 for the righthandside and condition 4 for the lefthandside implies

$$\vec{-v} \quad = \quad \vec{0} + (-1\vec{v})$$

Again by condition 3 we have

$$\vec{-v} \quad = \quad -1\vec{v}$$

□

Theorem 2.1.2. $\forall \alpha \in \mathbb{R}, \alpha\vec{0} = \vec{0}$

Proof. Let $\vec{u} \in V$ and $\alpha \in \mathbb{R}$. Then $\alpha\vec{u} = \alpha\vec{u}$. On the right hand side we have $\alpha\vec{u} = \alpha\vec{u} + \vec{0}$. On the left hand side using $\vec{u} + \vec{0} = \vec{u}$ we have $\alpha\vec{u} = \alpha(\vec{u} + \vec{0}) = \alpha\vec{u} + \alpha\vec{0}$. Thus

$$\alpha\vec{u} + \alpha\vec{0} = \alpha\vec{u} + \vec{0};$$

adding $-\alpha\vec{u}$ to both sides of the equation

$$\alpha\vec{u} + \alpha\vec{0} - \alpha\vec{u} = \alpha\vec{u} + \vec{0} - \alpha\vec{u} \qquad \Rightarrow \qquad \alpha\vec{0} + \vec{0} = \vec{0} + \vec{0}$$

and the desired result follows. □

Proof. We will use $0\vec{u} = \vec{0}$ for any vector $\vec{u}$:

$$\alpha\vec{0} = \alpha(0\vec{0}) = (\alpha 0)\vec{0} = 0\vec{0} = \vec{0}$$

□

2.2 Linear combinations

In the following we assume all sets of vectors are coming from the same vector space.

Definition 2.2.1. *A vector $\vec{u}$ is said to be linear combination of $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ if there exists constants $\alpha_1, \ldots, \alpha_n$ such that*

$$\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{a_i} = \alpha_1 \vec{a_1} + \cdots + \alpha_n \vec{a_n}$$

Theorem 2.2.1. *If $\vec{u}$ is linear combination of a subset of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ then it is linear combination of all the vectors.*

Example $\{(a,b) \mid a,b \in \mathbb{R}\}$ with the following operations:

$$\vec{x}+\vec{y} = (a,b)+(u,v) = (a+u-2, b+v)$$

$$\alpha\vec{x} = \alpha(a,b) = (\alpha a - 2\alpha + 2, \alpha b)$$

neutral is $(2,0)$ and inverse is $(-a+4,-b)$. Then $(2,0)$ is a linear combination of $(0,1),(0,2)$ are linearly dependent

Theorem 2.2.2. *If $\vec{u}$ is linear combination of $\vec{v_1},\vec{v_2},\dots,\vec{v_k}$ and each $\vec{v_i}$ is a linear combination of $\vec{a_1},\vec{a_2},\dots,\vec{a_k}$ then $\vec{u}$ is a linear combination of $\vec{a_1},\vec{a_2},\dots,\vec{a_k}$.*

2.3 Linear dependence and independence

Definition 2.3.1 (linear (in)dependence)**.** *Let $\vec{v_1},\vec{v_2},\dots,\vec{v_k}$ be a set of vectors. If*

$$a_1\vec{v_1}+a_2\vec{v_2}+\cdots+a_k\vec{v_k}=0 \quad \Rightarrow \quad a_1=a_2=\cdots=a_k=0$$

then the vectors $\vec{v_1},\vec{v_2},\dots,\vec{v_k}$ are called linearly independent *otherwise the are* linearly dependent*.*

Example: In $\mathbb{R}^2$ the vectors $\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent. In $\mathbb{R}^3$ the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v_3} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ are linearly dependent

Example: Let V be the vector space of continuous functions on $\mathbb{R}$. Then the functions $f_1(x) = x^2+\sin 2x$, $f_2(x) = \cos 2x$, $f_3(x) = \sin^2 x$, $f_4(x) = \cos^2 x$, and $f_5(x) = e^x$,$f_6(x) = e = 2.71\dots$, $f_7(x) = e^x$.

1. $f_2(x)$ and $f_3(x)$ are linearly independent because if $\alpha f_2(x) + \beta f_3(x) = 0(x)$ then evaluating at $x=0$ we get $0 = 0(0) = \alpha\cos(2\times 0) + \beta\sin^2 0 = \alpha$. Evaluating at $x = \frac{\pi}{2}$ we get $0\times\cos(2\times\frac{pi}{2}) + \beta\sin^2\frac{pi}{2} = \beta\sin^2\frac{pi}{2} = \beta = 0(\frac{\pi}{2}) = 0$.
2. The functions $f_3(x), f_4(x), f_6(x)$ are linearly dependent since $-2f_6(x)+2ef_3(x)+2ef_4(x) = 0(x)$.
3. The functions $f_3(x)$ and $f_7(x)$ are linearly independent

Theorem 2.3.1. *The standard basis vectors are linearly independent, in other words the columns and rows of I are linearly independent.*

Proof. Let $\vec{e_i}$ be the vector whose ith coordinate is one and the rest zero. Consider the system of linear equation whose vector form is $\sum_{i=1}^{n}\vec{e_i}x_i = \vec{0}$. This system is in row-reduced Echelon form and has a unique solution $x_1 = x_2 = \cdots = x_n = 0$. Thus the standard basis vectors are linearly independent. □

Theorem 2.3.2. *Let $\vec{v_1},\vec{v_2},\dots,\vec{v_k}$ be collection of vectors. If $k=1$ the system of vectors is linear dependent if and only if $\vec{v_1}=\vec{0}$.*

Proof. If $\vec{v} = \vec{0}$ then $1\vec{v} = 1\vec{0} = \vec{0}$ and therefore it is linearly dependent.

Assume $\vec{v}$ is linearly dependent then $\alpha\vec{v} = \vec{0}$ for some non-zero constant α. Multiplying both sides by α^{-1} we obtain $\vec{v} = \alpha^{-1}\vec{0}$. Or

$$\vec{v} = \alpha^{-1}\vec{0} = \alpha^{-1}(0\vec{0}) = \alpha^{-1}(0\vec{0}) = (\alpha^{-1}0)\vec{0} = (0)\vec{0} = 0\vec{0} = \vec{0}$$

$\square$

Theorem 2.3.3. *Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ be collection of vectors. If for some $1 \le i \le k$ we have that $\vec{v_i} = \vec{0}$ then the system of vectors is linear dependent.*

Theorem 2.3.4. *Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ be collection of vectors. If for some $1 \le i \ne j \le k$ we have that $\vec{v_i} = \vec{v_j}$ then the system of vectors is linear dependent.*

Theorem 2.3.5. *Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ be collection of linearly dependent vectors and $k > 1$. Then there is an index i such that $\vec{v_i}$ can be written as a linear combination of the remaining vectors.*

2.4 Main theorem

Theorem 2.4.1. *Let $A : \vec{a_1}, \vec{a_2}, \ldots, \vec{a_s}$ and $B : \vec{b_1}, \vec{b_2}, \ldots, \vec{b_k}$ be two system of vectors. Suppose that for each $1 \le i \le s$ we have that $\vec{a_i}$ is a linear combination of $\vec{b_1}, \vec{b_2}, \ldots, \vec{b_k}$. Suppose also $s > k$ then the vectors in A are linearly dependent.*

Proof. The argument proceeds by induction on k.

Base case $k = 1$: : Since $k = 1$ then $B = \{\vec{b_1}\}$. Then $\vec{a_1} = \gamma_1\vec{b_1}, \vec{a_2} = \gamma_2\vec{b_2}, \ldots, \vec{a_s} = \gamma_s\vec{b_s}$. If for any index i, $\gamma_i = 0$ then A contains the zero vector and therefore A is linearly dependent. Suppose now for all indices i, $\gamma_i \ne 0$ then since $s > k = 1$, there are at least two vectors $\vec{a_1}$ and $\vec{a_2}$ in A. Consider

$$\gamma_2\vec{a_1} - \gamma_1\vec{a_2} = \gamma_2\gamma_1\vec{b_1} - \gamma_1\gamma_2\vec{b_2} = 0\vec{b_1} = \vec{0}$$

Then $\vec{a_1}$ and $\vec{a_2}$ are linearly dependent. And since they are subsystem of A, then A itself is linearly dependent. This concludes the base case.

Inductive step: Let $k \ge 2$. By the theorem statement we have

$$\begin{array}{rcl}
\vec{a_1} & = & \gamma_{11}\vec{b_1} + \gamma_{12}\vec{b_2} + \cdots + \gamma_{1k}\vec{b_k} \\
\vec{a_2} & = & \gamma_{21}\vec{b_1} + \gamma_{22}\vec{b_2} + \cdots + \gamma_{2k}\vec{b_k} \\
 & \cdots & \\
\vec{a_{s-1}} & = & \gamma_{(s-1)1}\vec{b_1} + \gamma_{(s-1)2}\vec{b_2} + \cdots + \gamma_{(s-1)k}\vec{b_k}. \\
\vec{a_s} & = & \gamma_{s1}\vec{b_1} + \gamma_{s2}\vec{b_2} + \cdots + \gamma_{sk}\vec{b_k}.
\end{array}$$

If all $\gamma_{s1}, \gamma_{s2}, \ldots, \gamma_{sk}$ are zero then $\vec{a_s} = \vec{0}$ and therefore A is linearly dependent. Suppose now at least one of $\gamma_{s1}, \gamma_{s2}, \ldots, \gamma_{sk}$ is non-zero. Without loss of generality let $\gamma_{sk} \ne 0$. In this case we add $\frac{\gamma_{s1}}{\gamma_{sk}}$ the last equation to the first equation. Similarly, we add $\frac{\gamma_{s2}}{\gamma_{sk}}$ the last equation to the second equation and so forth until we add $\frac{\gamma_{(s-1)k}}{\gamma_{sk}}$ the last equation to equation $s-1$. We obtain

$$\begin{array}{rcl}
\vec{a'_1} = \vec{a_1} - \dfrac{\gamma_{1k}}{\gamma_{sk}}\vec{a_s} & = & \gamma'_{11}\vec{b_1} + \gamma'_{12}\vec{b_2} + \cdots + \gamma'_{1(k-1)}\vec{b}_{(k-1)} \\
\vec{a'_2} = \vec{a_2} - \dfrac{\gamma_{2k}}{\gamma_{sk}}\vec{a_s} & = & \gamma'_{21}\vec{b_1} + \gamma'_{22}\vec{b_2} + \cdots + \gamma'_{2(k-1)}\vec{b}_{(k-1)} \\
 & \cdots & \\
\vec{a'_{s-1}} = \vec{a_{s-1}} - \dfrac{\gamma_{(s-1)k}}{\gamma_{sk}}\vec{a_s} & = & \gamma'_{(s-1)1}\vec{b_1} + \gamma'_{(s-1)2}\vec{b_2} + \cdots + \gamma'_{(s-1)(k-1)}\vec{b}_{k-1}.
\end{array}$$

Since $s > k$, then $s-1 > k-1$. Furthermore, each $\vec{a'_1}, \ldots, \vec{a'_{s-1}}$ is a linear combination of the vectors $\vec{b'_1}, \ldots, \vec{a'_{k-1}}$. We apply the inductive hypothesis to conclude that $\vec{a'_1}, \ldots, \vec{a'_{s-1}}$ are linearly dependent. In other words there exists $\mu_1, \mu_2, \ldots, \mu_{s-1}$ not all zero such that

$$\begin{array}{rcl}
\vec{0} & = & \mu_1\vec{a'_1} + \mu_2\vec{a'_2} + \cdots + \mu_{s-1}\vec{a'_{s-1}} \\
 & = & \mu_1\left(\vec{a_1} - \dfrac{\gamma_{1k}}{\gamma_{sk}}\vec{a_s}\right) + \mu_2\left(\vec{a_2} - \dfrac{\gamma_{2k}}{\gamma_{sk}}\vec{a_s}\right) + \mu_{s-1}\left(\vec{a_{s-1}} - \dfrac{\gamma_{(s-1)k}}{\gamma_{sk}}\vec{a_s}\right) \\
 & = & \mu_1\vec{a_1} + \mu_2\vec{a_2} + \cdots + \mu_{s-1}\vec{a_{s-1}} + \tau\vec{a_s}.
\end{array}$$

Since at least one of μ_i's is non-zero, the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_s}$ are linearly dependent. □

2.5 Subspaces

Definition 2.5.1 (subspace)**.** *Let* **V** *be a vector space and let* **U** *be a subset of* **V***. If* **U** *is a vector space itself then* **U** *is called a* subspace *of* **V**

Examples:

1. x-axis is subspace of $\mathbb{R}^2$
2. line through points $(1,0)$ and $(0,1)$ is *not* subspace of $\mathbb{R}^2$
3. even degree polynomials are *not* subspace of polynomials
4. odd degree polynomials are *not* subspace of polynomials
5. polynomials of degree at most n are a subspace of all polynomials
6. $\{(a,b) \mid a,b \in \mathbb{R}\}$ with the following operations:

$$\vec{x} + \vec{y} = (a,b) + (u,v) = (a+u-2, b+v)$$

$$\alpha\vec{x} = \alpha(a,b) = (\alpha a - 2\alpha + 2, \alpha b)$$

   neutral is $(2,0)$ and inverse is $(-a+4, -b)$.

For the vector space $\mathbb{R}^n$ with the usual operations the set $(2,0)$ is not a subspace however for the vector space with operations $\{(a,b) \mid a,b \in \mathbb{R}\}$ with the following operations:

$$\vec{x} + \vec{y} = (a,b) + (u,v) = (a+u-2, b+v)$$

$$\alpha\vec{x} = \alpha(a,b) = (\alpha a - 2\alpha + 2, \alpha b)$$

with neutral element $(2,0)$ and inverse is $(-a+4,-b)$, the neutral element is a subspace.

Theorem 2.5.1. *In any vector space* $\mathbf{V}$ *the vector space itself is a subspace and the zero vector on its own is a vector space.*

Proof. TODO ☐

Theorem 2.5.2. *A set* $\mathbf{U}$ *is a subspace of* $\mathbf{V}$ *if and only if for all* $\vec{u}, \vec{w} \in \mathbf{U}$ *and for all* $s,t \in \mathbb{F}$ *we have that* $s\vec{u} + t\vec{w} \in \mathbf{U}$.

Proof. If $\mathbf{U}$ is a subspace of a vector space $\mathbf{V}$, then it is a vector space itself and since the operations are closed we have that $s\vec{u} + t\vec{w} \in \mathbf{U}$.

Suppose now $\forall \vec{u}, \vec{w} \in \mathbf{U}, \forall s,t \in \mathbb{C}, s\vec{u} + t\vec{v} \in \mathbf{U}$. We will verify all properties of vector spaces for $\mathbf{U}$.

closure of + from $\forall \vec{u}, \vec{w} \in \mathbf{U}, \forall s,t \in \mathbb{C}, s\vec{u} + t\vec{v} \in \mathbf{U}$ for $s = t = 1$ we conclude $\forall \vec{u}, \vec{w} \in \mathbf{U}, \vec{u} + \vec{v} = 1\vec{u} + 1\vec{v} \in \mathbf{U}$

closure of · from $\forall \vec{u}, \vec{w} \in \mathbf{U}, \forall s,t \in \mathbb{C}, s\vec{u} + t\vec{v} \in \mathbf{U}$ for $t = 0$ and $\vec{u} = \vec{w}$ we conclude $\forall \vec{u} \in \mathbf{U}, \forall s \in \mathbb{C}, s\vec{u} = s\vec{u} + 0\vec{u} \in \mathbf{U}$

commutativity since the operations are inherited from $\mathbf{V}$ the result follows

associativity since the operations are inherited from $\mathbf{V}$ the result follows

zero vector from $\forall \vec{u}, \vec{w} \in \mathbf{U}, \forall s,t \in \mathbb{C}, s\vec{u} + t\vec{v} \in \mathbf{U}$ for $s = t = 0$ $\vec{0} = \vec{0} + \vec{0} = 0\vec{u} + 0\vec{w} \in \mathbf{U}$; uniqueness and neutrality is inherited from $\mathbf{V}$;

neutral inverse from $\forall \vec{u}, \vec{w} \in \mathbf{U}, \forall s,t \in \mathbb{C}, s\vec{u} + t\vec{v} \in \mathbf{U}$ for $\vec{u} = \vec{w}$, $s = 0$ and $t = -1$ we have $\forall \vec{u} \in \mathbf{U}, \vec{-u} = \vec{0} + \vec{-u} = 0\vec{u} + (-1)\vec{u} \in \mathbf{U}$;

distributive properties are inherited from $\mathbf{V}$.

☐

2.6 Span

Definition 2.6.1 (span)**.** *Let* $\mathbf{S} = \{\vec{u}_1, \ldots, \vec{u}_m\}$ *be a set of vector the set of all linear combinations of the vectors in* $\mathbf{S}$ *is called the* span *of* $\mathbf{S}$ *and denoted by* $\langle \mathbf{S} \rangle$. *If* $\mathbf{S} = \emptyset$ *then* $\langle \mathbf{S} \rangle = \{\vec{0}\}$

1. span of $1, x, x^2$
2. span of standard basis vectors
3. span of infinite sequences where only one component is non-zero
4. span of all vectors in a vector space
5. span of columns of a matrix

Example:

$$\left|\begin{array}{rrrrrcr} x_1 & -2x_2 & +x_3 & +4x_4 & -2x_5 & = & 3 \\ 2x_1 & -4x_2 & +3x_3 & +8x_4 & -3x_5 & = & 5 \\ x_1 & -2x_2 & +x_3 & & -2x_5 & = & -1 \\ x_1 & -2x_2 & +2x_3 & +5x_4 & -x_5 & = & 3 \end{array}\right.$$

equivalently in vector form

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} -2 \\ -4 \\ -2 \\ -2 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} x_3 + \begin{pmatrix} 4 \\ 8 \\ 0 \\ 5 \end{pmatrix} x_4 + \begin{pmatrix} -2 \\ -3 \\ -2 \\ -1 \end{pmatrix} x_5 = \begin{pmatrix} 3 \\ 5 \\ -1 \\ 3 \end{pmatrix}$$

has solution (verify it)

$$\begin{array}{rcl} x_1 & = & 3s + 2t - w \\ x_2 & = & t - 2w \\ x_3 & = & -1 - s - w \\ x_4 & = & 1 \\ x_5 & = & s + w \end{array} \qquad \text{in vector form} \qquad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} w$$

Theorem 2.6.1. *A system $Ax = b$ has a solution if and only if b is in the span of the columns of A.*

Proof. TODO □

Observe

$$-2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -2 \\ -2 \end{pmatrix} \qquad -3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -2 \\ -1 \end{pmatrix}$$

So span of columns of $\begin{pmatrix} 1 & -2 & 1 & 4 & -2 \\ 2 & -4 & 3 & 8 & -3 \\ 1 & -2 & 1 & 0 & -2 \\ 1 & -2 & 2 & 5 & -1 \end{pmatrix}$ equals the span of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 0 \\ 5 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \\ -2 \\ -1 \end{pmatrix}$ and equals the span of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 0 \\ 5 \end{pmatrix}$ and equals the span of

Theorem 2.6.2. *$\langle \mathbf{S} \rangle = \langle \mathbf{S} \cup \vec{u} \rangle$ if and only if $\vec{u} \in \langle \mathbf{S} \rangle$.*

Proof. **if** assume $\vec{u} \in \mathbf{S}$ then $\langle \mathbf{S} \rangle \subseteq \langle \mathbf{S} \cup \vec{u} \rangle$ is by definition; $\langle \mathbf{S} \rangle \supseteq \langle \mathbf{S} \cup \vec{u} \rangle$ follows from transitivity of linear combinations

only if since $\vec{u} \in \langle \mathbf{S} \cup \vec{u} \rangle = \langle \mathbf{S} \rangle$.

□

Theorem 2.6.3. *The span of a set of vectors is a vector space.*

Proof. Subspace if and only if it is closed under linear combinations. □