1.9 Bipartite graphs

1.9.1 Basics

Definition 1.9.1 (bipartite graph). 1. *A graph G whose vertex set can be partitioned*[3] *into two sets A and B such that every edge in $E(G)$ is incident with one vertex in A and one vertex in B is called* bipartite *graph. The sets A and B are called a* bipartition.

2. *A graph G whose vertex set can be partitioned into two sets A and B such that A and B are independent sets is called* bipartite *graph. The sets A and B are called a* bipartition.

Comment: we will call the empty graph on one vertex bipartite.

Examples: $K_{m,n}$ and other bipartite graphs

Theorem 1.9.1. *Every tree is bipartite.*

Proof. Let u be a vertex in T and define $A \subseteq V(T)$ and $B \subseteq V(T)$, such that $A = \{w \in V(T) \mid d(u,w) = 2k\}$ where $d(u,w)$ denotes the length of the path between u and w. Recall that there is only one such path by Theorem 1.8.1. Likewise $B = \{w \in V(T) \mid d(u,w) = 2k+1\}$ where $d(u,w)$ denotes the length of the path between u and w. Since T is connected we have $A \cup B$ is the vertex set of T. Suppose there is an edge $e = (w,v)$ in T such that $w, v \in A$. Consider the path $uu_1u_2 \ldots u_{2k-1}w$ if $u_{n-1} = v$ then the path from u to v is of odd length contradicting the fact that $u \in A$. If $u \in \{u_1, \ldots, u_{2k-2}\}$ then we have a cycle in T contradiction T is a tree. Hence $uu_1u_2 \ldots u_{2k-1}wv$ is a path of odd length in T contradicting $v \in A$. Thus there is no edge among the vertices in A. Similarly there are no vertices among the edges of B thus A and B are bipartition of T implying that T is a bipartite graph. □

Theorem 1.9.2. *A graph is bipartite if and only if it has only even cycles.*

Proof. If the graph is bipartite it contains no odd cycles.

If the graph is not bipartite it contains an odd cycle. Indeed let H be a component of G, that is not bipartite (one must exist). Since H is connected it has a spanning tree T. Trees are bipartite and therefore T has a bipartition (A,B). Since H is not a bipartite graph then (A,B) is *not* a bipartition of H, and therefore there is an edge $e = \{u,v\}$ such that both u and v are in the same set A or B. Without loss of generality let $u, v \in A$. Since (A,B) is a bipartition of T then the edge e is not in the edge set of T. Since T is a connected there is a path $P = ux_1x_2, \ldots, x_nv$ in T that connects u and v. Since u and v are in the same bipartition the length of the path is even. Then adjoining e to P we obtain an odd length cycle in H. □

[3] A partition of a set S is a collection of sets $P_1, \ldots, P_n$ such that for $i \neq j$ $P_i \cap P_j = \emptyset$ and $S = P_1 \cup P_2 \cup \ldots \cup P_n$