Linear Algebra

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Gauss' Method

Gauss' theorem

Theorem (Gauss' method)

If a linear system S is changed to another S' by one of these operations:

1. *an equation is swapped with another*
2. *an equation has both sides multiplied by a non-zero constant*
3. *an equation is replaced by the sum of itself and a multiple of another*

then the two system of equations have the same set of solutions.

Elementary row operations

Definition (elementary row operations)

The *elementary row operations*, (also *row operations*, *Gaussian operations*) are

1. row swapping
2. rescaling (multiplication with a non-zero constant)
3. row combinations (adding a multiple of another row)

SLE S_0

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 + x_5 &= 7 \\ 3x_1 + 9x_2 - 6x_3 + 4x_4 + 3x_5 &= -7 \\ 2x_1 + 6x_2 - 4x_3 + 2x_4 + 2x_5 &= -4 \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 3 & 9 & -6 & 4 & 3 & -7 \\ 2 & 6 & -4 & 2 & 2 & -4 \end{array}\right)$$

$S_0 \to S_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 3 & 9 & -6 & 4 & 3 & -7 \\ 2 & 6 & -4 & 2 & 2 & -4 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 2 & 6 & -4 & 2 & 2 & -4 \end{array}\right)$$

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 + x_5 &= 7 \\ -15x_3 - 2x_4 &= -28 \\ 2x_1 + 6x_2 - 4x_3 + 2x_4 + 2x_5 &= -4 \end{aligned}$$

$S_1 \to S_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrrrr|r} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 2 & 6 & -4 & 2 & 2 & -4 \end{array}\right)$$

$$= \left(\begin{array}{rrrrr|r} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & -10 & -2 & 0 & -18 \end{array}\right)$$

$S_2 \to S_3$ in Echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & -10 & -2 & 0 & -18 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} \end{array}\right)$$

S_3 solution set

$$\left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} \end{array}\right)$$

$$\begin{array}{rcr} x_1 + 3x_2 + 3x_3 + 2x_4 + x_5 & = & 7 \\ -15x_3 - 2x_4 & = & -28 \\ -\frac{2}{3}x_4 & = & \frac{2}{3} \end{array}$$

$$\{(3 - 3t - s, t, 2, -1, s) \mid s, t \in \mathbb{C}\}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & -2 & 0 & -28 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)
$$

$$
= \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & 0 & 0 & -30 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)
$$

$S_5 \to S_6$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrrrr|r} 1 & 3 & 3 & 2 & 1 & 7 \\ 0 & 0 & -15 & 0 & 0 & -30 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$= \left(\begin{array}{rrrrr|r} 1 & 3 & 3 & 0 & 1 & 9 \\ 0 & 0 & -15 & 0 & 0 & -30 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$S_6 \to S_7$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{15} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 0 & 1 & 9 \\ 0 & 0 & -15 & 0 & 0 & -30 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 3 & 3 & 0 & 1 & 9 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$S_7 \to S_8$ Reduced Echelon Form

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccccc|r} 1 & 3 & 3 & 0 & 1 & 9 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|r} 1 & 3 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$Sols(S_8)$ in vector form

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$x_1 = 3 - 3t - s, x_2 = t, x_3 = 2, x_4 = -1, x_5 = s$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r, q \in \mathbb{C} \right\}$$

$$\begin{array}{rcl} x_1 + 3x_2 + 3x_3 + 2x_4 & = & 1 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 & = & 3 \\ -x_1 - 3x_2 + 3x_3 & = & 2 \end{array}$$

$$\left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 3 \\ -1 & -3 & 3 & 0 & 2 \end{array} \right)$$

$N_0 \to N_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 3 \\ -1 & -3 & 3 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ -1 & -3 & 3 & 0 & 2 \end{array}\right)$$

$N_1 \to N_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ -1 & -3 & 3 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 6 & 2 & 3 \end{array}\right)$$

$N_2 \to N_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 6 & 2 & 3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$N_3 \to N_4$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$N_4 \to N_5$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$N_4 \to N_5$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 3 \\ -1 & -3 & 3 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$N_0 \to N_5$

$$\begin{pmatrix} 3 & -1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ 5 & -2 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 3 \\ -1 & -3 & 3 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$N_0' \to N_5'$

$$\begin{pmatrix} 3 & -1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ 5 & -2 & 1 \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 3 \\ -1 & -3 & 3 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

N_0 and N_0'

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 3 \\ -x_1 - 3x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 3 \\ -x_1 - 3x_2 + 3x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{rrrr|rr} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 6 & 9 & 5 & 3 & 3 \\ -1 & -3 & 3 & 0 & 2 & 1 \end{array}\right)$$

N_0 and N_0'

$$\begin{pmatrix} 3 & -1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ 5 & -2 & 1 \end{pmatrix} \left(\begin{array}{rrrr|rr} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 6 & 9 & 5 & 3 & 3 \\ -1 & -3 & 3 & 0 & 2 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

U_0

$$\begin{aligned} 5x_1 - 8x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 + x_3 &= 0 \\ -4x_1 + 7x_2 - x_3 &= 0 \end{aligned} \qquad \begin{aligned} 5x_1 - 8x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 + x_3 &= 1 \\ -4x_1 + 7x_2 - x_3 &= 0 \end{aligned}$$

$$\begin{aligned} 5x_1 - 8x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 + x_3 &= 0 \\ -4x_1 + 7x_2 - x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$U_0 \to U_1$

$$
\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)
$$

$$
= \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)
$$

$U_1 \to U_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & 1 \end{pmatrix} \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & \frac{3}{5} & -\frac{1}{5} & \frac{4}{5} & 0 & 1 \end{array}\right)$$

$U_2 \to U_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & \frac{3}{5} & -\frac{1}{5} & \frac{4}{5} & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_3 \to U_4$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & -5 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_4 \to U_5$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & -5 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_5 \to U_6$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 5 & -8 & 0 & 2 & -3 & -1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_6 \to U_7$

$$\begin{pmatrix} 1 & 8 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{rrr|rrr} 5 & -8 & 0 & 2 & -3 & -1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 5 & 0 & 0 & 10 & 5 & 15 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_7 \to U_8$

$$\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 5 & 0 & 0 & 10 & 5 & 15 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_0 \to U_8$

$$\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 8 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$U_0 \to U_8$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 5 & -8 & 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

Representing invertible matrices

Theorem

If A is an invertible matrix then A can be written as a product of elementary matrices.

Representing invertible matrices

$$\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 8 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix}$$

$U_0 \to U_8$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \left(\begin{array}{rrr|rrr} 5 & -8 & 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 & 1 & 0 \\ -4 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 5 & -8 & 1 & 0 \\ 3 & -5 & 1 & 1 \\ -4 & 7 & -1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 5 & -8 & 1 & 0 \\ 3 & -5 & 1 & 1 \\ -4 & 7 & -1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{aligned} 5x_1 - 8x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 + x_3 &= 1 \\ -4x_1 + 7x_2 - x_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$S_0 \to S_8$ Reduced Echelon Form

$$\left(\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 1 & 7 \\ 3 & 9 & -6 & 4 & 3 & -7 \\ 2 & 6 & -4 & 2 & 2 & -4 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r, q \in \mathbb{C} \right\}$$

S_0 in matrix form

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ -4 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r, q \in \mathbb{C} \right\}$$

S_0 particular solution

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ -4 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r, q \in \mathbb{C} \right\}$$

S_0 homogeneous solution

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r, q \in \mathbb{C} \right\}$$

S_0 homogeneous solution

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \mid r, q \in \mathbb{C} \right\}$$

S_0 solution

$$\left\{ \underbrace{\begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q}_{\text{homogeneous solution}} \;\middle|\; r, q \in \mathbb{C} \right\}$$

S_0 particular solution

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \left[\begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 7 \\ -7 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ -4 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} r + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \mid r, q \in \mathbb{C} \right\}$$

S_0 homogenous solution

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 3 & 9 & -6 & 4 & 3 \\ 2 & 6 & -4 & 2 & 2 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -4 \end{pmatrix} r' + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \;\middle|\; r', q \in \mathbb{C} \right\}$$

S_0 homogenous solution

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -4 \end{pmatrix} r' + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{pmatrix} q \mid r', q \in \mathbb{C} \right\}$$

Solutions to a SLE

Theorem

Any linear system's solution set has the form

$$\left\{\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{C}\right\}$$

where $\vec{p}$ *is any particular solution and where the number of vectors* $\vec{\beta}_1, \ldots, \vec{\beta}_k$ *equals the number of free variables that the system has after a Gaussian reduction.*

Solutions to a SLE

Corollary

Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

Solutions to a SLE

		number of solutions of the homogeneous system	
		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
	no	no solutions	no solutions

$0 = 0$ number of solutions

$$\begin{array}{rrcr} x_1 & +3x_2 & = & 5 \\ 4x_1 & +12x_2 & = & k \end{array} \quad \rightarrow \quad \begin{array}{rrcr} x_1 & +3x_2 & = & 5 \\ 0x_1 & +0x_2 & = & k-20 \end{array}$$

1. $k = 20$ – infinitely many solutions
2. $k \neq 20$ – no solutions

$0 = 0$ number of solutions

$$\begin{array}{rrrrcr} x_1 & +3x_2 & +0x_3 & +0x_4 & = & 5 \\ 4x_1 & +12x_2 & +0x_3 & +0x_4 & = & 20 \\ 0x_1 & +0x_2 & +x_3 & +3x_4 & = & 5 \\ 0x_1 & +0x_2 & +4x_3 & +12x_4 & = & 21 \end{array}$$

$$\rightarrow \quad \begin{array}{rrrrcr} x_1 & +3x_2 & +0x_3 & +0x_4 & = & 5 \\ 0x_1 & +0x_2 & +0x_3 & +0x_4 & = & 0 \\ 0x_1 & +0x_2 & +x_3 & +3x_4 & = & 5 \\ 0x_1 & +0x_2 & +0x_3 & +0x_4 & = & 1 \end{array}$$

$0 = 0$ number of solutions

$$\left|\begin{array}{rrrrcr} x_1 & +3x_2 & & & = & 5 \\ 4x_1 & +12x_2 & & & = & 20 \\ & & +x_3 & +3x_4 & = & 5 \\ & & +4x_3 & +12x_4 & = & 21 \end{array}\right.$$

$$\rightarrow \quad \left|\begin{array}{rrrrcr} x_1 & +3x_2 & & & = & 5 \\ & & & 0 & = & 0 \\ & & +x_3 & +3x_4 & = & 5 \\ & & & 0 & = & 1 \end{array}\right.$$

$0 = 0$ number of solutions

$$\begin{array}{rrrrr} x_1 & +x_2 & +x_3 & = & 0 \\ & x_2 & +x_3 & = & 0 \end{array}$$

has infinitely many solutions