Math144
Finite Mathematics

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Determinants

Definition

Definition

A $n \times n$ *determinant* is a function $\det : \mathcal{M}_{n\times n} \to \mathbb{C}$ such that

$$\det(EA) = \det(E)\det A$$

where E is any matrix representing elementary row operation and

1. $\det(E) = 1$ if E is a linear combination matrix
2. $\det(E) = -1$ if E is a swap matrix
3. $\det(E) = k$ if E is a rescaling matrix
4. $\det(I) = 1$

Examples condition 1

for $i \neq j$ row combination

$$\det\begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det\begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_i \\ \vec{\rho}_i + k\vec{\rho}_j \\ \vec{\rho}_i \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

Examples condition 1

for $i = 2$, $j = 5$ and $k = 2$

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 12 & 0 & 3 & -1 & 5 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Examples condition 1

for $i = 2$, $j = 5$ and $k = 2$

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 12 & 0 & 3 & -1 & 5 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Examples condition 2

for $i \neq j$ swap

$$\det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_j \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = -\det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_i \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

Examples condition 2

for $i = 1$ and $j = 4$

$$\det\begin{pmatrix} 4 & 1 & 4 & 0 & -1 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Examples condition 2

for $i = 1$ and $j = 4$

$$\det \begin{pmatrix} 4 & 1 & 4 & 0 & -1 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Examples condition 3

$$\det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ k\vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = k \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

Examples condition 3

for $i = 4$ and $k = -2$

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ -8 & -2 & -8 & 0 & 2 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Examples condition 3

for $i = 4$ and $k = -2$

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ -8 & -2 & -8 & 0 & 2 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= -\,2 \det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 & 1 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 1 & 4 & 0 & -1 \\ 5 & 0 & 0 & 0 & 2 \end{pmatrix}$$

On condition 2

$$\begin{vmatrix} \vdots \\ \vec{\rho}_i \\ \vdots \\ \vec{\rho}_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{\rho}_i + \vec{\rho}_j \\ \vdots \\ \vec{\rho}_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{\rho}_i + \vec{\rho}_j \\ \vdots \\ \vec{\rho}_j - \vec{\rho}_i - \vec{\rho}_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{\rho}_i + \vec{\rho}_j \\ \vdots \\ -\vec{\rho}_i \\ \vdots \end{vmatrix}$$

$$= \begin{vmatrix} \vdots \\ \vec{\rho}_i + \vec{\rho}_j - \vec{\rho}_i \\ \vdots \\ -\vec{\rho}_i \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{\rho}_j \\ \vdots \\ -\vec{\rho}_i \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ \vec{\rho}_j \\ \vdots \\ \vec{\rho}_i \\ \vdots \end{vmatrix}$$

Results

Theorem

If a matrix A has a row of zeros then $\det(A) = 0$.

Results

Theorem

$\det(A) = 0$ *if and only if* $\vec{\rho}_1, \ldots, \vec{\rho}_n$ *are linearly dependent.*

Results

Theorem

$\det(A)$ *is unique.*

Results

Theorem

$$\det(AB) = \det(A)\det(B)$$

Results

Theorem

$$\det(A) = \det\left(A^T\right)$$

Results

Theorem

$$\det\begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{u}+\vec{v} \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det\begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{u} \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} + \det\begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{v} \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 3 & 2 & 1 & 5 \\ 2 & 3 & 3 & 2 \\ 1 & 8 & 1 & 5 \\ 0 & 2 & 7 & 7 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3 & 2 & 1 & 5 \\ 1 & 2 & 3 & 0 \\ 1 & 8 & 1 & 5 \\ 0 & 2 & 7 & 7 \end{pmatrix} + \det\begin{pmatrix} 3 & 2 & 1 & 5 \\ 1 & 1 & 0 & 2 \\ 1 & 8 & 1 & 5 \\ 0 & 2 & 7 & 7 \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & -8 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example top

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example top top

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example top

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example top middle

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 7 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example top

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example top bottom

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 7 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Example

$$
\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}
$$

$$
+\det\begin{pmatrix} 0 & -8 & 0 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}
$$

$$
+\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix}
$$

Example bottom

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example bottom top

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix} = \qquad \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example bottom

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example bottom middle

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 7 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example bottom

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+ \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ -4 & 7 & -1 \end{pmatrix}$$

$$+ \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix}$$

Example bottom bottom

$$
\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -4 & 7 & -1 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix}
$$

$$
+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 7 & 0 \end{pmatrix}
$$

$$
+\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}
$$

Example

$$
\begin{aligned}
&\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} \\
&= \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 7 & 0 \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & -8 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & -8 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ -4 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Example

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

Determinant

$$
\begin{aligned}
\det(A) &= \sum_{\text{permutations } \phi} a_{1\phi(1)} \dots a_{n\phi(n)} \det(P_\phi) \\
&= \sum_{i=1}^{n} (-1)^{r+i} a_{ri} \det\left(A(r|i)\right) \\
&= \sum_{i=1}^{n} (-1)^{i+c} a_{ic} \det\left(A(i|c)\right)
\end{aligned}
$$

Cofactors and minors

- $A(i|j)$ obtained from A by removing row i and column j
- $\det(A(i|j))$ is called the minor of a_{ij},
- $(-1)^{i+j}\det(A(i|j))$ is called the cofactor of a_{ij}.

Definition

$\phi = (\phi(1), \phi(2), \ldots, \phi(n))$. In a permutation matrix

$$P_\phi = \begin{pmatrix} \vdots \\ \rho_{\phi(k)} \\ \vdots \\ \rho_{\phi(l)} \\ \vdots \end{pmatrix}$$ two rows are an inversion if and only if

$\phi(k) > \phi(l)$.

expansion by first column

$$
\begin{aligned}
\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} &= (5)(-1)^{1+1}\det\begin{pmatrix} -5 & 1 \\ 7 & -1 \end{pmatrix} \\
&\quad +(3)(-1)^{2+1}\det\begin{pmatrix} -8 & 1 \\ 7 & -1 \end{pmatrix} \\
&\quad +(-4)(-1)^{3+1}\det\begin{pmatrix} -8 & 1 \\ -5 & 1 \end{pmatrix} \\
&= (5)(-1)^{1+1}(-2) \\
&\quad +(3)(-1)^{2+1}(1) \\
&\quad +(-4)(-1)^{3+1}(-3) \\
&= -1
\end{aligned}
$$

expansion by second row

$$
\begin{aligned}
\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} &= (3)(-1)^{2+1}\det\begin{pmatrix} -8 & 1 \\ 7 & -1 \end{pmatrix} \\
&\quad +(-5)(-1)^{2+2}\det\begin{pmatrix} 5 & 1 \\ -4 & -1 \end{pmatrix} \\
&\quad +(1)(-1)^{2+3}\det\begin{pmatrix} 5 & -8 \\ -4 & 7 \end{pmatrix} \\
&= (3)(-1)^{2+1}(1) \\
&\quad +(-5)(-1)^{2+2}(-1) \\
&\quad +(1)(-1)^{2+3}(3) \\
&= -1
\end{aligned}
$$