1.9 Trees

Definition 1.9.1 (tree)**.** *If a graph is* acyclic *it is called* forest*. A* tree *is a connected graph with no cycles.*

The following theorem established multiple ways to see a graphs

Theorem 1.9.1. *The following are equivalent for a graph T:*

1. *T is a tree;*
2. *any two vertices in T are connected by a unique path;*
3. *T is minimally connected, that is every edge is a cut edge;*
4. *T is maximally acyclic, that is $T + xy$ contains a cycle for any two non-adjacent vertices x and y in T;*

$(2 \Rightarrow 1)$. Since there is a path between any two vertices the graph is connected. If there is a cycle then there are two vertices that are connected with at least two distinct paths contradicting the assumption and hence the graph has no cycles. Thus it must be a tree. □

$(3 \Rightarrow 1)$. By assumption the graph is connected. Since every edge is a cut edge, no edge is in a cycle and thus there are no cycles hence the graph is a tree. □

$(4 \Rightarrow 1)$. Let u, v be two vertices. Suppose they are not adjacent. Then by assumption $T + (uv)$ contains a cycle and since T does not contain a cycle then the edge (uv) is in the cycle. Thus there are two paths from $T + (uv)$ from u to v. One of those paths does not contain (uv) and it must therefore lie all within T. Hence T is connected and acyclic thus a tree. □

$(1 \Rightarrow 2)$. Suppose vertices u, v are connected by two distinct paths $ue_1w_1e_2w_2 \ldots e_nv$ and $ue'_1w'_1e'_2w'_2 \ldots e'_nv$. Let w_k be such that for all $j \leq k$ we have that $e_jw_j = e'_jw'_j$. Let w_l be the first vertex on both paths such that $l > k$. Then there is a cycle in the graph from w_k to w_l along the two paths a contradiction. □

$(1 \Rightarrow 3)$. By Theorem 1.4.5 an edge is a bridge if it is in no cycles. Since a tree contains no cycles, none of the edges in a tree can be part of a cycle and therefore each edge is a bridge. □

$(1 \Rightarrow 4)$. Let u, v be any two non-adjacent vertices in T. Since T is connected there is a path from u to v in T. Along with the edge (uv) this forms a cycle. □

Motivation: at least how many edges you need to connect dots on the plane. If you have more do you necessarily have a cycle?

Theorem 1.9.2. *For any tree with q edges and p vertices we have that $q = p - 1$.*

Proof. By induction on the number of vertices.
Base case: $p = 1$: since there are no edge in a tree with one vertex $p = 1$ and $q = 0 = p - 1 = 1 - 1$ and the result holds.
Assumption: assume the statement holds for any tree with at most p vertices.
Inductive step: Let T be a tree with $p+1$ vertices. Take any edge e of T and consider $T - e$. Since e is a bridge and T is connected then $T - e$ has two components, say T_1 and T_2. Suppose that T_1 has p_1 vertices and T_2 has p_2 vertices. Both T_1 and T_2 contain no cycles, are connected and therefore they are trees. We apply the inductive hypothesis to these trees separately and obtain $q_1 = p_1 - 1$ and $q_2 = p_2 - 1$, where q_1 and q_2 are

the number of vertices in T_1 and T_2, respectively. By construction the number of vertices in T is $p_1 + p_2$, and number of edges is $q = q_1 + q_2 + 1$ where the additional 1 is for the edge e. Substituting with the values for q_1 and q_2 we obtain $q = q_1 + q_2 + 1 = p_1 - 1 + p_2 - 1 + 1 = p - 1$, which concludes the argument. □

Theorem 1.9.3. *A tree with at least two vertices has at least two vertices of degree one.*

Direct proof. Let $p = v_0 e_1 v_1 \dots v_{n-1} e_n v_n$ be a longest path in T. (A longest path is a path for which no other path in the tree has a longer length, for the result maximal path also suffices). This path has length at least one since it contains at least two vertices and hence at least one edge. Consider vertex v_0. If the degree of v_0 is greater than 1 then there must be another edge incident with v_0 other than e_1. Say this is the edge e'. Let e' be (v_0, w). There are two possible scenarios: either $w = v_i$ for some i or w is not a vertex that appears in the path p. In the first case $v_0 e_1 \dots v_{i-1} e_i w e' v$ is a cycle which contradicts the fact that the graph is a tree. In the second scenario $w e' v_0 e_1 v_1 \dots v_{n-1} e_n v_n$ is a path in the graph that has length strictly greater than the length of p contradicting the fact that p is a longest path. In either scenario we have a contradiction and therefore the degree of v_0 is one. Similar argument applies to v_n and hence in the three at least v_0 and v_n have degree one completing the argument. □

Proof by counting. Let n_i be the number of vertices of degree $i \geq 0$. Then for p then number of vertices in the tree we have

$$p = n_1 + n_2 + \cdots$$

By the Handshake lemma 1.1.3 we have

$$2q = \sum d(v) = n_1 + 2n_2 + 3n_3 + \cdots,$$

where q is the number of edges in the tree. By Theorem 1.9.2 we have that

$$q = -1 + p = -1 + n_1 + n_2 + n_3 + \dots.$$

Substituting p in the first equation and rearranging its terms we get

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots \geq 2$$

□

Proof by induction. □

Vertices of degree one are called *leaves*. By the above theorem every tree with at least two vertices has at least two leaves.

1.9.1 Spanning trees

Theorem 1.9.4. *Every connected graph has a spanning tree.*

Proof. Let G be a connected graph. Delete edges from the cycles of G one by one until the graph is acyclic. Since the edges that are deleted are in a cycle the subgraph is also connected. Thus it is a tree with the same vertex set meaning it is a spanning tree. □

Theorem 1.9.5. *If a graph has a spanning tree then the graph is connected.*

Proof. If T is a spanning tree of G then every path in T is also a path in G. Since T is a tree it is connected and thus between any pair of vertices there a T-path (a path that has only edge in T). Since every edge of T is also an edge in G any T path is also a G path and thus any pair of vertices in G is connected. Thus G is connected. $\square$

The above two theorems imply that a graph is connected if and only if it as a spanning tree.

Theorem 1.9.6. *If G is connected with p vertices and $p-1$ edges then G is a tree.*

Proof. By Theorem 1.9.5 the graph G has a spanning tree T. Since T is spanning tree it has p vertices and by Theorem 1.9.2 it has $p-1$ edges. Since G has $p-1$ edges every edge of G is in T and thus the edge set and vertex set of T and G coincide, thus they are the same graph. Hence G is a tree. $\square$