Linear Algebra

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Eigenvectors and Eigenvalues

Repeated application

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^{2n+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2^{2n+1} \end{pmatrix} \qquad A^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2n} \end{pmatrix}$$

$$A = \begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix}$$

$$A^n = ?$$

General LA

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -2 \\ 2 & -2 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

General LA

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Eigenvalues and eigenvectors

Definition (eigenvalues and eigenvectors)

Let A be a square matrix. A non-zero vector $\vec{u}$ is an *eigenvector* for A if $A\vec{u} = \lambda\vec{u}$ for some λ. The value λ is called *eigenvalue* for the eigenvector $\vec{u}$.

Eigenvalues and eigenvectors

$$A\vec{u} = \lambda\vec{u}$$

is

$$A\vec{u} = \lambda I\vec{u}$$

$$(A - \lambda I)\vec{u} = \vec{0}$$

Want linearly dependent columns!

Example

from

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

to

$$\begin{pmatrix} -11 & 14 & 4 \\ 3 & -2 & -2 \\ -18 & 22 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

from

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

to

$$\begin{pmatrix} -6 & 14 & 4 \\ 3 & 3 & -2 \\ -18 & 22 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

from

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

to

$$\begin{pmatrix} -10 & 14 & 4 \\ 3 & -1 & -2 \\ -18 & 22 & 8 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Goal

Given

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

are the columns of A linearly dependent.

Computing Eigenvalues and Eigenvectors

1. compute the determinant of $A - \lambda I$
2. compute the roots $\lambda_1, \ldots, \lambda_n$ of the resulting polynomial, the n (possibly repeated) roots are the eigenvalues
3. for each eigenvalue i find the corresponding eigenvectors by computing $(A - \lambda_i I)\vec{x} = \vec{0}$

Example

$$A = \begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^3 - 7\lambda + 6 = (\lambda - 2) \cdot (\lambda - 1) \cdot (\lambda + 3)$$

so

$$\begin{aligned} \lambda_0 &= 2 \\ \lambda_1 &= 1 \\ \lambda_2 &= -3 \end{aligned}$$

$A - \lambda_0 I$

$$= \begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} -2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -11 & 14 & 4 \\ 3 & -2 & -2 \\ -18 & 22 & 7 \end{pmatrix}$$

solve

$$\begin{pmatrix} -11 & 14 & 4 \\ 3 & -2 & -2 \\ -18 & 22 & 7 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

$A - \lambda_1 I$

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 14 & 4 \\ 3 & -1 & -2 \\ -18 & 22 & 8 \end{pmatrix}$$

solve

$$\begin{pmatrix} -10 & 14 & 4 \\ 3 & -1 & -2 \\ -18 & 22 & 8 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

$A - \lambda_2 I$

$$\begin{pmatrix} -9 & 14 & 4 \\ 3 & 0 & -2 \\ -18 & 22 & 9 \end{pmatrix} - (-3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 14 & 4 \\ 3 & 3 & -2 \\ -18 & 22 & 12 \end{pmatrix}$$

solve

$$\begin{pmatrix} -6 & 14 & 4 \\ 3 & 3 & -2 \\ -18 & 22 & 12 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

Example

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \lambda - 2 = (\lambda - 2) \cdot (\lambda + 1)$$

- *characteristic polynomial*:

$$\lambda^2 - \lambda - 2$$

- *characteristic equation*:

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda_0 = 2 \qquad \lambda_1 = -1$$

Example

$$A - \lambda_0 I = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix}$$

solve

$$\begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 5 \\ 2 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

Example

$$A - \lambda_1 I = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix}$$

solve

$$\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{\alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{C}\right\}$$

Example double root

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

so

$$\lambda_0 = 2$$
$$\lambda_1 = 2$$

Example double root

$$A - \lambda_0 I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

solve

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left\{ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid \alpha_0, \alpha_1 \in \mathbb{C} \right\}$$

Example double root again

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

so

$$\lambda_0 = 2$$
$$\lambda_1 = 2$$

Example

$$A - \lambda_0 I = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

solve

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left\{ \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

zero is an eigenvalue

$$A = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 2\lambda = (\lambda - 2) \cdot \lambda$$

- *characteristic polynomial*:

$$\lambda^2 - 2\lambda$$

- *characteristic equation*:

$$\lambda^2 - 2\lambda = 0$$

$$\lambda_0 = 2 \qquad \lambda_1 = 0$$

Example

$$A - \lambda_0 I = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}$$

solve

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$

Example

$$A - \lambda_1 I = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} - (0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}$$

solve

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\}$$