Chapter 3

Determinants

3.1 Definitions

Definition 3.1.1. *A* $n \times n$ determinant *is a function* $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ *such that*

1. *row combination* $\det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_i \\ \vec{\rho}_j + k\vec{\rho}_j \\ \vec{\rho}_i \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$ *for* $i \neq j$

$$\det\left[\begin{pmatrix} 1 & 0\dots & & & & & & & \dots 0 & 0 \\ & & & \ddots & & & & & & \\ 0 & \dots 0 & 0 & 0\dots & 1 & 0 & 0 & \dots 0 & 0 \\ 0 & \dots 0 & k & 0\dots & 0 & 1 & 0 & \dots 0 & 0 \\ 0 & \dots 0 & 0 & 0\dots & 0 & 0 & 1 & \dots 0 & 0 \\ & & & & & & & \ddots & \\ 0 & 0\dots & & & & & & \dots 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix}\right] = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_i \\ \vec{\rho}_j + k\vec{\rho}_j \\ \vec{\rho}_i \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

2. swap $\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_i\\ \vdots\\ \vec\rho_j\\ \vdots\\ \vec\rho_n\end{pmatrix} = -\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_j\\ \vdots\\ \vec\rho_i\\ \vdots\\ \vec\rho_n\end{pmatrix}$ *for* $i \neq j$

$$\det\left[\begin{pmatrix}
1 & 0\ldots & & & & & & & & & & & \ldots 0 & 0\\
 & & \ddots & & & & & & & & & & & \\
0 & \ldots 0 & 0 & 0\ldots & 1 & 0 & 0 & & & & & & \ldots 0 & 0\\
0 & \ldots 0 & 1 & 0\ldots & 0 & 0 & 0 & & & & & & \ldots 0 & 0\\
0 & \ldots 0 & 0 & 0\ldots & 0 & 0 & 1 & & & & & & \ldots 0 & 0\\
 & & & & & & & \ddots & & & & & & \\
0 & 0\ldots & & & & & & 1 & 0 & 0 & \ldots 0 & 0 & 0\ldots & \\
0 & 0\ldots & & & & & & 0 & 0 & 0 & \ldots 0 & 1 & 0\ldots & \\
0 & 0\ldots & & & & & & 0 & 0 & 1 & \ldots 0 & 0 & 0\ldots & \\
 & & & & & & & & & & \ddots & & & \\
0 & 0\ldots & & & & & & & & & & & \ldots 0 & 1
\end{pmatrix}\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\rho_i\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_{j-1}\\ \vec\rho_j\\ \vec\rho_{j+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}\right] = -\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\rho_j\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_{j-1}\\ \vec\rho_i\\ \vec\rho_{j+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$$

3. rescaling $\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ k\vec\rho_i\\ \vdots\\ \vec\rho_n\end{pmatrix} = k\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_i\\ \vdots\\ \vec\rho_n\end{pmatrix}$ *for any scalar* k

$$\det\left[\begin{pmatrix}
1 & 0\ldots & & & & \ldots 0 & 0\\
 & \ddots & & \vdots & & & \\
0 & 0\ldots & 1 & 0 & 0 & \ldots 0 & 0\\
0 & 0\ldots & 0 & k & 0 & \ldots 0 & 0\\
0 & 0\ldots & 0 & 0 & 1 & \ldots 0 & 0\\
 & & & \vdots & & \ddots & \\
0 & 0\ldots & & & & \ldots 0 & 1
\end{pmatrix}\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\rho_i\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}\right] = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ k\vec\rho_i\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$$

4. $\det(I) = 1$ *where* I *is an identity matrix*

(the $\vec\rho$ *'s are the rows of the matrix). We often write* $|T|$ *for* $\det(T)$*.*

In other words $\det(EA) = \det(E)\det A$ where E is any matrix representing elementary row operation with

1. $\det(E) = 1$ if E is linear combination

2. $\det(E) = -1$ if E is a swap

3. $\det(E) = k$ if E is rescaling

Theorem 3.1.1. *Condition 1 and Condition 3 imply Condition 2.*

Proof.

$$\begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_j \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i + \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_j \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i + \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_j - \vec{\rho}_i - \vec{\rho}_j \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i + \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ -\vec{\rho}_i \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_i + \vec{\rho}_j - \vec{\rho}_i \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ -\vec{\rho}_i \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ -\vec{\rho}_i \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix} = - \begin{vmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\rho}_j \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_{j-1} \\ \vec{\rho}_i \\ \vec{\rho}_{j+1} \\ \vdots \\ \vec{\rho}_n \end{vmatrix}$$

□

Theorem 3.1.2. *If a matrix A has a row of zeros then* $\det(A) = 0$.

Proof. Use scalar multiplication property with $k = 0$. □

Theorem 3.1.3. $\det(A) = 0$ *if and only if $\vec{\rho}_1, \dots, \vec{\rho}_n$ are linearly dependent.*

Proof. Suppose that $\vec{\rho}_1, \dots, \vec{\rho}_n$ are linearly dependent. Then there exist coefficients not all zero such that

$$\alpha_1 \vec{\rho}_1 + \cdots + \alpha_n \vec{\rho}_n = \vec{0}$$

let $\alpha_k \neq 0$. Then

$$\alpha_k \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_k \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \alpha_k \vec{\rho}_k \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \alpha_k \vec{\rho}_k + \sum_{i=1, i \neq k}^{n} \alpha_i \vec{\rho}_i \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{0} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = 0$$

Since $\alpha_k \neq 0$ it follows that the determinant is zero.

Assume now the rows are linearly independent (and so are the columns). By Theorem 2.9.4u sing the matrix representation of Gaussian operations we can write $A^{-1} = E_m \dots E_1$ where each E_i is an elemetary matrix and all scaling operations do not involve a scaling by zero. Then

$$1 = \det(I) = \det(A^{-1}A) = \det(E_m \dots E_1 A) = \det(E_m) \dots \det(E_1) \det(A)$$

since the right hand side is non-zero the left hand side is also non zero; and therefore $\det(A) \neq 0$.

□

Theorem 3.1.4. $\det(A)$ *is unique.*

Proof. If columns of A are linear dependent then $\det(A) = 0 = \tilde{\det}(A)$. If colums of A are linearly independent then from

$$\begin{aligned}
\tilde{\det}(A) &= \tilde{\det}(E_m \dots E_1) \\
&= \tilde{\det}(E_m) \dots \tilde{\det}(E_1) \\
&= \det(E_m) \dots \det(E_1) \\
&= \det(E_m \dots E_1) \\
&= \det(A)
\end{aligned}$$

□

Theorem 3.1.5. $\det(AB) = \det(A)\det(B)$

Proof. If $\det(B) = 0$ then its rows (and by the rank) its rows are linearly dependent. In $C = AB$ the rows of C are linear combinations of the rows of B. Then the span of the rows of C is a subset of the span of the rows of B and therefore the number of linearly independent rows of C cannot exceed the number of linearly independent rows of B. Thus the rows of C cannot be linearly independent. Thus $\det(C) = 0$ and the theorem holds in this case. Similarly if $\det(A) = 0$ then the columns of A are linearly dependent and by a similar argument $\det(C) = 0$. Assume now that $\det(A) \neq 0$ and $\det(B) \neq 0$. Then as in the above theorem we have

$$\det(A) = \det(E_m) \dots \det(E_1)$$

$$\det(B) = \det(E'_k) \dots \det(E'_1)$$

and

$$\det(AB) = \det(E_m \dots E_1 E'_k \dots E'_1) = \det(E_m) \dots \det(E_1)\det(E'_k) \dots \det(E'_1) = \det(A)\det(B)$$

Note that in the above we do *not* use the fact that $\det(AB) = \det(AB)$ we simply use the definition where $\det(EA) = \det(E)\det(A)$ for any elementary matrix E. □

Recall that by Theorem 2.9.4 we have that

> Suppose the rows (columns) of square A are linearly independent then A can be written as a product of elementary matrices.

Theorem 3.1.6. $\det(A) = \det(A^T)$

Proof. Homework □

The question is how do we know that from $A = E_m \dots E_1$ and $A = E'_k \dots E'_1$ we have that

$$\det(E_m) \dots \det(E_1) = \det(E'_k) \dots \det(E'_1)$$

For example for the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

how many swaps do we need? May be we can do it with 8 swaps and at the same time we can do it with 5 swaps. In one case we have determinant positive one and in the other negative one.

Theorem 3.1.7. $\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec u+\vec v\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec u\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} + \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec v\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$

Proof. If $\vec\rho_1,\ldots,\vec\rho_{i-1},\vec\rho_{i+1},\ldots,\vec\rho_n$ are linearly dependent then all determinants are zero and the result follows. Suppose now they are linearly independent. Then we can find a vector $\vec\beta$ such that $\vec\rho_1,\ldots,\vec\rho_{i-1},\vec\beta\vec\rho_{i+1},\ldots,\vec\rho_n$ are linearly independent. Since there are n of them they span all of $\mathbb{C}^n$ and therefore we have

$$\begin{aligned}
\vec u &= u_1\vec\rho_1+\cdots+u_{i-1}\vec\rho_{i-1}+u_i\vec\beta+u_{i+1}\vec\rho_{i+1}+\cdots+u_n\vec\rho_n\\
\vec v &= v_1\vec\rho_1+\cdots+v_{i-1}\vec\rho_{i-1}+v_i\vec\beta+v_{i+1}\vec\rho_{i+1}+\cdots+v_n\vec\rho_n\\
\vec u+\vec v &= (u_1+v_1)\vec\rho_1+\cdots+(u_{i-1}+v_{i-1})\vec\rho_{i-1}+(u_i+v_i)\vec\beta+(u_{i+1}+v_{i+1})\vec\rho_{i+1}+\cdots+(u_n+v_n)\vec\rho_n
\end{aligned}$$

Then for $j=1\ldots i-1$ and $j=i+1,\ldots n$ we have

$$\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec u+\vec v=\vec w_0\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec w_{j-1}-(u_j+v_j)\vec\rho_j\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$$

At the end we obtain

$$\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec u+\vec v\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ (u_i+v_i)\vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = (u_i+v_i)\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = u_i\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} + v_i\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$$

Consider

$$u_i\det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ u_i\vec\beta\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ u_i\vec\beta+u_1\vec\rho_1\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ u_i\vec\beta+u_1\vec\rho_1+u_2\vec\rho_2\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix} = \cdots = \det\begin{pmatrix}\vec\rho_1\\ \vdots\\ \vec\rho_{i-1}\\ \vec u\\ \vec\rho_{i+1}\\ \vdots\\ \vec\rho_n\end{pmatrix}$$

Similarly,

$$v_i \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{\beta} \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix} = \cdots = \det \begin{pmatrix} \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_{i-1} \\ \vec{v} \\ \vec{\rho}_{i+1} \\ \vdots \\ \vec{\rho}_n \end{pmatrix}$$

And substituting back we get the desired result. $\square$

Observer that we have the expansion

$$\begin{aligned}
\det\begin{pmatrix} 5 & -8 & 1 \\ 3 & -5 & 1 \\ -4 & 7 & -1 \end{pmatrix} &= \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 0 & -8 & 0 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} \\
&= \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & -7 & -1 \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 0 & -8 & 0 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} \\
&= \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & -7 & 0 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & -7 & -1 \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 0 & -8 & 0 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & 1 \\ 3 & -5 & 1 \\ -4 & -7 & -1 \end{pmatrix}
\end{aligned}$$

After the second equality observe that when we take the same element from the first row we get a matrix where the first two rows are linearly dependent hence the determinant is zero. So the only contribution comes from determinants where we take elements from different columns (and thus keep linearly independence). So we end up with the formula for the determinant

$$\begin{aligned}
\det(A) &= \sum_{\text{permutatins } \phi} a_{1\phi(1)} \dots a_{n\phi(n)} \det(P_\phi) \\
&= \sum_{i=1}^{n} (-1)^{k+i} a_{ki} \det\left(A(k|i)\right) \\
&= \sum_{i=1}^{n} (-1)^{k+i} a_{ik} \det\left(A(i|k)\right)
\end{aligned}$$

Here $A(i|j)$ is a submatrix obtained from A by removing row i and column j, where in general A need not be a square matrix. The value $\det(A(i|j)$ is called the minor of a_{ij}, and the value $(-1)^{k+i}\det(A(i|k))$ is called the cofactor of a_{ij}. Likewise, a permutation is a one-to-one and onto function from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$.

So from the last equation to show that determinant is well defined function we need to show it is well defined for permutations. These are matrices that have exactly one entry one in each column and row and all other entries are zero.

Definition 3.1.2. $\phi = (\phi(1), \phi(2), \ldots, \phi(n))$. *In a permutation matrix* $P_\phi = \begin{pmatrix} \vdots \\ \rho_{\phi(k)} \\ \vdots \\ \rho_{\phi(l)} \\ \vdots \end{pmatrix}$ *two rows are an inversion if and only if* $\phi(k) > \phi(l)$.

Examples

Definition 3.1.3. *The sign of a permutation is negative one if the number in inversions is odd, and positive one if the number of inversions is even.*

Theorem 3.1.8. $\det P_\phi = sign(\phi)$

Proof. Not part of the course material. $\square$