probability theory and random variables

contents

- introduction
- repeatable experiments, sample space, and events
- relative frequencies and basic concepts of probability
- finite sample spaces and events with equal frequencies
- determination of the number of events
- conditional probability
- Bayes theorem
- independent events
- random variables (discrete, continuous)
- concept of expected value

introduction

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- however more general focus is sampling and statistical interpretation
- sample based decisions on population have error rates because each sample differs from each other by chance
- probability deals with uncertainty about random events

repeatable experiments, sample space and events

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- they are all repeatable + observations vary for each different experiment

repeatable experiments, sample space and events

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- some experiments have limited number of elements in sample space while some have infinite number of elements

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- e.g. how many heads in a two-coins toss
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- events are combinations of elements that we care for in the sample space

repeatable experiments, sample space and events

- we can **never** know A event will happen or not after an experiment

repeatable experiments, sample space and events

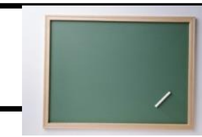
2.2

- we can **never** know A event will happen or not after an experiment
- however we can have an **expectation** about the probability that it will happen or not
- this expectation leads to a need for probability theory

relative frequencies and probability

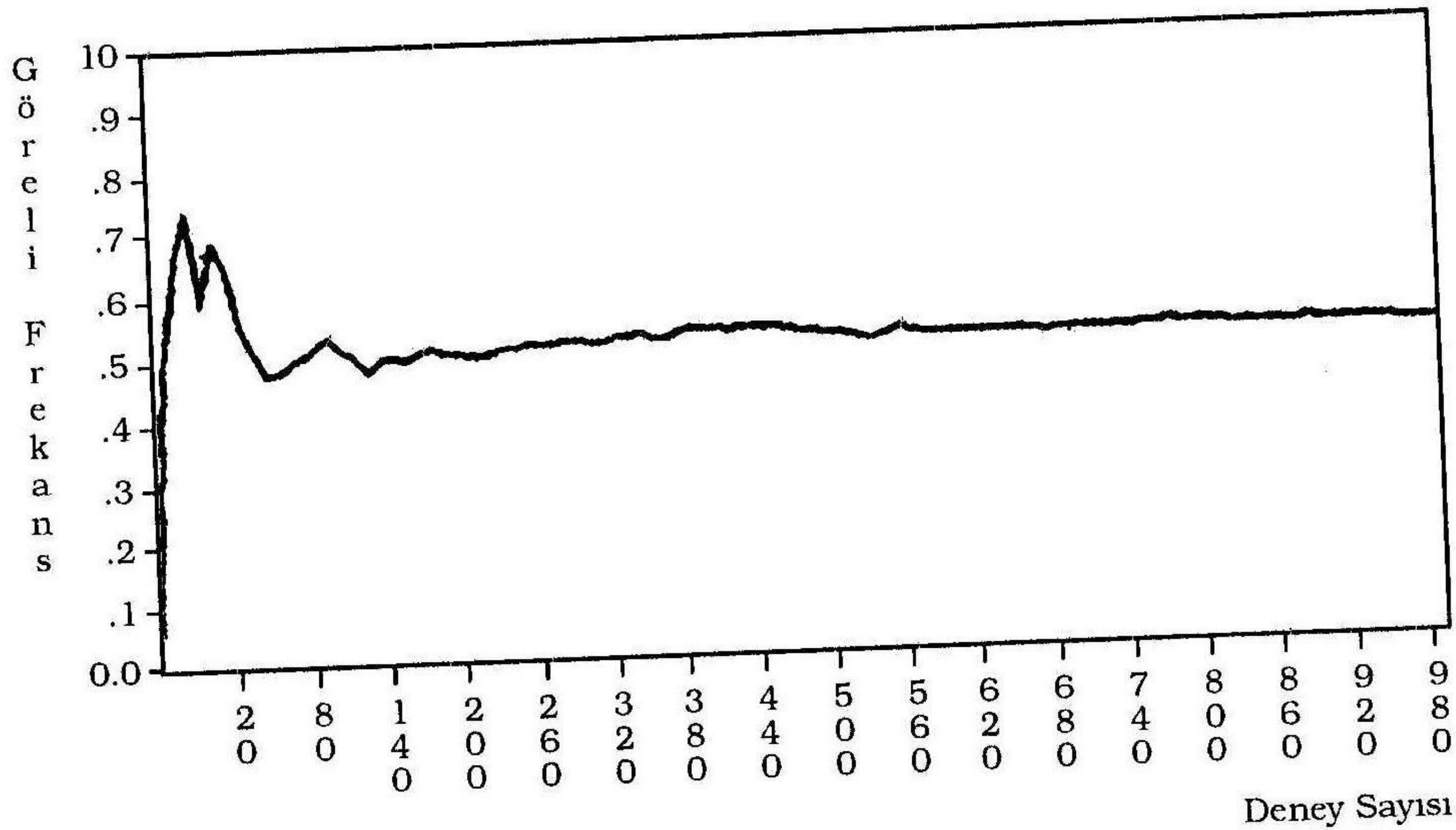
- relative frequency: event / all repetition of events
- properties of relative frequency

relative frequencies and probability



- relative frequency: event / all repetition of events
- properties of relative frequency
- and... **stability**

relative frequencies and probability

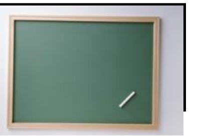


Şekil 2.3. Tekrarlanan bir deneyde göreli frekansların değişimi.

relative frequencies and probability

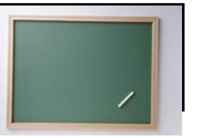
Deney No	1	2	3	4	5	10	20	50	100	200	300	500	1000
Gözlenen Yazı Sayısı	0	1	2	3	3	7	13	24	54	102	158	264	522
Göreli Frekanslar	0	0.50	0.67	0.75	0.60	0.70	0.65	0.48	0.54	0.51	0.53	0.53	0.52

relative frequencies and probability



- probability: the probability of an event -> when trials go to infinity
- properties of probability

relative frequencies and probability



- probability: the probability of an event -> when trials go to infinity
- properties of probability
- properties of P(A)

finite sample spaces and events with equal frequencies

- simple events: events consisting only a single element of sample space

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- simple events: events consisting only a single element of sample space
- events with equal frequencies *(must be careful! e.g. blood types differ – A, B, AB, O)*
- *1. be careful about equality of probabilities*
- *2. be careful about defining sample space (sometimes you can transform a sample space into an equal probability one -> taking 3 different sized bottles from a bag -> marking!) (and be careful about equality of different combinations of elements in an event)*

determining # of events

- # of elements in an event and total # of events of sample space have to be known to make the calculations, methods are required (e.g. drawing pills)

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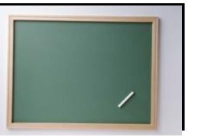
- multiplication method
- addition method
- permutation

determining # of events



- # of elements in an event and total # of events of sample space have to be known to make the calculations, methods are required (e.g. drawing pills)
- multiplication method
- addition method
- permutation
- combination

determining # of events

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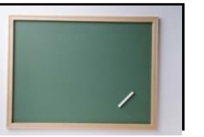
- addition method
- permutation
- combination
- mixed

conditional probability



- aspirin & LSD example, with or without putting back

conditional probability

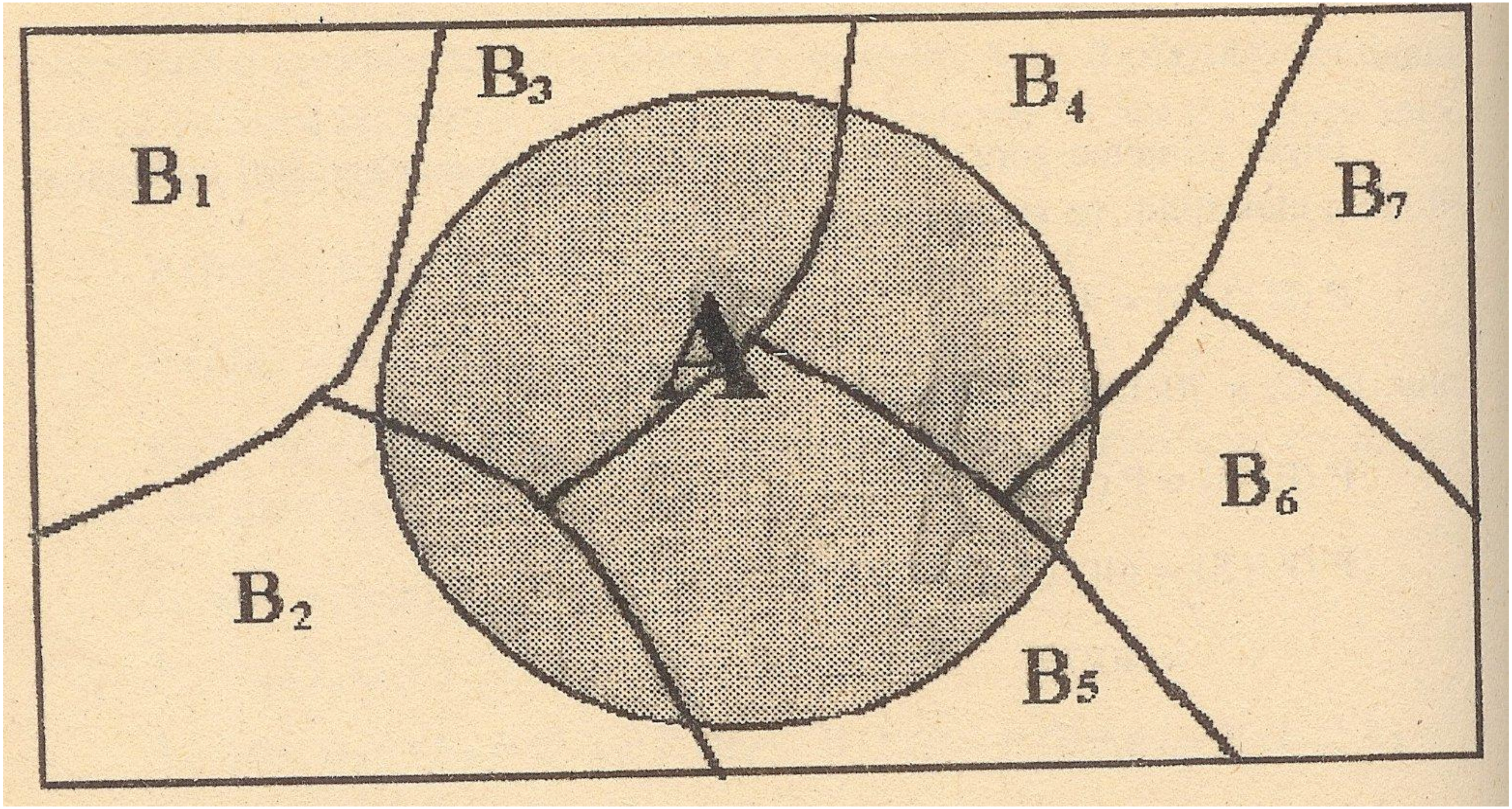
$$S = \begin{bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{bmatrix}$$

conditional probability

- properties of conditional probability

conditional probability

2.7



Bayes theorem

- return to LSD/aspirin example again:
  - some different queries e.g.
    - *realizing that the pill is defective, what's the probability that it has been produced in first factory?*
    - *realizing that second pill is LSD, what's the probability that first pill <u>was also</u> LSD*

Bayesian Probability

- The term *bayesian* came into use in 1950’s
- Contrasts with *frequency probability*

Thomas Bayes
(1702 – 1761)
British Mathematician

Bayesians v.s. Frequentists

- Frequentists assign probabilities to random events according to their <u>frequencies of occurence</u> or <u>subsets of populations as proportions of the whole</u>
- Bayesian assign probabilities to propositions that are <u>uncertain</u>

1) You have a box of white and black balls, bu[illegible] <u>knowledge as to the quantities</u> **0.5**
2) You have a box from wh[illegible]ave <u>drawn n balls, half black and th[illegible]e white</u> **0.5**
3) You [illegible]ox and you know that <u>there are same [illegible]mber of white and black balls</u> **0.5**

CRITISISM ON EVIDENCE

Bayes' Theorem

likelihood

conditional probability

standardized likelihood

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

prior probabilities

posterior probability

normalizing constant

$$\boxed{\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{normalizing constant}}}$$

$$\boxed{\text{posterior} = \text{standardised likelihood} \times \text{prior}.}$$

Derivation from Conditional Probability

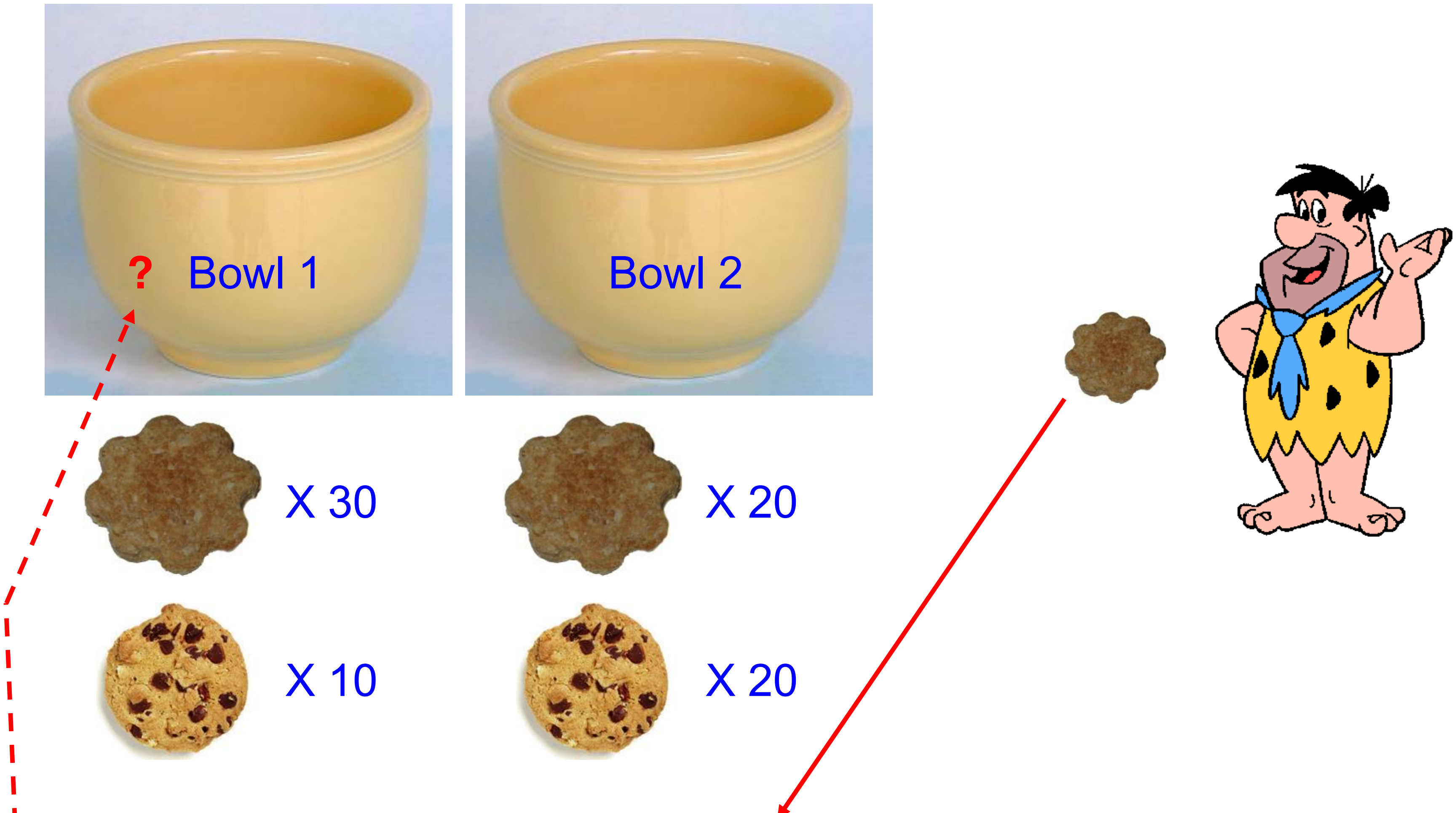
$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}. \qquad \Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}.$$

$$\Pr(A|B)\ \underline{\Pr(B)} = \Pr(A \cap B) = \Pr(B|A)\ \underline{\Pr(A)}.$$

Pr(B) Pr(B)

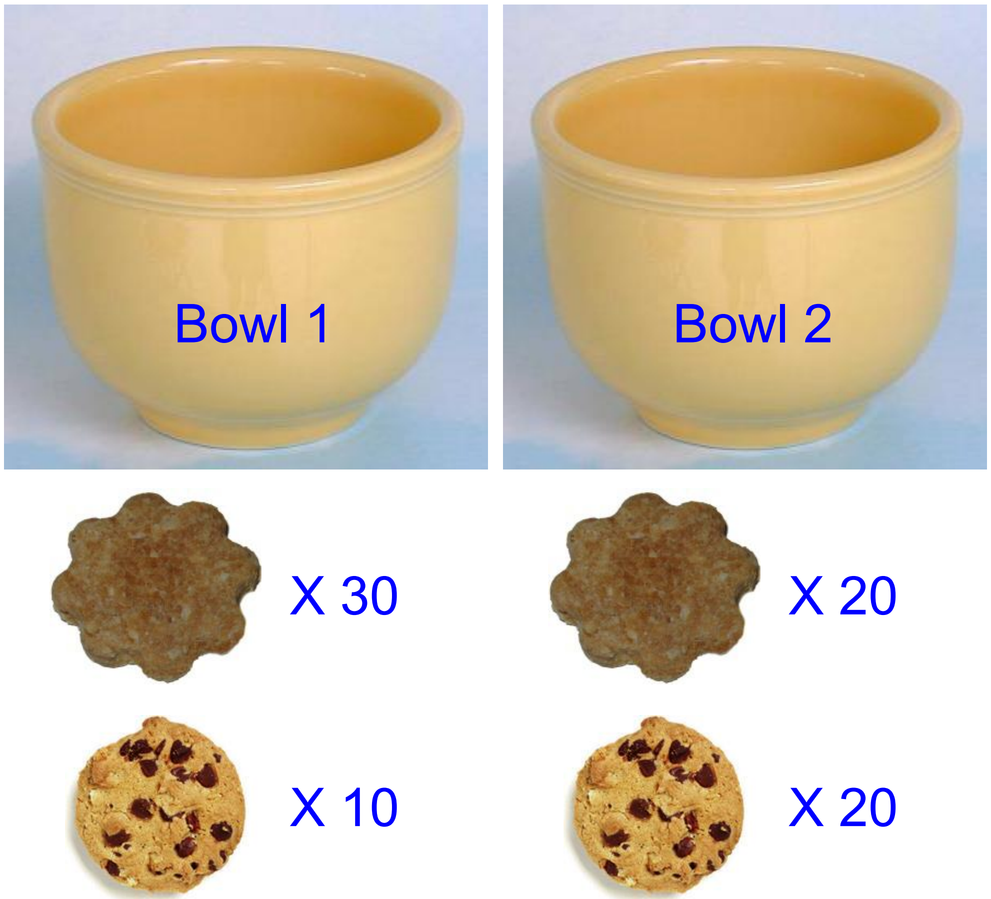
$$\Pr(A|B) = \frac{\Pr(B|A)\ \Pr(A)}{\Pr(B)}.$$

Example (Cookie)



- Fred picked a plain cookie
- What is the probability that he picked it from bowl 1?

Example (Cookie)



$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)} = \frac{0.75 \times 0.5}{0.625} = 0.6$$

Bayesian Inference

"For billions of years, the sun has risen after it has set. The sun has set tonight.

- *With very high probability (or <u>I strongly believe that or it is true that</u>) the sun will rise tomorrow.*
- *With very low probability (or <u>I do not at all believe that or it is false that</u>) the sun will not rise tomorrow."*

Very high degree of belief → TRUE

Very low degree of belief → FALSE

independent events

- previous topics covered:
  - $P(B|A)=0$ (mutually exclusive events)
  - $P(B|A)=1$ (event_A is subset of event_B)

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- in fact, many daily events do not have any effect on another (concept of independency) where
  - P(B|A)=P(B) (*e.g. rolling two dice*)

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- **statistical independence**

random variables

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- variables can be qualitative or quantitative (e.g. weather... daily production)
- observed variables are depicted with X,Y,Z etc... each element of sample space can correspond to only one X value i.e. any random variable can adress multiple elements in sample space
- in another words; random variable is a function that mathematically assigns values to elements of sample space

categorization of random variables

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categorization of random variables

- if there are infinitely many number of elements in sample space, it might be impractical to assign a probability value for each element (which would eventually add up to 1) (e.g. daily raindrop values)
- if the elements of sample space are **countable**, they are called “***discrete random variables***”
- if the elements of sample space are **uncountable**, they are called “***continuous random variables***”

discrete random variables

- if a random variable can have specific number of values (*e.g. # of bacteria in water, #of defectives in tens of samples, # of incoming calls to a call-center in a speific time, # of patients in the doctor's waiting room...*)

discrete random variables

• if a random variable can have specific number of values (*e.g. # of bacteria in water, #of defectives in tens of samples, # of incoming calls to a call-center in a speific time, # of patients in the doctor's waiting room...*)

• properties of the probability distribution of discrete random variables

continuous random variables

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- depends on sensitivity of measurement tool of course (height, weight...)

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• need to rely on relative frequencies of observations (the higher the # of samples, the more precise the distribution is)

continuous random variables

- how to choose $f(y)$?

continuous random variables

- how to choose f(y)?
- how to calculate the area below the curve? (integrals or readily available tables)

continuous random variables

- how to choose f(y)?
- how to calculate the area below the curve? (integrals or readily available tables)
- there are also methods to check whether selected f(y) and real relative frequency curve of population are relevant (covered in further topics)

expected value

- calculating the mean value of a function defined on a theoretic population

references

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