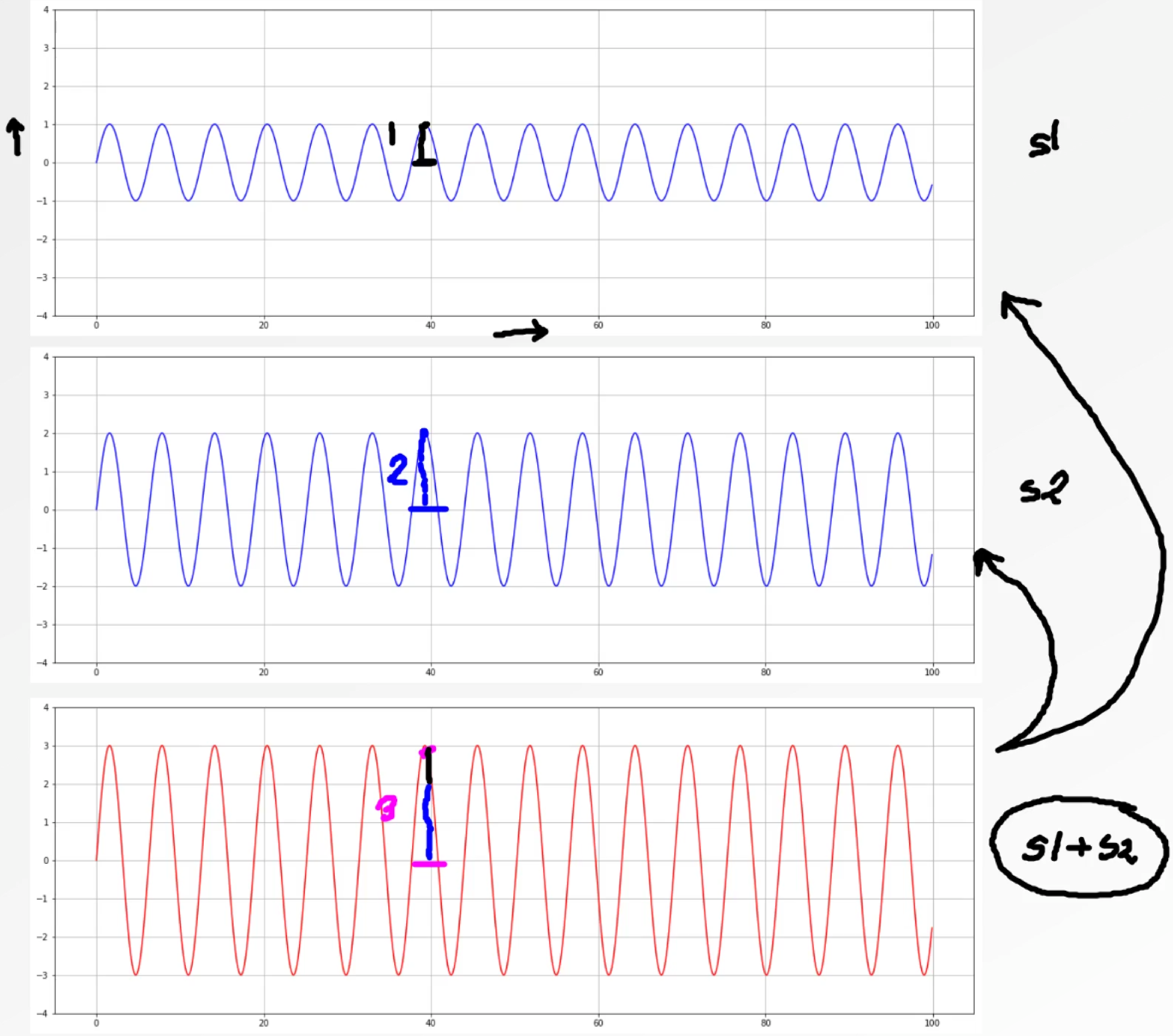
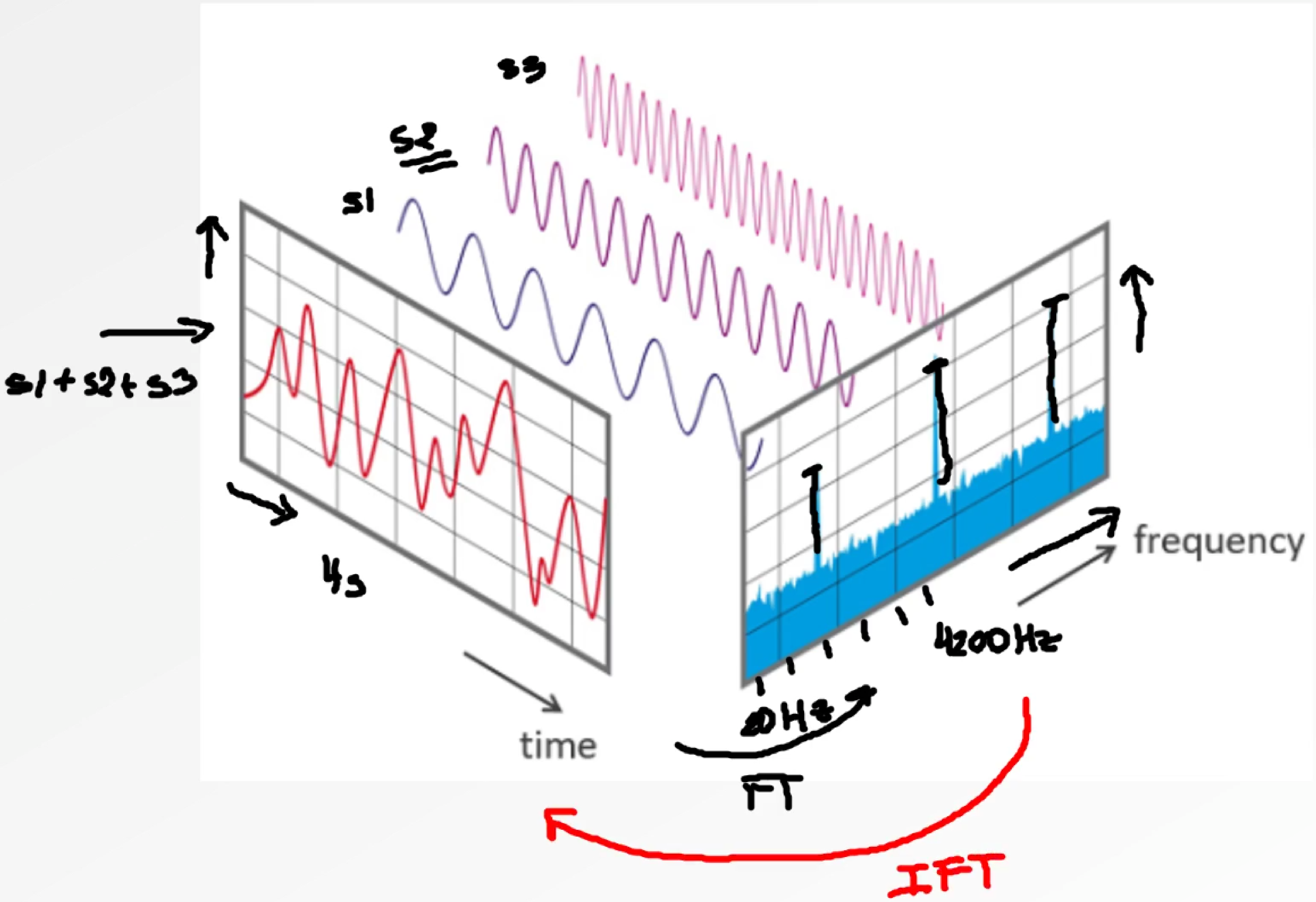
Fourier Transform

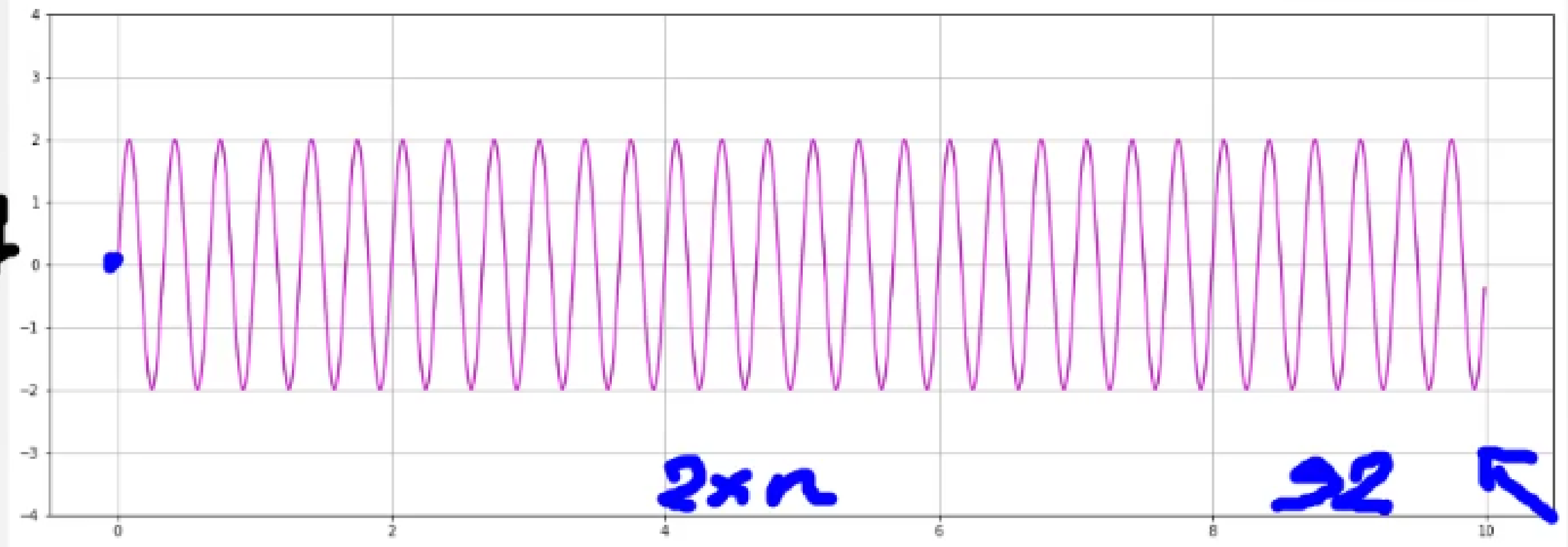
- The Fourier transform (FT) decomposes a signal into the frequencies that make it up



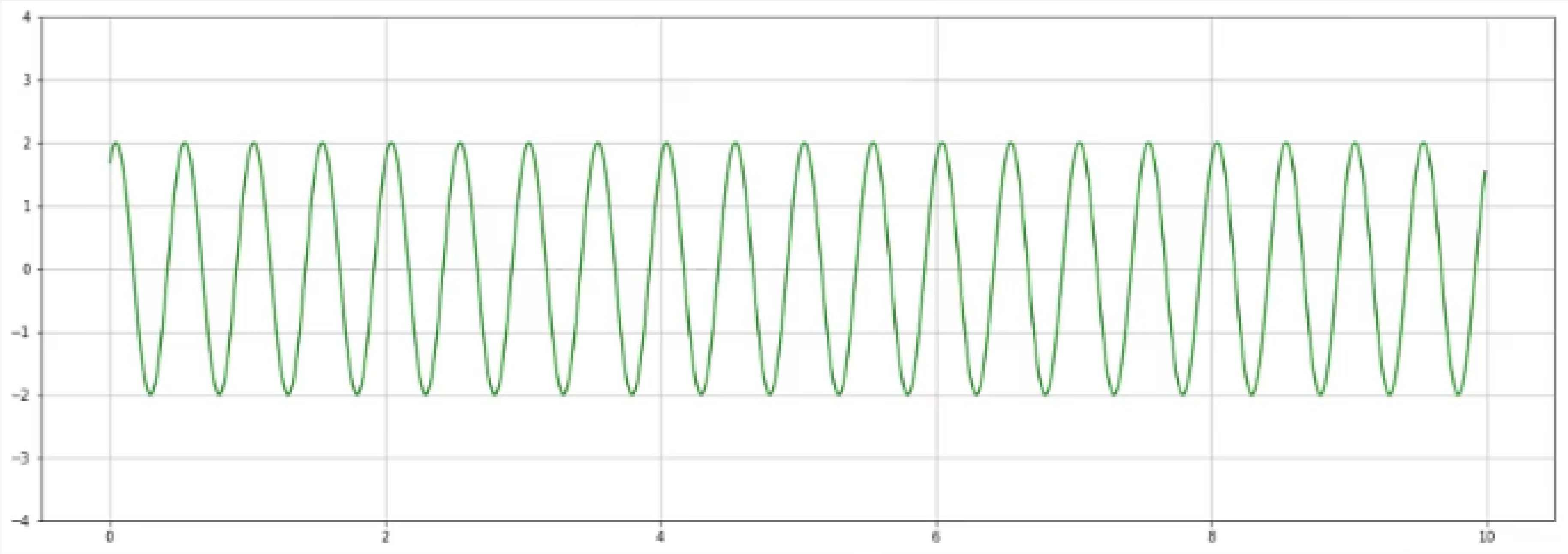
4 3 2 1 0 −1 −2 −3 −4
0 20 40 60 80 100
1
s1
2
s2
3
s1+s2


s3
s2
s1
s1+s2+s3
4s
time
frequency
20Hz
4200Hz
FT
IFT

y



x

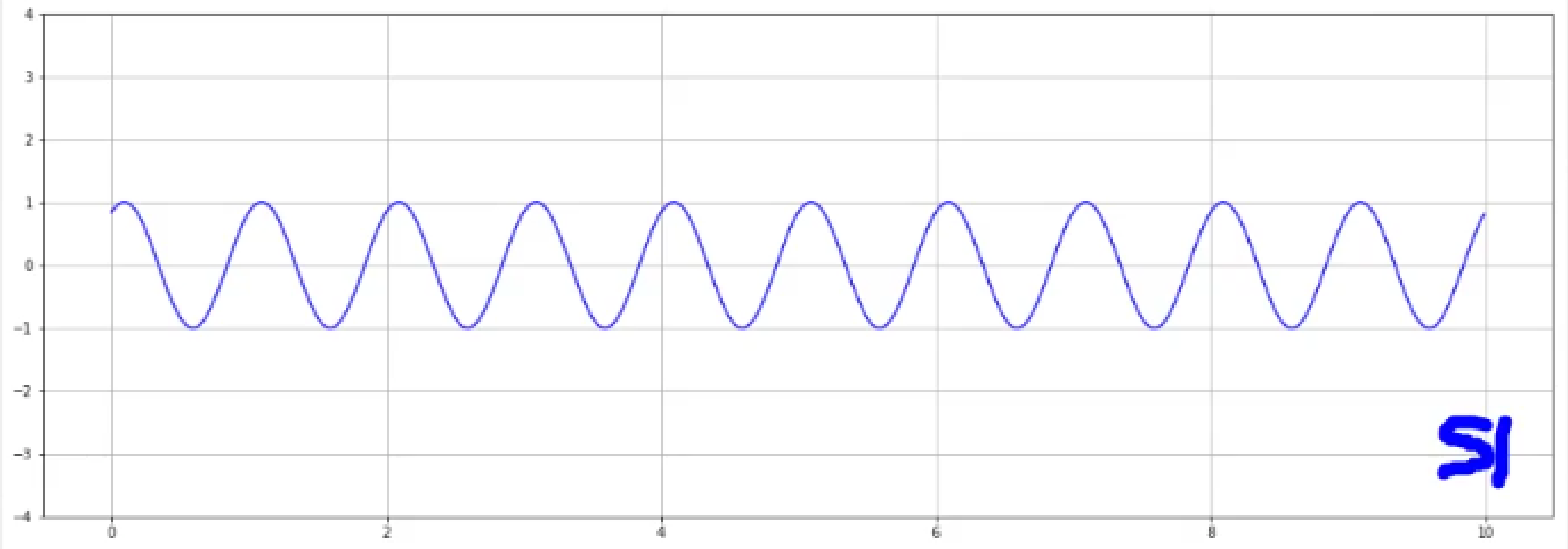


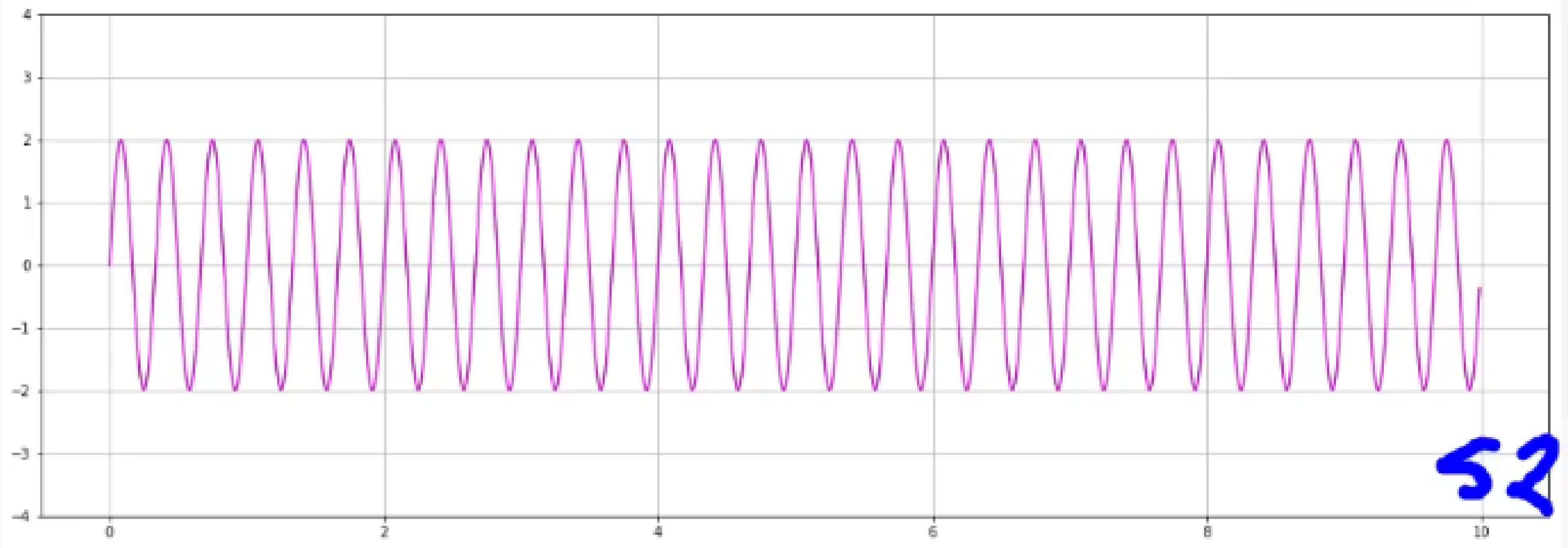
$$y(t) = A \cdot \sin(2\pi f t + \varphi)$$

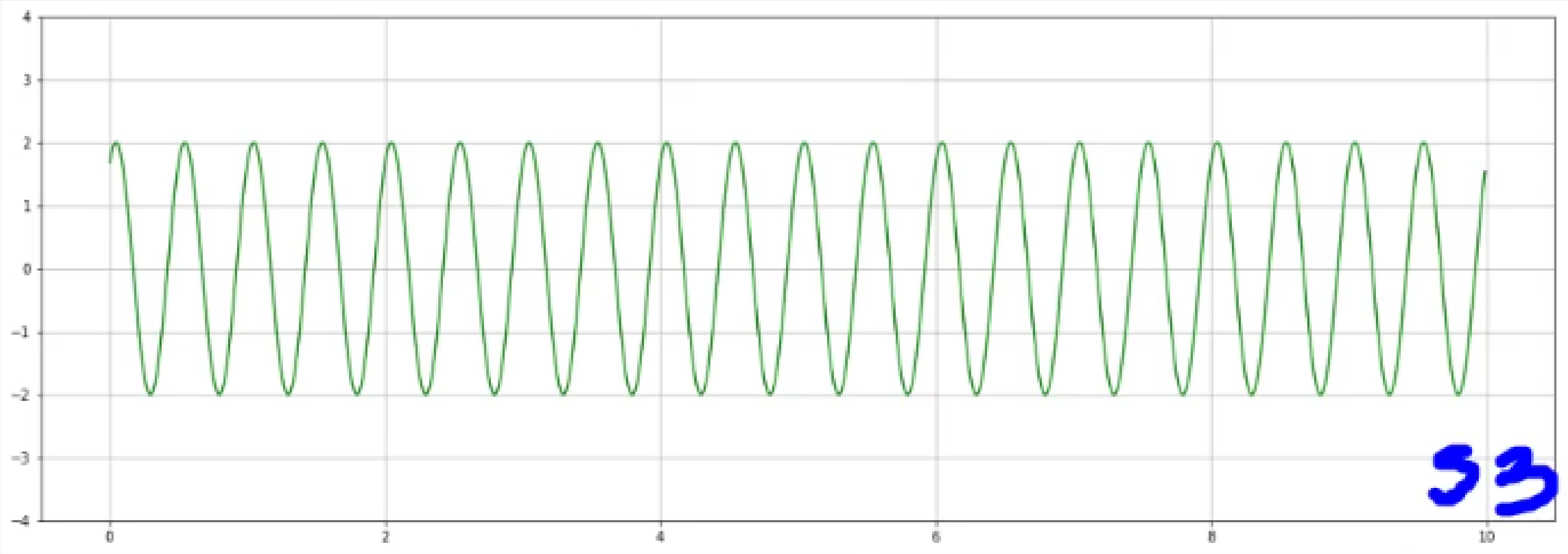
amplitude

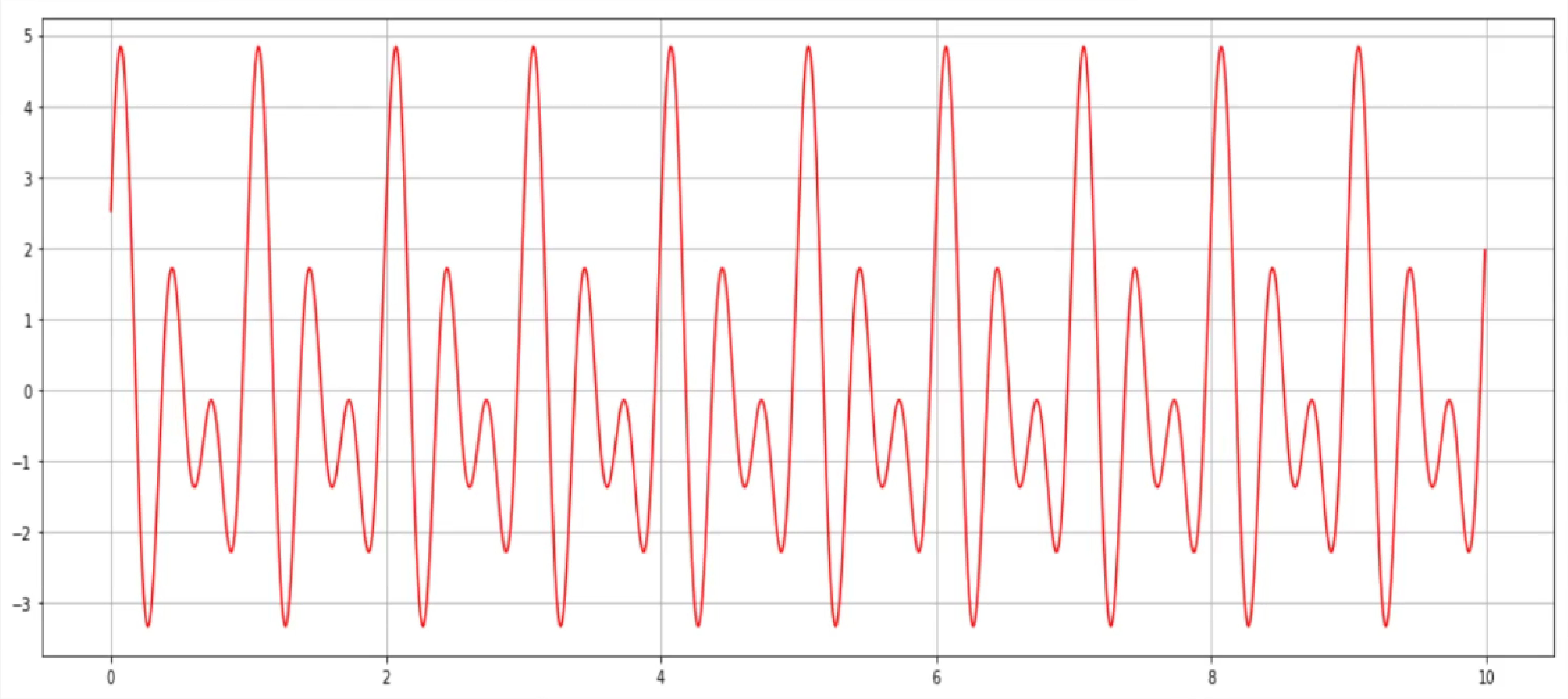
frequency

phase shift

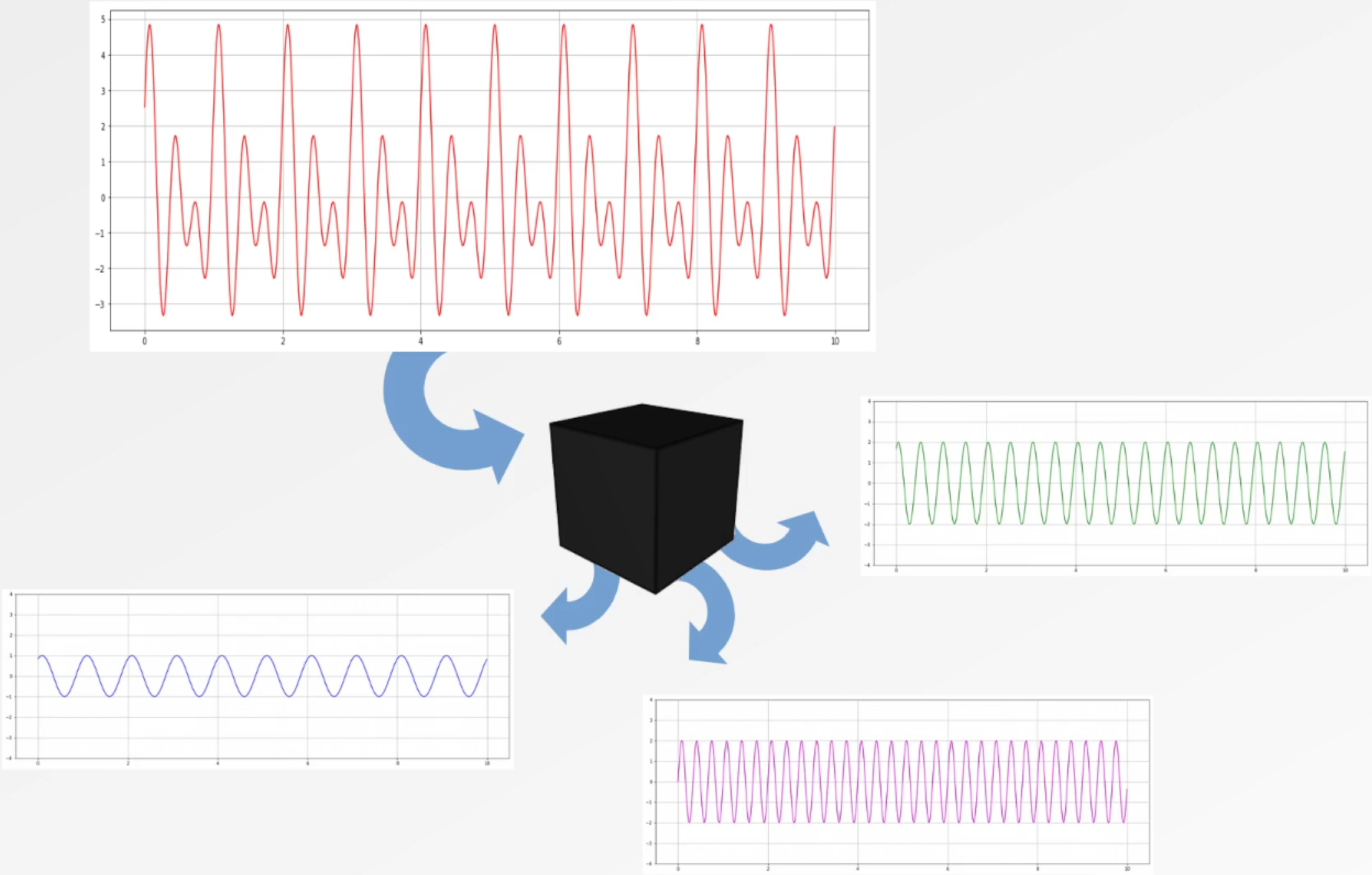




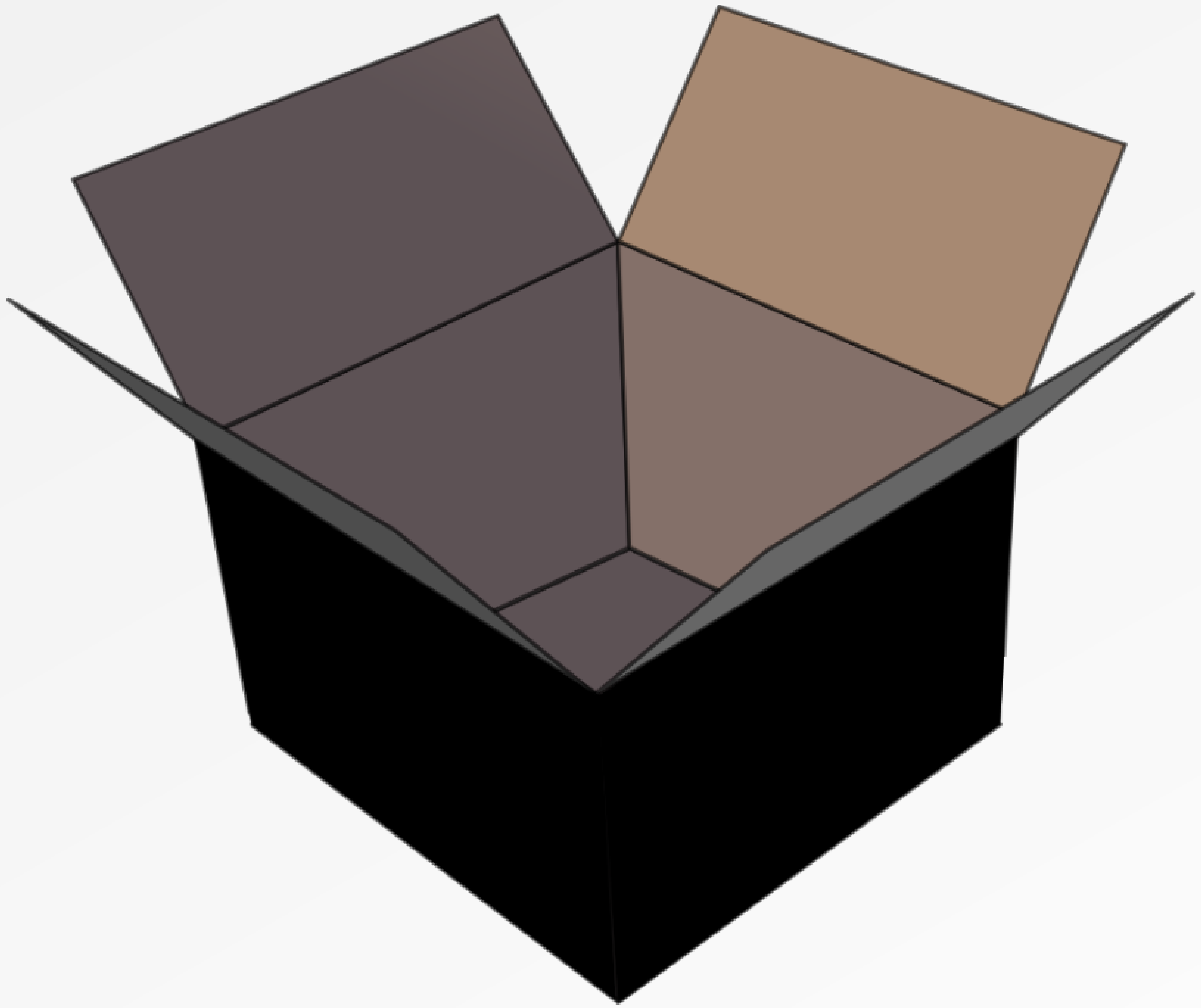




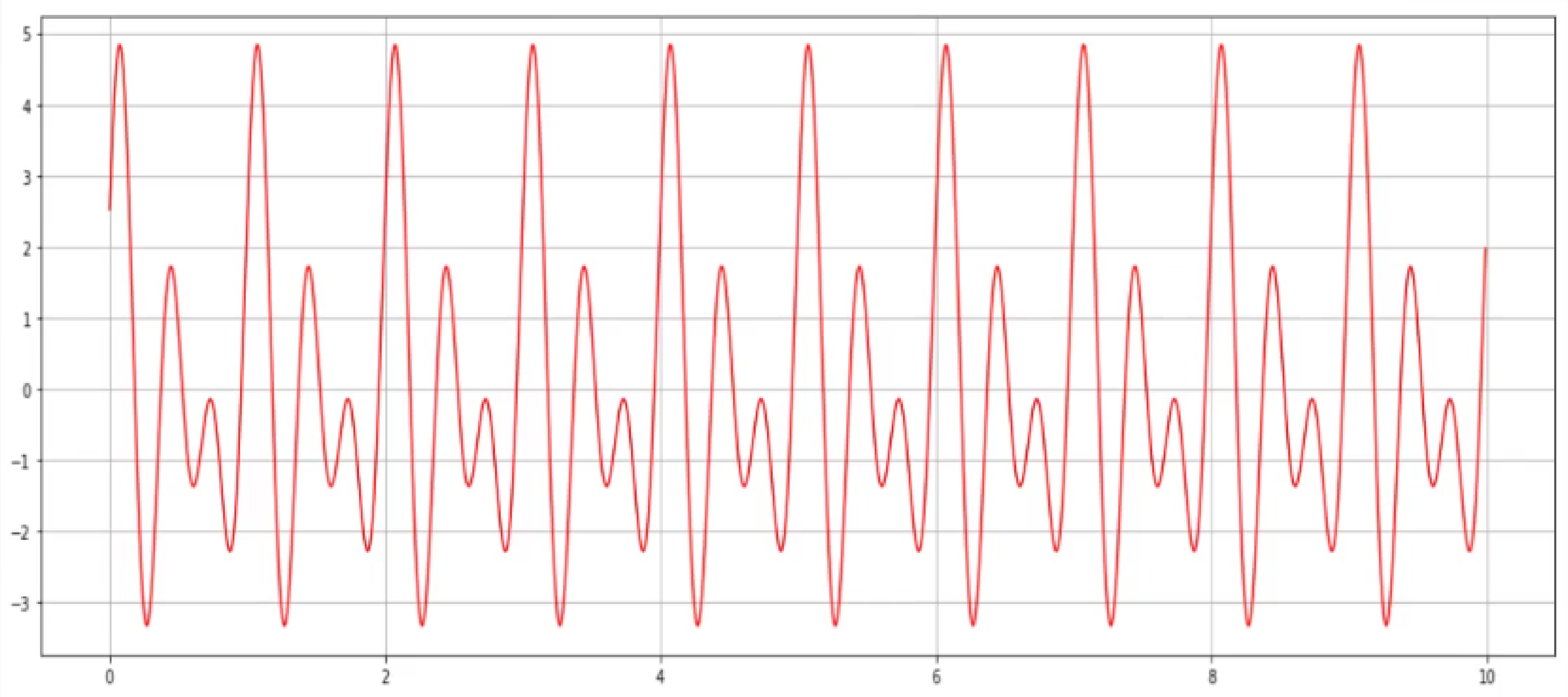
S1 + S2 + S3



5
4
3
2
1
0
-1
-2
-3
0
2
4
6
8
10



- Any continuous signal in the time domain can be represented uniquely and unambiguously by an infinite series of sinusoids



$x(t)$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos 2\pi k t + b_k \sin 2\pi k t\right)$$

$$X(F) = \int_{-\infty}^{+\infty} x(t) e^{-2\pi i F t} \, dt$$

$i^2 = -1$

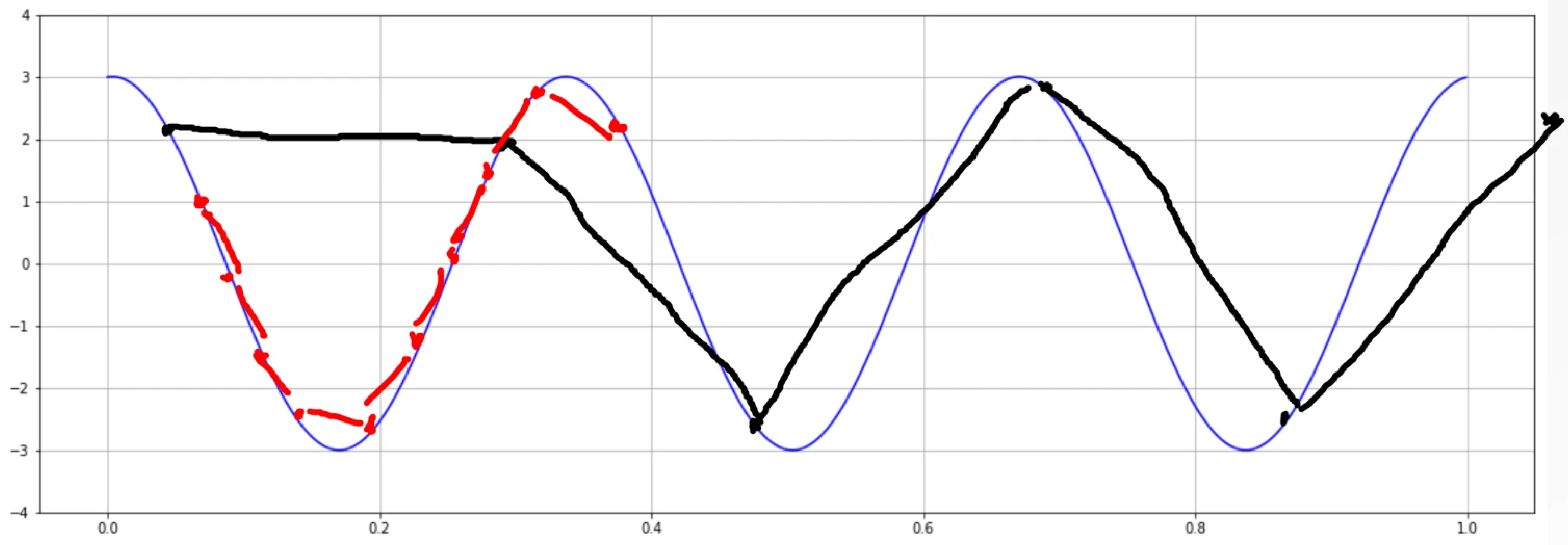
$\sqrt{-1} = i$

$a + bi$

1) How to implement FT
2) How to deal with i
3) How to deal with discrete signals

Sampling a signal

5 Hz



Discrete Fourier Transform

- Any sampled signal of length N can be represented uniquely and unambiguously by a finite series of sinusoids

$x(t)$

$\rightarrow \{x\} = x_0, x_1, \ldots, x_{N-1} \quad |x| = N$

$\rightarrow \{X\} = X_0, X_1, \ldots, X_{N-1}$

- Any sampled signal of length N can be represented uniquely and unambiguously by a finite series of sinusoids

$x(t)$

$\{x\} = x_0, x_1, \ldots, x_{N-1} \quad |x| = N$

$\{X\} = X_0, X_1, \ldots, X_{N-1}$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$a + bi$

$\cos(-\theta) = \cos(\theta)$

$\sin(-\theta) = -\sin(\theta)$

$$X_k = \sum_{n=0}^{N-1} x_n \left[\cos\left(-\frac{2\pi kn}{N}\right) + i\sin\left(-\frac{2\pi kn}{N}\right)\right]$$

$$X_k = \sum_{n=0}^{N-1} x_n \left[\cos\left(\frac{2\pi kn}{N}\right) - i\sin\left(\frac{2\pi kn}{N}\right)\right] =$$

$$= \underbrace{\sum_{n=0}^{N-1} x_n \cos\left(\frac{2\pi kn}{N}\right)}_{A_k} - i\underbrace{\sum_{n=0}^{N-1} \sin\left(\frac{2\pi kn}{N}\right)}_{B_k}$$

Magnitude and phase shift

$$|X_k| = \sqrt{A_k^2 + B_k^2}$$

$$\Phi = \tan^{-1}\left(\frac{B_k}{A_k}\right)$$

Fast Fourier Transform

Fast Fourier Transform

DFT $O(n^2)$

FFT $O(n\log n)$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i n k}{N}}$$

$$X_k = \sum_{n=0}^{\frac{N}{2}-1} x_{2n} e^{-\frac{2\pi i 2nk}{N}} + \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-\frac{2\pi i (2n+1)k}{N}}$$

$$X_k = \sum_{n=0}^{N/2-1} x_{2n} e^{-\frac{2\pi i 2nk}{N}} + \underbrace{e^{-\frac{2\pi i k}{N}}}_{\text{twiddle factor}} \sum_{n=0}^{N/2-1} x_{2n+1} e^{-\frac{2\pi i 2nk}{N}}$$

$$X_{k+\frac{N}{2}} =$$

Audio data classification