Fully Homomorphic Encryption using Ideal Lattices

by Craig Gentry

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Overview

1. Homomorphic Encryption
2. Somewhat Homomorphic Encryption
3. Bootstrapping
4. Fully Homomorphic Encryption

Homomorphic Encryption

Recall a regular PKE scheme: KeyGen(λ), Encrypt(pk, π), and Decrypt(sk, ψ).

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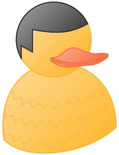

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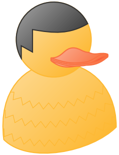

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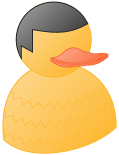

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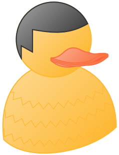

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Homomorphic Encryption

Type	Add/Multiply	Operations	Examples
Partially HE (PHE)	One	Unlimited	RSA, ElGamal, Paillier
Somewhat HE (SHE)	Both	Limited	[Gentry, 2009], Boneh-Goh-Nissim, Melchor-Gaborit-Herranz
Fully HE (FHE)	Both	Unlimited	[Gentry, 2009]

[Gentry, 2009] begins with an unstable lattice-based SHE scheme, and uses **bootstrapping** to derive a FHE scheme.

Somewhat Homomorphic Encryption

Given a **lattice** $\{\mathbf{Bx} : \mathbf{x} \in \mathbb{Z}^n\}$, its **half-open parallelepiped** is $\{\mathbf{Bx} : \mathbf{x} \in [-0.5, 0.5)^n\}$.

$\mathbf{v} \bmod \mathbf{B} = \mathbf{v} - \mathbf{B}\lceil \mathbf{B}^{-1}\mathbf{v} \rfloor$ reduces any point $\mathbf{v}$ onto the half-open parallelepiped.

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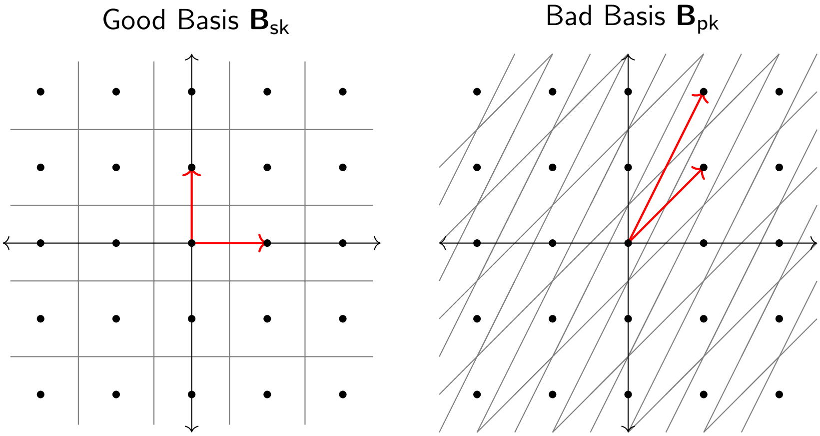

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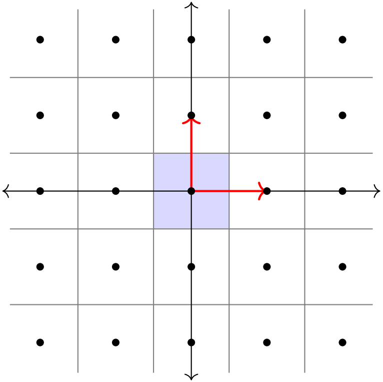

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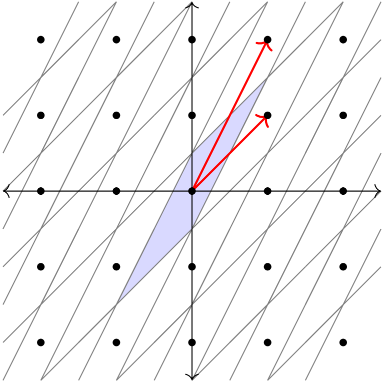

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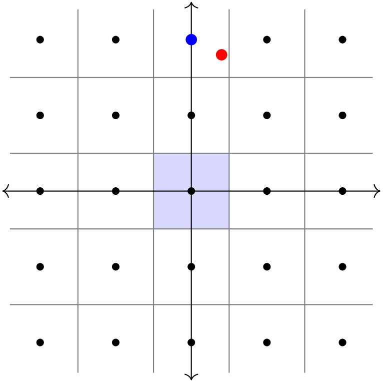

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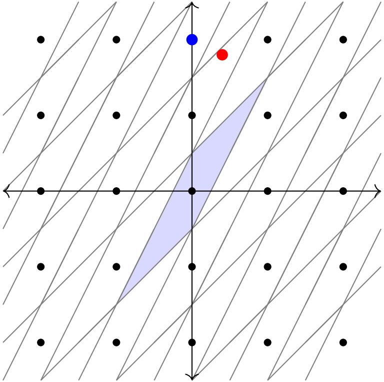

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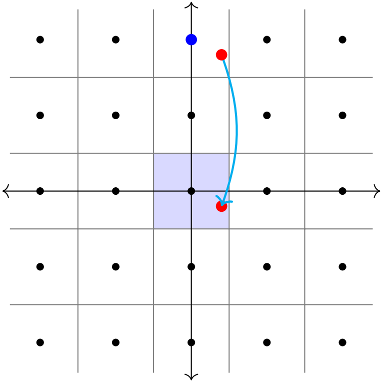

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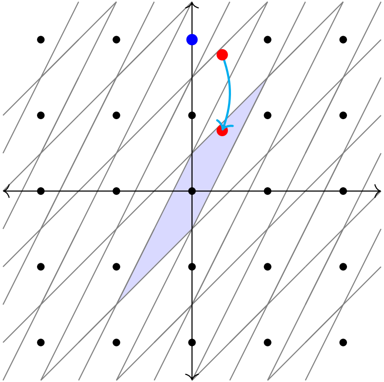

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Let $R = \mathbb{Z}[x]/f(x)$ be a **quotient ring**, where $f(x)$ is a monic polynomial of degree n.

Note that $R \cong \mathbb{Z}^n$, so R and any ideal $I \subset R$ can be viewed as a lattice.

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in R \quad \longleftrightarrow \quad (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}^n$$

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An **ideal lattice** is any subset $I \subset R$ that is closed under addition, and closed under multiplication with R.

Example: $I = 2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$ is an ideal lattice of the ring $\mathbb{Z}$.

Somewhat Homomorphic Encryption

Parameters: Ring $R = \mathbb{Z}[x]/f(x)$ and basis $\mathbf{B}_I$ of a "small" ideal lattice I. The plaintext space is R/I (cosets of I in R), meaning π is a coset ($\pi = r + I$ for some $r \in R$).

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- Choose random, small representative $\pi' \in \pi + I$ with $\|\pi'\| \leq r_{\mathsf{Enc}}$
- Compute $\psi = \pi' \bmod \mathbf{B}_{\mathsf{pk}}$, mapping to the half-open parallelepiped

$\pi \leftarrow \mathsf{Decrypt}(\mathbf{B}_{\mathsf{sk}}, \psi)$

- Compute $\pi' = \psi \bmod \mathbf{B}_{\mathsf{sk}}$, which recovers the small representative if $\|\pi'\| \leq r_{\mathsf{Dec}}$
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$\psi_+ \leftarrow \mathsf{Add}(\mathbf{B}_{\mathsf{pk}}, \psi_1, \psi_2)$ outputs $(\psi_1 + \psi_2) \bmod \mathbf{B}_{\mathsf{pk}}$
$\psi_\times \leftarrow \mathsf{Multiply}(\mathbf{B}_{\mathsf{pk}}, \psi_1, \psi_2)$ outputs $(\psi_1 \cdot \psi_2) \bmod \mathbf{B}_{\mathsf{pk}}$

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Encryption-decryption:

$$\begin{aligned}
\mathsf{Decrypt}(\mathbf{B}_{\mathsf{sk}}, \mathsf{Encrypt}(\mathbf{B}_{\mathsf{pk}}, \pi)) &= \mathsf{Decrypt}(\mathbf{B}_{\mathsf{sk}}, \pi' \bmod \mathbf{B}_{\mathsf{pk}}) \\
&= ((\pi' \bmod \mathbf{B}_{\mathsf{pk}}) \bmod \mathbf{B}_{\mathsf{sk}}) \bmod \mathbf{B}_I \\
&= \pi' \bmod \mathbf{B}_I && (\text{if } \|\pi'\| \leq r_{\mathsf{Enc}}, r_{\mathsf{Dec}}) \\
&= \pi + I
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$\mathrm{Add}(\mathbf{B}_{\mathrm{pk}}, \psi_1, \psi_2)$ can be decrypted if $\pi'_1 + \pi'_2$ is in the $\mathbf{B}_{\mathrm{sk}}$ parallelepiped.

$$\|\pi'_1 + \pi'_2\| \leq \|\pi'_1\| + \|\pi'_1\| \leq 2 \cdot r_{\mathrm{Dec}}$$

$\mathrm{Multiply}(\mathbf{B}_{\mathrm{pk}}, \psi_1, \psi_2)$ can be decrypted if $\pi'_1 \cdot \pi'_2$ is in the $\mathbf{B}_{\mathrm{sk}}$ parallelepiped.

$$\|\pi'_1 \cdot \pi'_2\| \leq \gamma_{\mathrm{Multiply}}(R) \cdot \|\pi'_1\| \cdot \|\pi'_1\|$$

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$$\begin{tikzcd}
 & \psi_{\mathsf{sk}} = \mathsf{Encrypt}(\mathsf{pk}, \mathsf{sk}) \arrow[d] & \\
\psi \arrow[r] & \boxed{\mathsf{Refresh}(\psi_{\mathsf{sk}}, \psi)} \arrow[r] & \psi' \\
\text{Saturated CT} & & \text{Error-Free CT}
\end{tikzcd}$$

Fully Homomorphic Encryption

3 modifications to SHE scheme:

1. Simplify $\text{Decrypt}(\mathbf{B}_{\text{sk}}, \psi) = (\psi \bmod \mathbf{B}_{\text{sk}}) \bmod \mathbf{B}_{\text{I}}$ to $\text{Decrypt}(\mathbf{v}_{\text{sk}}, \psi) = (\psi - \mathbf{v}_{\text{sk}} \times \psi) \bmod 2$
2. Reduce decryption radius $r_{\text{Dec}} \to r_{\text{Dec}}/2$. This means ψ is required to be within $r_{\text{Dec}}/2$ of a lattice point.
3. Use a hint to "squash" the decryption: the encrypter helps with computing the decryption even before messages are received.

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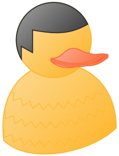

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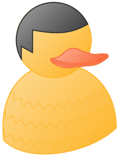

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$$\psi = \text{Evaluate}(\text{pk}, C_{i\dots i+k}, \boldsymbol{\psi})$$
$$\psi = \text{Refresh}(\psi_{\text{sk}}, \psi)$$
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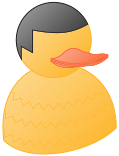

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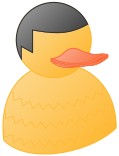

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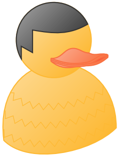

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References

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