CS419-Assignment 1

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November 12, 2023

Question 1

(a)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \tag{1}$$

Let $z = x - y$, then $dz = -dy$. Also since y ranges from $-\infty$ to ∞, z now ranges from ∞ to $-\infty$

$$\int_{\infty}^{-\infty} f(x-z)g(z)\,(-dz) = \int_{-\infty}^{\infty} f(x-z)g(z)\,dz \tag{2}$$

This concludes the following

$$\int_{\infty}^{-\infty} g(z)f(x-z)\,dz = (g * f)(x) \tag{3}$$

$$(f * g)(x) = (g * f)(x) \tag{4}$$

(b)

To prove that the cross-correlation operation is not commutative, we need to show that $(f*g)(x) \neq (g * f)(x)$ for some functions.

Let's define two partial functions f and g as follows:

$$f(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$g(t) = \begin{cases} 1 & \text{if } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the cross-correlation $(f * g)(x)$ is given by:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y) \cdot g(x+y)\,dy$$

Evaluating $(f * g)(x)$:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y) \cdot g(x+y)\,dy = f(0) \cdot g(x) = g(x)$$

$$(f * g)(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for $(g * f)(x)$:

$$(g * f)(x) = \int_{-\infty}^{+\infty} g(y) \cdot f(x + y)\, dy = g(1) \cdot f(x + 1) = f(x + 1)$$

$$(g * f)(x) = \begin{cases} 1 & \text{if } x = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(f * g)(x) \neq (g * f)(x)$ for all x, we have shown that the cross-correlation operation is not commutative.

(c)

$$((f * g) * h)(x) = \int_{-\infty}^{\infty} (f * g)(y) h(x - y)\, dy \tag{5}$$

$$((f * g) * h)(x) = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(z) g(y - z)\, dz) h(x - y)\, dy \tag{6}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) g(y - z) h(x - y)\, dz dy \tag{7}$$

By Fubini's Theorem, we can interchange the order of integration

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) g(y - z) h(x - y)\, dy dz \tag{8}$$

$$\int_{-\infty}^{\infty} f(z) (\int_{-\infty}^{\infty} g(y - z) h(x - y)\, dy) dz \tag{9}$$

Let $w = y - z$ therefore dy = dw

$$\int_{-\infty}^{\infty} f(z) (\int_{-\infty}^{\infty} g(w) h(x - z - w)\, dw) dz \tag{10}$$

$$\int_{-\infty}^{\infty} f(z) (g * h)(x - z) dz = (f(g * h))(x) \tag{11}$$

$$((f * g) * h)(x) = (f(g * h))(x) \tag{12}$$

Question 2

Given the rotated coordinates:

$$x' = x \cos(\theta) - y \sin(\theta) \tag{13}$$

$$y' = x \sin(\theta) + y \cos(\theta) \tag{14}$$

Now, differentiate f with respect to x and y in terms of x' and y':

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial x} = \frac{\partial f}{\partial x'} \cos(\theta) + \frac{\partial f}{\partial y'} \sin(\theta) \tag{15}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y} = \frac{\partial f}{\partial x'}(-\sin(\theta)) + \frac{\partial f}{\partial y'}\cos(\theta) \tag{16}$$

Calculate the second derivatives with respect to x and y:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\cos(\theta) + \frac{\partial f}{\partial y'}\sin(\theta)\right) = cos(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\right) + \sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) \tag{17}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos^2(\theta)\frac{\partial^2 f}{\partial x'^2} + \sin^2(\theta)\frac{\partial^2 f}{\partial y'^2} + 2\cos(\theta)\sin(\theta)\frac{\partial^2 f}{\partial x'\partial y'} \tag{18}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x'}(-\sin(\theta)) + \frac{\partial f}{\partial y'}\cos(\theta)\right) = (-\sin(\theta))\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x'}\right) + \cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right) \tag{19}$$

$$\frac{\partial^2 f}{\partial x^2} = \sin^2(\theta)\frac{\partial^2 f}{\partial x'^2} + \cos^2(\theta)\frac{\partial^2 f}{\partial y'^2} + 2\cos(\theta)\sin(\theta)\frac{\partial^2 f}{\partial x'\partial y'} \tag{20}$$

Combine the second derivatives with respect to x and y to obtain the Laplacian in the rotated coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (\sin^2(\theta) + \cos^2(\theta))\frac{\partial^2 f}{\partial x'^2} + (\cos^2(\theta) + \sin^2(\theta))\frac{\partial^2 f}{\partial y'^2} \tag{21}$$

Therefore we conclude that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \tag{22}$$

Question 3

A filter h is considered linear if it satisfies the two properties of linearity: additivity and homogeneity. Let's check these properties for the given filter function.

(a) Additivity

A filter is additive if $h((f+g)(x,y)) = h(f(x,y)) + h(g(x,y))$

$$\begin{aligned} h((f+g)(x,y)) = 3&(f+g)(x,y) \\ &+ 2(f+g)(x-1,y) \\ &+ 2(f+g)(x+1,y) \\ &- 17(f+g)(x,y-1) \\ &+ 99(f+g)(x,y+1) \end{aligned} \tag{23}$$

$$\begin{aligned} h((f+g)(x,y)) = 3&f(x,y) + 3g(x,y) \\ &+ 2f(x-1,y) + 2g(x-1,y) \\ &+ 2f(x+1,y) + 2g(x+1,y) \\ &- 17f(x,y-1) - 17g(x,y-1) \\ &+ 99f(x,y+1) + 99g(x,y+1) \end{aligned} \tag{24}$$

$$h((f+g)(x,y)) = (3f(x,y)+2f(x-1,y)+2f(x+1,y)-17f(x,y-1)+99f(x,y+1)) + (3g(x,y)+2g(x-1,y)+2g(x+1,y)-17g(x,y-1)+99g(x,y+1)) \quad (25)$$

This concludes the additivity of the filter

$$h((f+g)(x,y)) = h(f(x,y)) + h(g(x,y)) \quad (26)$$

(b) Homogeneity

A filter is homogeneous if $h(\alpha f) = \alpha h(f)$ for any image f and any scalar α

$$h(\alpha f) = 3(\alpha f)(x,y)+2(\alpha f)(x-1,y)+2(\alpha f)(x+1,y)-17(\alpha f)(x,y-1)+99(\alpha f)(x,y+1) \quad (27)$$

$$h(\alpha f) = \alpha(3f(x,y)+2f(x-1,y)+2f(x+1,y)-17f(x,y-1)+99f(x,y+1)) \quad (28)$$

This concludes the homogeneity of the filter

$$h(\alpha f) = \alpha h(f) \quad (29)$$

Since the mask is both additive and homogeneous, we can say it is linear.

The terms in the filters are actually the elements of convolution mask. In other terms, if we consider f(x, y) as the center of the mask, f(x-1, y) corresponds to the top neighbor pixel and f(x, y-1) corresponds to the left neighbor pixel. By considering this the convolution mask becomes like this:

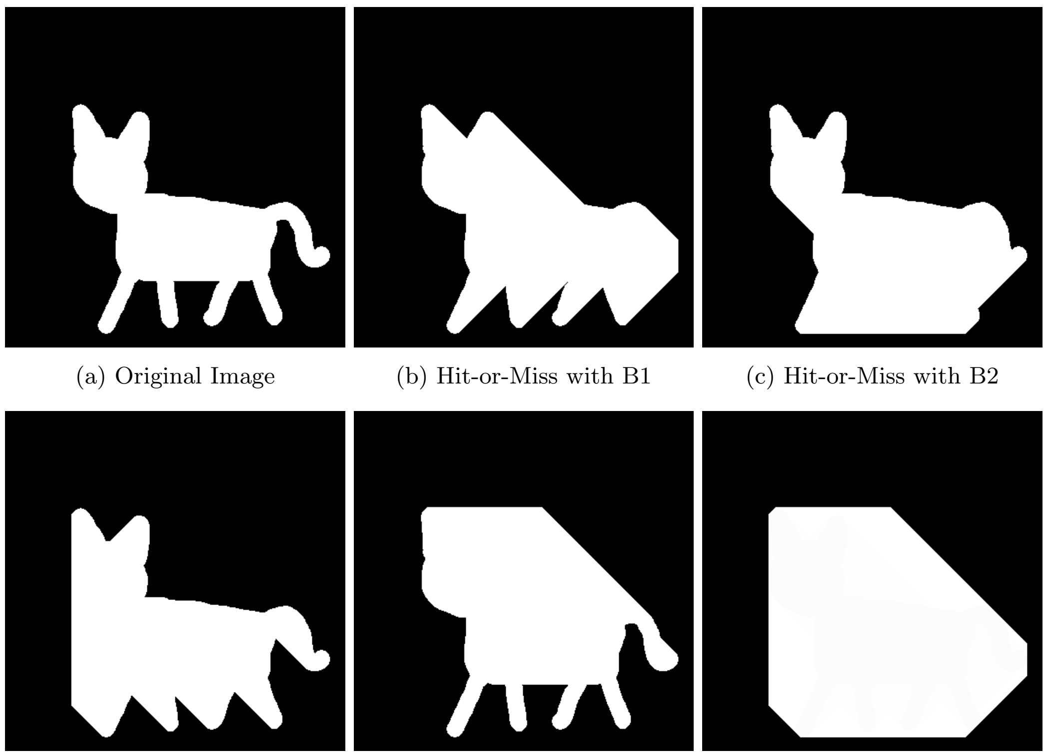
$$\text{w:} \begin{bmatrix} 0 & 2 & 0 \\ -17 & 3 & 99 \\ 0 & 2 & 0 \end{bmatrix}$$

The convolution mask is the rotated version of the filter by 180 degrees. Therefore the mask becomes:

$$\text{Convolution mask:} \begin{bmatrix} 0 & 2 & 0 \\ 99 & 3 & -17 \\ 0 & 2 & 0 \end{bmatrix}$$

Question 4

By implementing the algorithm stated in the book of Gonzalez and Woods, and using the structuring elements they provide, I created a convex hull of the perfectly_painted_cat.png. It is not a perfect example of the convex hull, but it still is. The results can be seen from the Figure 1



(a) Original Image (b) Hit-or-Miss with B1 (c) Hit-or-Miss with B2

(d) Hit-or-Miss with B3 (e) Hit-or-Miss with B4 (f) Result

Figure 1: Convex Hull Algorithm