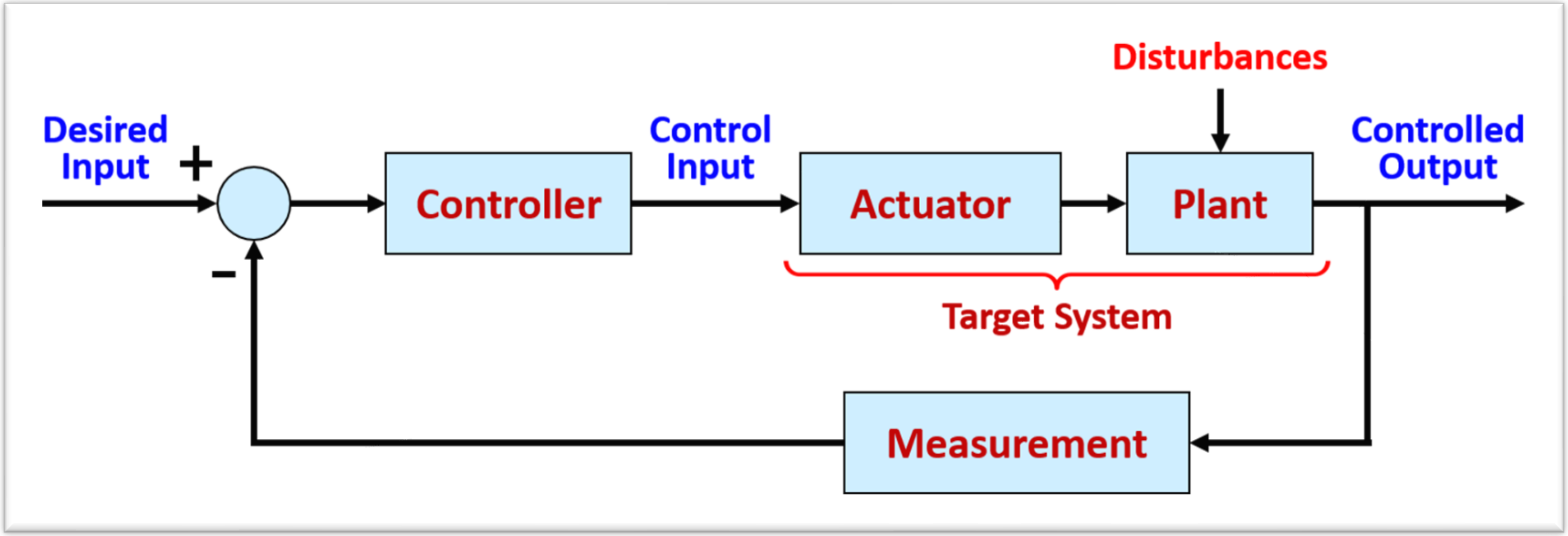
The Frequency-Response Design Method 1

Lecture 10:

- Frequency Response Analysis



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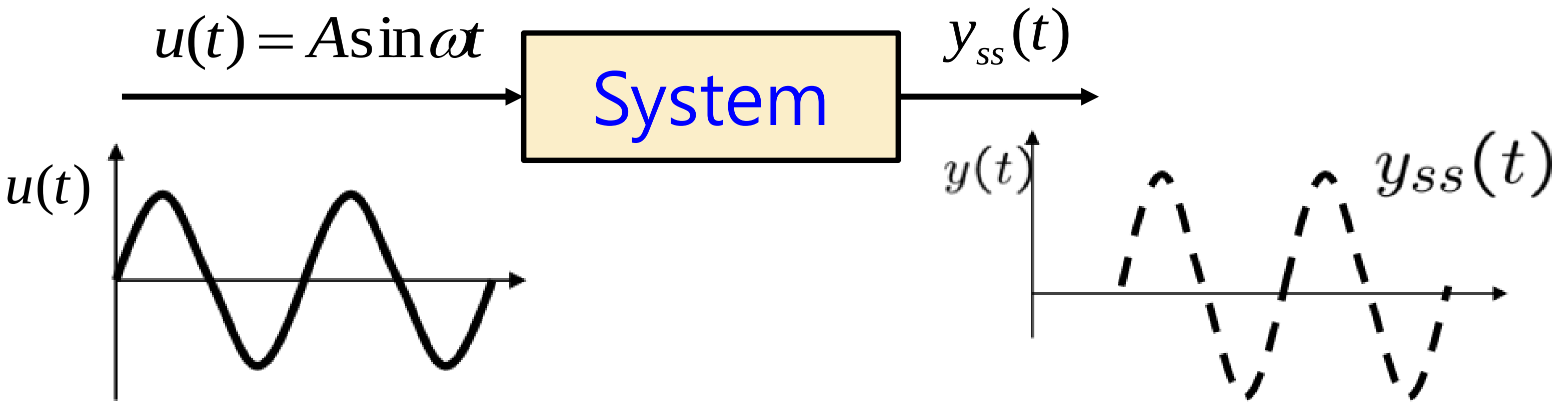
What is Frequency Response ??



- We would like to analyze a system property by applying a test sinusoidal input {u(t)} and observing a response {y(t)}.

- Steady-state response $y_{ss}(t)$ (after transient dies) of a system to sinusoidal inputs is called **frequency response**.

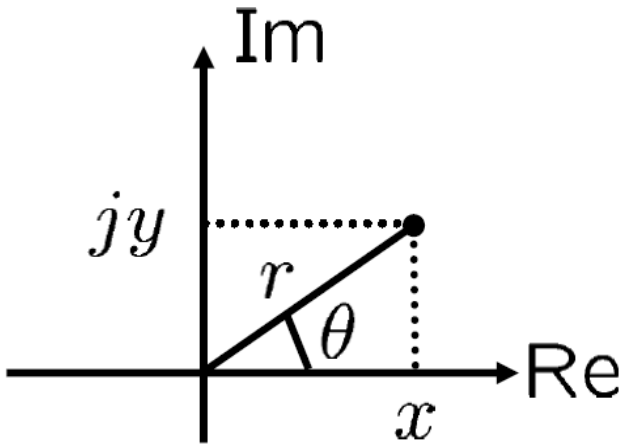
Complex Numbers (Review)

- **Euler's Identity (or Equation)**

$$\boxed{e^{j\theta} := \cos\theta + j\sin\theta} \quad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2},\ \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

- **Polar form** $\boxed{s := x + jy = re^{j\theta}}$

$$\boxed{\frac{1}{s} = s^{-1} = \frac{1}{x+jy} = \frac{1}{r}e^{-j\theta}}$$

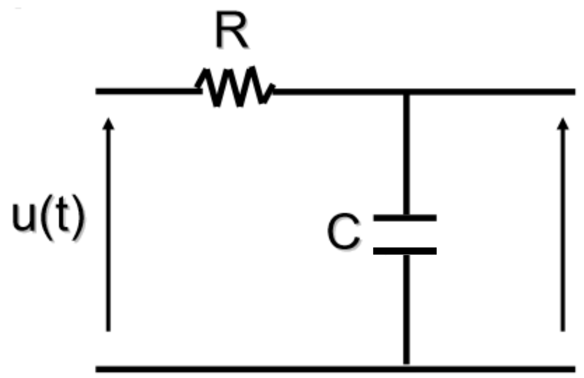


- **Magnitude** $r = \sqrt{x^2+y^2}$
- **Phase** $\theta = \tan^{-1}(y/x)$
- **Example** $s_1 = r_1e^{j\theta_1},\ s_2 = r_2e^{j\theta_2}$

$$\boxed{s_1s_2 = r_1r_2e^{j(\theta_1+\theta_2)}} \quad \boxed{\frac{s_1}{s_2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}}$$

A Simple Example

- **RC Circuit**



KCL & KVL

$$\begin{cases} i(t) = i_R(t) = i_C(t) \\ V_{input}(t) = V_R(t) + V_C(t) \end{cases}$$

Laplace Transform

$$\begin{cases} V_{input}(s) = (R + \frac{1}{Cs})I(s) \\ V_{output}(s) = V_C(s) = \frac{1}{Cs} I(s) \end{cases} \Rightarrow G(s) = \frac{V_{output}(s)}{V_{input}(s)} = \frac{1}{RCs + 1}$$

$$where,\ i(t) = C \frac{dV_C(t)}{dt}$$

- **Now, let us take a sinusoidal input voltage u(t).**
- **Then, what is the output voltage y(t) ??**

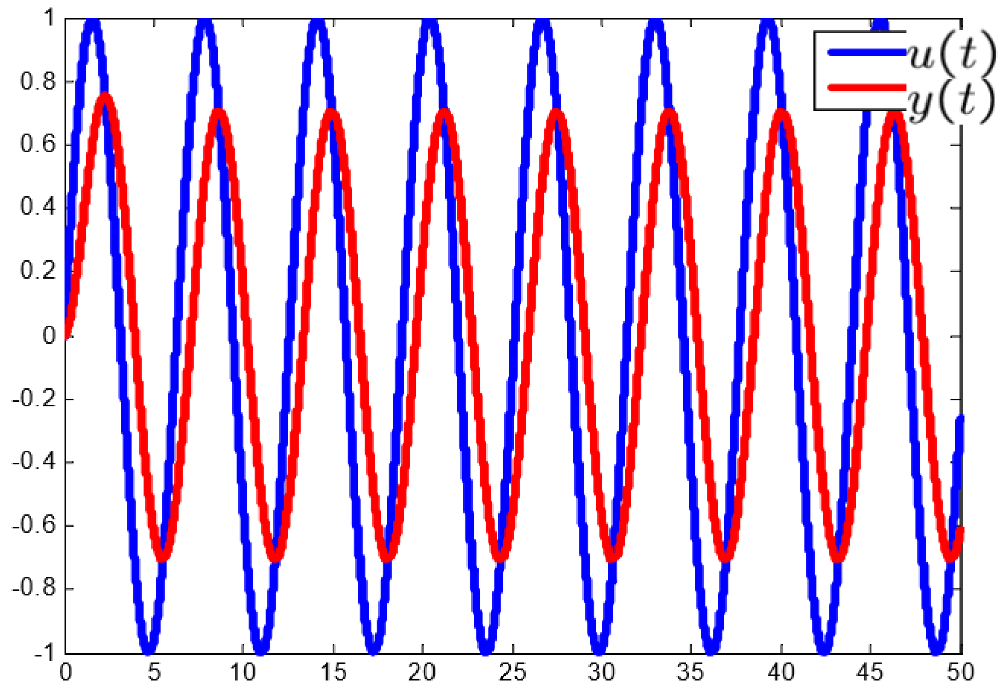
A Simple Example (cont'd)

$$G(s) = \frac{V_{output}(s)}{V_{input}(s)} = \frac{1}{RCs+1}$$

- **Let R = C = 1,**

$$G(s) = \frac{V_{output}(s)}{V_{input}(s)} = \frac{1}{s+1}$$

- **Let u(t) = sin(t)**



- **Important findings:**
  (1) At **steady-state**, u(t) and y(t) has the <u>**same frequency**</u> !!
  (2) But, <u>**different amplitude & phase**</u> !!

A Simple Example (cont'd)

- Derivation of y(t) for u(t) = sin(t)

$$Y(s) = G(s)\boxed{U(s)} = \underbrace{\frac{1}{s+1}\cdot\boxed{\frac{1}{s^2+1}}}_{\text{Partial fraction expansion}} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

$\mathcal{L}$

e^{-at}	$\leftrightarrow$	$\frac{1}{s+a}$
$\sin bt$	$\leftrightarrow$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\leftrightarrow$	$\frac{s}{s^2+b^2}$

- Then, taking inverse Laplace transform

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

0 (zero) as t → ∞

$$y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \boxed{\frac{1}{\sqrt{2}}}\sin(t\,\boxed{})$$

Amplitude Phase

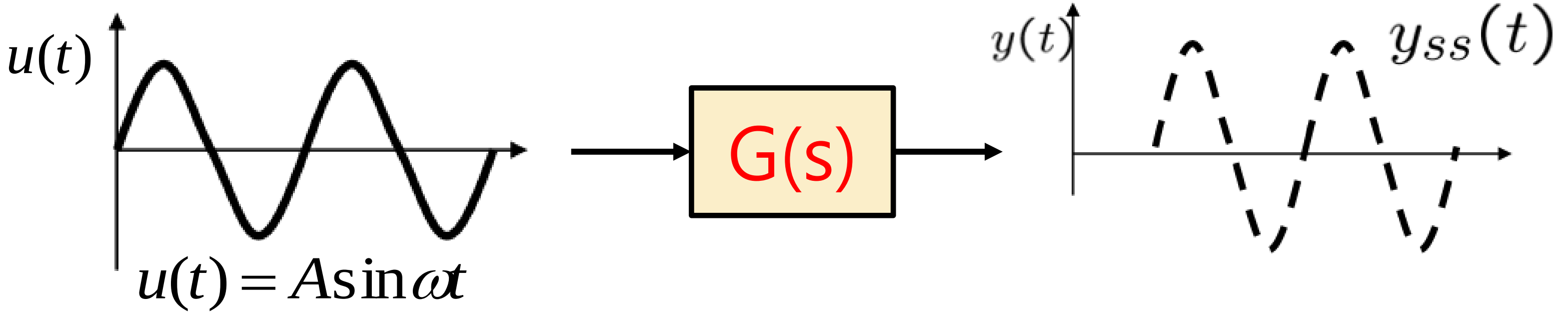
where, $a\sin t + b\cos t = \sqrt{a^2+b^2}\,\sin(t+\alpha)$

where, $\cos\alpha = a/\sqrt{a^2+b^2}$, $\sin\alpha = b/\sqrt{a^2+b^2}$

$\alpha = \tan^{-1}(b/a)$

Response to Sinusoidal Input

- How is the steady-state output of a linear system {G(s)} when the input is sinusoidal?



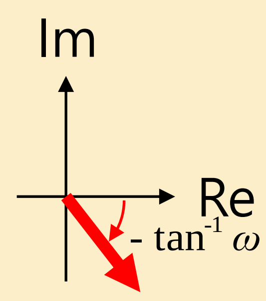
- **Steady-state output:** $y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$
  - **Frequency** is the same as the **input frequency ω**.
  - **Amplitude** is that of input (A) multiplied by $|G(j\omega)|$.
  - **Phase** shifts $\angle G(j\omega)$.

(or Gain)

Frequency Response Function (FRF)

- For a stable system G(s), **G(jω)** (ω is positive) is called **Frequency Response Function** (FRF)

- FRF is a complex number, and thus has (1) an **amplitude** and (2) a **phase**.



$$G(j\omega) = \frac{1}{1+\omega^2}(1 - j\omega)$$

- Example 1: First-order system,

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1} = \frac{1}{1 + j\omega}\frac{1 - j\omega}{1 - j\omega} = \frac{1}{1+\omega^2}(1 - j\omega)$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} & \textbf{: Amplitude} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$

Frequency Response Function (FRF)

- **Example 2: Second-order system**

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \qquad\qquad = -\tan^{-1}\dfrac{3\omega}{2-\omega^2} \end{cases}$$

First-Order System Example (Revisited)

- **FRF**

$$G(j\omega) = \frac{1}{j\omega + 1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -\tan^{-1}\omega$$

frequency ω	amplitude $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0	1	0°
0.5	0.894	-26.6°
1.0	0.707	-45°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°

- **Two graphs representing FRF**
  - Bode diagram (**Bode plot**)
  - Nyquist diagram (**Nyquist plot**)

Bode Diagram (or Plot) of G(jω)

Hendrik Wade Bode (/ˈboʊdi/ *boh-dee*; Dutch: [ˈbodə])[1] (December 24, 1905 – June 21, 1982)[1] was an American engineer, researcher, inventor, author and scientist, of Dutch ancestry. As a pioneer of modern control theory and electronic telecommunications he revolutionized both the content and methodology of his chosen fields of research.

He made important contributions to the design, guidance and control of anti-aircraft systems during World War II and, continuing post-World War II during the Cold War, to the design and control of missiles and anti-ballistic missiles.[2]

He also made important contributions to control system theory and mathematical tools for the analysis of stability of linear systems, inventing Bode plots, gain margin and phase margin.

Bode was one of the great engineering philosophers of his era.[3] Long respected in academic circles worldwide,[4][5] he is also widely known to modern engineering students mainly for developing the asymptotic magnitude and phase plot that bears his name, the Bode plot.

His research contributions in particular were not only multidimensional but far reaching as well, extending as far as the U.S. space program.[6][7][8]

Hendrik Wade Bode

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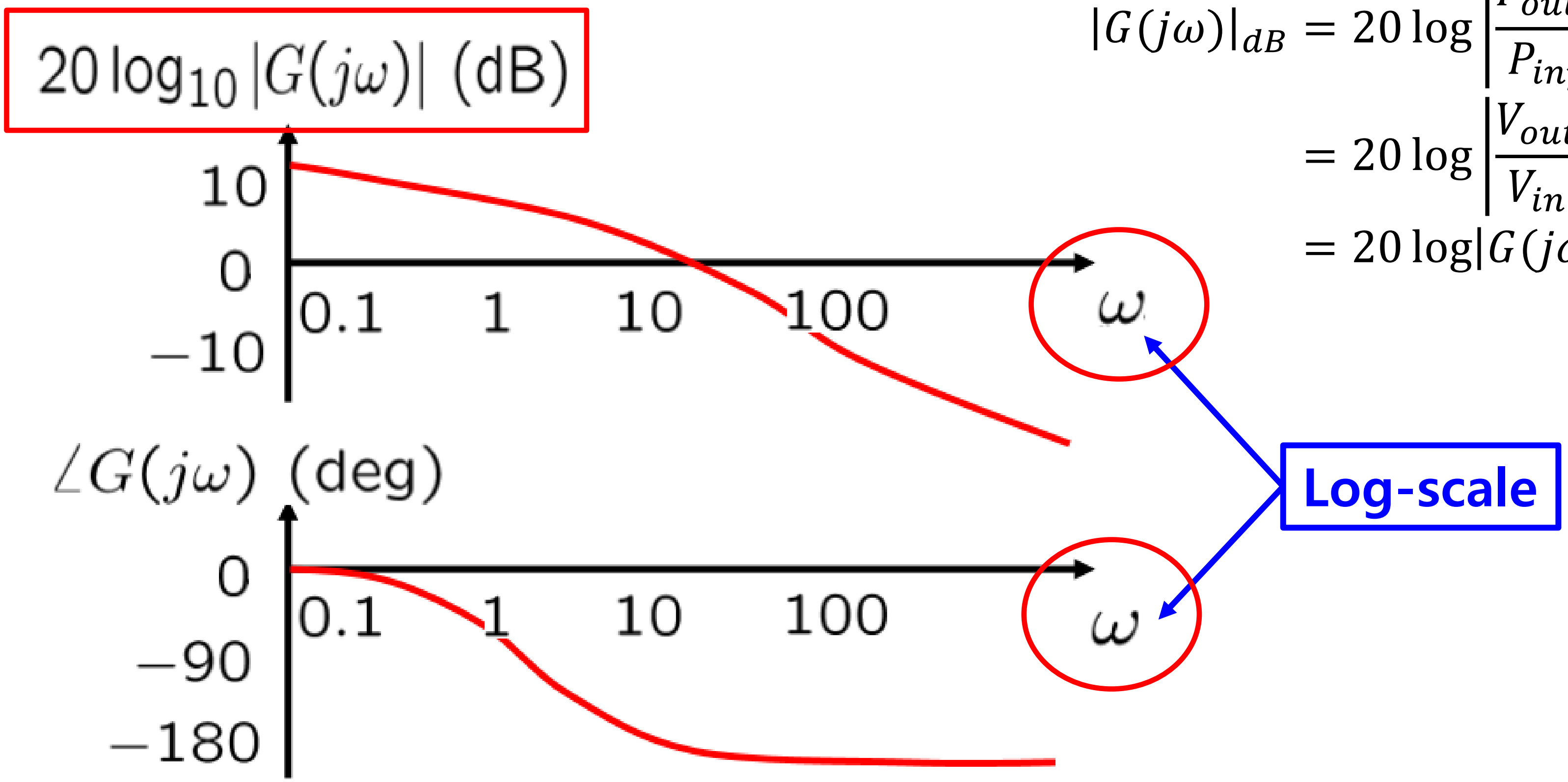
Born	December 24, 1905 Madison, Wisconsin
Died	June 21, 1982 (aged 76) Cambridge, Massachusetts
Residence	Cambridge, Massachusetts
Nationality	American
Alma mater	Ohio State University Columbia University
Known for	Bode plot, Control theory, Telecommunications

https://en.wikipedia.org/wiki/Hendrik_Wade_Bode

Bode Diagram of G(jω) (cont'd)

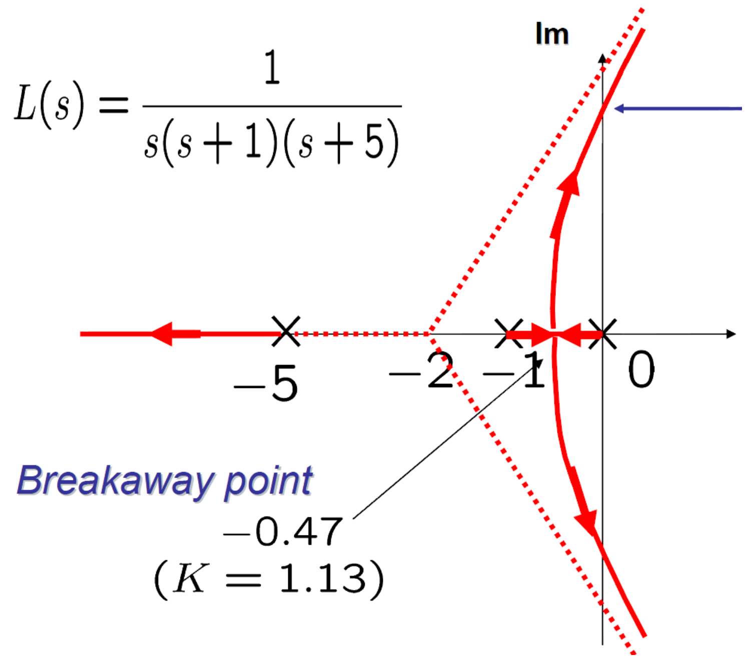
- **Bode plot consists of Gain Plot & Phase Plot w.r.t. ω**

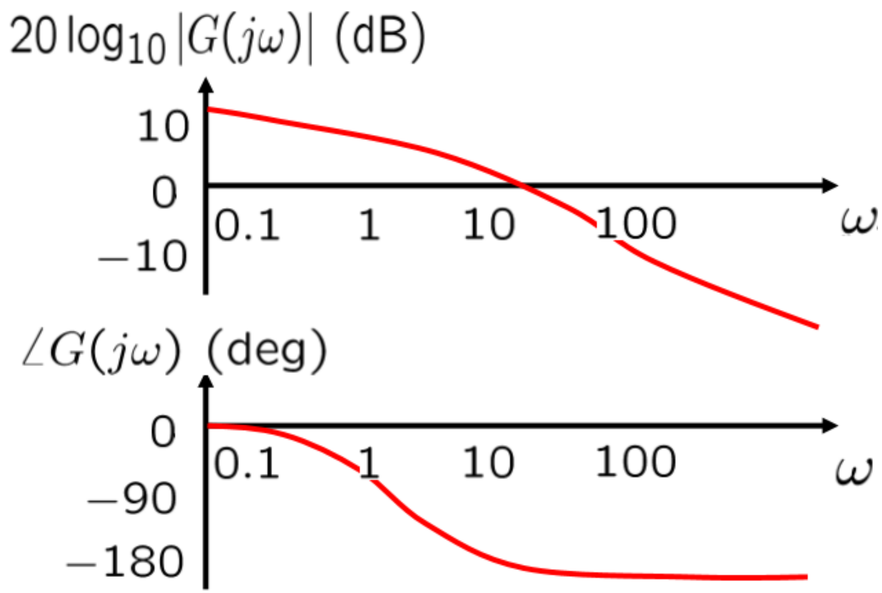


$$
\begin{aligned}
|G(j\omega)|_{dB} &= 20\log\left|\frac{P_{output}}{P_{input}}\right| \\
&= 20\log\left|\frac{V_{out}}{V_{in}}\right| \\
&= 20\log|G(j\omega)|
\end{aligned}
$$

Bode Diagram of G(jω) (cont'd)

- **Root Locus vs. Bode Diagram ??**





- **Important Questions?**
  - What are the same things?
  - But, what are the main Benefits of Bode Plots?

Remarks on Bode Diagram

- Bode diagram shows amplification and phase shift of a system output for sinusoidal inputs with various frequencies.

- It is very useful and important in analysis and design of control systems.

- The shape of Bode plot contains information of stability, time responses with respect to frequency!

- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)

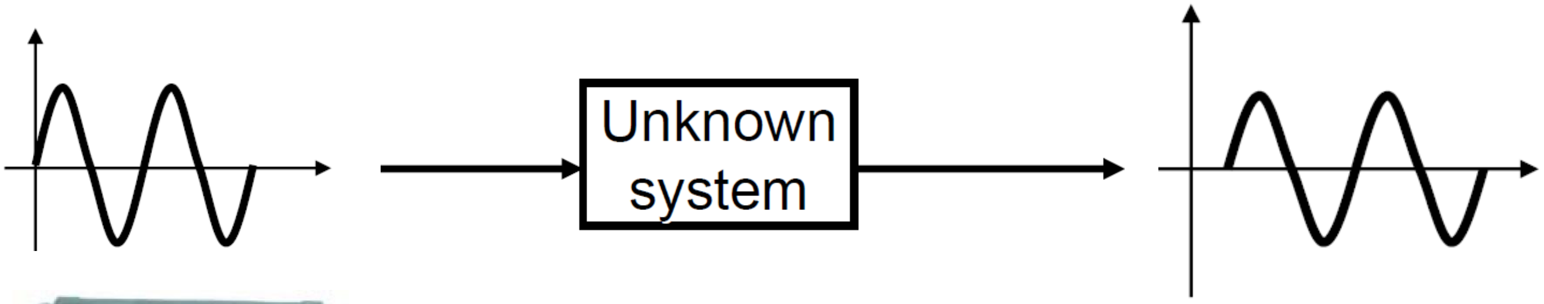
System Identification,

- Sweep frequencies of sinusoidal signals to unknown system and obtain FRF data (i.e., gain and phase).

- Select G(s) so that G(jω) fits the FRF data.

Generate sine signals
Sweep frequencies
(ex, 0.1Hz ~ 100Hz)

Collect FRF data
& Select G(s)



Sketching Bode Plot

- **Basic functions:**
  - Constant gains
  - Differentiator and integrator
  - Double integrator
  - First-order system and its inverse
  - Second-order system

- **Product of basic functions:**
  - Sketch Bode plot of each factor, and
  - Add the Bode plot graphically.

  ☞ **Main advantage of Bode plot !!**

Hendrik Wade Bode

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Residence	Cambridge, Massachusetts
Nationality	American
Alma mater	Ohio State University Columbia University
Known for	Bode plot, Control theory, Telecommunications
Awards	Richard E. Bellman Control Heritage Award (1979) Rufus Oldenburger Medal (1975) President's Certificate of Merit Edison Medal (1969) Ernest Orlando Lawrence Award (1960)

Bode Plot (1): Constant Gain

- **TF**

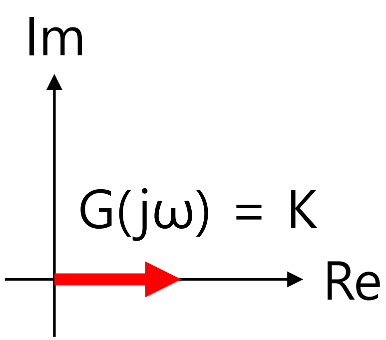
$$G(s) = K$$

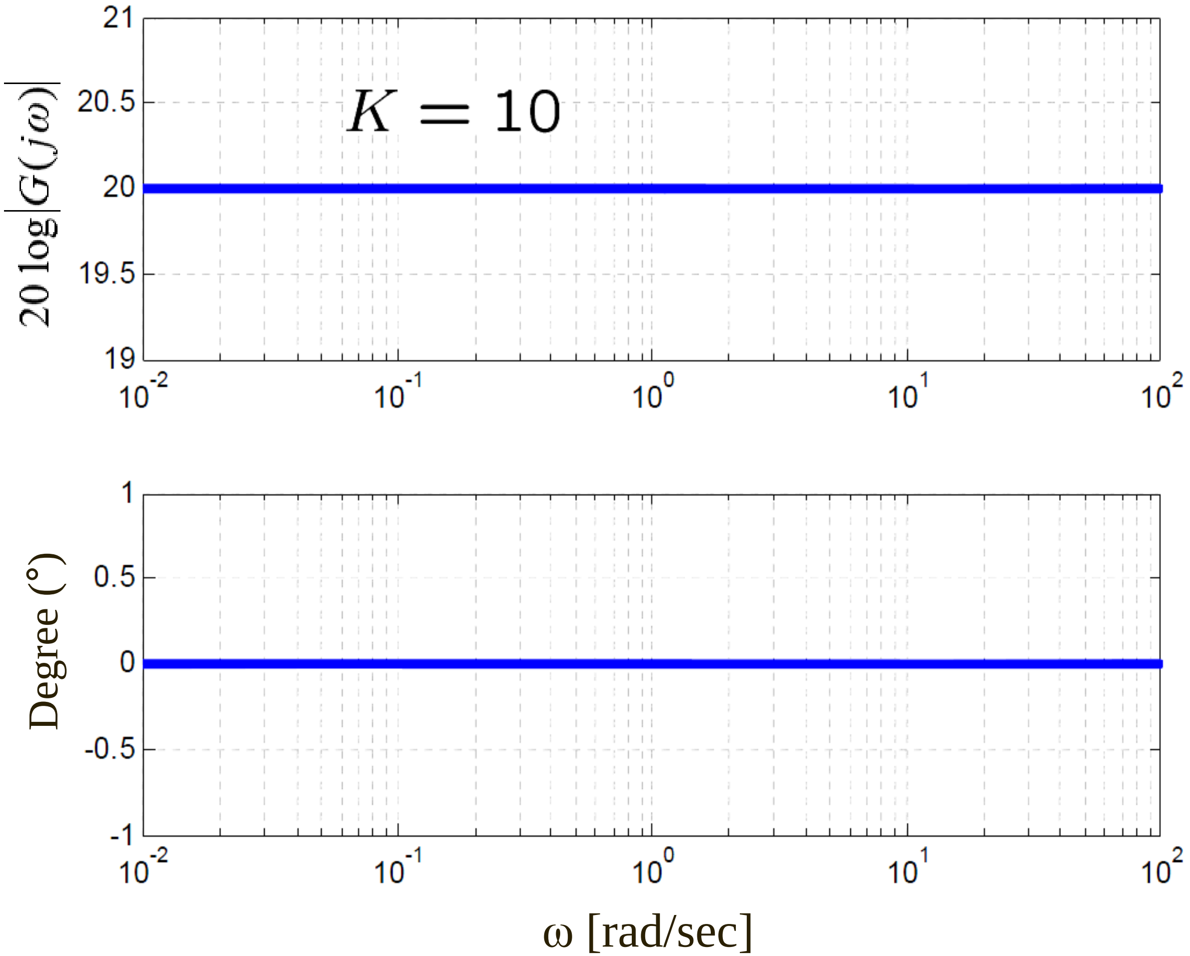
- **Amplitude (gain)**

$$|G(j\omega)| = K$$

- **Phase**

$$\angle G(j\omega) = 0^{\circ}$$





Bode Plot (2): Differentiator

- **TF**

  $$G(s) = s$$

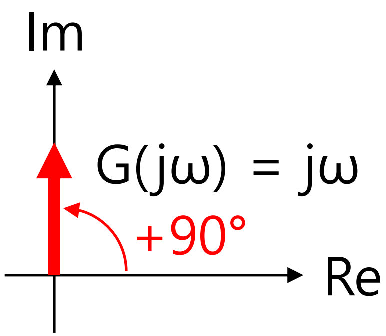
  - **Amplitude (gain)**

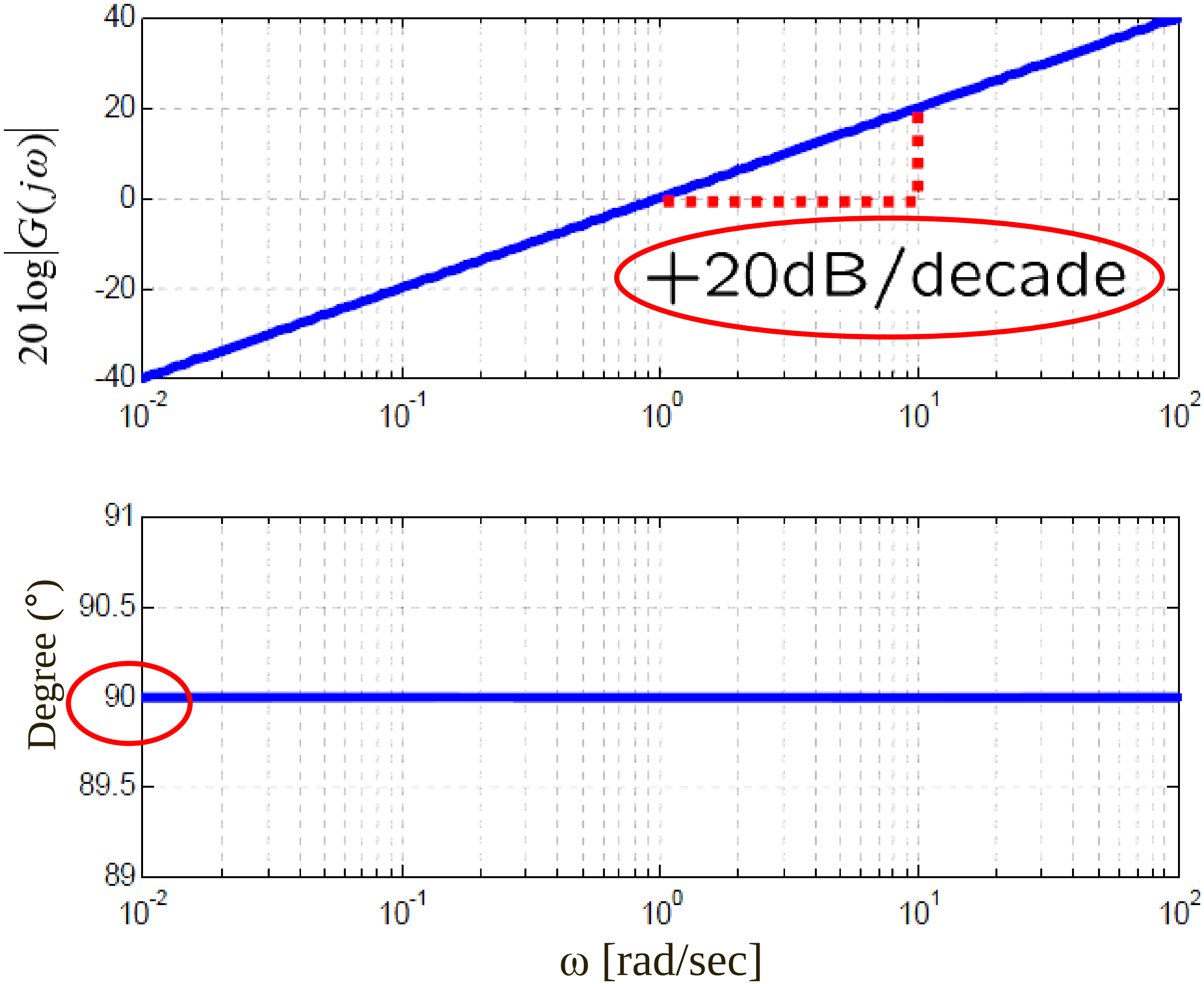
    $$|G(j\omega)| = \omega$$

  - **Phase**

    $$\angle G(j\omega) = 90^\circ$$

    (positive)





Bode Plot (3): Integrator

- **TF**

$$G(s) = \frac{1}{s}$$

- **Amplitude (gain)**

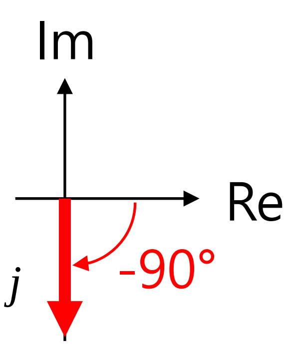
$$|G(j\omega)| = \frac{1}{\omega}$$

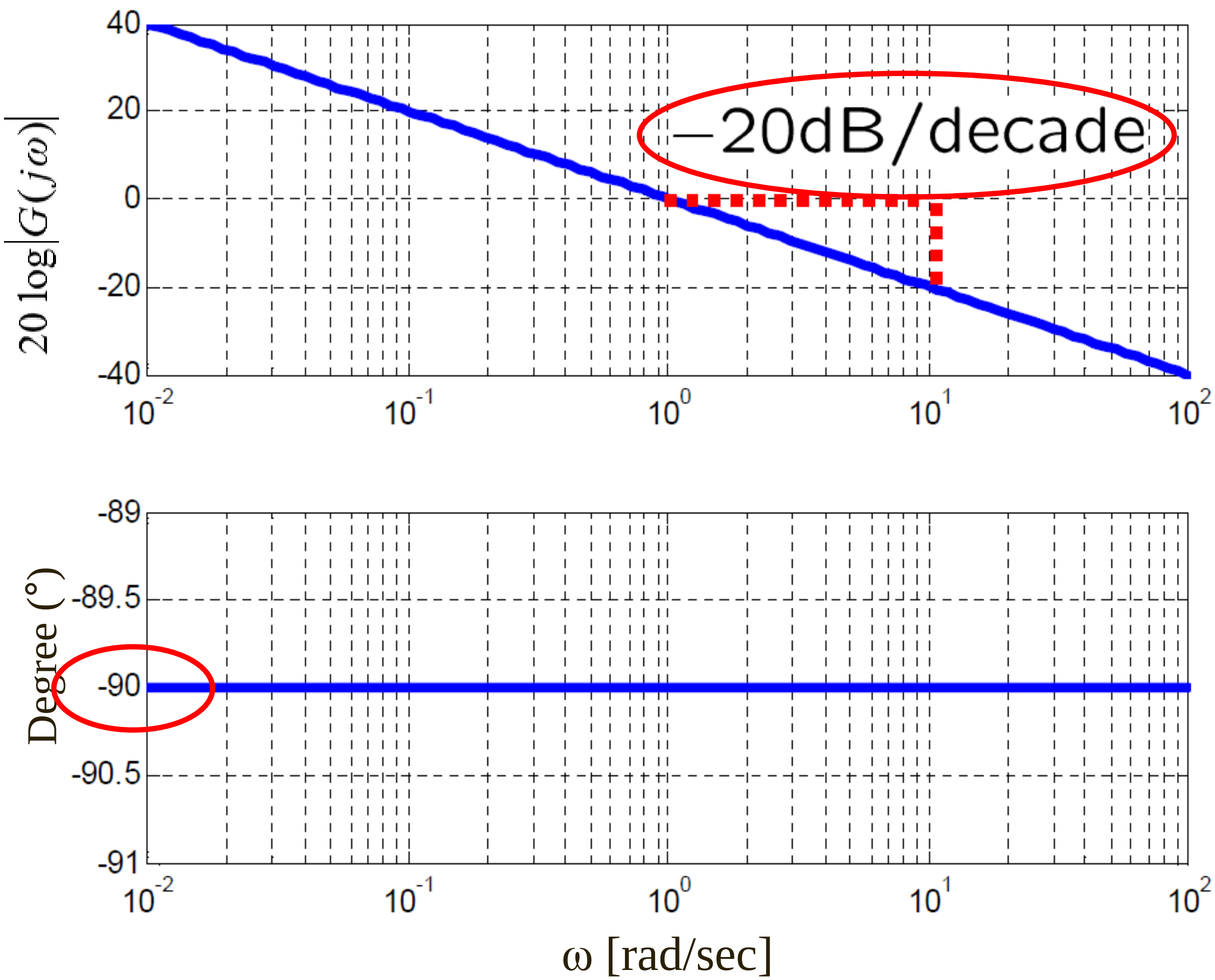
- **Phase**

$$\angle G(j\omega) = -90^\circ$$

(negative)

$$G(j\omega) = \frac{1}{j\omega} = -\frac{1}{\omega} j$$





Bode Plot (4): Double Integrator

- **TF**

$$G(s) = \frac{1}{s^2}$$

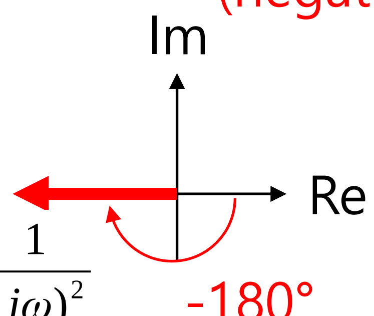
  - **Amplitude (gain)**

$$|G(j\omega)| = \frac{1}{\omega^2}$$

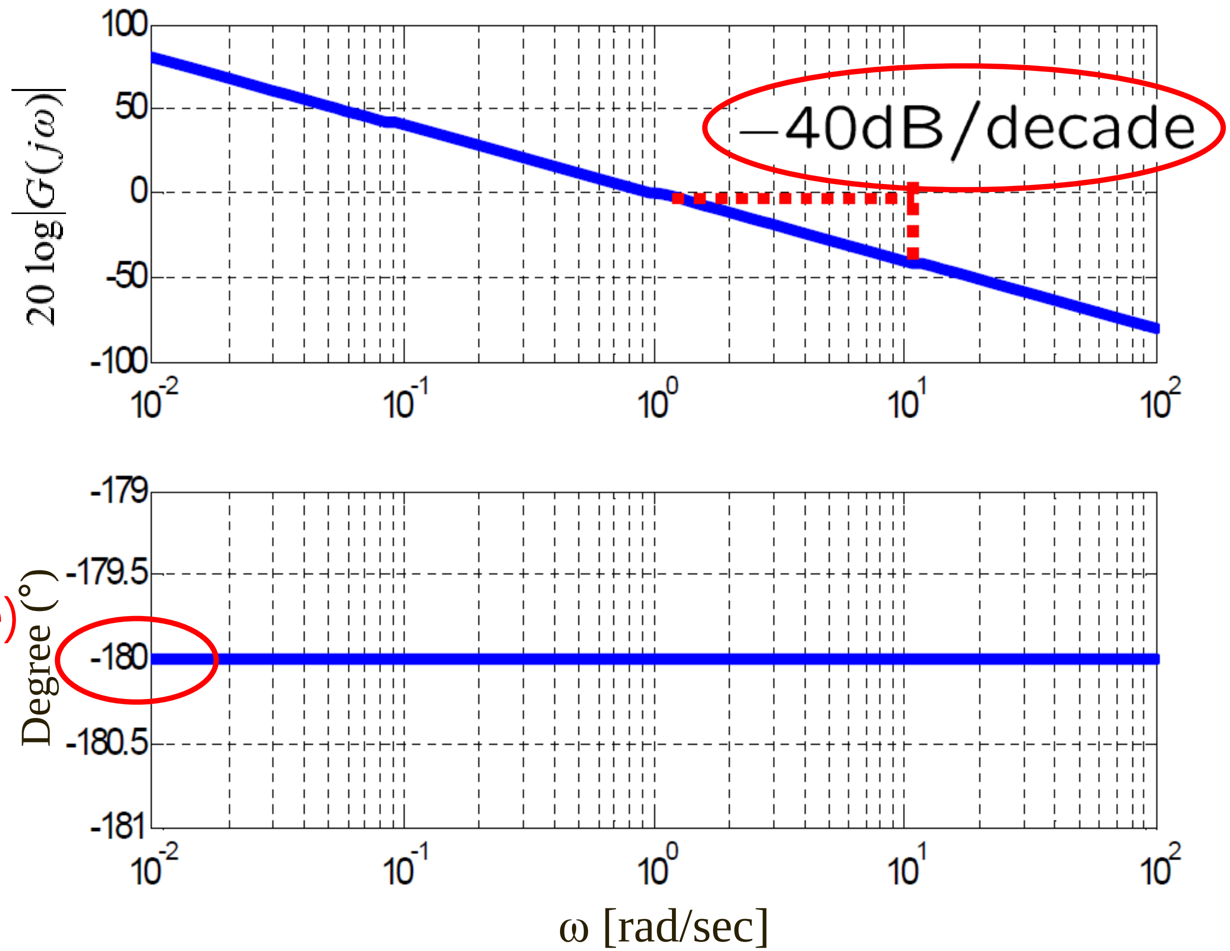
  - **Phase**

$$\angle G(j\omega) = -180^\circ$$

(negative)



$$G(j\omega) = \frac{1}{(j\omega)^2} = -\frac{1}{\omega^2}$$



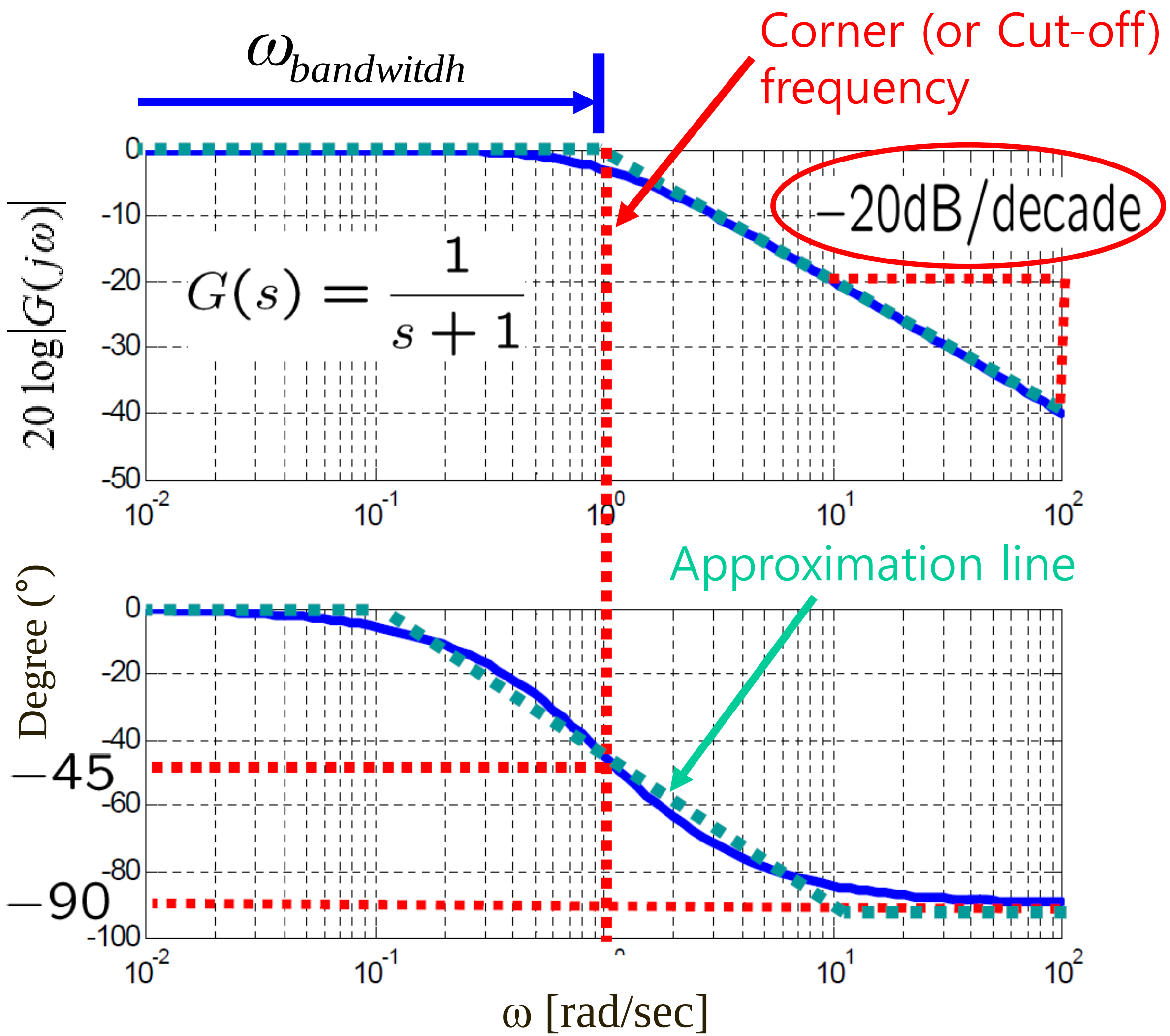
Bode Plot (5): First-Order System

- **TF**

$$G(s) = \frac{1}{s+1}$$

$$G(j\omega) = \frac{1}{j\omega + 1}$$

$$\approx \begin{cases} 1 & : if \ \ \omega << 1 \\ \dfrac{1}{j\omega} & : if \ \ \omega >> 1 \end{cases}$$



- What does bandwidth mean mathematically (-3dB) & physically??

Bode Plot (6): Inverse of First-Order System

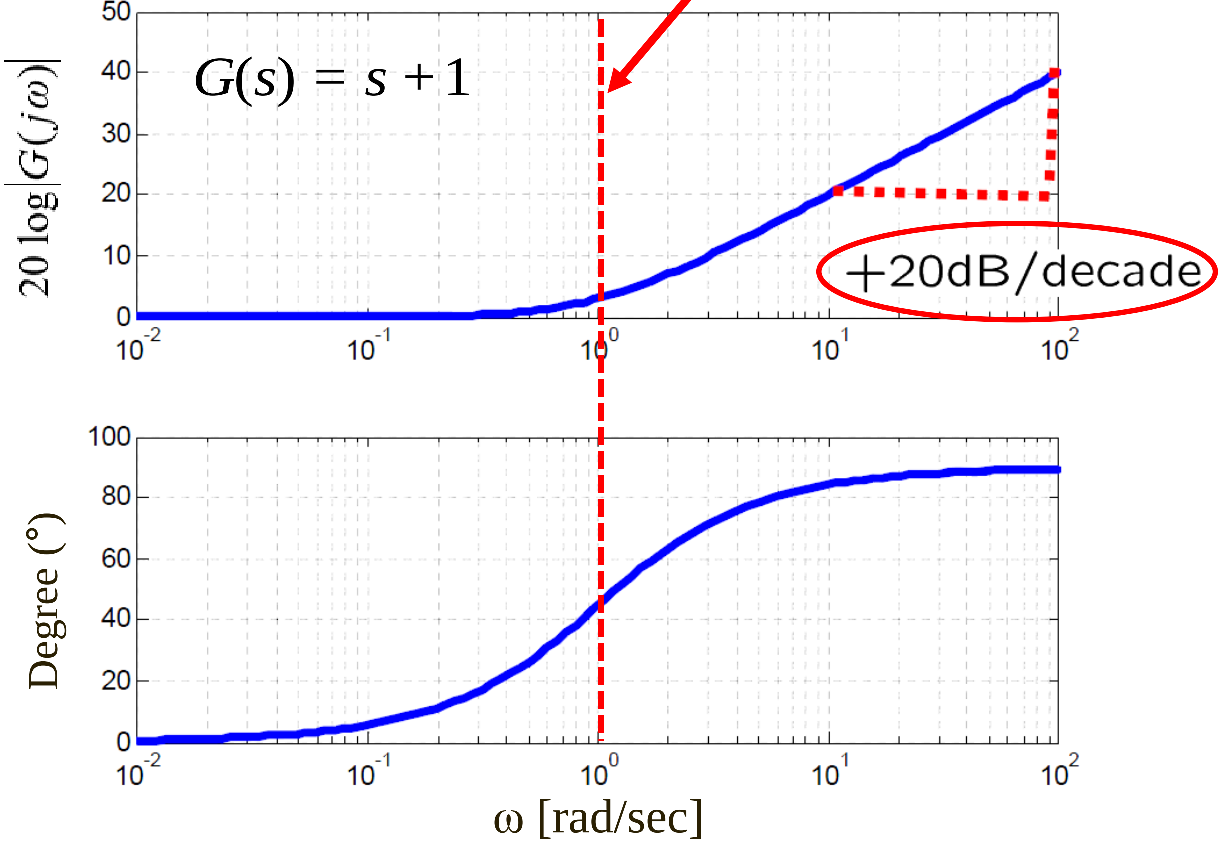
- **TF**

$$G(s) = \left(\frac{1}{s+1}\right)^{-1} = s + 1$$

$$G(j\omega) = j\omega + 1$$

$$\approx \begin{cases} 1 & : if \;\; \omega << 1 \\ j\omega & : if \;\; \omega >> 1 \end{cases}$$

Corner (or Cut-off) frequency



Bode Plot (7): Second-Order System

- **TF**

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- **Amplitude**

where, $\omega_n = 1$

$$|G(j\omega)| = \frac{\omega_n^{\ 2}}{\left|\omega_n^{\ 2} - \omega^2 + j2\zeta\omega_n\omega\right|}$$

$$= \omega_n^{\ 2}\left[(\omega_n^{\ 2} - \omega^2)^2 + 4\zeta^2\omega_n^{\ 2}\omega^2\right]^{-\frac{1}{2}} \approx \begin{cases} 1 & : if \ \ \omega << 1 \\ \dfrac{1}{2\zeta} & : if \ \ \omega = \omega_n \\ 0 & : if \ \ \omega >> 1 \end{cases}$$

- **Phase**

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^{\ 2} - \omega^2}\right) \approx \begin{cases} 0^\circ & : if \ \ \omega << 1 \\ -90^\circ & : if \ \ \omega = \omega_n \\ -180^\circ & : if \ \ \omega >> 1 \end{cases}$$

Bode Plot (7): Second-Order System (cont'd)

- **TF**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

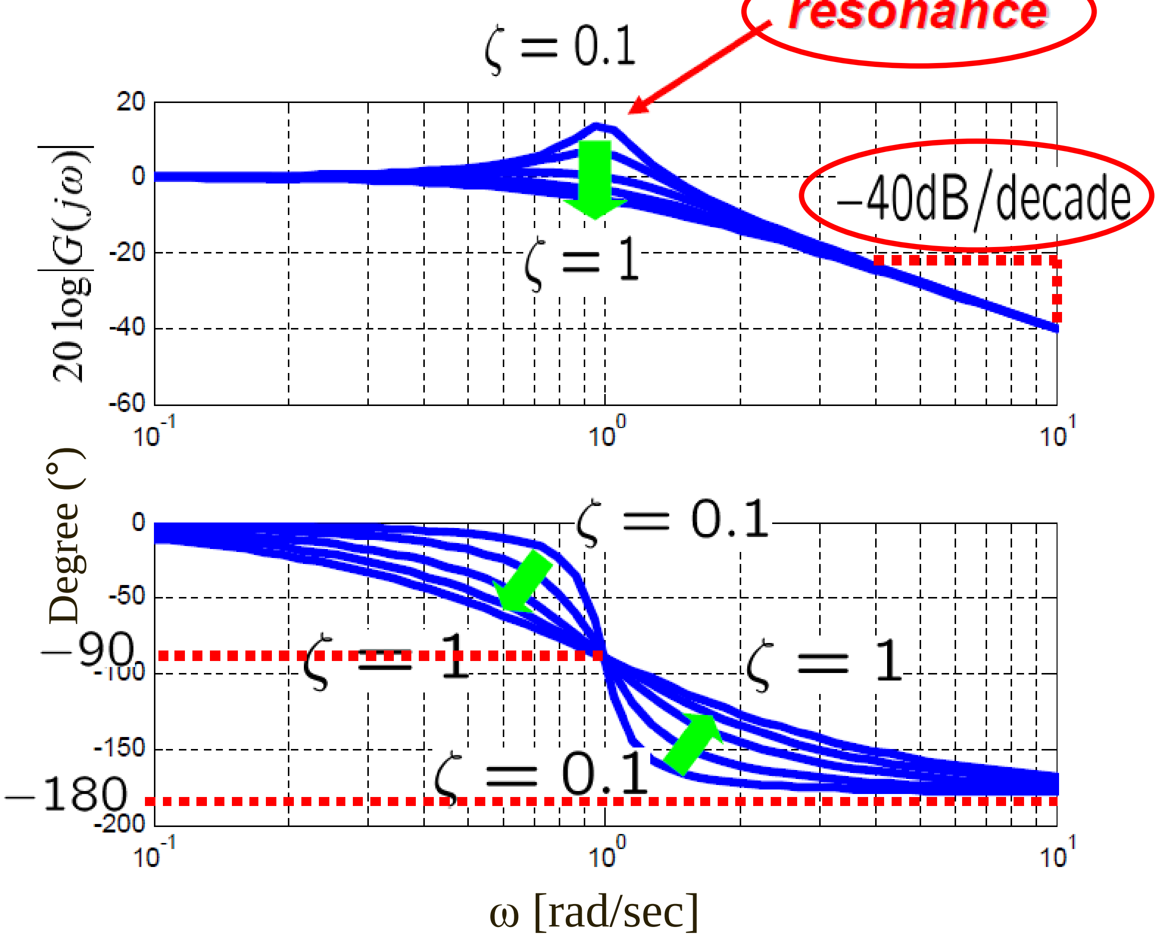
To find peak location,
(or <u>resonance</u>)

$$\frac{\partial}{\partial \omega}\left|G(j\omega)\right| = 0$$

where,

$$\left|G(j\omega)\right| = \frac{\omega_n^{\ 2}}{\left|\omega_n^{\ 2} - \omega^2 + j2\zeta\omega_n\omega\right|}$$

$$\omega = \omega_{peak} = \omega_n\sqrt{1 - 2\zeta^2}$$

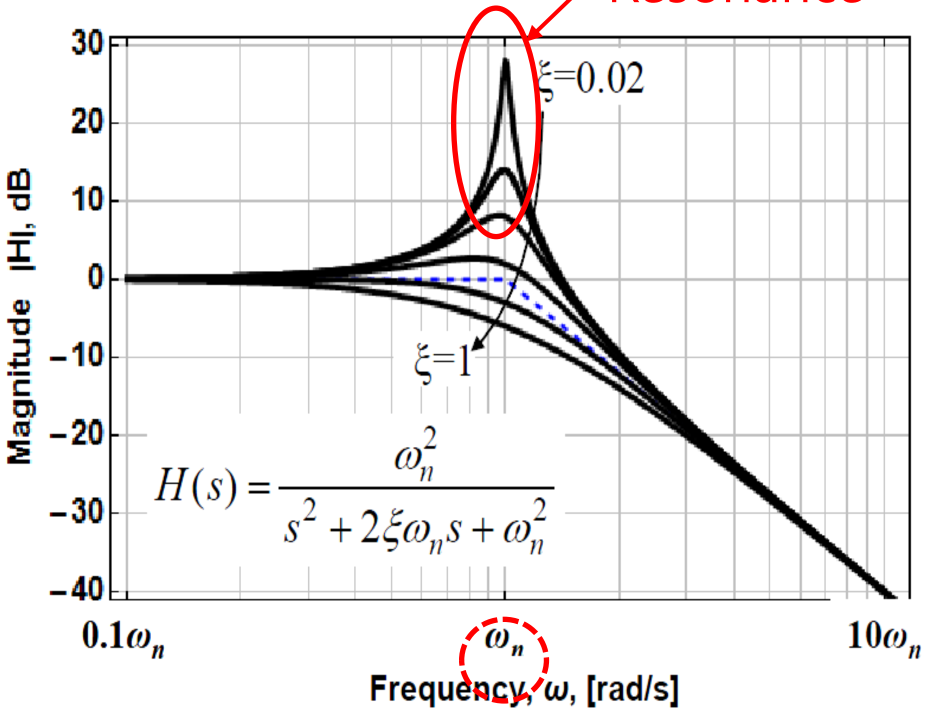


Bode Plot (7): Second-Order System (cont'd)

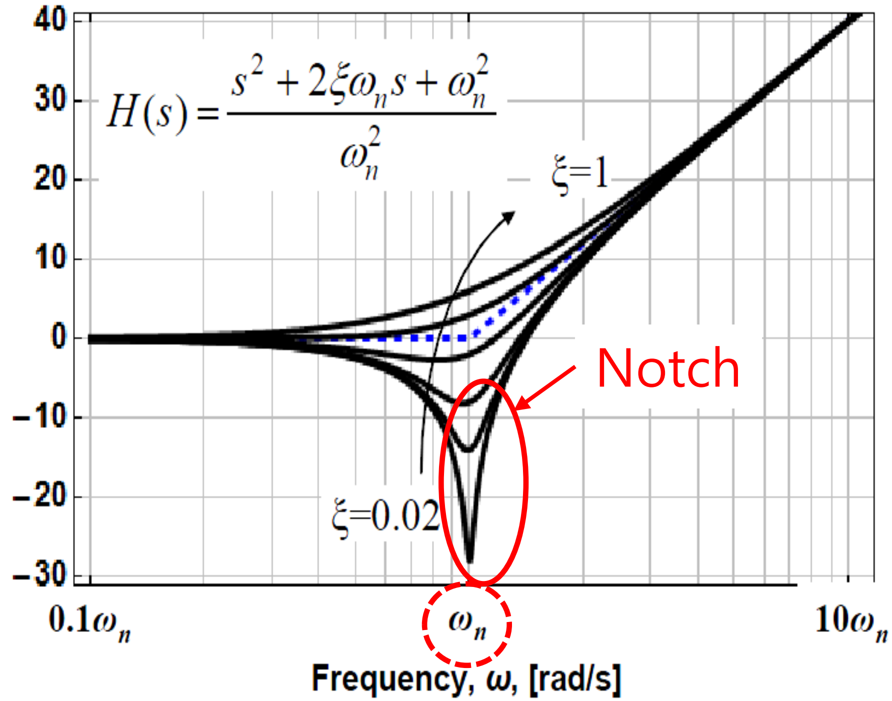
- **TF**

$$H(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \quad \textbf{VS.} \quad H(s)=\frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2}$$

Resonance



Notch



Advantage of Bode Plot

- Bode plot of a series connection $G_1(s) \cdot G_2(s)$ is the addition of each Bode plot of $G_1(s)$ and $G_2(s)$.
  - Amplitude (or Gain)

    $$20\log_{10}|G_1(j\omega)G_2(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$

  - Phase

    $$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

- We use this property to design $C(s)$ so that $C(s) \cdot G(s)$ has a "desired" shape of Bode plot.

Short Proofs:

Let us use polar representation,

$$G_1(j\omega) = |G_1(j\omega)|e^{j\angle G_1(j\omega)}$$

$$G_2(j\omega) = |G_2(j\omega)|e^{j\angle G_2(j\omega)}$$

Product two systems,

$$\begin{aligned} & G_1(j\omega)G_2(j\omega) \\ = \; & |G_1(j\omega)||G_2(j\omega)|e^{j\{\angle G_1(j\omega) + \angle G_2(j\omega)\}} \end{aligned}$$

Example 1

- Let us sketch the Bode plot of a transfer function:

$$\boxed{G(s) = \frac{10}{s}}$$

  - **Step 1:** Decompose G(s) into a product form:

$$G(s) = 10 \cdot \frac{1}{s}$$

  - **Step 2:** Sketch a Bode plot for each component on the same graph.

  - **Step 3:** Add them on both amplitude and phase plots.

Example 1 (cont'd)

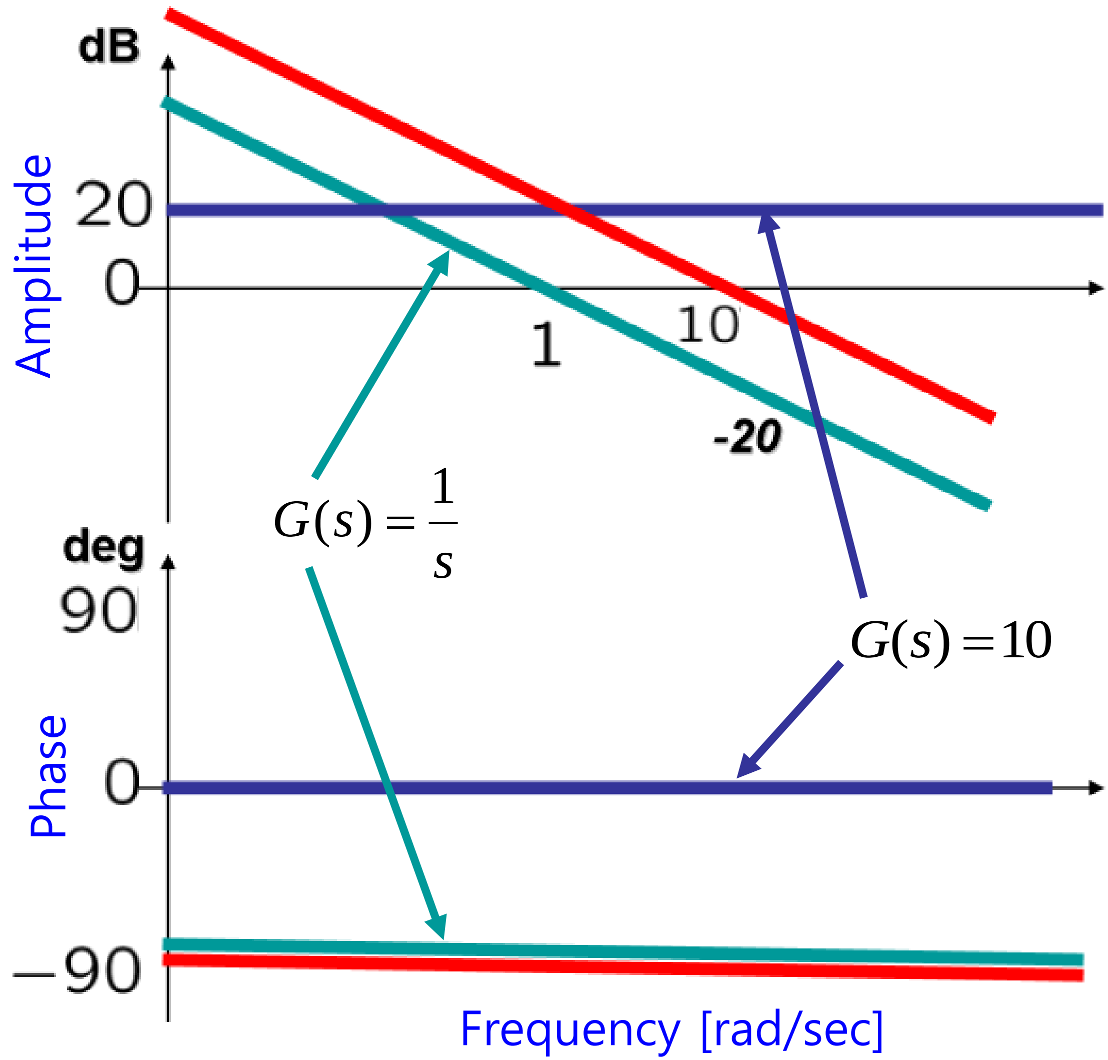
$$G(s) = 10$$

$$\times$$

$$G(s) = \frac{1}{s}$$

$$\Downarrow$$

$$G(s) = \frac{10}{s}$$



Example 2

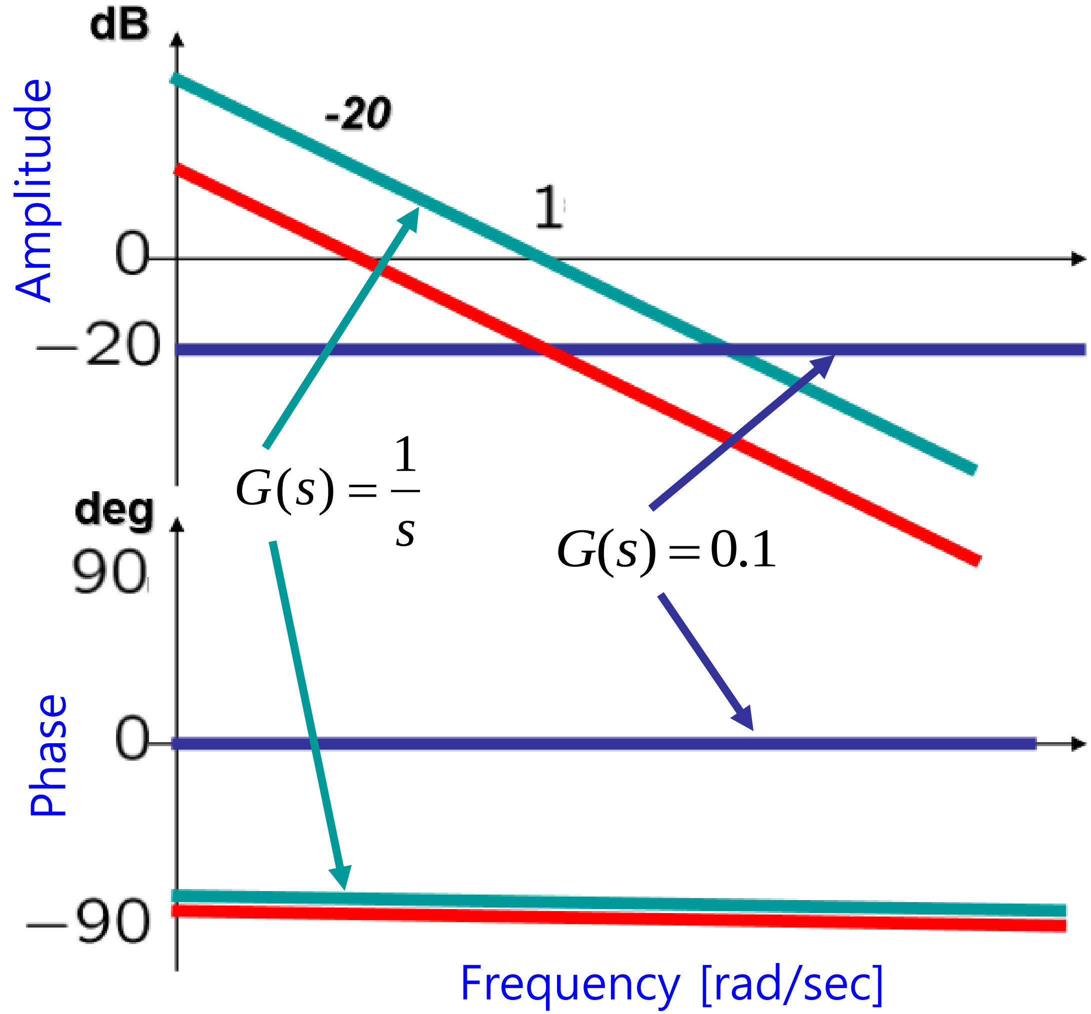
$$G(s) = 0.1$$

$$\times$$

$$G(s) = \frac{1}{s}$$

$$\Downarrow$$

$$G(s) = \frac{0.1}{s}$$



Example 3

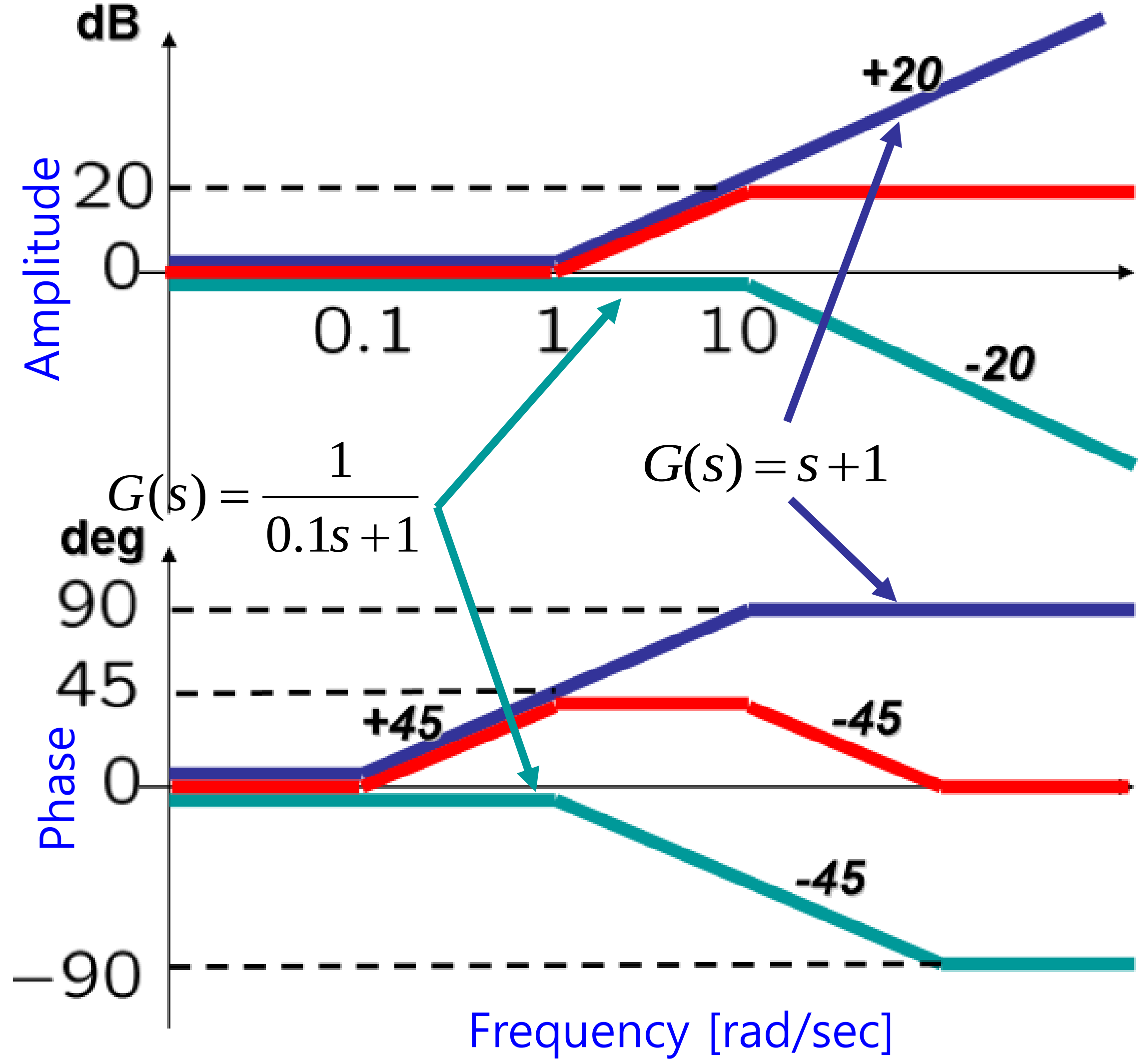
$$G(s) = s + 1$$

$$\times$$

$$G(s) = \frac{1}{0.1s + 1}$$

$$\Downarrow$$

$$G(s) = \frac{10(s+1)}{(s+10)}$$



Example 4

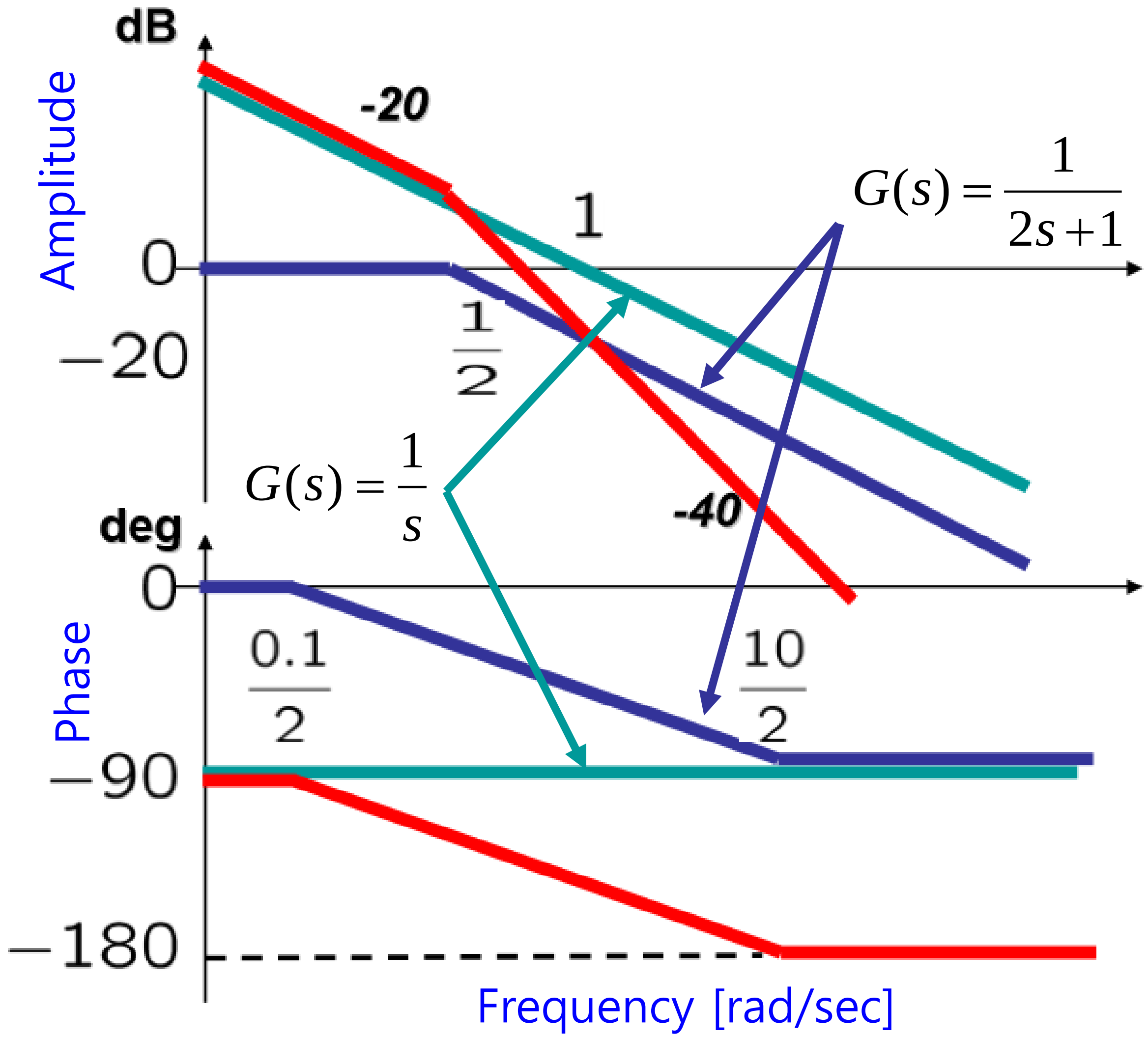
$$G(s) = \frac{1}{2s+1}$$

$$\times$$

$$G(s) = \frac{1}{s}$$

$$\Downarrow$$

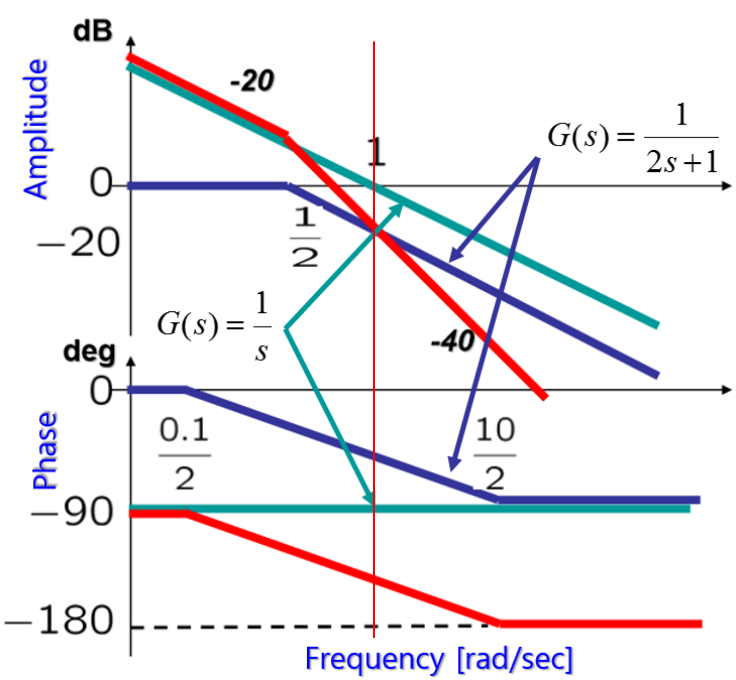
$$G(s) = \frac{1}{s(2s+1)}$$

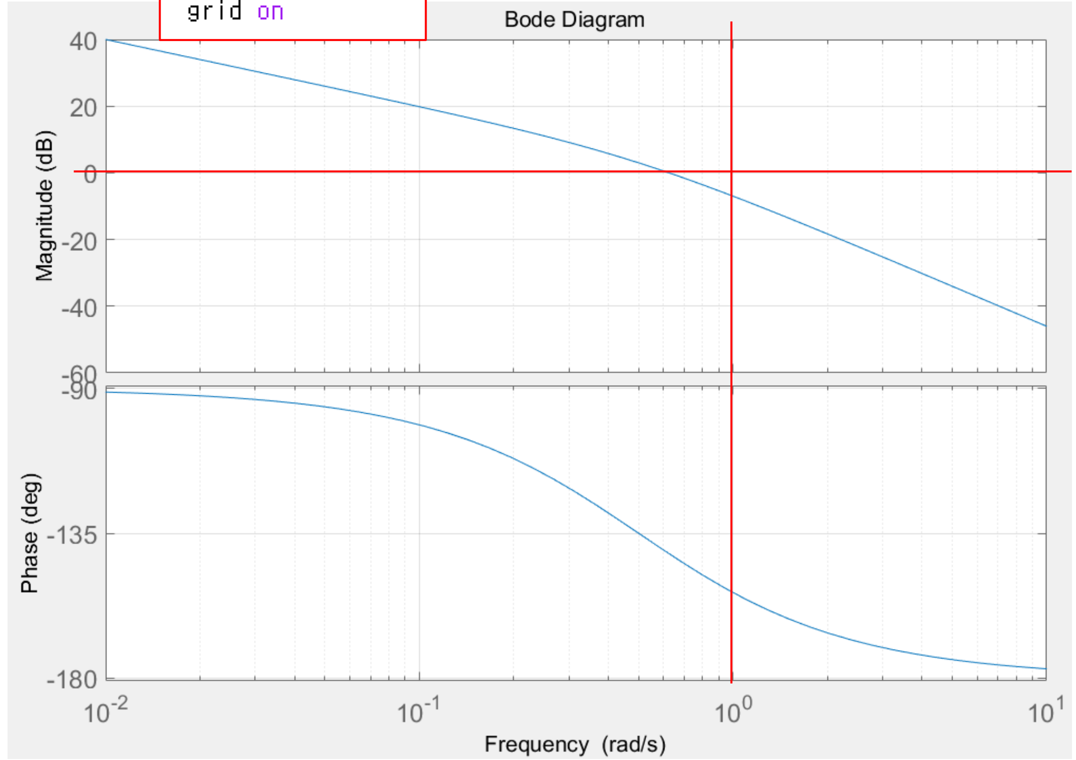


Example 4 (cont'd) by MATLAB

$$G(s) = \frac{1}{s(2s+1)} = \frac{1}{2s^2 + s}$$

```
s = tf('s');
num = [1];
den = [2 1 0];
G = tf(num,den);
bode(G)
grid on
```





Manual Sketch VS. **MATLAB (Bode Plot)**

Summary

- **Summary:**
  - Bode plot for frequency response analysis
    - Sketching of Basic functions
    - Connections with basic functions