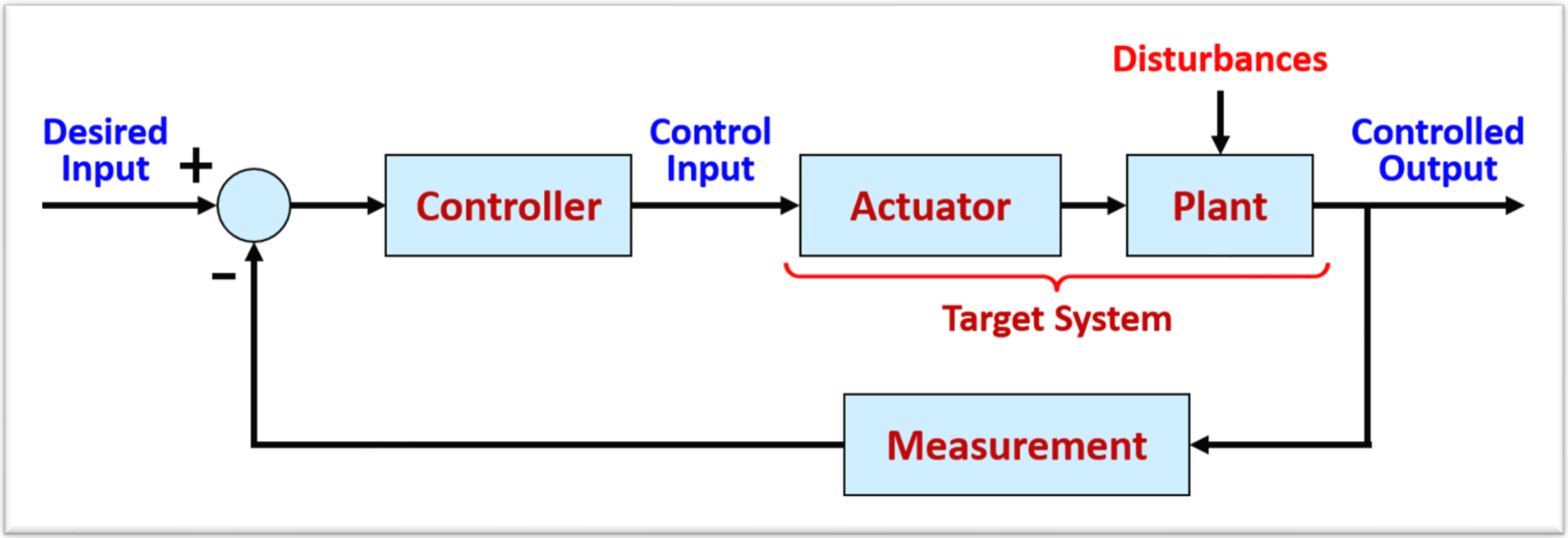
The Frequency-Response Design Method 2

Lecture 11:

- Relative Stability
- System Bandwidth



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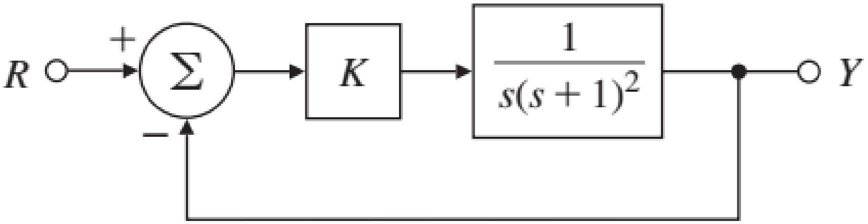
Nyquist Stability Criterion

- **Nyquist stability criterion**, discovered by Swedish-American electrical engineer **Harry Nyquist** at Bell Telephone Laboratories in 1932.

Harry Nyquist

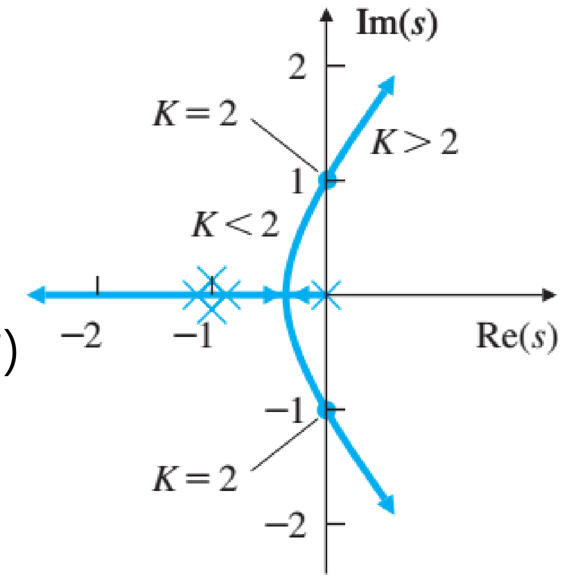
Harry Nyquist (1889–1976)

- **Neutral Stability**



$$|KG(j\omega)| = 1 \quad \& \quad \angle G(j\omega) = 180^\circ.$$

(or -180°)



$$where, \frac{Y(s)}{R(s)} = H(s) = \frac{KG}{1+KG}$$

$$Characteristic\ Eq. \Rightarrow 1+KG = 0$$

Nyquist Stability Criterion (cont'd)

- **Important Question:**

  Does increasing gain (K) increase or decrease the system's stability??

- **I: Stability Criteria (Stable)**

$$|KG(j\omega)| \,\square\, 1 \quad \text{at} \quad \angle G(j\omega) = -180^\circ.$$

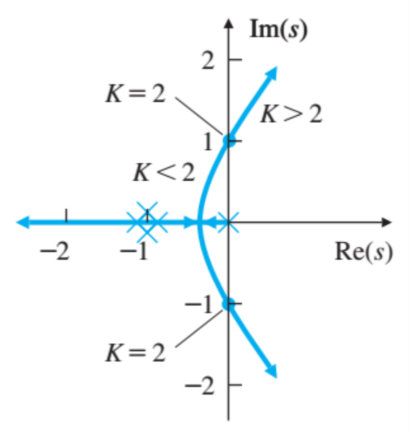
- **II: Neutral Stability Criteria (Marginal)**

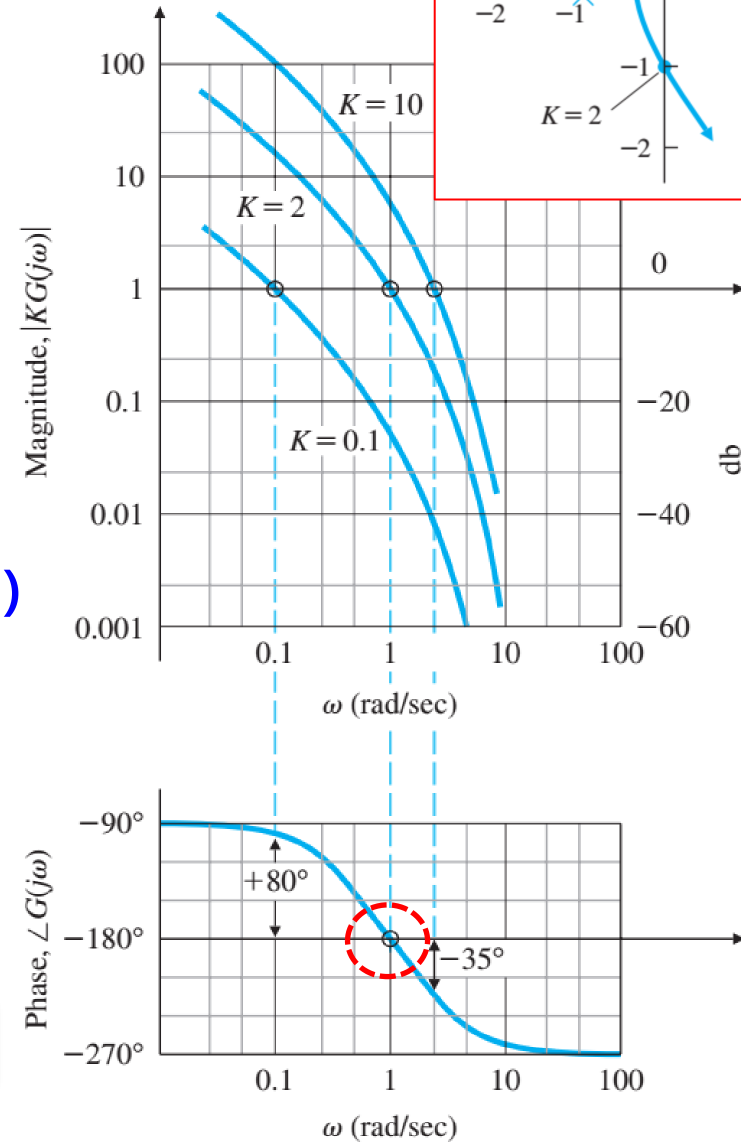
$$|KG(j\omega)| = 1 \quad \text{at} \quad \angle G(j\omega) = -180^\circ.$$

- **III: Instability Criteria (Unstable)**

$$|KG(j\omega)| \,\square\, 1 \quad \text{at} \quad \angle G(j\omega) = -180^\circ.$$

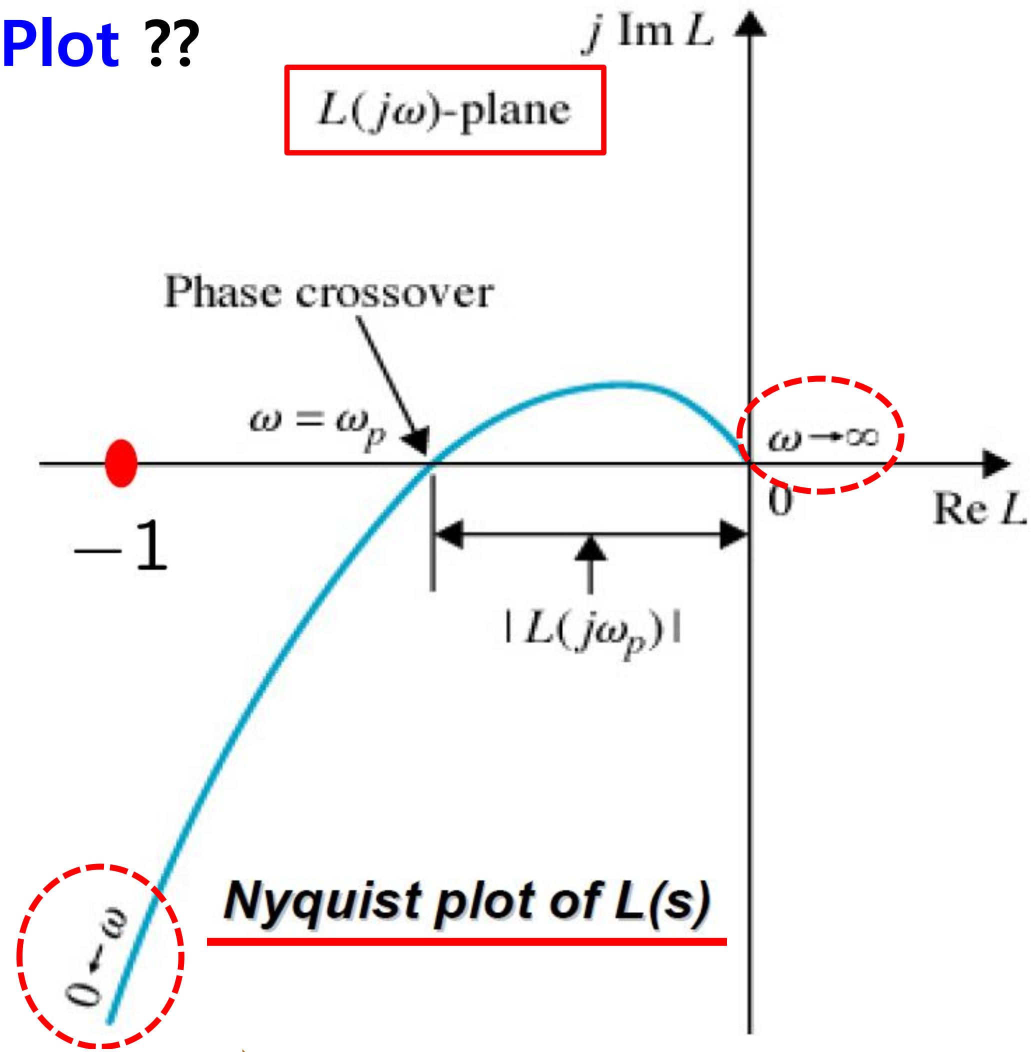
But, how to measure **stability margin** ?





Gain Margin (GM) on Nyquist Plot

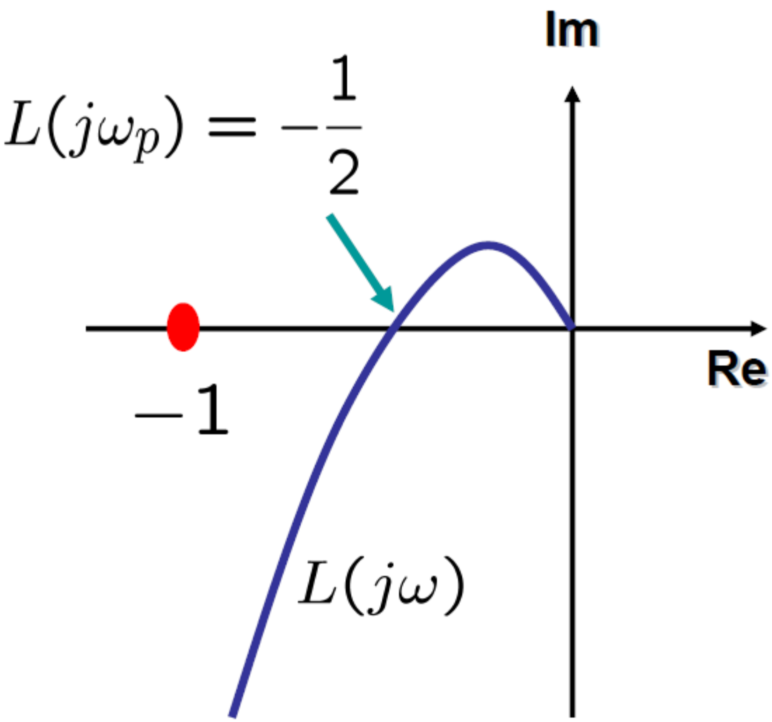
- **Bode Plot vs. Nyquist Plot ??**
- Phase (-180°) crossover frequency, ω_p :
  $\angle L(j\omega_p) = -180$
- Gain Margin (in dB)
  $$\boxed{GM = 20\log_{10}\frac{1}{|L(j\omega_p)|}}$$
- Indicates how much OL gain can be multiplied without violating CL stability.

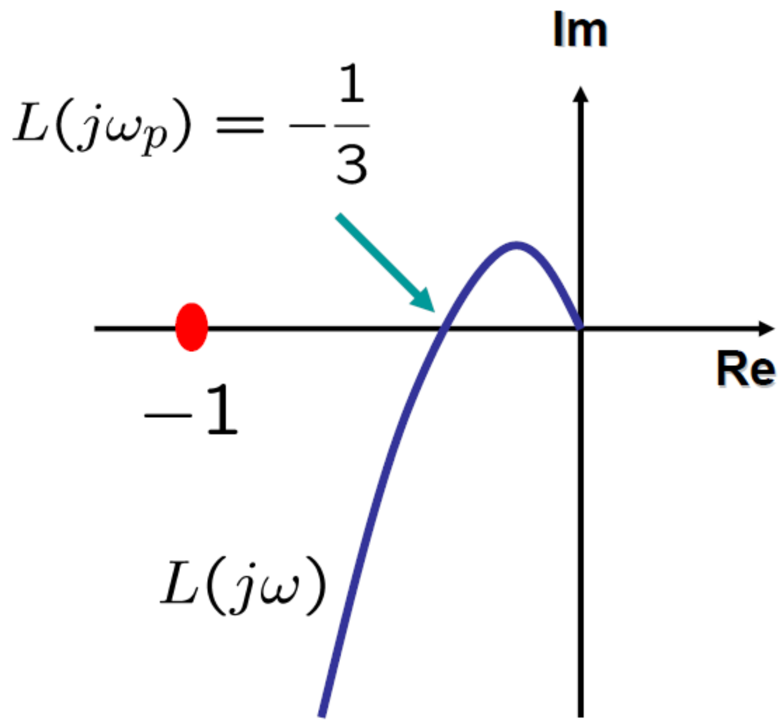


Nyquist plot of L(s)

$|KG(j\omega)| = 1 \quad \text{at} \quad \angle G(j\omega) = -180°.$ ⇒ **Neutral Stability!!**

Examples of Gain Margin (GM)

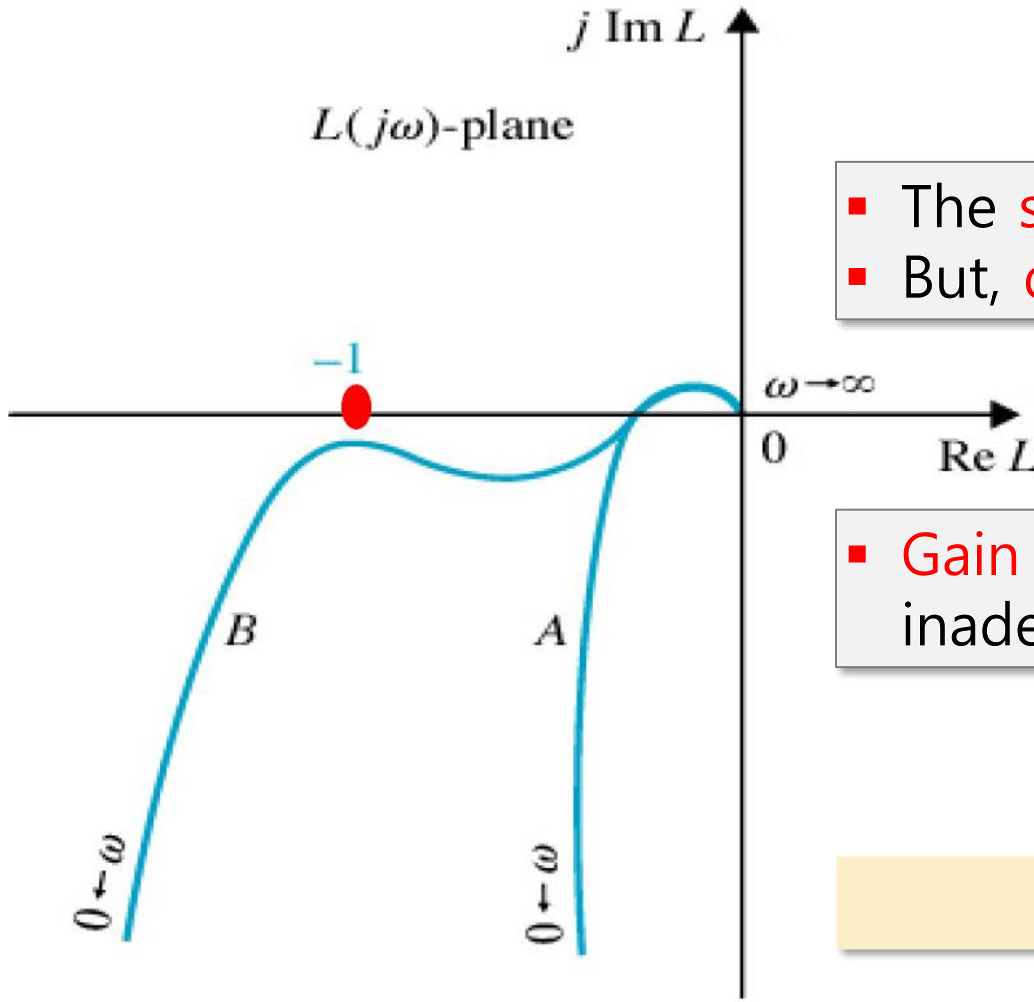




$$GM = 20\log_{10} \underbrace{\boxed{\frac{1}{|L(j\omega_p)|}}}_{2} \approx 6\text{dB}$$

$$GM = 20\log_{10} \underbrace{\boxed{\frac{1}{|L(j\omega_p)|}}}_{3} \approx 9.5\text{dB}$$

Reason Why GM is Inadequate ??



- The same gain margin,
- But, different relative stability

- Gain Margin is often inadequate to indicate stability.

Phase Margin !!

Phase Margin (PM)

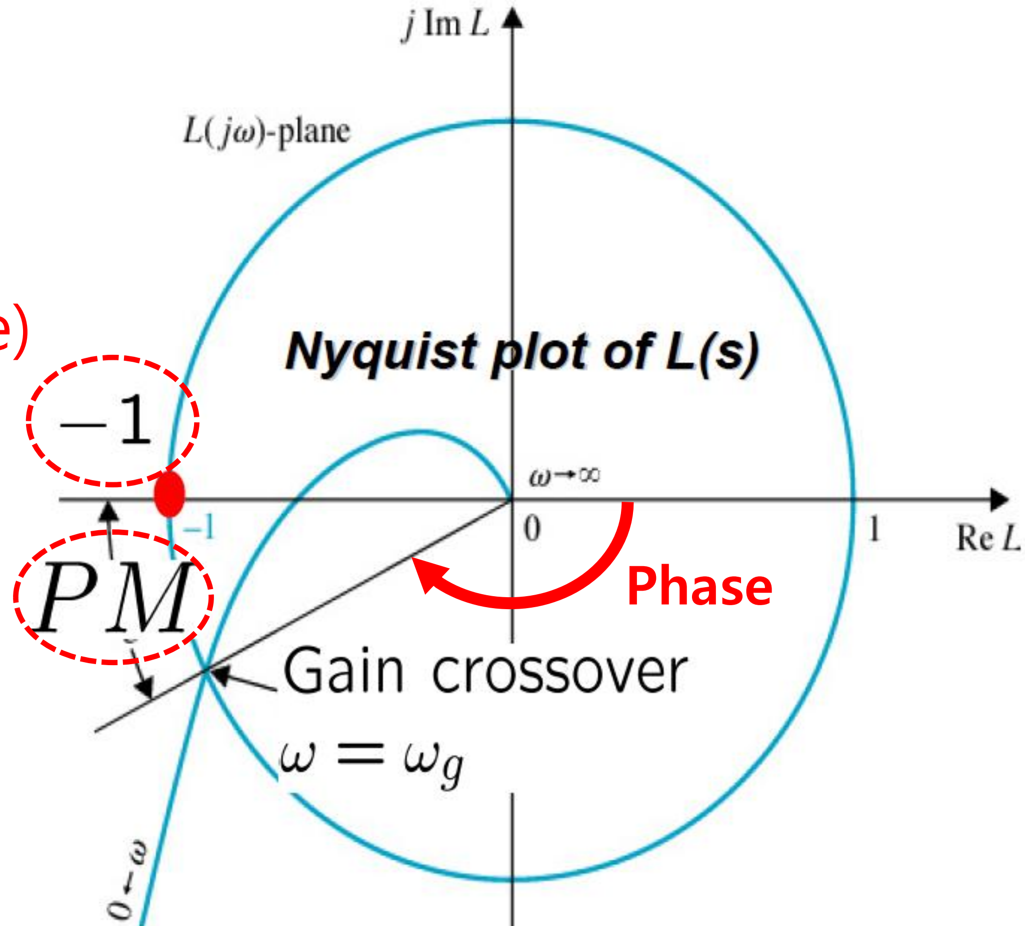
- Gain crossover frequency, ω_g :

  $|L(j\omega_g)| = 1$

- Phase Margin (in degree)

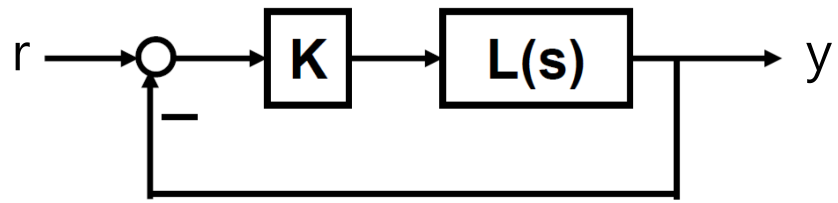
  $$\boxed{PM = 180^\circ + \angle L(j\omega_g)}$$

- Indicates how much OL phase-delay can be added without violating CL stability.



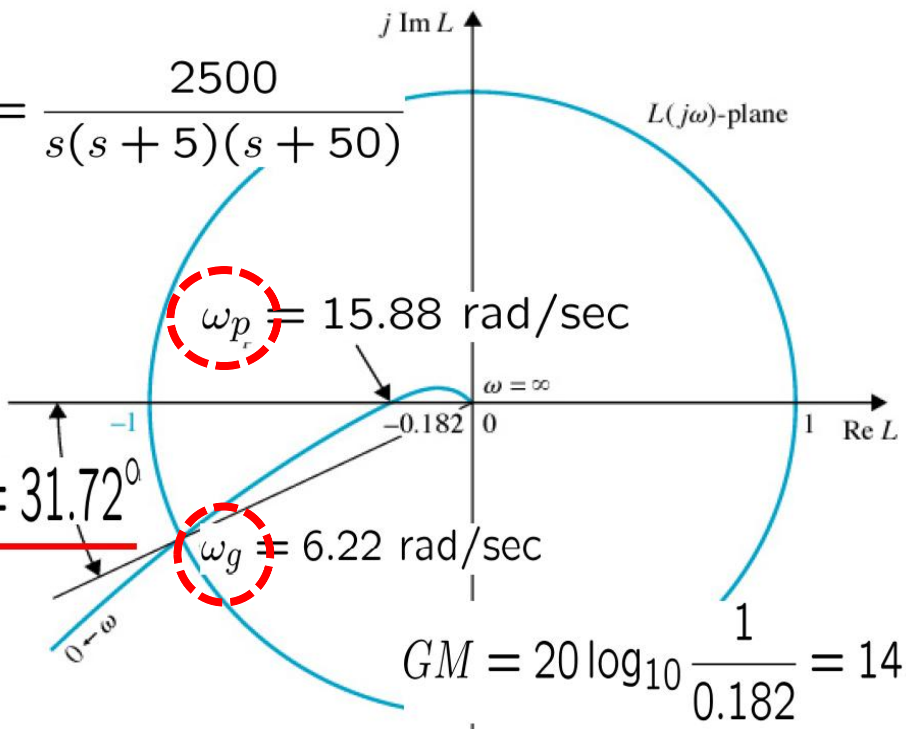
$|KG(j\omega)| = 1$ at $\angle G(j\omega) = -180^\circ$. Neutral Stability!!

Example on Nyquist Plot



$$K = 2500$$

$$L(s) = \frac{1}{s(s+5)(s+50)}$$



$$KL(s) = \frac{2500}{s(s+5)(s+50)}$$

$\omega_{p_r} = 15.88$ rad/sec

$\omega_g = 6.22$ rad/sec

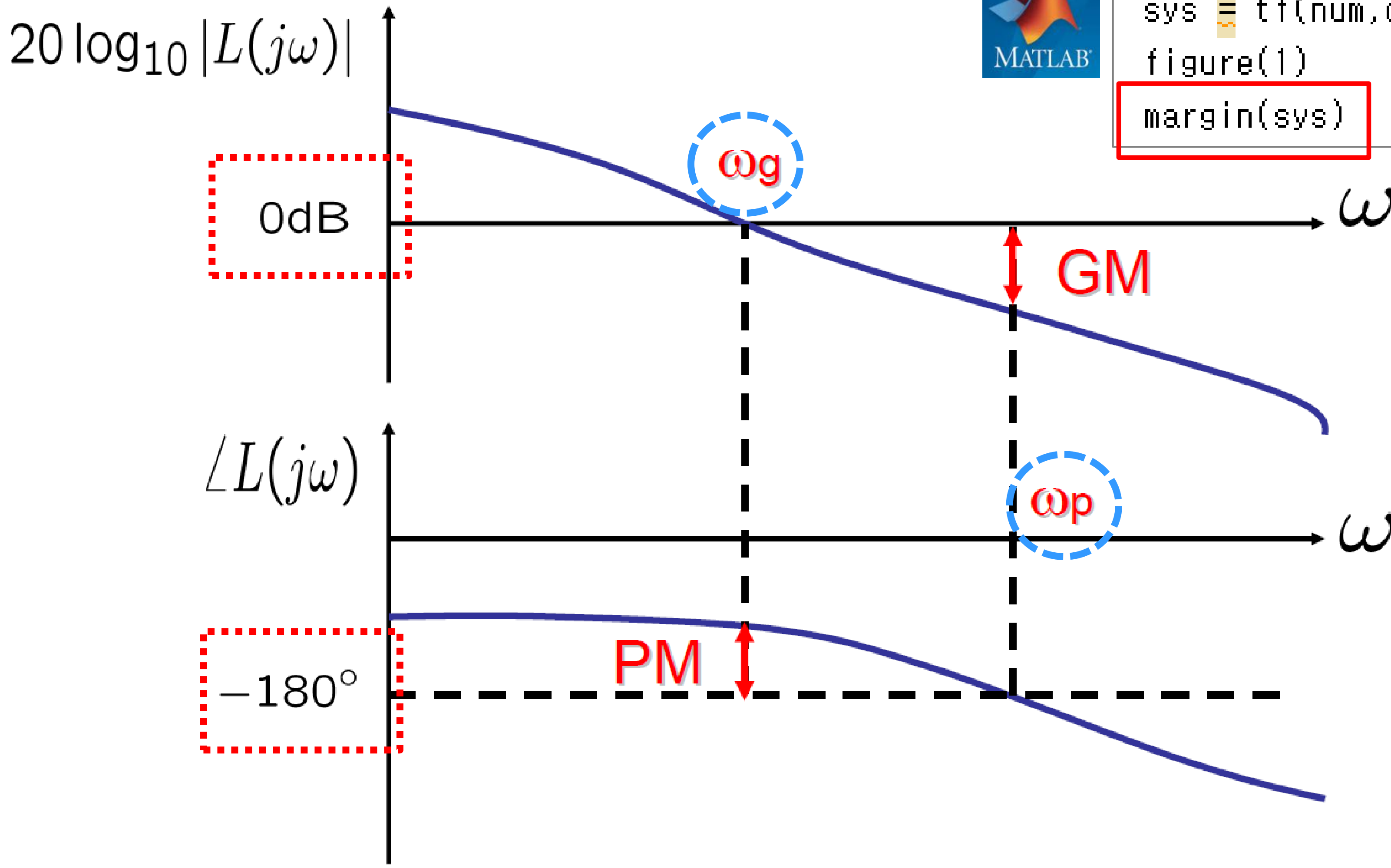
$$PM = 31.72^0$$

$$GM = 20\log_{10}\frac{1}{0.182} = 14.80\text{dB}$$

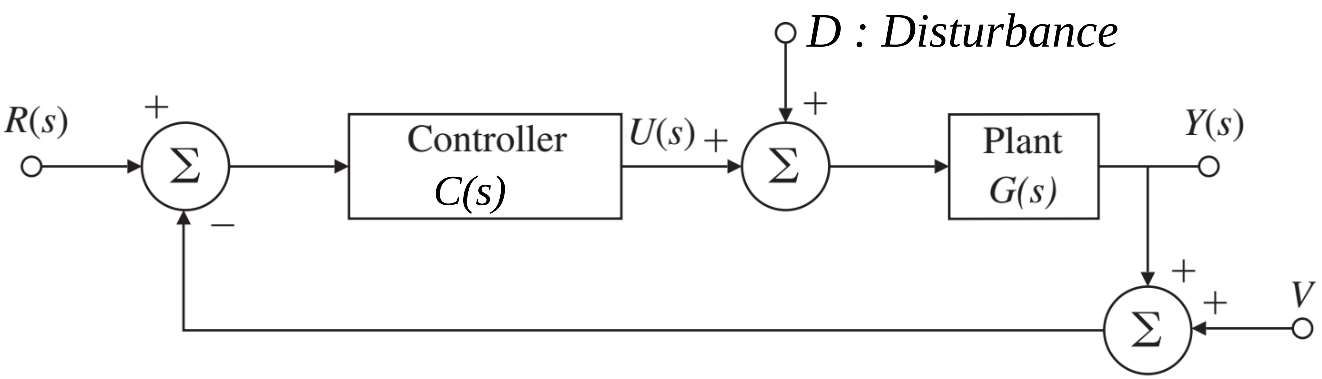
```
num = [2500];
den = [1 55 250 0];
s = tf('s')
sys = tf(num,den)
figure(1)
nyquist(sys)
```

Relative Stability on Bode Plot

```
num = [2500];
den = [1 55 250 0];
s = tf('s')
sys = tf(num,den)
figure(1)
margin(sys)
```



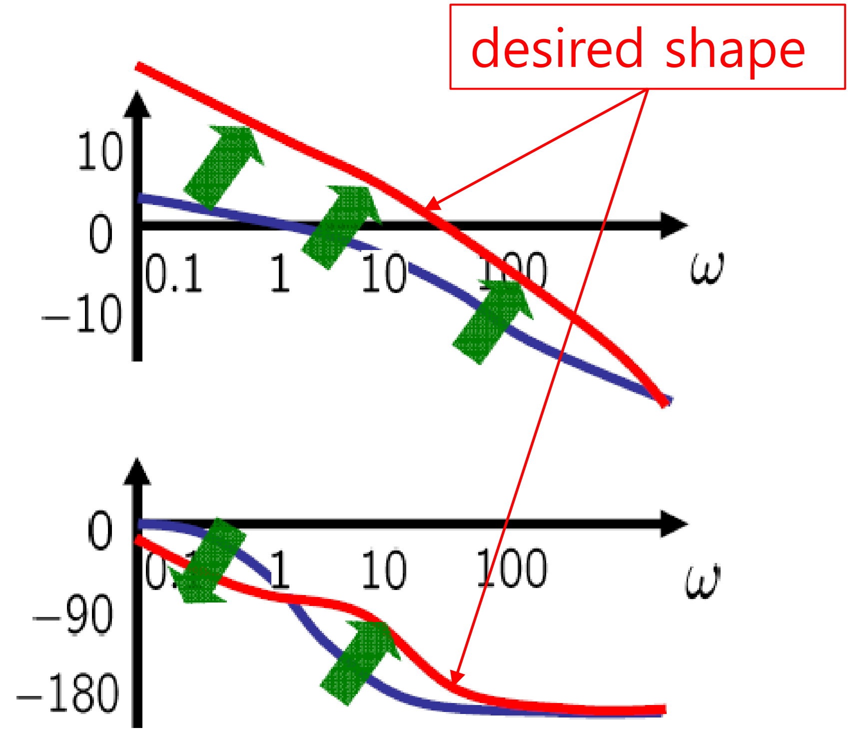
Frequency Shaping (Loop Shaping)



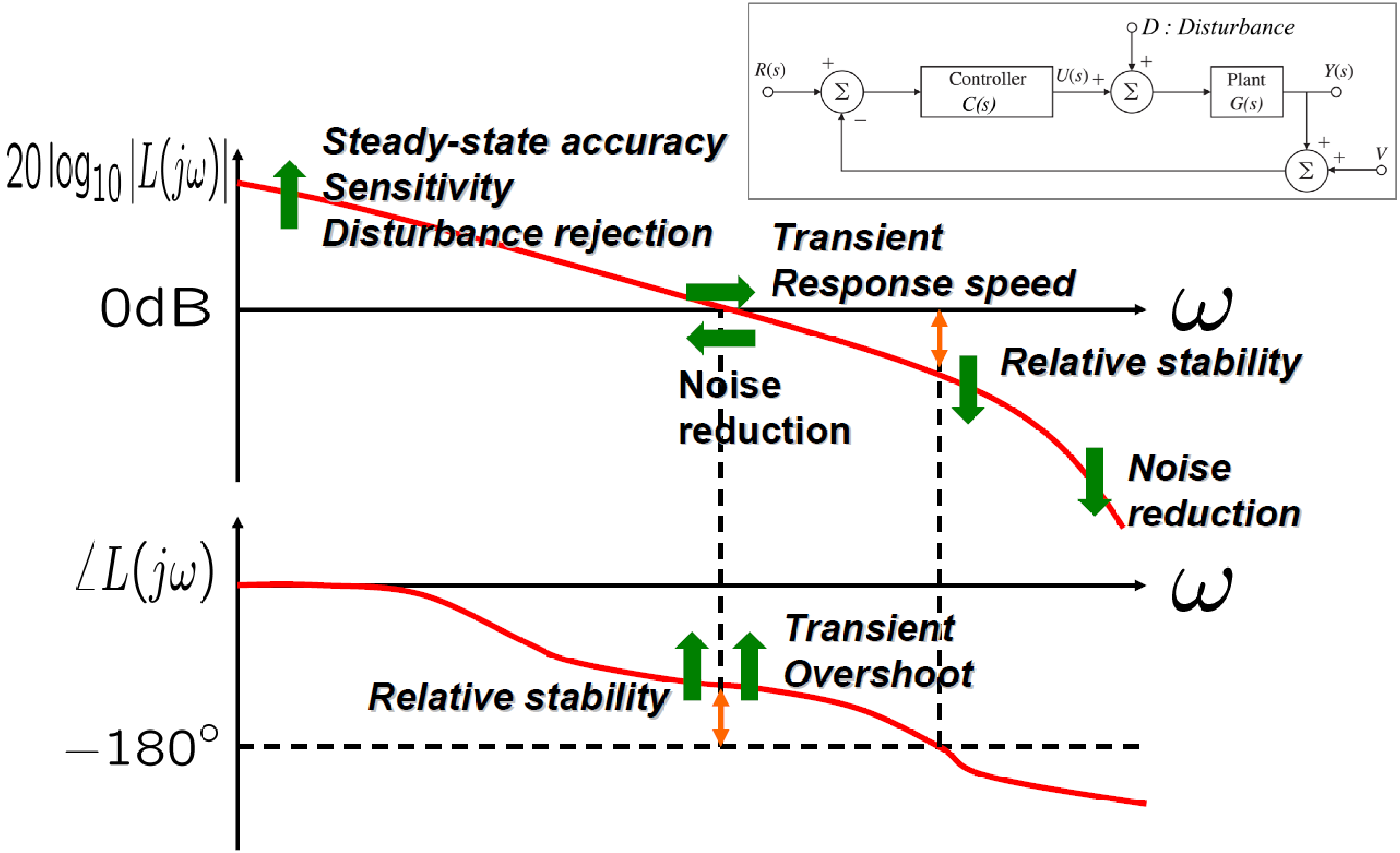
- Reshape Bode Plot of G(jω) into a "desired" shape of

$$L(j\omega) := G(j\omega)C(j\omega)$$

by a series connection of appropriate C(s).



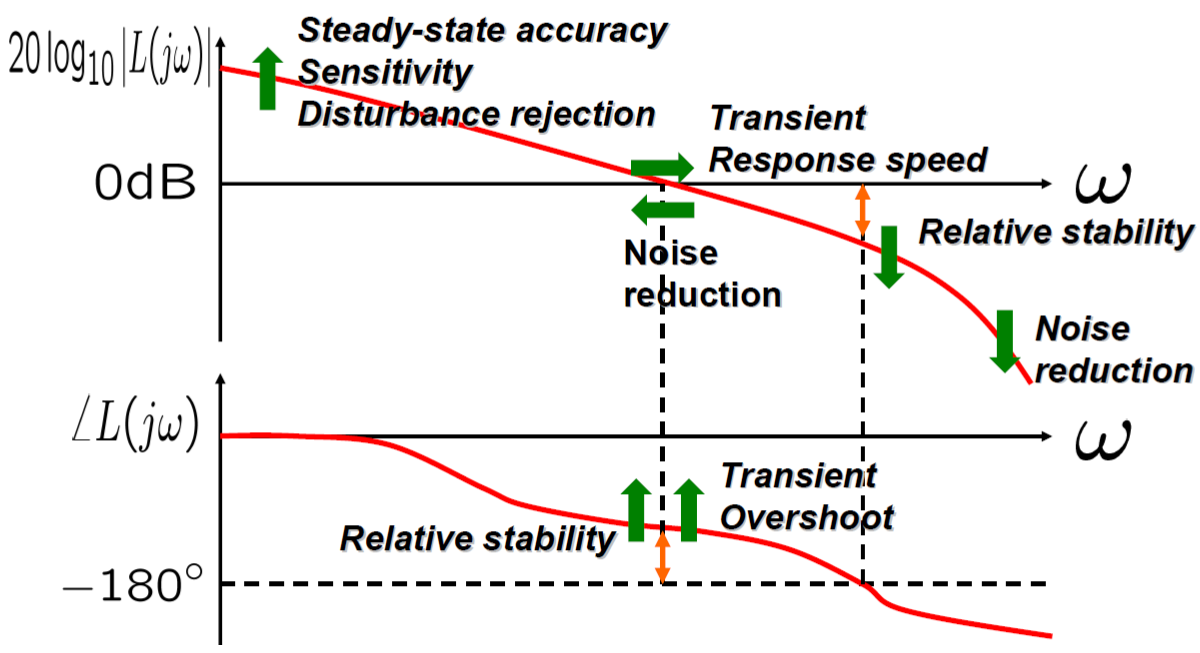
Typical Shaping Goal & Guideline



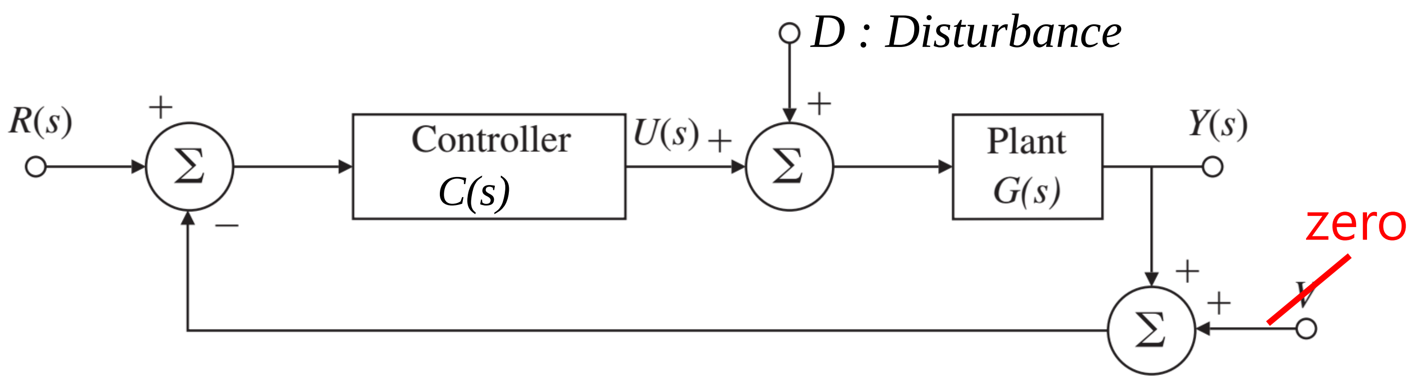
Important Notes on Bode Plot

Summary: Bode Plots

- Without computer, Bode plot can be sketched easily.
- Bode plot tells frequency region (or bandwidth) for notifying about
  - how much gain (amplitude, 이득)
  - how much sluggish (phase, 지연)
- For checking stability, the GM, PM and crossover frequencies are easily determined on Bode plot.



Stability Margin Comparison

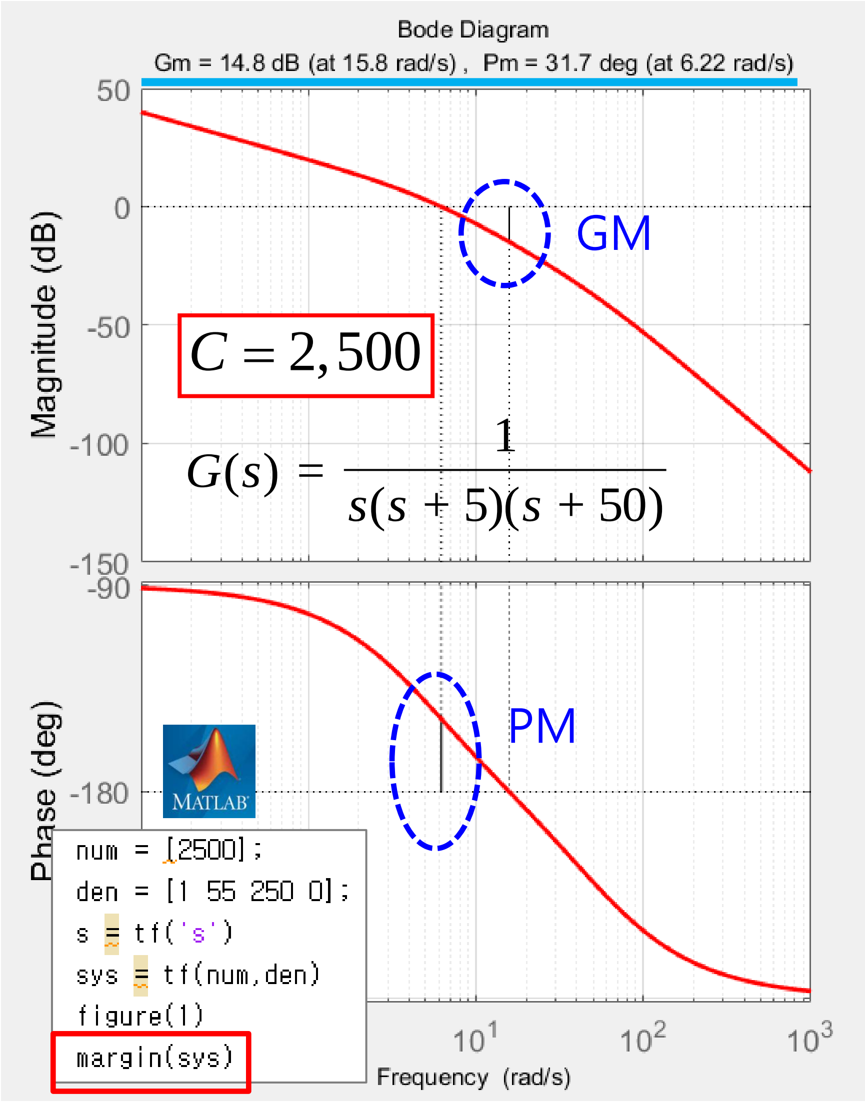


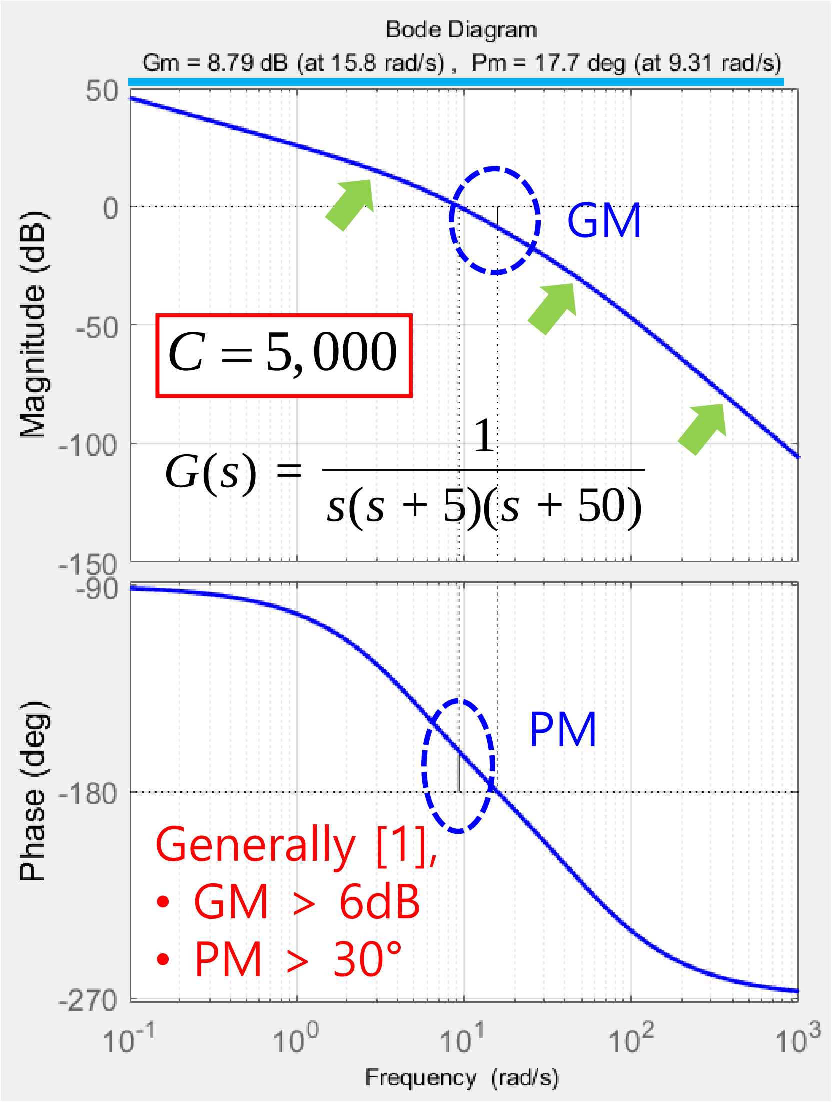
$$Y(s) = \frac{CG}{1+CG}R + \frac{G}{1+CG}D \quad \Rightarrow \quad E(s) = \frac{1}{1+CG}R + \frac{G}{1+CG}D$$

- C(s) should be larger for control performance, which means to minimize the tracking error.
- But, with larger C(s), stability margins can be reduced !!

Case 1: C(s) = 2,500 VS. **Case 2: C(s) = 5,000**

Stability Margin Comparison (cont'd)





[1] Safonov, M. G., 1980, *Stability and Robustness of Multivariable Feedback Systems*, MIT press.

The Frequency-Response Design Method 2

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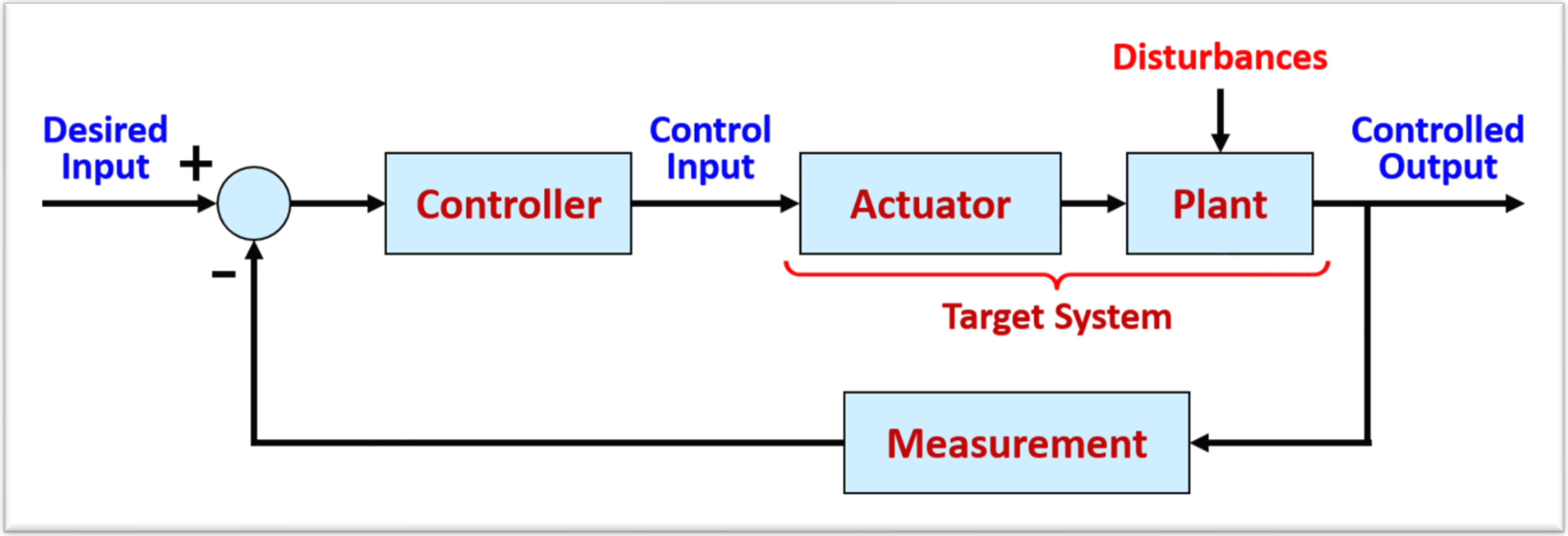
- **System Bandwidth**



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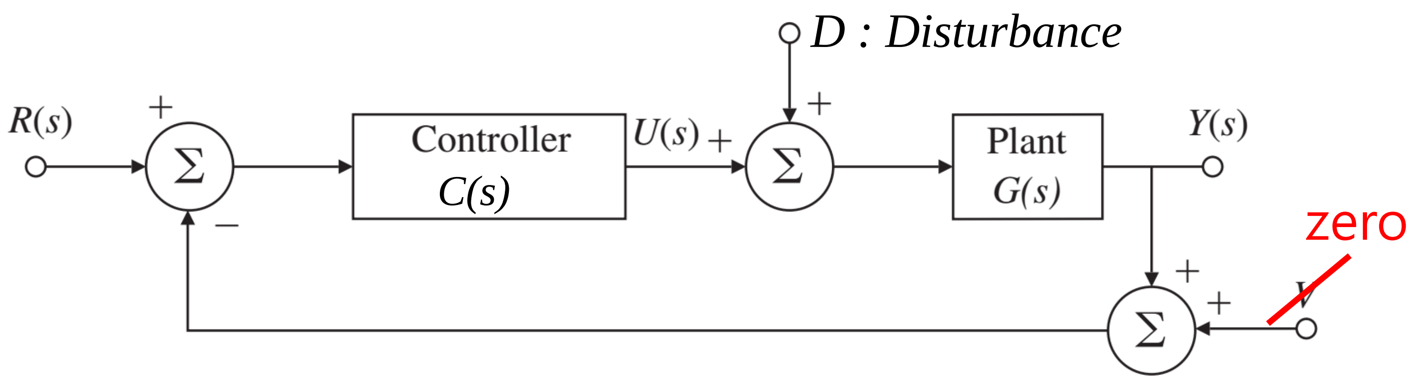
Frequency Shaping (Revisited)



$$Y(s) = \frac{CG}{1+CG}R + \frac{G}{1+CG}D \quad \Rightarrow \quad E(s) = \frac{1}{1+CG}R + \frac{G}{1+CG}D$$

- C(s) should be larger for control performance, which means to minimize the tracking error.
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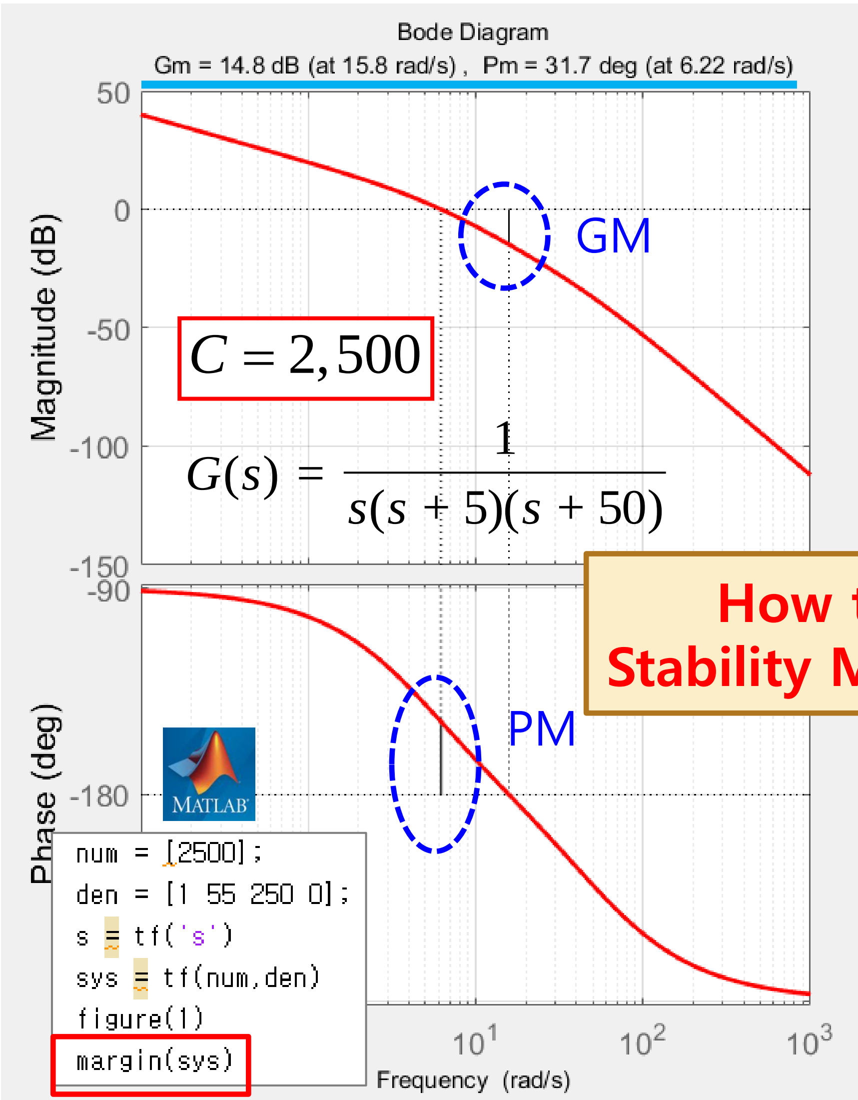
Case 1: C(s) = 2,500 VS. Case 2: C(s) = 5,000

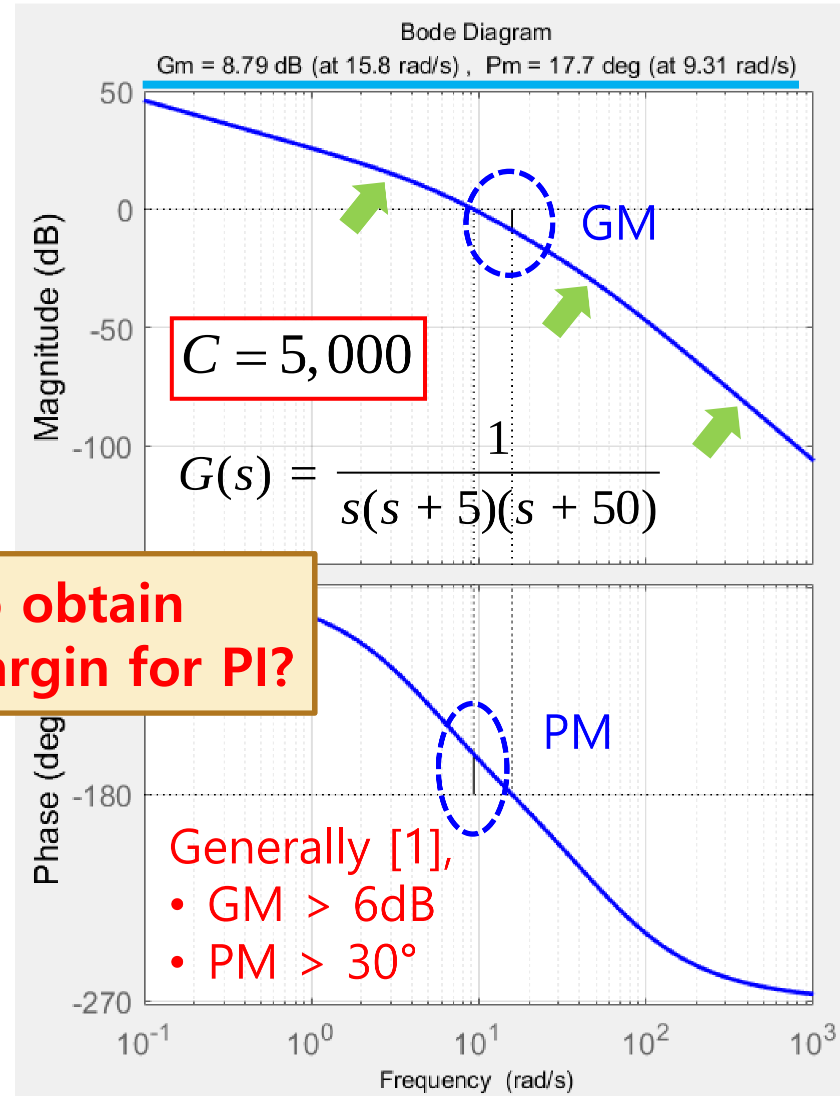
Frequency Shaping (Revisited) (cont'd)



$C = 2,500$

$$G(s) = \frac{1}{s(s+5)(s+50)}$$

```
num = [2500];
den = [1 55 250 0];
s = tf('s')
sys = tf(num,den)
figure(1)
margin(sys)
```



$C = 5,000$

$$G(s) = \frac{1}{s(s+5)(s+50)}$$

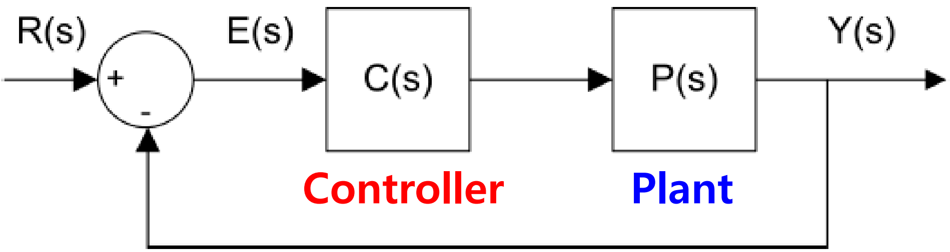
How to obtain Stability Margin for PI?

Generally [1],
- GM > 6dB
- PM > 30°

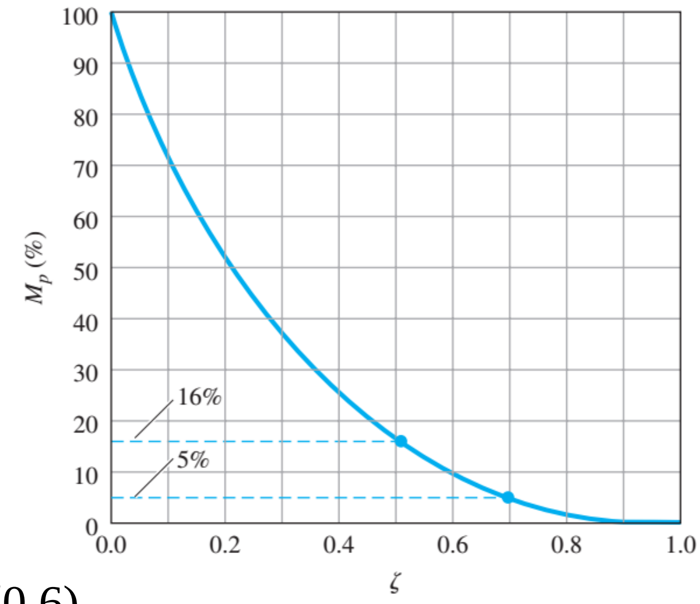
Reference: Safonov, M. G., 1980, *Stability and Robustness of Multivariable Feedback Systems*, MIT press.

Stability Margin with PI Controller

- Case 1: First-order System + PI control (Revisited)



$$C(s)_{PI} = k_p + \frac{k_i}{s} \quad P(s) = \frac{b}{s+a}$$



- Step 1: step response requirement

(1) Overshoot = 10%

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \Rightarrow \zeta \approx 0.6$$

(2) Rising time = 0.1 sec

$$T_r = \frac{\pi - \varphi}{\omega_n \sqrt{1-\varsigma^2}}$$

$$where \cos\varphi = \varsigma$$

$$\omega_n = \frac{3.14 - \cos^{-1}(0.6)}{0.1 \times \sqrt{1-0.6^2}} = 27.7\,[\text{rad/s}]$$

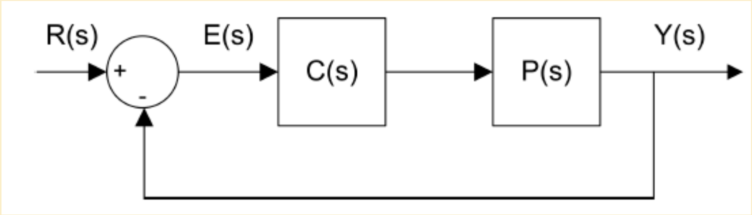
Stability Margin with PI Controller

- **Case 1: First-order System + PI control (Revisited)**
  - Step 2: Check desired pole locations

$$s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2} = 0$$

$$\Rightarrow s^2 + 2(0.6)(27.7)s + 27.7^2 = 0$$

$$s_{1,2} = -16.6 \pm 22.1j$$



$$C(s)_{PI} = k_p + \frac{k_i}{s} \qquad P(s) = \frac{b}{s+a}$$

  - Step 3: Calculate PI gains based on model matching

$$s^2 + (a + k_p b)s + k_i b$$

$$= s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}$$

$$s^1 \rightarrow k_p = \frac{1}{b}(2\zeta\omega_n - a)$$

$$s^0 \rightarrow k_i = \frac{\omega_n^{\ 2}}{b}$$

$a = 30$

$b = 30$

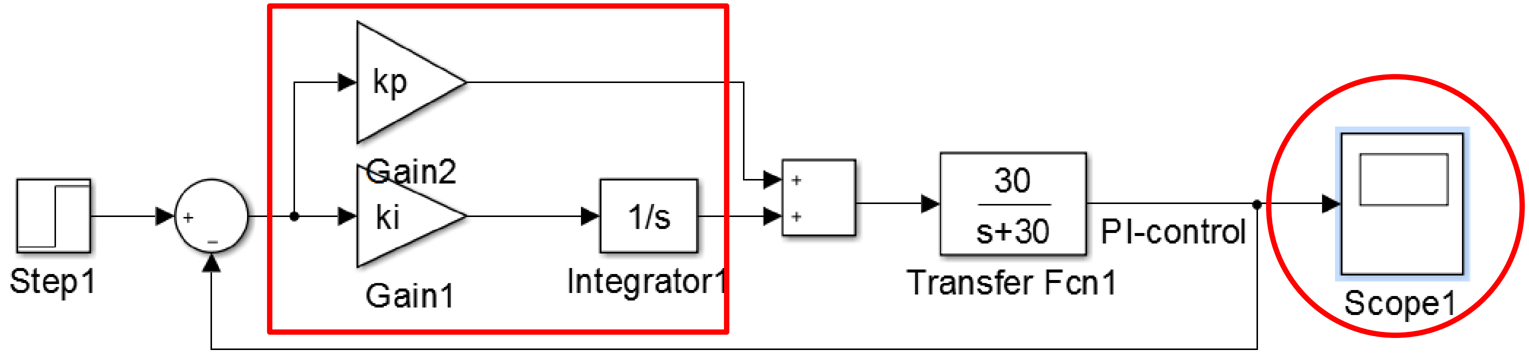
$\zeta \approx 0.6$

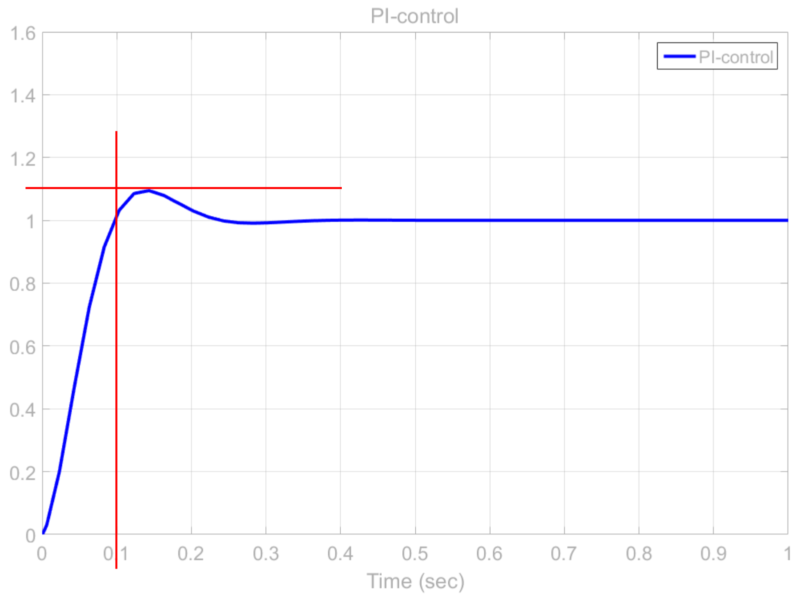
$\omega_n = 27.7\,[\text{rad/s}]$

$$\Rightarrow \quad k_p = 0.1, \quad k_i = 25.5$$

Stability Margin with PI Controller

- **Case 1: First-order System + PI control (Revisited)**
  - Step 4: Simulation model & result





Rising time = 0.1 sec ✓

Overshoot = 10 % ✓

Then, how to check the Stability Margin?

Stability Margin with PI Controller

```
s = tf('s');
a = 30;                 % plant model parameter
b = 30;                 % plant model parameter
plant = tf([b],[1 a]);  % plant model
```

$$P(s) = \frac{30}{s+30}$$

```
kp1 = 0.1;
ki1 = 25.5;
control_PI_1 = tf([kp1 ki1],[1 0]);
sys1 = control_PI_1 * plant;
subplot(121)
margin(sys1)
grid
```

$$C(s)_{PI,case1} = k_p + \frac{k_i}{s} = \frac{0.1s + 25.5}{s}$$

$$C(s)_{PI,case1} P(s) = \frac{0.1s + 25.5}{s} \cdot \frac{30}{s+30} \text{: Open Loop TF}$$

```
kp2 = 0.1;
ki2 = 50;
control_PI_2 = tf([kp2 ki2],[1 0]);
plant = tf([b],[1 a]);
sys2 = control_PI_2 * plant;
subplot(122)
margin(sys2)
grid
```

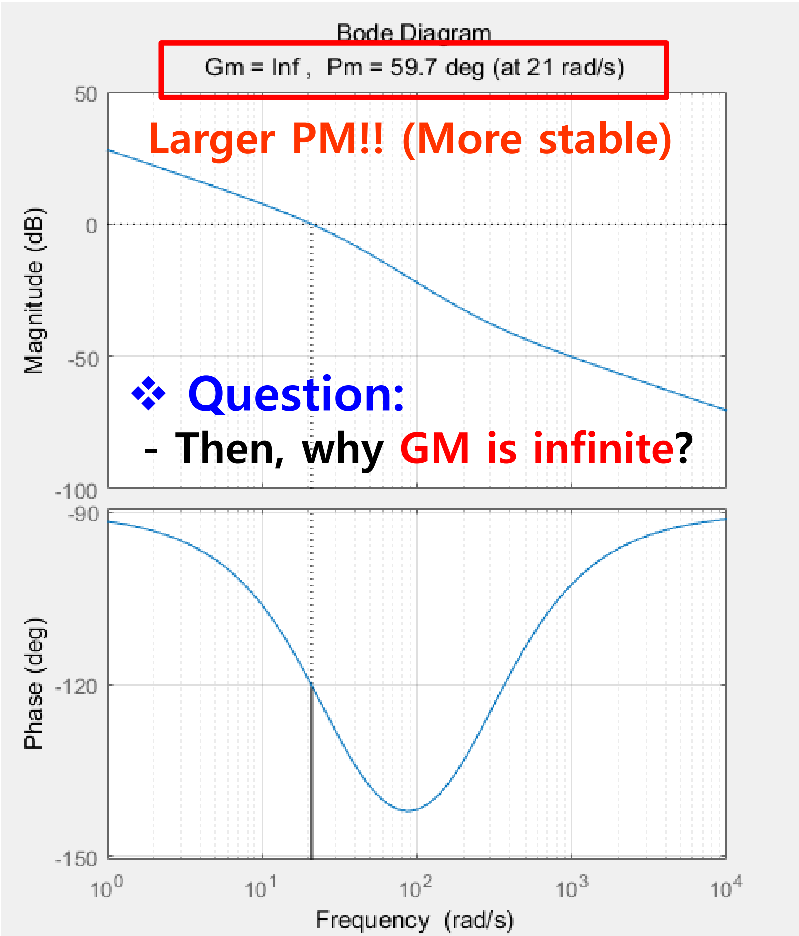
$$C(s)_{PI,case2} = k_p + \frac{k_i}{s} = \frac{0.1s + 50}{s}$$

$$C(s)_{PI,case2} P(s) = \frac{0.1s + 50}{s} \cdot \frac{30}{s+30} \text{: Open Loop TF}$$

Stability Margin should be checked with Open-Loop System [C(s)P(s)]!!

Stability Margin with PI Controller (cont'd)

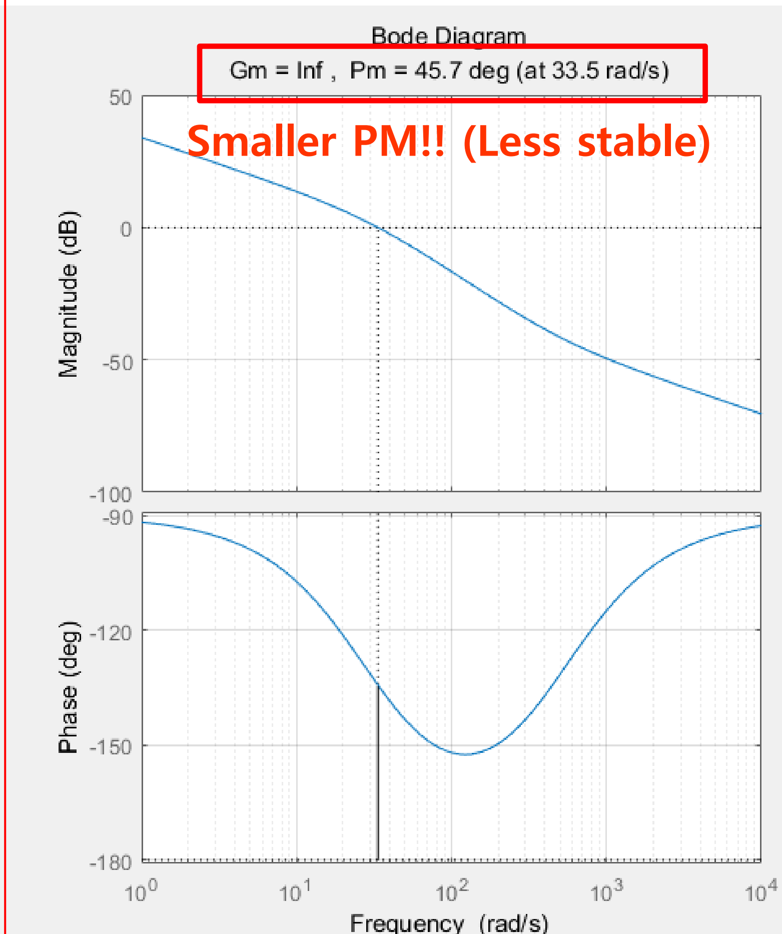
$$C(s)_{PI,case1}P(s) = \frac{0.1s+25.5}{s}\cdot\frac{30}{s+30} = \frac{s(3+765/s)}{s(s+30)}$$



Larger PM!! (More stable)

❖ **Question:**
- Then, why **GM is infinite**?

$$C(s)_{PI,case2}P(s) = \frac{0.1s+50}{s}\cdot\frac{30}{s+30} = \frac{s(3+1500/s)}{s(s+30)}$$



Smaller PM!! (Less stable)

Then, one more question:
- How to check the Bandwidth of Feedback Control System?

Bandwidth with PI Controller

```
s = tf('s');
a = 30;                    % plant model parameter
b = 30;                    % plant model parameter
plant = tf([b],[1 a]);     % plant model
```

$$P(s) = \frac{30}{s+30}$$

```
kp1 = 0.1;
ki1 = 25.5;
control_PI_1 = tf([kp1 ki1],[1 0]);
sys1 = control_PI_1 * plant;
sys1_CL = sys1/(1+sys1);
subplot(121)
bode(sys1_CL)
bandwidth(sys1_CL)
grid
```

$$C(s)_{PI,case1} = k_p + \frac{k_i}{s} = \frac{0.1s+25.5}{s}$$

$$C(s)_{PI,case1}P(s) = \frac{0.1s+25.5}{s}\cdot\frac{30}{s+30} : \text{Open Loop TF}$$

$$H(s)_{PI,case1} = \frac{C(s)_{PI,case1}P(s)}{1+C(s)_{PI,case1}P(s)} : \text{Closed Loop TF}$$

Bandwidth of Closed Loop TF with PI,Case1

```
kp2 = 0.1;
ki2 = 50;
control_PI_2 = tf([kp2 ki2],[1 0]);
plant = tf([b],[1 a]);
sys2 = control_PI_2 * plant;
sys2_CL = sys2/(1+sys2);
subplot(122)
bode(sys2_CL)
bandwidth(sys2_CL)
grid
```

$$C(s)_{PI,case2} = k_p + \frac{k_i}{s} = \frac{0.1s+50}{s}$$

$$C(s)_{PI,case2}P(s) = \frac{0.1s+50}{s}\cdot\frac{30}{s+30} : \text{Open Loop TF}$$

$$H(s)_{PI,case2} = \frac{C(s)_{PI,case2}P(s)}{1+C(s)_{PI,case2}P(s)} : \text{Closed Loop TF}$$

Bandwidth of Closed Loop TF with PI,Case2

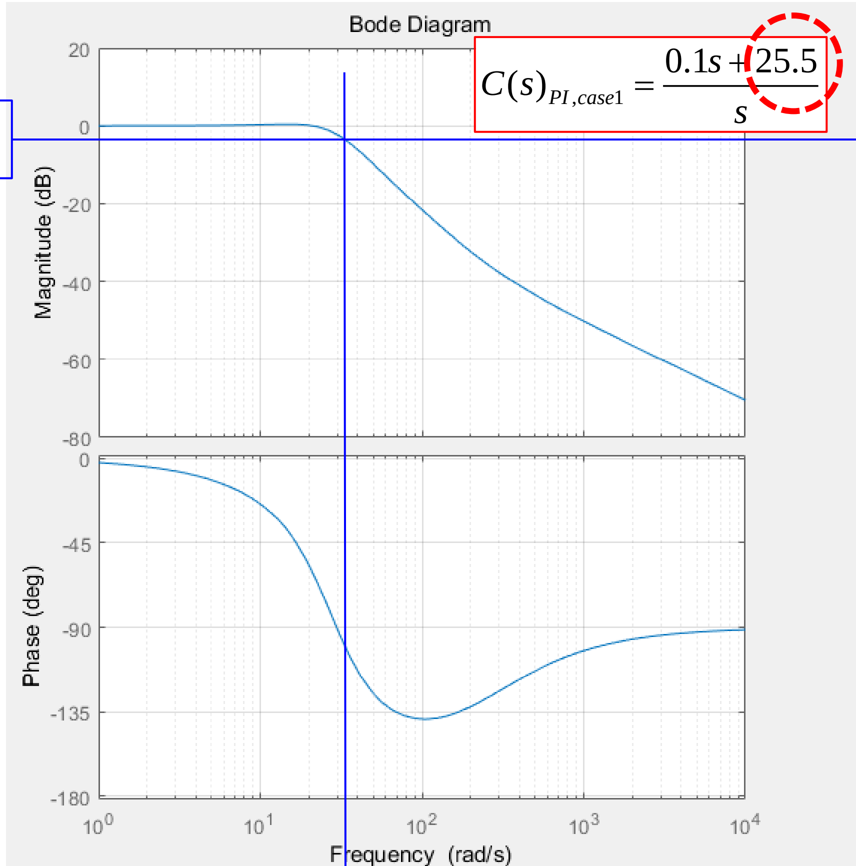
Bandwidth with PI Controller (cont'd)

$$H(s)_{PI,case1} = \frac{C(s)_{PI,case1}P(s)}{1+C(s)_{PI,case1}P(s)}$$

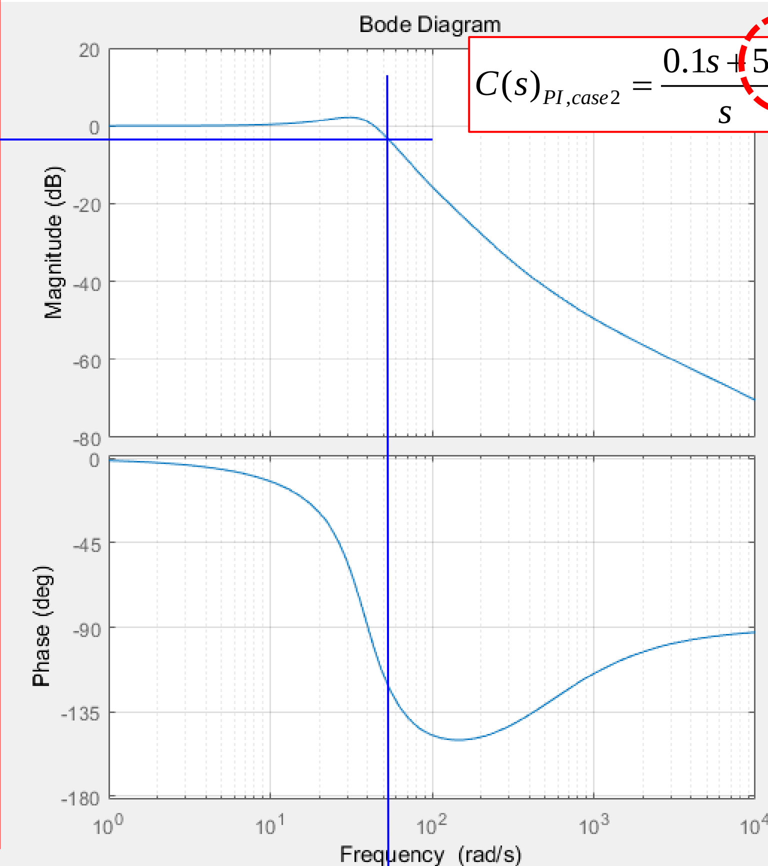
$$H(s)_{PI,case2} = \frac{C(s)_{PI,case2}P(s)}{1+C(s)_{PI,case2}P(s)}$$

-3dB

$$C(s)_{PI,case1} = \frac{0.1s + 25.5}{s}$$



$$C(s)_{PI,case2} = \frac{0.1s + 50}{s}$$



Bandwidth = 32.0 rad/s

Bandwidth = 52.4 rad/s

Bandwidth with PI Controller (cont'd)

- **Bode Plots: Open-Loop VS. Closed-Loop**

$$H(s) = C(s)P(s) = \frac{k_p s + k_i}{s}\frac{b}{s+a}$$

$$= \frac{\cancel{s}(k_p b + k_i b / s)}{\cancel{s}(s+a)}$$

$$H(j\omega) = \frac{\{k_p b + k_i b / (j\omega)\}}{(j\cancel{\omega}) + a}$$

$$if \;\; \omega \to 0 \;\; \text{then, } H(j\omega) \to \frac{k_i b}{j\omega} \to \infty$$

$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{\frac{k_p s + k_i}{s}\frac{b}{s+a}}{1 + \frac{k_p s + k_i}{s}\frac{b}{s+a}}$$

$$= \frac{k_p b s + k_i b}{s^2 + as + k_p bs + k_i b}$$

$$H(j\omega) = \frac{k_p b(j\cancel{\omega}) + k_i b}{(j\cancel{\omega})^2 + a(j\cancel{\omega}) + k_p b(j\cancel{\omega}) + k_i b}$$

$$if \;\; \omega \to 0 \;\; \text{then, } H(j\omega) \to 1$$

Summary

❖ **Summary**:

- Relative stability margin with Bode & Nyquist plot
  - Gain Margin & Phase Margin
  - Control design guidelines: stability, transient, steady-state error, noise reduction and etc.