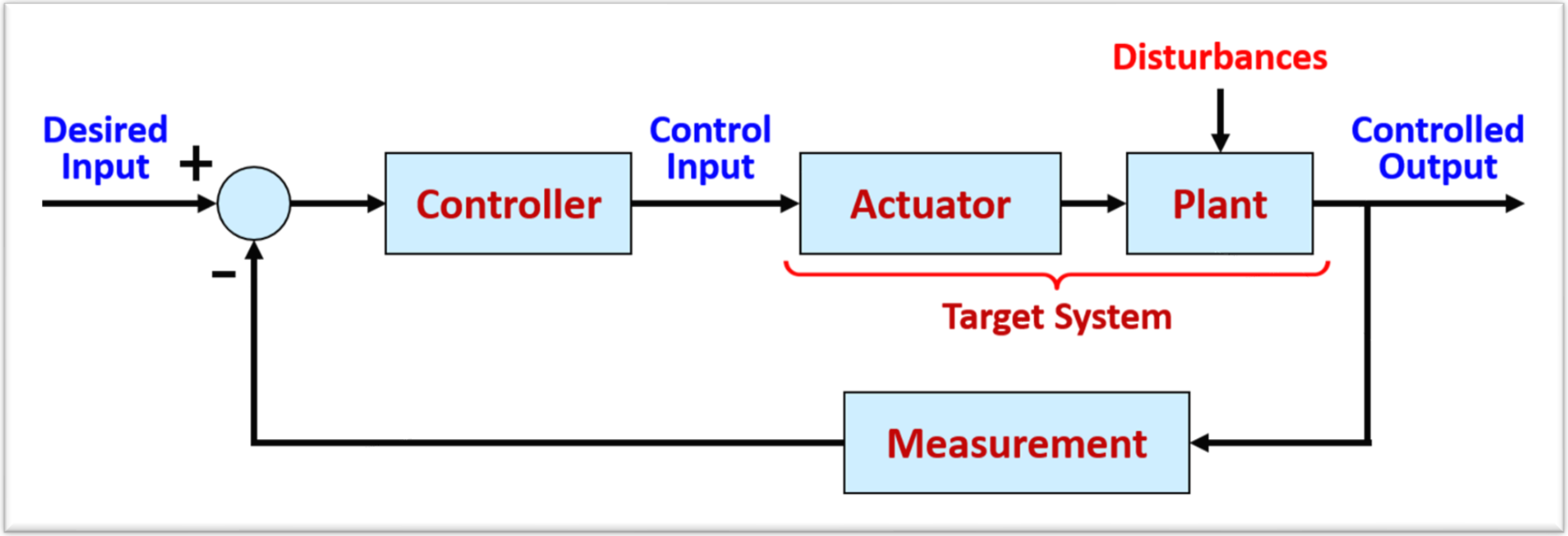
Laplace Transform and Transfer Function

Lecture 2:

- Laplace Transform
- Open-Loop vs. Closed-Loop System



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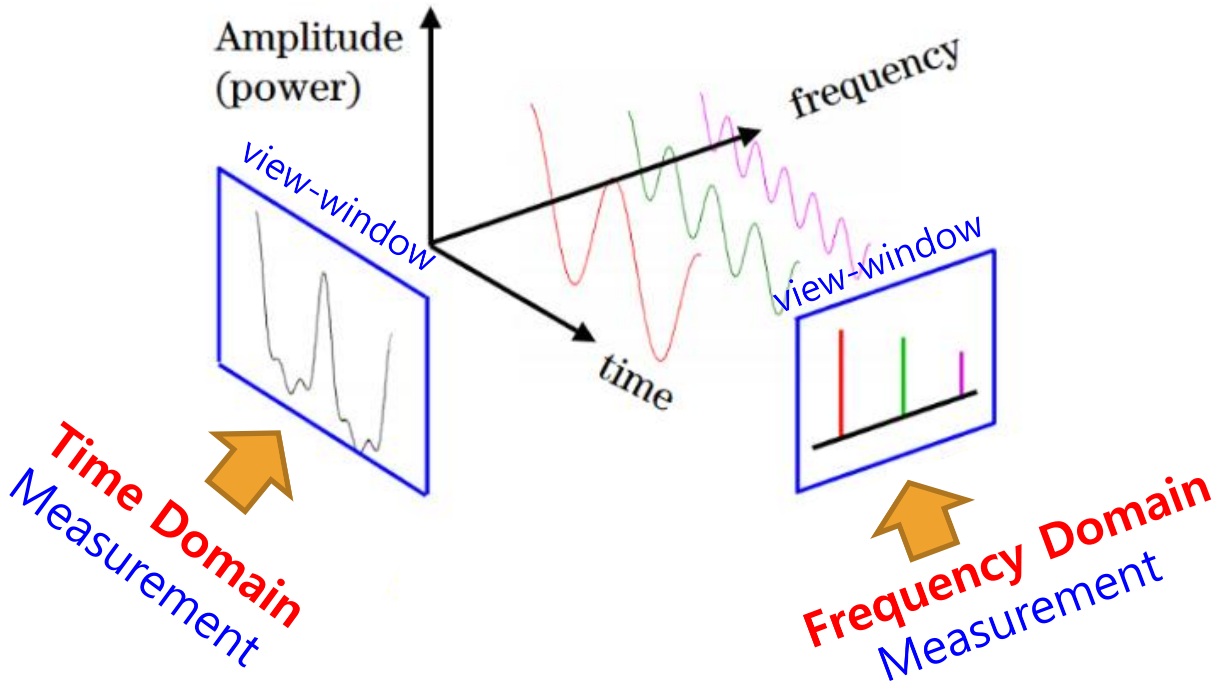
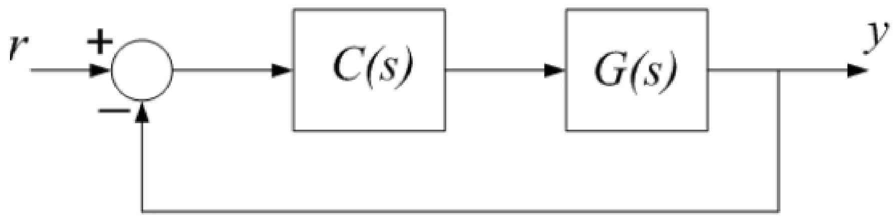
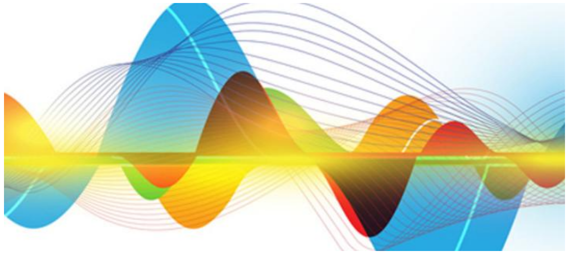
Review of Laplace Transform

❖ **Time Domain vs. Frequency Domain**







Review of Laplace Transform (cont'd)

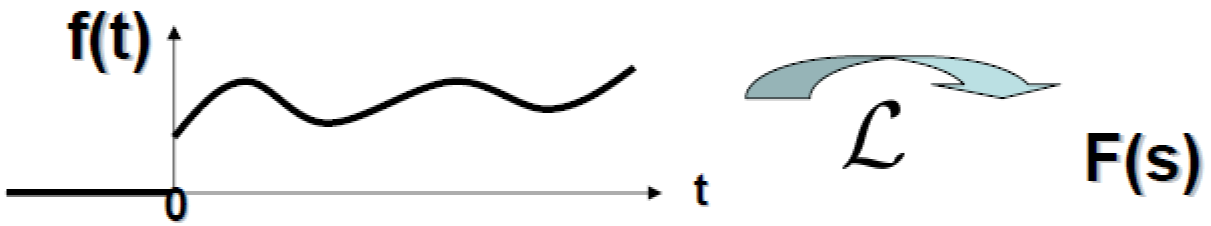
❖ **One of most important mathematics tool in the course !!**

❖ **Definition:**

- For a function f(t) (f(t)=0 for t<0),

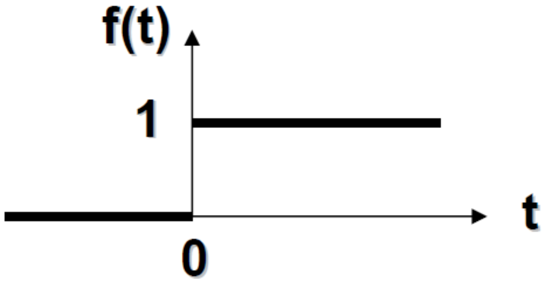
$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

※ s = σ + jω (complex variable)



Examples of Laplace Transform

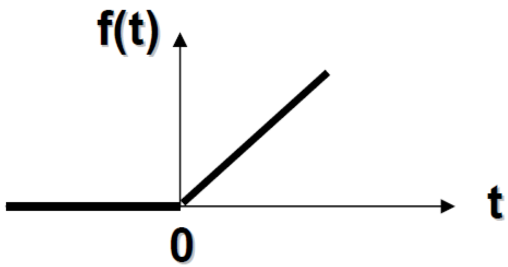
❖ Unit Step Function



$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$F(s) = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$$

❖ Unit Ramp Function



$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$F(s) = \int_0^\infty t e^{-st} dt \underset{\uparrow}{=} -\frac{1}{s}\left[t e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s^2}$$

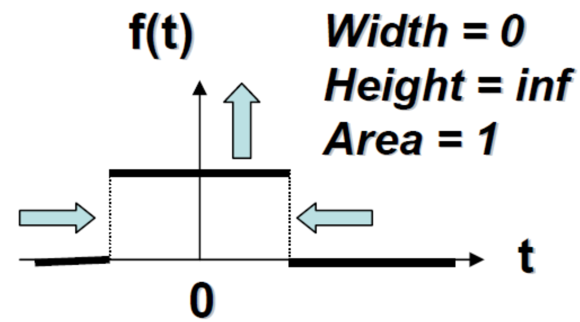
(Integration by parts)

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Examples of Laplace Transform (cont'd)

❖ Unit Impulse Function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t) g(t) dt = g(0)$$

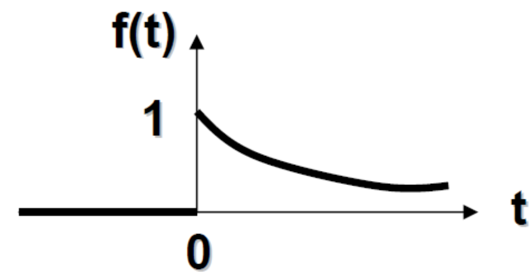


$$\Rightarrow \quad F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-s \cdot 0} = 1$$

❖ Exponential Function

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

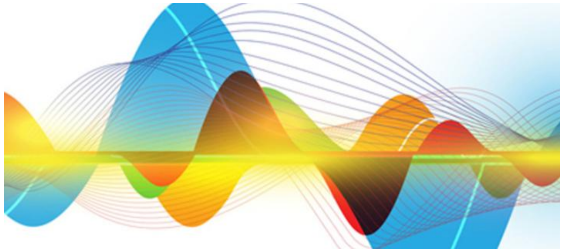
※ α : time constant

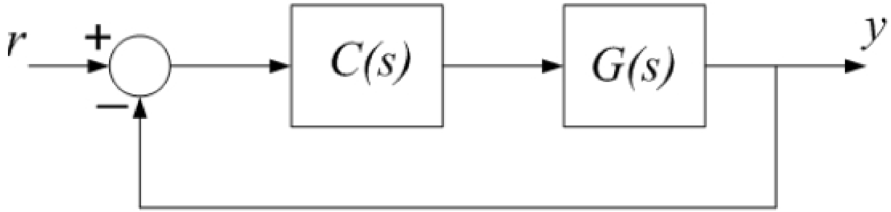


$$\Rightarrow \quad F(s) = \int_0^{\infty} e^{-\alpha t} \cdot e^{-st} dt = -\frac{1}{s+\alpha} \left[e^{-(s+\alpha)t} \right]_0^{\infty} = \frac{1}{s+\alpha}$$

Examples of Laplace Transform (cont'd)

❖ Sine & Cosine Functions:





$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \qquad \mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

❖ Notes:

- Instead of computing Laplace transform for each function, and/or memorizing complicated Laplace transform.
- We can use the Laplace transform table!!

Examples of Laplace Transform (cont'd)

(Forward) Laplace Transform

$f(t) \quad (t \geq 0) \xrightarrow{\mathcal{L}}$	$F(s)$
$\delta(t)$ (unit impulse)	1
$u(t)$ (unit step)	$\frac{1}{s}$
t (unit ramp)	$\frac{1}{s^2}$
t^n (n>-1)	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$

$\xleftarrow{\mathcal{L}^{-1}}$

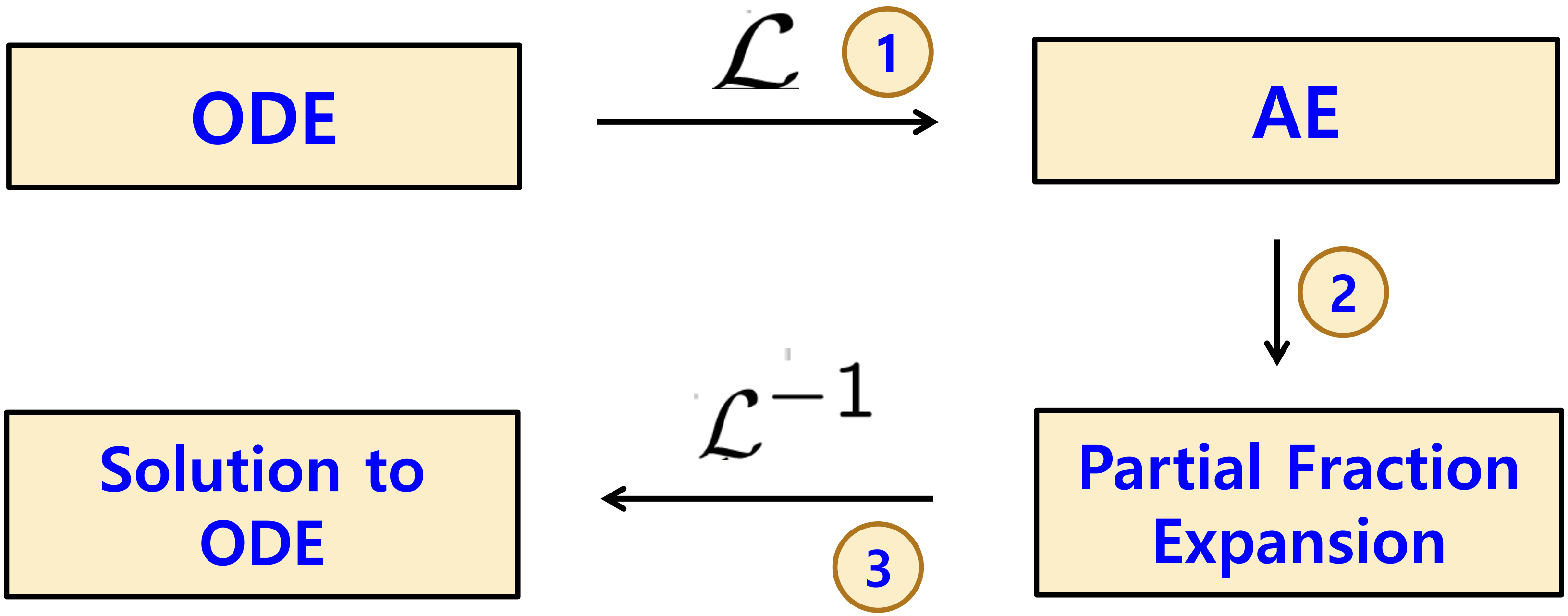
Inverse Laplace Transform

$f(t) \quad (t \geq 0) \xrightarrow{\mathcal{L}}$	$F(s)$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin \omega_n \sqrt{1-\xi^2}\, t \quad (\xi<1)$	$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$
$1-\frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}\, t+\theta\right)$ where $\theta=\cos^{-1}\xi \quad (\xi<1)$	$\frac{\omega_n^2}{s\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$
$\frac{df(t)}{dt}$	$sF(s)-f(0)$
$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
$f(t-a)$	$e^{-as}F(s)$

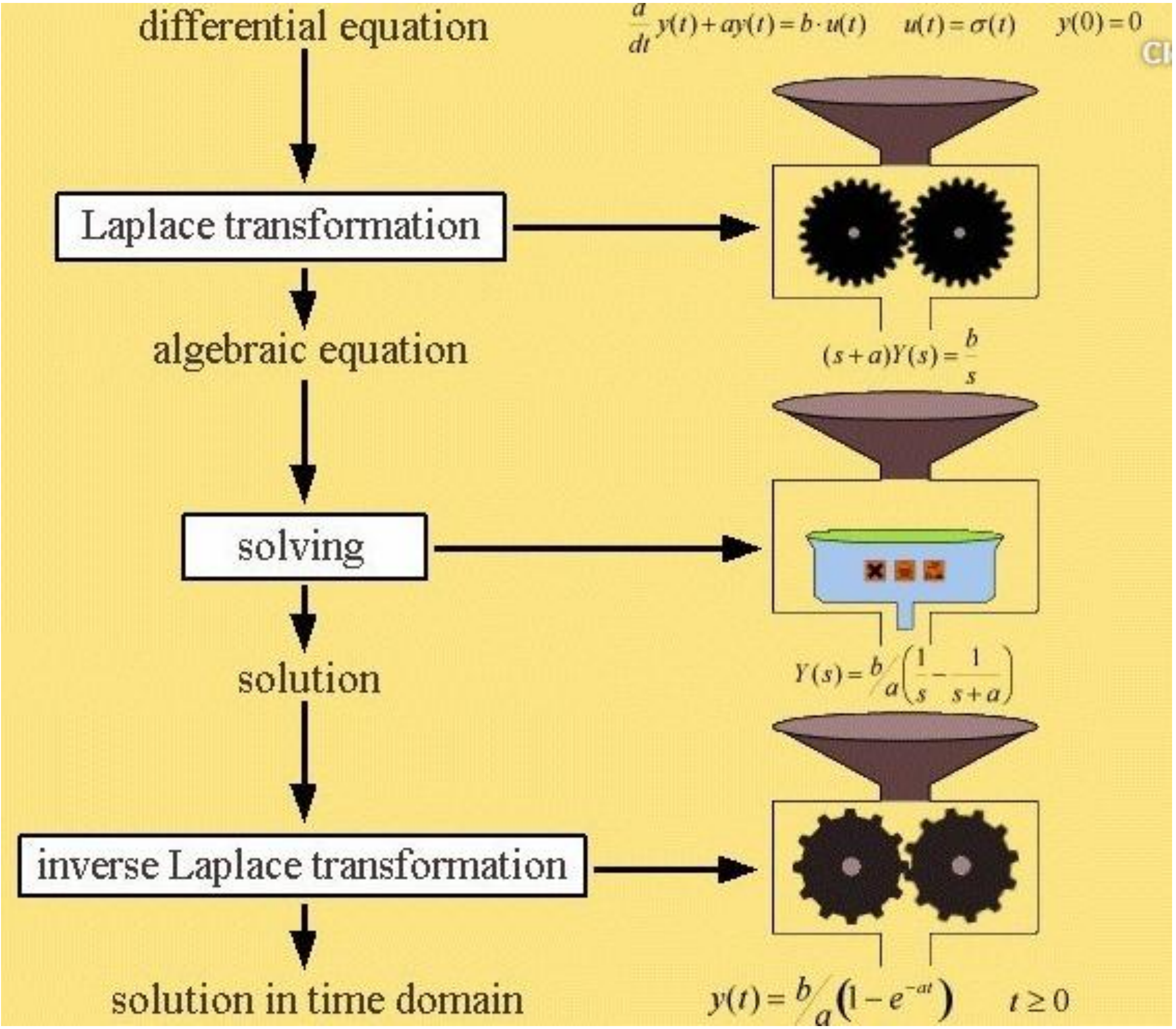
$\xleftarrow{\mathcal{L}^{-1}}$

Major Advantage of Laplace Transform

❖ Transform an Ordinary Differential Equation (ODE) into an Algebraic Equation (AE).



Major Advantage of Laplace Transform



https://www.quora.com/What-is-the-significance-of-the-Laplace-transform

Major Advantage of Laplace Transform

❖ Example 1: ODE with initial conditions (cont'd)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u_s(t),\ y(0)=-1,\ y'(0)=2$$

Step input

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\left\{y'(t)\right\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow \quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$

Major Advantage of Laplace Transform

❖ Example 1: ODE with initial conditions (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0} = A + s\frac{B}{s+1}\Big|_{s\to 0} + s\frac{C}{s+2}\Big|_{s\to 0} \Rightarrow A = sY(s)|_{s\to 0} = \frac{5}{2}$$

Similarly,

$$\begin{aligned} B &= (s+1)Y(s)|_{s\to -1} = \cdots = -5 \\ C &= (s+2)Y(s)|_{s\to -2} = \cdots = \frac{3}{2} \end{aligned}$$

Major Advantage of Laplace Transform

❖ Example 1: ODE with initial conditions (cont'd)

3. Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right\}$$

$$\Rightarrow \quad y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t}\right) u_s(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Properties of Laplace Transform (cont'd)

(1) Linearity (or Superposition)

$$\boxed{\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)}$$

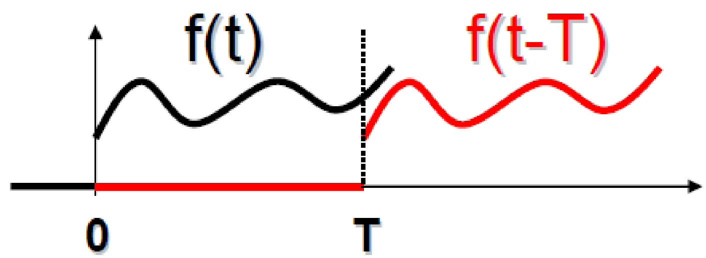
Proof.
$$\begin{aligned}\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} &= \int_0^\infty (\alpha_1 f_1(t) + \alpha_2 f_2(t)) e^{-st} dt \\ &= \alpha_1 \underbrace{\int_0^\infty f_1(t) e^{-st} dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t) e^{-st} dt}_{F_2(s)}\end{aligned}$$

Ex.
$$\mathcal{L}\left\{5u_s(t) + 3e^{-2t}\right\} = 5\mathcal{L}\{u_s(t)\} + 3\mathcal{L}\left\{e^{-2t}\right\} = \frac{5}{s} + \frac{3}{s+2}$$

Properties of Laplace Transform (cont'd)

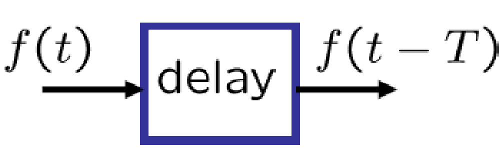
(2) Time Delay

$$\boxed{\mathcal{L}\{f(t-T)u_s(t-T)\} = e^{-Ts}F(s)}$$

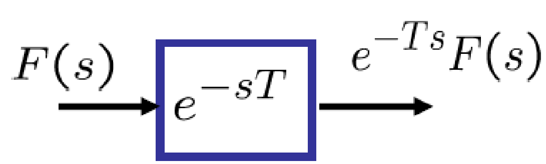


Proof.
$$\mathcal{L}\{f(t-T)u_s(t-T)\}$$
$$= \int_T^{\infty} f(t-T)e^{-st}dt$$
$$= \int_0^{\infty} f(\tau)e^{-s(T+\tau)}d\tau = e^{-sT}F(s)$$

Ex. $$\mathcal{L}\left\{e^{-0.5(t-4)}u_s(t-4)\right\} = \frac{e^{-4s}}{s+0.5}$$



t-domain



s-domain

Properties of Laplace Transform (cont'd)

(3) Differentiation

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)}$$

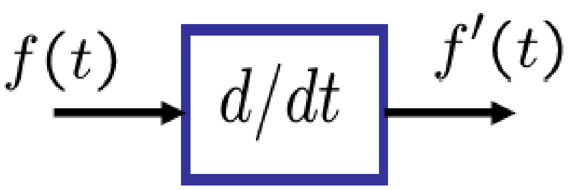
$$\int uv'\,dx = uv - \int u'v\,dx$$

Proof.

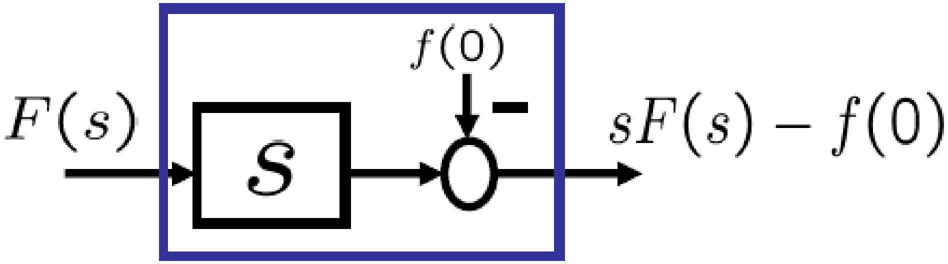
$$\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Ex. $\mathcal{L}\left\{(\cos 2t)'\right\} = s\mathcal{L}\{\cos 2t\} - 1 = \dfrac{s^2}{s^2+4} - 1 = \dfrac{-4}{s^2+4} (= \mathcal{L}\{-2\sin 2t\})$

t-domain



s-domain



Properties of Laplace Transform (cont'd)

(4) Integration

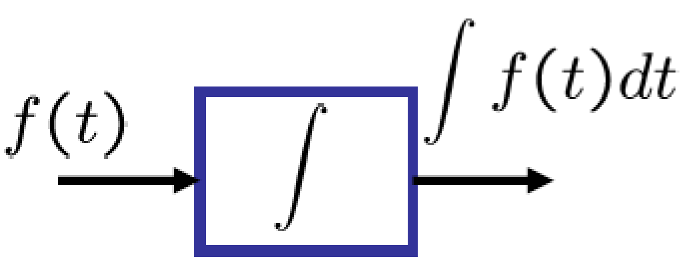
$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}}$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

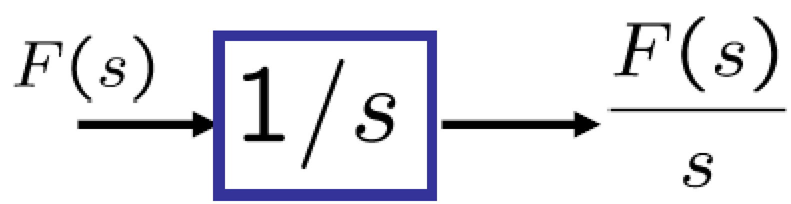
Proof. $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left(\int_0^t f(\tau)d\tau\right) e^{-st}dt$

$$= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right)e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{F(s)}{s}$$

t-domain



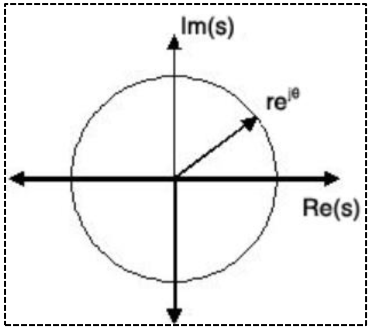
s-domain



Properties of Laplace Transform (cont'd)

(5) Initial Value Theorem

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s) \quad \text{if the limits exist.}$$



S-Plane

Remark: In this theorem, it does not matter if pole location is in LHS or not.

Roots @ LHS (Left Half Space) on S-plane

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s) = 0$

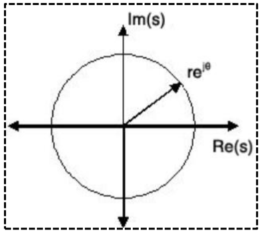
Root @ RHS (Right Half Space) on S-plane

Ex. $F(s) = \dfrac{4}{s^2-4} \Rightarrow \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s) = 0$

Properties of Laplace Transform (cont'd)

(6) Final Value Theorem = DC (direct current) Gain

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \quad \text{if all the poles of sF(s) are in the left half plane (LHP)}}$$



Roots @ LHP on S-plane

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} \dfrac{5}{s^2+s+2} = \dfrac{5}{2}$

Poles of sF(s) are in LHP, so final value thm applies.

Root @ RHP on S-plane

Ex. $Y(s) = \dfrac{3}{s(s-2)}. \Rightarrow y(\infty) = sY(s)|_{s=0} = -\dfrac{3}{2}.$

$\mathcal{L}^{-1} \Downarrow$

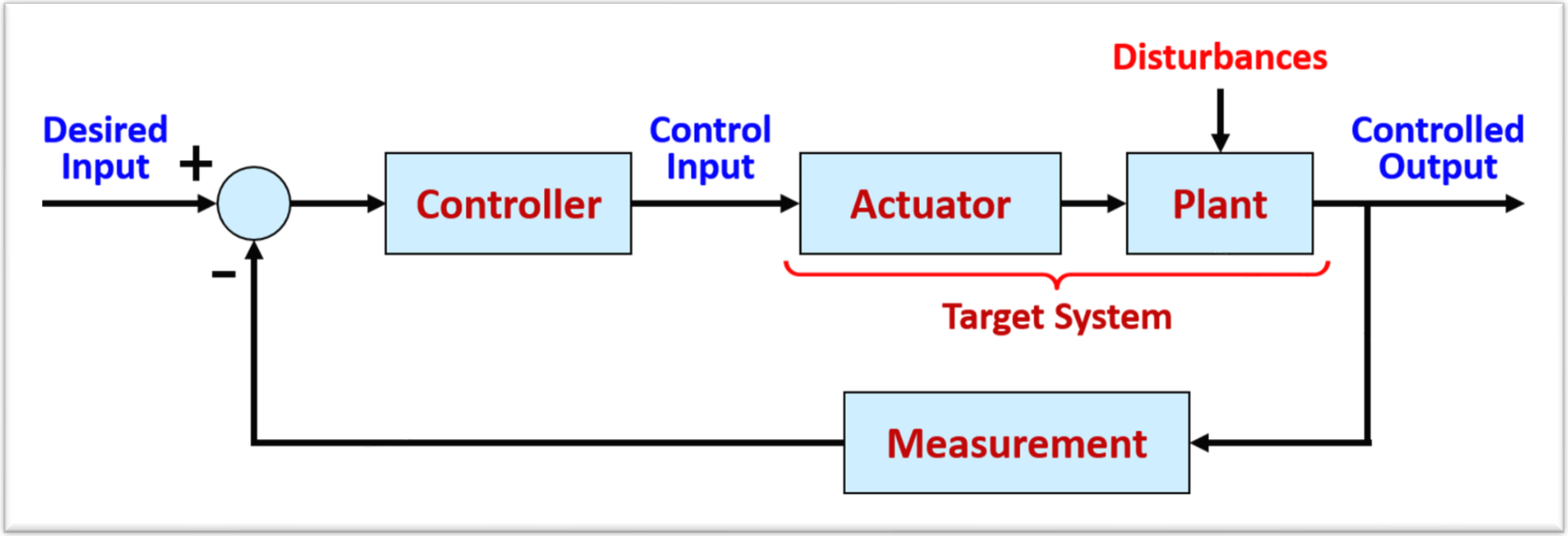
However, $y(t) = \left(-\dfrac{3}{2} + \dfrac{3}{2}e^{2t}\right)1(t),$

❖ Note: the true final value is **unbounded**, because of **unstable pole (root)** at s = 2.

Laplace Transform and Transfer Function

Lecture 2:

- Laplace Transform
- Open-Loop vs. Closed-Loop System



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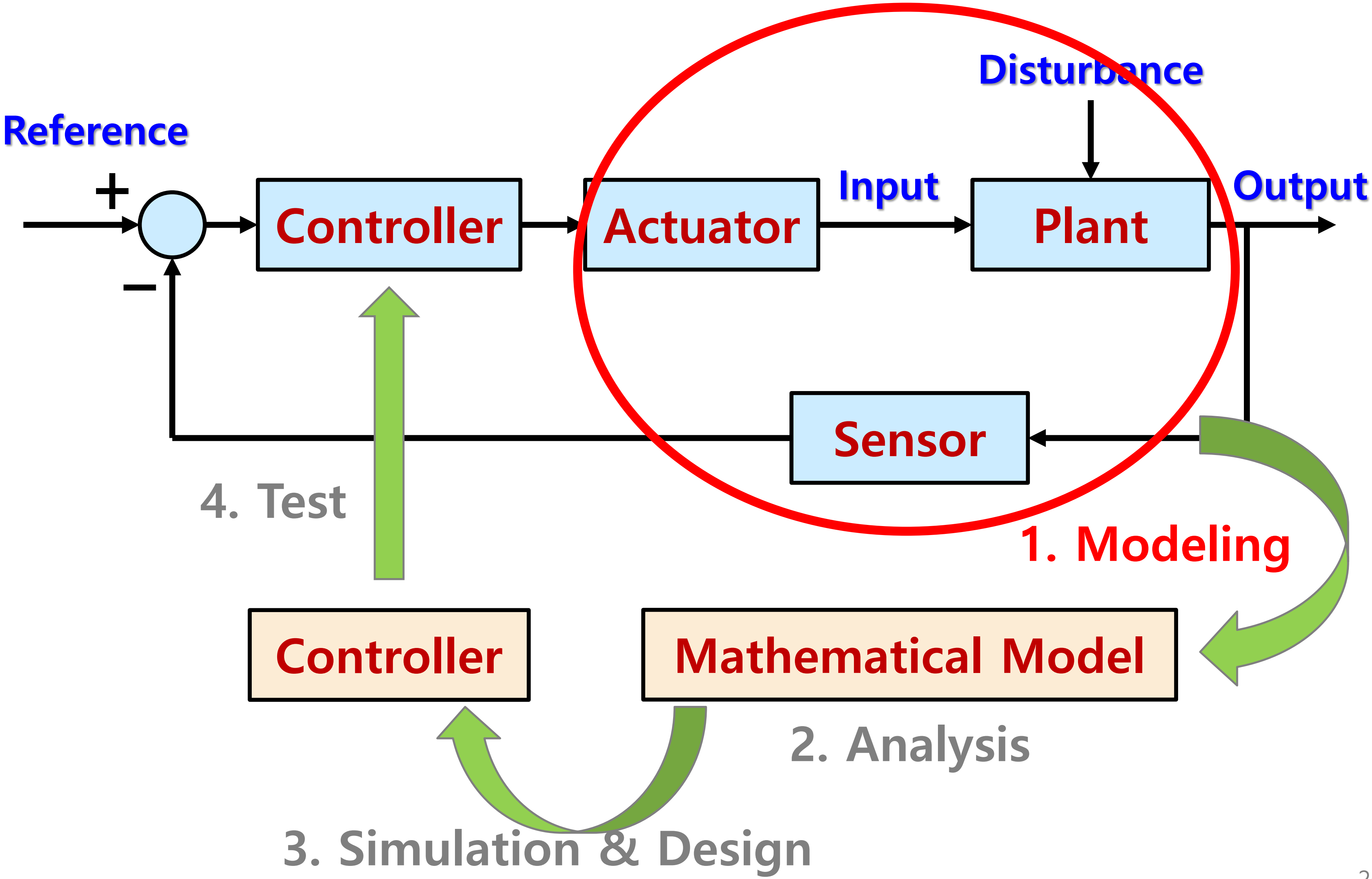
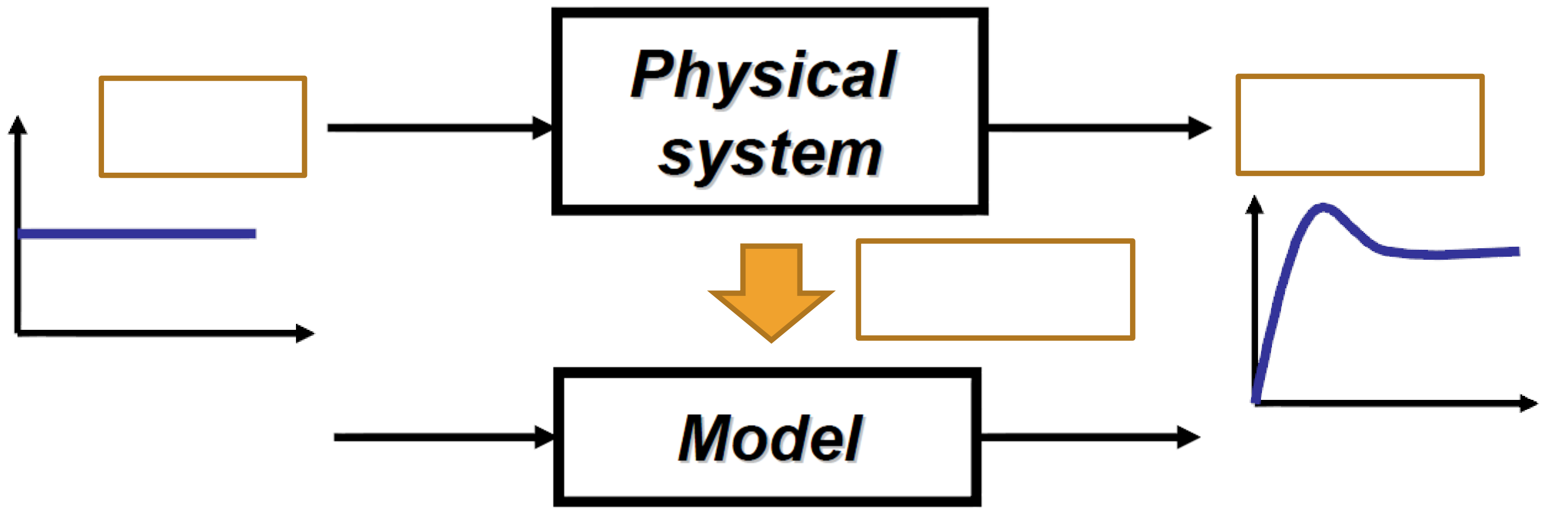
Controller Design Procedure (Revisited)



Mathematical Dynamic Model

- Representation of the input-output (signal) relation of a physical system.



- A mathematical dynamic model is used for the analysis and design of the control systems.

Important Remarks on Models

- Modeling is the one of the most important and difficult tasks in control system design..
- No mathematical model exactly represents a physical system.

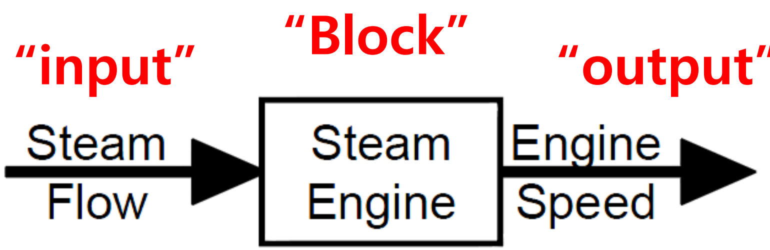
$$\text{Math model} \neq \text{Physical system}$$

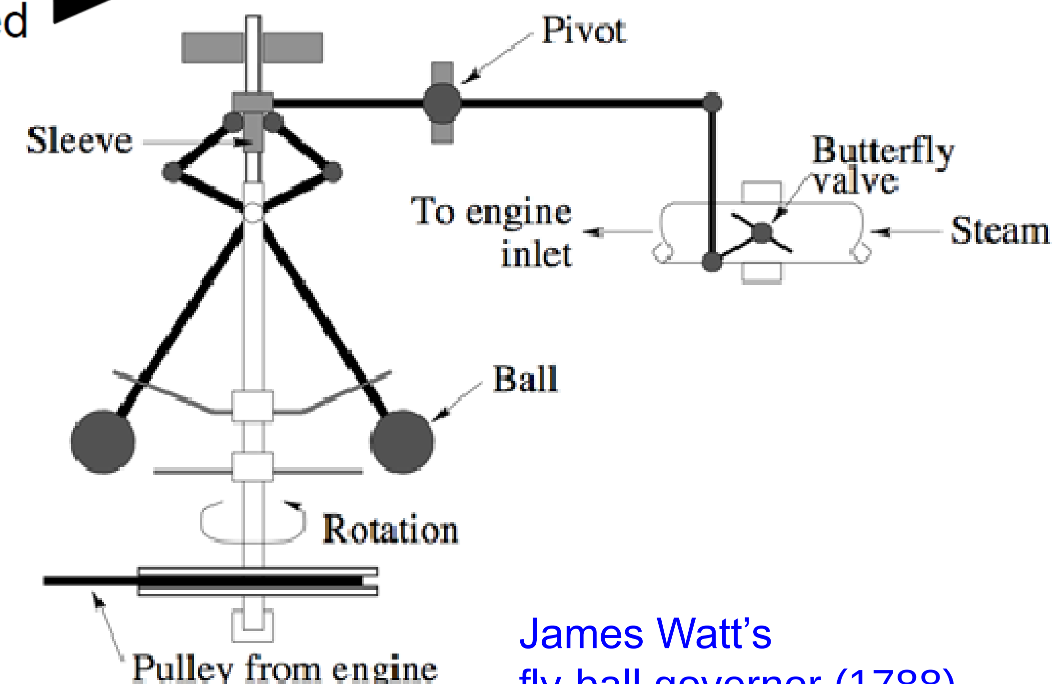
$$\text{Math model} \approx \text{Physical system}$$

- Do not confuse models with physical systems !!
- In this course, we may use the term “system” to mean a mathematical model.

The Block Diagram in Control

- **Communication tool for Engineering Systems.**
  - **Composed of Blocks with inputs and outputs**



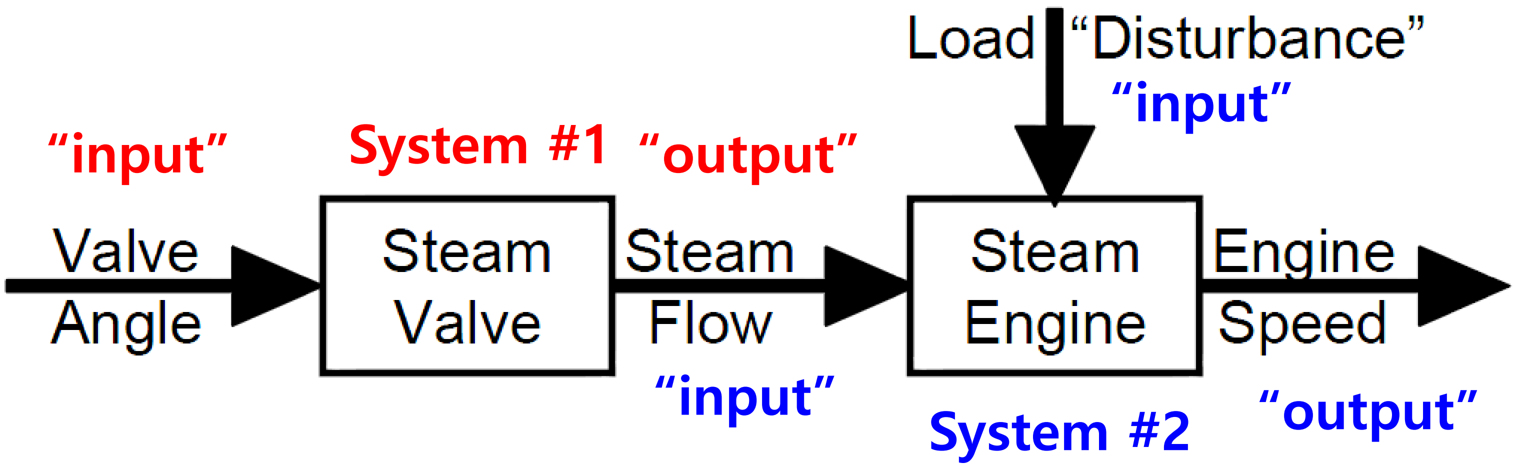


James Watt's
fly-ball governor (1788)

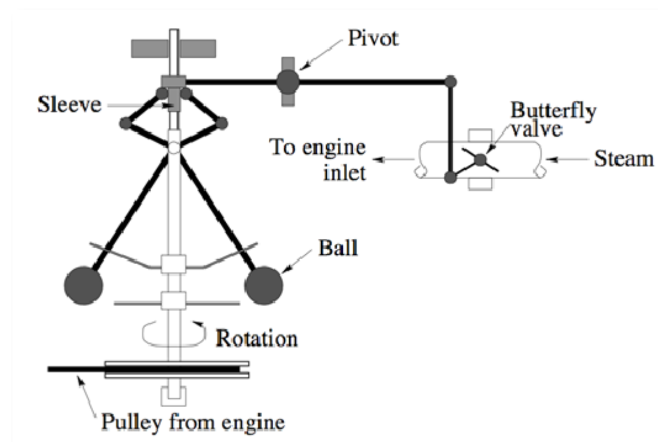
https://www.youtube.com/watch?v=rNN6DjB0N3Qv

The Block Diagram in Control (cont'd)

- **Blocks Connect to form systems.**
  - **Outputs of one block becomes input to another !!**



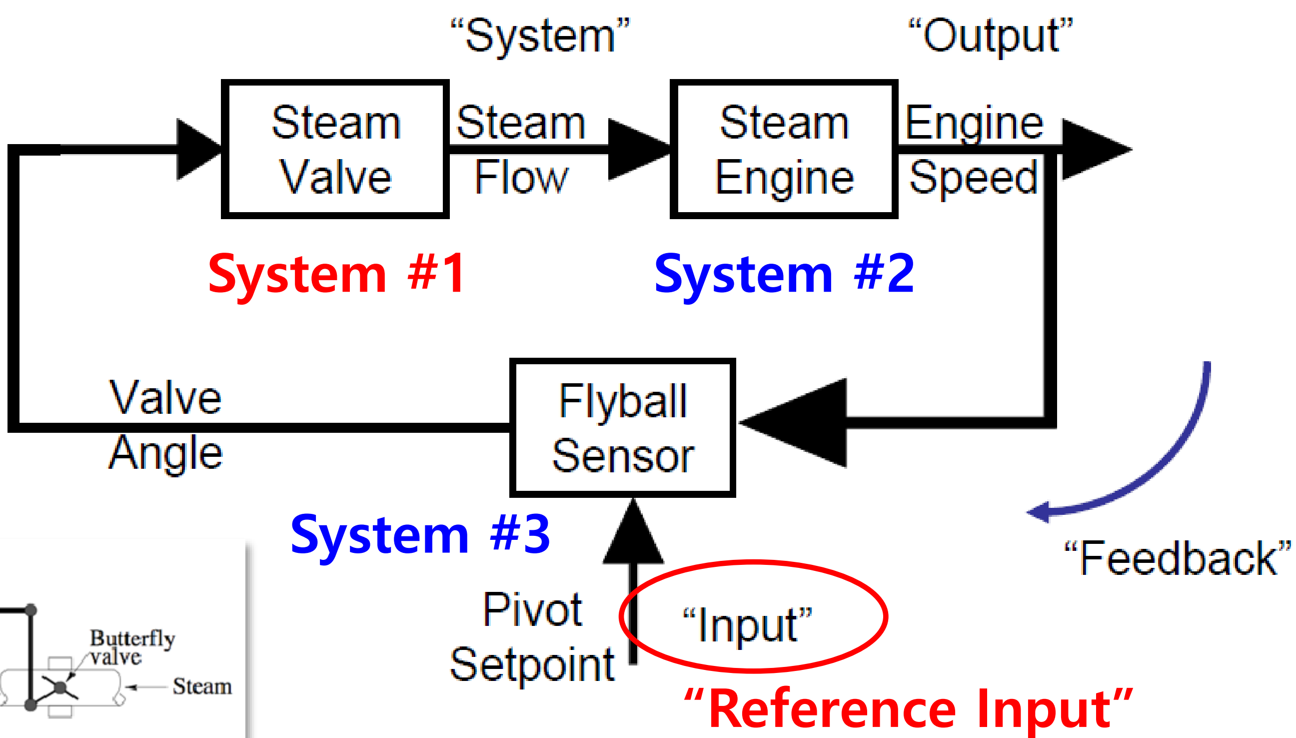
Open-Loop System

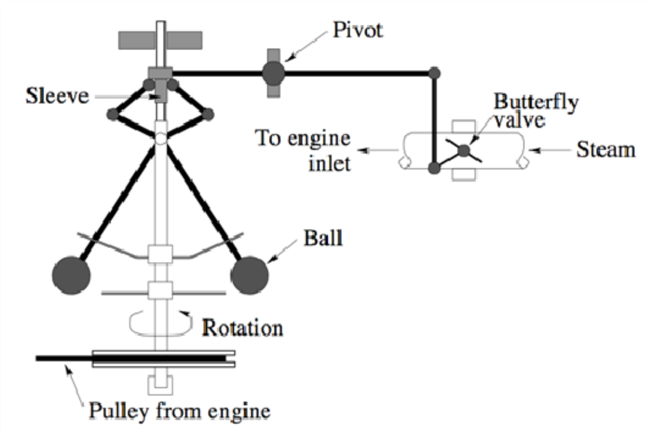


https://www.youtube.com/watch?v=HS_YGZXP2xY

The Block Diagram in Control (cont'd)

- Blocks Connect to form systems.
  - Outputs of one block becomes input to another !!





Closed-Loop System

Transfer Function in Control

- **A transfer function is defined by**

$$G(s) := \frac{Y(s)}{U(s)}$$

Laplace transform of system output

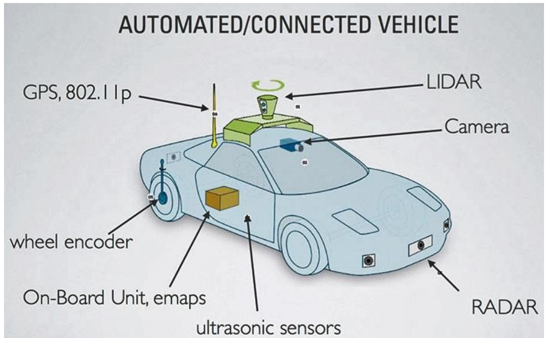
Laplace transform of system input

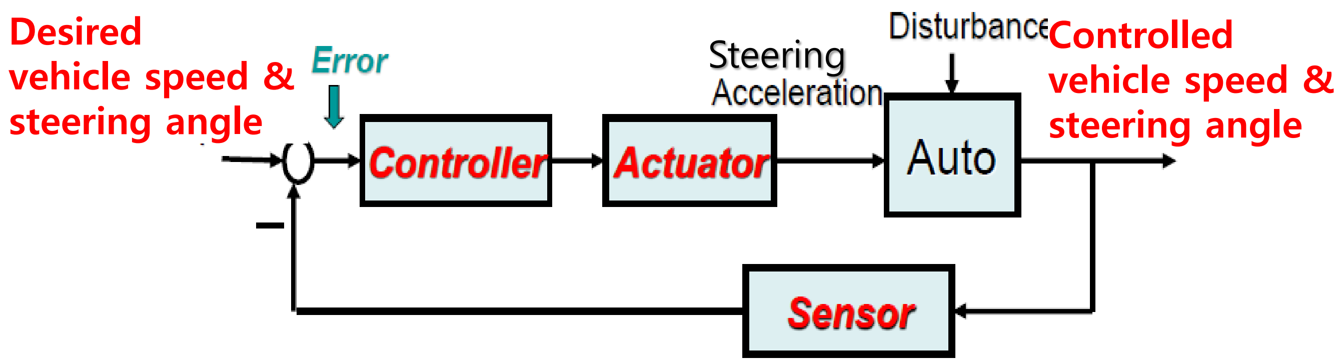
On block diagram representation,

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

- **In this class, all systems are assumed to be at rest. (zero initial condition)**

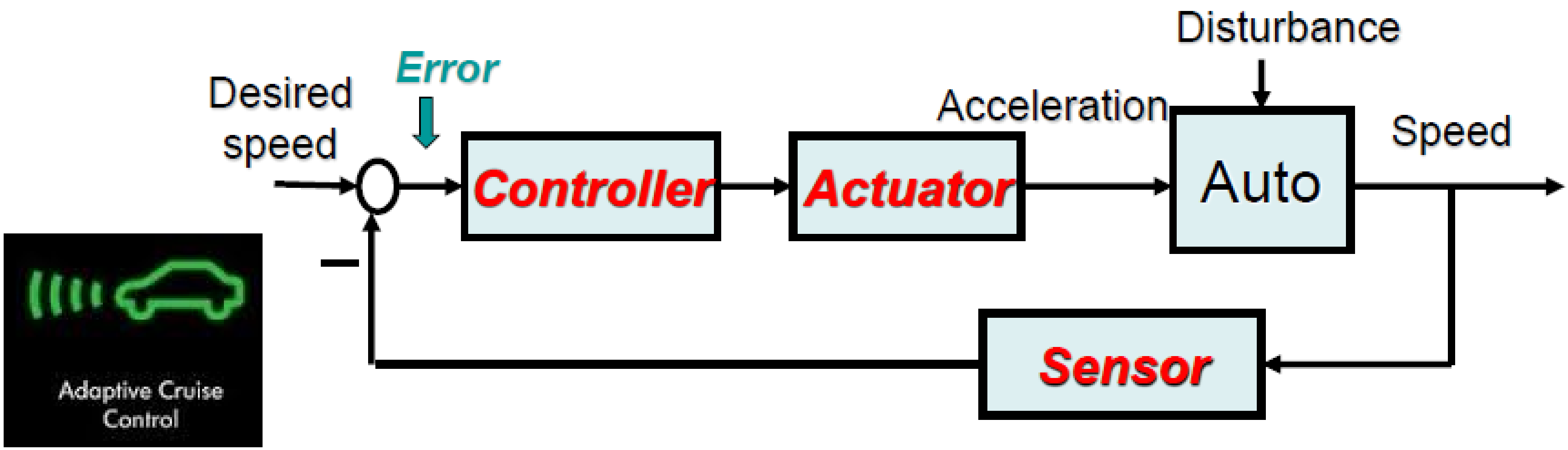
Ex: Fully Autonomous Vehicle Control (Revisited)





- Objective: to **maintain the vehicle speed and steering angle** of the automobile.
- Desired vehicle speed and steering angle should be determined by diverse sensors (LiDAR, Camera, GPS, radar and others) for self-driving.

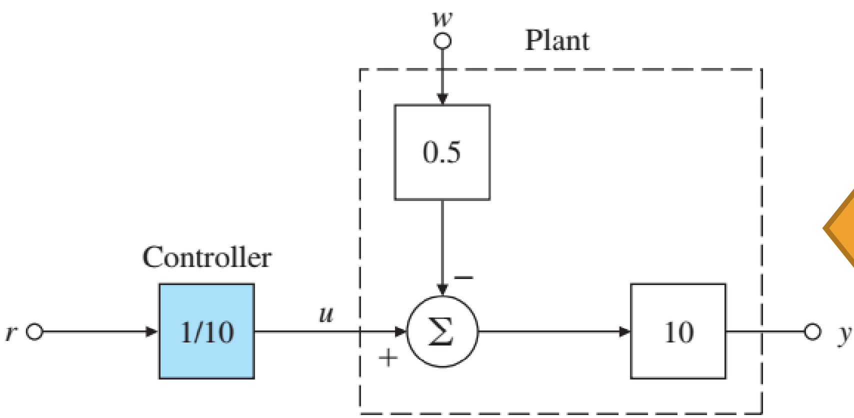
Ex: Automotive Cruise Control (Revisited)

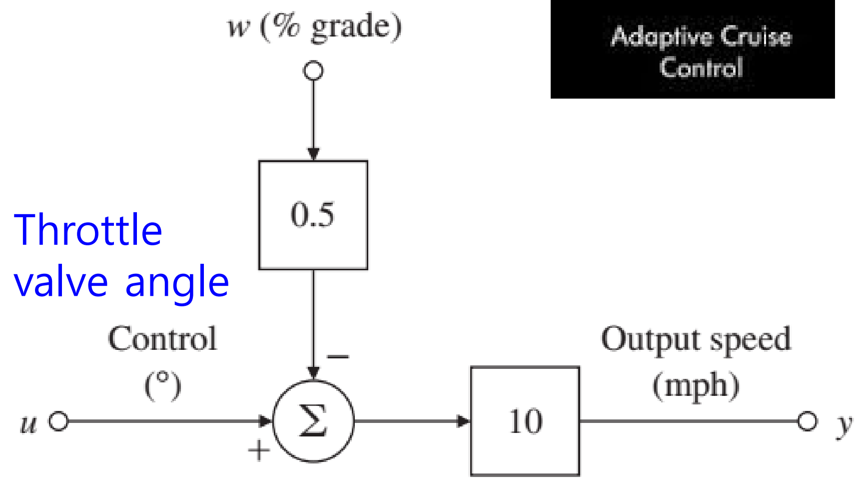


- Objective: to **maintain the speed** of the automobile.
- Cruise control can be both manual and automatic.
- Note the similarity of the diagram above to the diagram in the previous slide !!

Ex: Automotive Cruise Control (cont'd)

- **Question: IF Open-Loop Control??**



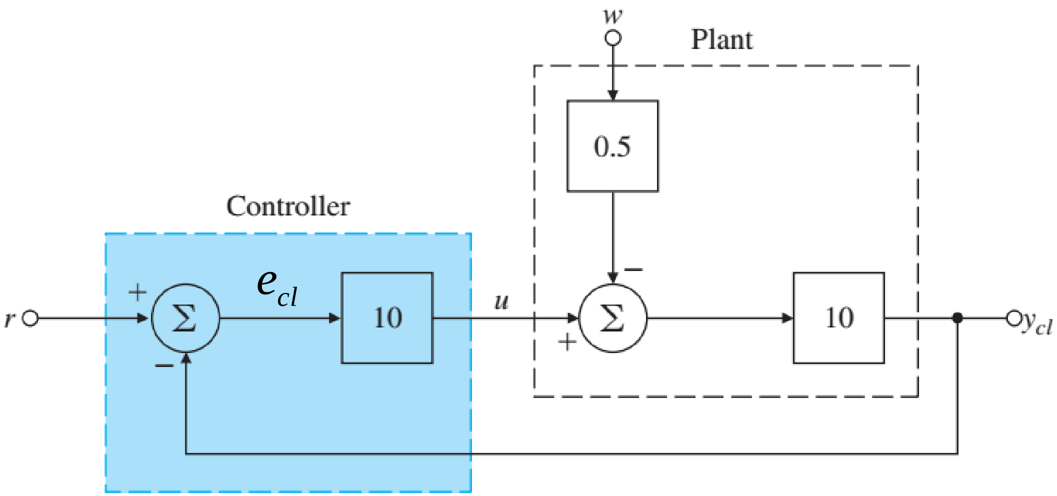


$$
\begin{aligned}
y_{ol} &= 10(u - 0.5w) \\
&= 10\left(\boxed{} - 0.5w\right) \\
&= \boxed{} - 5w.
\end{aligned}
$$

$$
\begin{aligned}
e_{ol} &= r - y_{ol} \\
&= r - (r - 5w) \\
&= 5w
\end{aligned}
$$

Ex: Automotive Cruise Control (Cont'd)

- **Question: IF Closed-loop control (Feedback)??**

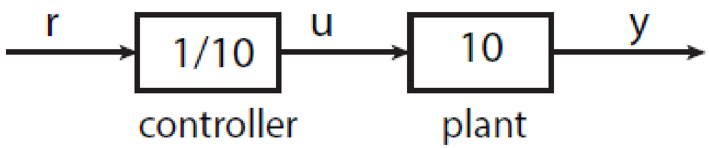


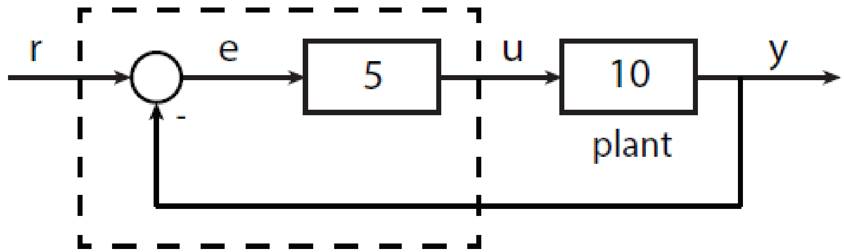
$$y_{cl} = \boxed{} - 5w,$$
$$u = 10(r - y_{cl}).$$

$$y_{cl} = 100r - 100y_{cl} - 5w,$$
$$101y_{cl} = 100r - 5w,$$
$$y_{cl} = \frac{100}{101}r - \frac{5}{101}w,$$

$$e_{cl} = r - y_{cl}$$
$$= r - \left(\frac{100}{101}r - \frac{5}{101}w\right)$$
$$= \frac{1}{101}r + \frac{5}{101}w$$

Summary: Open-Loop vs. Closed-Loop





- **Case 1: NO change in plant model (plant = 10)**

	Open Loop	Closed Loop
$\frac{y}{r}$	$\frac{1}{10} \cdot 10 = 1$	$\frac{5 \cdot 10}{1+5 \cdot 10} = \frac{50}{51} \approx 0.98$

- **Case 2: But, Some change in plant model (plant = 15)**

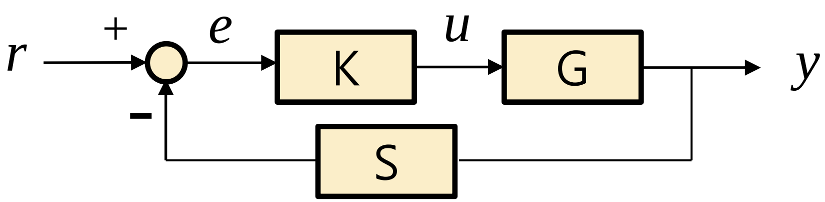
	Open Loop	Closed Loop
$\frac{y}{r}$	$\frac{1}{10} \cdot 15 = 1.5$	$\frac{5 \cdot 15}{1+5 \cdot 15} = \frac{75}{76} \approx 0.9868$

$\boxed{e = r - y = 50\%}$ $\boxed{e = r - y \approx 1.5\%}$

Key Concepts for Automatic Control

❖ **How to obtain the closed-loop transfer function**

- **Closed-Loop System**



$$y = Gu$$

$$u = Ke \qquad u = K(r - Sy)$$

$$e = r - Sy$$

$$y = KG(r - Sy)$$

$$(1 + KGS)y = KGr$$

$$\boxed{\frac{y}{r} = \frac{KG}{1 + KGS}}$$

Summary

❖ Summary

- Definition & properties of Laplace Transform
- Solution procedure to ODEs:
  ① Laplace Transform
  ② Partial fraction expansion
  ③ Inverse Laplace Transform
- Mathematic System Model & Transfer function
- Open Loop vs. Closed-Loop System