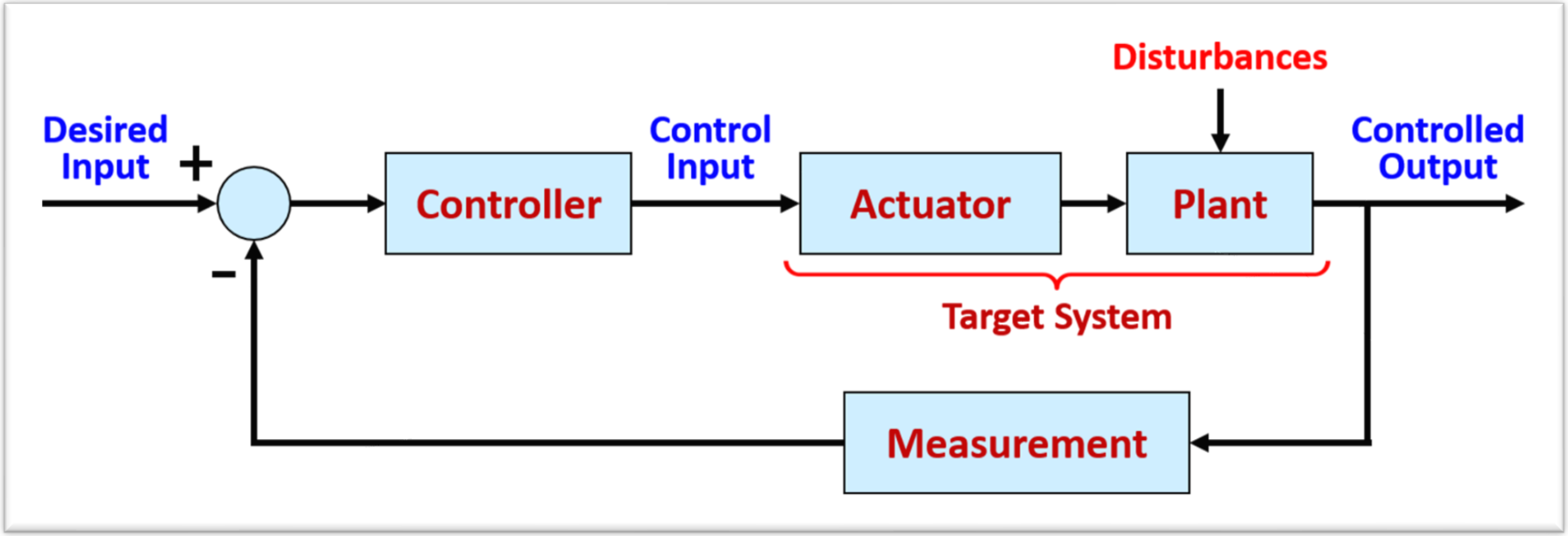
Dynamic Response 1

Lecture 4:

- Important Aspects of Dynamic Response
- Step Response Analysis



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Application of Control Engineering in Our Life (1)

❖ NAVER LABS Intelligence in Mobility

https://www.youtube.com/watch?v=B-RXhvcV0Pc

Important Aspects of Dynamic Response

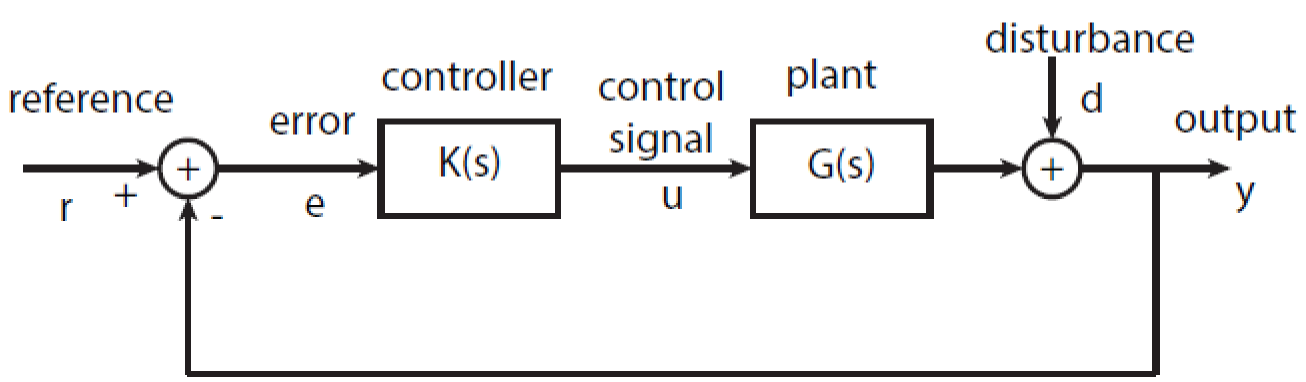
(1) Block diagram: Controller + Plant Model

(2) Poles and zeros

(3) Effect of pole locations

(4) Effect of zero locations

(5) Control Performance Measurement Index



Important Aspects of Dynamic Response

(1) Block Diagram with Single Transfer Function (Revisited)

- A transfer function is defined by,

$$G(s) := \frac{Y(s)}{U(s)}$$

Laplace transform of system output

Laplace transform of system input

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

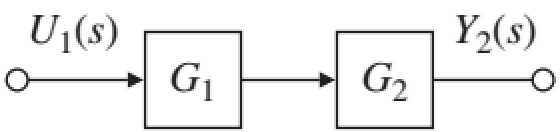
Block Diagram

- But, a system is assumed to be at rest. (zero initial condition)

Important Aspects of Dynamic Response

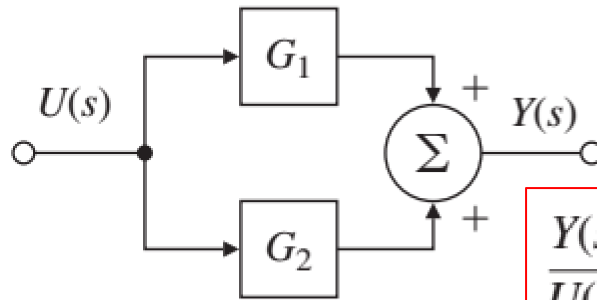
(1) Block Diagram with Multi-Transfer Functions (cont'd)

- **Serial Connection**



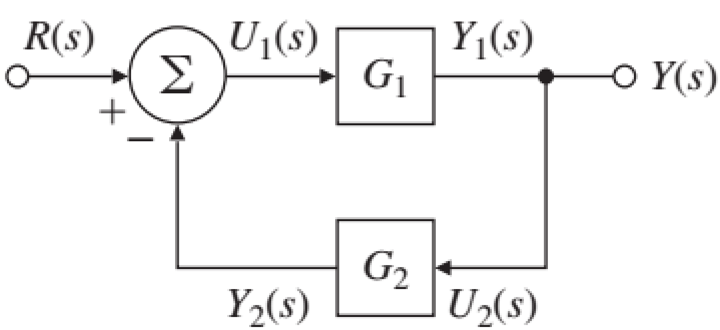
$$\boxed{\frac{Y_2(s)}{U_1(s)} = G_2 G_1}$$

- **Parallel Connection**



$$\boxed{\frac{Y(s)}{U(s)} = G_2 + G_1}$$

- **Feedback Connection (Type I)**



Proof:

$$U_1(s) = R(s) - Y_2(s),$$

$$Y_2(s) = \boxed{\qquad\qquad} U_1(s),$$

$$Y_1(s) = G_1(s) U_1(s),$$

$$where, Y_1(s) = Y(s)$$

$$U_1 = R - G_2 G_1 U_1$$

$$(1 + G_1 G_2) U_1 = R$$

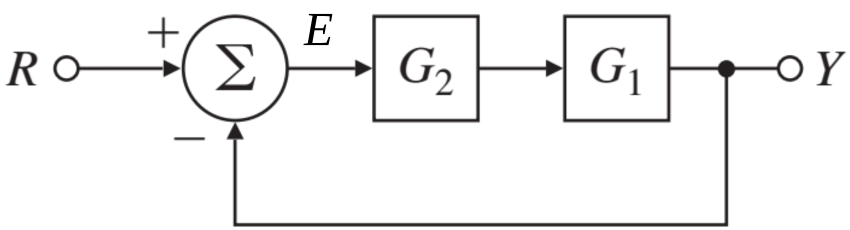
$$U_1 = \frac{R}{1 + G_1 G_2}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_2 G_1}}$$

Important Aspects of Dynamic Response

(1) Block Diagram with Multi-Transfer Functions (cont'd)

- **Feedback Connection (Type II)**



Proof:

$$E = \boxed{}$$

$$Y = \boxed{}\,E$$

$$Y = \boxed{}$$

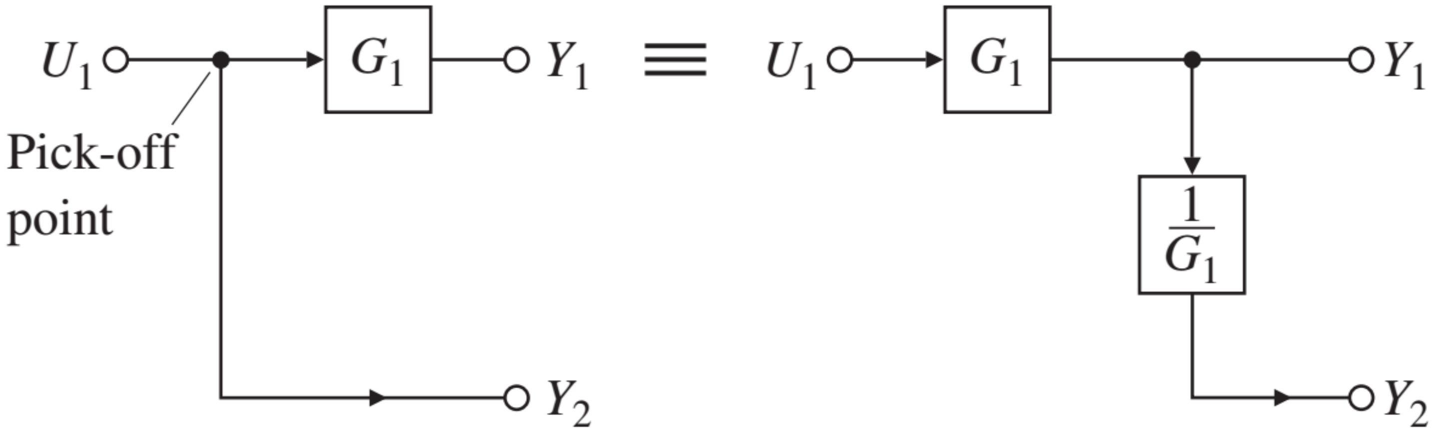
$$(1 + G_1 G_2)Y = (G_1 G_2)R$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2}}$$

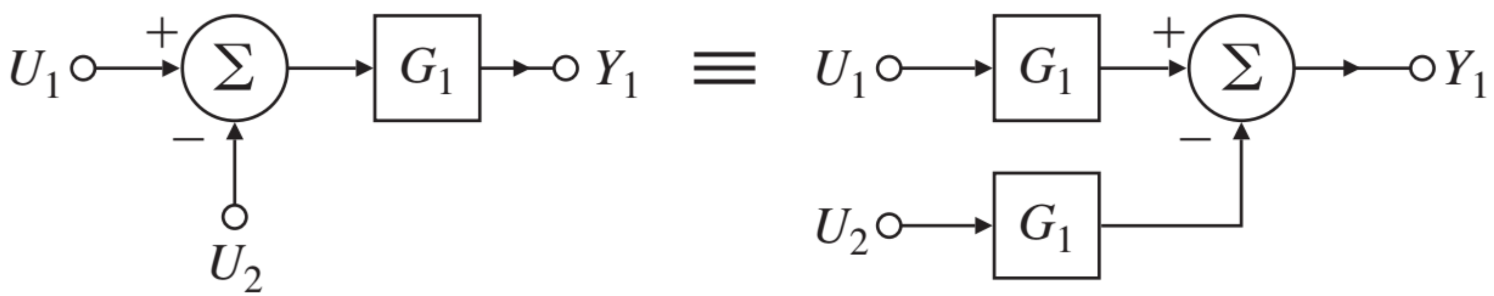
Important Aspects of Dynamic Response

(1) Block Diagram with Multi-Transfer Functions (cont'd)

- Block Diagram Algebra (1): Moving a pickoff point



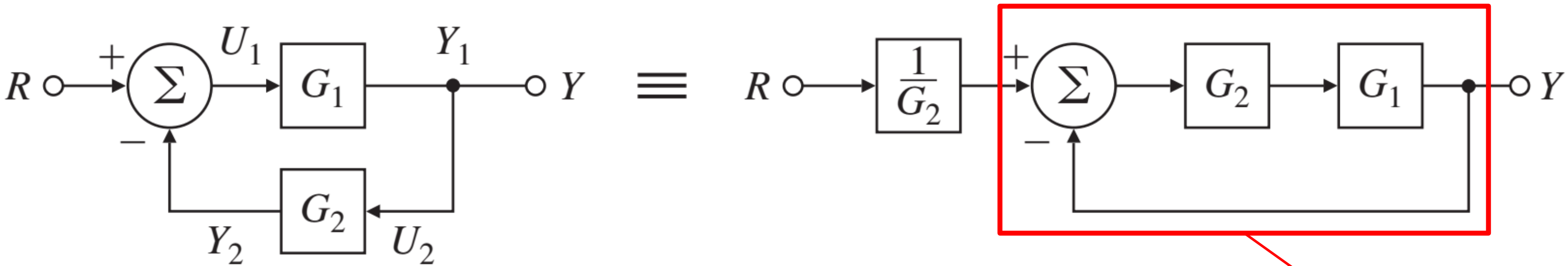
- Block Diagram Algebra (2): Moving a summer



Important Aspects of Dynamic Response

(1) Block Diagram with Multi-Transfer Functions (cont'd)

- **Block Diagram Algebra (3): Conversion to unity feedback**



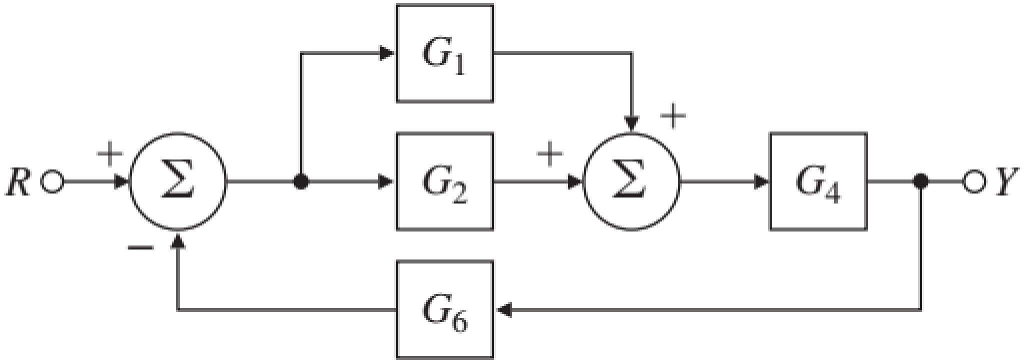
$$\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_1G_2}$$

$$\frac{Y(s)}{R(s)} = \frac{1}{\cancel{G_2}}\boxed{\frac{G_1\cancel{G_2}}{1 + G_1G_2}}$$

Important Aspects of Dynamic Response

(1) Block Diagram with Multi-Transfer Functions (cont'd)

- **Example using Matlab**



where,

$$G_1 = 2, G_2 = \frac{4}{s}, G_4 = \frac{1}{s}, G_6 = 1$$

```
s=tf('s'); % specify a TF in the Laplace variable s
sysG1=2; % define subsystem G1
sysG2=4/s; % define subsystem G2
sysG3=parallel(sysG1,sysG2); % parallel combination of G1 and G2
sysG4=1/s; % define subsystem G4
sysG5=series(sysG3,sysG4); % series combination of G3 and G4
sysG6=1;
sysCL=feedback(sysG5,sysG6,-1) % feedback combination of G5 and G6
```

```
sysCL =

     2 s + 4
  -------------
  s^2 + 2 s + 4
```

Important Aspects of Dynamic Response

(2) Poles and Zeros of Transfer Function

Parameters

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)},$$

Gain

$$H(s) = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}.$$

Roots

or Eigenvalues

or Zeros & Poles

- **Question: So, why we call these Zeros & Poles??**

Important Aspects of Dynamic Response

(2) Poles and Zeros of Transfer Function (cont'd)

$$H(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}.$$

If s = z_i, then ⇒ $H(s)|_{s=z_i} = 0.$ **Zeros**

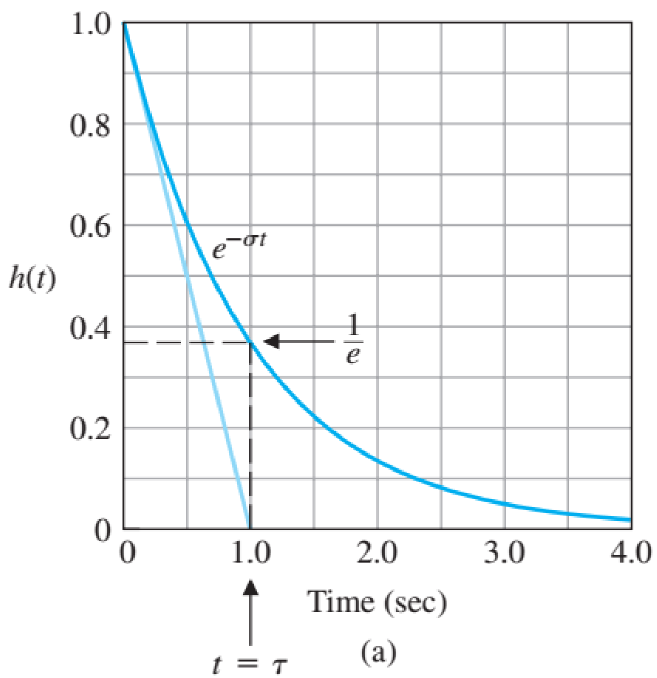
If s = p_i, then ⇒ $|H(s)|_{s=p_i} = \infty.$ **Poles**

Important Aspects of Dynamic Response

(3) Effect of Pole Locations

Ex) For a first-order pole with the impulse response,

$$H(s) = \frac{1}{s+\sigma}. \xrightarrow{\mathcal{L}^{-1}} h(t) = e^{-\sigma t}1(t).$$



$\sigma > 0 \Rightarrow$ pole location $s < 0 \Rightarrow$ stable

$\sigma < 0 \Rightarrow$ pole location $s > 0 \Rightarrow$ unstable

- **Thus, system stability depends on pole location!!**

Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- **Example 1: Real Poles**

$$H(s) = \frac{2s+1}{s^2+3s+2}.$$

Partial-fraction expansion gives,

$$H(s) = -\frac{1}{s+1} + \frac{3}{s+2}.$$

$$\Big\Downarrow \ \mathcal{L}^{-1}$$

$$h(t) = \begin{cases} -e^{-t} + 3e^{-2t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

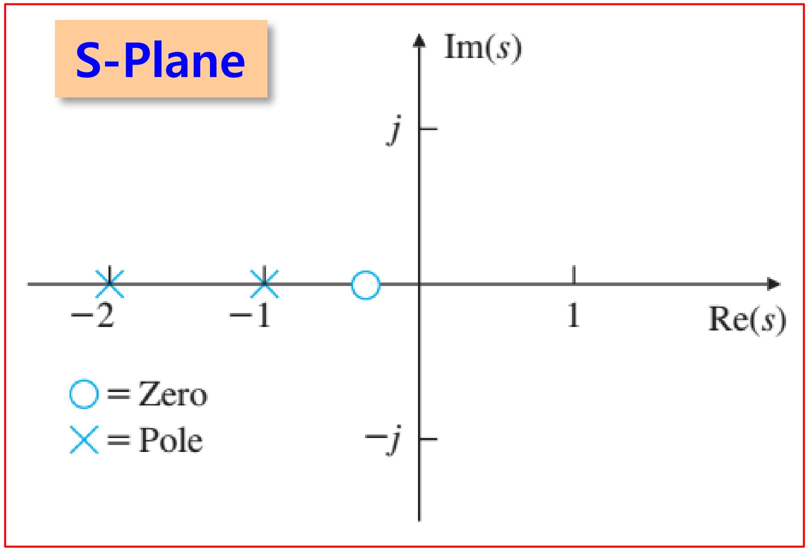
- **Example 1: Real Poles (cont'd)**

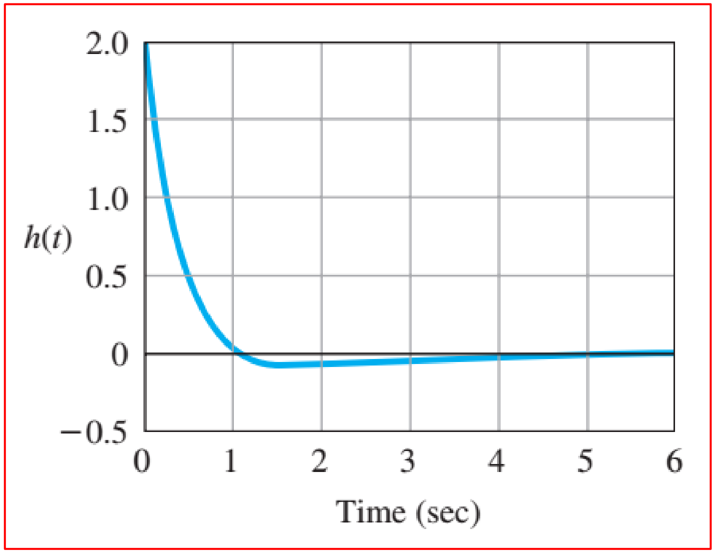
$$H(s) = \frac{2s+1}{s^2+3s+2}.$$

```
num = [2 1];
den = [1 3 2];
pzmap(num,den)
```

$$H(s) = -\frac{1}{s+1} + \frac{3}{s+2}.$$

$$h(t) = \begin{cases} -e^{-t} + 3e^{-2t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$





Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- Example 2: Complex (conjugate) Poles

$$H(s) = \frac{b(s)}{a(s)} \qquad s = -\sigma \pm j\omega_d.$$

$$a(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d) = (s+\sigma)^2 + \omega_d^2.$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

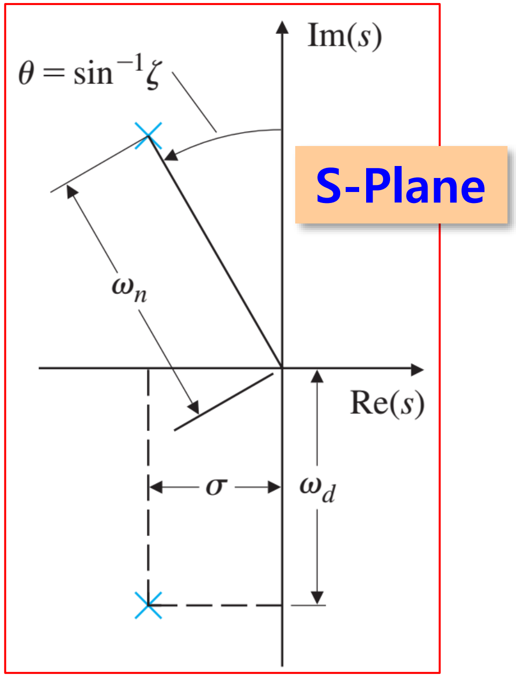
Damping ratio

Where, $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2},$$

Natural frequency

Damped frequency



Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- **Example 2: Complex (conjugate) Poles**

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Re-written

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}.$$

$\mathcal{L}^{-1}$

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t}(\sin \omega_d t)1(t).$$

$$\boxed{\frac{b}{(s+a)^2 + b^2} \xrightarrow{\mathcal{L}^{-1}} e^{-at} \sin bt}$$

$$where,\ a \Rightarrow \zeta\omega_n$$

$$b \Rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2}$$

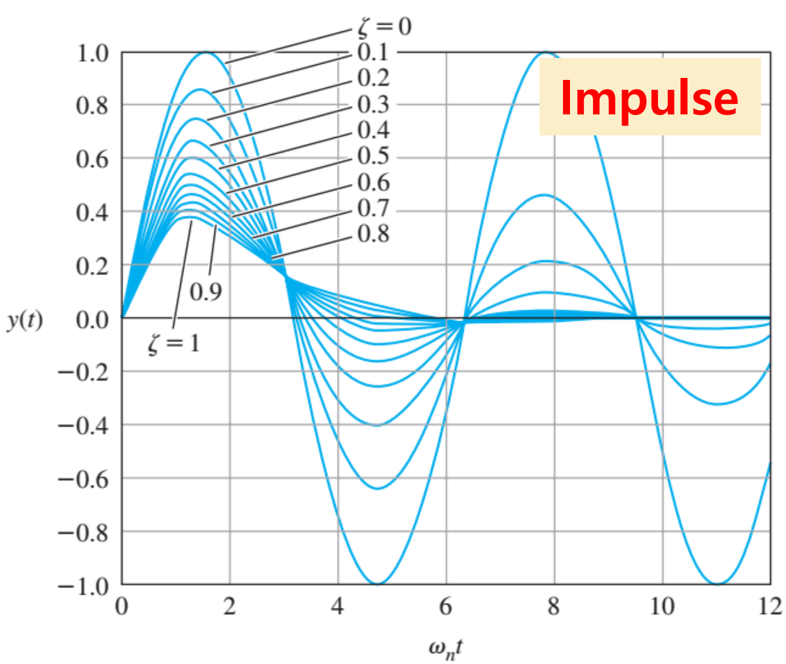
Important Aspects of Dynamic Response

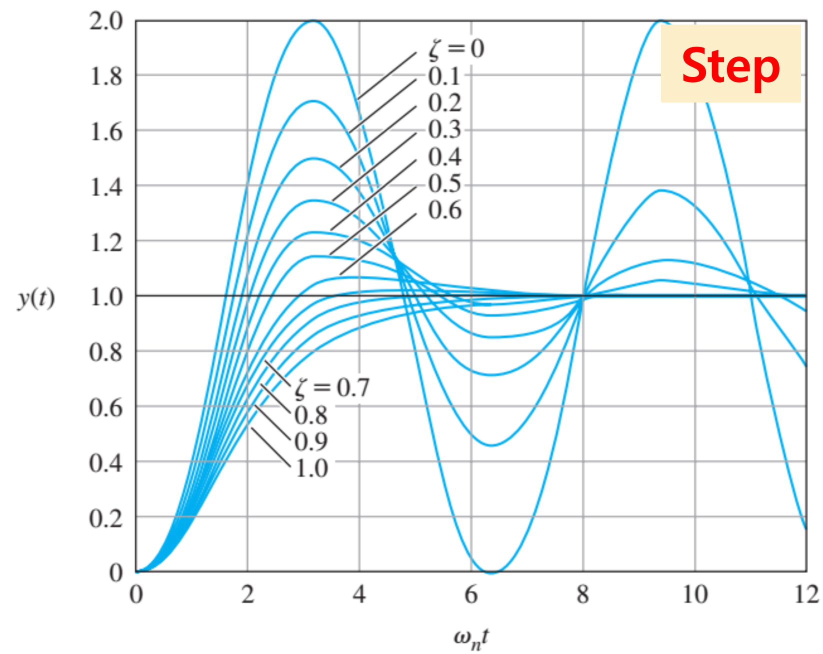
(3) Effect of Pole Locations (cont'd)

- Example 2: Complex (conjugate) Poles (cont'd)

Impulse
Step

u(t) → $$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$ → y(t)

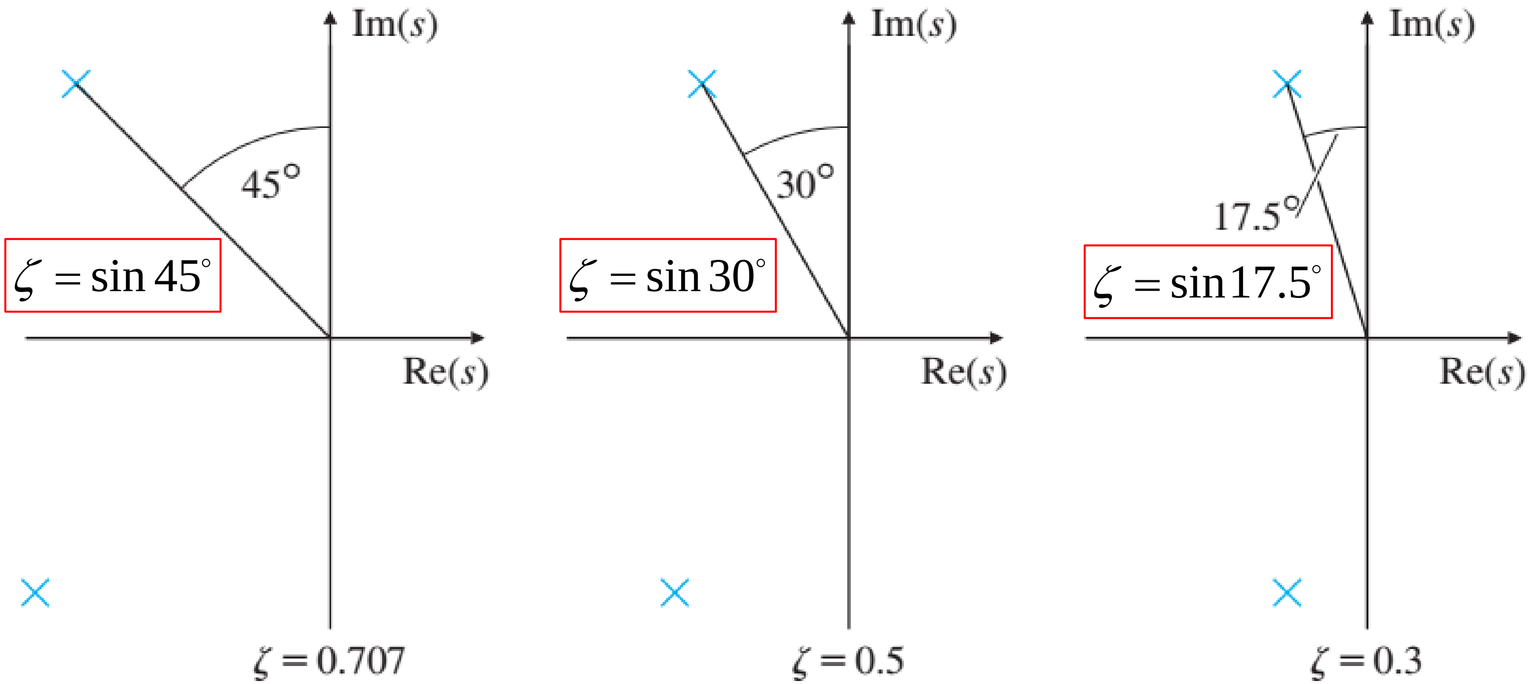




Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- **Summary 1: Damping Ratio vs. System Response (Oscillation)**

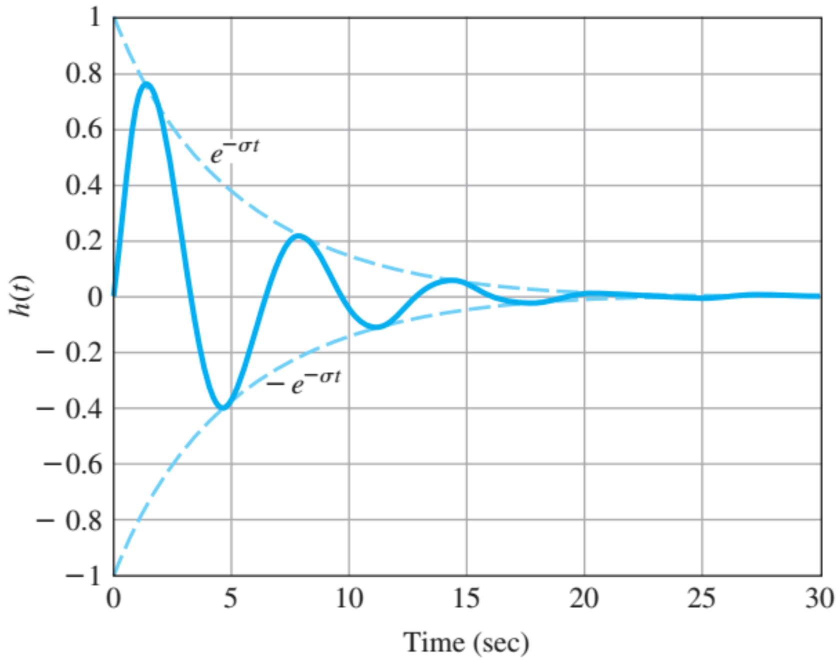


- **Question**: How can we understand the relationship btw **Damping Ratio (i.e., energy dissipation) vs. System Oscillation**?

Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- **Summary 2: Natural Frequency vs. System Response (Oscillation)**



$$\sigma = \zeta\omega_n$$

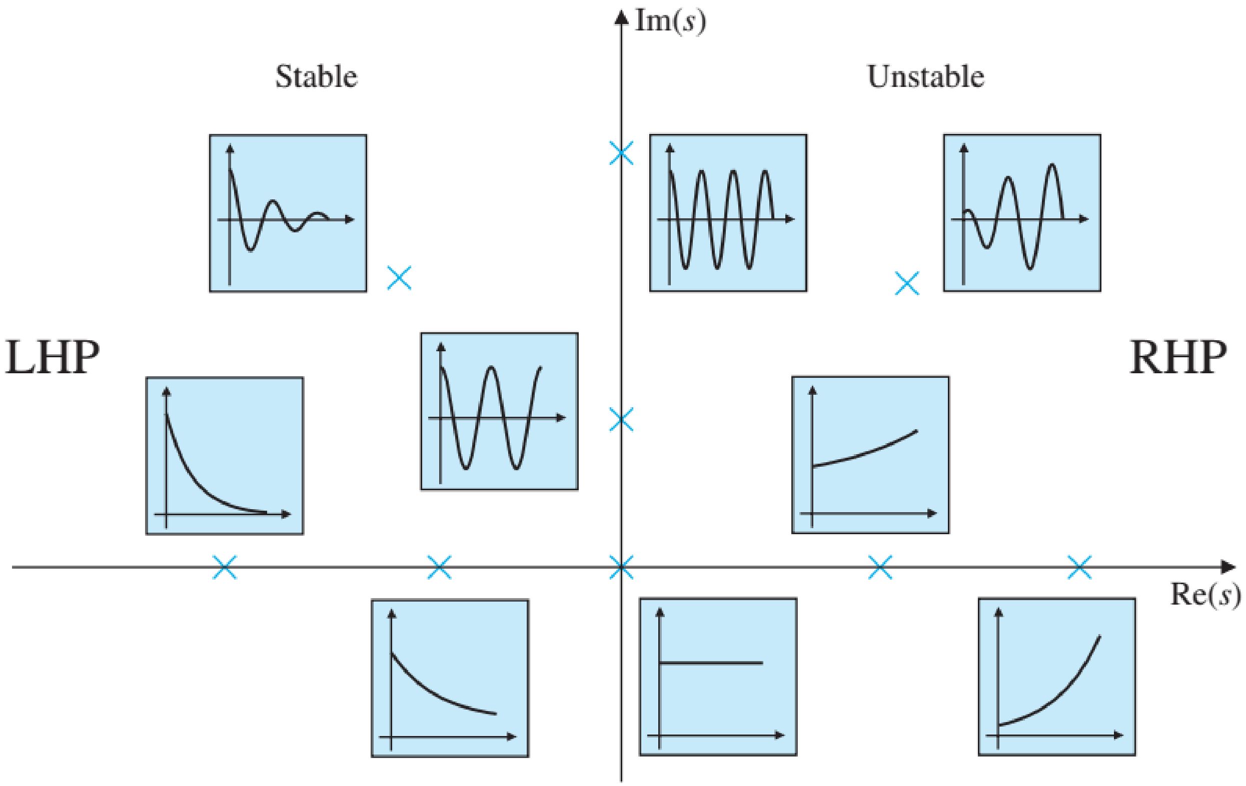
@ constant damping

- **Question**: How can we understand the relationship btw **Natural Frequency vs. System Oscillation**?

Important Aspects of Dynamic Response

(3) Effect of Pole Locations (cont'd)

- **Summary 3: Natural Frequency & Damping vs. System Response**



Important Aspects of Dynamic Response

(4) Effect of Zero Locations

- Case 1: No Zero TF

$$H_1(s) = \frac{2}{(s+1)(s+2)}$$

$$= \frac{2}{s+1} - \frac{2}{s+2},$$

- Case 2: One Zero near Pole in TF

$$H_2(s) = \frac{2\cancel{(s+1.1)}}{1.1\cancel{(s+1)}(s+2)}$$

Almost cancellation

$$= \frac{2}{1.1}\left(\frac{0.1}{s+1} + \frac{0.9}{s+2}\right)$$

$$= \frac{0.18}{s+1} + \frac{1.64}{s+2}.$$

Important Aspects of Dynamic Response

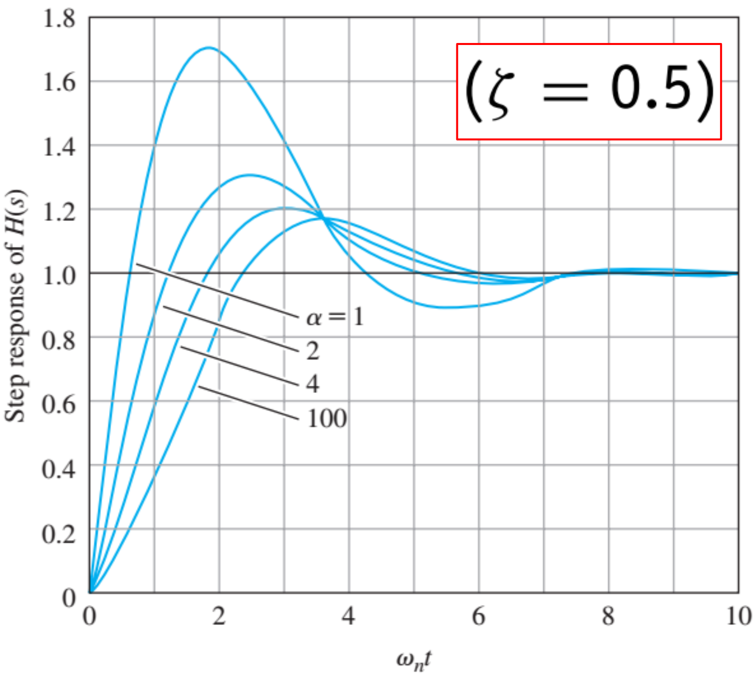
(4) Effect of Zero Locations (cont'd)

- **Normalized time and zero locations:**

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}. \Rightarrow H(s) = \frac{1}{s^2 + 2\zeta s + 1} + \frac{1}{\alpha\zeta}\frac{s}{s^2 + 2\zeta s + 1}.$$

- **Important Questions about effects on zero locations:**
  (1) Transient Response??
  (2) Overshoot??
  (3) Stability??

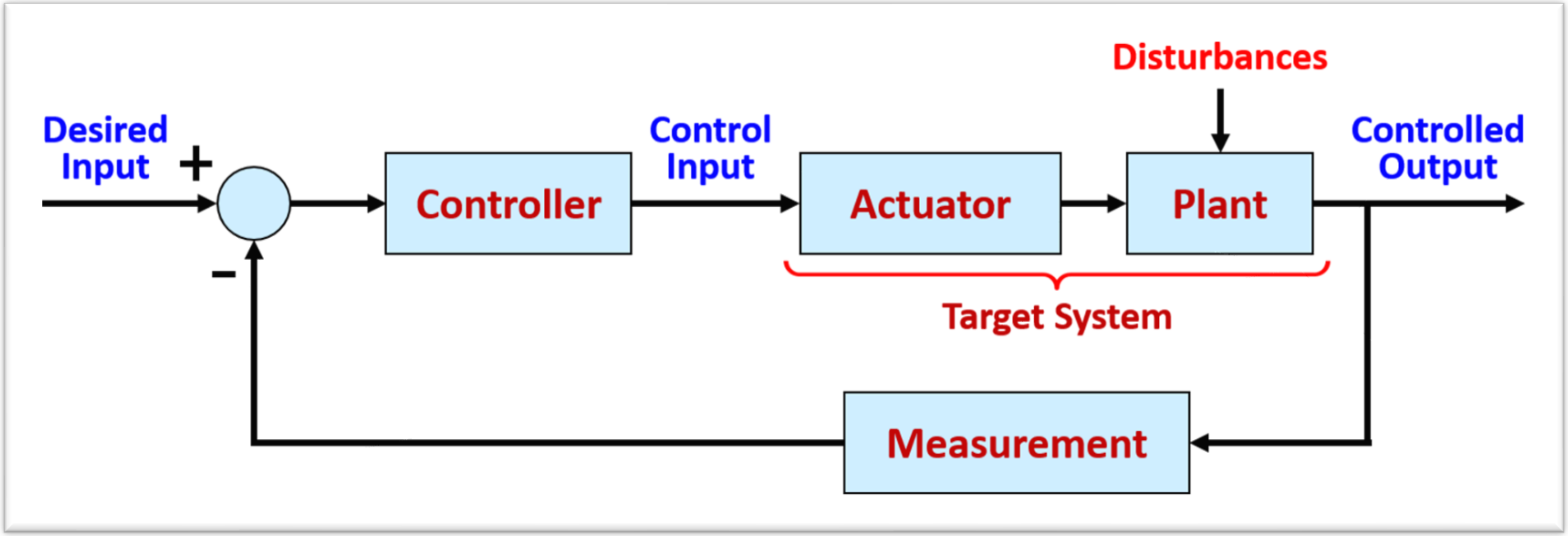
$$s_{zero} = -\alpha\zeta$$



Dynamic Response 1

Lecture 4:

- Important Aspects of Dynamic Response
- Step Response Analysis



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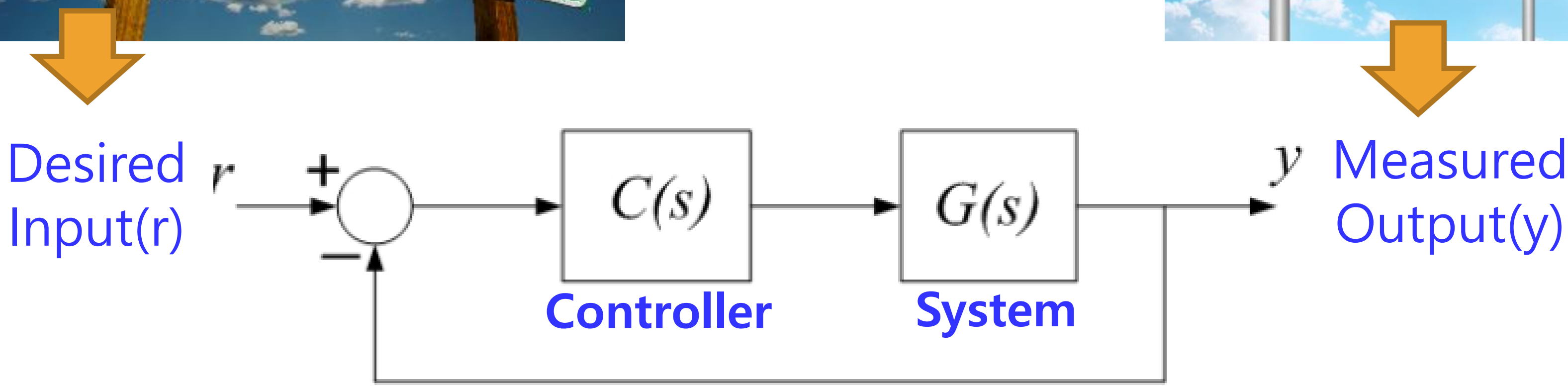
What is the Control Concept? (Revisited)

vs.

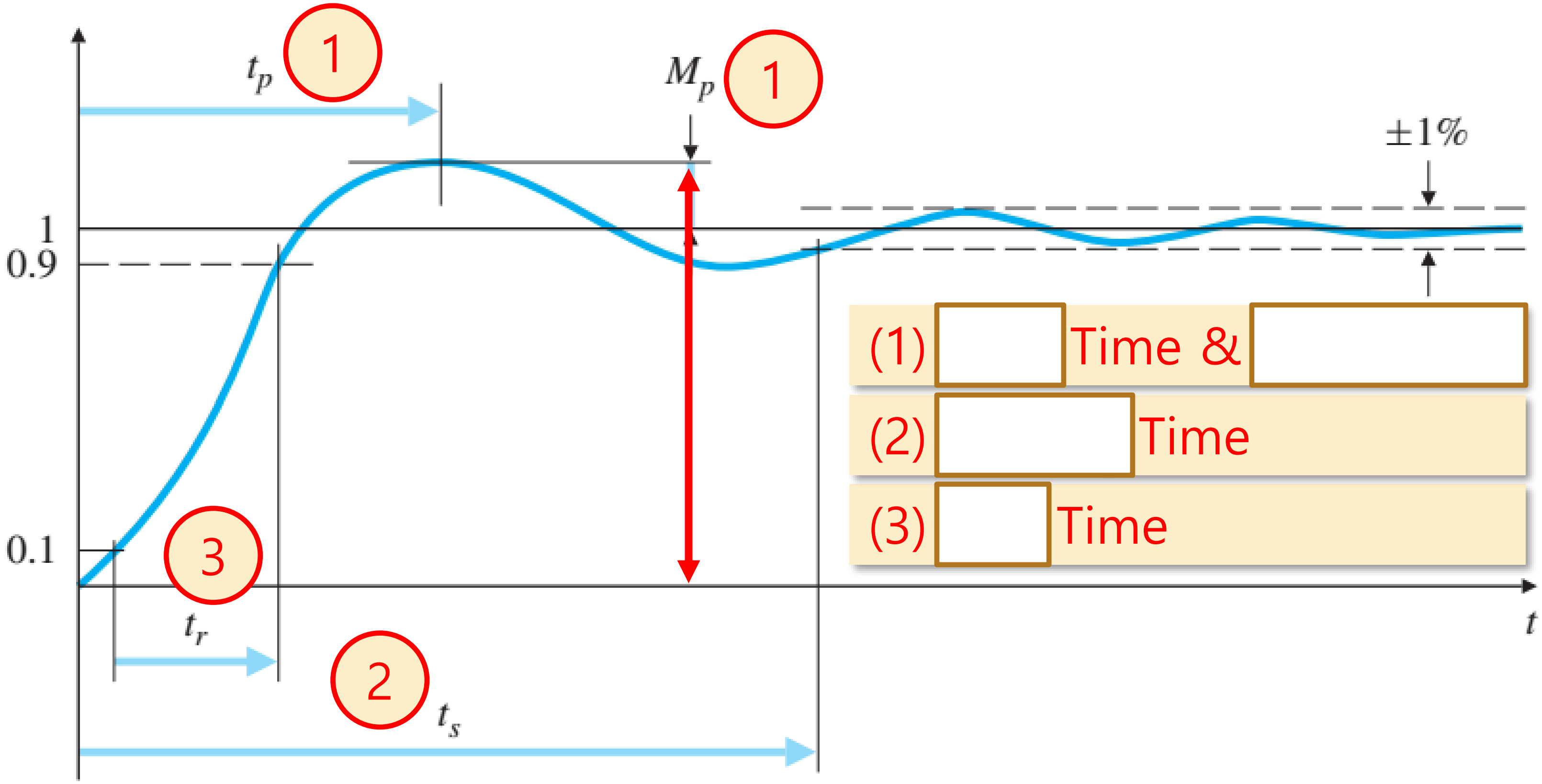


- Controller is to minimize error between the desired input (dream) between the measured output (reality) !!
- **Then, how to measure <u>the gap btw desire vs. reality</u> ??**

Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response

- **Key Three Attributes** for checking the control performance



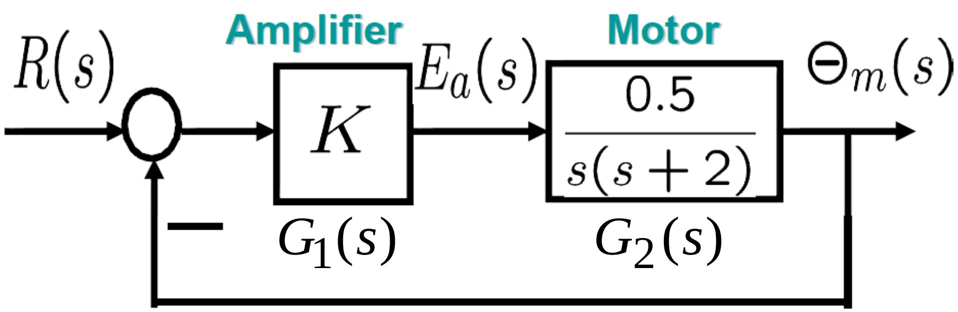
Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response

- **A standard form of the second-order system**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \begin{cases} \zeta : & \text{damping ratio} \\ \omega_n : & \text{undamped natural frequency} \end{cases}$$

- **Example: DC motor position control**



$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)}$$

Closed-Loop TF:

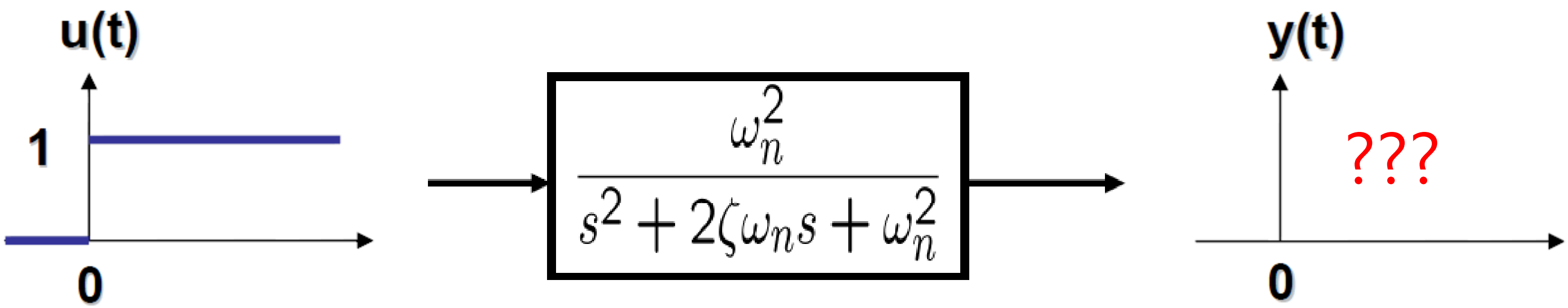
$$\frac{\Theta_m(s)}{R(s)} = \frac{K\dfrac{0.5}{s(s+2)}}{1 + K\dfrac{0.5}{s(s+2)}}$$

$$= \frac{0.5K}{s^2 + 2s + 0.5K}$$

Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response

- Input a unit step function to a 2nd-order system.
  What is the output?



Input: unit step G(s) Output

- We are interested in the error (i.e., input – output) !!

Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response

- Undamped

  $\zeta = 0$

- Underdamped

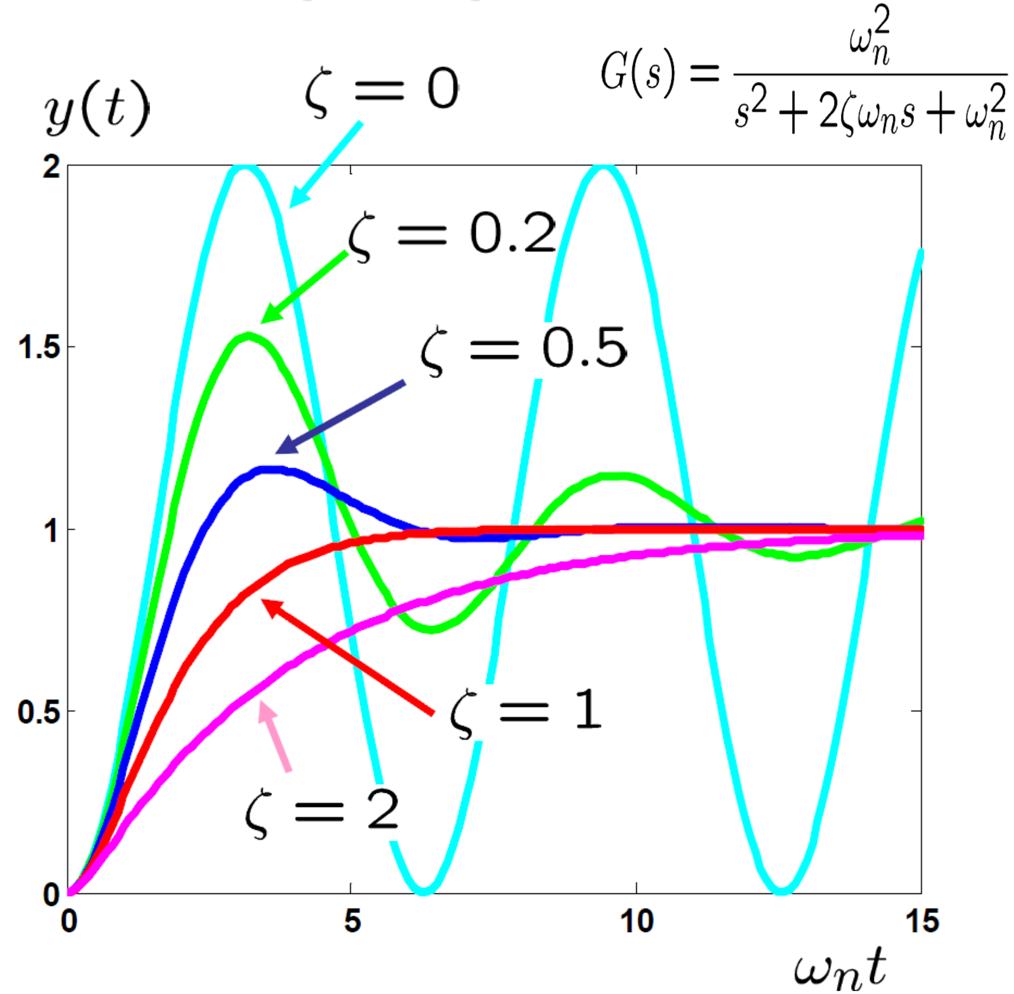
  $0 < \zeta < 1$

- Critically damped

  $\zeta = 1$

- Overdamped

  $\zeta > 1$



Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response

- Math expression of y(t) {or Y(s)} for underdamped case with the step input {R(s) = 1/s}

$0 < \zeta < 1$

(1) Peak Time & Overshoot ??

(2) Settling Time ??

(3) Rise Time ??

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \varphi\right)}$$

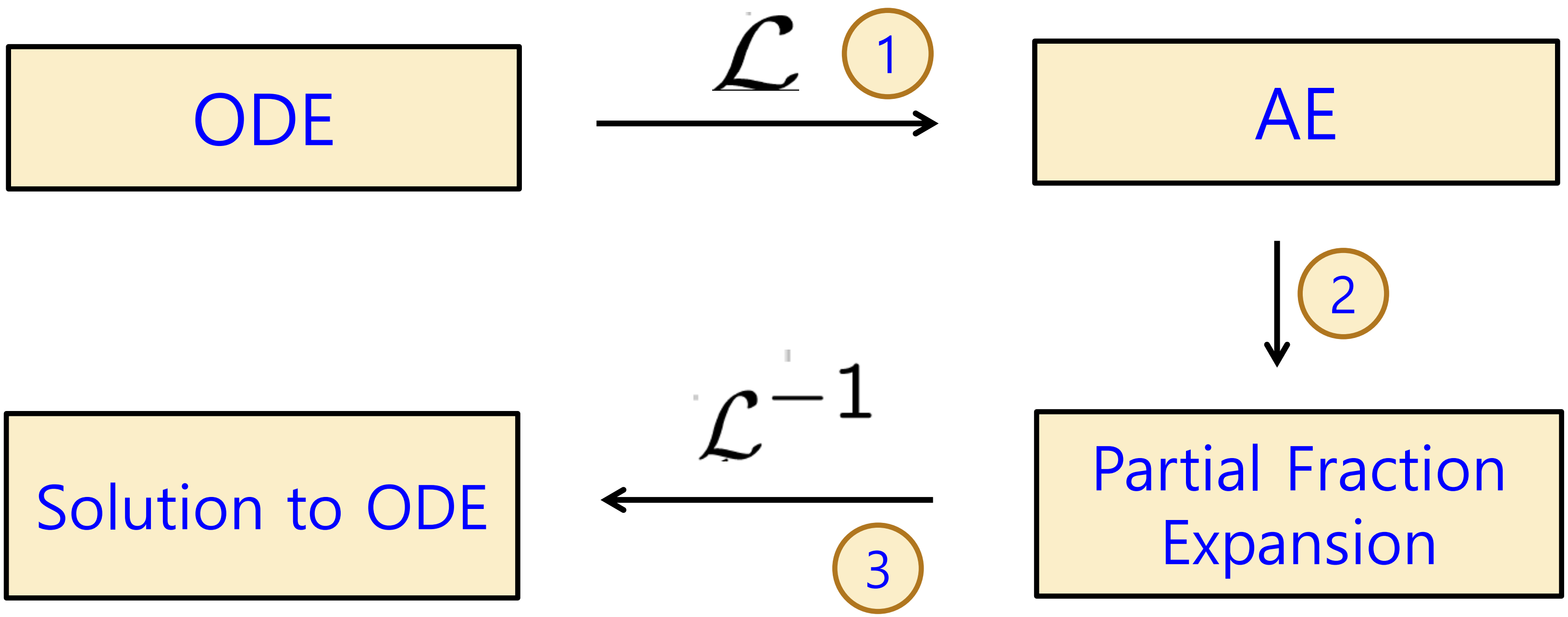
where,

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$: Damped Natural Frequency [rad/sec]

$\varphi = \cos^{-1}(\zeta)$: time delay [rad]

Major Advantage of Laplace Transform (Review)

❖ **Transform an ordinary differential equation (ODE) into an algebraic equation (AE).**



Major Advantage of Laplace Transform (Review)

❖ Example 1: ODE with initial conditions (cont'd)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u_s(t),\ y(0)=-1,\ y'(0)=2$$

Step input

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\left\{y'(t)\right\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow \quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$

Major Advantage of Laplace Transform (Review)

❖ Example 1: ODE with initial conditions (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0} = A + s\frac{B}{s+1}\bigg|_{s\to 0} + s\frac{C}{s+2}\bigg|_{s\to 0} \Rightarrow A = sY(s)|_{s\to 0} = \frac{5}{2}$$

Similarly,

$$\begin{aligned} B &= (s+1)Y(s)|_{s\to -1} = \cdots = -5 \\ C &= (s+2)Y(s)|_{s\to -2} = \cdots = \frac{3}{2} \end{aligned}$$

Major Advantage of Laplace Transform (Review)

❖ Example 1: ODE with initial conditions (cont'd)

3. Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right\}$$

$$\Rightarrow \quad y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B}e^{-t} + \underbrace{\frac{3}{2}}_{C}e^{-2t}\right)u_s(t)$$

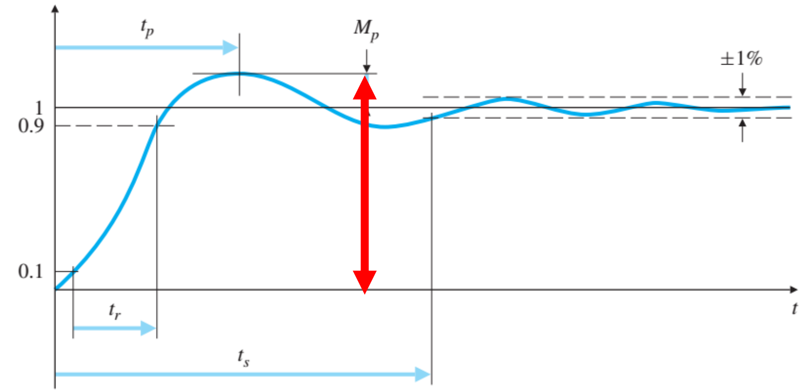
If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Important Aspects of Dynamic Response

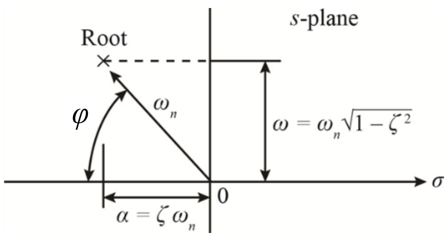
(5) Control Performance Index: Step Response (cont'd)

(1) Peak Time & Value



$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad \text{for } \zeta < 1$$

$$\text{peak value @ } \frac{dy}{dt} = 0$$



$$\frac{\varsigma\omega_n}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} \sin(\omega_d t + \varphi) - \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t} (\omega_d) \cos(\omega_d t + \varphi) = 0$$

$$\Rightarrow \varsigma\omega_n \sin(\omega_d t + \varphi) - (\omega_d) \cos(\omega_d t + \varphi) = 0$$

$$\Rightarrow \cos\varphi \sin(\omega_d t + \varphi) - \sin\varphi \cos(\omega_d t + \varphi) = 0$$

$$\therefore \sin(\omega_d t + \varphi - \varphi) = 0$$

$$T_p = \frac{\pi}{\omega_d}$$

$$M_p = 1 + \exp(-\zeta\omega_n T_p) = 1 + \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$where,\ \sin(\pi + \varphi) = -\sin(\varphi) = -\sqrt{1-\zeta^2}$$

$$where,\ \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\sin^2\varphi + \cos^2\varphi = 1$$

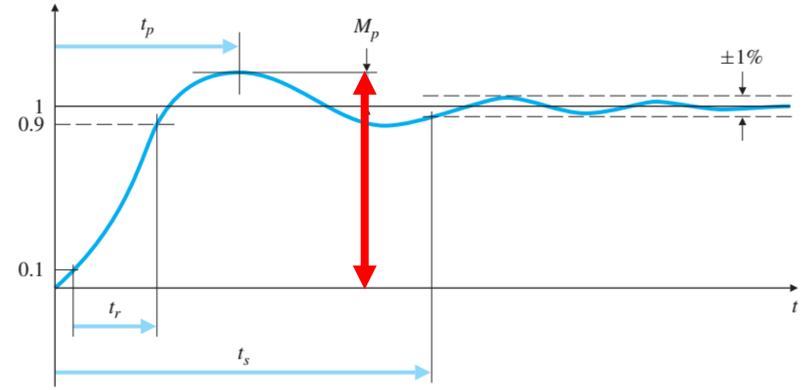
$$\sin\varphi = \sqrt{1-\zeta^2}$$

Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response (cont'd)

(2) Settling Time

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad \text{for} \quad \zeta < 1$$



peak value@settlingtime (T_s)

$$M_p = 1 + \exp(-\zeta\omega_n T_p)$$

$$\exp(-\varsigma\omega_n T_s) = 0.01\,(1\%)$$

$$-\varsigma\omega_n T_s = \ln(0.01) = -4.6$$

$$\therefore T_s \approx \frac{4.6}{\varsigma\omega_n} = \frac{4.6}{\sigma}$$

$$\exp(-\varsigma\omega_n T_s) = 0.02\,(2\%)$$

$$\therefore T_s \approx \frac{4.0}{\varsigma\omega_n} = \frac{4.0}{\sigma}$$

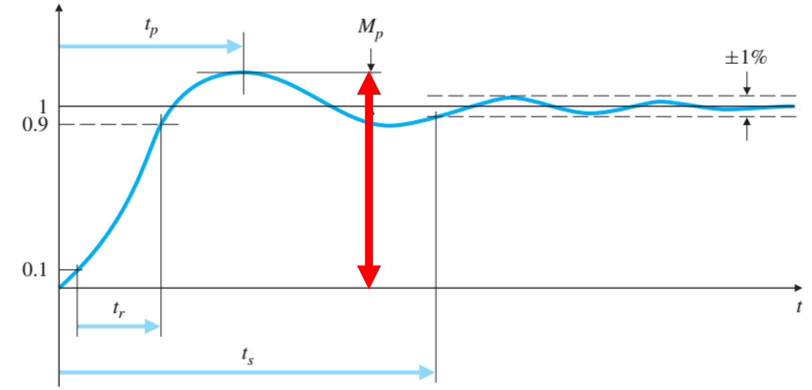
❖ **Important question**:
- How does damping ratio & natural frequency work for the settling time ??

Important Aspects of Dynamic Response

(5) Control Performance Index: Step Response (cont'd)

(3) Rise Time

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad \text{for } \zeta < 1$$



$$y(t) = 1 \text{ @ rising time } (T_r) \quad where, t_r \approx T_r$$

$$\sin(\omega_d T_r + \varphi) = 0$$

or

$$\therefore T_r = \frac{\pi - \varphi}{\omega_d}$$

where,

$$\omega_d = \omega_n \sqrt{1 - \varsigma^2}$$

$$\cos\varphi = \varsigma$$

$i)\ \varsigma = 0.5$

$\Rightarrow \varphi = \cos^{-1}(0.5) = 1.05$

$$\therefore T_r = \frac{3.14 - 1.05}{\omega_n \sqrt{1 - 0.5^2}} \approx \frac{2.4}{\omega_n}$$

$i)\ \varsigma = 0.7$

$\Rightarrow \varphi = \cos^{-1}(0.7) \approx 0.8$

$$\therefore T_r = \frac{3.14 - 0.8}{\omega_n \sqrt{1 - 0.7^2}} \approx \frac{3.3}{\omega_n}$$

❖ **Important question:**
- How does damping ratio & natural frequency work for the rise time ??

Important Aspects of Dynamic Response

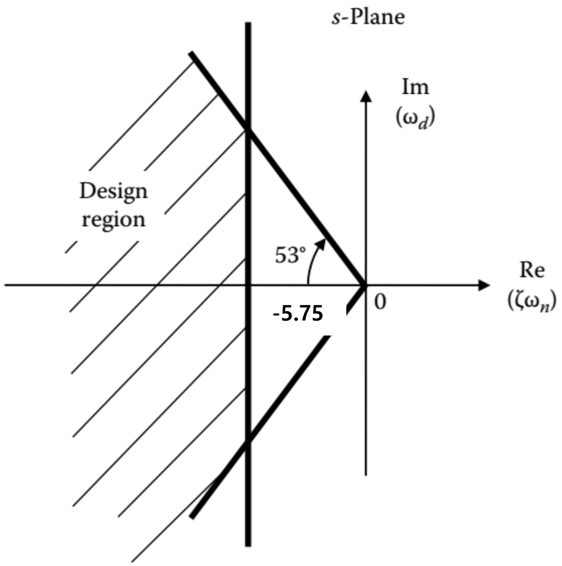
(5) Control Performance Index: Step Response (cont'd)

[Example]

Design requirement are for Controller + Plant (C-L)

(1) a percentage overshoot of less than 10%

(2) a settling time of less than 0.8 second



[Question]

Indicate this design specification as a region on the s-plane.

[Solution]

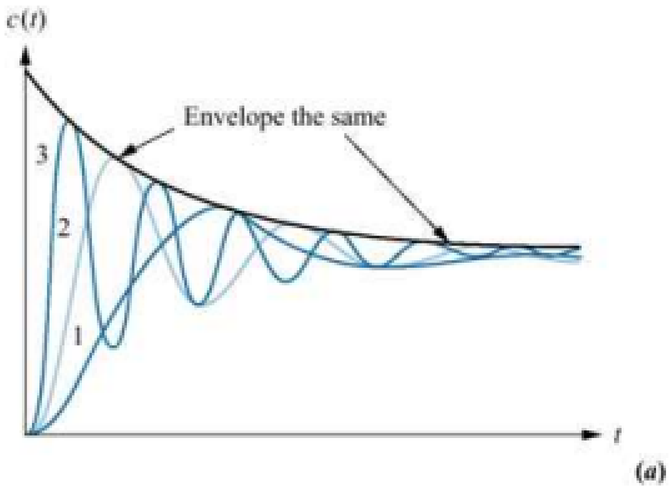
(1) 10% of overshoot means ⇨ $0.1 = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ ⇨ $0.6 \le \varsigma = \cos\varphi$, $\varphi \le 53(\text{deg})$

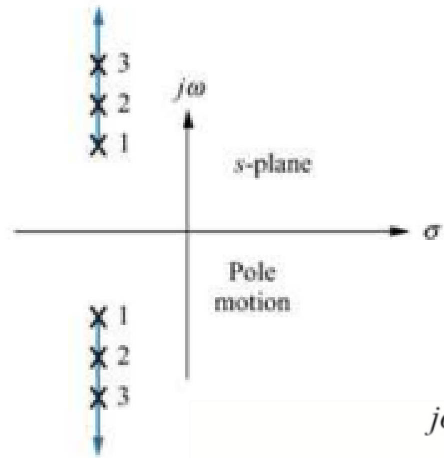
(2) T_s = 0.8 sec means ⇨ $\therefore T_s \approx \frac{4.6}{\varsigma\omega_n} \le 0.8 \Rightarrow \varsigma\omega_n \ge 5.75$

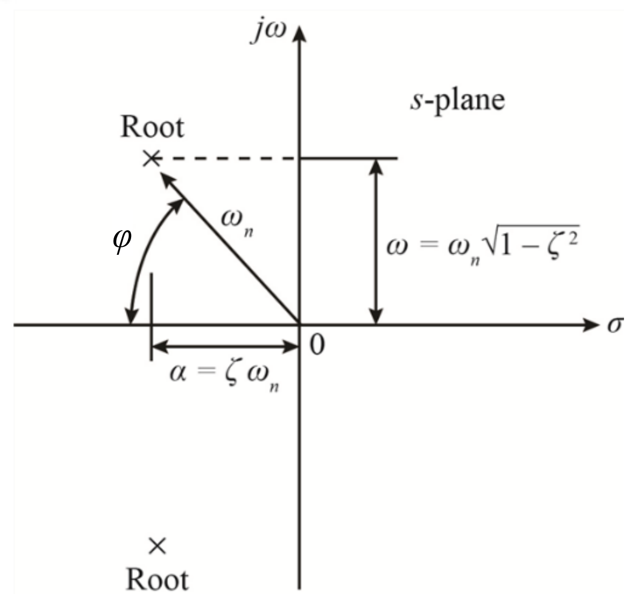
$$s = -\sigma \pm j\omega_d. \quad \sigma = \zeta\omega_n$$

Important Aspects of Dynamic Response

❖ Summary #1: Pole Placement vs. Step Response

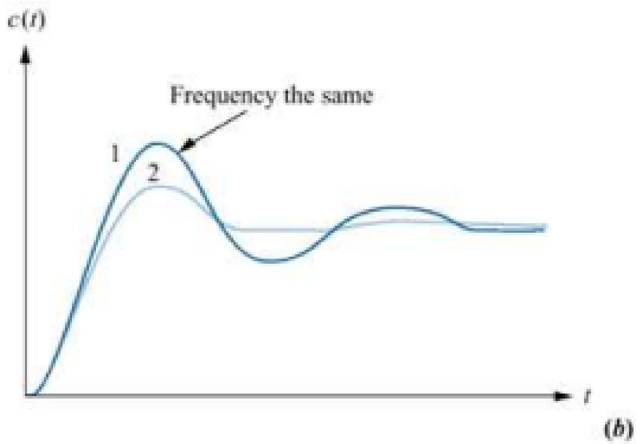


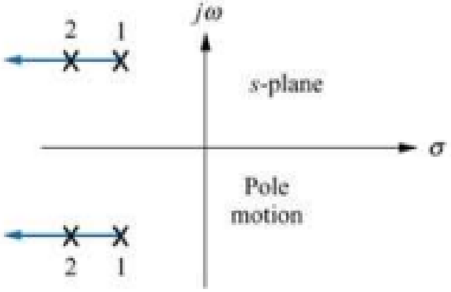


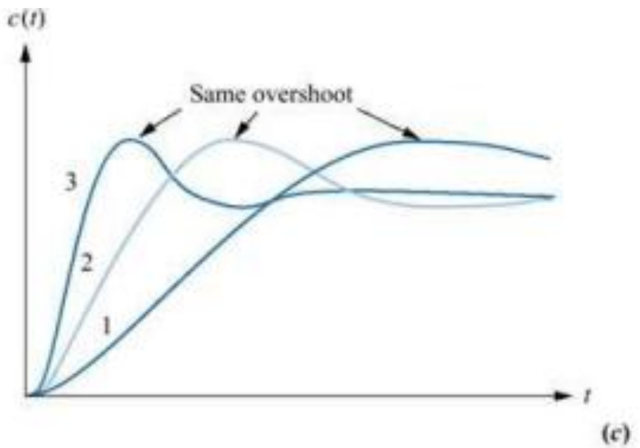


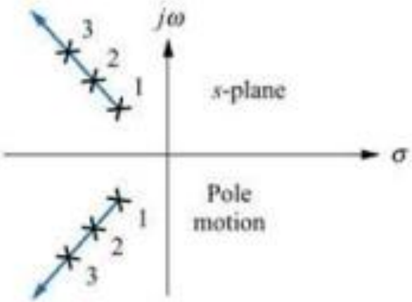
Important Aspects of Dynamic Response

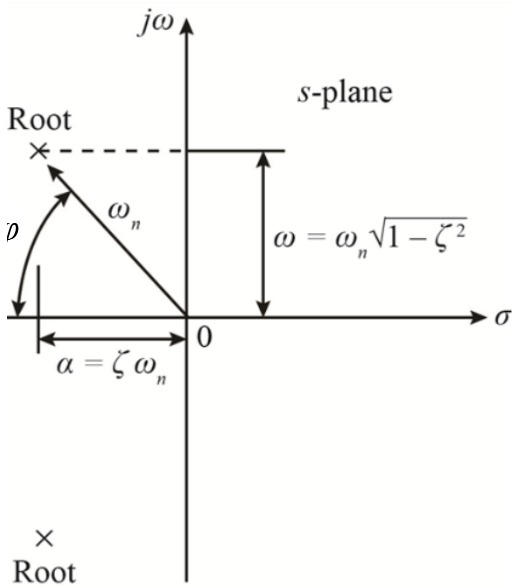
❖ Summary #1: Pole Placement vs. Step Response (cont'd)











Summary

❖ Summary:

- Important aspects of dynamic response for control system
- The specifications for the controller performance
- Step response by Pole & Zero locations