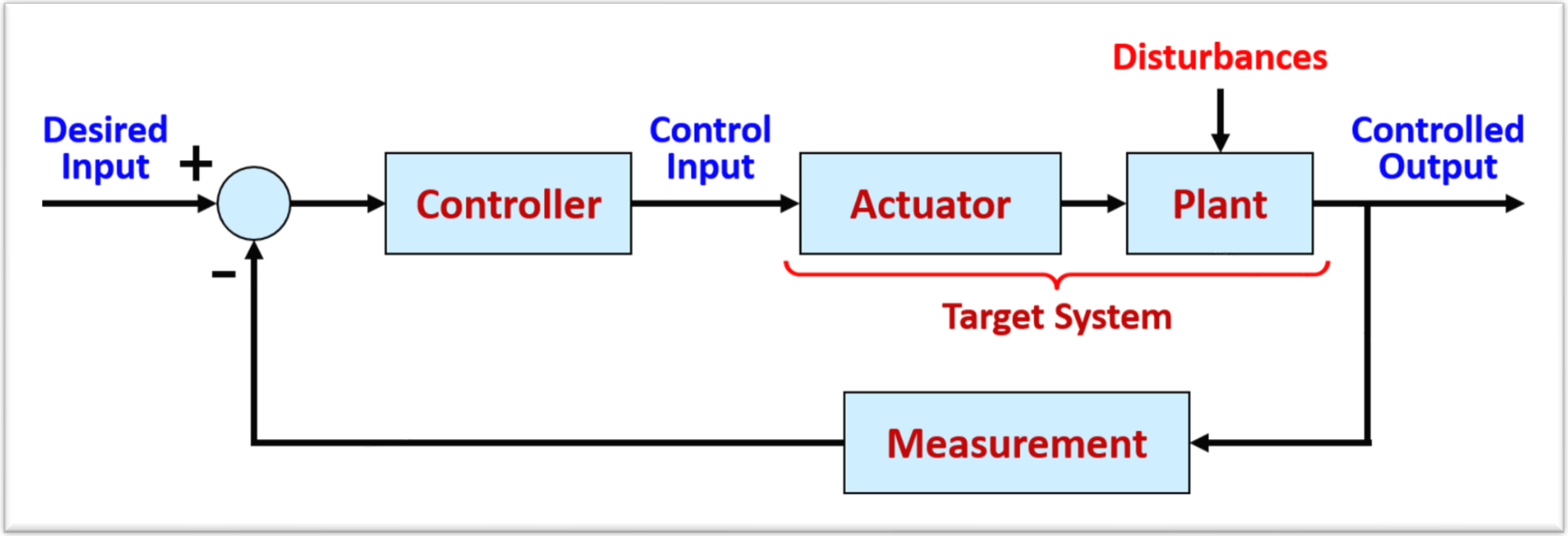
The Root-Locus Design Method 2

Lecture 9:

- Lead Compensator
- Lag Compensator



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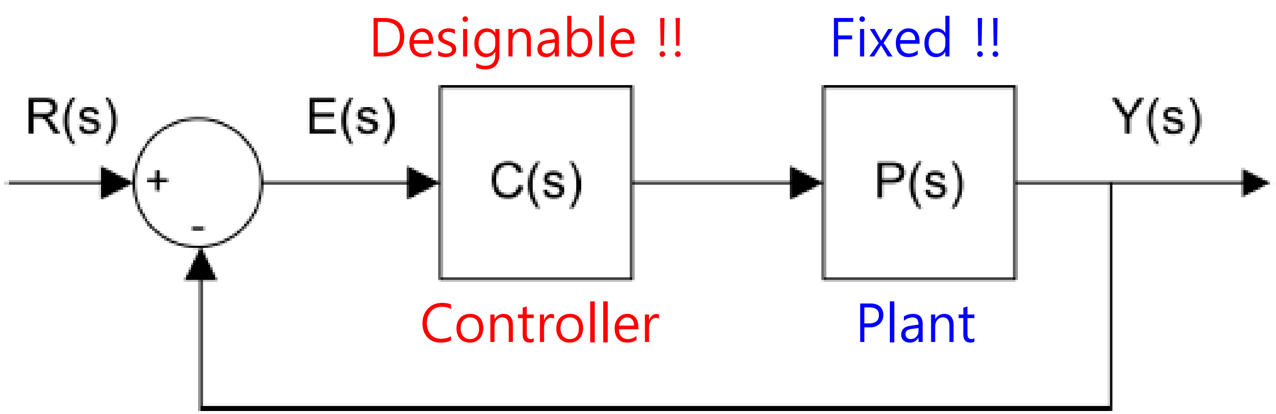
Closed-Loop Design by Root Locus



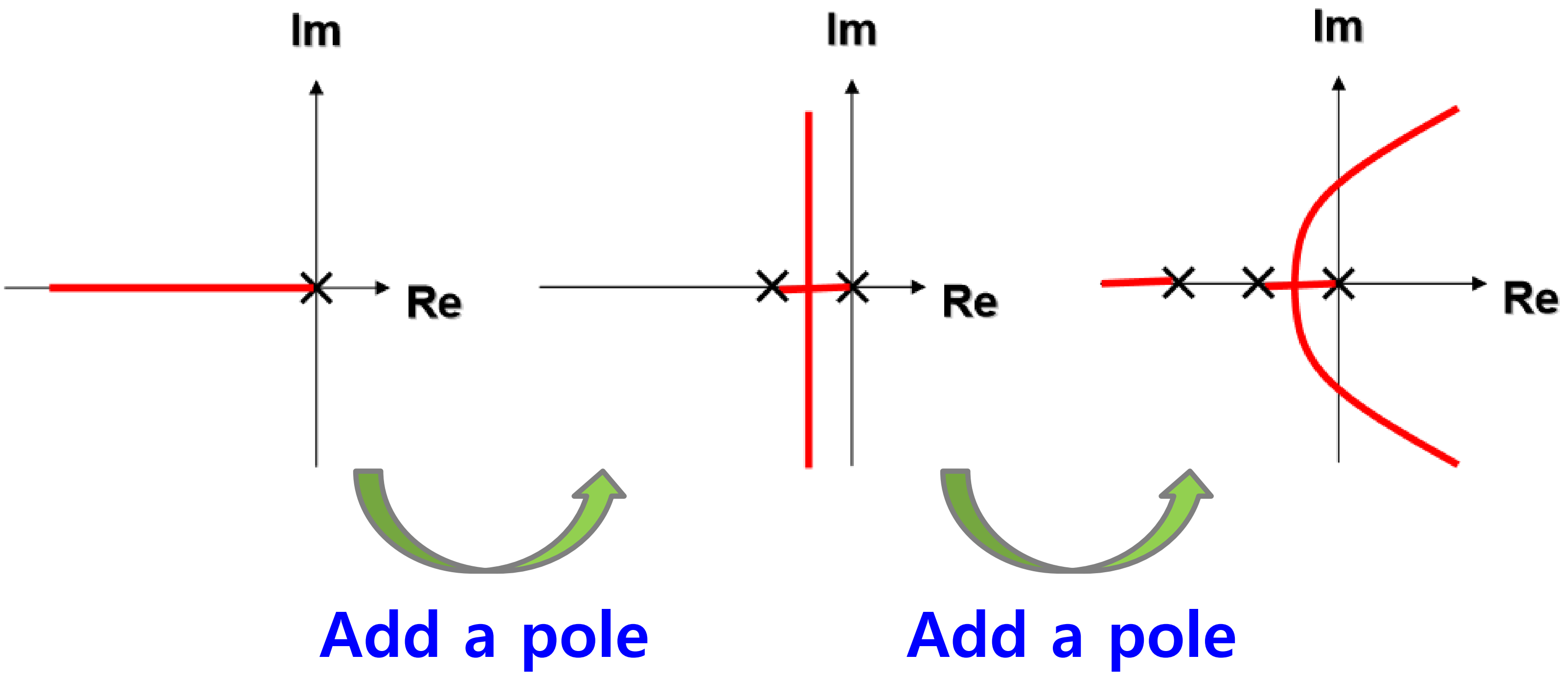
- Place closed-loop poles at desired locations,
  - By tuning the gain C(s) = K. (for time-domain specifications)

- If root locus does NOT pass the desired locations, then reshape the root locus
  - By adding poles and/or zeros to C(s).

    It is called "Compensation"

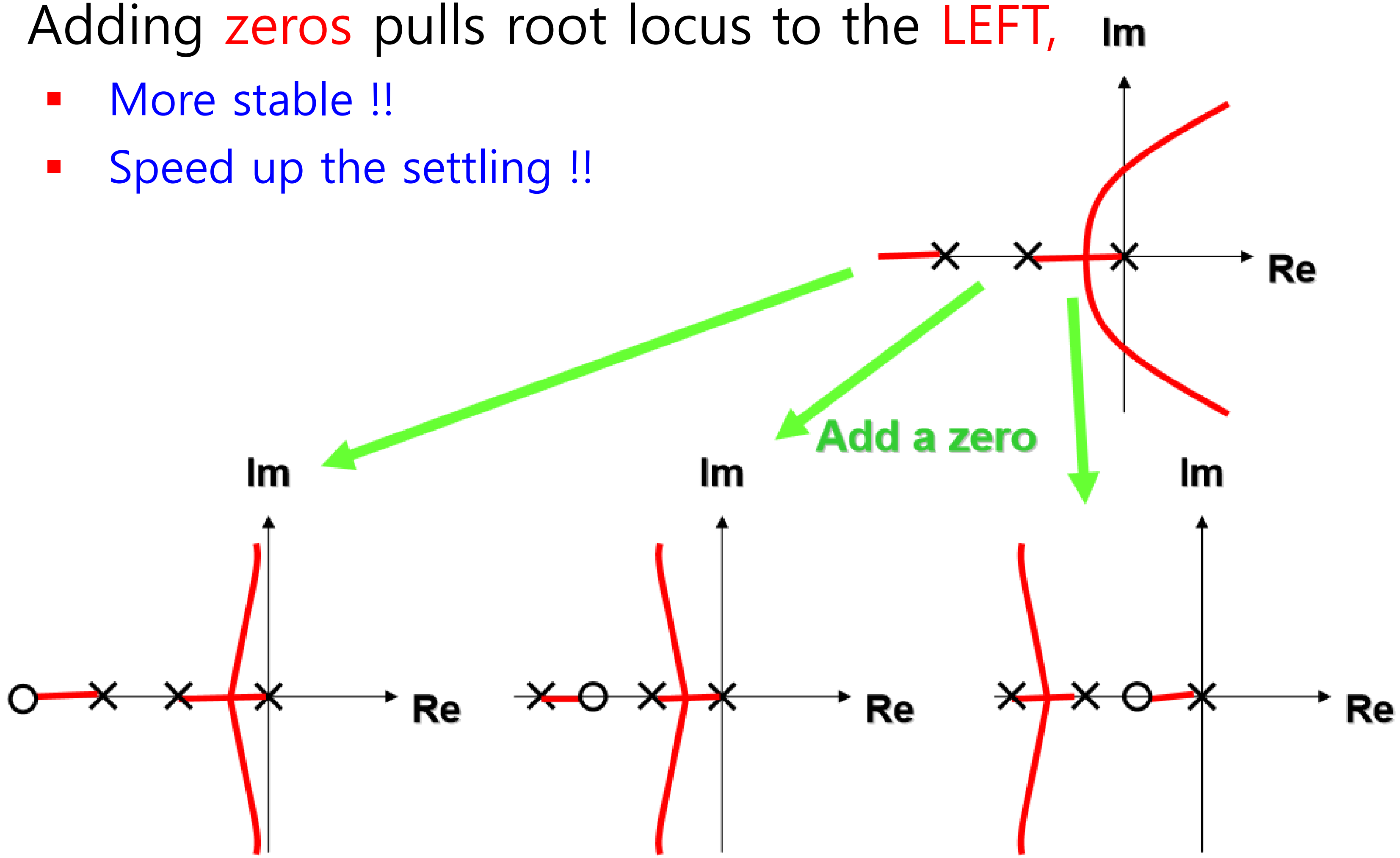
General Effect of Adding Poles

- Adding poles pulls root locus to the RIGHT,
  - Less stable !!
  - Slow down the settling !!



General Effect of Adding Zeros

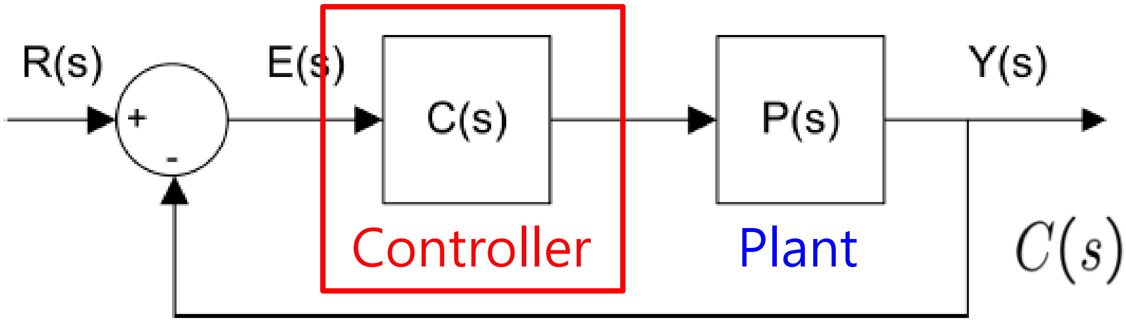
- Adding zeros pulls root locus to the LEFT,
  - More stable !!
  - Speed up the settling !!



Some Remarks about adding Poles/Zeros

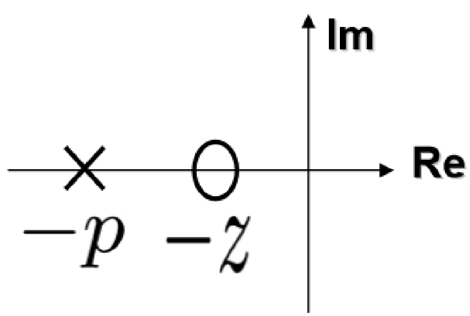
- Adding only zero, $C(s) = s + z, \ (z > 0)$
  - often gets side-effect because such controller amplifies high-frequency noise.
- Adding only pole, $C(s) = 1/(s + p), \ (p > 0)$
  - often gets side-effect because such controller makes a less stable system by moving the closed-loop poles to the right.
- There facts can be explained by using frequency response analysis (i.e., Bode plot) → Next Lecture
- Then, let us add Both Zero & Pole !!

Lead & Lag Compensators

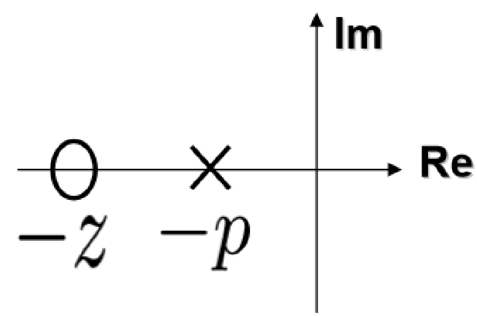


$$C(s) = K\frac{s+z}{s+p}, \quad (z > 0, p > 0)$$

- Lead Compensator



- Lag Compensator

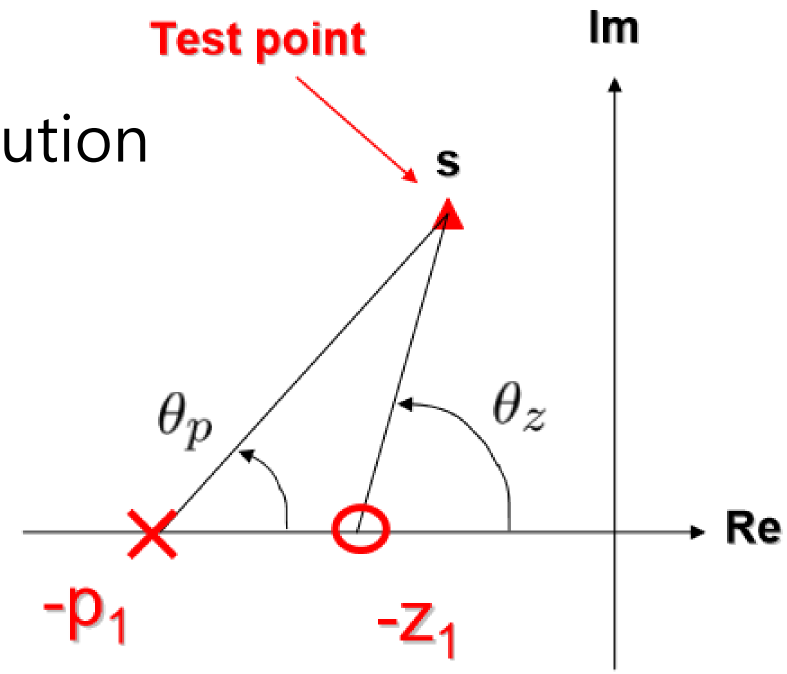


※ Why these are called "Lead" and "Lag" ??

→ We will see this from frequency response in next class.

Lead Compensators

- Positive (Lead) angle contribution



$$\angle C_{Lead}(s) = \theta_{Lead} > 0$$

$$\angle C_{Lead}(s) = \angle \frac{s+z_1}{s+p_1} = \angle(s+z_1) - \angle(s+p_1)$$

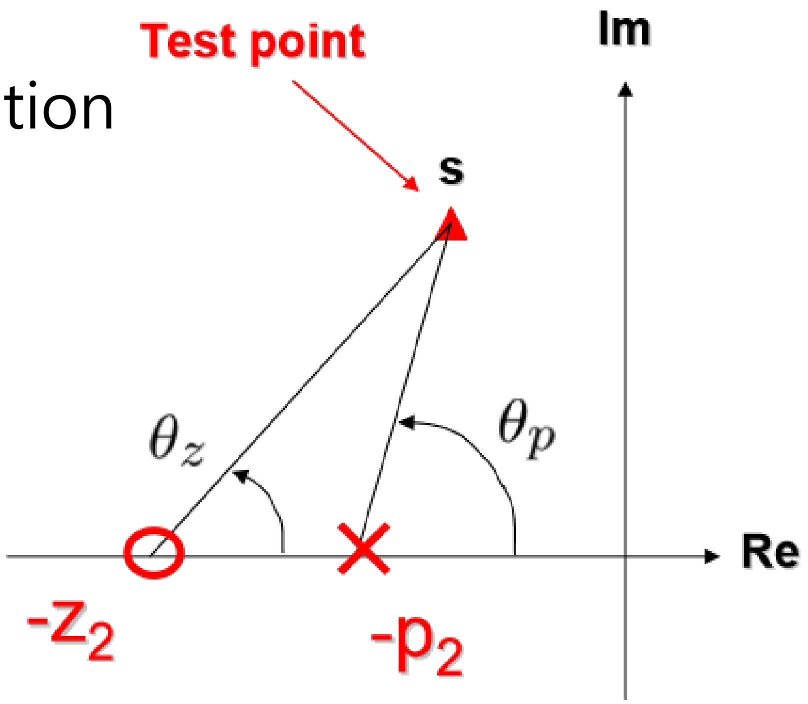
$$= \theta_z - \theta_p = \theta_{Lead} > 0$$

$$where, \quad \theta_z > \theta_p$$

Lag Compensators

- Negative (Lag) angle contribution

$$\angle C_{Lag}(s) = \theta_{Lag} < 0$$



$$\angle C_{Lag}(s) = \angle \frac{s+z_2}{s+p_2} = \angle(s+z_2) - \angle(s+p_2)$$

$$= \theta_z - \theta_p = \theta_{Lag} < 0$$

$$where, \ \theta_z < \theta_p$$

Roles of Lead and Lag Compensator

1. Lead Compensator

- Improve transient response.
- Improve stability.

$$C_{Lead}(s) = K_1 \frac{s + z_1}{s + p_1}$$

$$where, \quad p_1 > z_1$$

2. Lag Compensator

- Reduce steady state error.

$$C_{Lag}(s) = K_2 \frac{s + z_2}{s + p_2}$$

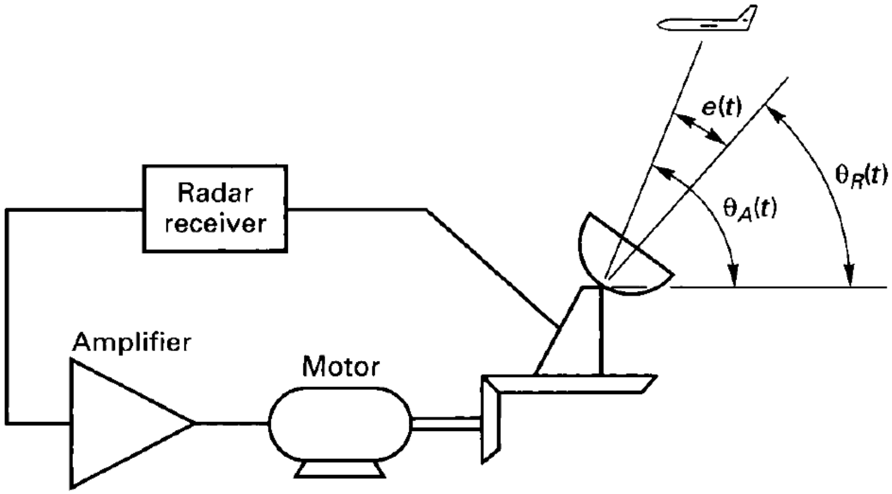
$$where, \quad p_2 < z_2$$

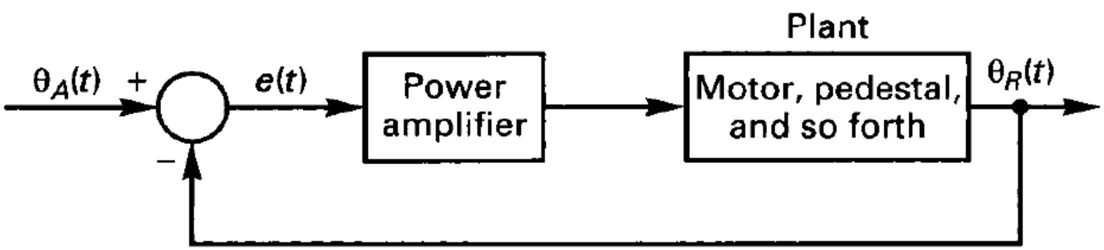
3. Lead-Lag Compensator

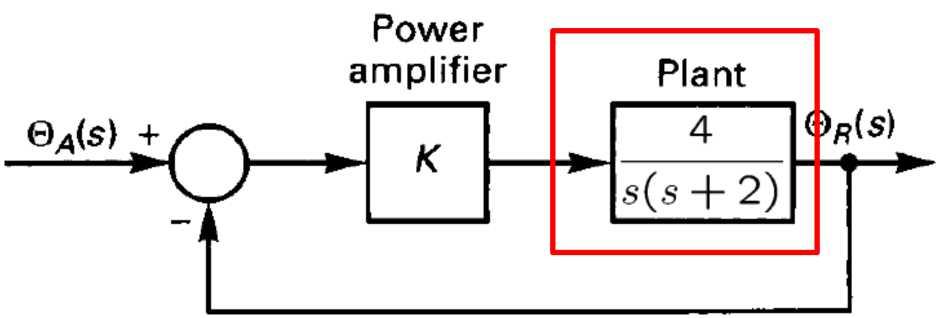
- Take into account all the above issues.

$$C_{LL}(s) = C_{Lead}(s) C_{Lag}(s)$$

Example: Radar Tracking System



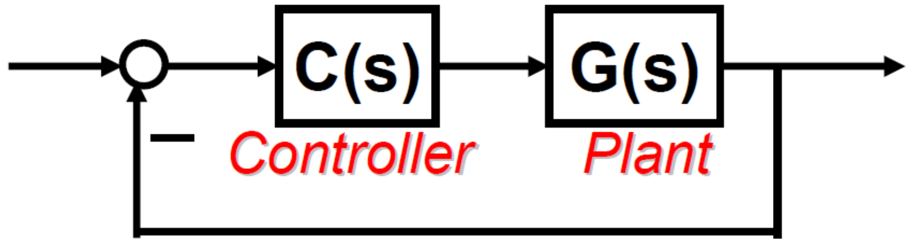




1. Lead Compensator Design

- **Consider system**

$$G(s) = \frac{4}{s(s+2)}$$



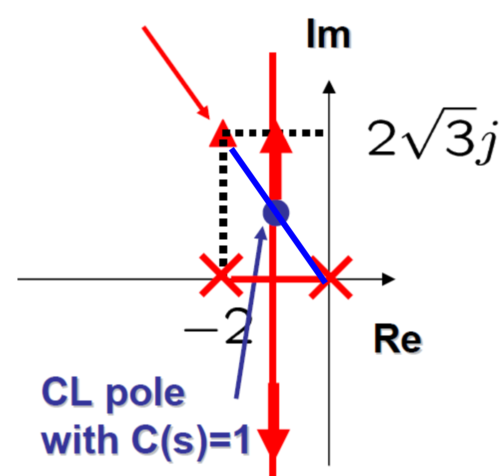
- **Analysis of C-L system for C(s) = 1** $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$

$$H(s)_{CL} = \frac{1 \times \dfrac{4}{s(s+2)}}{1 + 1 \times \dfrac{4}{s(s+2)}} = \frac{4}{s^2 + 2s + 4}$$

$$\boxed{\begin{aligned} \omega_n &= 2 \text{ [rad/sec]} \\ \zeta &= 0.5 \end{aligned}}$$

- **Performance Requirement Specification**
  - Damping ratio: $\zeta = 0.5$ (the same !!)
  - Natural frequency: $\omega_n = 4$ [rad/s] !!

$$H(s)_{CL,desired} = \frac{16}{s^2 + 4s + 16} = \frac{16}{(s + 2 + 2\sqrt{3}j)(s + 2 - 2\sqrt{3}j)}$$



Angle and Magnitude Conditions (Review)

- If a point s to be on root locus, then it satisfies
  - Step 1: Check angle condition

Odd number

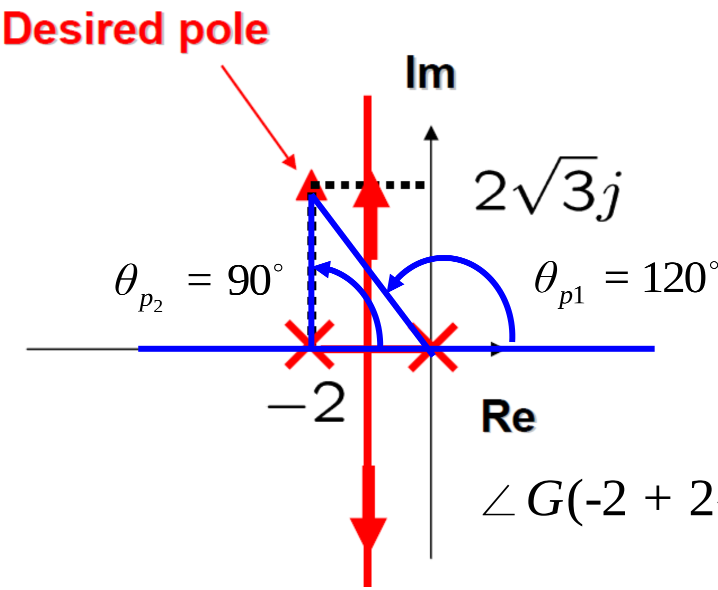
$$\angle L(s) = 180^{o} \times (2k+1), \;\; k = 0, \pm 1, \pm 2, \ldots$$

- For a point on root locus, gain K is obtained by
  - Step 2: Magnitude condition

$$1 + KL(s) = 0 \quad \Rightarrow \quad L(s) = -\frac{1}{K} \Rightarrow \boxed{\left|L(s)\right| = \frac{1}{K}}$$

1. Lead Compensator Design (cont'd)

- **First, Check whether angle condition is satisfied:**



$$\angle G(-2+2\sqrt{3}j) = \angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2)$$
$$= 0 - 120^\circ - 90^\circ = -210^\circ$$

- **But, angle condition is NOT satisfied !!**

Angle deficiency

$\varphi = +30^\circ$

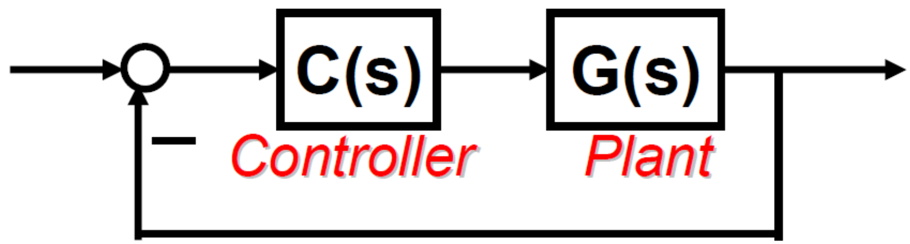
Lead or Lag Compensator ??

(positive)

1. Lead Compensator Design (cont'd)

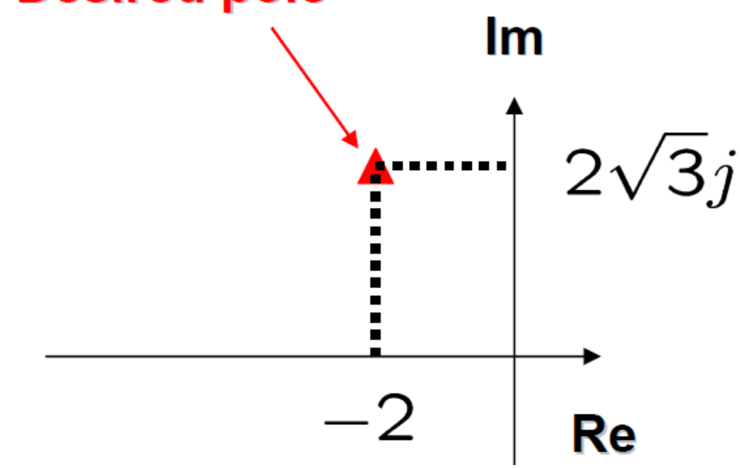
- To compensate angle deficiency,
  we need to design a lead compensator C(s)

$$C(s) = K\frac{s+z}{s+p}$$



satisfying

$$\angle C(-2+2\sqrt{3}j) = 30 (=: \phi)$$

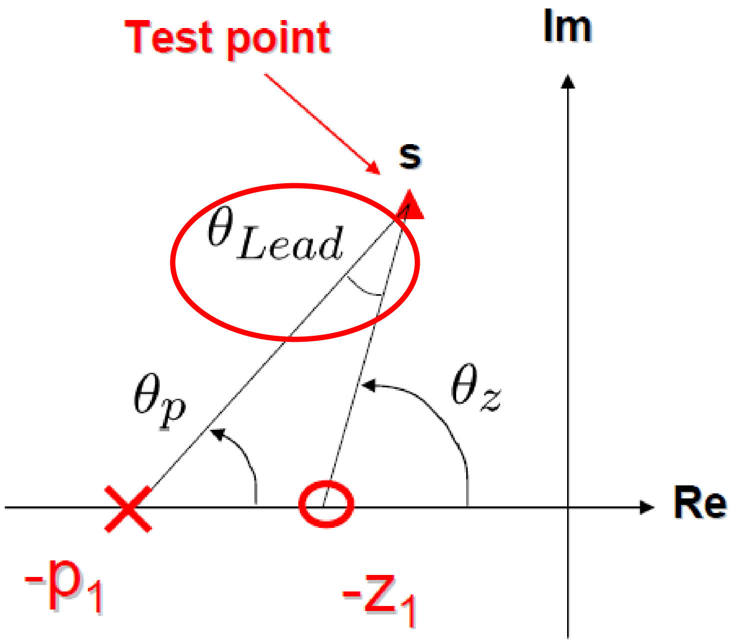


$$\Rightarrow \quad \angle H(s) = \angle CG(-2 + 2\sqrt{3}j) = -180^\circ$$

1. Lead Compensator Design (cont'd)

- Positive angle contribution

$$\angle C_{Lead}(s) = \theta_{Lead} > 0$$



- Triangle

$$\theta_p + \theta_{Lead} + (\pi - \theta_z) = \pi$$

$$\theta_z - \theta_p = \theta_{Lead}$$

How to Select Pole and Zero in Lead C(s)

- Step 1: Draw horizontal line PA
- Step 2: Draw line PO
- Step 3: Draw bi-sector PB

$$\angle APB = \angle BPO = \frac{1}{2}\angle APO$$

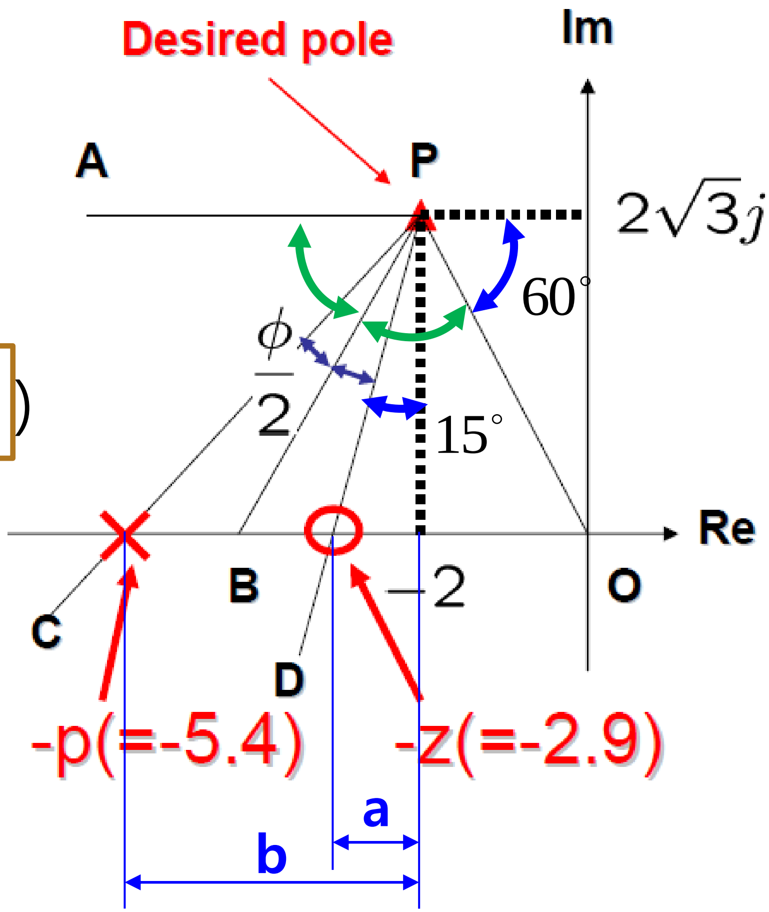
- Step 4: Draw PC & PD (Φ = $\boxed{}$)

$$\angle CPB = \angle BPD = \frac{\phi}{2}$$

- Step 5: Calculate a & b

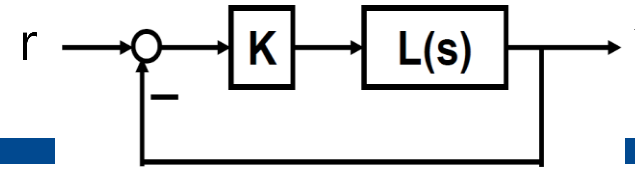
$$a = \tan(15^\circ) \times 2\sqrt{3} \approx 0.9$$

$$b = \tan(45^\circ) \times 2\sqrt{3} \approx 3.4$$

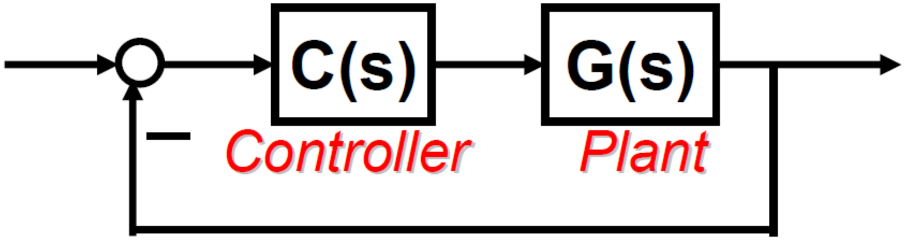


- Then, Pole and Zero of C(s) are shown in the figure.

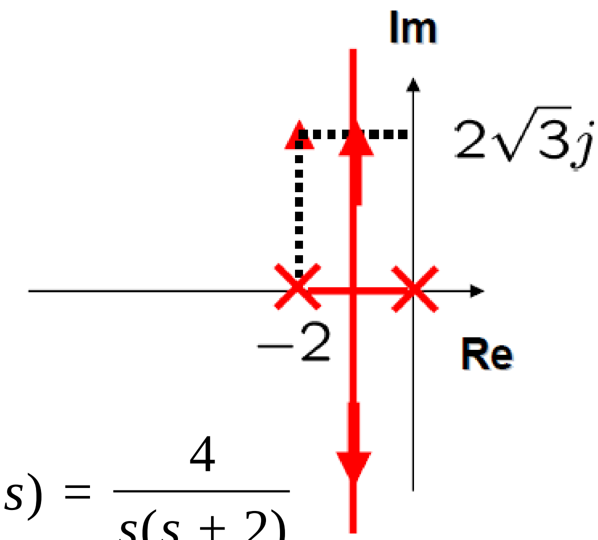
Comparison of Root Locus



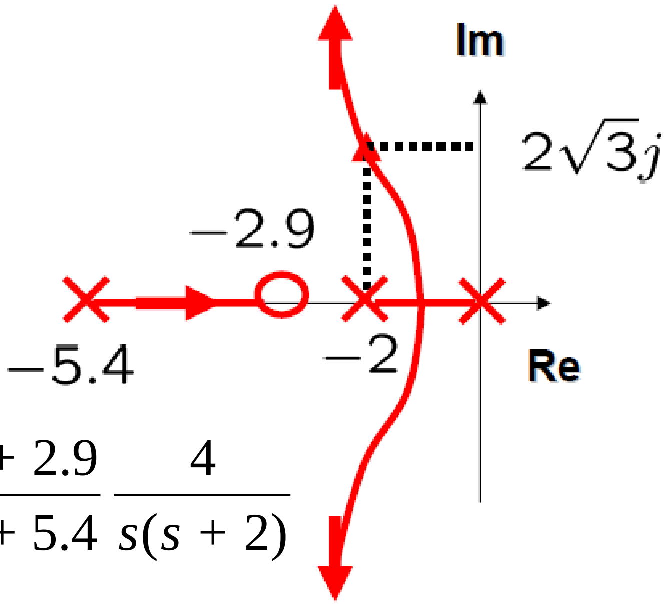
$$C(s) = K\frac{s+2.9}{s+5.4} \quad G(s) = \frac{4}{s(s+2)}$$



- G(s) – without Lead
- C(s)G(s) – with Lead Comp.



$$L(s) = \frac{4}{s(s+2)}$$



$$L(s) = \frac{s+2.9}{s+5.4}\,\frac{4}{s(s+2)}$$

Improved Stability !!

How to Design the Gain K ??

- **Lead Compensator** $C(s) = K\dfrac{s+2.9}{s+5.4}$

- **Open loop TF** $G(s)C(s) = K\dfrac{4(s+2.9)}{s(s+2)(s+5.4)}$

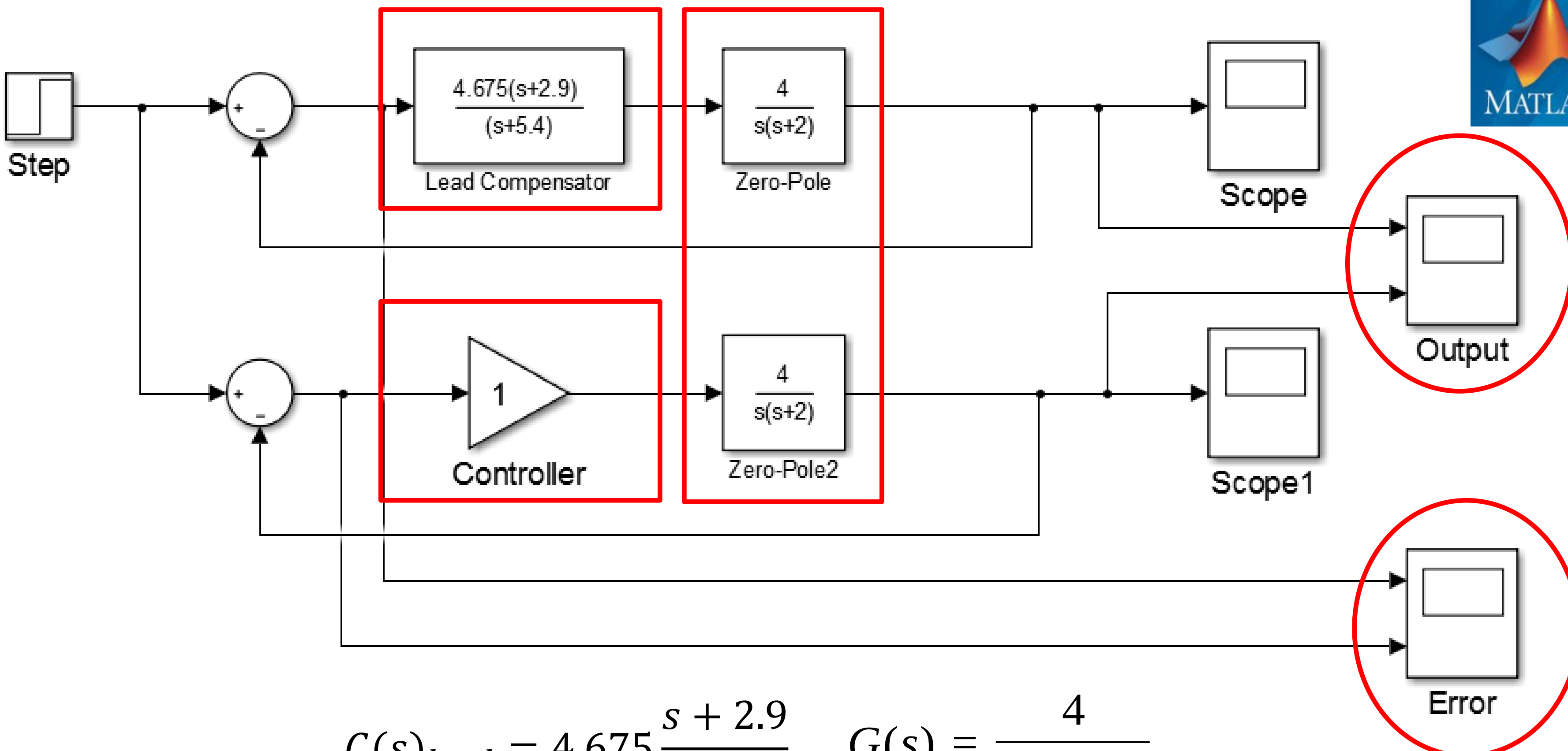
- **Magnitude condition**

$$|L(s)| = \frac{1}{K} \Rightarrow K\left|\frac{4(s+2.9)}{s(s+2)(s+5.4)}\right|_{s=-2+2\sqrt{3}j} = 1$$

$$\Downarrow$$

$$K = \frac{1}{|L(s)|} \cong 4.675$$

Comparison of Step Responses

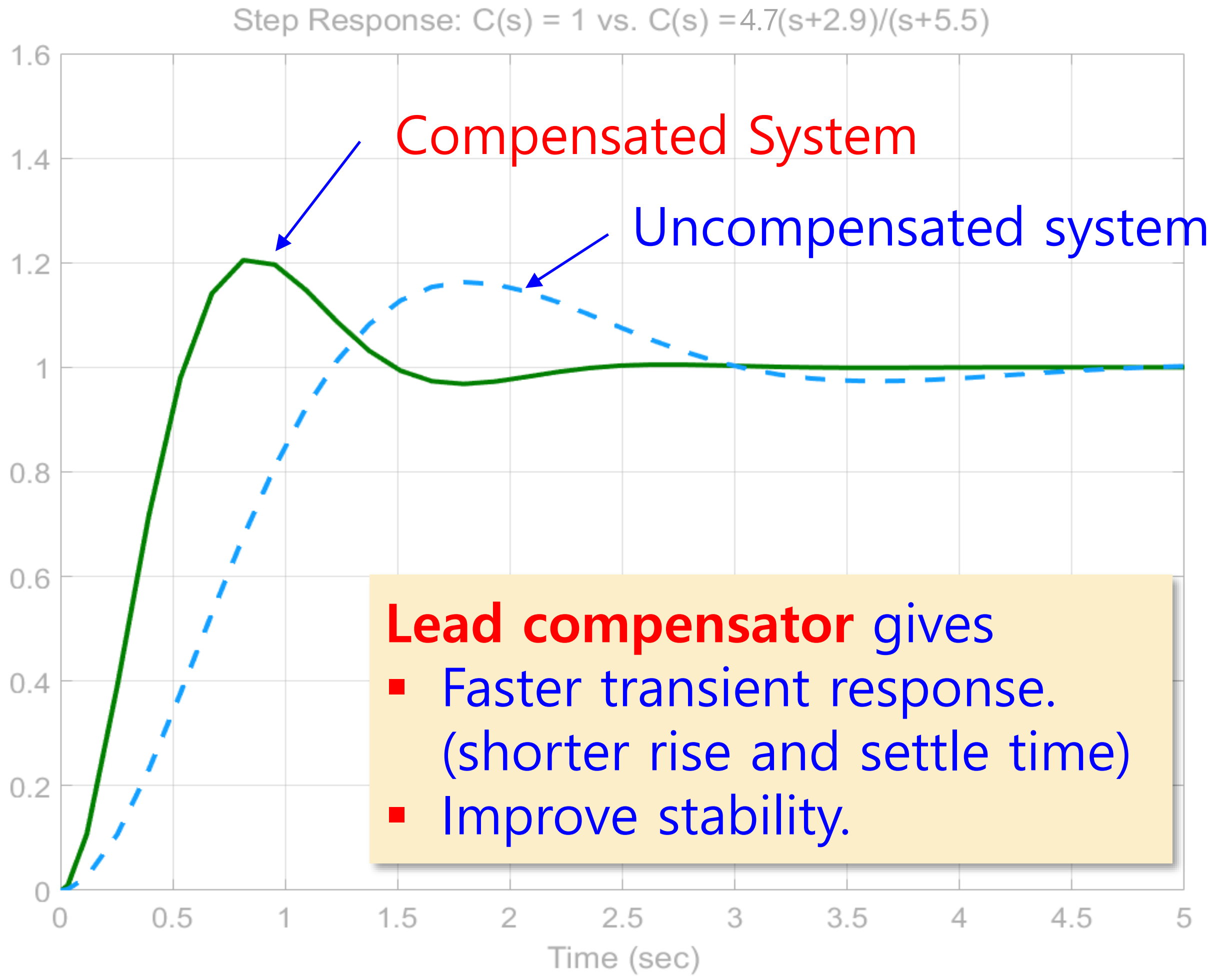


$$C(s)_{lead} = 4.675\frac{s+2.9}{s+5.4} \qquad G(s) = \frac{4}{s(s+2)}$$

$$C(s)_{uncompensated} = 1$$

Comparison of Step Responses (cont'd)

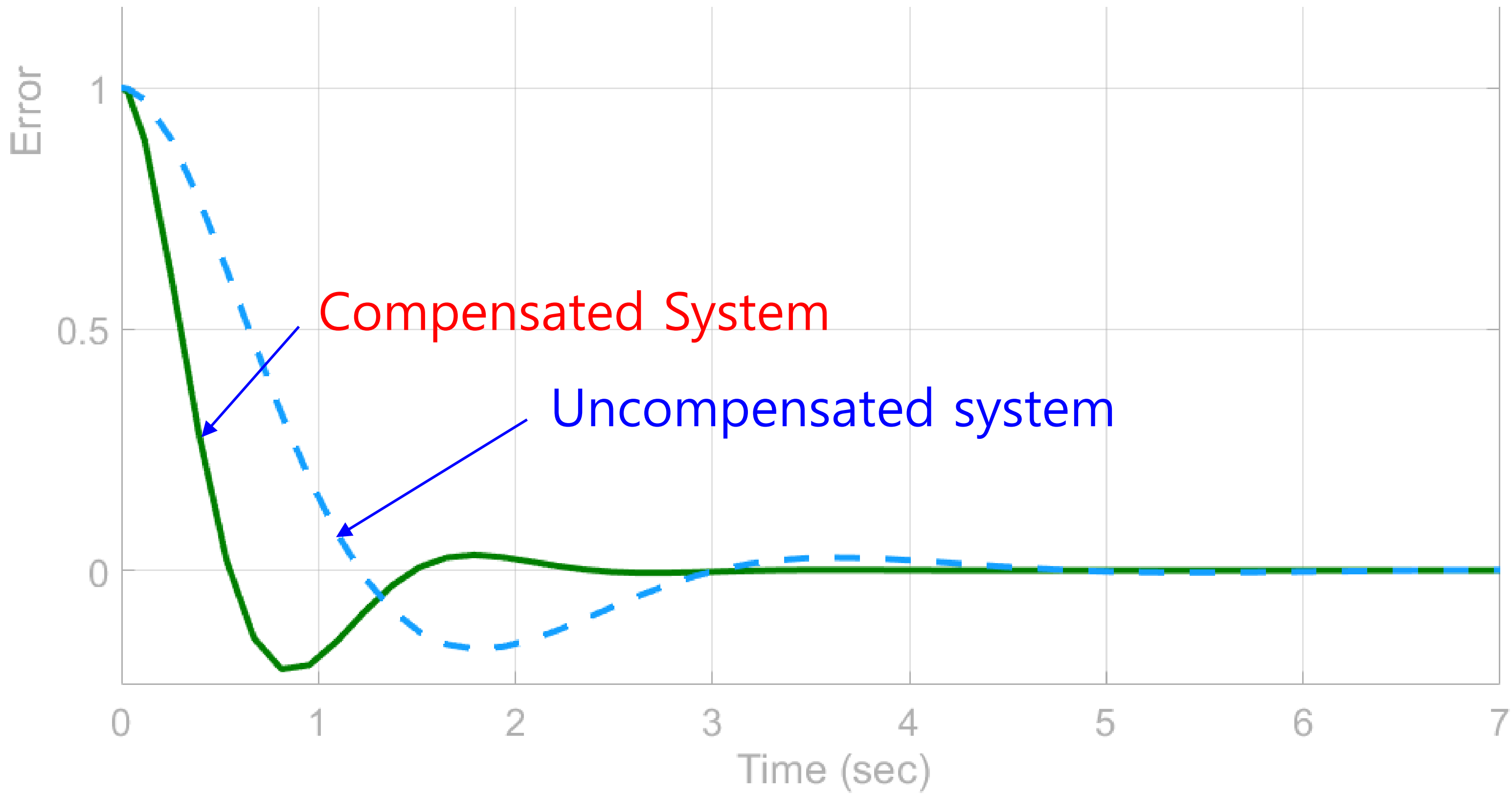
(1) Output comparison



Lead compensator gives
- Faster transient response. (shorter rise and settle time)
- Improve stability.

Comparison of Step Responses (cont'd)

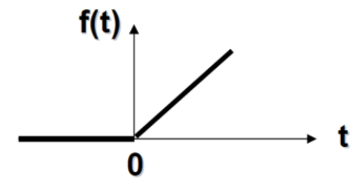
(2) Error comparison



Step response vs. Ramp response

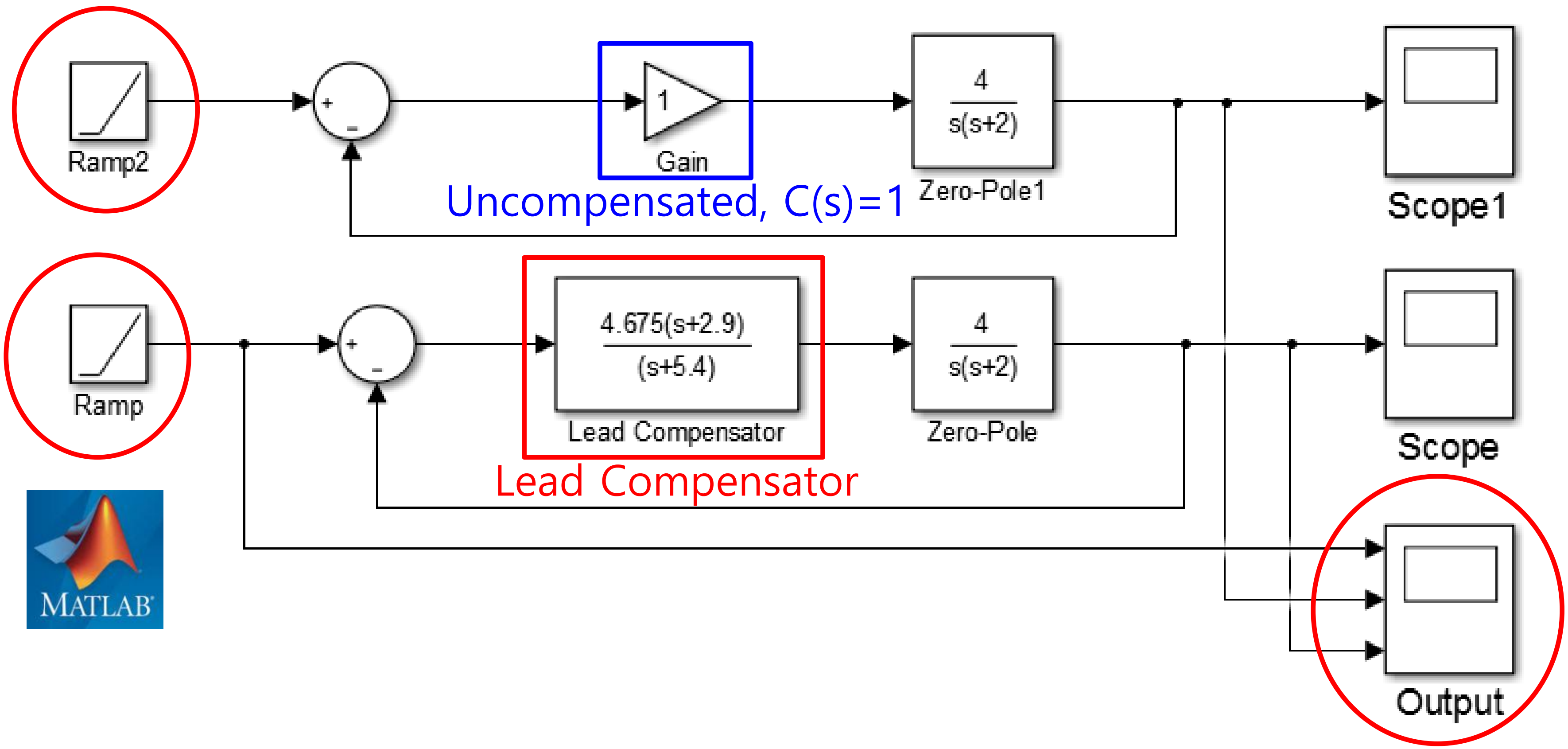
❖ Unit Ramp Function

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

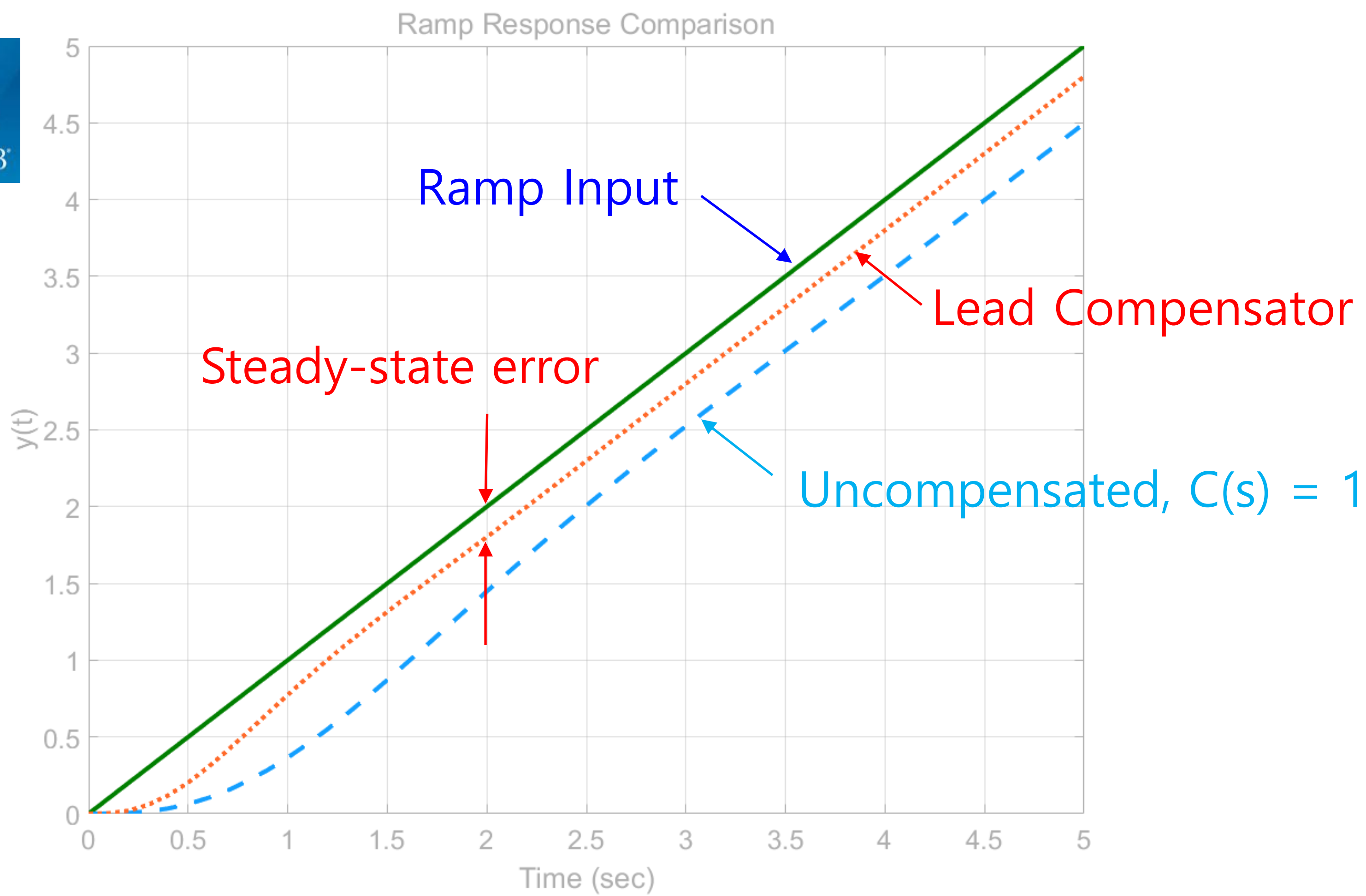


For Lead compensator,

But, what happened with ramp input ??



Step response vs. Ramp response (cont'd)

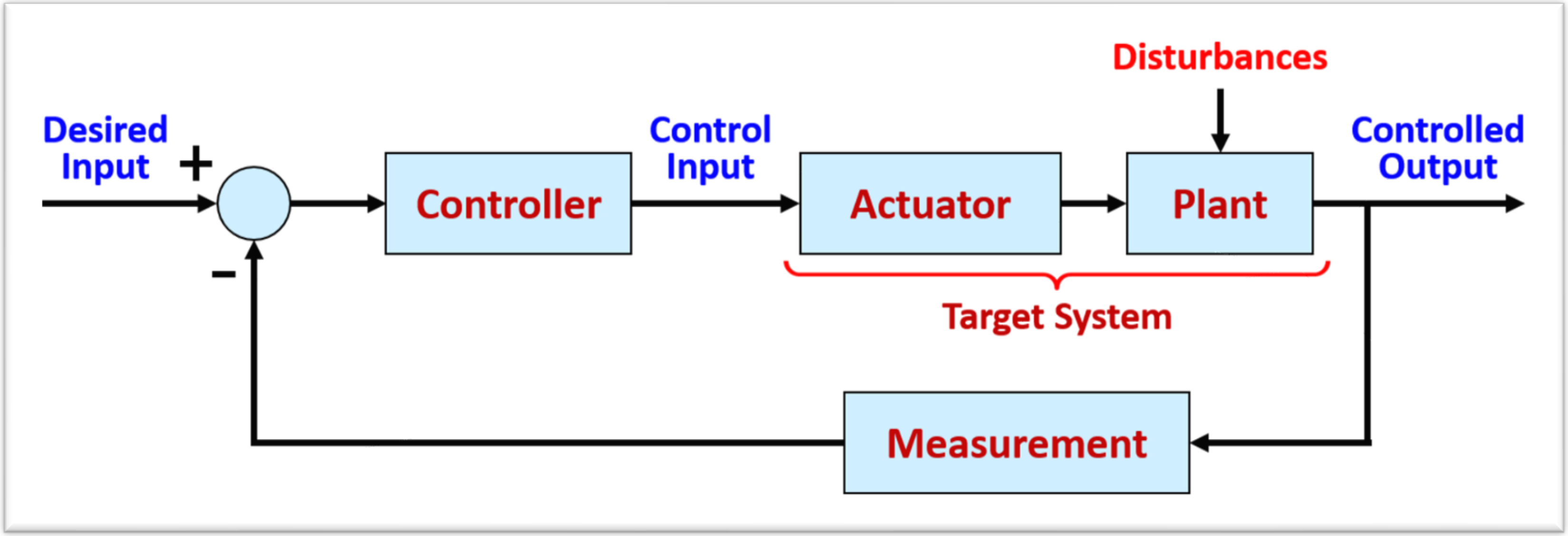


- Is it small enough steady-state error??
- Then, how to reduce something more about steady-state error??

The Root-Locus Design Method 2

Lecture 9:

- Lead Compensator
- **Lag Compensator**



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Roles of Lead and Lag Compensator (Review)

1. Lead Compensator ✓

- Improve transient response.
- Improve stability.

$$C_{Lead}(s) = K_1 \frac{s+z_1}{s+p_1}$$

$$where,\ \ p_1 > z_1$$

2. Lag Compensator

- Reduce steady state error.

$$C_{Lag}(s) = K_2 \frac{s+z_2}{s+p_2}$$

$$where,\ \ p_2 < z_2$$

3. Lead-Lag Compensator

- Take into account all the above issues.

$$C_{LL}(s) = C_{Lead}(s) C_{Lag}(s)$$

System Type for Tracking (revisited)

❖ Closed-Loop System

- The controlled output is given by

$$Y_{cl} = \frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}W - \frac{GD_{cl}}{1+GD_{cl}}V,$$

- The error, difference btw reference input and output is given by

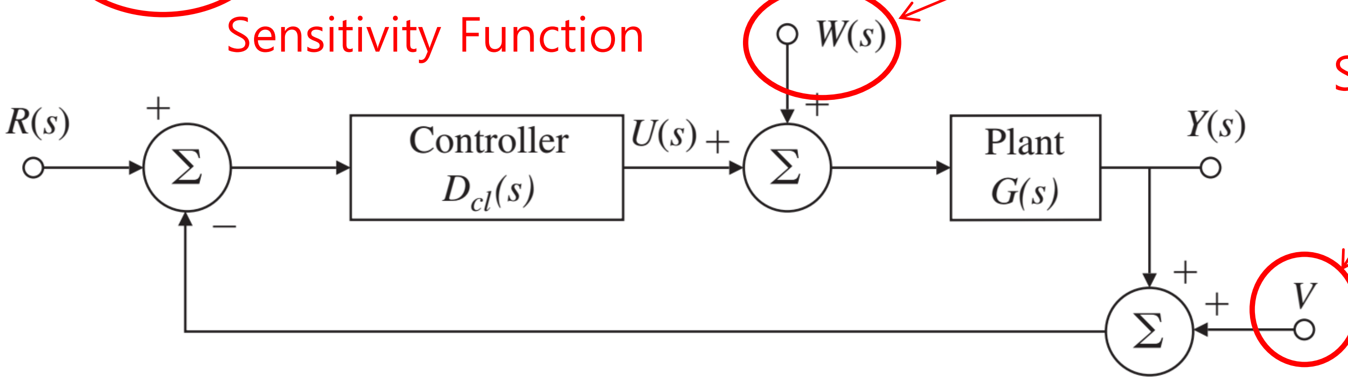
$$E_{cl} = R - \left[\frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}W - \frac{GD_{cl}}{1+GD_{cl}}V\right],$$

$$= \frac{1}{1+GD_{cl}}R - \frac{G}{1+GD_{cl}}W + \frac{GD_{cl}}{1+GD_{cl}}V.$$

Sensitivity Function

Disturbance

Sensor noise



System Type for Tracking (cont'd) (revisited)

- If we consider tracking the reference input alone, set W = V = 0, (where, G: plant model, D_{cl}: controller)

$$E = \frac{1}{1 + GD_{cl}} R = \mathcal{S} R, \quad \text{where} \quad \mathcal{S} = \frac{1}{1 + GD_{cl}}.$$

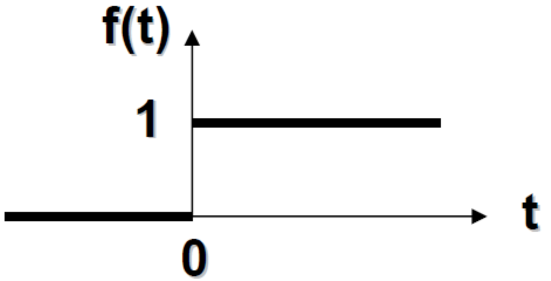
- For analyzing error {E(s)}, categorizing based on three types of reference inputs,

$$R(s) = \frac{1}{s^{k+1}} \begin{cases} k = 0 \Rightarrow R(s) = \dfrac{1}{s} & \text{Step Input (or position)} \\ \boxed{k = 1 \Rightarrow R(s) = \dfrac{1}{s^2} \quad \text{Ramp Input (or velocity)}} \\ k = 2 \Rightarrow R(s) = \dfrac{1}{s^3} & \text{Parabola Input (or acceleration)} \end{cases}$$

Examples of Laplace Transform (revisited)

❖ Unit Step Function

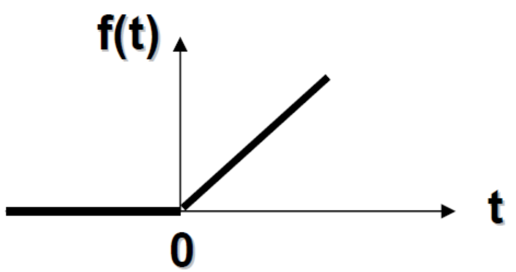
$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow \quad F(s) = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \boxed{\frac{1}{s}}$$

❖ Unit Ramp Function

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow \quad F(s) = \int_0^\infty t e^{-st} dt = -\frac{1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \boxed{\frac{1}{s^2}}$$

(Integration by parts)

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

System Type for Tracking (cont'd) (revisited)

- Apply the Final Value Theorem to the error formula,

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s\boxed{E(s)},$$

$$\boxed{R(s) = \frac{1}{s^{k+1}}}$$

$$= \lim_{s\to 0} s\boxed{\frac{1}{1+GD_{cl}}R(s)}, \quad = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s^{k+1}}.$$

- General form of GD_{cl} without the pole at the origin (s = 0)

$$GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}$$

where,

$GD_{clo}(0) = K_n$: DC gain (or constant)

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+\dfrac{GD_{clo}(s)}{s^n}}\frac{1}{s^{k+1}}, \quad = \lim_{s\to 0}\frac{s^n}{s^n+K_n}\frac{1}{s^k}.$$

System Type for Tracking (cont'd) (revisited)

- From the equation, $e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$. $where, R(s) = \frac{1}{s^{k+1}}$

  then, we have Five Cases for checking error-constant:

✓ Case 1: n > k ⇨ $e_{ss} = 0$

✓ Case 2: n < k ⇨ $e_{ss} = \infty$

✓ Case 3: n = k = 0 (type 0) ⇨ $e_{ss} = \frac{1}{1 + K_p}$ Step error-constant

$R(s) = \frac{1}{s}$

✓ Case 4: n = k = 1 (type 1) ⇨ $e_{ss} = \frac{1}{K_v}$ Ramp error-constant

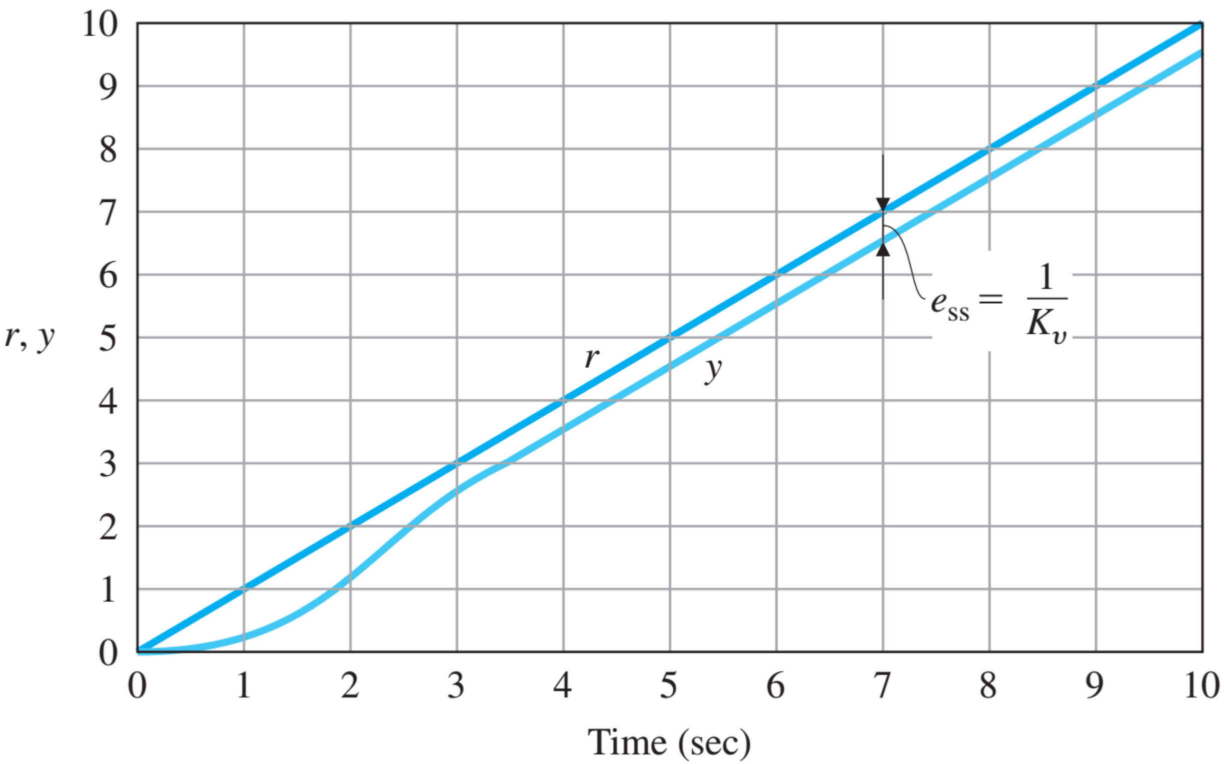
$R(s) = \frac{1}{s^2}$

✓ Case 5: n = k = 2 (type 2) ⇨ $e_{ss} = \frac{1}{K_a}$ Parabola error-constant

$R(s) = \frac{1}{s^3}$

System Type for Tracking (cont'd)

- Ramp response vs. K_v (ramp error-constant)



$$R(s) = \frac{1}{s^2}$$

- Important question:
  - What value of K_v is better to minimize steady-state error (SSE) ??

System Type for Tracking

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s),$$
$$= \lim_{s\to 0} s \frac{1}{1+GD_{cl}} R(s),$$

- **Summary: Error-Constant**

$$[\text{Step}]: \frac{1}{e_{ss}} = \cancel{1} + K_p = \lim_{s\to 0}\frac{1}{sE(s)} = \lim_{s\to 0}\frac{1+GD_{cl}}{s} s = \cancel{1} + \lim_{s\to 0} GD_{cl}$$

$$[\text{Ramp}]: \frac{1}{e_{ss}} = K_v = \lim_{s\to 0}\frac{1}{sE(s)} = \lim_{s\to 0}\frac{1+GD_{cl}}{s} s^2 = 0 + \lim_{s\to 0} sGD_{cl}$$

$$[\text{Parabola}]: \frac{1}{e_{ss}} = K_a = \lim_{s\to 0}\frac{1}{sE(s)} = \lim_{s\to 0}\frac{1+GD_{cl}}{s} s^3 = 0 + \lim_{s\to 0} s^2 GD_{cl}$$

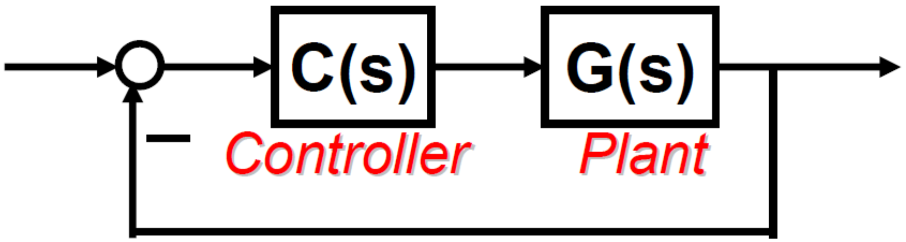
Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

2. Lead-Lag Compensator Design

- **Consider system,**

$$G(s) = \frac{4}{s(s+2)}$$



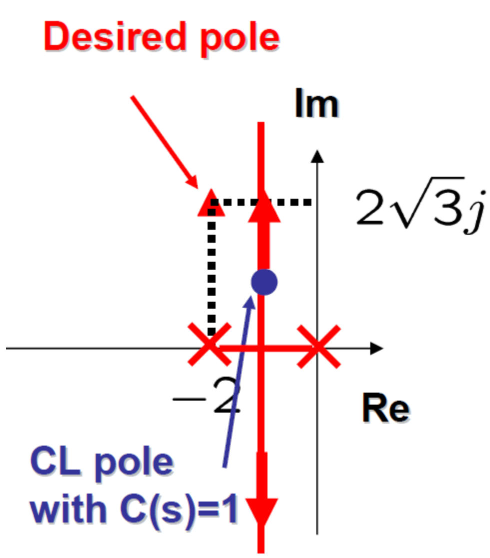
- **Analysis of C-L system for C(s) = 1**

$$H(s)_{CL} = \frac{CG}{1+CG} = \frac{4}{s^2+2s+4}$$

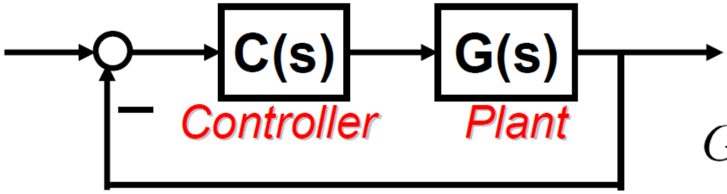
  - Damping ratio: $\zeta = 0.5$ (the same !!)
  - (Undamped) Natural frequency: $\omega_n = 2$ [rad/s] !!
  - Ramp-error constant: $K_v = 2$ → two small !!

$$K_v = \lim_{s \to 0} sC(s)G(s) = s\frac{4}{s(s+2)} = 2$$

- **Performance Requirement Specification**
  - Damping ratio: $\zeta = 0.5$ (the same !!)
  - (Undamped) natural frequency: $\omega_n = 4$ [rad/s] !!
  - Ramp-error constant: $K_v = 50$!! (Target !!)



Error Constants (after Lead Compensator only)



$$G(s)C_{Lead}(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

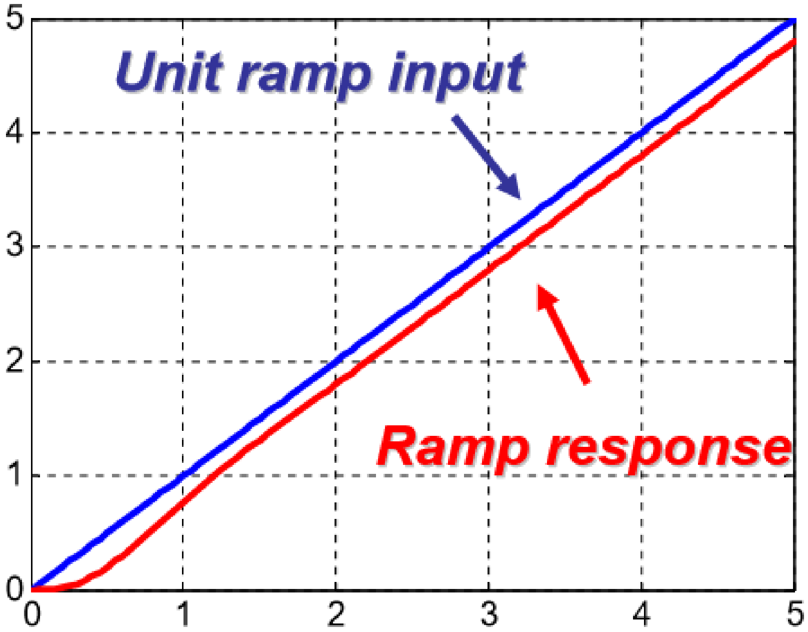
- **Step-error constant**

$$K_p = \lim_{s \to 0} G(s)C_{Lead}(s) = \infty$$

- **Ramp-error constant**

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s) = 5.02$$

Not satisfied !!
(K_v > 50)



- **Lag compensator** can reduce steady-state error (SSE).

How to Design Lag Compensator

- Lag compensator $C_{Lag}(s) = \dfrac{s+z}{s+p}$
- Objectives: increasing ramp-error constant

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s)\boxed{C_{Lag}(s)} = 5.02 \cdot \boxed{\frac{z}{p}} > 50$$

Take, for example, z = 10p

- We do NOT want to change CL pole location s_1 so much.
(because we already had satisfactory transient !!)

$$\left.\begin{aligned} 1 + G(s_1)C_{Lead}(s_1) = 0 \\ C_{Lag}(s_1) \approx 1 \end{aligned}\right\} \Rightarrow 1 + G(s_1)C_{Lead}(s_1)C_{Lag}(s_1) \approx 0$$

Guidelines to choose z and p (Lag comp.)

- The zero and the pole of a lag compensator should be close to each other, for

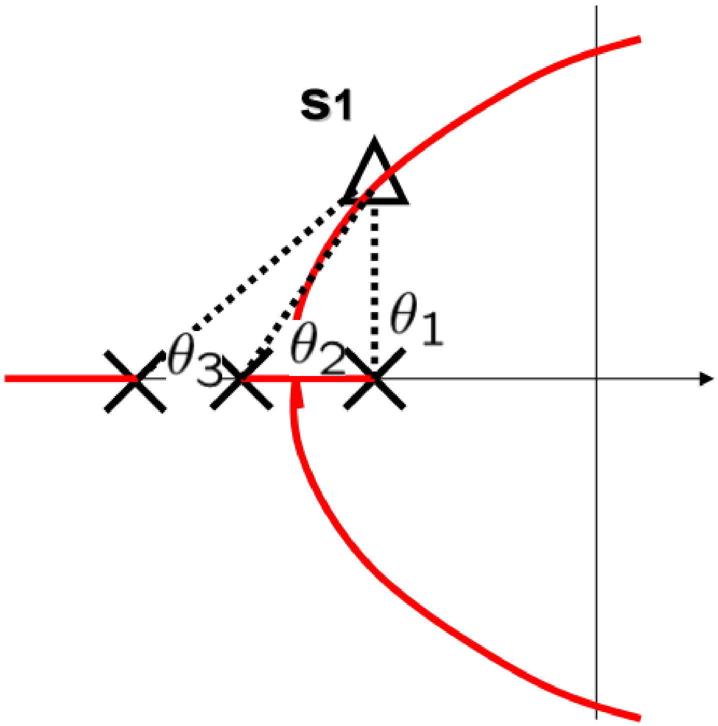
$$C_{Lag}(s_1) \approx 1$$

- The pole of a lag compensator should be close to the origin, to have a large ratio z/p, leading to a large ramp-error constant K_v

- However, the pole of a lag compensator too close to the origin may be problematic:
  - Slow settling (due to closed-loop pole near the origin)

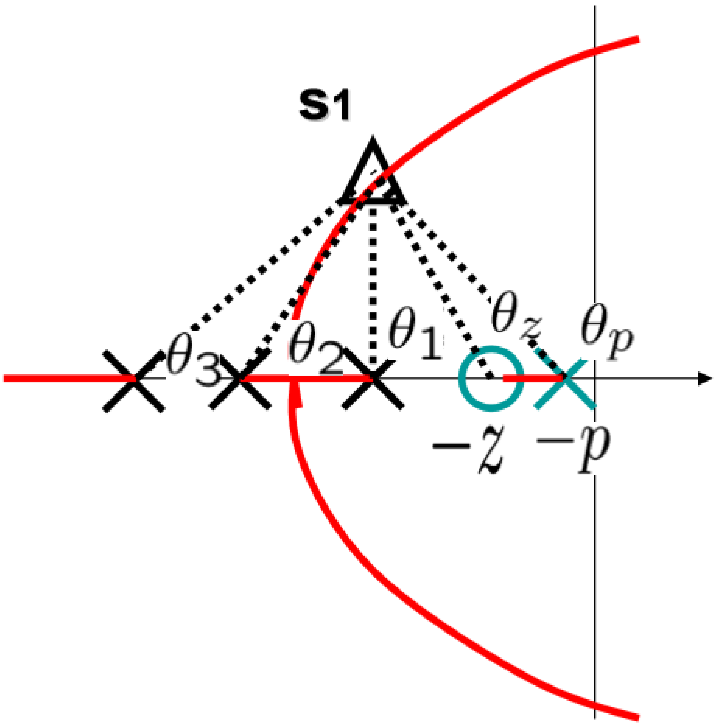
Root Locus with Lag Compensator

- **Without compensator**



$$-\theta_1 - \theta_2 - \theta_3 = 180$$

- **With compensator (Lag Compensator)**



$$-\theta_1 - \theta_2 - \theta_3 + \underbrace{\boxed{\theta_z - \theta_p}}_{\approx 0} \approx 180$$

How to Design Lag Compensator (cont'd)

- For the desired CL pole: $s_1 = -2 + 2\sqrt{3}j$

$$C_{Lag}(s_1) \approx 1 \Leftrightarrow \left|\frac{s_1 + 10p}{s_1 + p}\right| \approx 1 \ \ \& \ \ \angle\left(\frac{s_1 + 10p}{s_1 + p}\right) \approx 0$$

- Take a small pole (by trial-and-error !!)

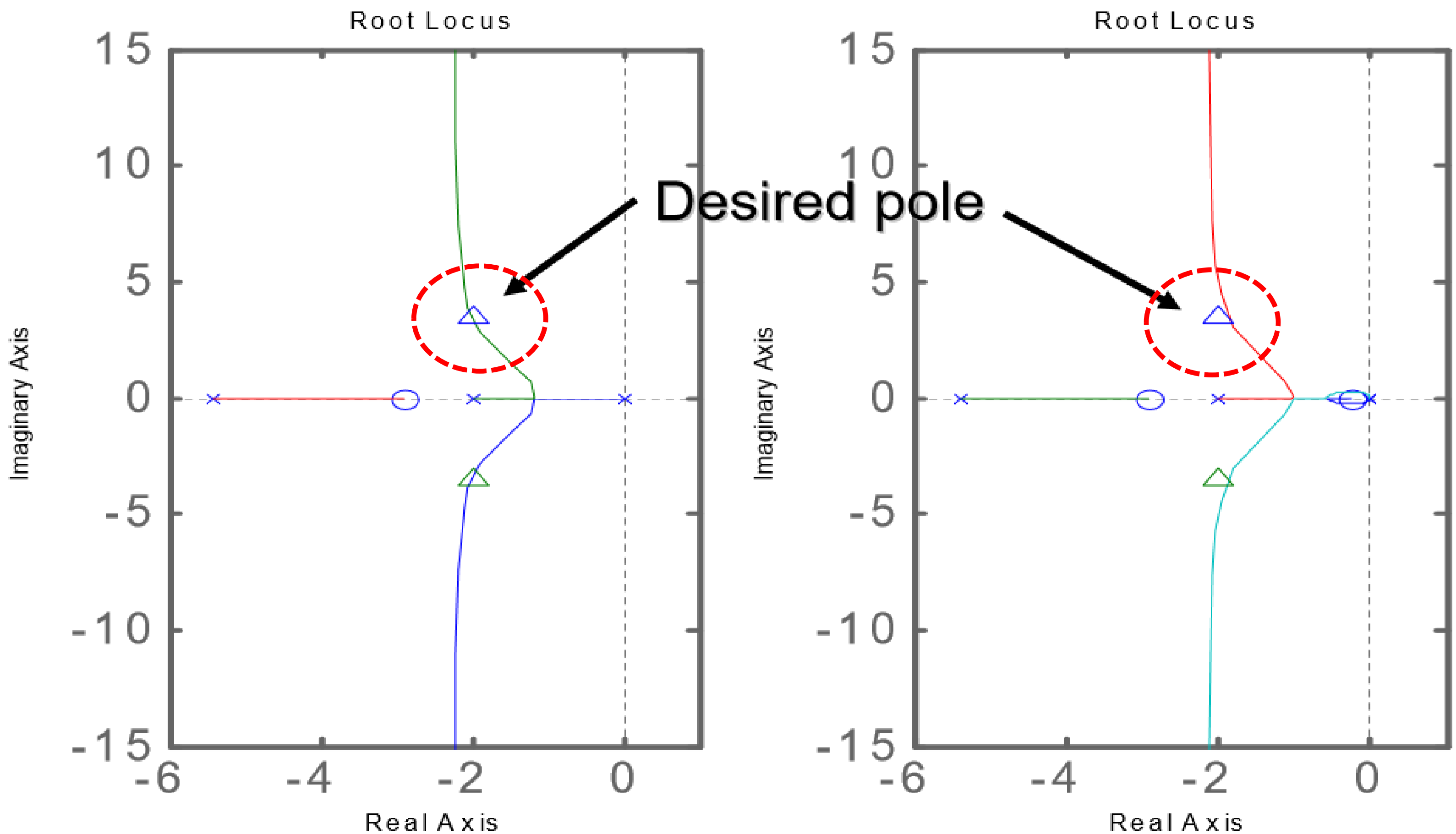
$$p = 0.025 \Rightarrow \left|\frac{s_1 + 10p}{s_1 + p}\right| = 0.97 \ \ \angle\left(\frac{s_1 + 10p}{s_1 + p}\right) \approx -2.88^o$$

- Lead-Lag Controller

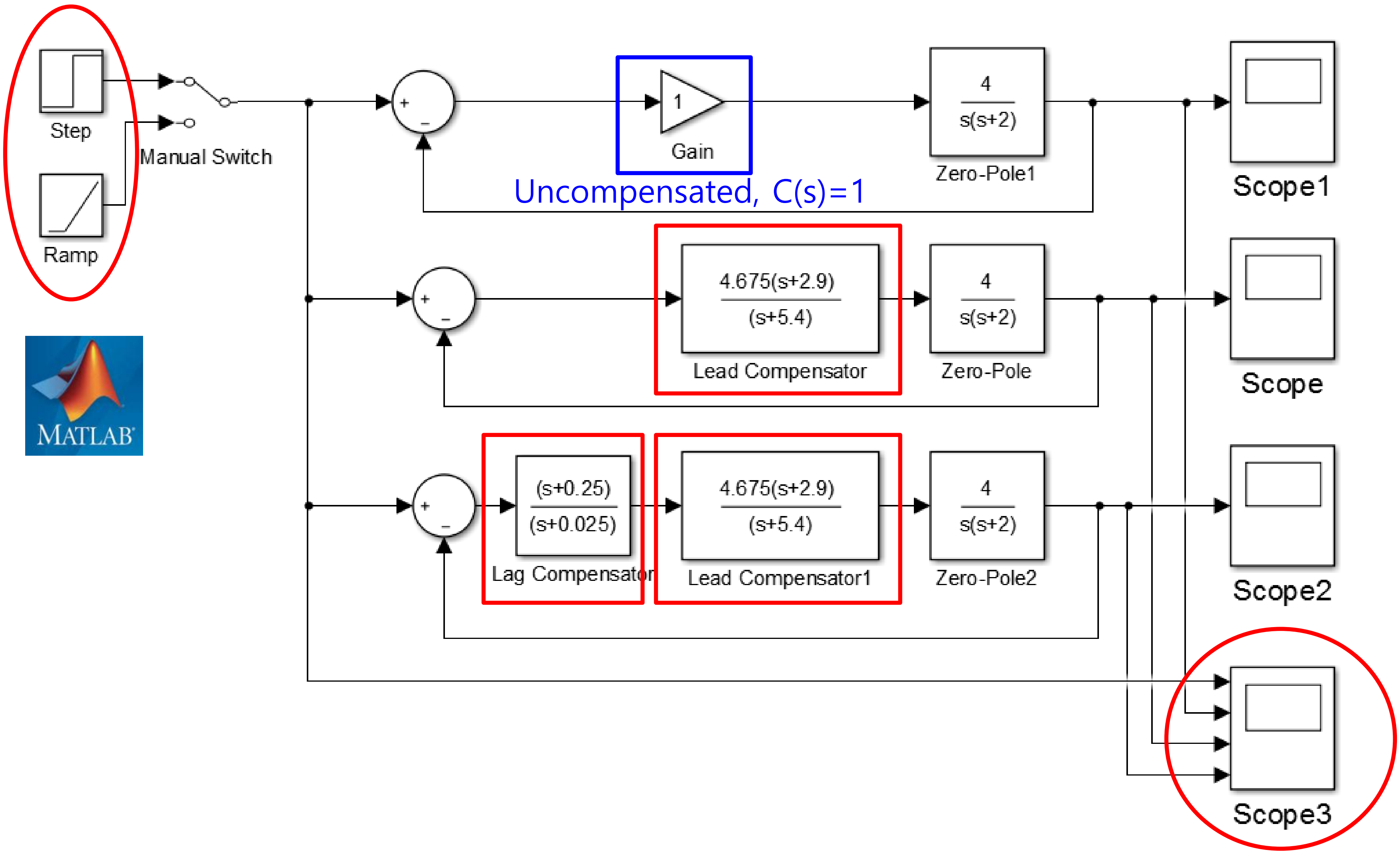
$$\boxed{C_{LL}(s) = 4.675\frac{s + 2.9}{s + 5.4} \cdot \frac{s + 0.25}{s + 0.025}}$$

Root Locus Comparison with Compensators

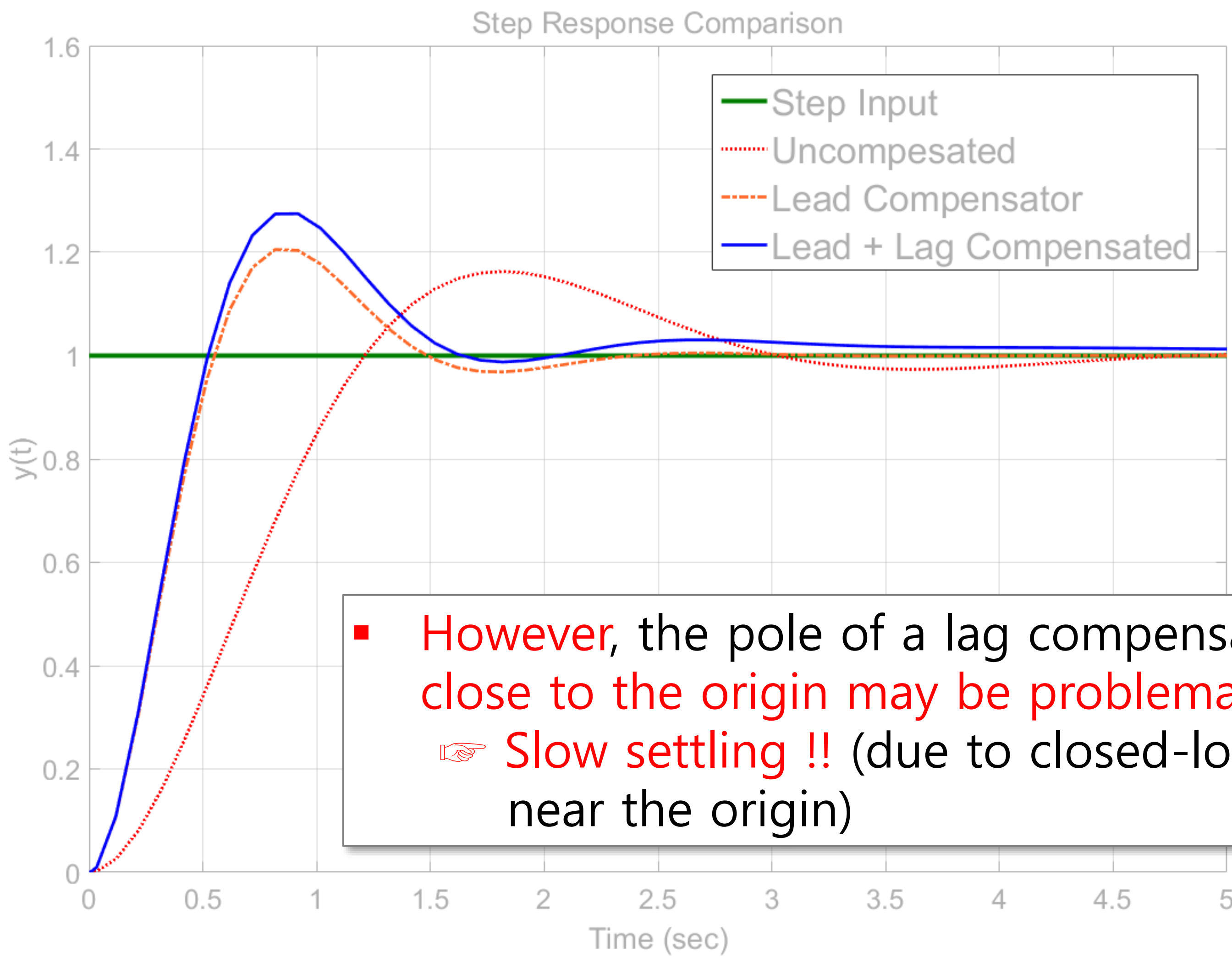
- With lead compensator
- With lead-lag compensator



Comparison: Step vs. Ramp Response

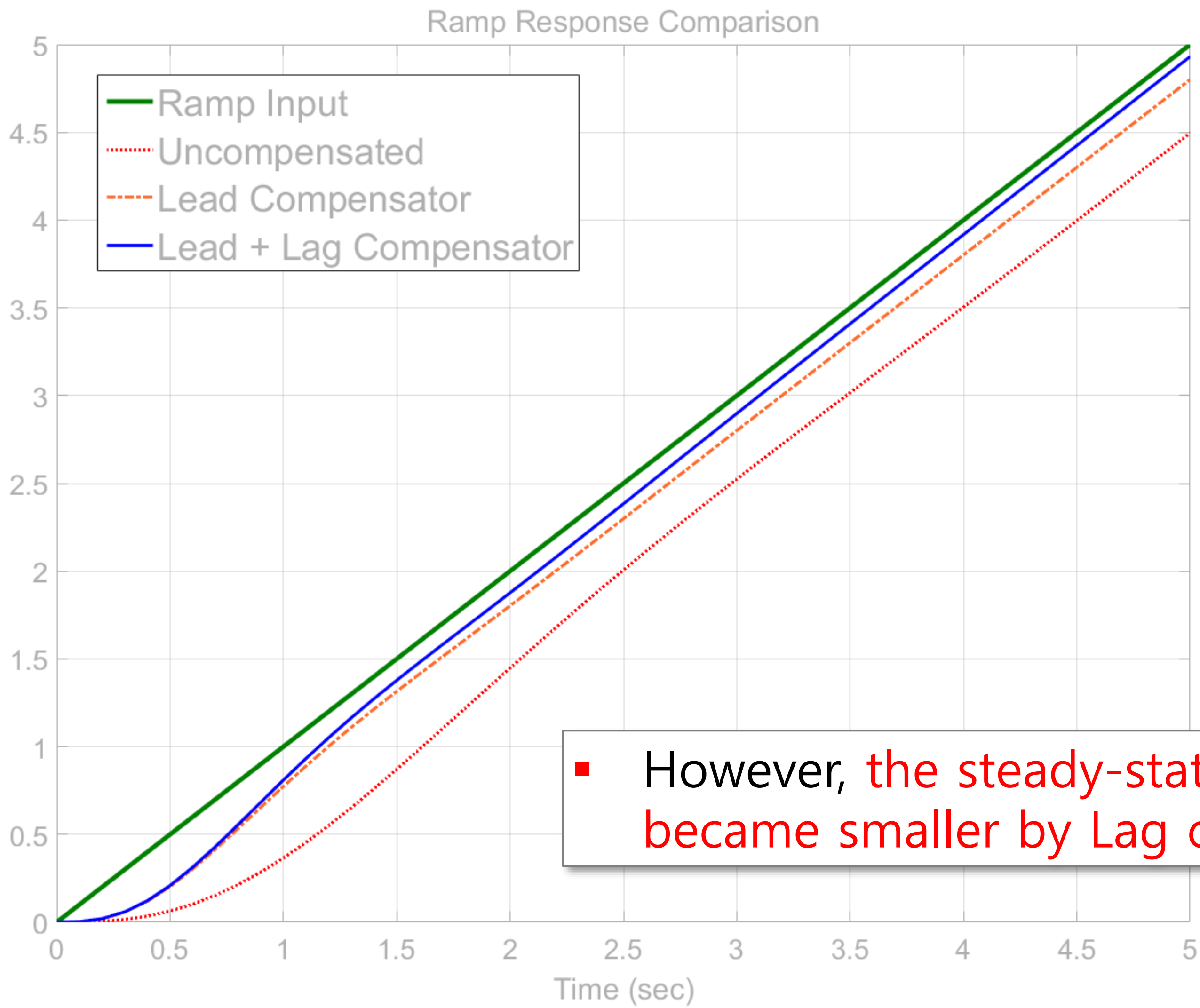


Comparison (1): Step Response



- However, the pole of a lag compensator too close to the origin may be problematic:
  - ☞ Slow settling !! (due to closed-loop pole near the origin)

Comparison (2): Ramp Response



- However, the steady-state error (SSE) became smaller by Lag compensator !!

Summary

❖ **Summary**:

- Controller design based on root locus.
  - Lead compensator improve stability and transient response.
  - Lag compensator improves steady-state error.
    ※ Important: checking error-constant (or system type) !!
  - Lead-Lag compensator improves stability, transient and steady-state response at the same time.