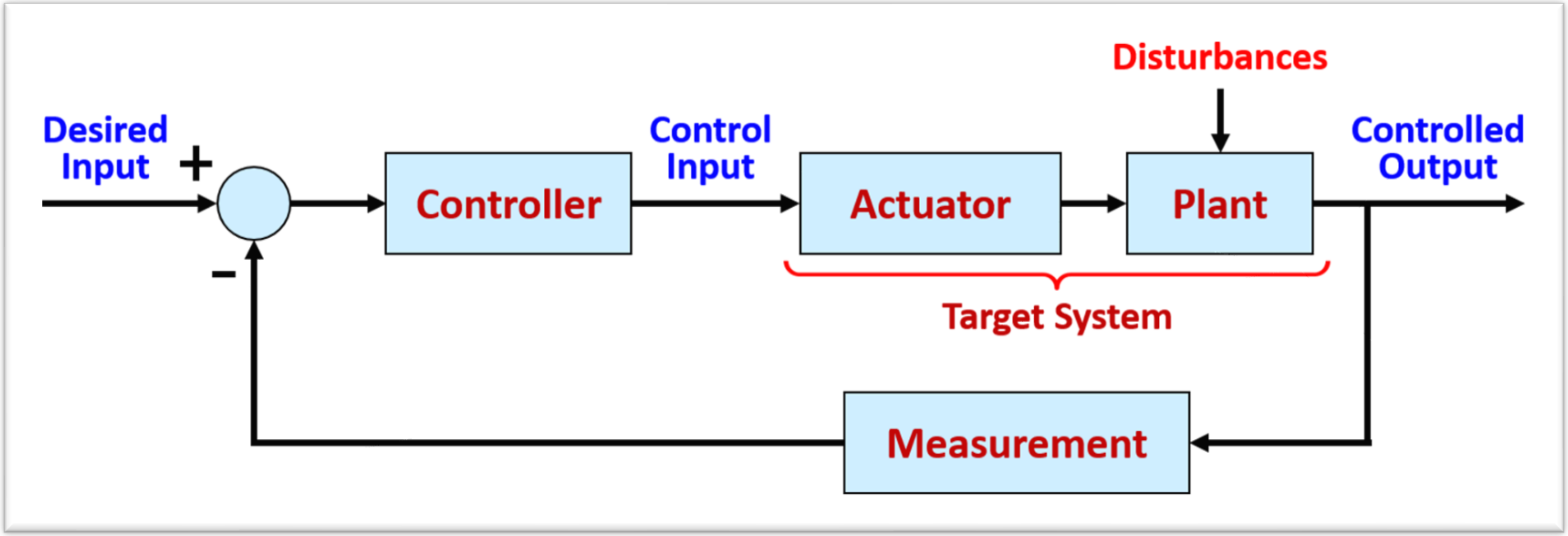
The Root-Locus Design Method 1

Lecture 8:

- Root-Locus Drawing Rules
- Multi-Parameter Controller Design



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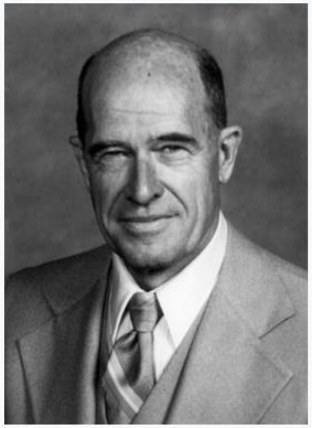
What is Root-Locus ??

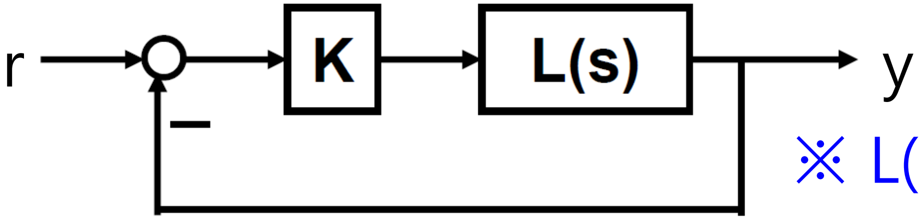
Walter R. Evans



Born	January 15, 1920
Died	July 10, 1999 (aged 79)
Residence	United States

- Walter R. Evans developed in 1948.
- Poles location of the feedback control system characterizes stability and transient properties.

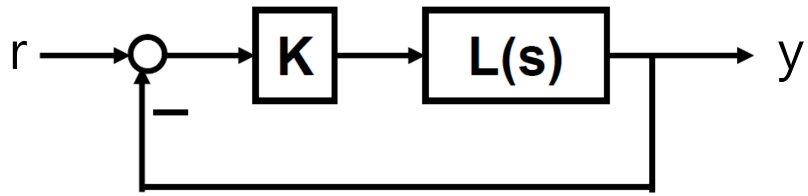
Consider a feedback system with one parameter (gain) K>0



※ L(s): Open-Loop TF

- Root-locus graphically shows poles locations of closed-loop system as K varies from 0 to infinity.

Characteristic Equation & Root-Locus



- First, the closed-loop transfer function (R vs. Y) is given by

$$\frac{Y(s)}{R(s)} = \frac{KL(s)}{1 + KL(s)}$$

- Then, what is the Characteristic Equation??

$$1 + KL(s) = 0 \quad \Leftrightarrow \quad K = -\frac{1}{L(s)} \quad \Leftrightarrow \quad L(s) = -\frac{1}{K}$$

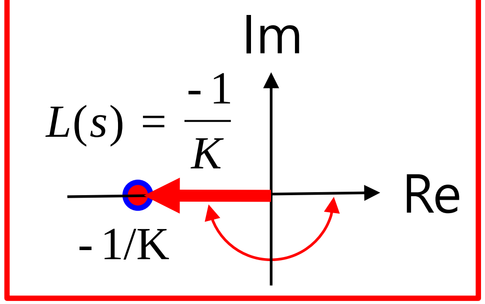
- Root-locus is obtained by
  - ✓ for a fixed K>0, finding roots of the characteristic equation, and

- A point s is on the root locus, if and only if L(s) evaluated for that s is a negative real number.

Magnitude & Angle Conditions

- Characteristic eq. can be split into **two conditions**:

$$1 + KL(s) = 0 \iff L(s) = -\frac{1}{K}$$



(I) Magnitude Condition

$$\boxed{|L(s)| = \frac{1}{K}} \longrightarrow$$

For any point s,
this condition holds
for some positive K.

(II) Angle Condition

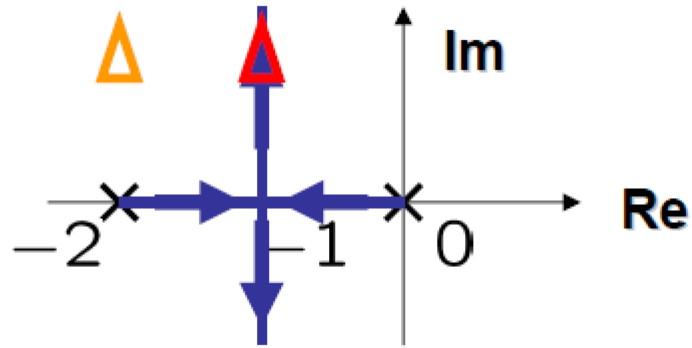
Odd number

$$\boxed{\angle L(s) = 180^o \times (2k+1), \;\; k = 0, \pm 1, \pm 2, \ldots}$$

(Practically, physical meaning of angle is how [ ] system is !!)

A Simple Example

$$L(s) = \frac{1}{s(s+2)}$$



- **Select a point s = -1+j**

$$\begin{aligned} L(s) &= \frac{1}{s(s+2)} \\ &= \frac{1}{(-1+j)(1+j)} = -\frac{1}{2} \end{aligned}$$

Negative **real** number !!

$\Rightarrow$ $\angle L(s) = 180$

$\Rightarrow$ Thus, **s** is ON root locus !!

- **Select a point s = -2+j**

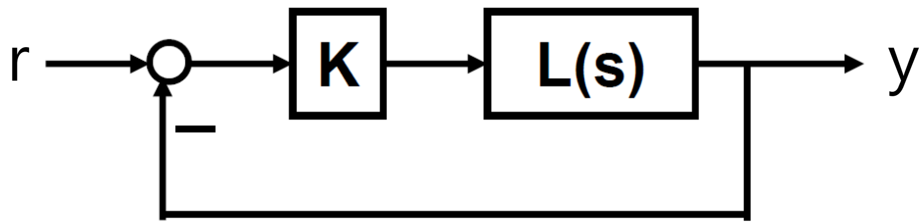
$$\begin{aligned} L(s) &= \frac{1}{s(s+2)} \\ &= \frac{1}{(-2+j)j} \\ &= \frac{1}{-2j-1} \end{aligned}$$

Negative **complex** number !!

$\Rightarrow$ $\angle L(s) \neq 180$

$\Rightarrow$ Thus, **s** is NOT on root locus !!

A Simple Example

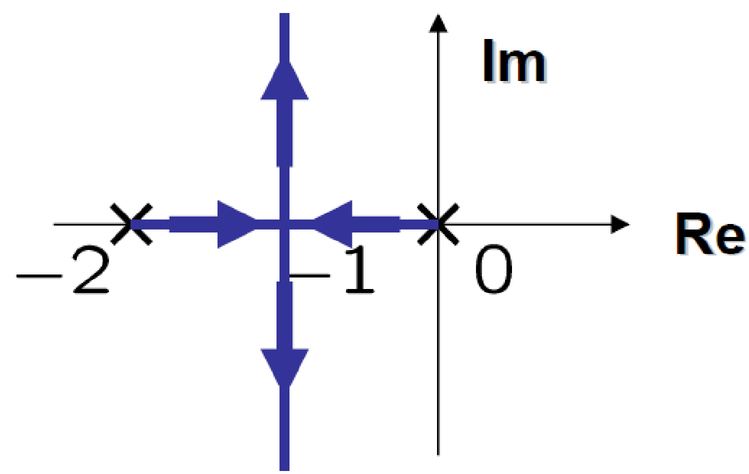


- Characteristic eq. $1 + K\dfrac{1}{s(s+2)} = 0$

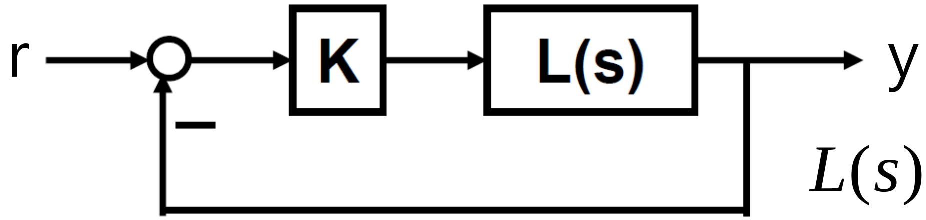
Closed-Loop Poles !!

$$\Rightarrow \; s^2 + 2s + K = 0 \; \Rightarrow \; s = -1 \pm \sqrt{1 - K}$$

- K=0: s=0,-2
- K=1: s=-1,-1
- K>1: complex numbers



A More Complicated Example



$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

- Characteristic eq. $1 + K\dfrac{s+1}{s(s+2)(s+3)} = 0$

$$\Rightarrow \quad s(s+2)(s+3) + K(s+1) = 0 \quad \Rightarrow \quad s = ???$$

- It is hard to solve this analytically for each K.
- Is there some way to sketch roughly root-locus by hand? (In Matlab, use fn. "rlocus(sys)".)

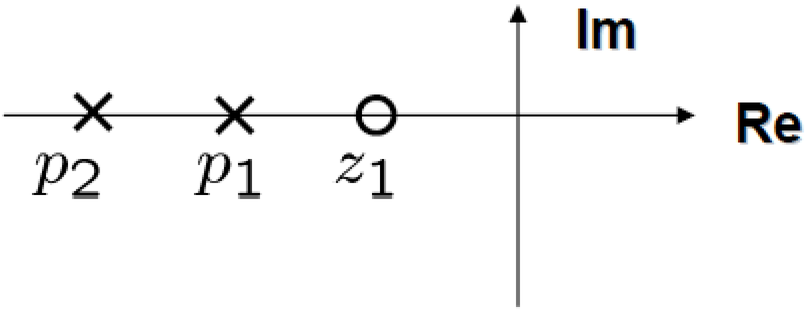
Root-Locus: Rule 0

- Root-locus is symmetric w.r.t. the real axis.
  - Characteristic equation is an equation with real coefficients. Hence, if a complex number is a root, its complex conjugate is also a root.

- The number of branches = order of L(s)

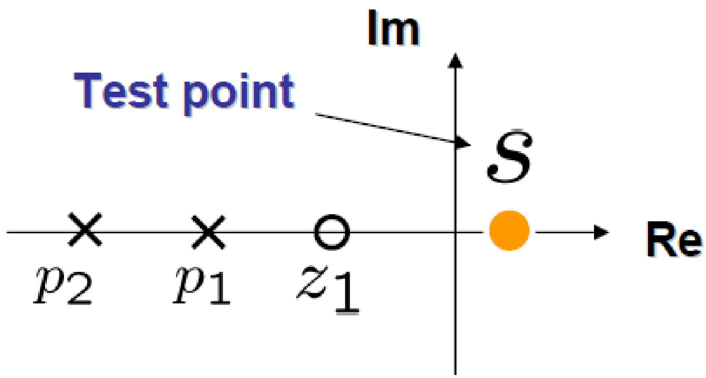
- Mark poles of L(s) with "x" and zeros of L with "o".

$$L(s) = \frac{s - z_1}{(s - p_1)(s - p_2)}$$



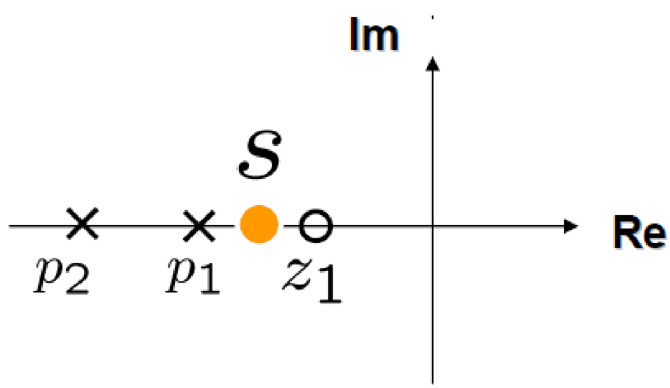
Root-Locus: Rule 1-1

- RL includes all points on real axis to the left of an odd number of real poles & zeros.



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{0} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

Not satisfy angle condition!

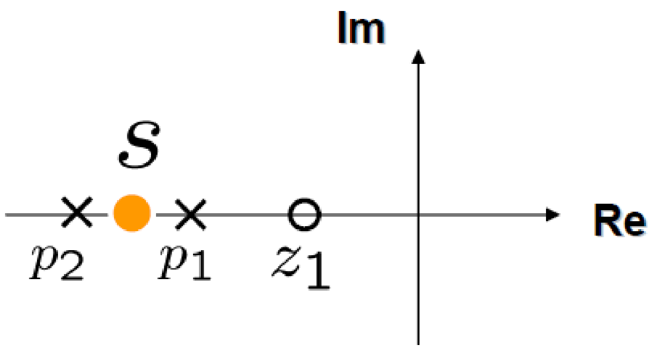


$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

Satisfy angle condition!

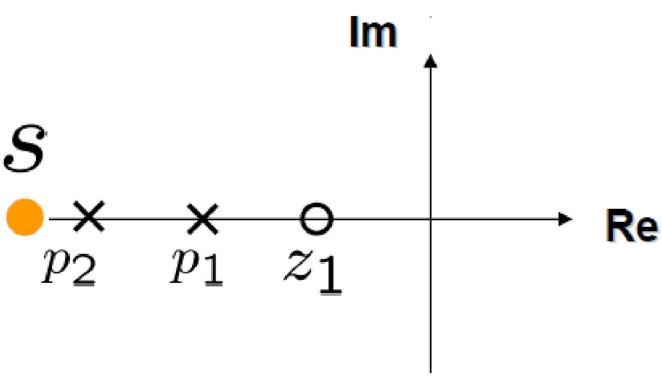
Root-Locus: Rule 1-1 (cont'd)

- RL includes all points on real axis to the left of an odd number of real poles & zeros.



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{\boxed{}} - \underbrace{\angle(s-p_1)}_{\boxed{}} - \underbrace{\angle(s-p_2)}_{\boxed{}}$$

Not satisfy angle condition!



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{180} - \underbrace{\angle(s-p_2)}_{180}$$

Satisfy angle condition!

Root-Locus: Rule 1-2

- RL originates from the poles of L(s) and terminates at the zeros of L(s), including infinity zeros.

$$1+K\underbrace{\frac{n(s)}{d(s)}}_{L(s)}=0 \Leftrightarrow d(s)+Kn(s)=0 \Leftrightarrow \frac{1}{K}+\frac{n(s)}{d(s)}=0$$

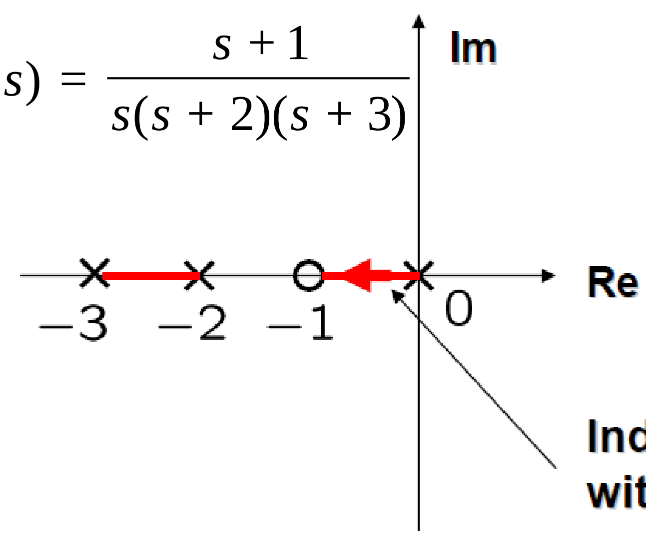
$K = 0$ $\qquad$ $K = \infty$

$$d(s)=0 \qquad \frac{n(s)}{d(s)}=0$$

s: Poles of L(s) (**Start** Point !!)

s: Zeros of L(s) (**End** Point !!)

$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$



Indicate the direction with an arrowhead.

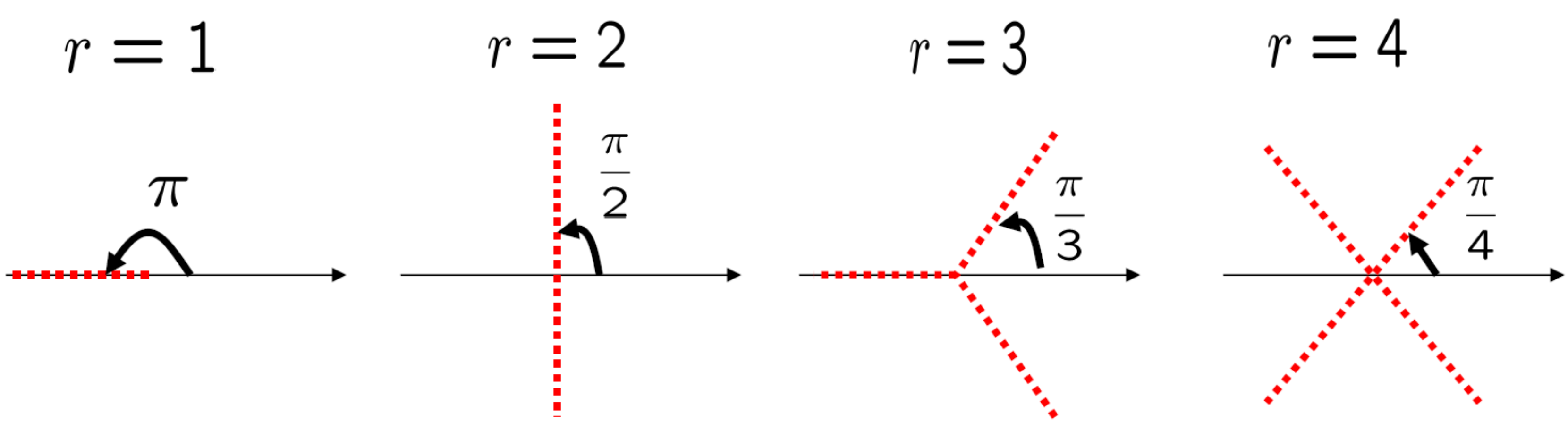
Root-Locus: Rule 2 (Asymptotes)

- Number of asymptotes = relative degree (r) of L:

$$L(s)=\frac{b(s)}{a(s)}=\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n} \qquad r:=\underbrace{n}_{\text{deg(den)}}-\underbrace{m}_{\text{deg(num)}}$$

- Angles of asymptotes are

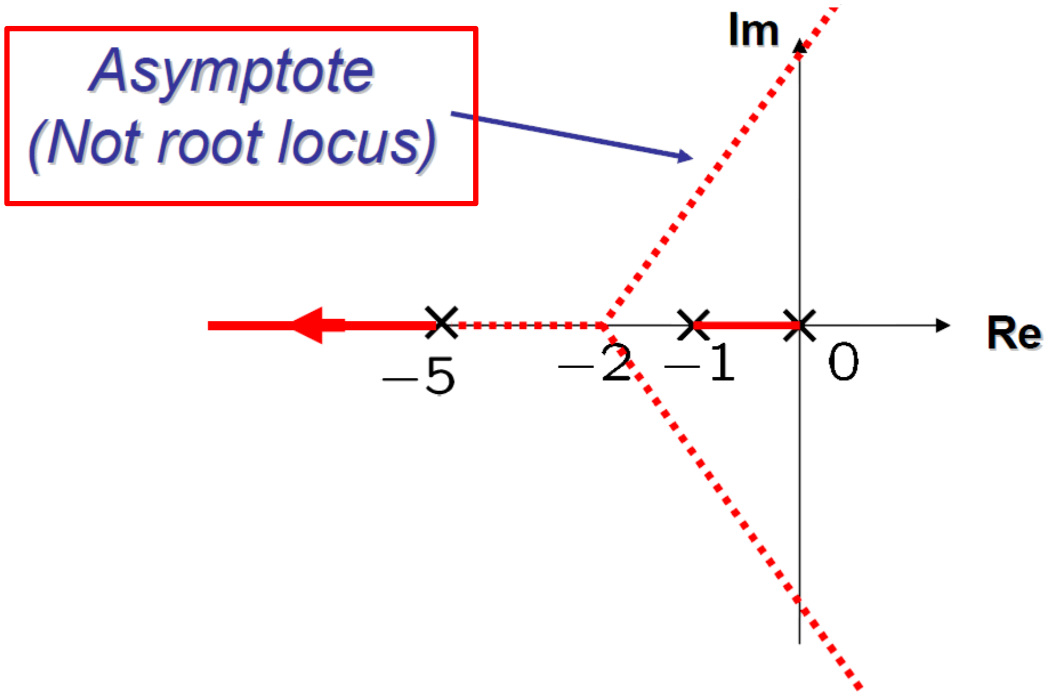
$$\frac{\pi}{r}\times(2k+1),\ \ k=0,1,\ldots$$

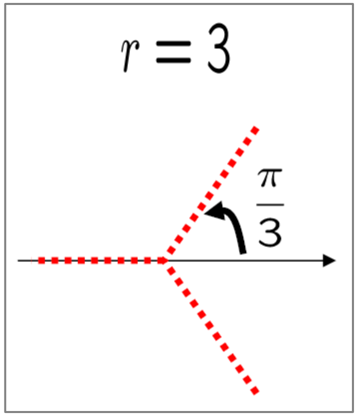


Root-Locus: Rule 2 (Asymptotes)

- Intersections of asymptotes: $\dfrac{\sum Poles - \sum Zeros}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{\sum Poles - \sum Zeros}{r} \Rightarrow \frac{0+(-1)+(-5)}{3} = -2$$





Root-Locus: Rule 3 (Breakaway Points)

- Breakaway points are among roots of $\dfrac{dL(s)}{ds} = 0$

$$L(s) = \frac{1}{s(s+1)(s+5)}$$

$$\Rightarrow \frac{dL(s)}{ds} = -\frac{3s^2+12s+5}{(*)} = 0$$

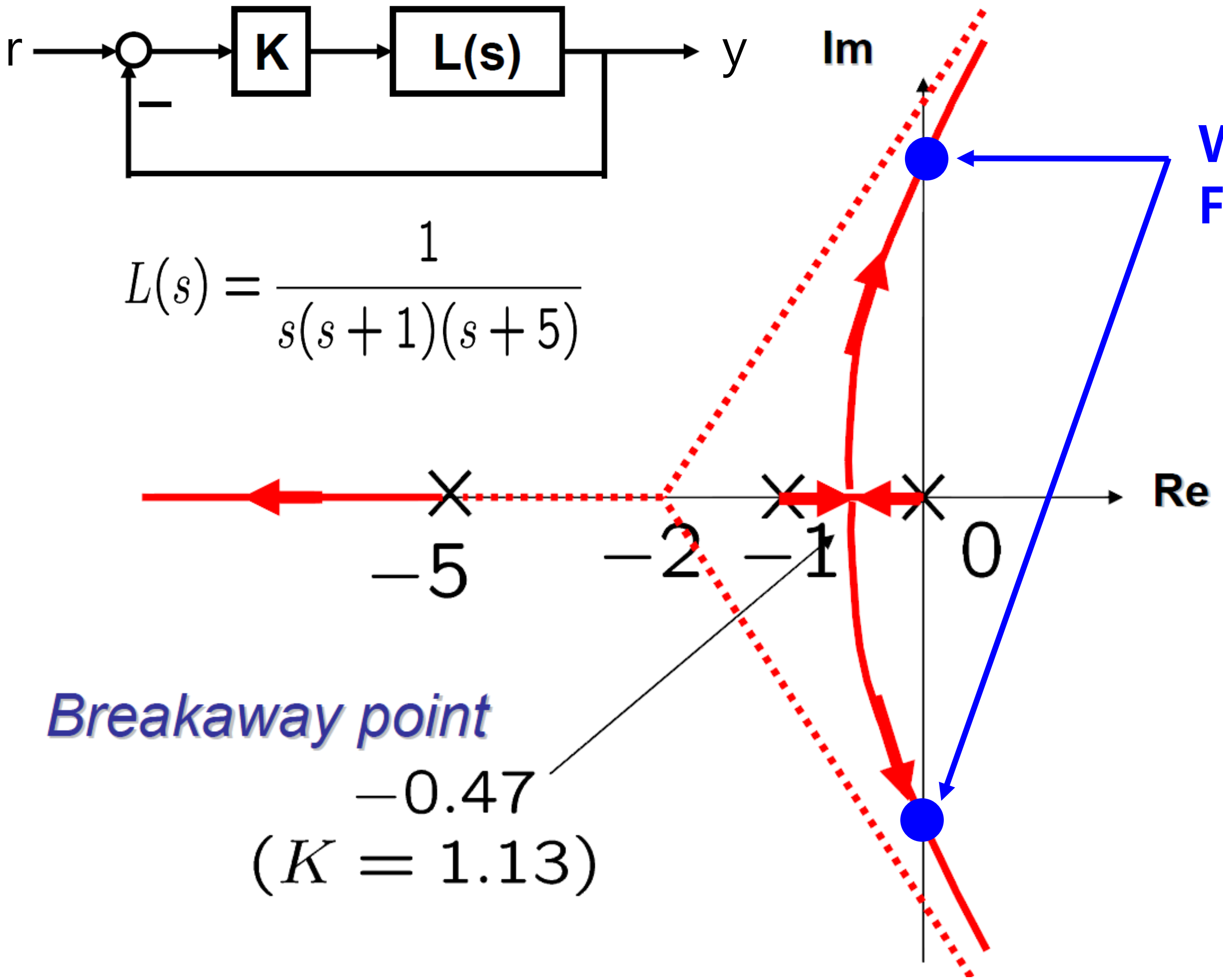
$$\Rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, check the positivity of

$$1 + KL(s) = 0 \Rightarrow K = -\frac{1}{L(s)}$$

$$\Rightarrow \begin{cases} s = -2 + \dfrac{\sqrt{21}}{3} \approx -0.47 & \longrightarrow K \approx 1.13 \\ s = -2 - \dfrac{\sqrt{21}}{3} \approx -3.52 & \longrightarrow \cancel{K \approx \text{-}13.1} \end{cases}$$

Root-Locus: Rule 3 (Breakaway Points)



$$L(s) = \frac{1}{s(s+1)(s+5)}$$

What is the value ??
For what K ??

Routh-Hurwitz !!

Finding K for Critical Stability

- **Characteristic equation**

$$1+\frac{K}{s(s+1)(s+5)}=0 \Leftrightarrow s^3+6s^2+5s+K=0$$

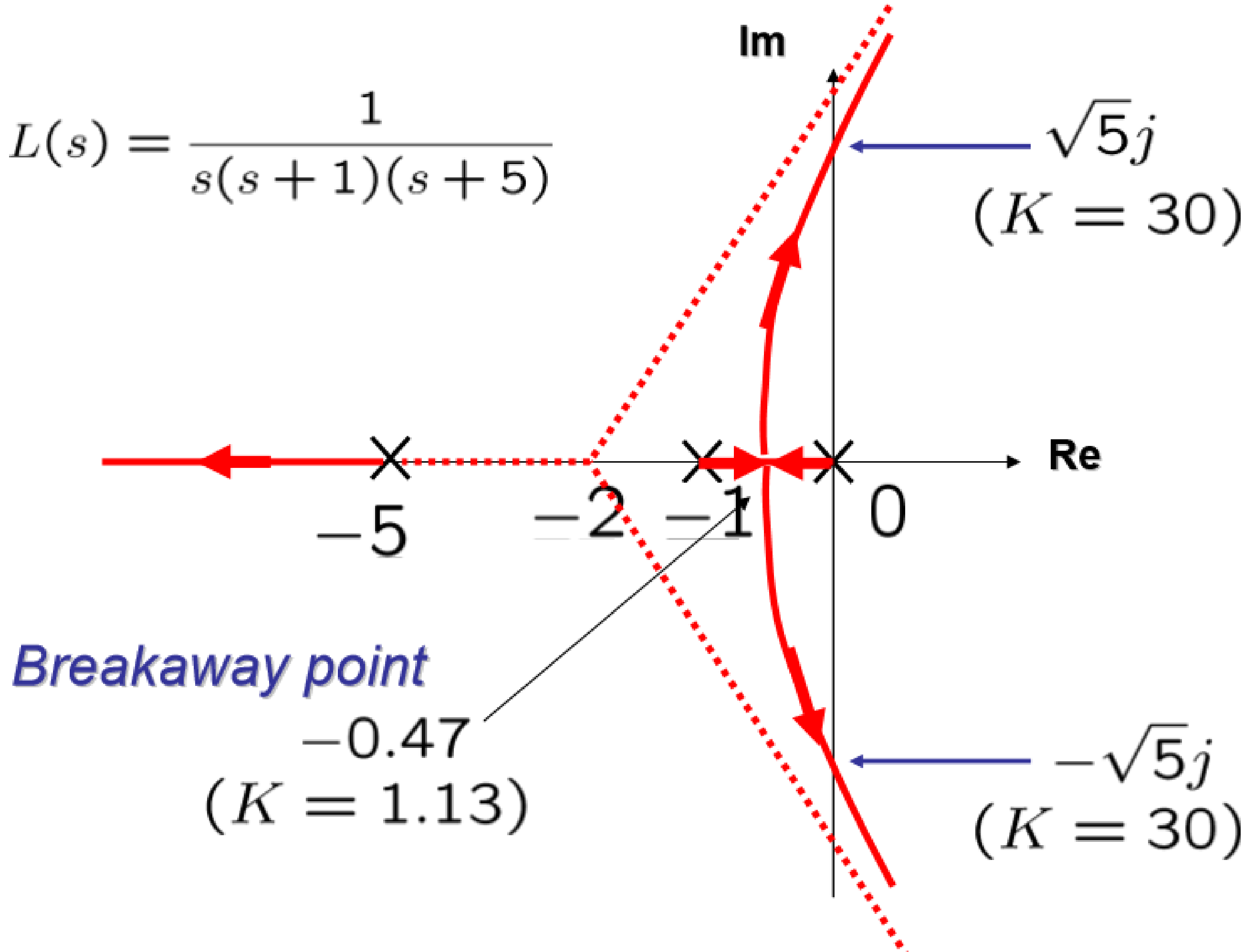
- **Routh Array**

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 6 & K \\ s^1 & \boxed{} & \\ s^0 & K & \end{array} \Bigg\} \Rightarrow \boxed{\begin{array}{c}\text{Stability condition} \\ 0 < K < 30\end{array}}$$

- **When K = 30,**

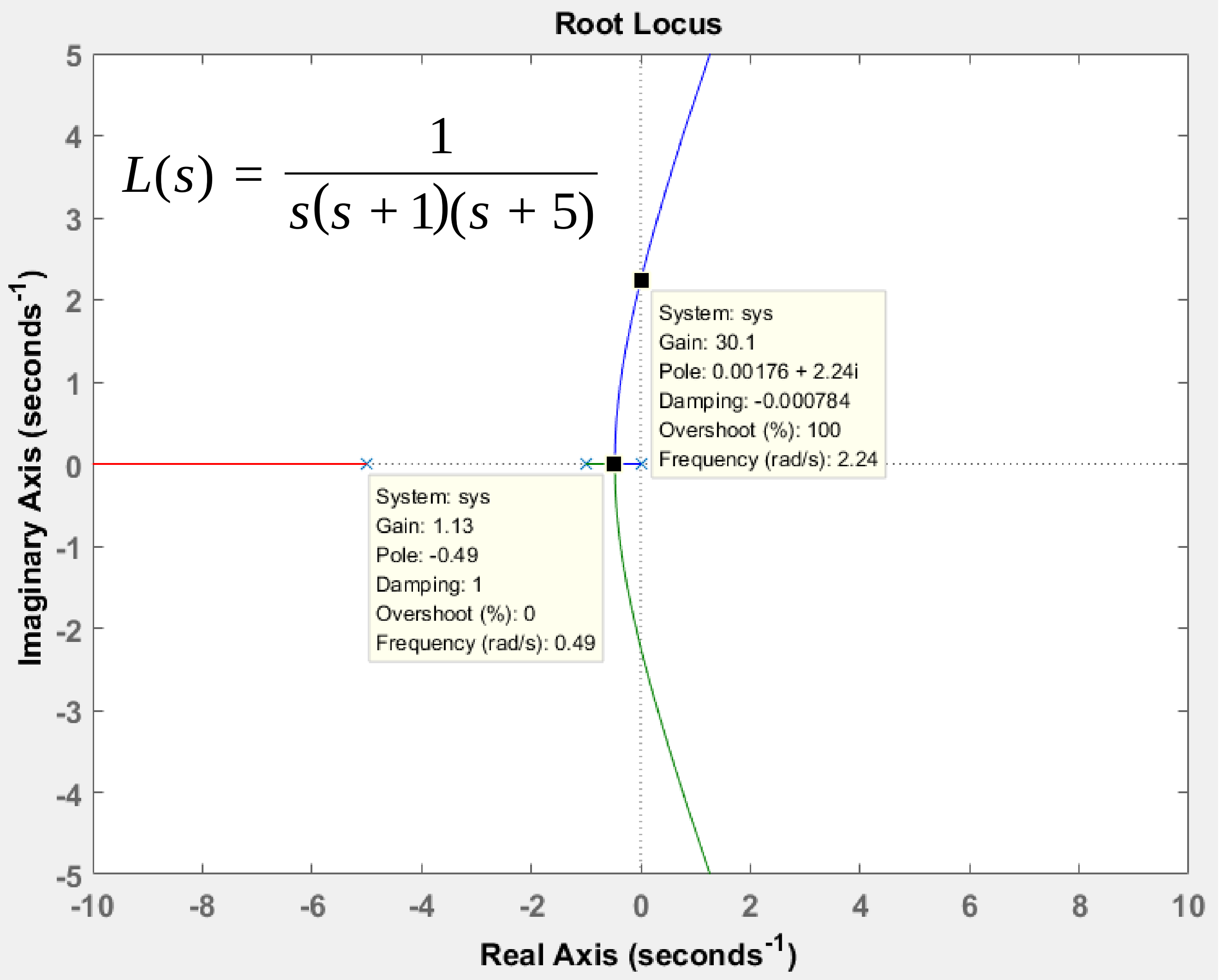
$$s^3+6s^2+5s+30=0 \Rightarrow (s+6)(s^2+5)=0 \Rightarrow s=\pm\sqrt{5}j$$

Root-Locus: Rule 3 (Breakaway Points)



Use of Matlab Command "rlocus"

```
num = [1];
den = [1 6 5 0];
sys = tf(num,den);
rlocus(sys)
```



Example with Complex Poles

$$L(s) = \frac{s}{s^2 + s + 1}$$

- **Rules 0 & 1 (Poles & Zeros)**

zero $\quad 0$

pole $\quad -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

- **Rules 2 (Asymptotes)**
  → r = 1 → π

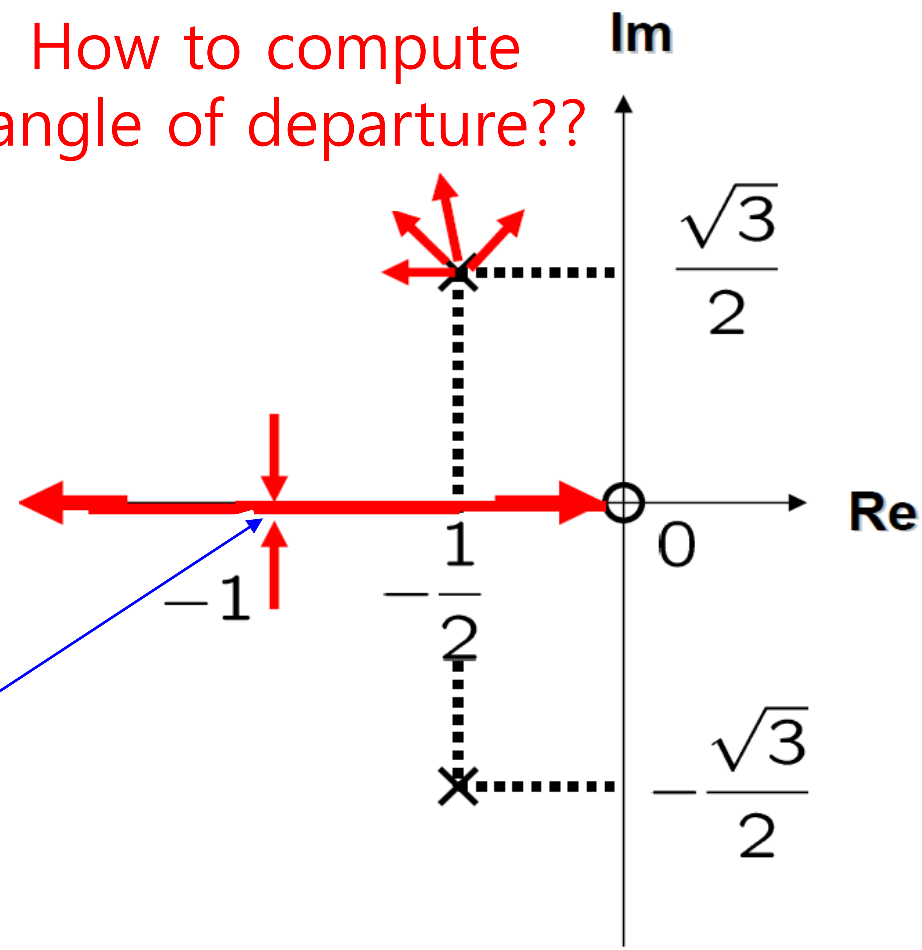
- **Rules 3 (Breakaway point)**

$$\frac{dL(s)}{ds} = 0$$

$$s^2 + s + 1 - s(2s+1) = 0$$

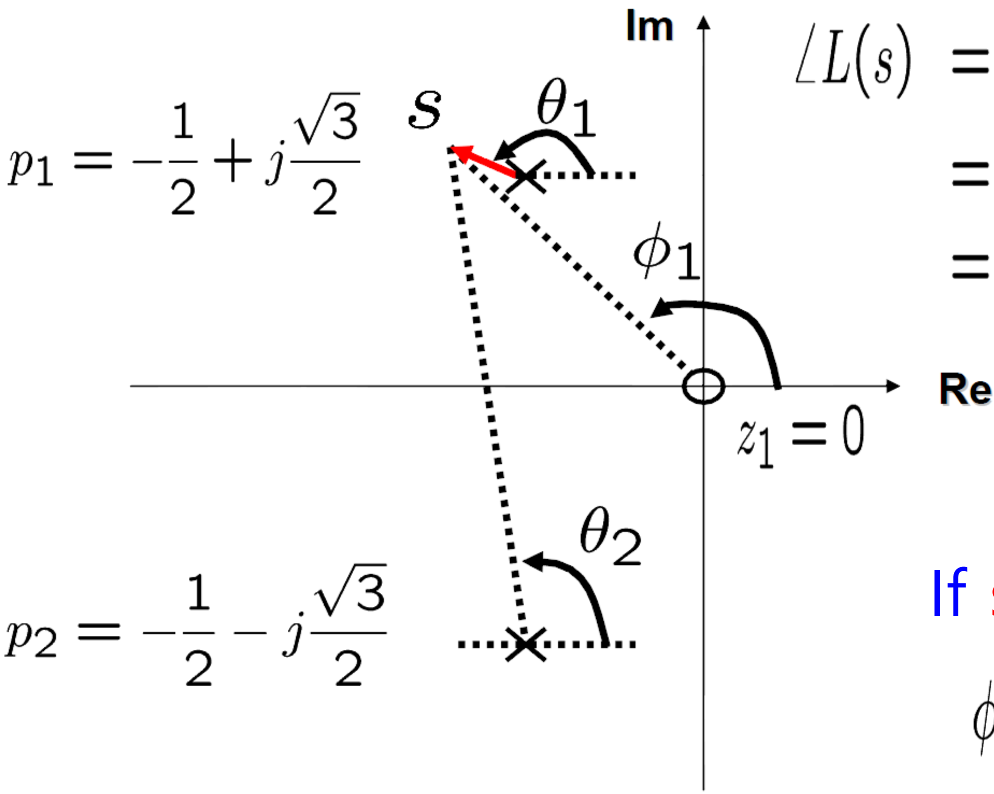
$$\Rightarrow s = \pm 1$$

How to compute angle of departure??



Root-Locus: Rule 4 (Angle of Departure)

- **Angle condition: For s to be on Root-Locus,**



$$\begin{aligned} \angle L(s) &= \angle \frac{s-z_1}{(s-p_1)(s-p_2)} \\ &= \angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2) \\ &= \phi_1 - \theta_1 - \theta_2 = \underline{180} \end{aligned}$$

Pole angle

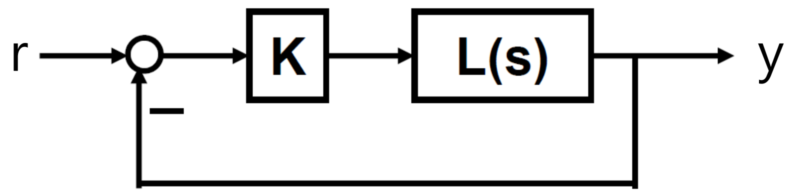
Zero angle

If s is close to p_1

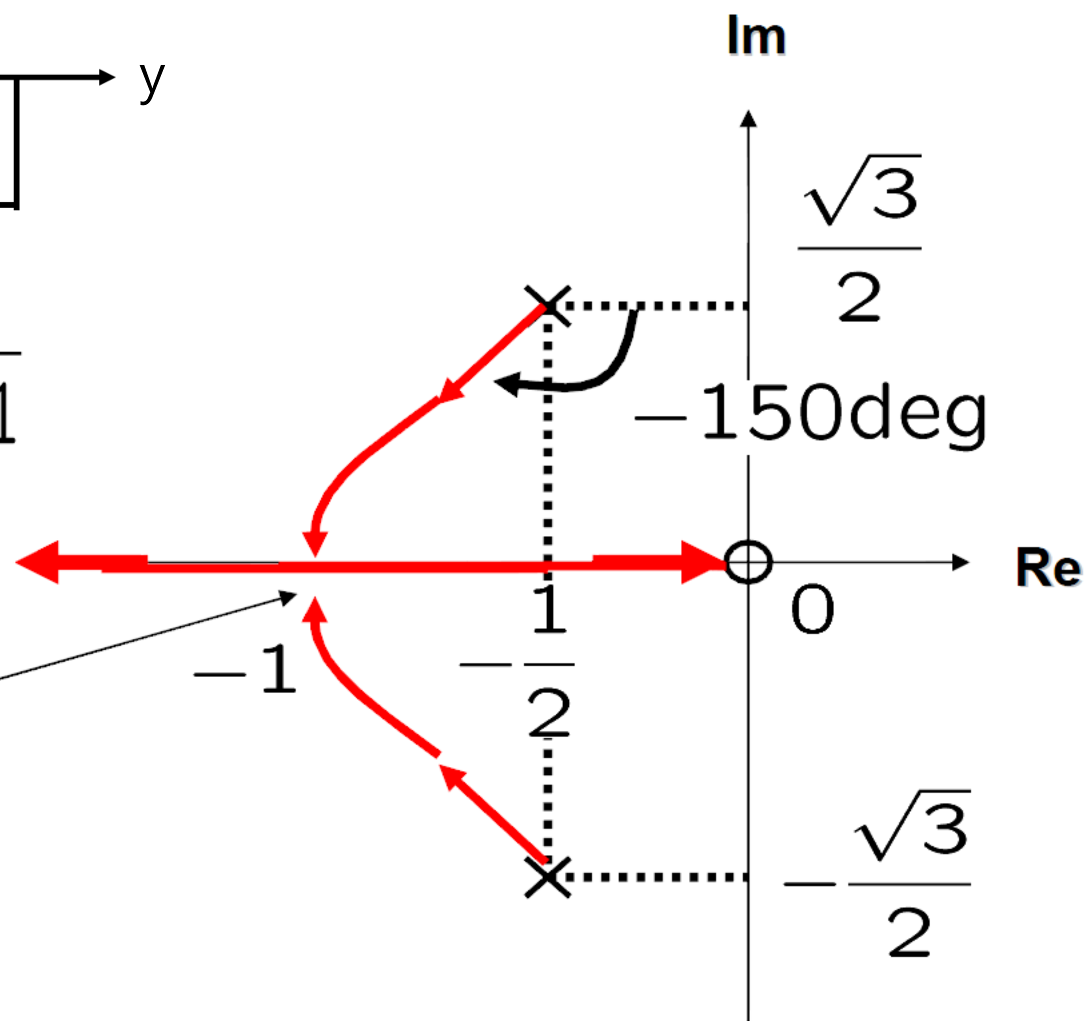
$$\phi_1 \approx 120, \ \theta_2 \approx 90$$

$$\Rightarrow \theta_1 = -150$$

Root-Locus: Rule 4 (Angle of Departure)



$$L(s) = \frac{s}{s^2 + s + 1}$$

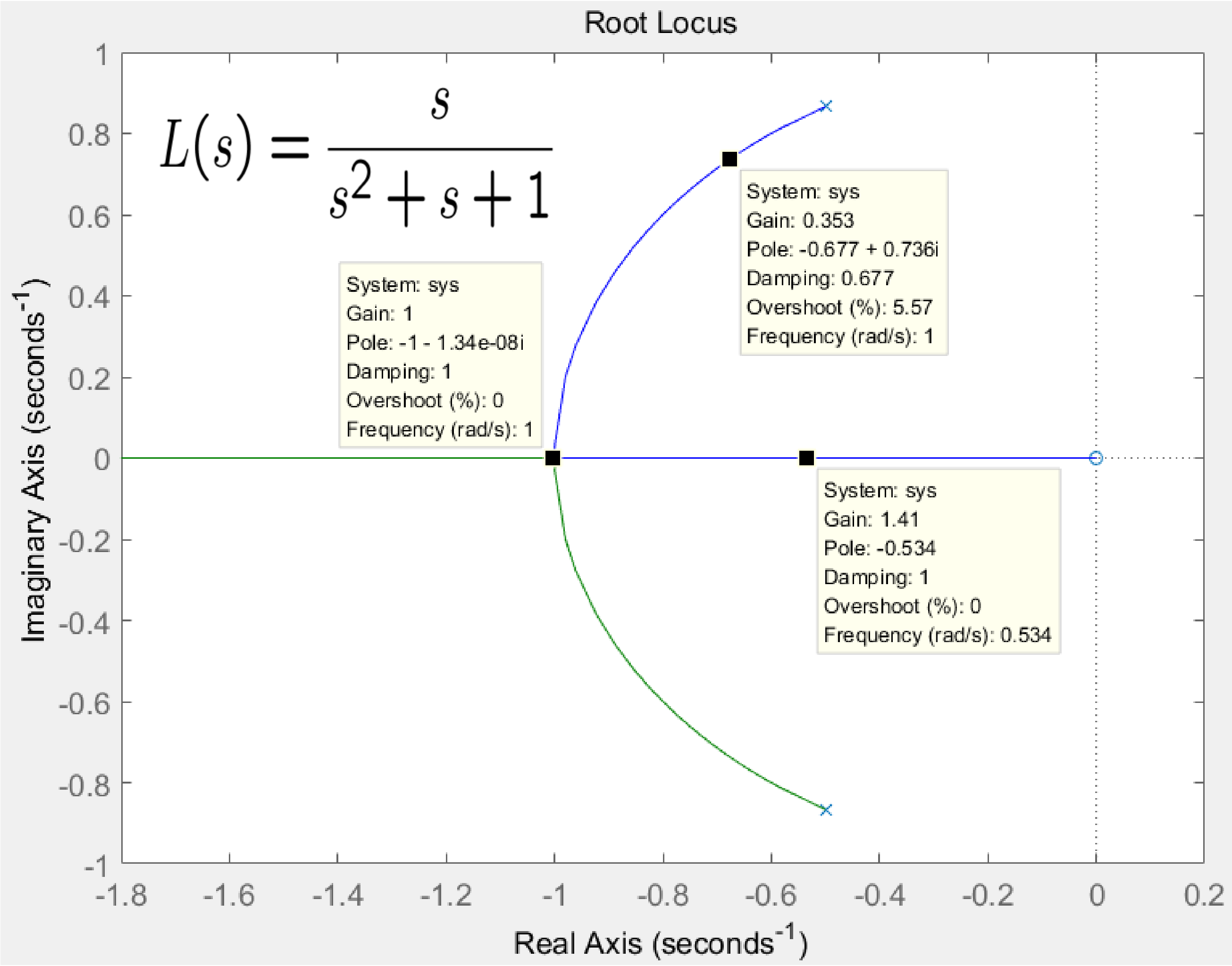


Breakaway point

$$s = -1 \ (K = 1)$$

Use of Matlab Command "rlocus"

```
num = [1 0];
den = [1 1 1];
sys = tf(num,den);
rlocus(sys)
```



The Root-Locus Design Method 1

Lecture 8:

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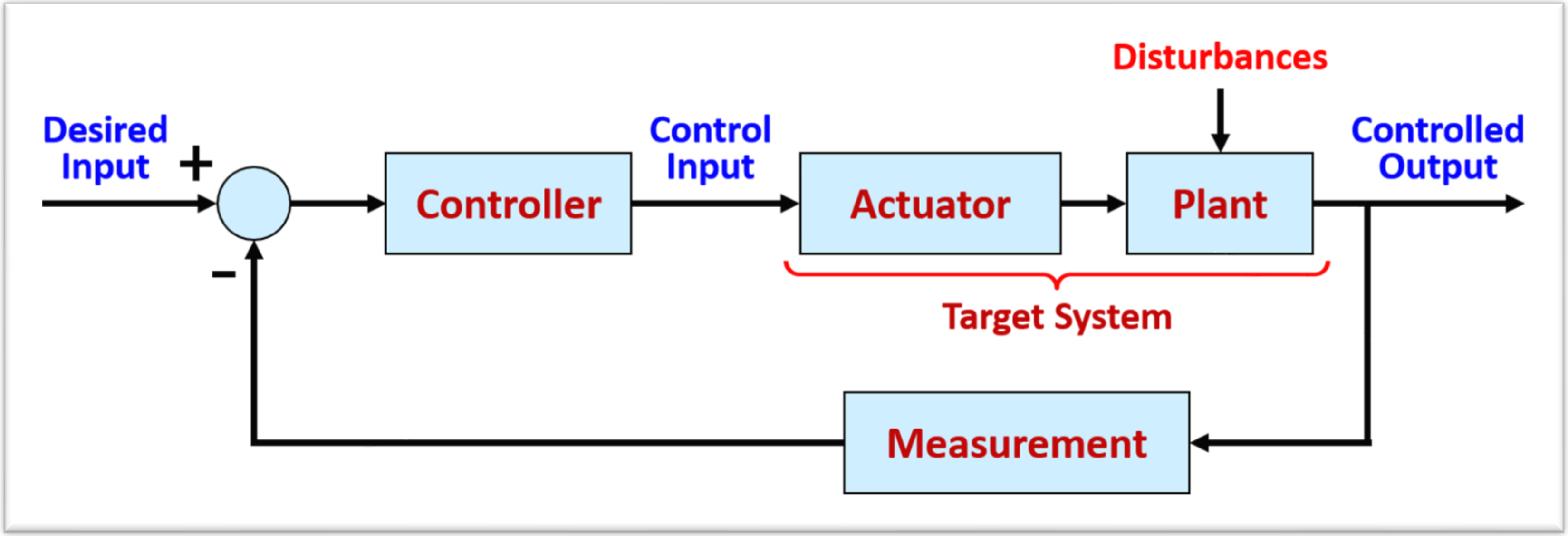

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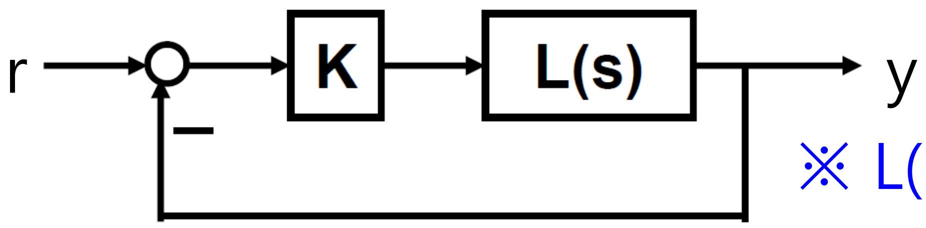
What is Root-Locus ?? (Review)

- Root-locus graphically shows poles locations of closed-loop system as K varies from 0 to infinity.

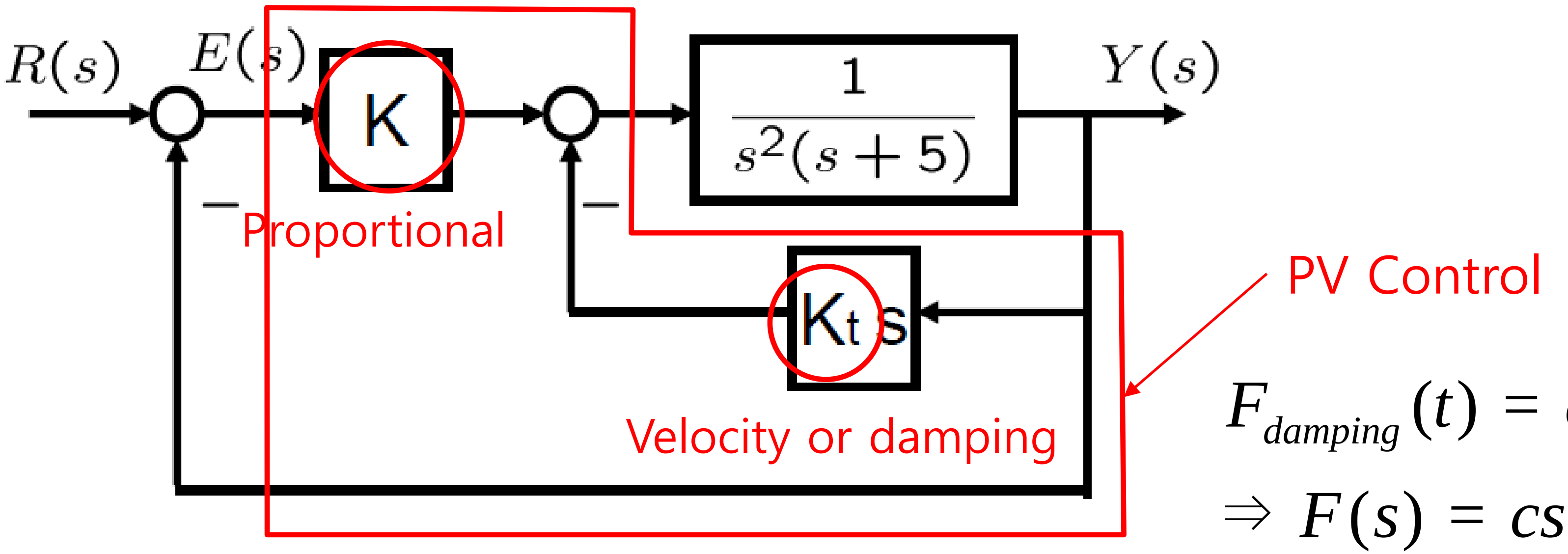


※ L(s): Open-Loop TF

- Consider a feedback system that has <u>one parameter (gain)</u> $K>0$ to be designed

- <u>Now, multiple</u> design parameters !!

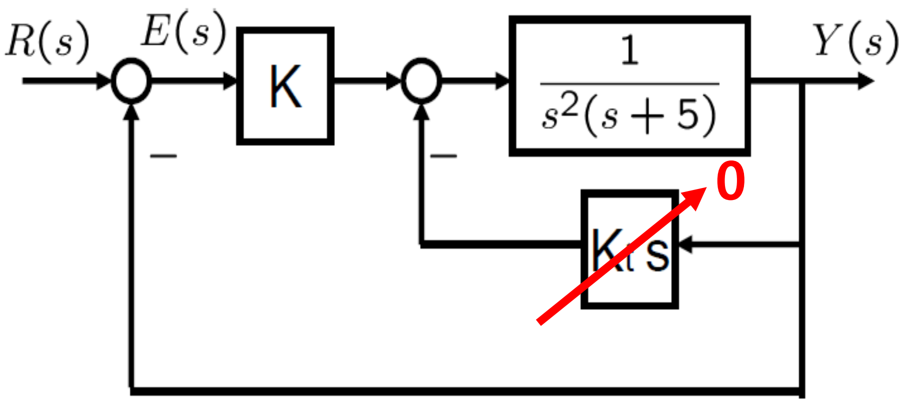
Example 1: Multi Parameters

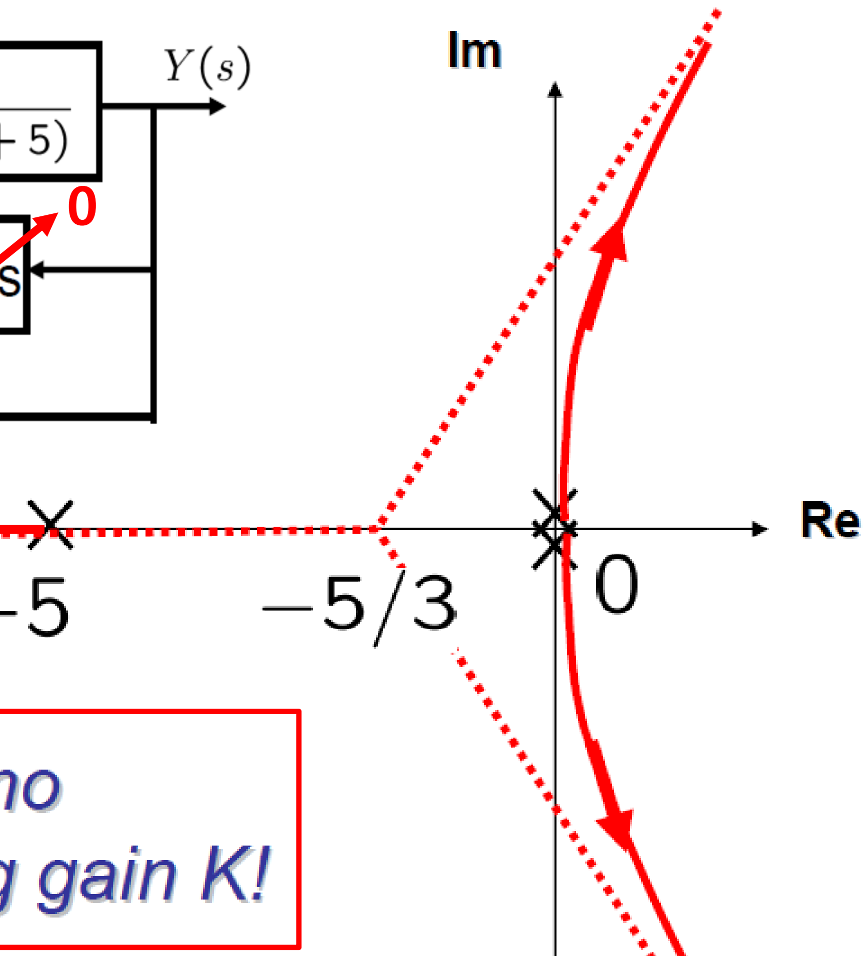


$$F_{damping}(t) = c\dot{y}(t)$$
$$\Rightarrow F(s) = csY(s)$$

- **Case 1: Set K_t = 0, draw root-locus for K > 0.**
- **Case 2: Set K = 10, draw root-locus for K_t > 0.**
- **Case 3: Set K = 5, draw root-locus for K_t > 0.**

Example 1) Case 1: $K_t = 0$

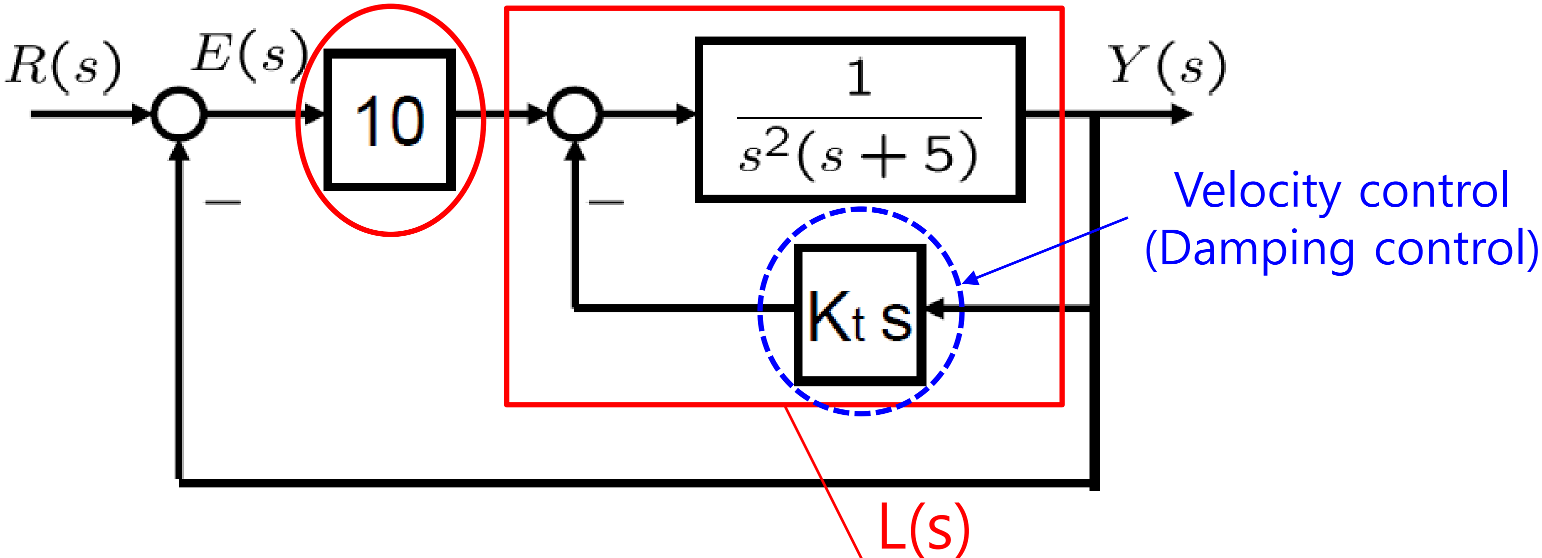




There is no stabilizing gain K!

❖ **That is why the damping factor is needed for many systems !!**

Example 1) Case 2: K = 10



- **Characteristic Eq.**

$$1+10\boxed{\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right)}=0$$

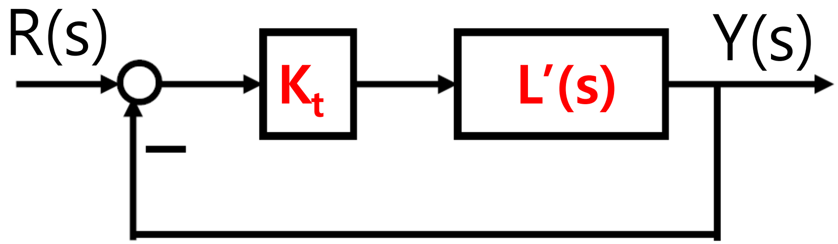
$\Leftrightarrow$ $s^2(s+5)+K_t s+10=0$ $\Leftrightarrow$ $1+K_t\boxed{\frac{s}{s^3+5s^2+10}}=0$

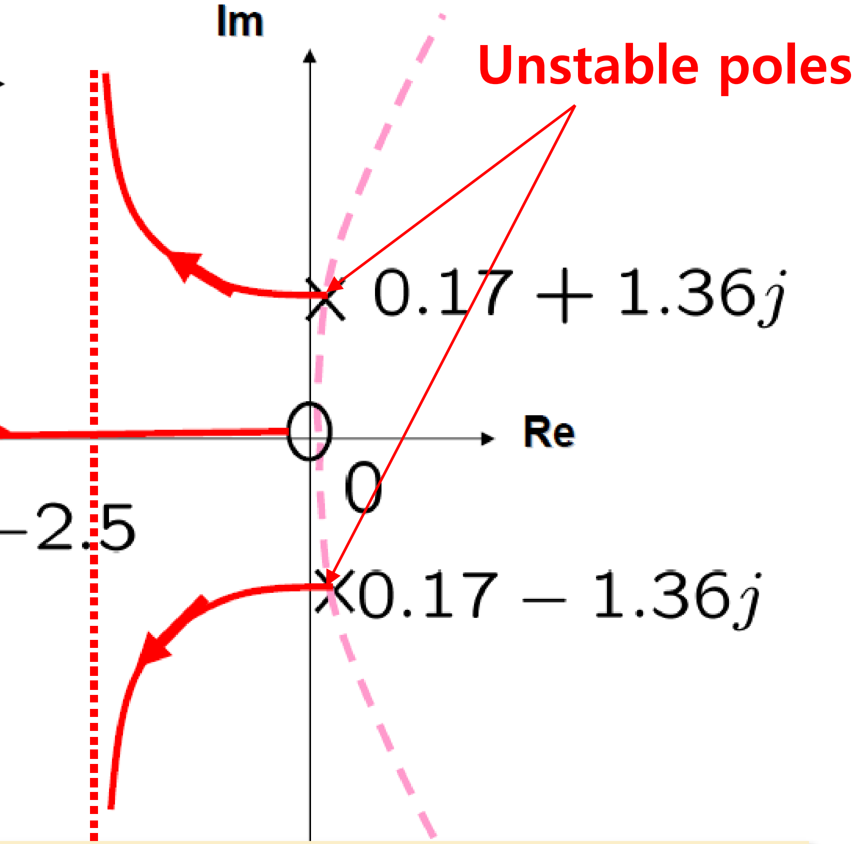
L′(s)

Example 1) Case 2: K = 10 (cont'd)

$$1 + K_t \boxed{\underbrace{\frac{s}{s^3 + 5s^2 + 10}}_{L(s)}} = 0$$

L′(s)





By increasing Kt, we can stabilize the CL system.

- **The damping factor is working for system stability !!**

Finding K_t for Marginal Stability

- **Characteristic Eq.**

$$1+\frac{K_t s}{s^3+5s^2+10}=0 \Leftrightarrow s^3+5s^2+K_t s+10=0$$

- **Routh Array**

$$\begin{array}{c|cc} s^3 & 1 & K_t \\ s^2 & 5 & 10 \\ s^1 & \boxed{} & \\ s^0 & 10 & \end{array} \rightarrow \boxed{\begin{array}{c}\text{Stability condition} \\ K_t > 2\end{array}}$$

- **When $K_t = 2$,**

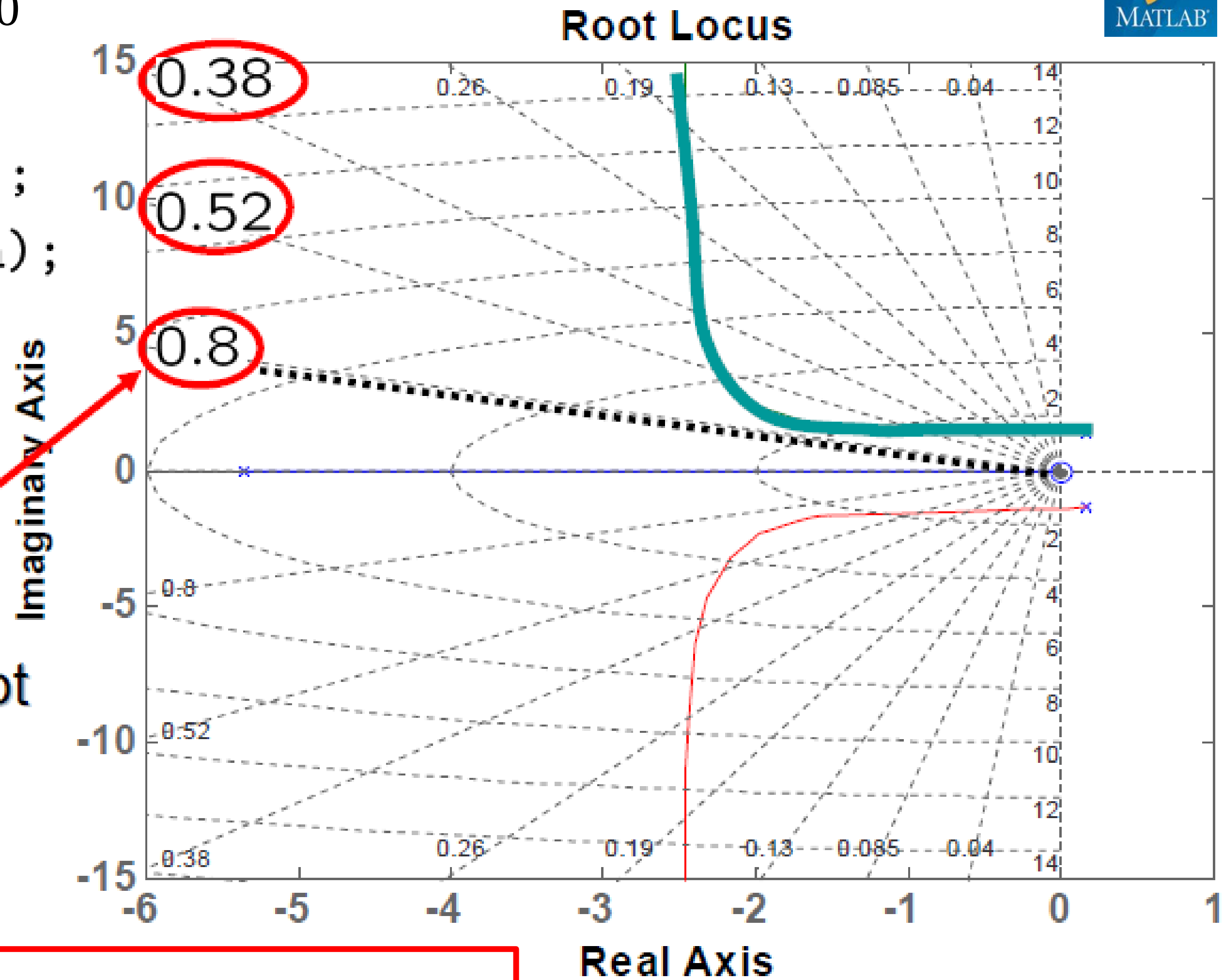
$$s^3+5s^2+2s+10=(s+5)(s^2+2)=0$$

$$\Rightarrow 5s^2+10=0 \Rightarrow s=\pm\sqrt{2}j$$

Use of Matlab Command "rlocus"

$$L(s) = \frac{s}{s^3 + 5s^2 + 10}$$

```
num=[1 0];
den=[1 5 0 10];
sys=tf(num,den);
rlocus(sys)
grid on
```



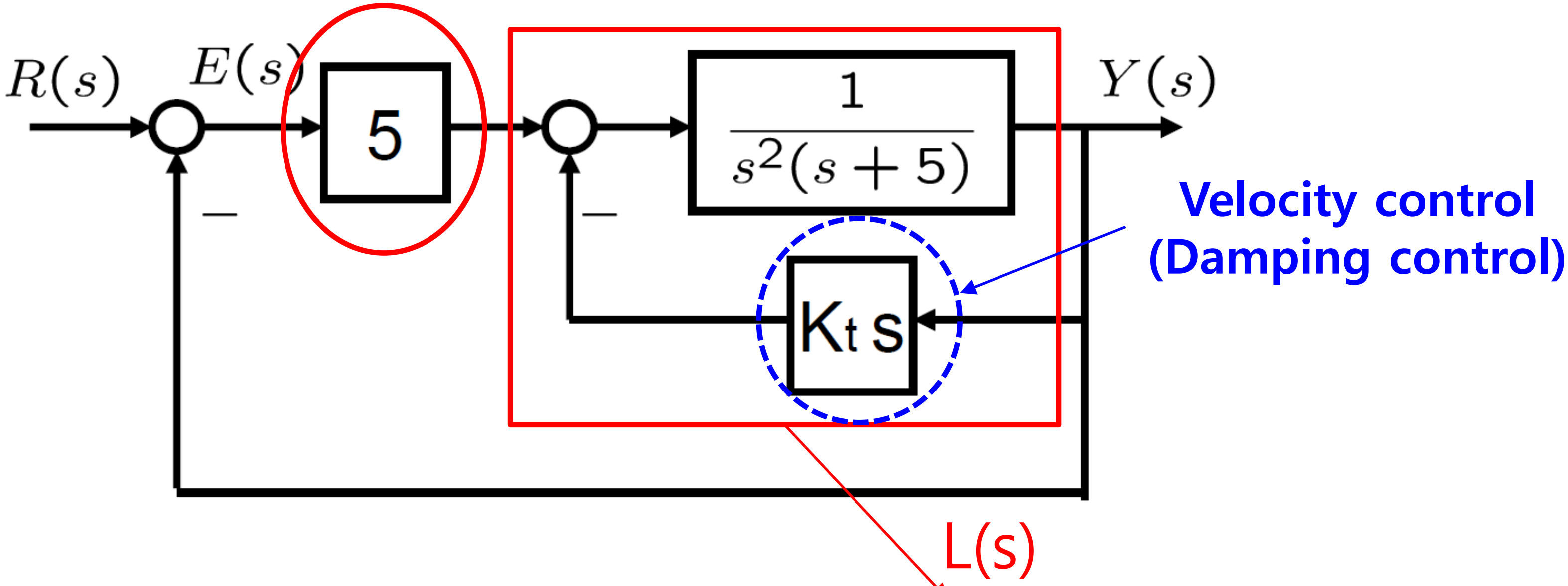
Damping ratio

If K=10, we cannot achieve $\zeta = 0.8$

for any Kt>0.

$\boxed{0<\zeta<0.8 \text{ for any } K_t > 0 \text{ !!}}$

Example 1) Case 3: K = 5



- **Characteristic Eq.**

$$1+5\boxed{\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right)}=0$$

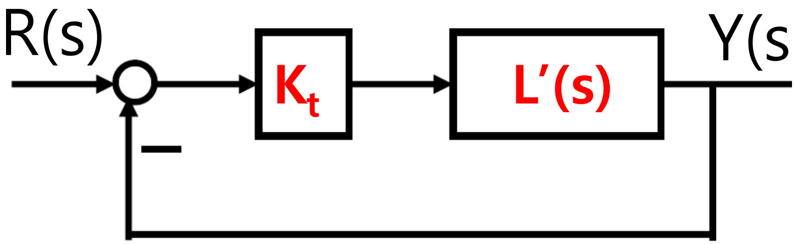
$$\Leftrightarrow \quad s^2(s+5)+K_t s+5=0 \quad \Leftrightarrow \quad 1+K_t\boxed{\frac{s}{s^3+5s^2+5}}=0$$

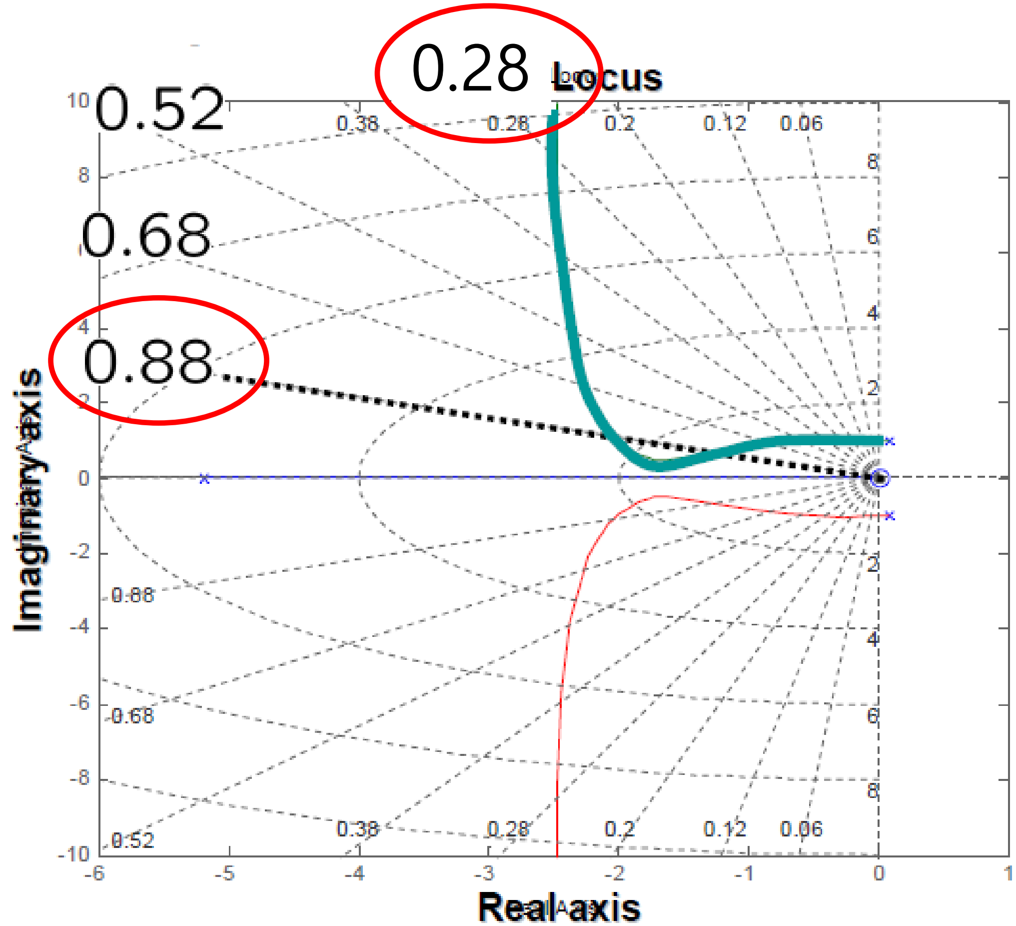
L′(s)

Example 1) Case 3: K = 5 (cont'd)

$$1 + K_t \boxed{\underbrace{\frac{s}{s^3+5s^2+5}}_{L(s)}} = 0$$

L′(s)





Now, we CAN achieve

$0<\zeta<1.0$ for any $K_t > 0$!!

Summary

- **Summary**:
  - What is the Root-Locus?
  - How to roughly **sketch root-locus**.
  - Multiple-parameter control design examples.