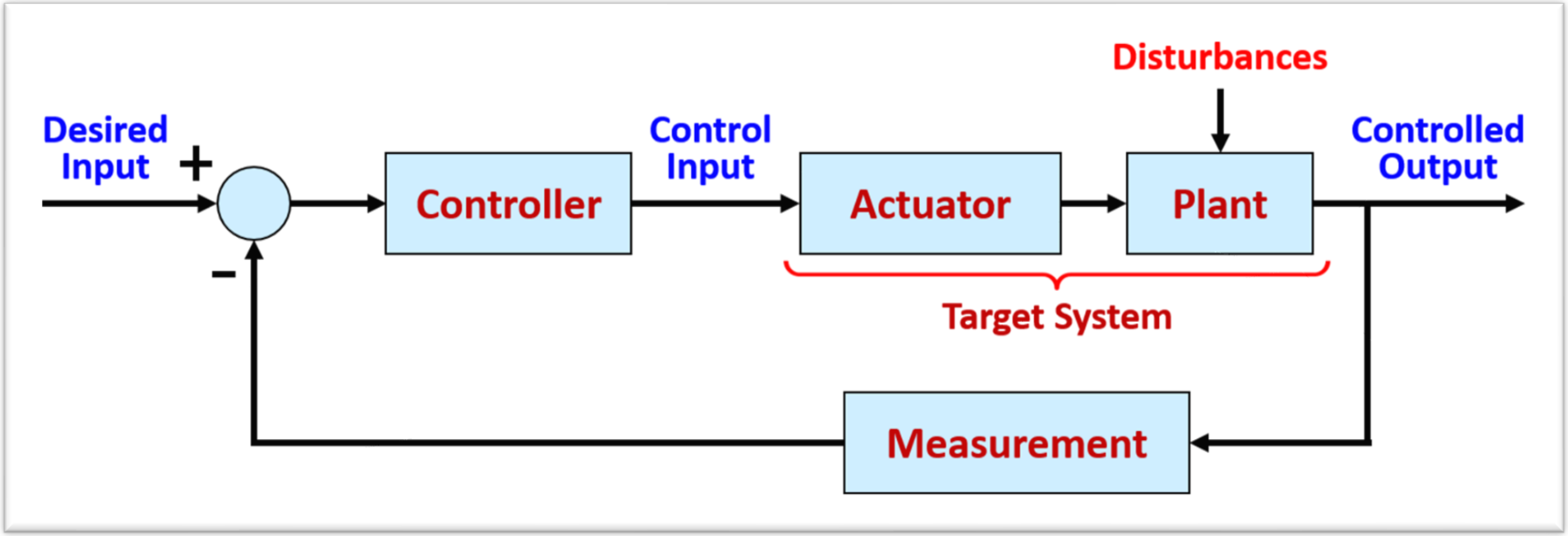
Dynamic Models

Lecture 3:

- **Modeling of Mechanical & Electrical Systems**
- Linearization & Electro-Mechanical Systems



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Important Remarks on Models (Revisited)

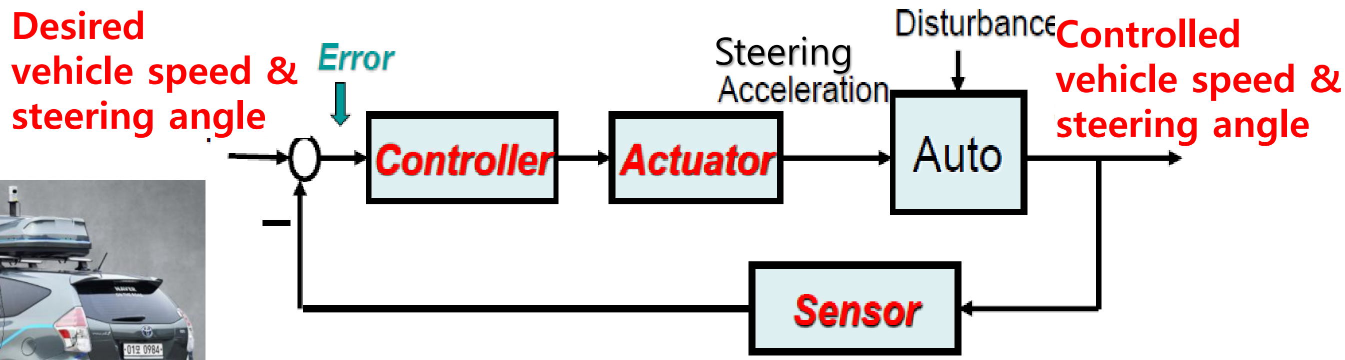
- Modeling is the one of the most important and difficult tasks in control system design..
- No mathematical model exactly represents a physical system.

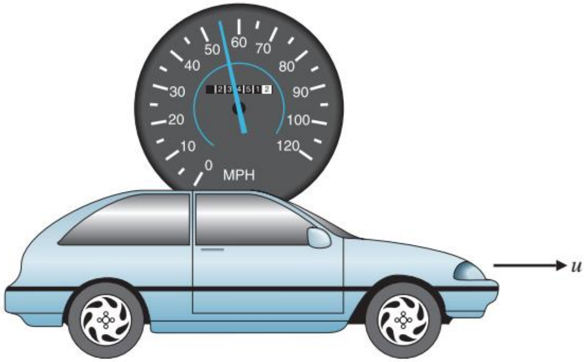
$$\text{Math model} \neq \text{Physical system}$$

$$\text{Math model} \approx \text{Physical system}$$

- Do not confuse models with physical systems !!
- In this course, we may use the term “system” to mean a mathematical model.

Important Remarks on Models (Revisited)





Modeling

(Vehicle Velocity)

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}.$$

(Force by Engine)

Physical System

Mathematical Model

Steps of Math Model Development

❖ **Typical Processes of Dynamic (Math) Model Development:**

- **Step 1:** Identify the system of interest by defining its purpose and the system boundary.
- **Step 2:** Identify the variables - inputs (forcing functions or excitations) vs. outputs (response).
- **Step 3:** Approximate various segments (components or processes or phenomena) in the system by ideal elements that are suitably interconnected.
- **Step 4:** Draw a free-body diagram for the system where the individual elements are isolated or separated, as appropriate.
- **Step 5:** Write physical laws for the elements in Time Domain.
- **Step 6:** Take Laplace Transform for analysis on Frequency Domain.

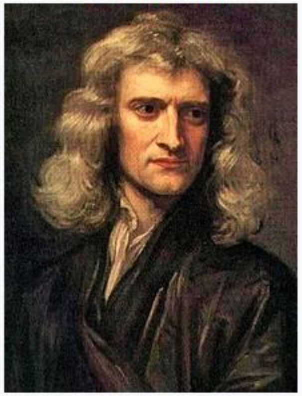
Newton's & Euler's Laws of Motion

- 1^{st} law:
  - A particle remains at rest or continues to move in a straight line with a constant velocity if there is no unbalancing force acting on it.
- 2^{nd} law:
  - $\sum F_i(t) = m\frac{d^2x}{dt^2}$: translational
  - $\sum \tau_i(t) = I\frac{d^2\theta}{dt^2}$: rotational

Euler's 2^{nd} Laws

- 3^{rd} law:
  - For every action has an equal and opposite reaction

Sir Isaac Newton



Portrait of Newton at 46 in 1689 by Godfrey Kneller

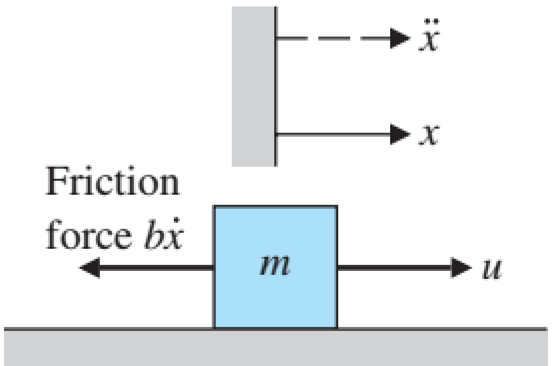
Leonhard Euler

Portrait by Jakob Emanuel Handmann (1753)

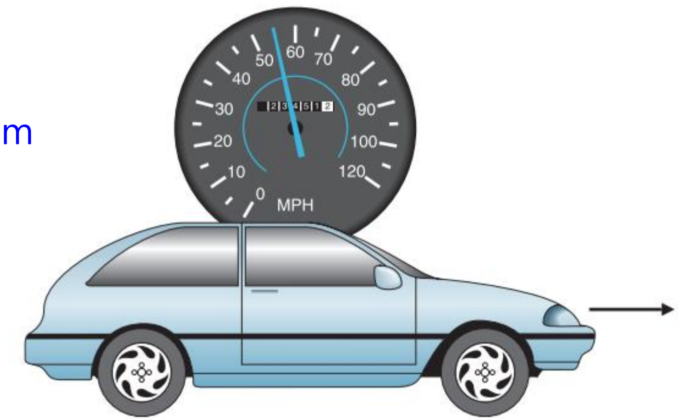
Born	15 April 1707 Basel, Switzerland
Died	18 September 1783 (aged 76)

Modeling of Mechanical Systems

(1) 1-D Translational Motion: Cruise Control Model



Free-Body-Diagram (FBD)



$$u - b\dot{x} = m\ddot{x},$$

$$\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}.$$

$\mathcal{L}$

$$\left(s + \frac{b}{m}\right) V_o e^{st} = \frac{1}{m} U_o e^{st}.$$

where,

$$\dot{x} = v = V_0 e^{st}$$

$$u = U_0 e^{st}$$

$$\boxed{\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}.}$$

Source: Franklin, Gene F.; Powell, J; Emami-Naeini, Abbas. *Feedback Control of Dynamic Systems*

Modeling of Mechanical Systems

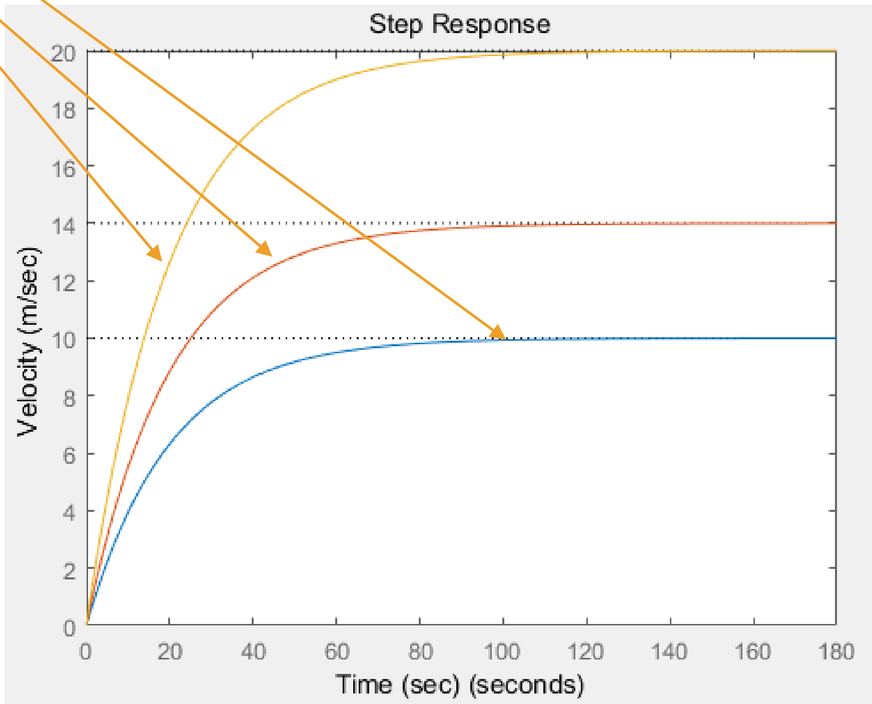
(1) 1-D Translational Motion: Cruise Control Model

```
s=tf('s');
sys = (1/1000)/(s + 50/1000);
figure(1)
hold on
grid on
step(500*sys);  % plots the step response for u = 500.
step(700*sys);  % plots the step response for u = 700.
step(1000*sys); % plots the step response for u = 1000.
ylabel('Velocity (m/sec)');
xlabel('Time (sec)');
```

m = 1,000 [kg]
b = 50 [N·sec/m]

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}.$$

MATLAB
The Language of Technical Computing

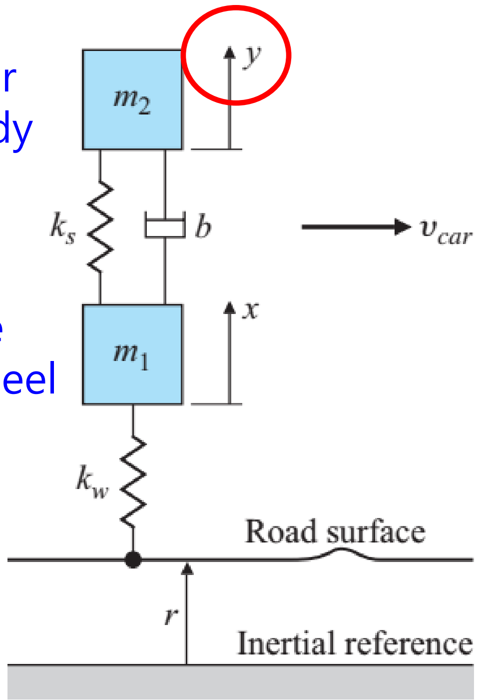


Modeling of Mechanical Systems

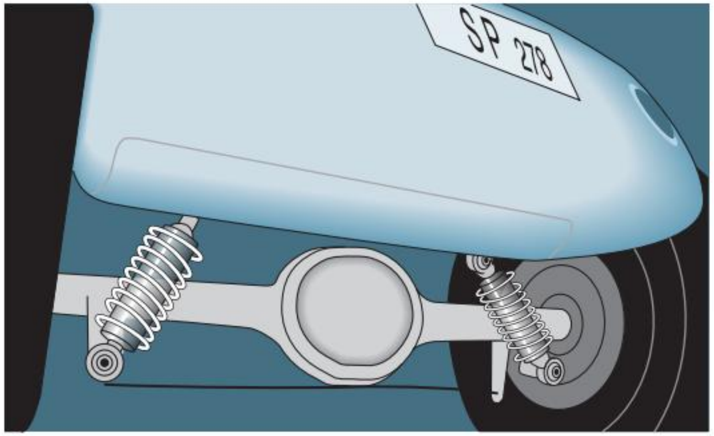
(2) Two-Mass Translational Motion: Suspension Model

Car body

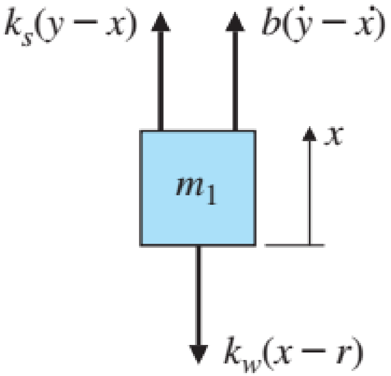
Tire & wheel

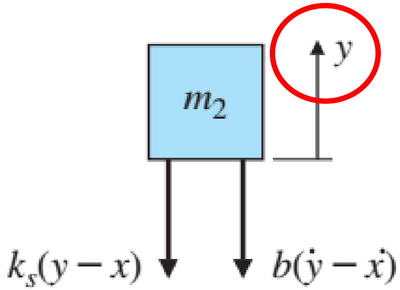


Quarter-car Model (1/4)



FBD

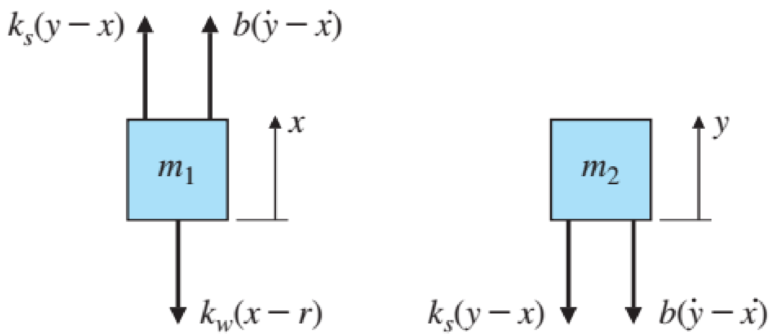




Modeling of Mechanical Systems

(2) Two-Mass Translational Motion : Suspension Model (cont'd)

$$b(\dot{y}-\dot{x})+k_s(y-x)-k_w(x-r)=m_1\ddot{x},$$
$$-k_s(y-x)-b(\dot{y}-\dot{x})=m_2\ddot{y}.$$



rearranging

$$\ddot{x}+\frac{b}{m_1}(\dot{x}-\dot{y})+\frac{k_s}{m_1}(x-y)+\frac{k_w}{m_1}x=\frac{k_w}{m_1}r,$$
$$\ddot{y}+\frac{b}{m_2}(\dot{y}-\dot{x})+\frac{k_s}{m_2}(y-x)=0.$$

$\mathcal{L}$

$$s^2X(s)+s\frac{b}{m_1}(X(s)-Y(s))+\frac{k_s}{m_1}(X(s)-Y(s))+\frac{k_w}{m_1}X(s)=\frac{k_w}{m_1}R(s),$$
$$s^2Y(s)+s\frac{b}{m_2}(Y(s)-X(s))+\frac{k_s}{m_2}(Y(s)-X(s))=0,$$

Eliminating X(s)

Modeling of Mechanical Systems

(2) Two-Mass Translational Motion : Suspension Model (cont'd)

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

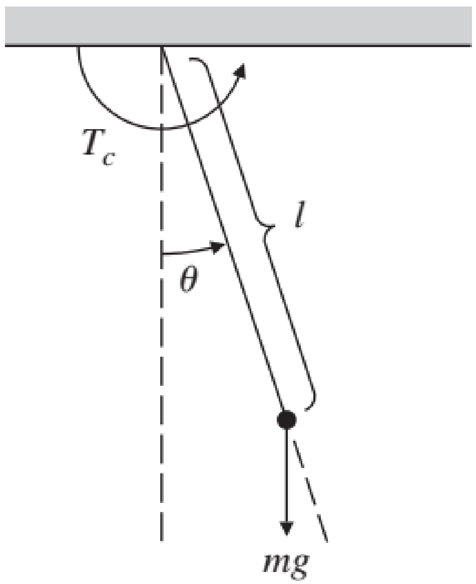
- To determine numerical values:
  - Total vehicle mass = 1580kg → (x 1/4) → m2 = 375kg
  - Each wheel mass (m1) = 20kg
  - k_s = 130,000 N/m, k_w = 1,000,000 N/m, b = 9,800 N·sec/m

$$\boxed{\frac{Y(s)}{R(s)} = \frac{1.31e06(s + 13.3)}{s^4 + (516.1)s^3 + (5.685e04)s^2 + (1.307e06)s + 1.733e07}.}$$

❖ Now, we are ready to perform simulation for designing the controller !!

Modeling of Mechanical Systems

(3) 1-D Rotational Motion: Pendulum Model



$$T_c - mgl\sin\theta = I\ddot{\theta},$$

rearranging

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}.$$

Linearization

θ is small enough,
Then, $\sin\theta \approx \theta$

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2}.$$

$\mathcal{L}$

$$\sum \tau_i(t) = I\frac{d^2\theta}{dt^2}$$: rotational

$$\boxed{\frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}.}$$

$$\frac{\Theta(s)}{T_c(s)} = \frac{K}{s^2 + \omega_n^2}$$

$\omega_n = \sqrt{\frac{g}{l}}$: natural frequency $[rad/\sec]$

Source: Franklin, Gene F.; Powell, J; Emami-Naeini, Abbas. *Feedback Control of Dynamic Systems*

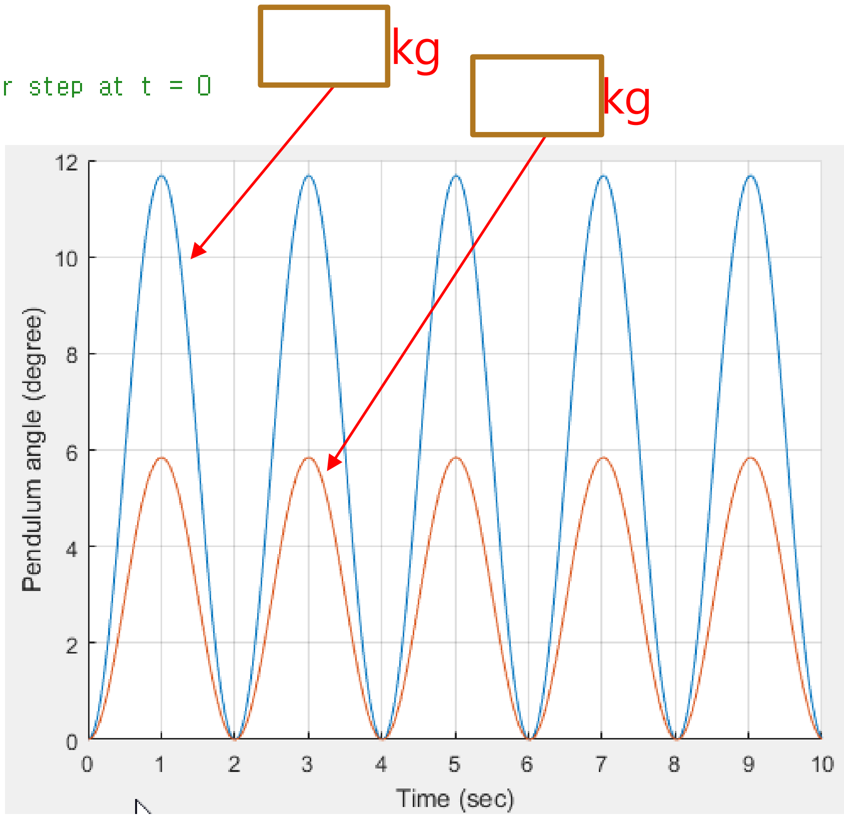
Modeling of Mechanical Systems

(3) 1-D Rotational Motion: Pendulum Model (cont'd)

```
t = 0:0.02:10;
m1 = 1;         % value of mass (Kg)
m2 = 2;         % value of mass (Kg)
L = 1;          % value of length (m)
g = 9.81;       % value of gravity, g (m/sec 2 )
s = tf('s');    % sets up transfer function input mode
sys1 = (1/(m1*L^2))/ (s^2 + g/L);
sys2 = (1/(m2*L^2))/ (s^2 + g/L);
y1 = step(sys1,t);  % step responses at times given by t for step at t = 0
y2 = step(sys2,t);
Rad2Deg = 57.3;     % converts radians to degrees
figure(1)
hold on; grid on;
plot(t, Rad2Deg*y1)  % plots step response
plot(t, Rad2Deg*y2)  % plots step response
ylabel('Pendulum angle (degree)');
xlabel('Time');
```

$$\omega_n = \sqrt{\frac{g}{l}} \text{ : natural frequency } [rad\,/\,\mathrm{sec}]$$

$$\frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}.$$



Modeling of Mechanical Systems

(3) 1-D Rotational Motion: Pendulum Model (cont'd)

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}.$$

$$\begin{cases} \ddot{\theta} = -9.81 * \theta + 1, \\ \ddot{\theta} = -9.81 * \sin\theta + 1, \end{cases}$$

Linear model

Non-linear model

θ is small enough,

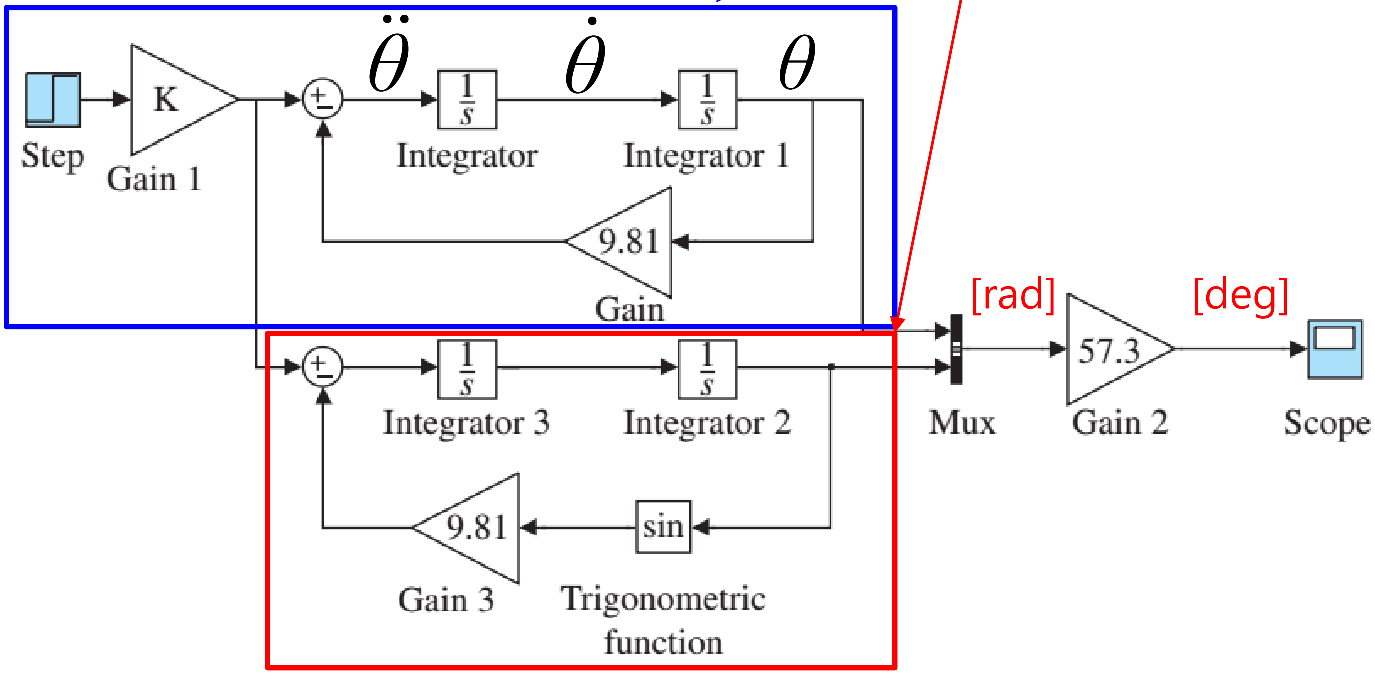
Then, $\sin\theta \approx \theta$

m =1[kg]

l = 1[m]

T_c = 1[Nm]

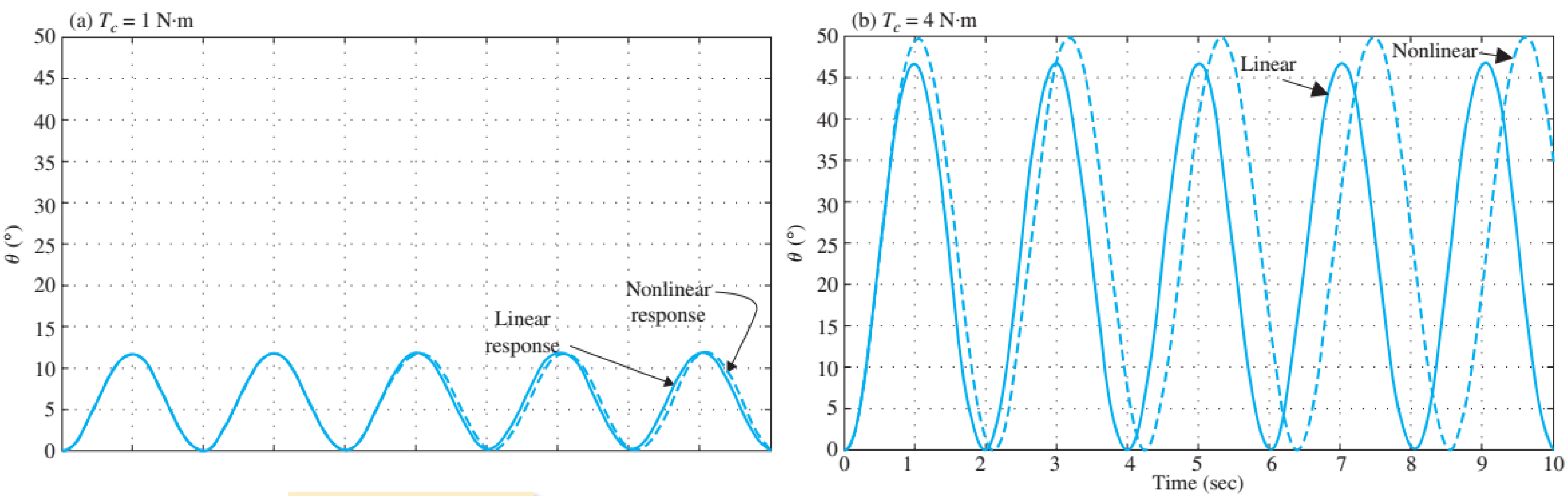
MATLAB SIMULINK



Modeling of Mechanical Systems

(3) 1-D Rotational Motion: Pendulum Model (cont'd)

- Simulation Results by Matlab/Simulink



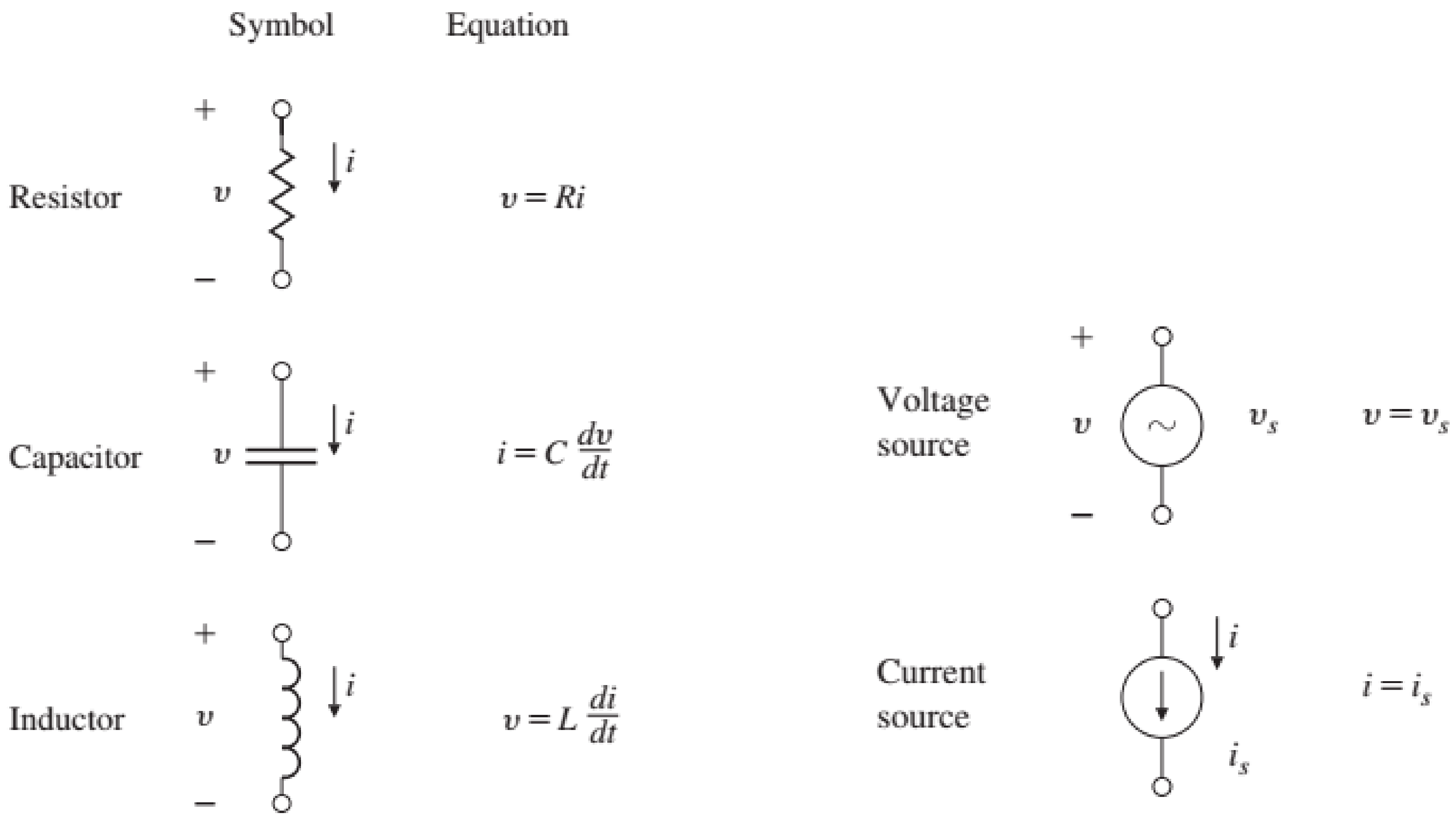
T_c = 1 Nm

T_c = 4 Nm

❖ Smaller input torque means small amount of pendulum angle (almost linear) !!

Introduction: Modeling of Electrical Systems

❖ Models of Electric Circuits



Linear circuit elements

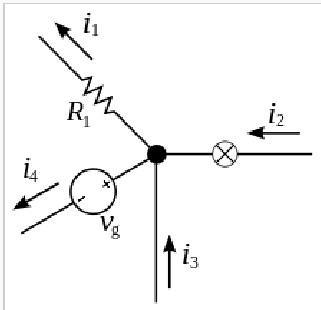
Linear circuit sources

Basic Equations of Electric Circuits

❖ Kirchhoff's current law (KCL) – 1st Law

At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node

$$\sum_{k=1}^{n} I_k = 0$$

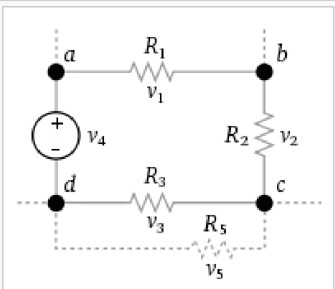


The current entering any junction is equal to the current leaving that junction. $i_2 + i_3 = i_1 + i_4$

❖ Kirchhoff's voltage law (KVL) – 2nd Law

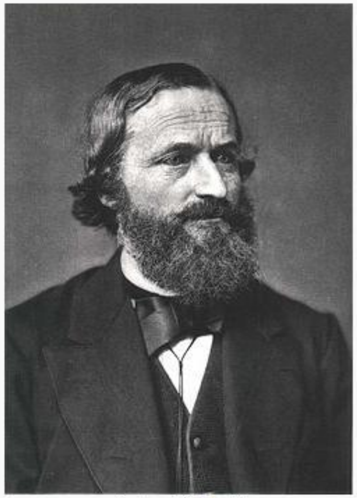
The directed sum of the electrical potential differences (voltage) around any closed network is zero

$$\sum_{k=1}^{n} V_k = 0$$



The sum of all the voltages around a loop is equal to zero. $v_1 + v_2 + v_3 - v_4 = 0$

Gustav Kirchhoff


Gustav Kirchhoff

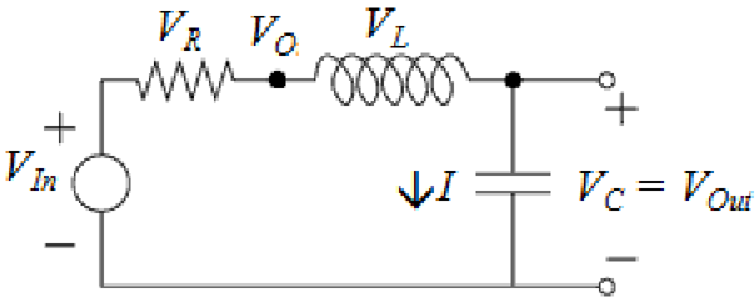
Born	Gustav Robert Kirchhoff 12 March 1824 Königsberg, Kingdom of Prussia (present-day Russia)
Died	17 October 1887 (aged 63) Berlin, Prussia, German Empire (present-day Germany)

Photo from https://en.wikipedia.org/wiki/Gustav_Kirchhoff

Modeling of Electrical Systems

(1) Simple Electrical Circuit: RLC Model

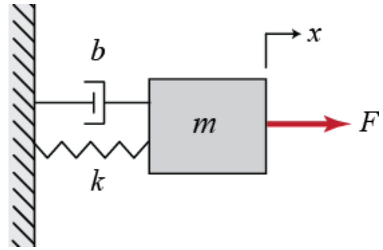
Using Kirchhoff's Voltage Law KVL



$$V_{In} = V_R + V_L + V_{C,out}$$

$$V_{In}(t) = i(t)R + L\frac{di(t)}{dt} + V_C(t)$$

But, the same current in the loop will be given as



$$i(t) = C\frac{dV_C}{dt},$$

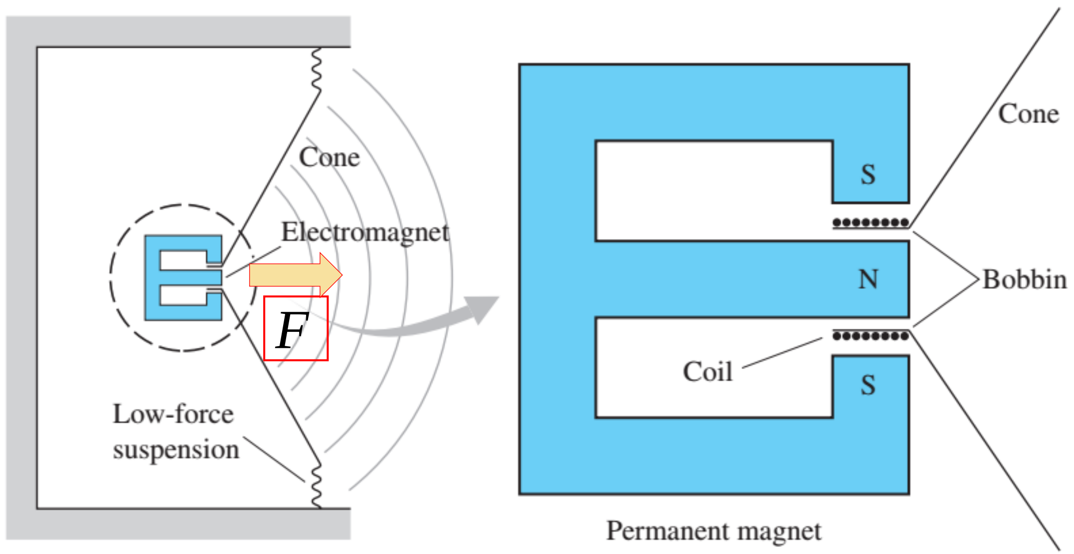
$$\frac{di(t)}{dt} = C\frac{d(dV_C/dt)}{dt} = C\frac{d^2V_C}{dt^2}$$

$$\therefore LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) = V_{In}(t) \Leftrightarrow m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F(t)$$

$$\xrightarrow{\mathcal{L}} \frac{V_{out}(s)}{V_{in}(s)} = \frac{1/LC}{s^2 + (R/L)s + 1/LC}$$

Modeling of Electro-Mechanical Systems

(1) Modeling of Loudspeaker: Part I – Mechanical motion



Law of motor :

$$F = Bil \text{ [N]}$$

where,

B : Magnetic field [T]

i : Current [A]

l : coil length [m]

$$M\ddot{x} + b\dot{x} = F = Bil$$

where,

M : cone mass [kg]

b : damping coeff. [N.s/m]

$\mathcal{L}$

$$TF = \frac{\text{Displacement}}{\text{Current}} = \frac{X(s)}{I(s)} = \frac{Bl}{Ms^2 + bs}$$

Source: Franklin, Gene F.; Powell, J; Emami-Naeini, Abbas. *Feedback Control of Dynamic Systems*

Modeling of Electro-Mechanical Systems,

(1) Modeling of Loudspeaker: Part II – Motion on Voltage

Law of generator :

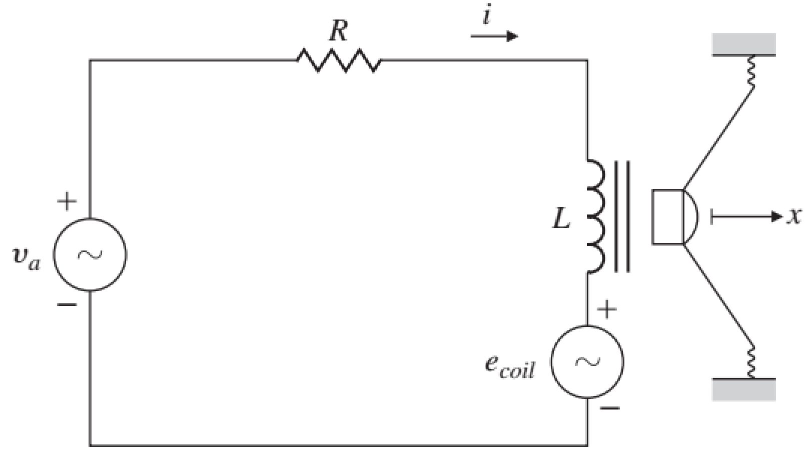
$e = Blv$ [V]

where,

B : Magnetic field [T]

l : coil length [m]

v : velocity [m/sec]



Based on the KVL for the circuit,

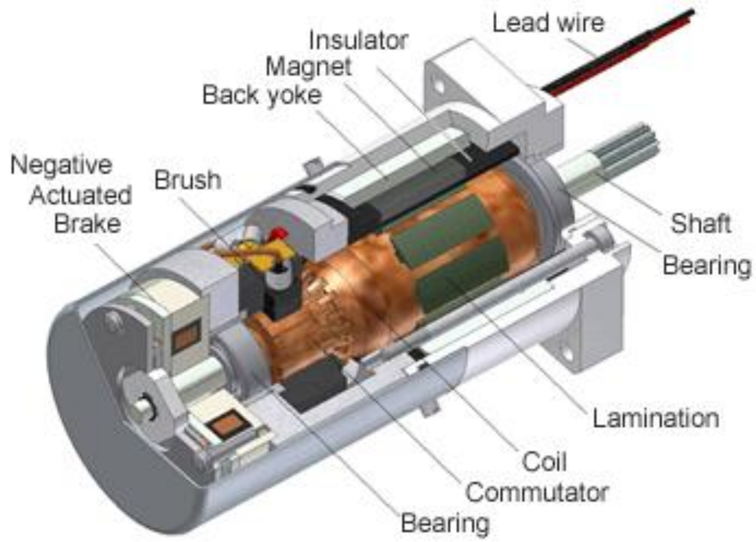
$$L\frac{di}{dt} + Ri = v_a - Bl\frac{dx}{dt} \quad \xrightarrow{\mathcal{L}} \quad (Ls + R)I(s) = V_a(s) - BlsX(s)$$

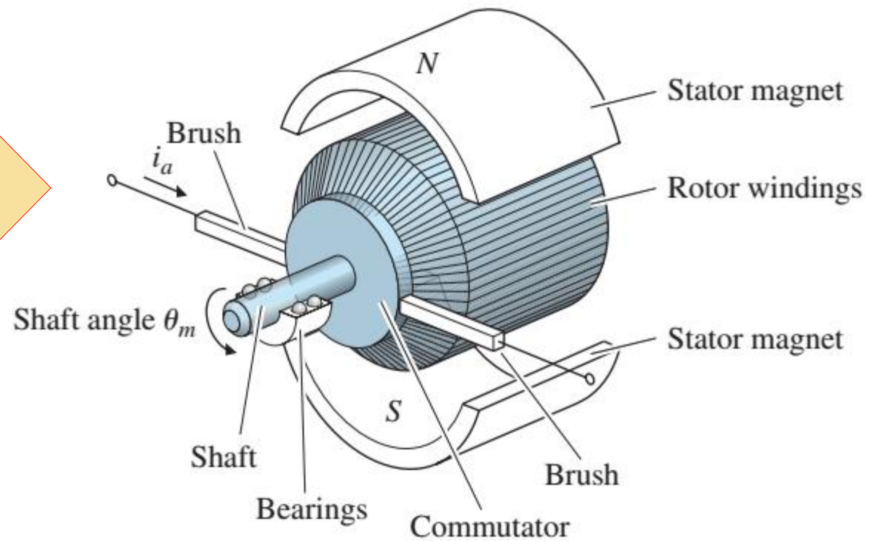
$$I(s) = \frac{Ms^2 + bs}{Bl}X(s)$$

$$\boxed{TF = \frac{\text{Displacement}}{\text{Input voltage}} = \frac{X(s)}{V_a(s)} = \frac{Bl}{s\{(Ms + b)(Ls + R) + (Bl)^2\}}}$$

Modeling of Electro-Mechanical Systems

(2) Modeling of DC motor: Background





Motor torque :

$$T = K_t i_a \text{ [Nm]}$$

Back - EMF volotage :

$$e = K_e \dot{\theta}_m \text{ [V]}$$

where,

K_t : motor torque constant [Nm/A]

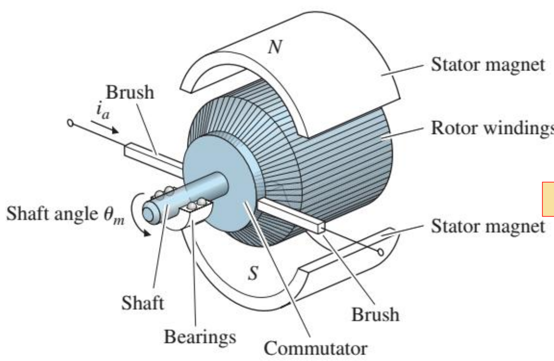
K_e : electric constant [V.s]

But, practrically $K_t = K_e$

Source: Franklin, Gene F.; Powell, J; Emami-Naeini, Abbas. *Feedback Control of Dynamic Systems*

Modeling of Electro-Mechanical Systems

(2) Modeling of DC motor



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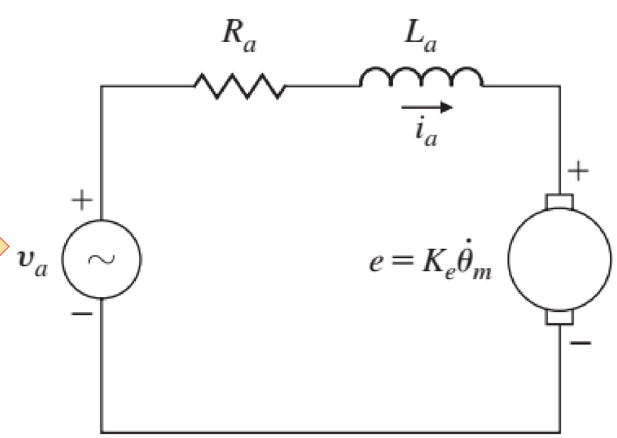
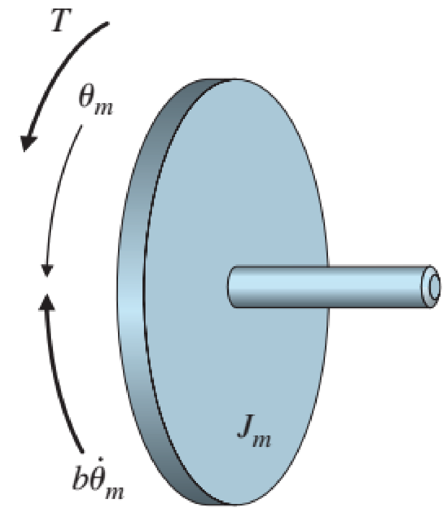




- Part I: Mechanical motion

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a.$$

$$\Downarrow \mathcal{L}$$

$$\frac{\theta_m(s)}{I_a(s)} = \frac{K_t}{J_m s^2 + bs}$$

- Part II: Electric circuit equation

$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m.$$

$$\Downarrow \mathcal{L}$$

$$(L_a s + R_a)I_a(s) = V_a(s) - K_e s\theta_m(s)$$

Modeling of Electro-Mechanical Systems

(2) Modeling of DC motor (cont'd)

$$(L_a s + R_a) I_a(s) = V_a(s) - K_e s\theta_m(s)$$

$$I_a(s) = \frac{J_m s^2 + bs}{K_t}\,\theta_m(s)$$

Arranging

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}.$$

But, practically $L_a << J_m$ or R_a, Thus it is negligible !!

Re-arranging

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + \left(b + \frac{K_t K_e}{R_a}\right)s} = \frac{K}{s(\tau s + 1)},$$

Where,

$$K = \frac{K_t}{bR_a + K_t K_e},$$

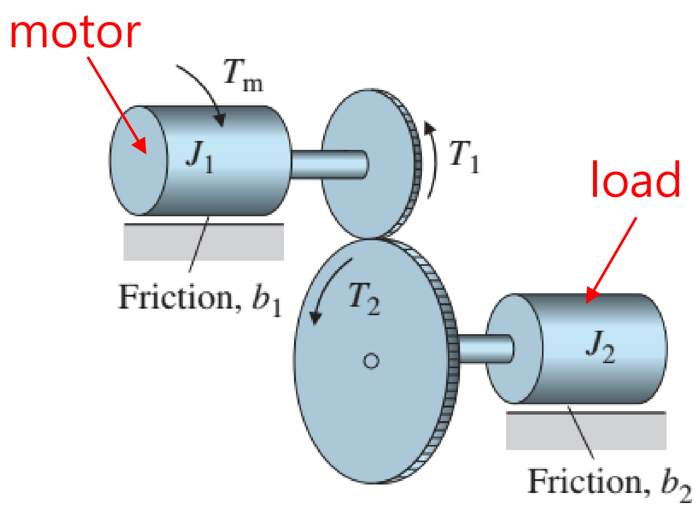
$$\tau = \frac{R_a J_m}{bR_a + K_t K_e}.$$

$$\boxed{\frac{\Omega(s)}{V_a(s)} = s\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{\tau s + 1}.}$$

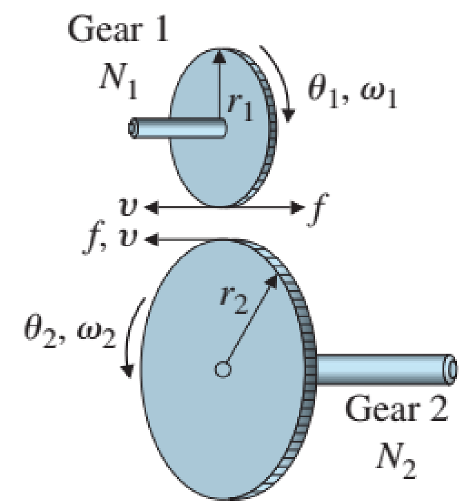
Velocity Model

Modeling of Electro-Mechanical Systems

(3) Modeling of DC motor with Gears



FBD



- Gear Ratio(n) vs. Torque (T)

$$\frac{T_2}{T_1} = \frac{r_2}{r_1} = \frac{N_2}{N_1} = n,$$

- Gear Ratio(n) vs. Angle (θ)

$$\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} = n.$$

Modeling of Electro-Mechanical Systems

(3) Modeling of DC motor with Gears (cont'd)

- System I – motor part modeling

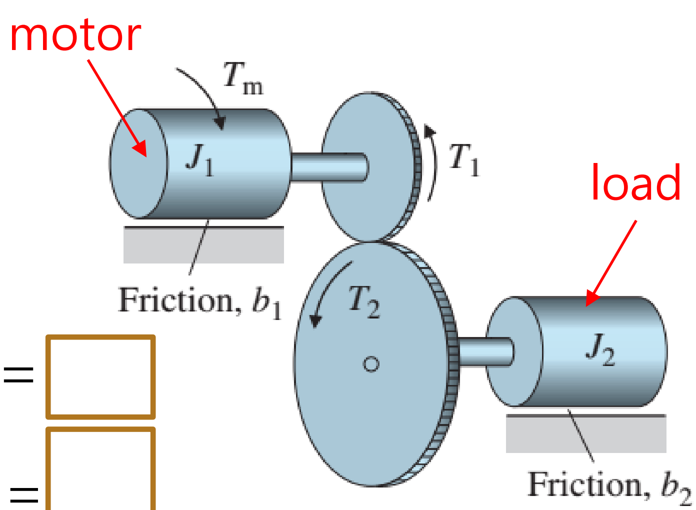
$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 = T_m - T_1,$$

- System II – load part modeling

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 = T_2,$$

$$\theta_1 = \boxed{}$$

$$T_1 = \boxed{}$$



From the system I,

$$J_1 n\ddot{\theta}_2 + b_1 n\dot{\theta}_2 = T_m - \frac{T_2}{n}$$

$$\Rightarrow T_2 = nT_m - J_1 n^2\ddot{\theta}_2 - b_1 n^2\dot{\theta}_2$$

$$(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 = nT_m.$$

$$\frac{\Theta_2(s)}{T_m(s)} = \frac{n}{J_{eq}s^2 + b_{eq}s},$$

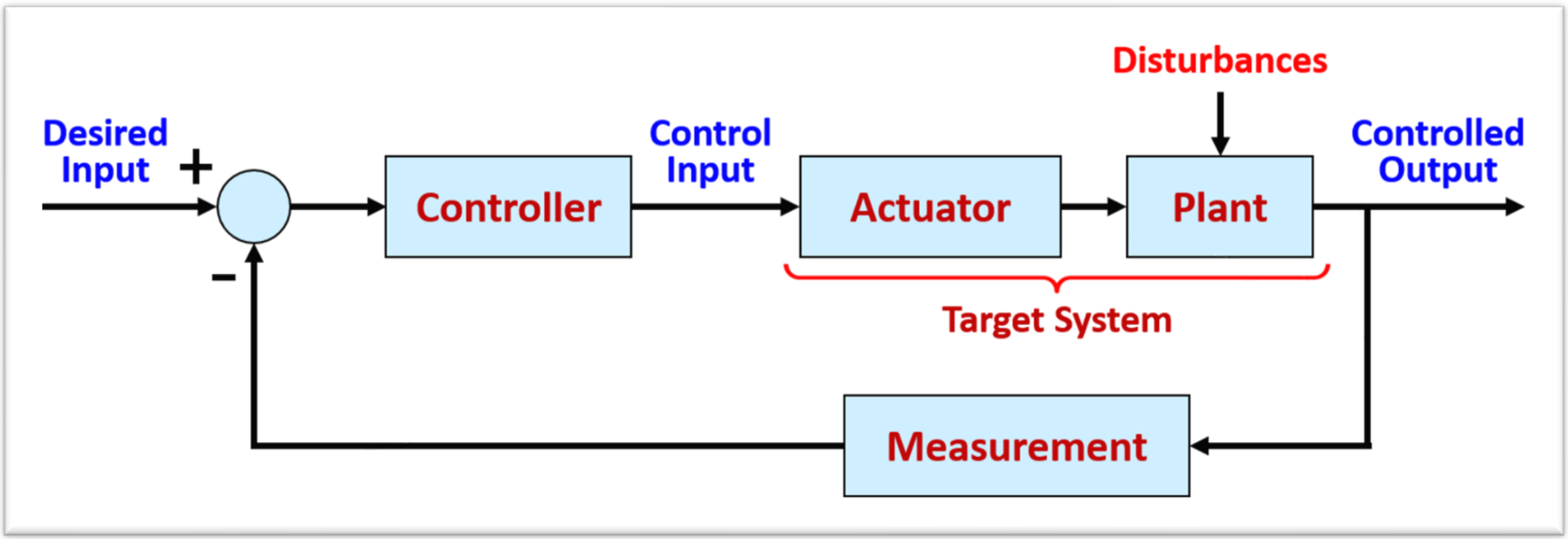
Where,

$$J_{eq} = J_2 + J_1 n^2, \text{ and } b_{eq} = b_2 + b_1 n^2.$$

Dynamic Models

Lecture 3:

- Modeling of Mechanical & Electrical Systems
- **Linearization & Electro-Mechanical Systems**



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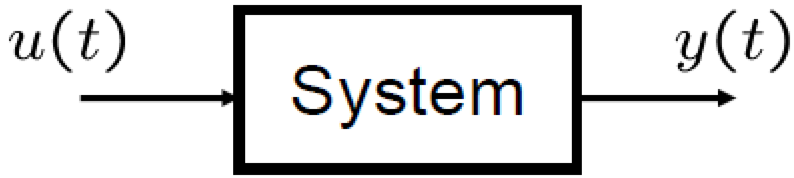
What is a Linear System?

❖ A system having **Principle of Superposition**



$$\left.\begin{array}{l} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{array}\right\} \Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

$$\forall \alpha_1, \alpha_2 \in \mathbb{R}$$

❖ But, a nonlinear system does NOT satisfy the principle of superposition.

Why Linearization?

- Easier to understand and obtain solutions

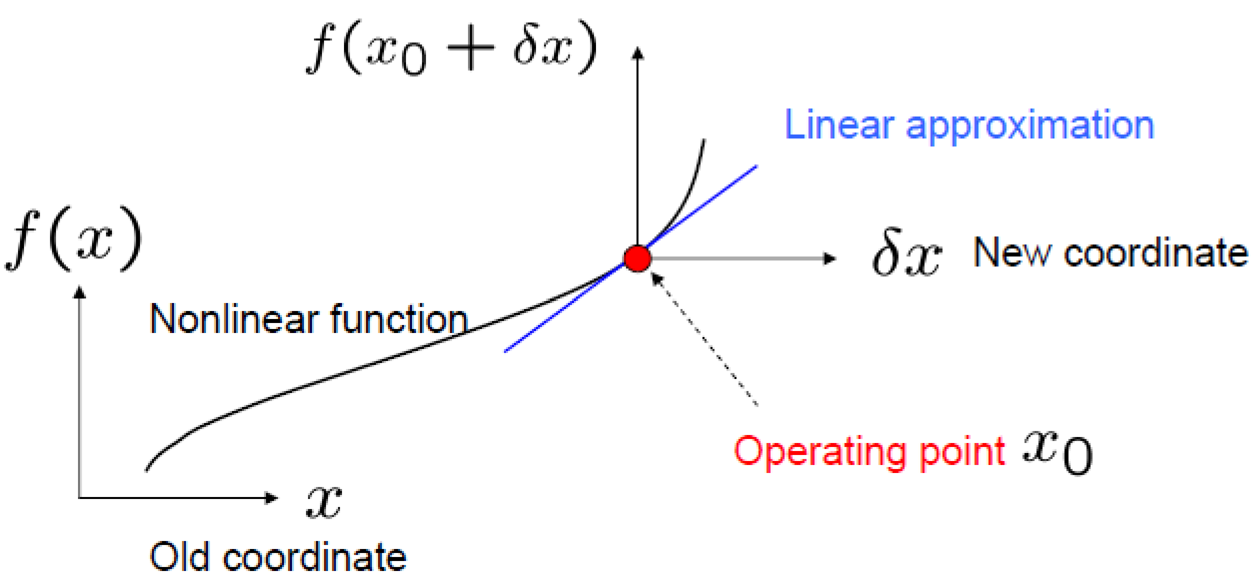
- Linear ordinary differential equations (ODEs),
  - Homogeneous solution & particular solution
  - Solution caused by initial values & forced solution
  - Transient solution & steady state solution

- Add many simple solutions to get more complex ones (Use superposition !!)

- Easy to check the stability of stationary states (by using Laplace Transform & Bode Plot)

How to linearize it?

- Nonlinearity can be approximated by a linear function for small deviations (δx) around an operating point (x_0)
- Taylor series expansion is very useful !!



How to linearize it? (cont'd)

- Nonlinear system: $\dot{x} = f(x, u)$
- Let u_0 be a nominal input and let the resultant state be x_0
- Perturbation: $u(\cdot) = u_o(\cdot) + \delta u(\cdot)$
- Resultant perturb: $x(\cdot) = x_o(\cdot) + \delta x(\cdot)$
- Taylor series expansion:

$$f(x,u) = f(x_0,u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x$$

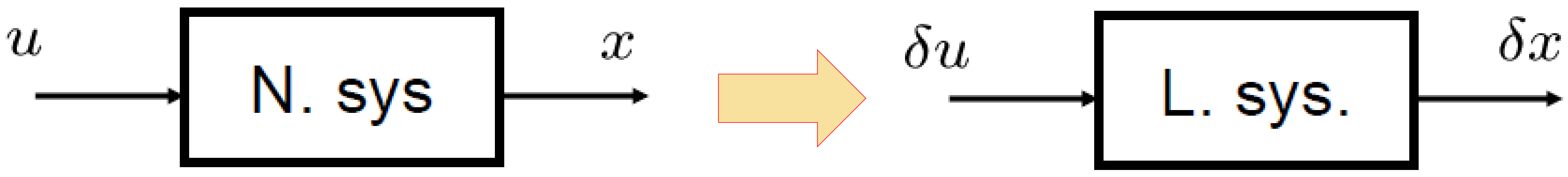
$$+ \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u + \underbrace{\text{H.O.T.}}_{\approx 0}$$

How to linearize it? (cont'd)

$$\dot{x}_0 + \delta\dot{x} = f(x_0, u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u$$

notice that $\dot{x}_0 = f(x_0, u_0)$; hence

$$\boxed{\delta\dot{x} = \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u}$$

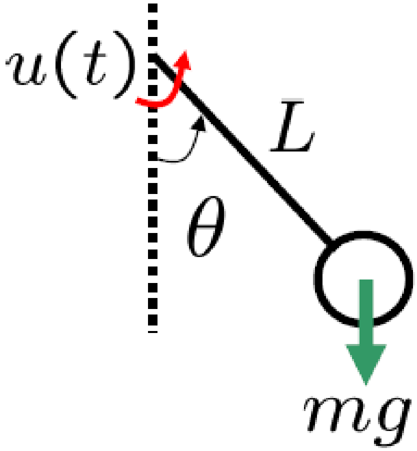


How to linearize it? (cont'd)

❖ Example: Inverted Pendulum

$$mL^2\ddot{\theta}(t) + mgL\sin\theta(t) = u(t)$$

$$\ddot{\theta}(t) + \underbrace{\frac{g\sin\theta(t)}{L} - \frac{u(t)}{mL^2}}_{f(\theta,u)} = 0$$



- Linearize it at $\theta_0 = \pi$ **Inverted !!**
- Find u_0 $\quad \ddot{\pi} + \frac{g\sin\pi}{L} - \frac{u_0}{mL^2} = 0 \rightarrow u_0 = 0$
- New coordinates:

$$\theta = \theta_0 + \delta\theta = \pi + \delta\theta$$
$$u = u_0 + \delta u = 0 + \delta u$$

How to linearize it? (cont'd)

$$mL^2\ddot{\theta}(t) + mgL\sin\theta(t) = u(t)$$

$$\ddot{\theta}(t) + \underbrace{\frac{g\sin\theta(t)}{L} - \frac{u(t)}{mL^2}}_{f(\theta,u)} = 0$$

❖ Example: Inverted Pendulum (cont'd)

- Taylor series expansion of $f(\theta, u)$ at $\theta = \pi, u = 0$

$$\frac{\partial f(\theta,u)}{\partial \theta}\Big|_{\substack{\theta=\pi\\u=0}} = \frac{g\cos\theta}{L}\Big|_{\theta=\pi} = -\frac{g}{L}$$

$$\frac{\partial f(\theta,u)}{\partial u}\Big|_{\substack{\theta=\pi\\u=0}} = -\frac{1}{mL^2}$$

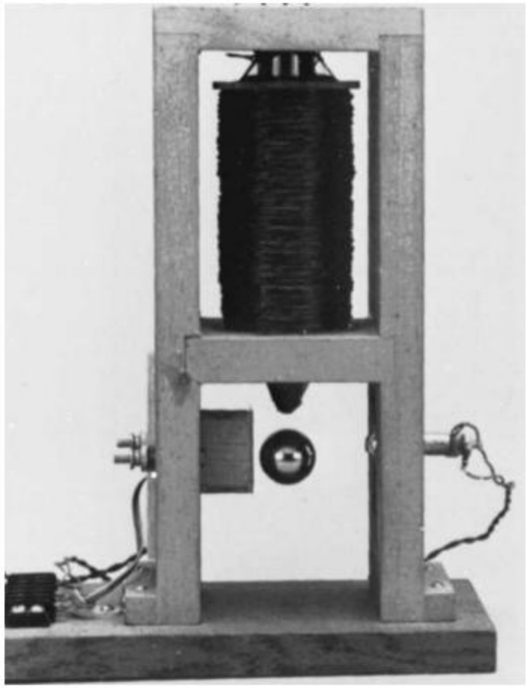
Arranging & replacing terms

$$\delta\ddot{\theta} + \frac{\partial f(\theta,u)}{\partial \theta}\Big|_{\substack{\theta=\pi\\u=0}}\delta\theta + \frac{\partial f(\theta,u)}{\partial u}\Big|_{\substack{\theta=\pi\\u=0}}\delta u = 0$$

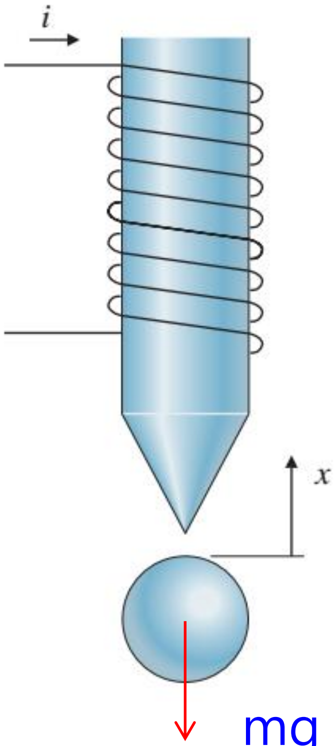
$$\boxed{\delta\ddot{\theta} - \frac{g}{L}\delta\theta - \frac{1}{mL^2}\delta u = 0}$$

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System



FBD



- Mechanical motion by the field of the electromagnet

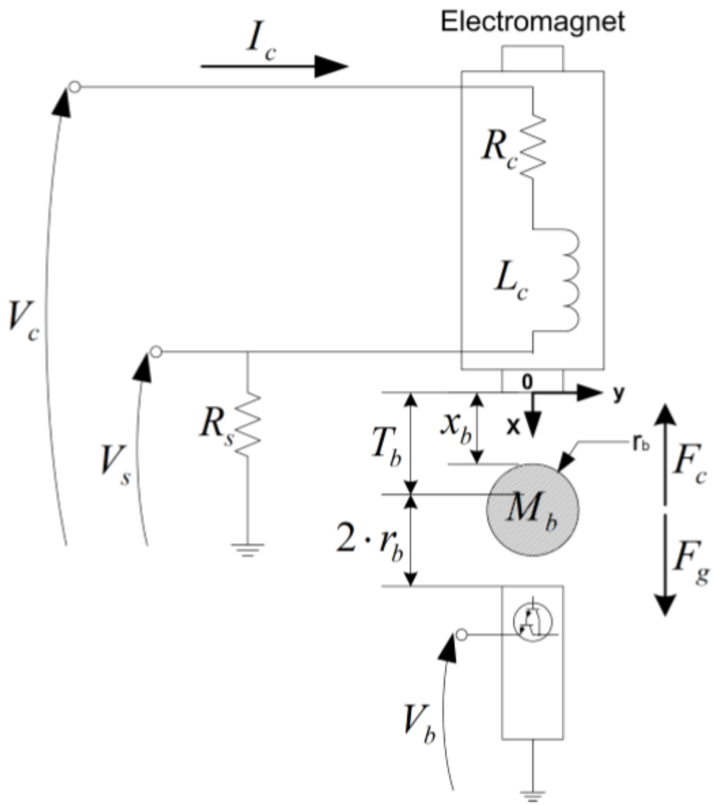
$$m\ddot{x} = f_m(x, i) - mg,$$ where, $f_m(x,i)$: electromagnetic force

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System (cont'd)

Lab Class:
Magnetic Levitation System

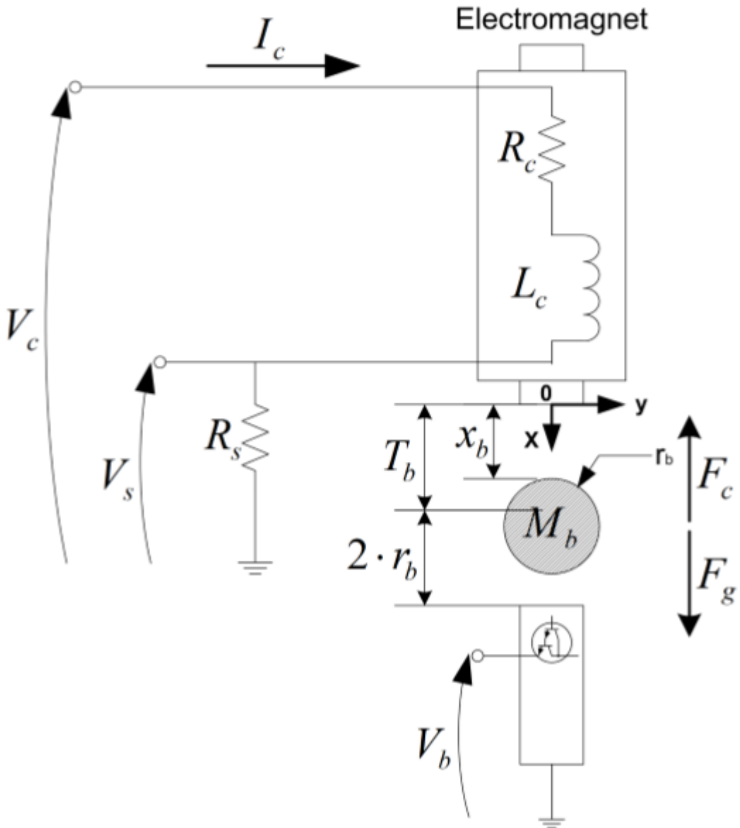
FBD



Two parts combined:
Electrical & electro-mechanical

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System (cont'd)



- **Part I: Electrical model equations**

Using KVL,

$$v_c(t) = (R_c + R_s)i_c(t) + L_c \frac{di_c(t)}{dt}$$

$$\Downarrow \mathcal{L}$$

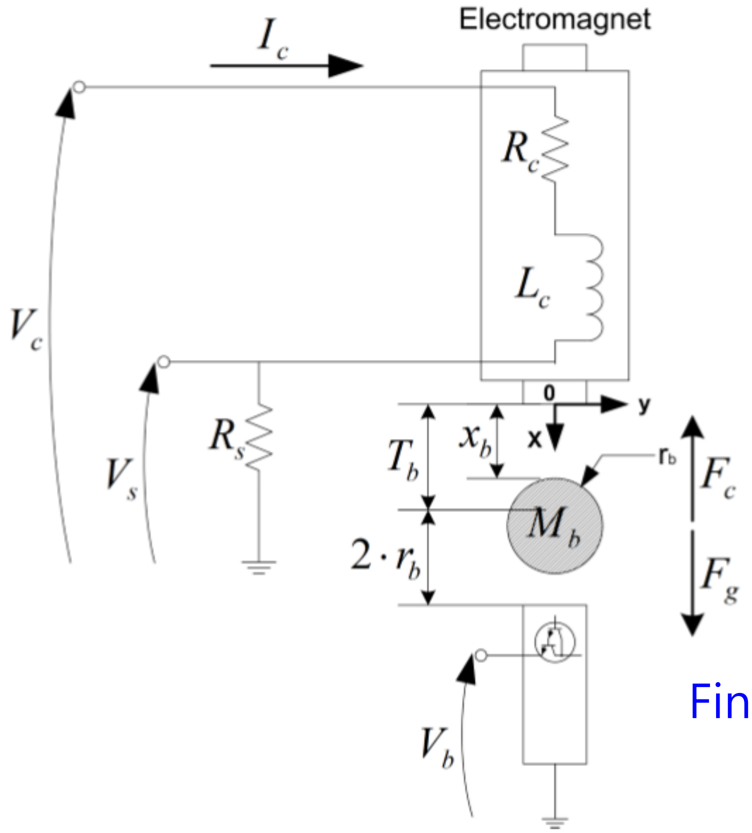
$$\frac{I_c(s)}{V_c(s)} = \frac{1}{L_c s + R_c + R_s}$$

$$= \frac{1/(R_c + R_s)}{\tau_c s + 1} \quad \text{where,}$$

$$\tau_c = \frac{L_c}{R_c + R_s}$$

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System (cont'd)



- **Part II: Electro-Mechanical model**

(1) Electro-magnetic force on ball is,

$$F_c = \frac{K_m i_c(t)^2}{x_b{}^2}$$

(2) Gravity force on the ball is

$$F_g = M_b g.$$

Total external force = (1) + (2),

$$F_{ext} = -F_c + F_g = -\frac{K_m i_c(t)^2}{x_b(t)^2} + M_b g.$$

Finally, inertia motion = external force $M_b \ddot{x}_b = F_{ext}$

$$M_b \ddot{x}_b(t) = \boxed{-\frac{K_m i_c(t)^2}{x_b(t)^2}} + M_b g$$ **Nonlinear !!**

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System (cont'd)

- **Part II: Electro-Mechanical model (nonlinear → linear model)**
  - ✓ Static equilibrium at a nominal operating point: (x_{b0}, i_{c0})
  - ✓ Small perturbed current : δi_c
  - ✓ Small perturbed ball position: δx_b
  - ✓ Then, ball position (x_b) and current (i_c) can be expressed as following:

$$\begin{cases} x_b = x_{b0} + \delta x_b \\ i_c = i_{c0} + \delta i_c \end{cases}$$

  - ✓ Here, linearized function by using Tylor Series Expansion is,

$$f(x,i) = f(x_{b0}, i_{c0}) + \left.\frac{\partial f(x,i)}{\partial x}\right|_{\substack{x = x_{b0} \\ i = i_{c0}}} \delta x_b + \left.\frac{\partial f(x,i)}{\partial i}\right|_{\substack{x = x_{b0} \\ i = i_{c0}}} \delta i_c$$

Modeling of Electro-Mechanical Systems

(4) Modeling of Magnetic Levitation System (cont'd)

- **Part II: Electro-Mechanical model (nonlinear → linear model)**

$$M_b \ddot{x}_b(t) = -\frac{K_m i_c(t)^2}{x_b(t)^2} + M_b g$$

Tylor series expansion around operation points

$$M_b \delta\ddot{x}_b(t) \approx \boxed{-\frac{K_m i_{c0}^2}{x_{b0}^2} + \left(\frac{2K_m i_{c0}^2}{x_{b0}^3}\right)\delta x_b - \left(\frac{2K_m i_{c0}}{x_{b0}^2}\right)\delta i_c} + M_b g$$

where,

$$x_b = x_{bo} + \delta x_b \Rightarrow \ddot{x}_b = \delta\ddot{x}_b$$

$$f(x_{b0}, i_{b0}) = -\frac{K_m i_{c0}^2}{x_{b0}^2}$$

but, at equilibrium point without incremental terms,

$$M_b g - \frac{K_m i_{c0}^2}{x_{b0}^2} = 0$$

$\mathcal{L}$

$$TF = \frac{\text{perturved displacement}}{\text{perturbed current}}$$

$$= \frac{\delta X_b(s)}{\delta I_c(s)} = -\frac{\omega_n^2}{s^2 - \omega_b^2}$$

where,

$$\omega_b = \sqrt{\frac{2K_m i_{c0}^2 / x_{b0}^3}{M_b}}$$

$$\omega_n = \sqrt{\frac{2K_m i_{c0}}{M_b x_{b0}^2}}$$

http://web.mit.edu/2.737/www/MagLev/linearized/

Summary

❖ **Summary:**

- Examples of **mechanical & electrical system modeling**
- Linearization method application
- Several cases of **electro-mechanical system modeling**