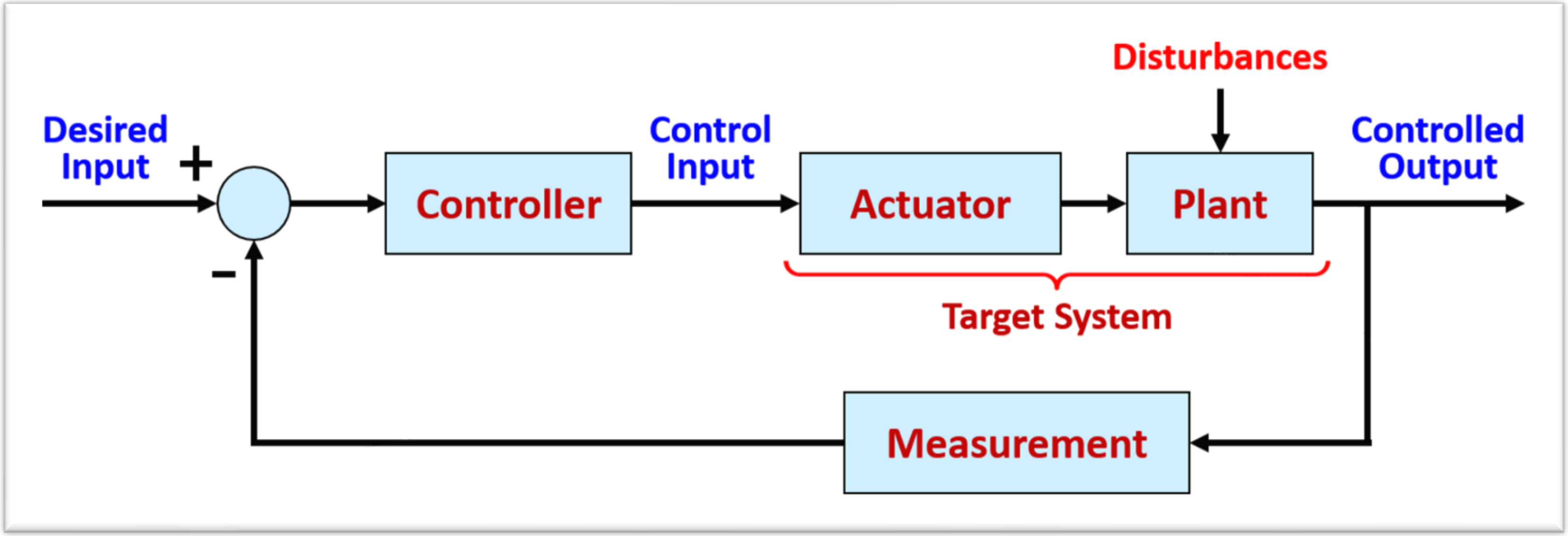
Analysis of Feedback 2

Lecture 7:

- Control of Steady-State Error
- Feedforward Control System



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System Type for Tracking

❖ Closed-Loop System

- The controlled output is given by

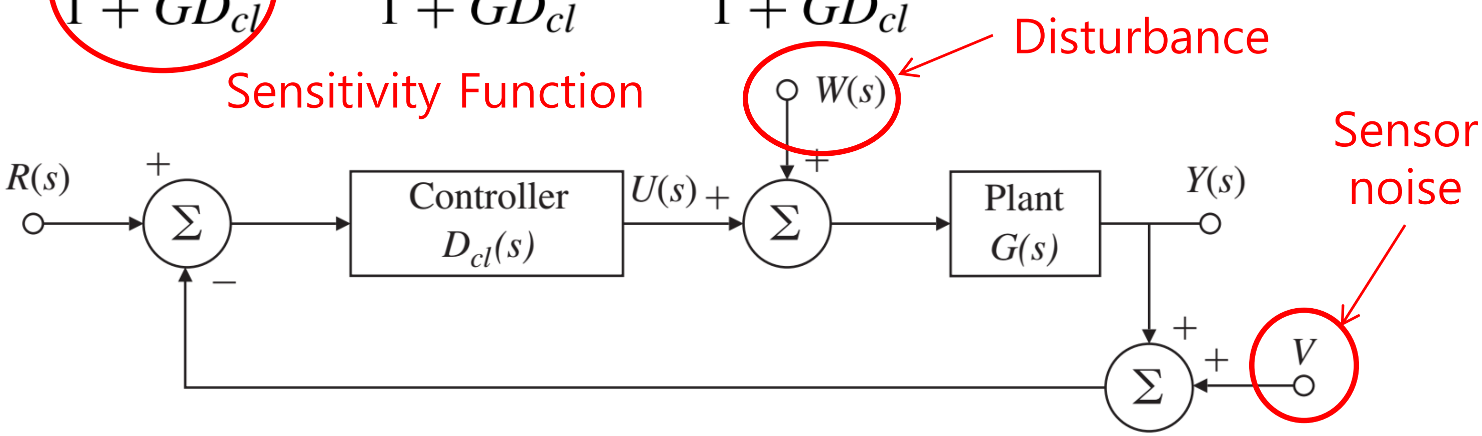
$$Y_{cl} = \frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}W - \frac{GD_{cl}}{1+GD_{cl}}V,$$

- The error, difference btw reference input and output is given by

$$E_{cl} = R - \left[\frac{GD_{cl}}{1+GD_{cl}}R + \frac{G}{1+GD_{cl}}W - \frac{GD_{cl}}{1+GD_{cl}}V\right],$$

$$= \frac{1}{1+GD_{cl}}R - \frac{G}{1+GD_{cl}}W + \frac{GD_{cl}}{1+GD_{cl}}V.$$

Sensitivity Function



System Type for Tracking (cont'd)

- If we consider tracking the reference input alone,
  set W = V = 0, (where, G: plant model, D_{cl}: controller)

$$E = \frac{1}{1+GD_{cl}} R = \mathcal{S} R, \quad \text{where} \quad \mathcal{S} = \frac{1}{1+GD_{cl}}.$$

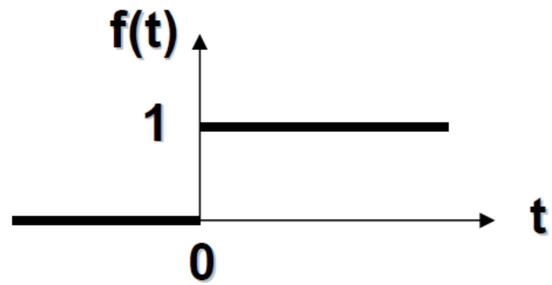
- For analyzing error {E(s)},
  categorizing based on three types of reference inputs,

$$R(s) = \frac{1}{s^{k+1}} \begin{cases} k = 0 \Rightarrow R(s) = \dfrac{1}{s} & \Rightarrow \text{Step Input (or position)} \\ \boxed{k = 1 \Rightarrow R(s) = \dfrac{1}{s^2} \quad \Rightarrow \text{Ramp Input (or velocity)}} \\ k = 2 \Rightarrow R(s) = \dfrac{1}{s^3} & \Rightarrow \text{Parabola Input (or acceleration)} \end{cases}$$

Examples of Laplace Transform (Review)

❖ Unit Step Function

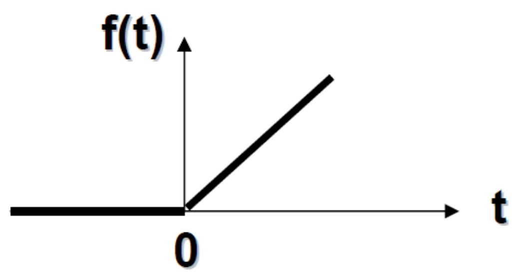
$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow \quad F(s) = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \boxed{\frac{1}{s}}$$

❖ Unit Ramp Function

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow \quad F(s) = \int_0^\infty t e^{-st} dt = -\frac{1}{s}\left[t e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \boxed{\frac{1}{s^2}}$$

(Integration by parts)

$$\boxed{\int u v' \, dx = uv - \int u' v \, dx}$$

System Type for Tracking (cont'd)

- Apply the Final Value Theorem to the error formula,

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s\boxed{E(s)},$$

$$\boxed{R(s) = \frac{1}{s^{k+1}}}$$

$$= \lim_{s\to 0} s\boxed{\frac{1}{1+GD_{cl}}R(s)}, \quad = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s^{k+1}}.$$

- General form of GD_{cl} without the pole at the origin (s = 0)

$$GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}$$

where,

$GD_{clo}(0) = K_n$: DC gain (or constant)

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+\dfrac{GD_{clo}(s)}{s^n}}\frac{1}{s^{k+1}}, \quad = \lim_{s\to 0}\frac{s^n}{s^n+K_n}\frac{1}{s^k}.$$

System Type for Tracking (cont'd)

- From the equation, $e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$. $where, R(s) = \frac{1}{s^{k+1}}$

then, we have Five Cases for checking error-constant:

- ✓ Case 1: n > k ⇒ $e_{ss} = 0$
- ✓ Case 2: n < k ⇒ $e_{ss} = \infty$

- ✓ Case 3: n = k = 0 (type 0) ⇒ $e_{ss} = \frac{1}{1 + K_p}$ Step error-constant

  $R(s) = \frac{1}{s}$

- ✓ Case 4: n = k = 1 (type 1) ⇒ $e_{ss} = \frac{1}{K_v}$ Ramp error-constant

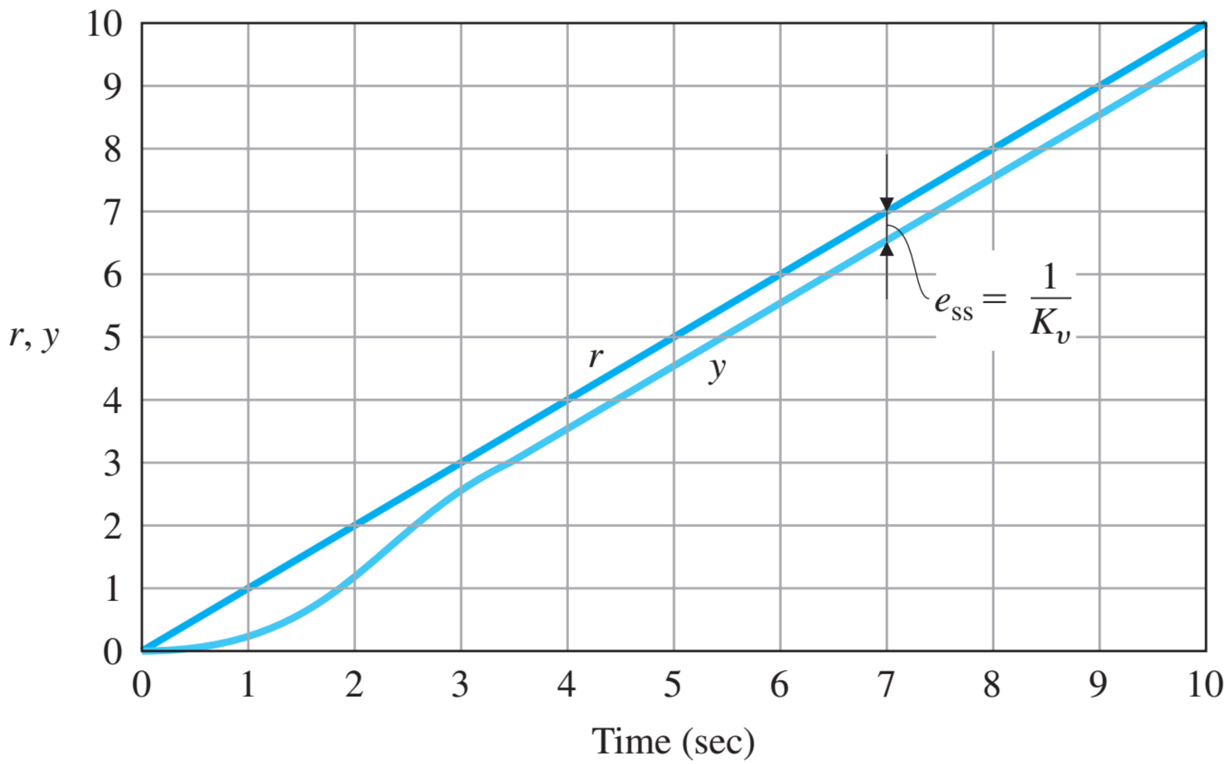
  $R(s) = \frac{1}{s^2}$

- ✓ Case 5: n = k = 2 (type 2) ⇒ $e_{ss} = \frac{1}{K_a}$ Parabola error-constant

  $R(s) = \frac{1}{s^3}$

System Type for Tracking (cont'd)

- Ramp response vs. K_v (ramp error-constant)



$$R(s) = \frac{1}{s^2}$$

- Important question:
  - What value of K_v is better to minimize steady-state error (SSE) ??

System Type for Tracking

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s),$$
$$= \lim_{s\to 0} s \frac{1}{1+GD_{cl}} R(s),$$

- **Summary: Error-Constant**

$$[\text{Step}]: \frac{1}{e_{ss}} = \cancel{1} + K_p = \lim_{s\to 0} \frac{1}{sE(s)} = \lim_{s\to 0} \frac{1+GD_{cl}}{s} s = \cancel{1} + \lim_{s\to 0} GD_{cl}$$

$$[\text{Ramp}]: \frac{1}{e_{ss}} = K_v = \lim_{s\to 0} \frac{1}{sE(s)} = \lim_{s\to 0} \frac{1+GD_{cl}}{s} s^2 = 0 + \lim_{s\to 0} sGD_{cl}$$

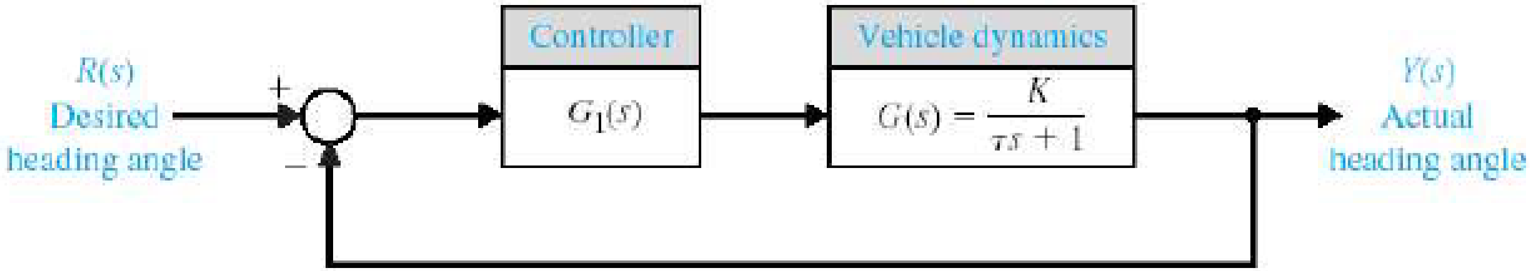
$$[\text{Parabola}]: \frac{1}{e_{ss}} = K_a = \lim_{s\to 0} \frac{1}{sE(s)} = \lim_{s\to 0} \frac{1+GD_{cl}}{s} s^3 = 0 + \lim_{s\to 0} s^2 GD_{cl}$$

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

System Type for Tracking (cont'd)

- **Ex. steering control system** ($G_1(s) = K_1 + K_2/s$)



1) when $K_2 = 0$

$$G_1 G = \frac{K_1 K}{\tau s + 1} \quad \Rightarrow \quad \text{type-0 system}$$

$$K_p = \lim_{s \to 0} G_1(s) G(s) = K_1 K$$

for step input, $e_{ss} = \dfrac{A}{1+K_p} = \dfrac{A}{1+K_1 K}$

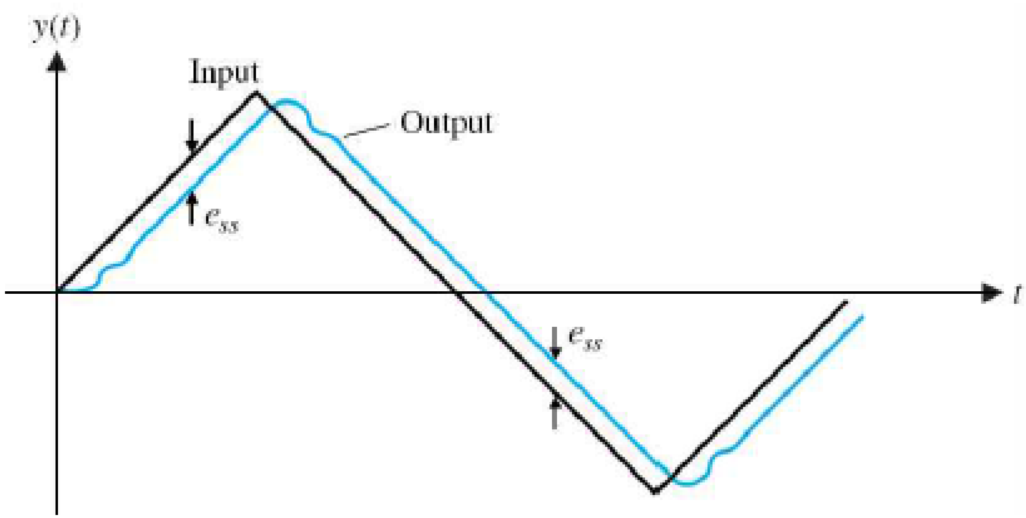
System Type for Tracking (cont'd)

2) when $K_2 > 0$

$$G_1 G = \frac{(K_1 s + K_2)K}{s(\tau s + 1)} \quad \Rightarrow \quad \text{type-1 system}$$

$$K_v = \lim_{s \to 0} s G_1(s) G(s) = K_2 K$$

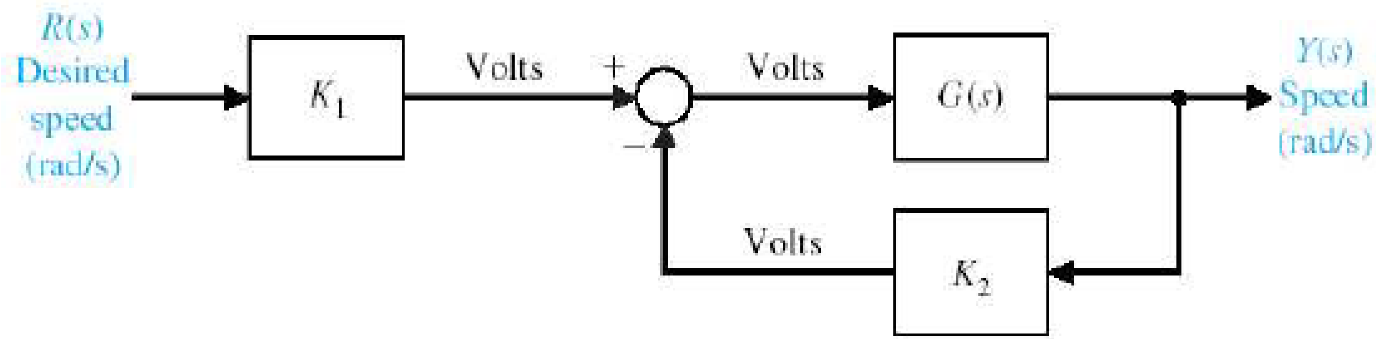
for ramp input, $e_{ss} = \dfrac{A}{K_v} = \dfrac{A}{K_2 K}$



Note: error constants K_p, K_v, K_a show system ability to follow input (the bigger, the better)

System Type for Tracking (cont'd)

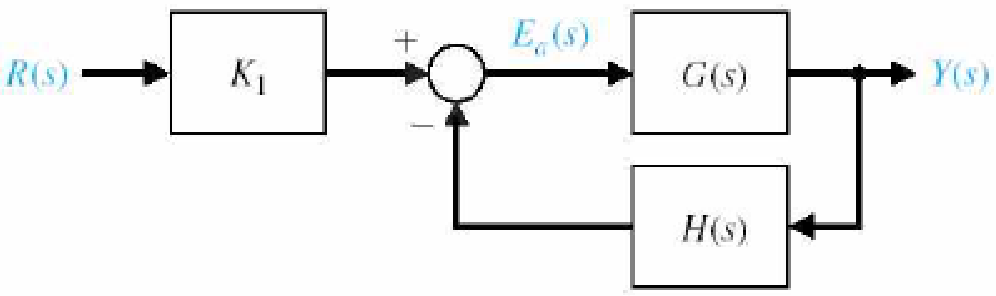
- **Ex. nonunity feedback system 1**



⇨ K_1 , K_2 : conversion-of-unit factors

⇨ for correct unit conversion, $K_1 = K_2$

System Type for Tracking (cont'd)



$$G(s) = \frac{40}{s+5}, \quad H(s) = \frac{20}{s+10}$$

unknown: K_1, e_{ss} for unit step input

sol)

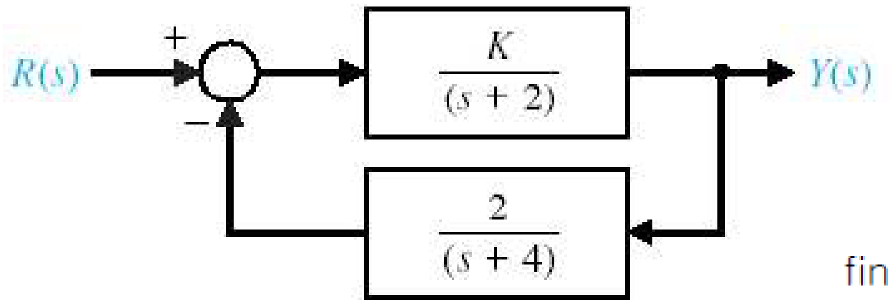
$$H(s) = \frac{20}{s+10} = \frac{2}{0.1s+1} \quad \Rightarrow \quad \text{DC gain of H(s) = 2} \quad \Rightarrow \quad K_1 = 2$$

$$E = R - Y = R - K_1 \frac{G}{1+GH} R = \frac{1+GH-K_1 G}{1+GH} R$$

for unit step input, $e_{ss} = \lim_{s \to 0} sE(s) = \dfrac{1+G(0)H(0)-K_1 G(0)}{1+G(0)H(0)} = \dfrac{1+16-2*8}{1+16} = \dfrac{1}{17}$

System Type for Tracking (cont'd)

- **Ex. nonunity feedback system 2**



find K so that e_{ss} to a unit step input is minimized

Sol)

$$E = R - Y = R - \frac{G}{1+GH}R = \frac{1+GH-G}{1+GH}R$$

for unit step, $e_{ss} = \lim_{s\to 0} sE(s) = \frac{1+G(0)H(0)-G(0)}{1+G(0)H(0)} = \frac{1+K/4-K/2}{1+K/4} = \frac{4-K}{4+K}$

⇨ zero e_{ss} when K = 4

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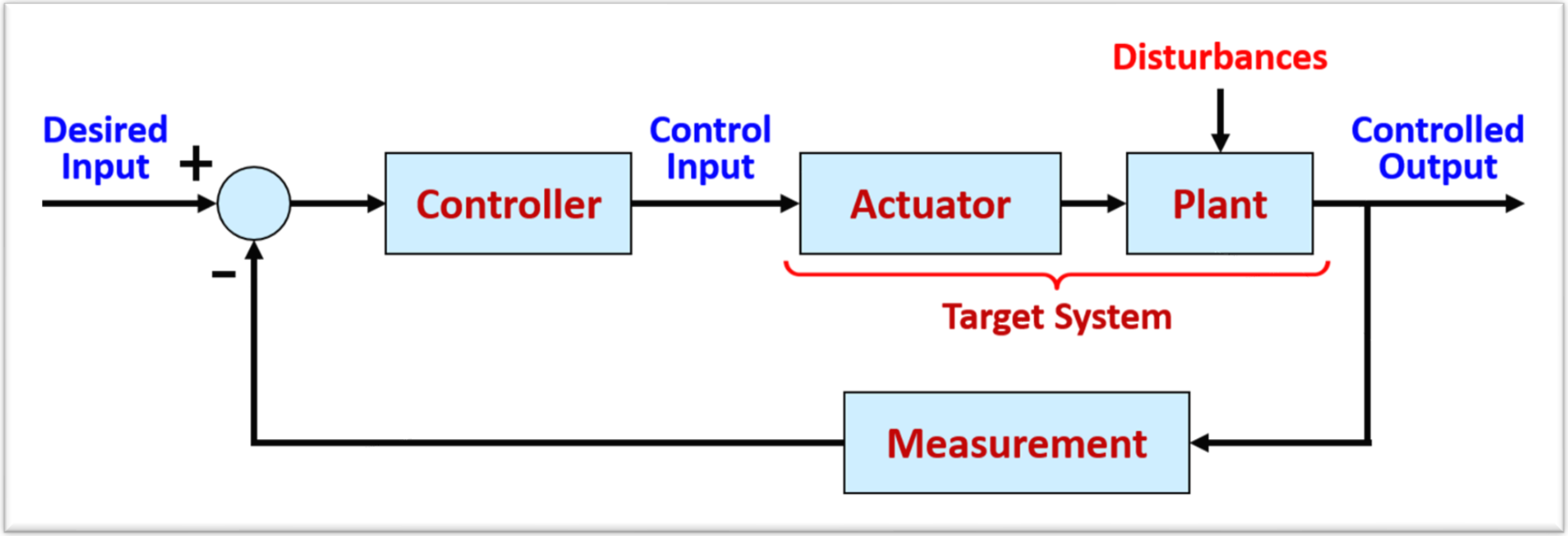

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Limitations of PID & Modifications (Revisited)

- **PID is Feedback !! ☞ Thus, response is reactive!!**
  ✓ Feed-forward control with the knowledge about the process model will be able to increase responding performance !!

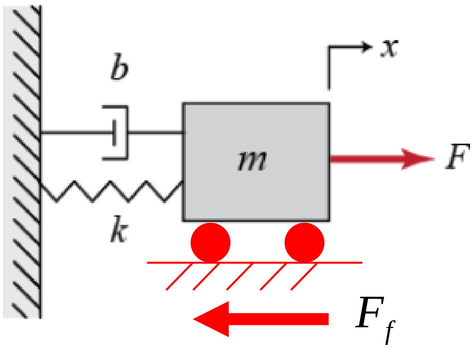
- **PID has constant parameters (or gains) !!**
  ☞ Thus, if system variations exist, performance gets worse !!
  ✓ Changing the control parameters (i.e., gains) based on the process variations (e.g., gain scheduling and adaptive control)

- **PID control is no direct knowledge of the process !!**
  ✓ Identify the process model and optimize controller gain (e.g., by using Matlab tool(e.g., PID tuner), auto-tuning method, model matching condition and etc)

- **Integral wind-up & High-frequency noise amplification on Derivative !!**
  (1) Anti-windup schemes (e.g., temporally stopping integral action)
  (2) Low-pass-filter !! but, slow response is following.

Feedforward Control (I)

- **Example 2: Spring-Damper-System w/ Friction**



- Step 1 : Equation of Motion (EOM) of this system $m\ddot{x}+b\dot{x}+kx=F-F_f$
- Step 2 : Taking Laplace transform of EOM with friction force

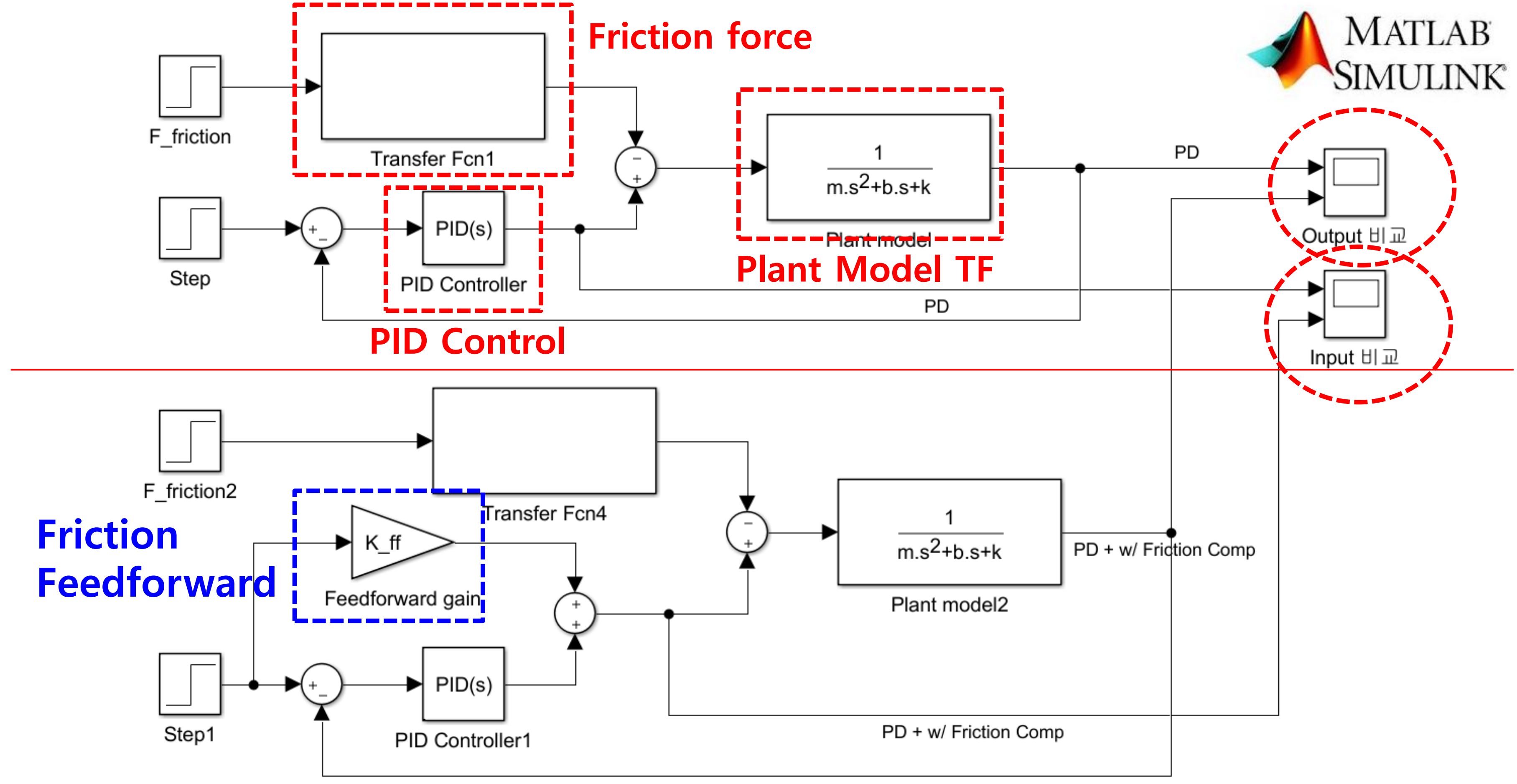
$$\left(ms^2+bs+k\right)X(s)=F(s)-F_f(s)$$

- Step 3 : TF btw the displacement {X(s)} and force {F(s)}

$$X(s)=\frac{1}{ms^2+bs+k}F(s)-\frac{1}{ms^2+bs+k}F_f(s)$$

Feedforward Control (cont'd)

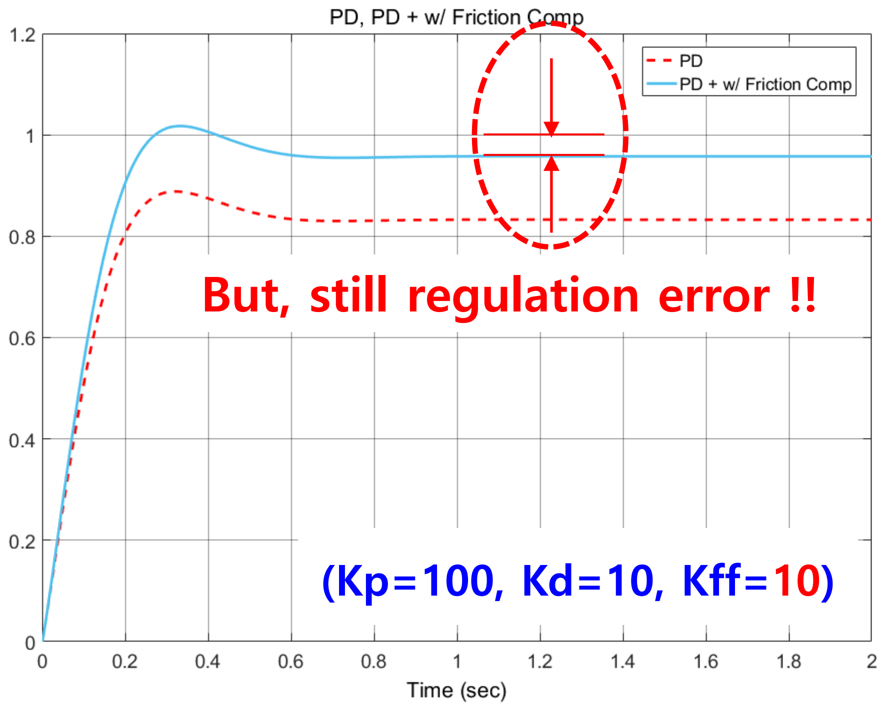
- Example 2: Spring-Damper-System w/ Friction (cont'd)



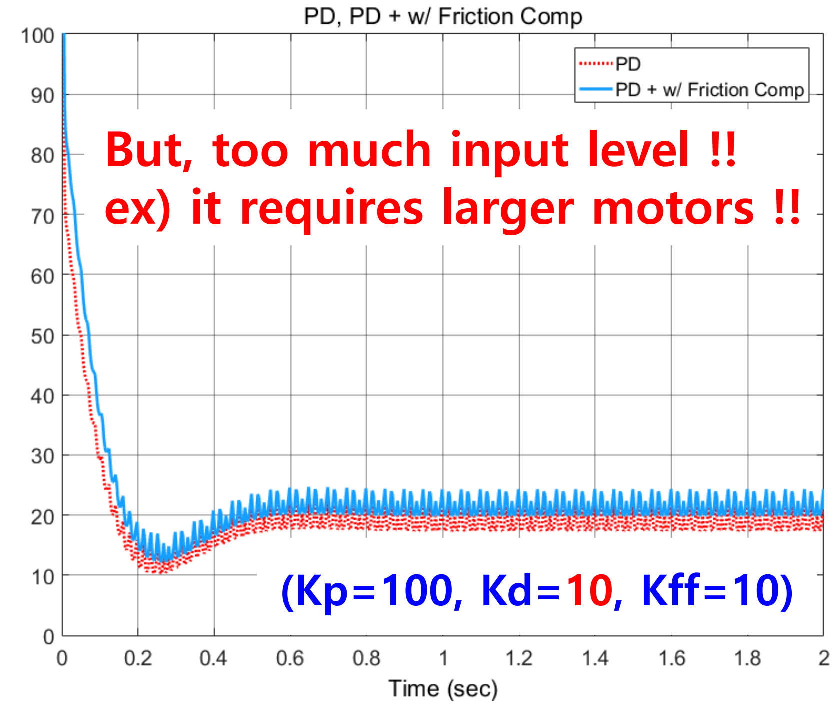
Feedforward Control (cont'd)

- **Example 2: Spring-Damper-System w/ Friction (cont'd)**
  - **Case 1: PD vs. PD + w/ Friction Compensator (Feedforwad)**

Simulation Result: Outputs



Simulation Result: Inputs

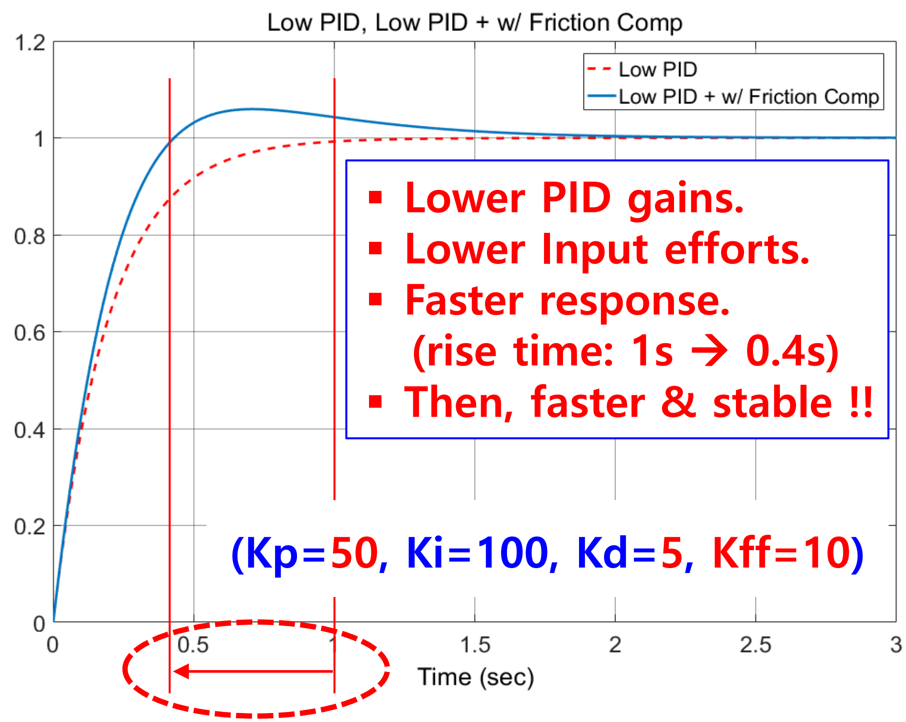


$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{de(t)}{dt}$$

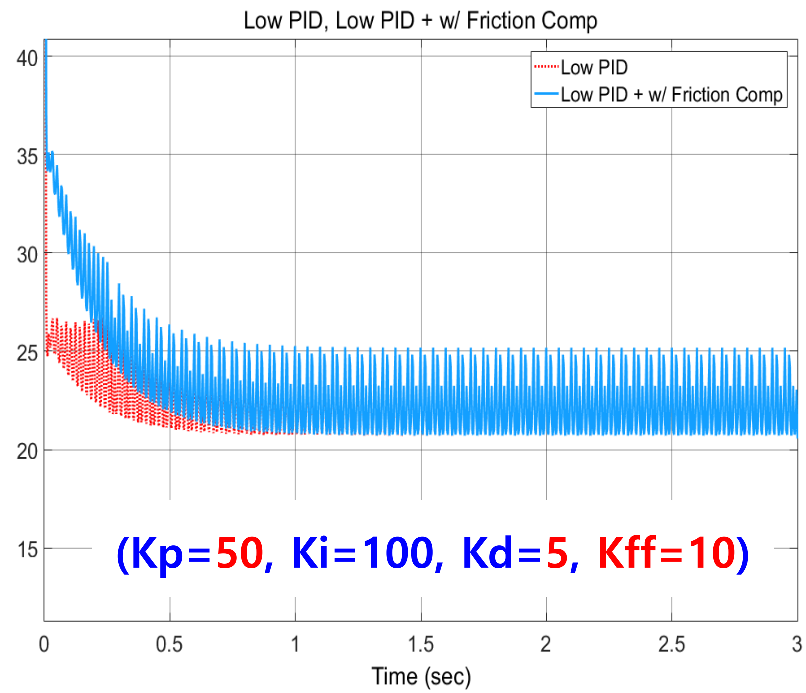
Feedforward Control (cont'd)

- **Example 2: Spring-Damper-System w/ Friction (cont'd)**
  - **Case 2: Low PID gains + w/ Friction Compensator (Feedforwad)**

Simulation Result: Outputs



Simulation Result: Inputs

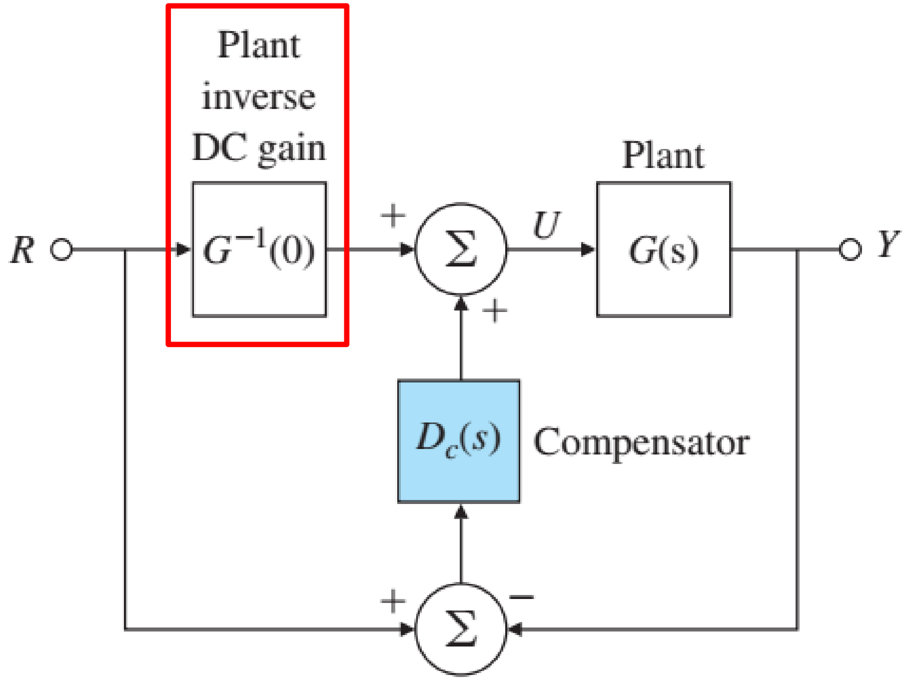


$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{de(t)}{dt}$$

Feedforward Control (II)

❖ Feedforward Control (or Plant Inverse DC gain)

- Integral control was to reduce errors to zero for steady disturbances or constant reference commands,
- But, integral typically decreases the damping or stability of system.
- Thus, feedforward (plant inverse DC gain) of the control that will eliminate the steady-state errors due to command inputs.



Other PID Tuning: Feedforward Control

❖ Feedforward Control (or Plant Inverse DC gain)

- Example: Feedforward Control for DC Motor

Assume DC motor model {G(s)} is $G(s) = \dfrac{1}{s^2 + 2s + 1}$

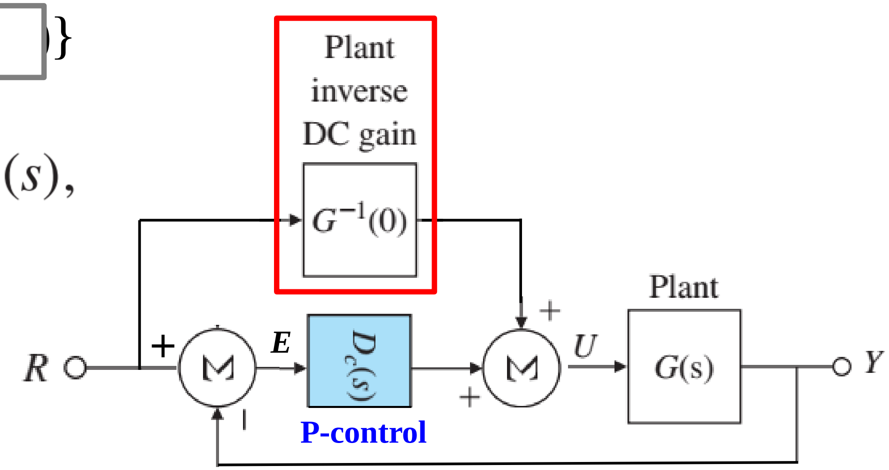
Then, we have the plant inverse DC gain is $\boxed{G^{-1}(0) = 1}$

Next, the closed-loop transfer function with P-control is,

$$Y(s) = G(s)\{\underline{D_c(s)E(s)} + \boxed{\quad\quad\quad}\}$$

$$E(s) = R(s) - Y(s),$$
$$D_c(s) = k_P$$

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{(1 + k_P)G(s)}{1 + k_P G(s)}.$$



The same as the previous figure !!

Other PID Tuning: Feedforward Control

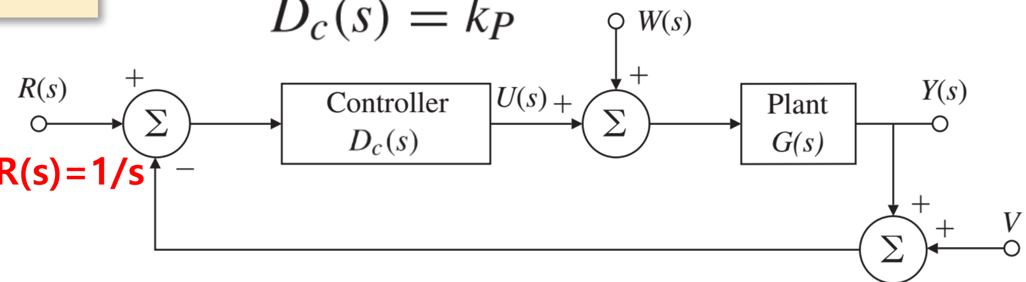
❖ Feedforward Control (or Plant Inverse DC gain)

- Example: Feedforward Control for DC Motor (cont'd)

P-control without Feedforward

$$D_c(s) = k_P$$

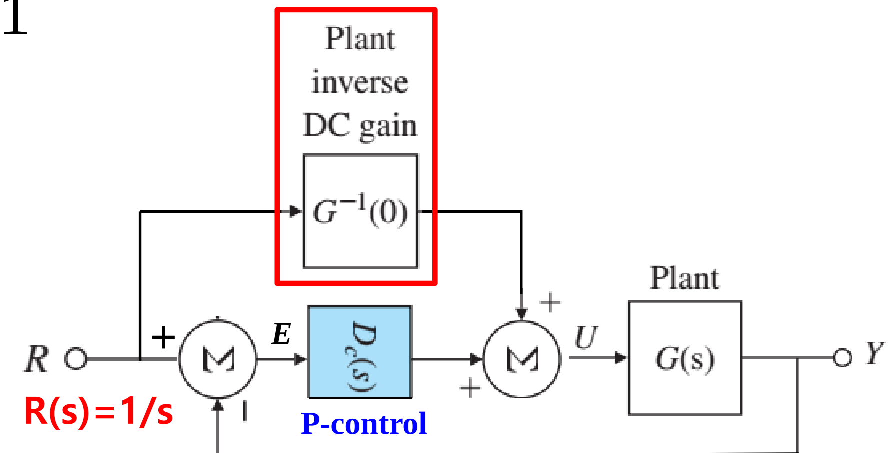
$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_p G(s)}{1 + k_p G(s)}$$



⇨ $T(0) \square 1 \quad where,\ G(0) = 1$

P-control with Feedforward

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{(1 + k_P)G(s)}{1 + k_P G(s)}.$$



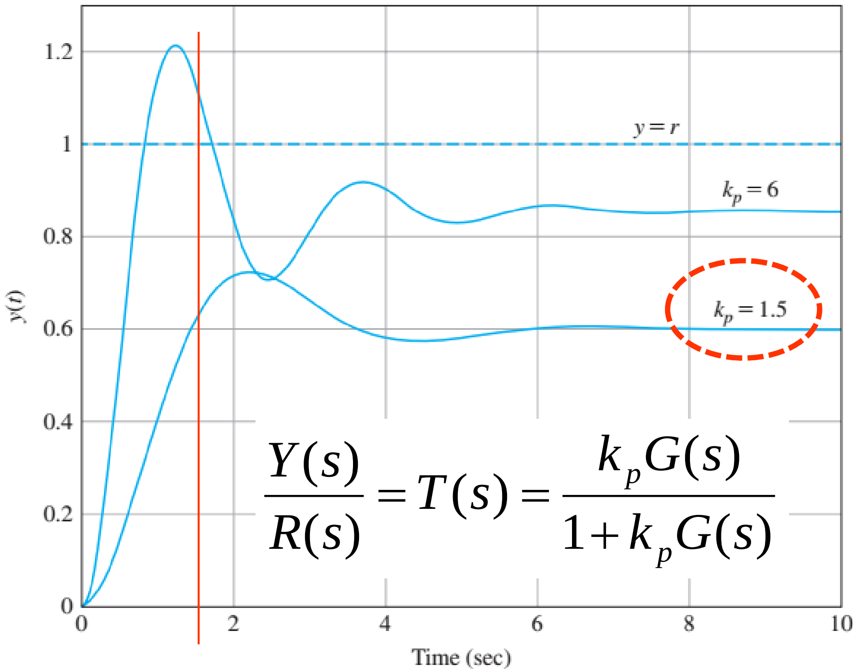
⇨ $T(0) \square 1 \quad where,\ G(0) = 1$

Other PID Tuning: Feedforward Control

❖ Feedforward Control (or Plant Inverse DC gain)

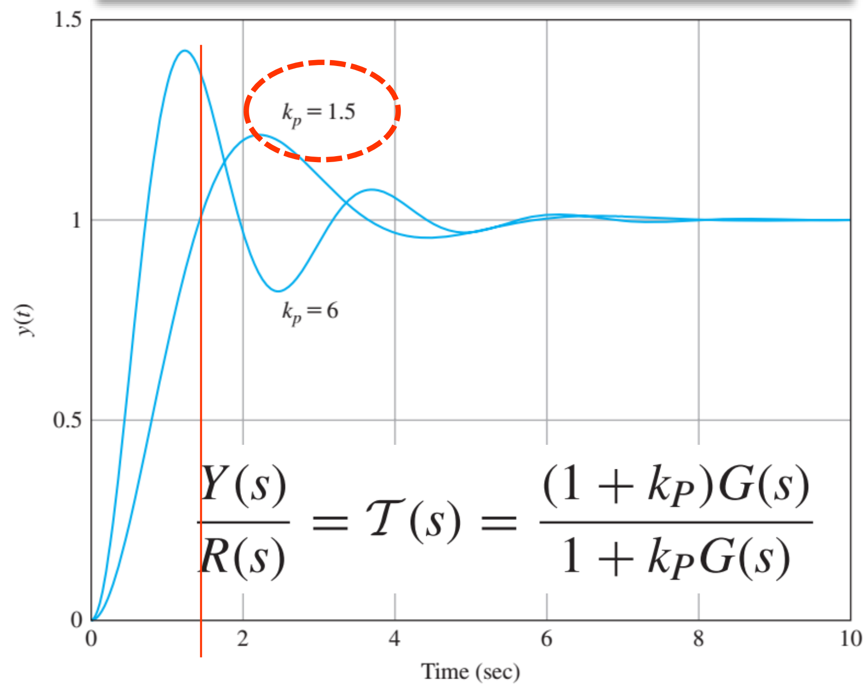
- **Example: Feedforward Control for DC Motor (cont'd)**

P-control without Feedforward



$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_p G(s)}{1 + k_p G(s)}$$

P-control with Feedforward



$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{(1 + k_P)G(s)}{1 + k_P G(s)}$$

- What is another benefit with feedforward control??
- But, in this method, what are some challenges in real world??

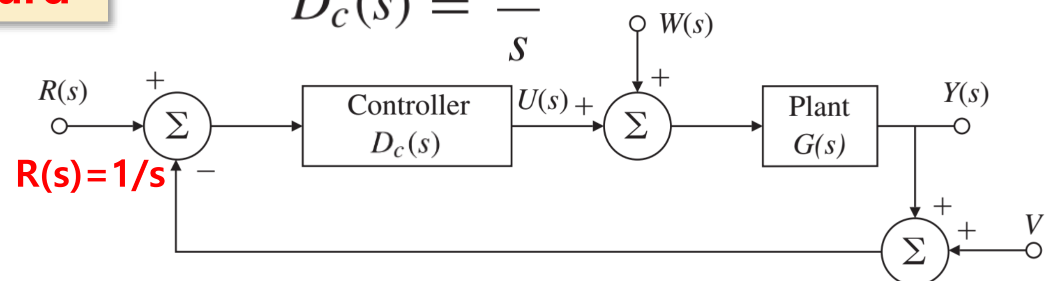
Other PID Tuning: Feedforward Control

❖ Feedforward Control (or Plant Inverse DC gain)

- Example: Feedforward Control for DC Motor (cont'd)

I-control without Feedforward

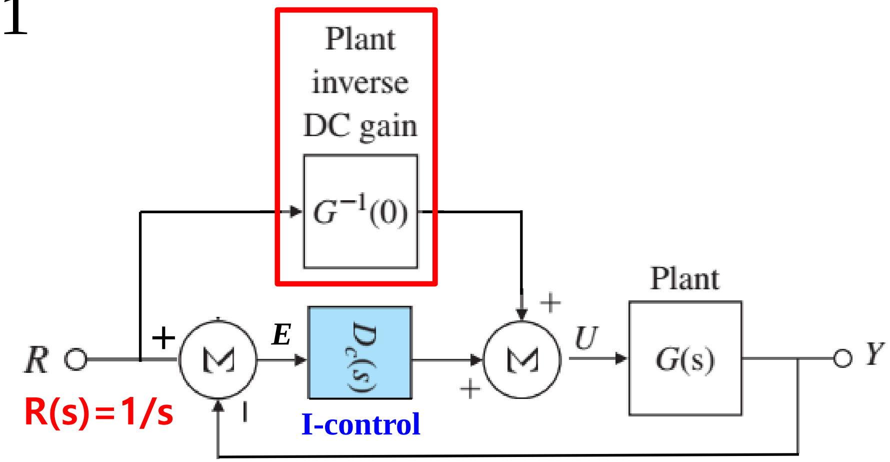
$$D_c(s) = \frac{k_I}{s}$$



$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_I G(s)}{s + k_I G(s)}$$

$$T(0)\,\square\,1 \quad where,\ G(0) = 1$$

I-control with Feedforward



$$\frac{Y(s)}{R(s)} = T(s) = \frac{(s + k_I)G(s)}{s + k_I G(s)}$$

$$T(0)\,\square\,1 \quad where,\ G(0) = 1$$

Summary

❖ **Summary**:

- Steady-state error with respect to system type
- Feedforward controller design method