Chapter 11
Direct Adaptive Control

11.1 Introduction

Direct adaptive control covers those schemes in which the parameters of the controller are directly updated from a signal error (adaptation error) reflecting the difference between attained and desired performance. Direct adaptive control schemes are generally obtained in two ways.

1. Define an equation for a signal error (adaptation error) which is a function of the difference between the tuned controller parameters and the current controller parameters. Use this adaptation error to generate a PAA for the controller parameters.
2. Use an indirect adaptive control approach with an adaptive predictor of the plant output reparameterized in terms of the controller parameters and force the output of the adaptive predictor to follow exactly the desired trajectory.

The second approach allows the direct adaptation of the parameters of the controller without solving an intermediate "design equation". As it will be shown the prediction error used in the PAA is in fact an image of the difference between the nominal and the attained performance because of closed-loop operation. Furthermore, the resulting schemes are governed by the same equations as those obtained by the first approach.

Although direct adaptive control is very appealing, it cannot be used for all types of plant model and control strategies. In fact, the situations where direct adaptive control can be used are limited. The typical use of this approach covers:

1. adaptive tracking and regulation with independent objectives (Sect. 11.2);
2. adaptive tracking and regulation with weighted input (Sect. 11.3);
3. adaptive minimum variance tracking and regulation (Sect. 11.4).

As will be shown, the basic hypothesis on the plant model is that, for any possible values of the parameters either

- *the finite zeros of the plant model are inside the unit circle* (this hypothesis is used for schemes 1 and 3),

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or

- *there exists a linear combination of the denominator and numerator of the plant transfer function which is an asymptotically stable polynomial* (this hypothesis is used by scheme 2).

For other direct adaptive control schemes dedicated to plant models with non-necessarily stable zeros see M'Saad et al. (1985).

It also has to be mentioned that even if the zeros of the plant model are asymptotically stable, it is not possible (or it becomes very complicated) to develop direct adaptive control schemes for pole placement, linear quadratic control or generalized predictive control. The reason is that it is not possible to obtain an adaptation error equation which is linear in the difference between the nominal and estimated controller parameters. For various attempts to develop direct adaptive control schemes for pole placement see Eliott (1980), Lozano and Landau (1982), Åström (1980).

At first glance, the derivation of PAA for direct adaptive control schemes looks quite close to the one used in recursive identification and adaptive prediction. The PAA structures considered in Chaps. 3 and 4 will be used as well as the synthesis and analysis tools presented in Chap. 3 (for deterministic environment) and Chap. 4 (for stochastic environment). The objective will be that the resulting adaptive control schemes achieve asymptotically the desired performance for the case of known parameters. However, the analysis of adaptive control schemes is much more involved than the analysis of recursive identification and adaptive predictor schemes since in this case the plant input (the control) will depend upon the parameters' estimates. Showing that the plant input and output remain bounded during the adaptation transient is one of the key issues of the analysis. In the analysis of the various schemes, it will be assumed that the unknown parameters are either constant or piece-wise constant and that their variations from one value to another is either slow or is subject to sparse step changes. In the stochastic case, in order to assure the convergence of the adaptive controller toward the optimal one, it is necessary to assume that the parameters are unknown but constant and therefore a decreasing adaptation gain will be used. The robustness aspects related to the violation of the basic hypotheses will be examined in detail in Sect. 11.5.

11.2 Adaptive Tracking and Regulation with Independent Objectives

11.2.1 Basic Design

The tracking and regulation with independent objectives for the case of known plant model parameters has been discussed in detail in Sect. 7.4. For the development of a direct adaptive control scheme, the time-domain design discussed in Sect. 7.4.2 is useful.

The plant model (with unknown parameters) is assumed to be described by:

$$\begin{aligned} A(q^{-1})y(t) &= q^{-d}B(q^{-1})u(t) \\ &= q^{-d-1}B^*(q^{-1})u(t) \end{aligned} \tag{11.1}$$

where $u(t)$ and $y(t)$ are the input and the output of the plant respectively.

In the case of known parameters, the objective is to find a control

$$u(t) = f_u[y(t), y(t-1), \dots, u(t-1), u(t-2), \dots]$$

such that

$$\varepsilon^0(t+d+1) = P(q^{-1})[y(t+d+1) - y^*(t+d+1)] = 0 \tag{11.2}$$

where $P(q^{-1})$ is an asymptotically stable polynomial defined by the designer. For the case of unknown plant model parameters the objective will be to find a control

$$u(t) = f_u[\hat{\theta}_c(t), y(t), y(t-1), \dots, u(t-1), u(t-2), \dots]$$

with $\{u(t)\}$ and $\{y(t)\}$ bounded, such that:

$$\lim_{t\to\infty} \varepsilon^0(t+d+1) = 0 \tag{11.3}$$

where $\hat{\theta}_c(t)$ denotes the current controller parameters estimates. Observe that $\varepsilon^0(t+d+1)$ is a measure of the discrepancy between the desired and achieved performance, and as such is a potential candidate to be the adaptation error. This quantity can be generated at each sample ($y^*(t+d+1)$ is known $d+1$ steps ahead).

The next step toward the design of an adaptive control scheme is to replace the fixed controller given in (7.107) or (7.111) by an adjustable controller. Since the performance error for a certain value of plant model parameters will be caused by the misalignment of the controller parameters values, it is natural to consider an adjustable controller which has the same structure as in the linear case with known plant parameters and where the fixed parameters will be replaced by adjustable ones. It will be the task of the adaptation algorithm to drive the controller parameters towards the values assuring the satisfaction of (11.3). Therefore, the control law in the adaptive case will be chosen as:

$$\hat{S}(t,q^{-1})u(t) + \hat{R}(t,q^{-1})y(t) = P(q^{-1})y^*(t+d+1) \tag{11.4}$$

where:

$$\hat{S}(t,q^{-1}) = \hat{s}_0(t) + \hat{s}_1(t)q^{-1} + \cdots + \hat{s}_{nS}(t)q^{-nS} = 1 + q^{-1}\hat{S}^*(t,q^{-1}) \tag{11.5}$$

$$\hat{R}(t,q^{-1}) = \hat{r}_0(t) + \hat{r}_1(t)q^{-1} + \cdots + \hat{r}_{n_R}(t)q^{-n_R} \tag{11.6}$$

which can be written alternatively as (see also (7.111)):

$$\hat{\theta}_C^T(t)\phi_C(t) = P(q^{-1})y^*(t+d+1) \tag{11.7}$$

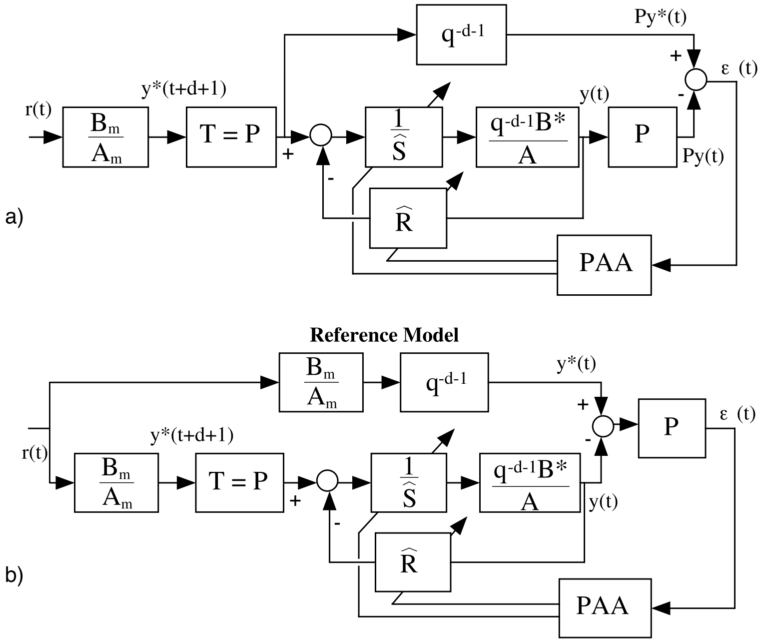


Fig. 11.1 Adaptive tracking and regulation with independent objective; (**a**) basic configuration, (**b**) a model reference adaptive control interpretation

where:

$$\hat{\theta}_C^T(t) = [\hat{s}_0(t), \ldots, \hat{s}_{n_S}(t), \hat{r}_0(t), \ldots, \hat{r}_{n_R}(t)]; \quad \hat{s}_0(t) = \hat{b}_1(t) \tag{11.8}$$

$$\phi_C^T(t) = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R)] \tag{11.9}$$

and the effective control input will be computed as:

$$u(t) = \frac{1}{\hat{s}_0(t)}[P(q^{-1})y^*(t+d+1) - \hat{S}^*(t, q^{-1})u(t-1) - \hat{R}(t, q^{-1})y(t)] \tag{11.10}$$

The choice made for the adaptation error (11.2) and the structure of the adjustable controller leads to the block diagram of the adaptive control system shown in Fig. 11.1a. Figure 11.1a allows a rapprochement with model reference adaptive control (MRAC) to be made, where the output of the reference model generates the quantity $P(q^{-1})y^*(t)$ which is compared to the filtered plant output $P(q^{-1})y(t)$. Alternatively, one can consider (as shown in Fig. 11.1b) that the adaptation error is defined by the difference between the desired trajectory $y^*(t)$ generated by the tracking reference model and the plant output $y(t)$ filtered by $P(q^{-1})$. As explained in Landau and Lozano (1981), $P(q^{-1})$ can be interpreted as a "reference model" for regulation.

Since we have selected a candidate for the adaptation error and a structure for the adjustable controller, the next step will be to give an expression for $\varepsilon^0(t+d+1)$

as a function of the controller parameters misalignment. This will allow one to see to what extent $\varepsilon^0(t+d+1)$ can be considered as an adaptation error to be used in a PAA of the forms discussed in Chap. 3. We note first that for the case of known parameters where the controller parameters are computed by solving the polynomial equation (7.91), one has $\varepsilon^0(t+d+1) \equiv 0$ and using (7.111), one can conclude that the filtered predicted plant output is given by:

$$\theta_C^T \phi_C(t) = P(q^{-1}) y(t+d+1) \tag{11.11}$$

since in the case of known parameters:

$$P(q^{-1}) y^*(t+d+1) = P(q^{-1}) y(t+d+1); \quad \forall t > 0 \tag{11.12}$$

where:

$$\theta_C^T = [s_0, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}] \tag{11.13}$$

defines the parameter vector of the tuned controller (unknown). Subtracting (11.7) which contains the parameters of the adjustable controller from (11.11), one obtains:

$$\begin{aligned}\varepsilon^0(t+d+1) &= P(q^{-1}) y(t+d+1) - P(q^{-1}) y^*(t+d+1) \\ &= [\theta_C - \hat{\theta}_C(t)]^T \phi_C(t)\end{aligned} \tag{11.14}$$

which has the desired form (linear in the parameter error).

For the PAA synthesis, we will use Theorem 3.2.

The case $d = 0$: One sees immediately that for $d = 0$, $\varepsilon^0(t+d+1) = \varepsilon^0(t+1)$ can be interpreted as an a priori adaptation error and one can associate an a posteriori adaptation error governed by:

$$\nu(t+1) = \varepsilon(t+1) = [\theta_C - \hat{\theta}_C(t+1)]^T \phi_C(t) \tag{11.15}$$

and Theorem 3.2 can simply be used to derive the PAA assuring

$$\lim_{t \to \infty} \varepsilon(t+1) = 0 \tag{11.16}$$

The case $d > 0$: As in the case of j-steps ahead adaptive predictors (see Sect. 6.2.1), one will associate to (11.14) an a posteriori adaptation error equation:

$$\varepsilon(t+d+1) = [\theta_C - \hat{\theta}_C(t+d+1)]^T \phi_C(t) \tag{11.17}$$

Equation (11.17) has the form of the a posteriori adaptation error equation considered in Theorem 3.2. Applying Theorem 3.2, the PAA to be used for assuring:

$$\lim_{t \to \infty} \varepsilon(t+d+1) = 0 \tag{11.18}$$

is:

$$\hat{\theta}_C(t+d+1) = \hat{\theta}_C(t+d) + F(t)\phi_C(t)\varepsilon(t+d+1) \tag{11.19}$$

$$F(t+1)^{-1} = \lambda_1(t)F(t)^{-1} + \lambda_2(t)\phi_C(t)\phi_C^T(t)$$
$$0 < \lambda_1(t) \leq 1;\ 0 \leq \lambda_2(t) < 2;\ F(0) > 0 \tag{11.20}$$

Notice that because of the form of (11.15) and (11.17), the positive real condition of Theorem 3.2 is automatically satisfied. To make the above PAA implementable, we have to now give an expression of $\varepsilon(t+d+1)$ in terms of the a priori adaptation error $\varepsilon^0(t+d+1)$ and the parameters $\hat{\theta}(t+i)$ up to and including $i=d$. As with the direct adaptive prediction, (6.18), one can rewrite (11.17) as:[1]

$$\begin{aligned}\varepsilon(t+d+1) &= \varepsilon^0(t+d+1) + [\hat{\theta}_C(t) - \hat{\theta}_C(t+d+1)]^T\phi_C(t)\\ &= \varepsilon^0(t+d+1) - \phi_C^T(t)F(t)\phi_C(t)\varepsilon(t+d+1)\\ &\quad + [\hat{\theta}_C(t) - \hat{\theta}_C(t+d)]^T\phi_C(t)\end{aligned} \tag{11.21}$$

from which one obtains:

$$\varepsilon(t+d+1) = \frac{\varepsilon^0(t+d+1) - [\hat{\theta}_C(t+d) - \hat{\theta}_C(t)]^T\phi_C(t)}{1+\phi_C^T(t)F(t)\phi_C(t)} \tag{11.22}$$

which can also be expressed as:

$$\varepsilon(t+d+1) = \frac{Py(t+d+1) - \hat{\theta}_C^T(t+d)\phi_C(t)}{1+\phi_C^T(t)F(t)\phi_C(t)} \tag{11.23}$$

Analysis

The fact that the PAA of (11.19), (11.20) and (11.22) assures that (11.18) holds, does not guarantee that the objective of the adaptive control scheme defined by (11.3) will be achieved (i.e., the a priori adaptation error should goes to zero) and that $\{u(t)\}$ and $\{y(t)\}$ will be bounded. This is related to the boundedness of $\phi_C(t)$. Effectively from (11.22), one can see that $\varepsilon(t+d+1)$ can go to zero without $\varepsilon^0(t+d+1)$ going to zero if $\phi_C(t)$ becomes unbounded. Since $\phi_C(t)$ contains the input and the output of the plant, it is necessary to show that the PAA achieves the objective of (11.3) while assuring that $\phi_C(t)$ remains bounded for all t.

An important preliminary remark is that the plant model (11.1) is a difference equation with constant coefficients and therefore $y(t+1)$ cannot become unbounded for finite t if $u(t)$ is bounded. Therefore if $u(t)$ is bounded, $y(t)$ and respectively $\phi_C(t)$ can become unbounded only asymptotically.

Bearing in mind the form of (11.10), it is also clear that in order to avoid $u(t)$ becoming unbounded the adjustable parameters must be bounded and $\hat{s}_0(t) =$

[1]The term $[\hat{\theta}_C(t) - \hat{\theta}_C(t+d+1)]^T\phi_C(t)$ is sometimes termed "auxiliary error" and $\varepsilon(t+d+1)$ is sometimes termed "augmented error" in MRAC literature.

$\hat{b}_1(t) \neq 0$ for all t. This will assure $\phi_C(t)$ can become eventually unbounded only asymptotically.

In order to assure that $\hat{b}_1(t) \neq 0$ for any t, knowing that b_1 cannot be zero and its sign is assumed to be known, one fixes a minimum value $|\hat{b}_1(t)| \geq \delta > 0$. If $|\hat{b}_1(t)| < \delta$ for some t, one either simply uses the value δ (with the appropriate sign), or one takes advantage of the weighting sequences $\lambda_1(t), \lambda_2(t)$ and recomputes $\hat{\theta}_C(t)$ for different values of $\lambda_1(t-d-1), \lambda_2(t-d-1)$ (for example: $\lambda_1' = \lambda_1 + \Delta\lambda_1, \lambda_2' = \lambda_2 + \Delta\lambda_2$) such that $|\hat{b}_1(t)| > \delta$.

After these preliminary considerations, the next step is to show that (11.3) is true and $\phi_C(t)$ is bounded. Two approaches are possible:

1. Use a "bounded growth" lemma which will allow to straightforwardly conclude.
2. Derive a dynamic equation for $\phi_C(t)$ and use it for boundedness analysis (Landau and Lozano 1981).

We will use the first approach by outlining the "bounded growth" lemma (Goodwin and Sin 1984).

Lemma 11.1

(1) *Assume that*:

$$\|\phi(t)\|_{F(t)} = [\phi^T(t)F(t)\phi(t)]^{1/2} \leq C_1 + C_2 \max_{0\leq k\leq t+d+1} |\varepsilon^0(k)|$$
$$0 < C_1,\ C_2 < \infty,\ F(t) > 0 \tag{11.24}$$

(2) *Assume that*:

$$\lim_{t\to\infty} \frac{[\varepsilon^0(t+d+1)]^2}{1 + \|\phi(t)\|^2_{F(t)}} = 0 \tag{11.25}$$

then:

$$\|\phi(t)\| \textit{ is bounded} \tag{11.26}$$

$$\lim_{t\to\infty} \varepsilon^0(t+d+1) = 0 \tag{11.27}$$

Proof The proof is trivial if $\varepsilon^0(t)$ is bounded for all t. Assume now that $\varepsilon^0(t+d+1)$ is asymptotically unbounded. Then there is a particular subsequence such that: $\lim_{t_n\to\infty} \varepsilon^0(t_n) = \infty$ and $|\varepsilon^0(t+d+1)| < |\varepsilon^0(t_n)|;\ t+d+1 \leq t_n$. For this sequence one has:

$$\frac{|\varepsilon^0(t+d+1)|}{(1 + \|\phi(t)\|^2_F)^{1/2}} \geq \frac{\varepsilon^0(t+d+1)}{1 + \|\phi(t)\|_{F(t)}} \geq \frac{|\varepsilon^0(t+d+1)|}{1 + C_1 + C_2|\varepsilon^0(t+d+1)|}$$

but:

$$\lim_{t\to\infty} \frac{|\varepsilon^0(t+d+1)|}{1 + C_1 + C_2|\varepsilon^0(t+d+1)|} = \frac{1}{C_2} > 0$$

which contradicts Assumption 2 and proves that neither $\varepsilon^0(t+d+1)$ nor $\|\phi(t)\|$ can become unbounded and that (11.27) is true. □

The application of this lemma is straightforward in our case. From (11.12) one has:

$$P(q^{-1})y(t+d+1) = P(q^{-1})y^*(t+d+1) + \varepsilon^0(t+d+1) \tag{11.28}$$

Since $P(q^{-1})$ has bounded coefficients and y^* is bounded, one gets immediately from (11.28) that:

$$|y(t)| \leq C_1' + C_2' \max_{0\leq k\leq t+d+1} |\varepsilon^0(k)| \tag{11.29}$$

Using the assumption that $B^*(z^{-1})$ has all its zeros inside the unit circle, the inverse of the system is asymptotically stable and one has:

$$|u(t)| \leq C_1'' + C_2'' \max_{0\leq k\leq t+d+1} |y(t+d+1)| \tag{11.30}$$

From (11.29) and (11.30), one concludes that:

$$\|\phi_C(t)\|_{F(t)} \leq C_1 + C_2 \max_{0\leq k\leq t+d+1} |\varepsilon^0(k)| \tag{11.31}$$

and, on the other hand, from Theorem 3.2 one has (11.25). Therefore the assumptions of Lemma 11.1 are satisfied allowing to conclude that (11.3) is satisfied and that $\{u(t)\}$ and $\{y(t)\}$ are bounded. The results of the previous analysis can be summarized under the following form:

Theorem 11.1 *Consider a plant model of the form* (11.1) *controlled by the adjustable controller* (11.4) *whose parameters are updated by the PAA of* (11.19), (11.20) *and* (11.23) *where*:

$$\varepsilon^0(t+d+1) = Py(t+d+1) - Py^*(t+d+1)$$

Assume that:

(1) *The integer delay d is known.*
(2) *Upper bounds on the degrees of the polynomials A and B^* are known.*
(3) *For all possible values of the plant parameters, the polynomial B^* has all its zeros inside the unit circle.*
(4) *The sign of b_1 is known.*

Then:

- $\lim_{t\to\infty} \varepsilon^0(t+d+1) = 0$;
- *The sequences $\{u(t)\}$ and $\{y(t)\}$ are bounded.*

Remarks

- Integral + Proportional adaptation can also be used instead of the integral PAA considered in (11.19) and (11.20) (see Sect. 3.3.4, (3.270) through (3.274)).
- Various choices can be made for $\lambda_1(t)$ and $\lambda_2(t)$ as indicated in Sect. 3.2.3. This will influence the properties and the performances of the scheme. In particular for $\lambda_1(t) \equiv 1$ and $\lambda_2(t) > 0$, one will obtain a PAA with decreasing adaptation gain. The *constant trace algorithm* is probably the most used algorithm for obtaining an adaptive control scheme which is continuously active.
- Monitoring the eigenvalues of $F(t)$ is recommended in order to assure $0 < \delta < \|F(t)\| < \infty$ for all t.
- The choice of $P(q^{-1})$ which influences the robustness of the linear design also influences the adaptation transients. Taking $P(q^{-1}) = 1$, one gets oscillatory adaptation transients (and the linear design will be very sensitive to parameters change) while $P(q^{-1})$ in the range of the band pass of the plant will both improve the robustness of the linear design and the smoothness of the adaptation transient (Landau and Lozano 1981). See Sect. 11.6 for details.

Alternative Direct Adaptive Control Design via Adaptive Prediction

The basic idea is to use an indirect adaptive control approach as follows:

Step 1: Design an adaptive predictor for the filtered plant output.
Step 2: Force the output of the adaptive predictor to be equal to the desired filtered trajectory.

Remember that using the time-domain design discussed in Sect. 7.4.2, one has the $d+1$ steps ahead predictor given by:

$$P(q^{-1})\hat{y}(t+d+1) = F(q^{-1})y(t) + G(q^{-1})u(t) = \theta_C^T \phi_C(t) \tag{11.32}$$

which has the property that:

$$[P(q^{-1})y(t+d+1) - P(q^{-1})\hat{y}(t+d+1)] = 0 \tag{11.33}$$

Forcing the output of the prediction to be:

$$P(q^{-1})\hat{y}(t+d+1) = P(q^{-1})y^*(t+d+1) = \theta_C^T \phi_C(t) \tag{11.34}$$

one gets the desired result. In the case of the unknown plant parameters, we will use a similar procedure but replacing the linear predictor with fixed coefficient by an adaptive predictor.

Step 1: Design of an adaptive predictor. Using the results of Sect. 6.2.1, one has:

$$P(q^{-1})\hat{y}^0(t+d+1) = \hat{F}(t)y(t) + \hat{G}(t)u(t) = \hat{\theta}_C^T(t)\phi_C(t) \tag{11.35}$$

which has the property that:

$$\lim_{t\to\infty} \varepsilon^0(t+d+1) = \lim_{t\to\infty} [P(q^{-1})y(t+d+1) - P(q^{-1})\hat{y}^0(t+d+1)] = 0$$

when using the PAA of (6.16), (6.17) and (6.19) (assuming that $u(t)$ and $y(t)$ are bounded).

Step 2: Use the ad-hoc separation theorem and compute a control such that the output of the adaptive predictor follows the desired filtered trajectory, i.e.:

$$P(q^{-1})\hat{y}^0(t+d+1) = \hat{\theta}_C^T(t)\phi_C(t) = P(q^{-1})y^*(t+d+1) \tag{11.36}$$

Notice that the control resulting from (11.36) has exactly the same structure as the one given by (11.7).

Furthermore, using (11.36), the equation of the prediction error becomes:

$$\begin{aligned}\varepsilon^0(t+d+1) &= [P(q^{-1})y(t+d+1) - P(q^{-1})y^*(t+d+1)]\\ &= [\theta_C - \hat{\theta}_C(t)]^T\phi_C(t)\end{aligned} \tag{11.37}$$

In other terms, by the choice of the control law, the prediction error becomes the measure of the control performance error and the equation of the prediction error in terms of controller parameter difference is exactly the same as (11.14). One concludes that both schemes are identical and that in the adaptive predictor, one estimates the controller parameters.

Note also that with this particular strategy, the output of the adjustable predictor at each t is equal to the filtered desired trajectory, i.e., it behaves exactly as the filtered value of the reference model used in Fig. 11.1. For this reason, this type of direct adaptive control design was called in the past "implicit model reference adaptive control" (Landau 1981; Landau and Lozano 1981).

Note that this scheme, despite a formal similarity with adaptive predictors, does not operate in open loop and as a consequence $\phi_C(t)$ is not independent of $\hat{\theta}(t), \hat{\theta}(t-1), \ldots$ as for the scheme discussed in Sect. 6.1. This explains why a complementary specific analysis has to be considered. However, since this scheme is equivalent to the previous one, Theorem 11.1 holds.

11.2.2 Extensions of the Design

Filtering of the Measurement Vector and of the Adaptation Error

In order to guarantee certain convergence conditions (which occur for example when using this scheme in a stochastic environment) or to filter the measurement vector $\phi_C(t)$ outside the closed-loop bandpass (for robustness improvement as well as for finding various other designs considered in the literature), it is useful to consider a more general PAA which uses a filtered measurement vector as observation vector:

$$L(q^{-1})\phi_{Cf}(t) = \phi_C(t) \tag{11.38}$$

where:

$$L(q^{-1}) = 1 + l_1 q^{-1} + \cdots + l_{n_L} q^{-n_L} \tag{11.39}$$

is an asymptotically stable monic polynomial and:

$$\begin{aligned}\phi_{Cf}^T(t) &= \frac{1}{L(q^{-1})}[u(t), \ldots, u(t-n_B-d+1), y(t), \ldots, y(t-n+1)] \\ &= [u_f(t), \ldots, u_f(t-n_B-d+1), y_f(t), \ldots, y_f(t-n+1)]\end{aligned} \tag{11.40}$$

Notice first that in the linear case with known parameters, one can alternatively write the controller equation as:

$$u(t) = L(q^{-1})u_f(t) \tag{11.41}$$

where $u_f(t)$ satisfies the equation:

$$P(q^{-1})y_f^*(t+d+1) = Ry_f(t) + Su_f(t) = \theta_C^T \phi_{Cf}(t) \tag{11.42}$$

in which:

$$L(q^{-1})y_f^*(t+d+1) = y^*(t+d+1) \tag{11.43}$$

The natural extension for the adaptive case is to replace (11.42) by:

$$P(q^{-1})y_f^*(t+d+1) = \hat{\theta}_C^T(t)\phi_{Cf}(t) \tag{11.44}$$

We will now give an expression for the adaptation error. From (11.11) one has:

$$P(q^{-1})y(t+d+1) = \theta_C^T \phi_C(t) \tag{11.45}$$

Remembering that θ_C is a constant vector one can write (11.45) as:

$$P(q^{-1})y(t+d+1) = L(q^{-1})[\theta_C^T \phi_{Cf}(t)] \tag{11.46}$$

On the other hand, from (11.44), one obtains:

$$P(q^{-1})y^*(t+d+1) = L(q^{-1})[\hat{\theta}_C^T(t)\phi_{Cf}(t)] \tag{11.47}$$

Therefore, the error between the desired filtered trajectory and the achieved filtered output is given by:

$$\begin{aligned}\varepsilon^0(t+d+1) &= Py(t+d+1) - Py^*(t+d+1) \\ &= L(q^{-1})[\theta_C - \hat{\theta}_C(t)]^T \phi_{Cf}(t)\end{aligned} \tag{11.48}$$

The associated a posteriori error equation is defined as:

$$\varepsilon(t+d+1) = L(q^{-1})[\theta_C - \hat{\theta}_C(t+d+1)]^T \phi_{Cf}(t) \tag{11.49}$$

Defining the a posteriori adaptation error as:

$$\nu(t+d+1) = \frac{H_1(q^{-1})}{H_2(q^{-1})}\varepsilon(t+d+1) \tag{11.50}$$

where $H_1(q^{-1})$ and $H_2(q^{-1})$ are asymptotically stable monic polynomials defined as:

$$H_j(q^{-1}) = 1 + (q^{-1})H_j^*(q^{-1}); \quad j = 1, 2 \tag{11.51}$$

one gets:

$$\nu(t+d+1) = H(q^{-1})[\theta_C - \hat{\theta}_C(t+d+1)]^T \phi_{Cf}(t) \tag{11.52}$$

where:

$$H(q^{-1}) = \frac{H_1(q^{-1})L(q^{-1})}{H_2(q^{-1})} \tag{11.53}$$

Applying Theorem 3.2,

$$\lim_{t\to\infty} \nu(t+d+1) = 0 \tag{11.54}$$

will be assured using the PAA

$$\hat{\theta}(t+d+1) = \hat{\theta}(t+d) + F(t)\phi_{Cf}(t)\nu(t+d+1) \tag{11.55}$$

$$F^{-1}(t+1) = \lambda_1(t)F^{-1}(t) + \lambda_2(t)\phi_{Cf}(t)\phi_{Cf}^T(t)$$

$$0 < \lambda_1(t) \le 1;\ 0 \le \lambda_2(t) < 2;\ F(0) > 0 \tag{11.56}$$

provided that there is $2 > \lambda \ge \max_t \lambda_2(t)$ such that:

$$H'(z^{-1}) = H(z^{-1}) - \frac{\lambda}{2} \tag{11.57}$$

is strictly positive real.

It remains to give an expression for $\nu(t+d+1)$ in terms of $\varepsilon^0(t+d+1)$ and $\hat{\theta}(t+i)$ for i up to and including d. Using similar developments as in Sect. 3.3, one has from (11.50):

$$\nu(t+d+1) = \varepsilon(t+d+1) + H_1^*\varepsilon(t+d) - H_2^*\nu(t+d)$$

but:

$$\begin{aligned}\varepsilon(t+d+1) = \varepsilon^0(t+d+1) &- L^*(q^{-1})[\hat{\theta}_C(t+d) - \hat{\theta}_C(t+d-1)]^T \phi_{Cf}(t-1)\\ &- L(q^{-1})[\hat{\theta}_C(t+d) - \hat{\theta}_C(t)]^T \phi_{Cf}(t)\\ &- \phi_{Cf}^T(t)F(t)\phi_{Cf}^T \nu(t+d+1)\end{aligned}$$

from which one obtains:

$$\nu(t+d+1) = \frac{\nu^0(t+d+1)}{1+\phi_{Cf}^T(t)F(t)\phi_{Cf}(t)} \tag{11.58}$$

where:

$$\begin{aligned}
&\nu^0(t+d+1) \\
&\quad = \varepsilon^0(t+d+1) - L^*(q^{-1})[\hat{\theta}_C(t+d) - \hat{\theta}_C(t+d-1)]^T \phi_{Cf}(t-1) \\
&\qquad - L(q^{-1})[\hat{\theta}_C(t+d) - \hat{\theta}_C(t)]^T \phi_{Cf}(t) + H_1^*\varepsilon(t+d) - H_2^*\nu(t+d) \\
&\quad = \varepsilon'(t+d+1) - L^*(q^{-1})[\hat{\theta}_C(t+d) - \hat{\theta}_C(t+d-1)]^T \phi_{Cf}(t-1) \\
&\qquad + H_1^*\varepsilon(t+d) - H_2^*\nu(t+d)
\end{aligned} \tag{11.59}$$

with:

$$\varepsilon'(t+d+1) = P(q^{-1})y(t+d+1) - L(q^{-1})\hat{\theta}_C^T(t+d)\phi_{Cf}(t) \tag{11.60}$$

and:

$$\begin{aligned}
\varepsilon(t+d) &= L(q^{-1})[\theta - \hat{\theta}(t+d)]^T \phi_{Cf}(t-1) \\
&= Py^*(t+d) - L(q^{-1})\hat{\theta}_C(t+d)\phi_{Cf}(t-1)
\end{aligned} \tag{11.61}$$

Taking into Account Measurable Disturbances

In a number of applications, measurable disturbances act upon the output of the process through unknown dynamics. The knowledge of these dynamics would be useful for compensating the effect of the disturbance by using an appropriate controller. This problem has been discussed for the case of known parameters in Sect. 7.4.2.

The plant output is described by (7.130):

$$A(q^{-1})y(t+d+1) = B^*(q^{-1})u(t) + C^*(q^{-1})v(t) \tag{11.62}$$

where $v(t)$ is the measurable disturbance and $C^*(q^{-1})$ is a polynomial of order $n_c - 1$.

In the case of unknown parameters, one uses the same controller structure as in (7.134) but with adjustable parameters, i.e.:

$$P(q^{-1})y^*(t+d+1) = \hat{\theta}_C^T(t)\phi_C(t) \tag{11.63}$$

where:

$$\hat{\theta}_C^T(t) = [\hat{s}_0(t), \ldots, \hat{s}_{n_S}(t), \hat{r}_0(t), \ldots, \hat{r}_{n_R}(t), \hat{w}_0(t), \ldots, \hat{w}_{n_W}(t)] \tag{11.64}$$

$$\phi_C^T(t) = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R), v(t), \ldots, v(t-n_W)] \tag{11.65}$$

Then one proceeds exactly as for the case without measurable disturbances. The PAA will be given by (11.19), (11.20) and (11.23) with the remark that in this case $\hat{\theta}_C(t)$ and $\phi_C^T(t)$ will be of higher dimension.

11.3 Adaptive Tracking and Regulation with Weighted Input

To develop an adaptive version of this control strategy presented in Sect. 7.5, we will proceed in a similar way as for adaptive tracking and regulation with independent objectives. In Sect. 7.5, it was pointed out that the objective in the case of known parameters is that:

$$\begin{aligned}\varepsilon^0(t+d+1) &= P(q^{-1})y(t+d+1) + \lambda Q(q^{-1})u(t) - P(q^{-1})y^*(t+d+1)\\ &= 0; \quad \forall t > 0\end{aligned} \tag{11.66}$$

In the case of unknown parameters, the objective will be:

$$\lim_{t\to\infty} \varepsilon^0(t+d+1) = \lim_{t\to\infty} [\bar{y}(t+d+1) - P(q^{-1})y^*(t+d+1)] = 0 \tag{11.67}$$

where:

$$\bar{y}(t+d+1) = Py(t+d+1) + \lambda Qu(t) \tag{11.68}$$

defines a so called "augmented output" or "generalized output".

The adjustable controller will have the same structure as the one given for the case of known parameters, but with adjustable parameters. Therefore from (7.143), one chooses the following adjustable controller:

$$Py^*(t+d+1) - \lambda Qu(t) = \hat{S}(t)u(t) + \hat{R}(t)y(t) = \hat{\theta}_C^T(t)\phi_C(t) \tag{11.69}$$

where:

$$\hat{\theta}_C^T(t) = [s_0, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}] \tag{11.70}$$

$$\phi_C(t) = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R)] \tag{11.71}$$

On the other hand (see Sect. 11.2):

$$P(q^{-1})y(t+d+1) = \theta_C^T \phi_C(t) \tag{11.72}$$

Combining (11.66), (11.69) and (11.72), one gets:

$$\varepsilon^0(t+d+1) = [\theta_C - \hat{\theta}_C(t)]^T \phi_C(t) \tag{11.73}$$

which allows to use the same PAA as for adaptive tracking and regulation ((11.19), (11.20) and (11.22)) where $\varepsilon^0(t+d+1) = \bar{y}(t+d+1) - Py^*(t+d+1)$. This will assure that the a posteriori adaptation error:

$$\varepsilon(t+d+1) = [\theta_C - \hat{\theta}_C(t+d+1)]^T \phi_C(t) \tag{11.74}$$

goes asymptotically to zero. It remains to check that using the PAA of (11.19), (11.20) and (11.23), (11.67) will also be satisfied and that $\{u(t)\}$ and $\{y(t)\}$ will be bounded. From (11.67), one has that:

$$|\bar{y}(t+d+1)| \leq C_1' + C_2' \max_{0\leq k\leq t+d+1} |\varepsilon^0(k)| \tag{11.75}$$

On the other hand, $\bar{y}(t+d+1)$ can be expressed as the output of an "augmented plant":

$$\bar{y}(t+d+1) = \frac{\lambda QA + B^*P}{A} u(t) \tag{11.76}$$

From this expression, one concludes that if $\lambda QA + B^*P$ is asymptotically stable, for $\bar{y}(t+d+1)$ bounded we will have a bounded $u(t)$ as well as $y(t)$ (as a consequence of (11.67)). Using now the "bounded growth" lemma (Lemma 11.1), one concludes that (11.67) is true and that $\{y(t)\}$ and $\{u(t)\}$ are bounded. This analysis can be summarized as:

Theorem 11.2 *Consider a plant model of the form* (11.1) *with not necessarily stable zeros, controlled by the adjustable controller* (11.69) *whose parameters are updated by the PAA of* (11.19), (11.20) *and* (11.23) *where*:

$$\varepsilon^0(t+d+1) = P(q^{-1})y(t+d+1) + \lambda Q(q^{-1})u(t) - P(q^{-1})y^*(t+d+1)$$

Assume that:

(1) *The integer delay d is known.*
(2) *Upper bounds on the degrees of the polynomial A and B^* are known.*
(3) *For all possible values of the plant parameters, the polynomial $(\lambda QA + B^*P)$ has all its zeros inside the unit circle where λ, Q and P are fixed and chosen by the designer.*
(4) *The sign of b_1 is known.*

Then:

- $\lim_{t\to\infty} \varepsilon^0(t+d+1) = 0$;
- *the sequences $\{u(t)\}$ and $\{y(t)\}$ are bounded.*

Remarks

- The resulting closed-loop poles are defined by the polynomial $\lambda AQ + B^*P$ and they will depend upon A and B^*. Therefore the location of the closed loop poles cannot be guaranteed.
- The hypothesis 3 is crucial. If it is not satisfied, $\varepsilon^0(t+d+1)$ can go to zero but $u(t)$ and $y(t)$ become unbounded.

11.4 Adaptive Minimum Variance Tracking and Regulation

The minimum variance tracking and regulation control algorithm for the case of known plant and disturbance model parameters is presented in Sect. 7.6. The objective of this section is to present the direct adaptive version of this control strategy for the case of unknown plant and disturbance model parameters, to analyze it and to examine several important special cases and in particular the case of adaptive minimum variance regulation (Åström and Wittenmark 1973).

As in the case of parameter identification and adaptive prediction in stochastic environment, due to the presence of noise, one typically uses decreasing adaptation gain in order to have convergence of the estimated parameters toward constant values. The resulting schemes are currently called *self-tuning minimum variance tracking and regulation*.

In the case of known parameters, the strong relationship between tracking and regulation with independent objectives and minimum variance tracking and regulation was pointed out. This strong resemblance carry over in the adaptive case, where the adaptive minimum variance tracking and regulation can be viewed as the stochastic version of tracking and regulation with independent objectives (called also stochastic model reference adaptive control) (Landau 1981, 1982a; Dugard et al. 1982).

As with the development of the adaptive tracking and regulation with independent objectives scheme, one can consider two approaches for deriving the direct adaptive version of the minimum variance tracking and regulation

(1) Use the difference between the plant output $y(t)$ and the reference trajectory $y^*(t)$ as an adaptation error and find an adaptation mechanism for the controller parameters which drive this quantity asymptotically (in stochastic sense) towards the optimal value obtained in the case of known parameters. For the case $d = 0$, this can be expressed for example as:

$$\text{Prob}\left\{\lim_{t\to\infty} \varepsilon^0(t+1) = \lim_{t\to\infty} [y(t+1) - y^*(t+1)] = e(t+1)\right\} = 1$$

where $e(t+1)$ is a white noise sequence, or:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\varepsilon^0(t+1) - e(t+1)]^2 = 0 \quad \text{a.s.}$$

(2) Use an indirect adaptive approach by building an adaptive predictor reparameterized in terms of the controller parameters and force the predictor output to follow exactly the reference trajectory by an appropriate choice of the control. (This corresponds to the use of the ad-hoc separation theorem.)

Both approaches lead to the same scheme. The corresponding block diagram of adaptive minimum variance tracking and regulation is shown in Fig. 11.2. The major difference with regard to adaptive tracking and regulation with independent

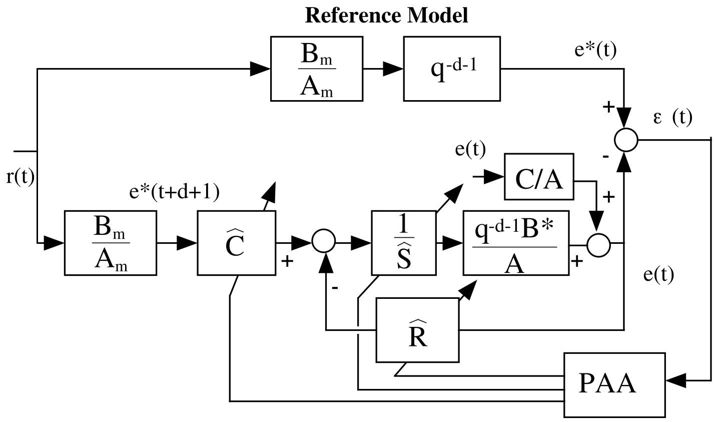


Fig. 11.2 Adaptive minimum variance tracking and regulation

objectives is the fact that the closed-loop poles and the parameters of the pre-compensator T, depend on the noise model (polynomial C) which is unknown and should be estimated. Therefore with respect to the deterministic case, there are more parameters to be adapted.

We will use the second approach for explicitly obtaining the algorithms. We will start with the case $d = 0$ and then extend the algorithms to the case $d \geq 1$. (For the use of the first approach see Landau 1981.) We will proceed then to the analysis of the asymptotic properties of the algorithms using the averaging method presented in Chap. 4, under the hypothesis that the parameter estimates give at each time t a stabilizing controller. The convergence toward the optimal error will depend on a positive real condition on the noise model (like in recursive identification). In connection with this analysis, an adaptive minimum variance tracking and regulation algorithm with filtering will be presented which allows to relax the positive real condition for convergence. We will then present a more complete analysis of self-tuning minimum variance tracking and regulation using martingales.

The plant and the disturbance are described by the following ARMAX model:

$$y(t + d + 1) = -A^* y(t + d) + B^* u(t) + Ce(t + d + 1) \qquad (11.77)$$

where $u(t)$ and $y(t)$ are the input and output of the plant and $e(t + 1)$ is a zero mean white noise sequence. For the case $d = 1$, this model takes the form:

$$y(t + 1) = -A^* y(t) + B^* u(t) + Ce(t + 1) \qquad (11.78)$$

11.4.1 The Basic Algorithms

Exact Adaptive Minimum Variance Tracking and Regulation

The case $d = 0$ We will start by building an adaptive predictor.

Step 1: Adaptive prediction

In the case of known parameters, using the polynomial equation:

$$C = AE + q^{-1}F \tag{11.79}$$

which gives for $d = 0 : E = 1$, $F = C^* - A^*$, the one step ahead predictor associated with the model (11.78) takes the form:

$$\begin{aligned}\hat{y}(t+1) &= -C^*\hat{y}(t) + Fy(t) + B^*u(t)\\ &= -C^*\hat{y}(t) + Ry(t) + Su(t) = \theta_C^T\phi(t)\end{aligned} \tag{11.80}$$

where $R = F$, $S = B^*$ and C^* correspond to the parameters of the minimum variance controller and:

$$\theta_C^T = [s_0, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}, c_1, \ldots, c_{n_C}] \tag{11.81}$$

$$\begin{aligned}\phi^T(t) = [&u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R),\\ &-\hat{y}(t), \ldots, -\hat{y}(t-n_C)]\end{aligned} \tag{11.82}$$

The corresponding adaptive predictor will have the form:

$$\hat{y}^0(t+1) = -\hat{C}^*(t)\hat{y}(t) + \hat{R}(t)y(t) + \hat{S}(t)u(t) = \hat{\theta}_C^T(t)\phi(t) \tag{11.83}$$

$$\hat{y}(t+1) = -\hat{\theta}_C^T(t+1)\phi(t) \tag{11.84}$$

where $\hat{y}^0(t+1)$ and $\hat{y}(t+1)$ define the a priori and a posteriori outputs of the adaptive predictor and:

$$\hat{\theta}_C^T(t) = [\hat{s}_0(t), \ldots, \hat{s}_{n_S}(t), \hat{r}_1(t), \ldots, \hat{r}_{n_R}(t), \hat{c}_1(t), \ldots, \hat{c}_{n_C}(t)] \tag{11.85}$$

Defining the a priori and a posteriori prediction errors as:

$$\varepsilon^0(t+1) = y(t+1) - \hat{y}^0(t+1) \tag{11.86}$$

$$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) \tag{11.87}$$

the appropriate PAA with decreasing adaptation gain will take the form:

$$\hat{\theta}_C(t+1) = \hat{\theta}_C(t) + F(t)\phi(t)\varepsilon(t+1) \tag{11.88}$$

$$\begin{aligned}&F(t+1)^{-1} = F(t)^{-1} + \lambda_2(t)\phi(t)\phi^T(t)\\ &0 < \lambda_2(t) < 2;\ \ F(0) > 0\end{aligned} \tag{11.89}$$

$$\varepsilon(t+1) = \frac{\varepsilon^0(t+1)}{1 + \phi^T(t)F(t)\phi(t)} \tag{11.90}$$

Step 2: Use of the ad-hoc separation Theorem

Compute $u(t)$ such that $\hat{y}^0(t+1) = y^*(t+1)$. This yields:

$$y^*(t+1) = \hat{y}^0(t+1) = \hat{\theta}_C^T(t)\phi(t) \tag{11.91}$$

from which one obtains the controller expression:

$$\hat{\theta}_C^T(t)\phi(t) = \hat{S}(t)u(t) + \hat{R}(t)y(t) - \hat{C}^*(t)\hat{y}(t) = y^*(t+1) \tag{11.92}$$

or:

$$u(t) = \frac{1}{\hat{b}_1(t)}[y^*(t+1) + \hat{C}^*(t)\hat{y}(t) - \hat{S}^*(t)u(t-1) - \hat{R}(t)y(t)] \tag{11.93}$$

This adjustable controller has almost the same structure as the controller used in the case of known parameters. The only difference is the replacement of $y^*(t) = \hat{y}^0(t)$; $y^*(t-1) = \hat{y}^0(t-1), \ldots$ by the a posteriori predicted outputs $\hat{y}(t), \hat{y}(t-1), \ldots$. Taking into account (11.86), (11.87) and (11.91), $\hat{y}(t)$ can be expressed as:

$$\hat{y}(t) = y^*(t) + [\hat{y}(t) - \hat{y}^0(t)] = y^*(t) + [\hat{\theta}_C(t) - \hat{\theta}_C(t-1)]^T\phi(t) \tag{11.94}$$

This allows (11.92) and (11.93) to be rewritten as follows:

$$\begin{aligned}&\hat{S}(t)u(t) + \hat{R}(t)y(t) - \hat{C}^*(t)y^*(t) - \hat{C}^*(t)[\hat{\theta}_C(t) - \hat{\theta}_C(t-1)]^T\phi(t)\\&\quad = y^*(t+1)\end{aligned} \tag{11.95}$$

or:

$$\begin{aligned}u(t) = \frac{1}{\hat{b}_1(t)}&[y^*(t+1) + \hat{C}^*(t)y^*(t) - \hat{S}^*u(t) - R(t)y(t)\\&+ C^*(t)[\hat{\theta}_C(t) - \hat{\theta}_C(t-1)]^T\phi(t)]\end{aligned} \tag{11.96}$$

where the term depending upon $[\hat{\theta}_C(t) - \hat{\theta}_C(t-1)]$ accounts in a certain sense for the speed of adaptation and will become null asymptotically if $\phi(t)$ is bounded (one uses a decreasing adaptation gain algorithm). Therefore asymptotically $\hat{y}(t)$ tends towards $y^*(t)$ and the controller equation (11.96) will have the structure of the fixed controller for the known parameter case.

To summarize, the *exact adaptive minimum variance tracking and regulation* for $d = 0$ is obtained using the adjustable controller defined by (11.92) whose parameters are updated by the PAA of (11.88) through (11.90) where:

$$\varepsilon^0(t+1) = y(t+1) - y^*(t+1) \tag{11.97}$$

As it can be seen the resulting direct adaptive control uses as primary source of information the error between the measured plant output and the desired output (similar to the deterministic case). For this reason this scheme is also called *stochastic model reference adaptive control* (Landau 1981, 1982a; Dugard et al. 1982).

Because of the division by $\hat{b}_1(t)$ (see (11.96)), the sign of b_1 should be assumed known, and the estimate of $\hat{b}_1(t)$ used is defined as:

$$\hat{b}_1'(t) = \begin{cases}\hat{b}_1(t) & |\hat{b}_1(t)| > \delta > 0\\ |\hat{b}_1'(t)| = \delta & |\hat{b}_1(t)| < \delta\end{cases} \tag{11.98}$$

in order to avoid division by zero.

Approximate Adaptive Minimum Variance Tracking and Regulation

The case $d = 0$ Taking into account (11.94), one can consider approximating $\hat{y}(t), \hat{y}(t-1), \ldots$ by $y^*(t), y^*(t-1), \ldots$ both in the controller equation and in the observation vector used in the PAA of (11.88) through (11.90). This corresponds to the use of a priori predictions instead of a posteriori predictions in the observation vector. In this case the adjustable controller is given by:

$$\hat{\theta}_C^T(t)\phi_C(t) = \hat{S}(t)u(t) + \hat{R}(t)y(t) - \hat{C}^*(t)y^*(t) = y^*(t+1) \tag{11.99}$$

where $\hat{\theta}_c(t)$ is given by (11.85) and

$$\phi_C^T(t) = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R), -y^*(t), \ldots, -y^*(t-n_C+1)] \tag{11.100}$$

and the PAA is given by:

$$\hat{\theta}_C(t+1) = \hat{\theta}_C(t) + F(t)\phi_C(t)\varepsilon(t+1) \tag{11.101}$$

$$F(t+1)^{-1} = F(t)^{-1} + \lambda_2(t)\phi_C(t)\phi_C^T(t)$$

$$0 < \lambda_2(t) < 2; \ \ F(0) > 0 \tag{11.102}$$

$$\varepsilon(t+1) = \frac{y(t+1) - y^*(t+1)}{1 + \phi_C^T(t)F(t)\phi_C(t)} \tag{11.103}$$

The case $d > 0$ One considers a $d+1$ steps ahead adaptive predictor of the form (see Sect. 6.3):

$$\hat{y}^0(t+d+1) = -\hat{C}^*(t)\hat{y}^0(t+d) + \hat{F}(t)y(t) + \hat{G}(t)u(t) \tag{11.104}$$

The control $u(t)$ is obtained by forcing:

$$\hat{y}^0(t+d+1) = y^*(t+d+1) \tag{11.105}$$

which yields:

$$\theta_C^T(t)\phi_C(t) = \hat{R}(t)y(t) + \hat{S}(t)u(t) - C^*(t)y^*(t+d) = y^*(t+d+1) \tag{11.106}$$

with:

$$\theta_C^T(t) = [\hat{s}_0(t), \ldots, \hat{s}_{n_S}(t), \hat{r}_0(t), \ldots, \hat{r}_{n_R}(t), \hat{c}_1(t), \ldots, \hat{c}_{n_C}(t)] \tag{11.107}$$

$$\begin{aligned}\phi_C^T(t) = [&u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_F), \\ &-y^*(t+d), \ldots, -y^*(t+d-n_C+1)]\end{aligned} \tag{11.108}$$

In the above equations, $\hat{F}(t)$ and $\hat{G}(t)$ have been replaced by $\hat{R}(t)$ and $\hat{S}(t)$ similar to the known parameter case and $\hat{y}(t+d), \hat{y}^0(t+d-1), \ldots$ by $y^*(t+d), y^*(t+d-1), \ldots$ taking into account (11.105). The used PAA is given by:

$$\hat{\theta}_C(t+d+1) = \hat{\theta}_C(t+d) + F(t)\phi_C(t)\varepsilon(t+d+1) \tag{11.109}$$

$$F^{-1}(t+1) = F^{-1}(t) + \lambda_2(t)\phi_C(t)\phi_C^T(t)$$
$$0 < \lambda_2(t) < 2; \; F(0) > 0 \tag{11.110}$$

$$\varepsilon(t+d+1) = \frac{y(t+d+1) - \hat{\theta}_C^T(t+d)\phi_C(t)}{1 + \phi_C^T(t)F(t)\phi(t)} \tag{11.111}$$

Adaptive Minimum Variance Regulation

In the case of regulation only $y^*(t) \equiv 0$ and both the controller and the PAA become simpler. Therefore the approximate (or the exact) algorithm for $d = 0$ will take the form:

Adjustable controller

$$\hat{S}(t)u(t) + \hat{R}(t)y(t) = \hat{\theta}_R^T(t)\phi_R(t) \tag{11.112}$$

or

$$u(t) = -\frac{\hat{R}(t)y(t) + \hat{S}^*(t)u(t-1)}{\hat{b}_1(t)} \tag{11.113}$$

where

$$\hat{\theta}_R^T(t) = [\hat{s}_0(t), \ldots, \hat{s}_{n_S}(t), \hat{r}_0(t), \ldots, \hat{r}_{n_R}(t)] \tag{11.114}$$

$$\phi_R^T(t) = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R)] \tag{11.115}$$

and the PAA takes the form

$$\hat{\theta}_R(t+1) = \hat{\theta}_R(t) + F(t)\phi_R(t)\varepsilon(t+1) \tag{11.116}$$

$$F^{-1}(t+1) = F^{-1}(t) + \lambda_2(t)\phi_R(t)\phi_R^T(t)$$
$$0 < \lambda_2(t) < 2; \; F(0) > 0 \tag{11.117}$$

$$\varepsilon(t+1) = \frac{y(t+1)}{1 + \phi_R^T(t)F(t)\phi_R(t)} \tag{11.118}$$

From (11.113) one can see that in the particular case of regulation, the control law has one redundant parameter and there is an infinite number of solutions leading to a good result. In practice, one often replaces $\hat{b}_1(t)$ by a "fixed" a priori estimate $\hat{b}_1$ satisfying $\hat{b}_1 > b_1/2$ (Egardt 1979).

Remark The algorithm is close to the original Åström and Wittenmark self-tuning minimum variance controller (Åström and Wittenmark 1973) and it can be shown that in this case the resulting biased plant parameters estimates converge toward a controller assuring the minimum variance regulation asymptotically. This algorithm can be obtained straightforwardly by

(1) making a least squares one step ahead predictor (i.e., neglecting the term $Ce(t+1)$) which estimates only $\hat{A}$, $\hat{B}$. The estimates will be biased;
(2) compute the input by forcing $\hat{y}^0(t+1) = 0$.

11.4.2 Asymptotic Convergence Analysis

In this section, we will examine the asymptotic properties of the various algorithms using the averaging method presented in Sect. 4.2 (and in particular Theorem 4.1) and we will present an extension of the basic algorithms through filtering the observation vector. While this analysis will give conditions for asymptotic convergence toward minimum variance control, it assumes that during adaptation transient, one has at each t a stabilizing controller (a more complete analysis will be discussed in Sect. 11.4.3).

The analysis begins by deriving an equation for the a priori or a posteriori adaptation error (since the averaging method does not make a distinction between them) and setting the adjustable parameters to a constant value. Then application of Theorem 4.1 will give us the convergence conditions.

Exact Adaptive Minimum Variance Tracking and Regulation

Using the polynomial equation (11.79), the plant output at $t+1$ can be expressed as:

$$\begin{aligned} y(t+1) &= -C^*y(t) + Ry(t) + Su(t) + Ce(t+1) \\ &= \theta_C^T\phi(t) - C^*[y(t) - \hat{y}(t)] + Ce(t+1) \end{aligned} \tag{11.119}$$

where θ_C defines the vector of the nominal parameters of the minimum variance controller given in (11.70) and $\hat{y}(t)$ is the a posteriori prediction given by (11.84). Subtracting (11.91) from (11.119) one gets:

$$\varepsilon^0(t+1) = [\theta_C - \hat{\theta}_C(t)]^T\phi(t) - C^*\varepsilon(t) + Ce(t+1) \tag{11.120}$$

and subtracting (11.84) from (11.119) one gets:

$$\varepsilon(t+1) = [\theta_C - \hat{\theta}_C(t+1)]^T\phi(t) - C^*\varepsilon(t) + Ce(t+1) \tag{11.121}$$

from which one obtains:

$$\varepsilon(t+1) = \frac{1}{C(q^{-1})}[\theta_C - \hat{\theta}_C(t+1)]^T\phi(t) + e(t+1) \tag{11.122}$$

Making $\hat{\theta}_C(t) = \hat{\theta}_C$, both (11.120) and (11.121) lead to:

$$\varepsilon^0(t+1,\hat{\theta}_C) = \varepsilon(t+1,\hat{\theta}_C) = \frac{1}{C(q^{-1})}[\theta_C - \hat{\theta}_C]^T\phi(t,\hat{\theta}_C) + e(t+1) \tag{11.123}$$

and since the image of the disturbance in (11.123) is a white noise, Theorem 4.1 can be applied. Assuming that $\hat{\theta}_C(t)$ does not leave the domain D_S of controller parameters for which the closed loop is asymptotically stable and that $\max_t \lambda_2(t) \leq \lambda_2 < 2$, it results immediately from Theorem 4.1 that if:

$$H'(z^{-1}) = \frac{1}{C(z^{-1})} - \frac{\lambda_2}{2} \tag{11.124}$$

is a strictly positive real transfer function one has:

$$\text{Prob}\left\{\lim_{t\to\infty} \hat{\theta}_C(t) \in D_C\right\} = 1 \tag{11.125}$$

where D_C is a set of all possible convergence points characterized by:

$$D_C\{\hat{\theta}_C | [\theta_C - \hat{\theta}_C]^T \phi(t, \hat{\theta}) = 0\} \tag{11.126}$$

Since $\hat{\theta}_c(\infty) \in D_C$ w.p.1, it results from (11.123) that asymptotically:

$$\text{Prob}\left\{\lim_{t\to\infty} \varepsilon^0(t+1) = \lim_{t\to\infty} \varepsilon(t+1) = e(t+1)\right\} = 1 \tag{11.127}$$

which is the desired control objective in the adaptive case. Note that (11.125) does not imply the convergence of $\hat{\theta}_C$ towards θ_C (θ_C is just one possible convergence point).

Approximate Adaptive Minimum Variance Tracking and Regulation

This algorithm can be analyzed in a similar way and same positive real conditions will result. We will subsequently examine the case $d > 0$. Using the results of Sect. 2.2, the output of the plant at $t + d + 1$ is given by:

$$y(t+d+1) = -C^*(t)y(t+d) + Fy(t) + Gu(t) + CEe(t+d+1) \tag{11.128}$$

where E and F are solutions of the polynomial equation:

$$C = AE + q^{-d-1}F \tag{11.129}$$

and:

$$G = B^*E \tag{11.130}$$

However, from Sect. 7.6, one has in the case of minimum variance control

$$S = G \quad \text{and} \quad R = F \tag{11.131}$$

which allows one to rewrite (11.128) as:

$$y(t+d+1) = \theta_C^T \phi_C(t) - C^*[y(t+d) - \hat{y}^0(t+d)] + CEe(t+d+1) \tag{11.132}$$

Taking into account (11.105) and subtracting (11.106) from (11.132), one gets:

$$\begin{aligned}\varepsilon^0(t+d+1) &= y(t+d+1) - \hat{y}^0(t+d+1)\\ &= [\theta_C - \hat{\theta}_C(t)]^T \phi_C(t) - C^* \varepsilon^0(t+d) + CEe(t+d+1)\end{aligned} \tag{11.133}$$

which, after passing the term $C^*\varepsilon^0(t+d)$ on the left hand side, can be put under the form:

$$\varepsilon^0(t+d+1) = \frac{1}{C(q^{-1})}[\theta_C - \hat{\theta}_C(t)]^T \phi_C(t) + Ee(t+d+1) \tag{11.134}$$

Making $\hat{\theta}_C(t) = \hat{\theta}_C$ one gets:

$$\varepsilon^0(t+d+1, \hat{\theta}_C) = \frac{1}{C(q^{-1})}[\theta_C - \hat{\theta}_C]^T \phi_C(t, \hat{\theta}_C) + Ee(t+d+1) \tag{11.135}$$

The term $Ee(t+d+1)$ contains $e(t+d+1)e(t+d)\cdots e(t+1)$ which all are uncorrelated with the components of $\phi_c(t, \hat{\theta}_c)$ ($e(t+i)$ is a white noise) and therefore:

$$\mathbf{E}\{\phi_C(t, \hat{\theta}_C), Ee(t+d+1)\} = 0 \tag{11.136}$$

The form of (11.135) and (11.136) allows to directly apply the results of Theorem 4.1 yielding the same positive real condition given in (11.124). Same result holds for the case of adaptive minimum variance regulation. The result of the above analysis can be summarized as follows:

Theorem 11.3 *For the adaptive minimum variance tracking and regulation algorithms given either by* (11.106) *through* (11.111), *or by* (11.92), (11.88) *through* (11.90) *and* (11.97) (*for* $d = 0$) *and for adaptive minimum variance regulation algorithm given by* (11.112) *through* (11.118) *provided that*:

(1) *An upper bound for the orders of the polynomials* A, B, C *is known.*
(2) *The integer time delay* d *is known.*
(3) *For all possible values of the plant model parameters,* $B(z^{-1})$ *has its zeros inside the unit circle.*
(4) *The sign of* b_1 *is known.*
(5) *The estimated controllers parameters belong infinitely often to the domain for which the closed-loop system is asymptotically stable.*

One has:

$$\left.\begin{aligned}&\text{Prob}\Big\{\lim_{t\to\infty}[y(t+1) - y^*(t+1)] = e(t+1)\Big\} = 1;\ d = 0\\ &\text{Prob}\Big\{\lim_{t\to\infty}[y(t+d+1) - y^*(t+d+1)] = Ee(t+d+1)\Big\} = 1\\ &\deg E = d\end{aligned}\right\} \tag{11.137}$$

if there is $\sup_t \lambda_2(t) \leq \lambda_2 < 2$ *such that*:

$$H'(z^{-1}) = \frac{1}{C(z^{-1})} - \frac{\lambda_2}{2} \tag{11.138}$$

is strictly positive real.

Assumptions 1, 2, 3 are just assuring that a minimum variance control can be computed. Assumption 4 is necessary in order to use a mechanism for avoiding eventual division by zero during adaptation transient (see (11.98)). Assumption 5 is the typical limitation of the averaging method.

The result of Theorem 11.3 indicates that when $C(q^{-1}) \neq 1$, the adaptive control may not converge towards a value assuring asymptotic optimality of the controller if $\frac{1}{C(z^{-1})} - \frac{\lambda_2}{2}$ is not strictly positive real. Such an example of non convergence can be found in Ljung (1977b). One way to remove this restriction is by over parameterization (Shah and Franklin 1982) but it has the disadvantage of augmenting the number of parameters to be adapted. If one has some a priori knowledge of the domain of variation of $C(q^{-1})$ one can use a filtered observation vector (similar to the method described in the deterministic case—Sect. 11.2) in order to relax the positive real condition (11.138).

11.4.3 Martingale Convergence Analysis

The use of the averaging method allows a straightforward analysis of the asymptotic properties of the *adaptive minimum variance tracking and regulation* schemes. However, this analysis does not provide any information on the boundedness of the input and output of the plant during the adaptation transient. The objective of the martingale convergence analysis is to show that not only the asymptotic objectives are achieved but also that all the signals (including input and output of the plant) remain bounded (in a mean square sense). Of course the same positive real condition which occurred in the analysis through the averaging method will be encountered. The martingale convergence analysis of self-tuning minimum variance tracking and regulation has been the subject of a large number of papers (Goodwin et al. 1980b, 1980c; Guo 1993, 1996; Johansson 1995; Fuchs 1982; Chen and Guo 1991; Kumar and Moore 1982; Ren and Kumar 2002; Goodwin and Sin 1984). In particular the convergence in the case of matrix adaptation gain is still a subject of research.

In this section, we will prove the following results for the exact adaptive minimum variance tracking and regulation with $d = 0$.

Theorem 11.4 (Goodwin and Sin 1984) *For the adaptive minimum variance tracking and regulation algorithm given by* (11.92), (11.97) *and* (11.88) *through* (11.90) *provided that*:

(1) *An upper bound for the orders of the polynomials A, B and C is known.*
(2) *The integer delay $d = 0$.*
(3) *For all possible values of the plant model parameters $B^*(z^{-1})$ has its zeros inside the unit circle.*
(4) *The sign of b_1 is known.*
(5)

$$\lim_{N\to\infty} \sup \frac{\lambda_{\max} F(N)}{\lambda_{\min} F(N)} \le K < \infty \tag{11.139}$$

where $\lambda_{\max} F(N)$ and $\lambda_{\min} F(N)$ are the maximum and minimum values of the eigenvalues of the matrix adaptation gain (bounded condition number assumption).
(6) *The sequence $\{e(t)\}$ is a martingale difference sequence defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and adapted to the sequence of increasing algebras $\mathscr{F}_t$ generated by the observations up to and including time t. The sequence $\{e(t+1)\}$ is assumed to satisfy the following:*

$$\mathbf{E}\{e(t+1)|\mathscr{F}_t\} = 0 \tag{11.140}$$

$$\mathbf{E}\{e^2(t+1)|\mathscr{F}_t\} = \sigma^2 \tag{11.141}$$

$$\lim_{N\to\infty} \sup \frac{1}{N}\sum_{t=1}^{N} e^2(t) < \infty \tag{11.142}$$

one has:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} [\varepsilon^0(t+1) - e(t+1)]^2 = 0 \quad a.s. \tag{11.143}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} y^2(t) < \infty \quad a.s. \tag{11.144}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u^2(t) < \infty \quad a.s. \tag{11.145}$$

if there is: $\sup_t \lambda_2(t) \le \lambda_2 < 2$ *such that:*

$$H'(z^{-1}) = \frac{1}{C(z^{-1})} - \frac{\lambda_2}{2} \tag{11.146}$$

is strictly positive real.

Equation (11.143) tells us that the difference between the tracking (regulation) error $\varepsilon^0(t) = y(t) - y^*(t)$ and the optimal minimum variance error goes asymptotically to zero in mean square sense with probability 1. While the hypothesis (11.139) is of technical nature for the convergence proof with matrix adaptation gain, it is in any way taken into account in practical implementation of the adaptation algorithm. This may lead to a slight modification of the $F(t)$ given by (11.89) in order to satisfy the condition number requirement (see Chap. 16 for details).

Proof The proof of Theorem 11.4 is based on the use of Theorem 4.3 (Chap. 4) plus some additional steps involving the analysis of

- the properties of $[\varepsilon^0(t+1) - e(t+1)]$;
- the relationship between, $u(t)$, $y(t)$, $\hat{y}(t)$ and $[\varepsilon^0(t) - e(t)]$.

From (11.122) one has:

$$\varepsilon(t+1) = \frac{1}{C(q^{-1})}[\theta_C - \hat{\theta}_C(t+1)]^T \phi(t) + e(t+1) \tag{11.147}$$

which combined with the PAA of (11.88) through (11.90) allows the straightforward application of the results of Theorem 4.3 where:

$$r(t) = r(t-1) + \lambda_2(t)\phi^T(t-1)\phi(t-1); \quad 0 < \lambda_2(t) < 2 \tag{11.148}$$

with the observation that $r(0) = \operatorname{tr} F^{-1}(0)$ which implies $\operatorname{tr} F^{-1}(t) = r(t)$. Provided that (11.146) is strictly positive real one has:

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{[\varepsilon(t+1) - e(t+1)]^2}{r(t)} < \infty \tag{11.149}$$

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{\phi^T(t)F(t)\phi(t)}{r(t)} \varepsilon^2(t+1) < \infty \tag{11.150}$$

Then one can prove the following result:

Lemma 11.2 *Under the hypotheses of Theorem* 11.4 *provided that* (11.146) *is strictly positive real, one has*:

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{[\varepsilon^0(t+1) - e(t+1)]^2}{r(t)} < \infty \tag{11.151}$$

This is a property of the measured tracking and regulation error $\varepsilon^0(t+1)$ (a priori adaptation error).

Proof Multiplying both terms of (11.88) from the left by $-\phi^T(t)$ one has:

$$\begin{aligned} -\phi^T(t)\hat{\theta}_C(t+1) &= -\phi^T(t)\hat{\theta}_C(t) - \phi^T(t)F(t)\phi(t)\varepsilon(t+1) \\ &= -\hat{y}(t+1) = -\hat{y}^0(t+1) - \phi^T(t)F(t)\phi(t)\varepsilon(t+1) \end{aligned} \tag{11.152}$$

Adding in both sides $[y(t+1)-e(t+1)]$ one gets:

$$\varepsilon(t+1)-e(t+1)=[\varepsilon^0(t+1)-e(t+1)]-\phi^T(t)F(t)\phi(t)\varepsilon(t+1) \quad (11.153)$$

Using the Schwarz inequality one has:

$$\begin{aligned}[\varepsilon^0(t+1)-e(t+1)]^2 \leq{}& 2[\varepsilon(t+1)-e(t+1)]^2 \\ &+2[\phi^T(t)F(t)\phi(t)]^2\varepsilon^2(t+1)\end{aligned} \quad (11.154)$$

but taking into account the condition number hypothesis (11.139), one can write:

$$\begin{aligned}[\phi^T(t)F(t)\phi(t)]^2 &\leq [\phi^T(t)\phi(t)][\phi(t)F^2(t)\phi(t)] \\ &\leq \phi^T(t)\phi(t)\frac{\phi^T(t)F(t)\phi(t)}{r(t)}\end{aligned} \quad (11.155)$$

and therefore from (11.154) one finally obtains:

$$\begin{aligned}&\sum_{t=1}^{N}\frac{[\varepsilon^0(t+1)-e(t+1)]^2}{r(t)} \\ &\quad\leq 2\sum_{t=1}^{N}\frac{[\varepsilon(t+1)-e(t+1)]^2}{r(t)} \\ &\qquad+2\sum_{t=1}^{N}\frac{\phi^T(t)\phi(t)}{r(t)}\frac{\phi^T(t)F(t)\phi(t)}{r(t)}\varepsilon^2(t+1)\end{aligned} \quad (11.156)$$

Now taking into account that $[\phi^T(t)\phi(t)/r(t)] \leq 1$ together with (11.149) and (11.150), one concludes that (11.151) is true. □

The next step is to use a "bounded growth" lemma similar to a certain extend with the one used in the deterministic case (Lemma 11.1, Sect. 11.2).

Lemma 11.3 (Stochastic Bounded Growth Lemma Goodwin and Sin 1984) *If* (11.151) *holds, then if there are constants* K_1, K_2 *and* $\bar{N}$: $0 \leq K_1 < \infty$; $0 < K_2 < \infty$; $0 < \bar{N} < \infty$ *such that*:

$$\frac{1}{N}r(N) \leq K_1 + \frac{K_2}{N}\sum_{t=1}^{N}[\varepsilon^0(t+1)-e(t+1)]^2; \quad N \geq \bar{N} \quad (11.157)$$

one has:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}[\varepsilon^0(t+1)-e(t+1)]^2 = 0 \quad a.s. \quad (11.158)$$

$$\lim_{N\to\infty}\sup\frac{1}{N}r(N) < \infty \quad a.s. \quad (11.159)$$

Note that (11.158) and (11.159) correspond to the desired results (11.143) through (11.145) and it will remain to show that indeed for the algorithm considered, (11.157) is verified.

Proof of Lemma 11.3 For the case $r(t) < K < \infty$ for all t, (11.158) results immediately from (11.157) and (11.151). Suppose now that $r(t)$ may become unbounded, then one can apply the Kronecker Lemma (see Appendix D) to (11.151) and conclude that:

$$\lim_{N\to\infty} \frac{N}{r(N)} \frac{1}{N} \sum_{t=1}^{N} [\varepsilon^0(t+1) - e(t+1)]^2 = 0 \tag{11.160}$$

Taking into account (11.157), one gets:

$$\lim_{N\to\infty} \frac{\frac{1}{N}\sum_{t=1}^{N} [\varepsilon^0(t+1) - e(t+1)]^2}{K_1 + \frac{K_2}{N}\sum_{t=1}^{N} [\varepsilon^0(t+1) - e(t+1)]^2} = 0 \tag{11.161}$$

and (11.158) follows by contradiction arguments. Since (11.158) is true it results from (11.160) that also (11.159) is true. □

To complete the proof of Theorem 11.4, it remains to show now that for the considered algorithm $r(t)$ given by (11.148) satisfies the condition (11.157) of the Lemma 11.3. Notice that:

$$\begin{aligned}\frac{1}{N} r(N) = \frac{1}{N} \operatorname{tr} F^{-1}(0) + \sum_{i=0}^{n_B-1} \left[\frac{1}{N}\sum_{t=1}^{N} u^2(t-i)\right] \\ + \sum_{i=0}^{n_A-1} \left[\frac{1}{N}\sum_{t=1}^{N} y^2(t-i)\right] + \sum_{i=0}^{n_C-1} \left[\frac{1}{N}\sum_{t=1}^{N} \hat{y}^2(t-i)\right]\end{aligned} \tag{11.162}$$

Therefore, it will be sufficient to show that $\frac{1}{N}\sum_{t=1}^{N} u^2(t)$, $\frac{1}{N}\sum_{t=1}^{N} y^2(t)$, $\frac{1}{N}\sum_{t=1}^{N} \hat{y}^2(t)$ satisfy a condition of the form (11.157). Notice that $y(t+1)$ can be expressed as:

$$\begin{aligned}y(t+1) &= \varepsilon^0(t+1) + y^*(t+1) \\ &= [\varepsilon^0(t+1) - e(t+1)] + e(t+1) + y^*(t+1)\end{aligned} \tag{11.163}$$

Using again the Schwarz inequality one has:

$$y^2(t+1) \le 3[\varepsilon^0(t+1) - e(t+1)]^2 + 3e^2(t+1) + 3{y^*}^2(t+1) \tag{11.164}$$

Bearing in mind (11.142) and the boundedness of y^* one gets:

$$\frac{1}{N}\sum_{t=1}^{N} y^2(t+1) \leq \frac{C_1}{N}\sum_{t=1}^{N}[\varepsilon^0(t+1) - e(t+1)]^2 + C_2$$
$$0 < C_1 < \infty;\ 0 \leq C_2 < \infty \tag{11.165}$$

On the other hand, since in (11.78) $B^*(z^{-1})$ is asymptotically stable (i.e., the inverse of the system is stable) and (11.142) holds, one can express $u(t)$ as a function of $y(t)$ and $e(t)$ and one gets:

$$\frac{1}{N}\sum_{t=1}^{N} u^2(t+1) \leq \frac{C_3}{N}\sum_{t=1}^{N} y^2(t) + C_4; \quad 0 < C_3 < \infty;\ 0 \leq C_4 < \infty \tag{11.166}$$

which taking into account (11.165) allows one to conclude that $u(t)$ will satisfy a relationship of the form (11.157). Consider now the expression of $\hat{y}(t+1)$:

$$\begin{aligned}
\hat{y}(t+1) &= y(t+1) - \varepsilon(t+1) = y(t+1) - \frac{\varepsilon^0(t+1)}{1+\phi^T(t)F(t)\phi(t)} \\
&= y^*(t+1) + \varepsilon^0(t+1) - \frac{\varepsilon^0(t+1)}{1+\phi^T(t)F(t)\phi(t)} \\
&= y^*(t+1) + \frac{\phi^T(t)F(t)\phi(t)}{1+\phi^T(t)F(t)\phi(t)}[\varepsilon^0(t+1) - e(t+1)] \\
&\quad + \frac{\phi^T(t)F(t)\phi(t)}{1+\phi^T(t)F(t)\phi(t)}e(t+1)
\end{aligned} \tag{11.167}$$

and using now the Schwarz inequality one has:

$$\hat{y}^2(t+1) \leq 3y^{*^2}(t+1) + 3[\varepsilon^0(t+1) - e(t+1)]^2 + 3e^2(t+1) \tag{11.168}$$

from which one obtains:

$$\frac{1}{N}\sum_{t=1}^{N} \hat{y}^2(t+1) \leq \frac{C_5}{N}\sum_{t=1}^{N}[\varepsilon^0(t+1) - e(t+1)]^2 + C_6 \tag{11.169}$$

Therefore (11.165), (11.166) and (11.169) allow one to conclude that $r(t)$ satisfies the condition (11.157) and therefore that (11.143), (11.144) and (11.145) are true if (11.146) is strictly positive real. This ends the proof of Theorem 11.4. □

11.5 Robust Direct Adaptive Control

11.5.1 The Problem

In the previous sections, the design of the direct adaptive controllers has been discussed under the assumptions that (called also the *ideal case*):

- the integer delay d is known;
- upper bounds on the degrees of the polynomials A and B^* are known;
- the sign of b_1 is known.

When dealing with disturbances, it was assumed that they have a stochastic model of known structure (in particular an ARMAX model has been considered).

The fact that:

1. the true plant model order is assumed to be less or equal to the model order used for design (i.e., the plant model is in the *model set*),
2. the unmeasurable disturbances have a clear (nice) stochastic model,

constitute two major drawbacks of the analysis discussed in the previous section from a practical point of view. Examples are available in the literature (Egardt 1979; Rohrs et al. 1981; Ioannou and Kokotovic 1983) showing that either particular form of unmodeled dynamics or special types of disturbances can cause instability of direct adaptive control schemes. For a detailed review see Ortega and Tang (1989). Therefore, it is extremely important from a practical point of view, to assess the robustness of direct adaptive controller in the presence of:

- unmodeled dynamics (the plant model is not in the model set);
- bounded disturbances (without a specific stochastic model).

Two points of view can be considered:

1. analysis of the robustness of the "ideal case" designs,
2. introduction of modifications in the parameter adaptation algorithms in order to improve the robustness of the "ideal case" designs (robust adaptation).

The first approach is extremely interesting for a deep understanding of the mechanisms leading to the instability in direct adaptive control in the presence of unmodeled dynamics and bounded disturbances (see Ortega and Tang 1989). We will focus on the use of robust parameter adaptation algorithms discussed in Chap. 10, in order to enhance the robustness of direct adaptive control schemes. While one can argue that these modifications are not always necessary (in theory), they definitely enlarge the set of situations which can be handled. In particular, two techniques will be used: PAA with dead zone and PAA with normalization. The presentation is mainly based on Ortega and Lozano-Leal (1987), Lozano and Ortega (1987), Ortega et al. (1985), Ortega (1993), Kreisselmeier and Anderson (1986). Use of PAA with projection can also be considered as shown in Ydstie (1989).

11.5.2 Direct Adaptive Control with Bounded Disturbances

Consider the plant model (11.1) in the presence of a disturbance $w'(t+1)$:

$$A(q^{-1})y(t) = q^{-d-1}B^*(q^{-1})u(t) + w'(t+1) \tag{11.170}$$

To simplify the presentation, we will consider the case $d = 0$ and we will use a fixed adaptation gain F. Using the polynomial equation:

$$A(q^{-1})S'(q^{-1}) + q^{-1}R(q^{-1}) = P(q^{-1}) \tag{11.171}$$

The filtered predicted output in the presence of disturbances can be written (see also (11.11) through (11.13)) as:

$$\begin{aligned} P(q^{-1})y(t+1) &= \theta_C^T\phi_C(t) + S'(q^{-1})w'(t+1) \\ &= \theta_C^T\phi_C(t) + w(t+1) \end{aligned} \tag{11.172}$$

where:

$$\theta_C^T = [s_0, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}] \tag{11.173}$$

$$\phi_C^T = [u(t), \ldots, u(t-n_S), y(t), \ldots, y(t-n_R)] \tag{11.174}$$

θ_C defines the parameter vector of the tuned controller. Its components are the coefficients of the polynomials $R(q^{-1})$ and $S(q^{-1}) = S'(q^{-1})B^*(q^{-1})$. We will assume that a bound for the disturbance $w(t+1)$ is known, i.e.:

$$|w(t+1)| < \Delta \tag{11.175}$$

The controller parameters according to (11.7) will be given by:

$$\hat{\theta}_C^T(t)\phi_C(t) = P(q^{-1})y^*(t+1) \tag{11.176}$$

The tracking error is defined as in (11.14):

$$\varepsilon^0(t+1) = Py(t+1) - Py^*(t+1) = \tilde{\theta}_C^T(t)\phi_C(t) + w(t+1) \tag{11.177}$$

where:

$$\tilde{\theta}_C(t) = \theta_C - \hat{\theta}_c(t) \tag{11.178}$$

Consider the following PAA (Ortega and Lozano-Leal 1987):

$$\hat{\theta}_C(t+1) = \hat{\theta}_C(t) + \frac{F\phi_C(t)}{1+\phi_C^T(t)F\phi_C(t)} f[\varepsilon^0(t+1)]; \quad F > 0 \tag{11.179}$$

$$f[\varepsilon^0(t+1)] = \begin{cases} \varepsilon^0(t+1) - \Delta & \text{if } \varepsilon^0(t+1) > \Delta \\ 0 & \text{if } |\varepsilon^0(t+1)| \le \Delta \\ \varepsilon^0(t+1) + \Delta & \text{if } \varepsilon^0(t+1) < -\Delta \end{cases} \tag{11.180}$$

where $\varepsilon^0(t+1)$ is given by (11.177). This is a slight modification of the algorithm considered in Sect. 10.4. Note also that for $\Delta = 0$ it is exactly the one used in the ideal case (Sect. 11.2). The basic property of this algorithm which will be used for the analysis of the scheme is that:

$$[\varepsilon^0(t+1) - w(t+1)] f[\varepsilon^0(t+1)] \geq f^2[\varepsilon^0(t+1)]; \quad \forall t \tag{11.181}$$

Using this algorithm one has the following results summarized in Theorem 11.5.

Theorem 11.5 *Consider the plant model* (11.170) *and its predictor form* (11.172), *the adjustable controller* (11.176) *and the PAA given by* (11.179) *and* (11.180). *Assume that*:

1. *The integer delay d is known.*
2. *Upper bounds on the degrees of the polynomials A and B^* are known.*
3. *For all possible values of the plant parameters*, *the polynomial B^* has all its zeros inside the unit circle.*
4. *The sign of b_1 is known.*
5. *An upper bound for the disturbance $w(t+1)$ in* (11.172) *is known.*

Then:

$$\text{(i)} \qquad \lim_{t\to\infty} \sup \|\varepsilon^0(t)| \leq \Delta \tag{11.182}$$

$$\text{(ii)} \qquad \|\tilde{\theta}_C(t)\|_{F^{-1}} < M < \infty; \quad \forall t \tag{11.183}$$

$$\text{(iii)} \qquad \lim_{t\to\infty} \|\hat{\theta}_C(t+1) - \hat{\theta}_C(t)\| = 0 \tag{11.184}$$

$$\text{(iv)} \qquad \|\phi_C(t)\| < M < \infty \tag{11.185}$$

Proof In the proof we will need an equation for the evolution of $\phi_C(t)$. This was already established in Sect. 11.2 taking into account that the system has all its zeros inside the unit circle (stable inverse). From (11.31) one has:

$$\|\phi_C(t)\|_F \leq c_1 + c_2 \max_{0<k<t+1} |\varepsilon^0(k+1)|; \quad 0 < c_1,\ c_2 < \infty \tag{11.186}$$

Let us now consider the evolution of the parameter error $\tilde{\theta}_C(t)$. From (11.178) and (11.179) one has:

$$\tilde{\theta}_C(t+1) = \tilde{\theta}_C(t) - \frac{F\phi_C(t)}{1 + \phi_C^T(t) F \phi_C(t)} f[\varepsilon^0(t+1)] \tag{11.187}$$

which allows to write:

$$\tilde{\theta}_C^T(t+1)F^{-1}\tilde{\theta}_C(t+1) = \tilde{\theta}_C^T(t)F^{-1}\tilde{\theta}_C(t) - 2\frac{\tilde{\theta}_C^T(t)\phi_C(t)f[\varepsilon^0(t+1)]}{1+\phi_C^T(t)F\phi_C(t)} + \frac{\phi_C^T(t)F\phi_C(t)}{[1+\phi_C^T(t)F\phi_C(t)]^2}f^2[\varepsilon^0(t+1)] \tag{11.188}$$

But from (11.177):

$$\tilde{\theta}_C^T(t)\phi_C(t) = \varepsilon^0(t+1) - w(t+1) \tag{11.189}$$

and using (11.181), one gets:

$$\tilde{\theta}_C^T(t)\phi_C(t)f[\varepsilon^0(t+1)] \geq f^2[\varepsilon^0(t+1)] \tag{11.190}$$

Therefore:

$$\|\tilde{\theta}_C^T(t+1)\|_{F^{-1}}^2 \leq \|\tilde{\theta}_C^T(t)\|_{F^{-1}}^2 - \frac{f^2[\varepsilon^0(t+1)]}{1+\phi_C^T(t)F\phi_C(t)} \tag{11.191}$$

As a consequence, the sequence $\{\|\hat{\theta}_C(t+1)\|_{F^{-1}}^2\}$ is a nonincreasing positive sequence bounded below by zero and thus converges. Since F is positive definite, (11.183) results. Equation (11.191) also implies that:

$$\lim_{t\to\infty}(\|\tilde{\theta}_C(t+1)\|_{F^{-1}}^2 - \|\tilde{\theta}_C(t)\|_{F^{-1}}^2) = 0 \tag{11.192}$$

and therefore (11.184) holds. From (11.191) one also has that:

$$\lim_{t\to\infty}\frac{f^2[\varepsilon^0(t+1)]}{1+\phi_C^T(t)F\phi_C(t)} = 0 \tag{11.193}$$

Taking into account (11.180), (11.186) can be expressed as:

$$\begin{aligned}\|\phi_C^T(t)\|_F &\leq c_1 + c_2 \max_{0<k<t+1}[|f[\varepsilon^0(t+1)]| + \Delta] \\ &\leq c_1' + c_2 \max_{0<k<t+1}|f[\varepsilon^0(k+1)]|\end{aligned} \tag{11.194}$$

Using now (11.193), (11.194) and the "bounded growth" lemma (Lemma 11.1) one concludes that (11.182) and (11.185) are true, which ends the proof. □

Remark The result can be extended for the case $d > 1$ (Ortega and Lozano-Leal 1987).

11.5.3 Direct Adaptive Control with Unmodeled Dynamics

Consider again the equation of the filtered predicted output given in (11.172). In this case, it will be assumed that the disturbance is defined by:

$$|w(t+1)| \leq d_1 + d_2\eta(t), \quad 0 < d_1,\ d_2 < \infty \tag{11.195}$$

$$\eta^2(t) = \mu^2\eta^2(t-1) + \|\phi_C(t)\|^2 \tag{11.196}$$

where d_1 accounts for a bounded external disturbance and $d_2\eta(t)$ accounts for the equivalent representation of the effect of unmodeled dynamics. More details about the description of unmodeled dynamics can be found in Sect. 10.6 (the chosen representation for the unmodeled dynamics corresponds to Assumption B in Sect. 10.6). In this case, due to the effect of feedback the unmodeled dynamics in (11.172) are characterized by $H'^T(q^{-1})\phi(t)$ where $H'^T(q^{-1}) = S'(q^{-1})H^T(q^{-1})$ and it is assumed that $H'^T(z^{-1})$ is analytic in $|z| < \mu < 1$ and $d_2 = \|H'(\mu^{-1}z^{-1})\|_\infty$.

It should be noted that even if the parameters of the reduced order plant model are perfectly known, there is no fixed linear controller that could stabilize the plant for all possible value of d_2 (which corresponds to the μ-scaled infinity norm of a transfer function depending on the system parameters and design objectives). Nevertheless one can design a controller that stabilizes the plant provided that $d_2 \leq d_2^*$ where d_2^* is a threshold that again depends on the system parameters.

The assumptions made upon the system are:

1. The delay d is known.
2. Upper bounds on the degrees of the polynomials A and B^* are known.
3. For all possible values of the plant parameters, the polynomial B^* has all its zeros inside the unit circle.
4. The sign of b_1 is known.
5. The disturbances upper bounds d_1 and d_2 are known.

Define the normalized input-output variables as:

$$\bar{y}(t) = \frac{y(t+1)}{m(t)}; \qquad \bar{u}(t) = \frac{u(t)}{m(t)}; \qquad \bar{\phi}_C(t) = \frac{\phi_C(t)}{m(t)} \tag{11.197}$$

where:

$$m^2(t) = \mu^2 m^2(t-1) + \max(\|\phi_C(t)\|^2, 1), \qquad m(0) = 1, \quad 0 < \mu < 1 \tag{11.198}$$

The normalized predicted plant output is given by:

$$P(q^{-1})\bar{y}(t+1) = \theta_C\bar{\phi}_C(t) + \bar{w}(t+1) \tag{11.199}$$

where:

$$\bar{w}(t+1) = \frac{w(t+1)}{m(t)} \tag{11.200}$$

The normalized a priori filtered tracking error is given by:

$$\bar{\varepsilon}^0(t+1) = P(q^{-1})\bar{y}(t+1) - \hat{\theta}_C^T(t)\bar{\phi}_C(t) + \bar{w}(t+1)$$
$$= \tilde{\theta}_C^T(t)\bar{\phi}_C(t) + \bar{w}(t+1) \tag{11.201}$$

Consider the following PAA:

$$\hat{\theta}_C(t+1) = \hat{\theta}_C(t) + \frac{F\bar{\phi}_C(t)}{1+\bar{\phi}_C^T(t)F\bar{\phi}_C(t)} f[\bar{\varepsilon}^0(t+1)], \quad F > 0 \tag{11.202}$$

$$f[\bar{\varepsilon}^0(t+1)] = \begin{cases} \bar{\varepsilon}^0(t+1) - \bar{\delta}(t+1) & \text{if } \bar{\varepsilon}^0(t+1) > \bar{\delta}(t+1) \\ 0 & \text{if } |\bar{\varepsilon}^0(t+1)| \le \bar{\delta}(t+1) \\ \varepsilon^0(t+1) + \bar{\delta}(t+1) & \text{if } \bar{\varepsilon}^0(t+1) < -\bar{\delta}(t+1) \end{cases} \tag{11.203}$$

where $\bar{\delta}(t+1)$ is given by:

$$\bar{\delta}(t+1) = d_2 + \frac{d_1}{m(t)} \ge |\bar{w}(t+1)| \tag{11.204}$$

and $\varepsilon^0(t+1)$ is given by (11.201).

Theorem 11.6 *For the system* (11.170), (11.172), (11.176) *and* (11.177) *where the disturbances and the unmodeled dynamics are defined by* (11.195) *and* (11.196), *using the PAA given by* (11.202), (11.203) *and* (11.204) *with the normalized variables defined in* (11.197) *and* (11.198), *one has*:

(i) $$\lim_{t\to\infty} \sup |\bar{\varepsilon}^0(t)| \le \bar{\delta}(t) \tag{11.205}$$

(ii) $$\|\tilde{\theta}_c(t)\| < M < \infty; \quad \forall t \tag{11.206}$$

(iii) $$\lim_{t\to\infty} \|\hat{\theta}_C(t+1) - \hat{\theta}_C(t)\| = 0 \tag{11.207}$$

(iv) $$\varepsilon^0(t+1) = \bar{\varepsilon}^0(t+1)m(t) \tag{11.208}$$

(v) *there is a value* $d_2 \le d_2^*$ *such that*:

$$\|\phi_C(t)\| < M_1 \quad \text{and} \quad |\varepsilon^0(t+1)| < M_2, \quad M_1, M_2 < \infty$$

Proof The proof of properties (i), (ii) and (iii) is similar to that of Theorem 11.5, except that one operates on normalized variables. One has:

$$\|\tilde{\theta}_C(t+1)\|^2_{F^{-1}} \le \|\tilde{\theta}_C(t)\|^2_{F^{-1}} - \frac{f^2[\bar{\varepsilon}^0(t+1)]}{1+\bar{\phi}_C^T(t)F\bar{\phi}_C(t)} \tag{11.209}$$

Similar to Theorem 11.5, one concludes that (ii) and (iii) hold. From (11.209) one also has:

$$\lim_{t\to\infty} \frac{f^2[\bar{\varepsilon}^0(t+1)]}{1+\bar{\phi}_C^T(t)F\bar{\phi}_C(t)} = 0 \tag{11.210}$$

and since $\|\bar{\phi}_C(t)\| \leq 1$, one concludes that:

$$\lim_{t\to\infty} f[\bar{\varepsilon}^0(t+1)] = 0 \tag{11.211}$$

which implies (11.205). From the definition of the normalized variables, one gets (11.208). Equation (11.186) can also be rewritten as:

$$\begin{aligned}\|\phi_C(t+1)\|^2 &\leq c_1' + c_2' \max_{0<k<t+1} |\varepsilon^0(k+1)|^2 \\ &= c_1' + c_2' \max_{0<k<t+1} |\bar{\varepsilon}^0(k+1)|^2 m^2(k)\end{aligned} \tag{11.212}$$

Assume now that $\phi_C(t+1)$ diverges. It follows that there exists a subsequence such that for this subsequence $t_1, t_2, \dots, t_n$, one has:

$$\|\phi_C(t_1)\| \leq \|\phi_C(t_2)\| \leq \cdots \leq \|\phi_C(t_n)\| \tag{11.213}$$

or, equivalently:

$$\|\phi_C(t)\| \leq \|\phi_C(t_n)\|; \quad \forall t \leq t_n \tag{11.214}$$

In the meantime as $\phi_C(t)$ diverges, it follows that $\bar{\delta}^2(t) \to d_2^2$. Therefore:

$$|\varepsilon^0(t+1)|^2 \leq d_2^2 m^2(t) \leq d_2^2 m^2(t+1) \tag{11.215}$$

Introducing this result in (11.212), one gets:

$$\|\phi_C(t+1)\|^2 \leq c_1' + c_2' d_2^2 m^2(t+1) \tag{11.216}$$

and, respectively:

$$\frac{\|\phi_C(t+1)\|^2 - c_1'}{m^2(t+1)} \leq c_2' d_2^2 \tag{11.217}$$

The LHS of (11.217) converges toward a positive number $\rho \leq 1$ as $\phi_C(t+1)$ diverges $(m^2(t+1) \leq \gamma + \|\phi(t+1)\|^2, \gamma > 0)$. Therefore, there are values of $d_2 \leq d_2^*$ such that $c_2' d_2^2 < 1$ which leads to a contradiction. Therefore, for $d_2 \leq d_2^*$, $\|\phi_C(t+1)\|$ is bounded and from (11.208), it results also that $\varepsilon^0(t+1)$ is bounded. □

Since d_2 corresponds to the "μ-scaled infinity norm" of the unmodeled dynamics $H(z^{-1})$ filtered by $S'(z^{-1})$, one gets the condition that boundedness of $\phi_C(t+1)$ is assured for:

$$\|S'(\mu^{-1}z^{-1}) \cdot H(\mu^{-1}z^{-1})\|_\infty < d_2^* \tag{11.218}$$

In the above approach, the unmodeled response of the plant to be controlled has been represented as an equivalent disturbance. Using normalization it was possible to convert the problem to the bounded disturbance case and using a PAA with dead zone, boundedness of the various signals is assured under a certain condition upon the unmodeled dynamics.

One may question if the PAA with dead zone is really necessary in order to handle unmodeled dynamics in direct adaptive control (even if a small dead zone is always useful in practice). The answer is that it is not always necessary and this can be shown using an equivalent feedback representation of the adaptive systems in the presence of unmodeled dynamics (Ortega et al. 1985; Ortega 1993).

In what follows we will concentrate on the main steps of the analysis and on the interpretation of the results for the design of robust direct adaptive controllers. The proofs of some intermediate steps are omitted. The reader is invited to consult (Ortega et al. 1985; Ortega 1993) for further details. To simplify the presentation we will consider:

- delay $d = 0$;
- no filters on the regressor vector or on the tracking error;
- PAA with constant adaptation gain;
- the unmodeled dynamics is represented by a multiplicative uncertainty on the input.

The plant model will be represented by:

$$A(q^{-1})y(t+1) = B^*(q^{-1})[1 + G(q^{-1})]u(t) \tag{11.219}$$

where $G(q^{-1})$ accounts for the unmodeled dynamics (multiplicative uncertainty). The filtered predicted output is given by:

$$Py(t+1) = (AS' + q^{-1}R)y(t+1) \tag{11.220}$$

where S' and R are solutions of the polynomial equation (11.171). Taking into account (11.219), one has from (11.220):

$$Py(t+1) = Su(t) + SGu(t) + Ry(t) \tag{11.221}$$

where $S = S'B^*$. From the definition of $\varepsilon^0(t+1)$ given in (11.177) and using (11.176), one gets:

$$\varepsilon^0(t+1) = (1+G)Su(t) + Ry(t) - \hat{\theta}_C^T(t)\phi_C(t) \tag{11.222}$$

Bearing in mind that:

$$Su(t) + Ry(t) = \theta_C^T \phi_C(t) \tag{11.223}$$

and adding and subtracting the terms: $\pm GRy(t), \pm G\hat{\theta}_C^T(t)\phi_C(t)$, (11.222) becomes:

$$\varepsilon^0(t+1) = -[1+G]\tilde{\theta}_C^T(t)\phi(t) - GRy(t) + GPy^*(t+1) \tag{11.224}$$

where:

$$\tilde{\theta}_C(t) = \hat{\theta}_C(t) - \theta_C \tag{11.225}$$

From (11.177), one has that:

$$Py(t+1) = \varepsilon^0(t+1) + Py^*(t+1) \tag{11.226}$$

and, therefore:

$$y(t+1) = \frac{1}{P}\varepsilon^0(t+1) + y^*(t+1) \tag{11.227}$$

Using this in (11.224), one gets:

$$\begin{aligned}\varepsilon^0(t+1) = &-[1+G]\tilde{\theta}_C^T(t)\phi(t) - GR_P\varepsilon^0(t+1)\\ &- GR_PPy^*(t+1) + GPy^*(t+1)\end{aligned} \tag{11.228}$$

where:

$$R_P(q^{-1}) = \frac{q^{-1}R}{P} \tag{11.229}$$

Rearranging the various terms in (11.228), it results that:

$$\begin{aligned}\varepsilon^0(t+1) &= -\left(\frac{1+G}{1+GR_P}\right)\tilde{\theta}_C^T(t)\phi(t) + G\frac{[1-R_P]}{1+GR_P}Py^*(t+1)\\ &= -H_1(q^{-1})\tilde{\theta}_C^T(t)\phi(t) + e^*(t+1)\end{aligned} \tag{11.230}$$

where:

$$H_1(q^{-1}) = \frac{1+G}{1+GR_P}; \qquad e^*(t+1) = (H_1-1)Py^*(t+1) \tag{11.231}$$

Equation (11.230) defines a linear block for an equivalent feedback representation of the system (since $\tilde{\theta}_C(t)$ is a function of $\varepsilon^0(t)$) where $H_1(q^{-1})$ represents the transfer operator of this linear block and $e^*(t+1)$ is an exogenous input which is bounded if (H_1-1) is asymptotically stable. One considers the following normalized parameter adaptation algorithm:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + f\bar{\phi}_C(t)\bar{\varepsilon}^0(t+1); \quad 0 < f < 2 \tag{11.232}$$

where:

$$\bar{\phi}_C(t) = \frac{\phi_C(t)}{m(t)}; \qquad \bar{\varepsilon}^0(t+1) = \frac{\varepsilon^0(t+1)}{m(t)} \tag{11.233}$$

$$\begin{aligned}m^2(t) &= \mu^2 m^2(t-1) + \max(\|\phi_C(t)\|^2, 1)\\ m(0) &= 1; \quad 0 < \mu < 1\end{aligned} \tag{11.234}$$

Associated with (11.232) one can define a system with input $\bar{\varepsilon}^0(t+1)$ and output $\tilde{\theta}_C^T(t)\bar{\phi}_C(t)$:

$$\tilde{\theta}_C^T(t)\bar{\phi}_C(t) = H_2\varepsilon^0(t+1) \tag{11.235}$$

where H_2 defines the input-output operator. Using the normalized adaptation algorithm one can establish the following result:

Lemma 11.4 *For the PAA of* (11.232), *one has*:

$$\sum_{0}^{t_1} \tilde{\theta}_C^T(t)\bar{\phi}_C(t)\bar{\varepsilon}^0(t+1) \geq -\frac{f}{2}\sum_{0}^{t_1}\bar{\varepsilon}^0(t+1)^2 - \gamma_2^2; \quad \gamma_2^2 < \infty \tag{11.236}$$

Remark The system with the input $\bar{\varepsilon}^0(t+1)$ and the output $\tilde{\theta}_C^T(t)\bar{\phi}_C(t)$ has a lack of input passivity. However, the system with output $\tilde{\theta}_C^T(t)\bar{\phi}_C(t) + (f/2)\varepsilon^0(t+1)$ is passive.

Proof From (11.225) and (11.232) one has:

$$\begin{aligned}\|\tilde{\theta}_C(t+1)\|^2 &= \|\tilde{\theta}_C(t)\|^2 + 2f\tilde{\theta}_C^T(t)\bar{\phi}_C(t)\bar{\varepsilon}^0(t+1) \\ &\quad + f^2\|\bar{\phi}_C(t)\|^2\bar{\varepsilon}^0(t+1)^2 \end{aligned} \tag{11.237}$$

$$\begin{aligned}\tilde{\theta}_C^T(t)\phi_C(t)\varepsilon^0(t+1) &= -\frac{f}{2}\|\bar{\phi}_C(t)\|^2\bar{\varepsilon}^0(t+1)^2 + \frac{1}{2f}\|\tilde{\theta}_C(t+1)\|^2 \\ &\quad - \frac{1}{2f}\|\tilde{\theta}_C(t)\|^2 \end{aligned} \tag{11.238}$$

Summing up from 0 to t_1 and taking into account that $\|\bar{\phi}_C(t)\| \leq 1$ one gets (11.236). □

One can now consider a normalized version of the system (11.230) with input $-\tilde{\theta}_C^T(t)\bar{\phi}_C(t)$ and output $\bar{\varepsilon}^0(t+1)$, i.e.:

$$\bar{\varepsilon}^0(t+1) = -\bar{H}_1\tilde{\theta}_C^T(t)\bar{\phi}_C(t) + \bar{e}^*(t+1) \tag{11.239}$$

where $\bar{H}_1$ is an input-output operator whose properties are related to those of $H_1(q^{-1})$. Systems (11.239) and (11.235) form a feedback interconnection. This is shown in Fig. 11.3 where the detailed structure of (11.239) is emphasized. One observes the presence of multipliers in the feedforward path, as well as the presence of a delay in the feedback path (this is why the PAA is not passive). The use of multipliers in analysis of stability of feedback systems goes back to the Popov criterion (Popov 1960) and is part of the "loop transformations" used in the analysis of feedback systems (Desoer and Vidyasagar 1975). Provided that the multipliers and their inverse are bounded causal operators, the stability of the equivalent system will imply the stability of the system without multipliers, from which it will be possible to conclude upon the boundedness of $\phi_C(t)$ and $\varepsilon^0(t+1)$.

The proof is divided in two major steps:

1. Analysis of the stability of the normalized equivalent feedback system (Fig. 11.3).
2. Proof of boundedness of $\phi_C(t)$ and of the multipliers and their inverse which implies in fact the stability of the feedback system without multipliers.

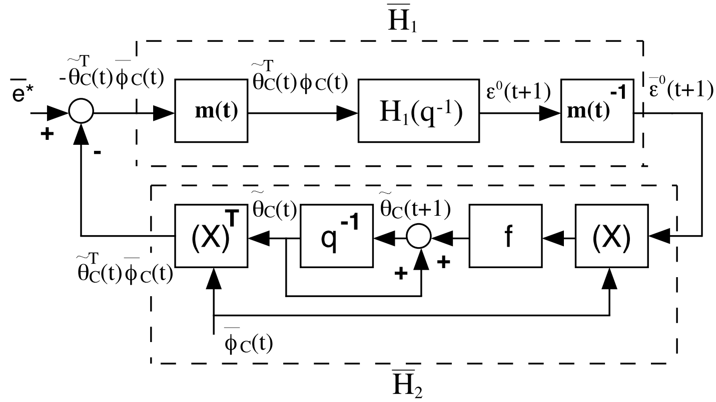


Fig. 11.3 Equivalent feedback representations of the direct adaptive control scheme in the presence of unmodeled dynamics when using data normalization

We will limit ourselves to the point 1 of the proof. To do this we need an intermediate result upon the passivity of the system (11.239).

Lemma 11.5 *For the system* (11.239) *with input* $-\tilde{\theta}^T(t)\bar{\phi}_C(t)$ *and output* $\bar{\varepsilon}^0(t+1)$, *one has for* $\bar{e}^*(t) = 0$:

$$-\sum_0^t \tilde{\theta}_C^T(t)\bar{\phi}_C(t)\bar{\varepsilon}^0(t+1) \geq \delta \sum_0^t \bar{\varepsilon}^0(t+1)^2 + \kappa \sum_0^t \hat{\theta}_C^T(t)\bar{\phi}_C(t) - \gamma_1^2$$

$$\delta > 0,\ \kappa > 0,\ \gamma_1^2 < \infty \tag{11.240}$$

if $H_1(\mu^{-1}z^{-1})$ *defines a very strictly passive system, i.e.*:

$$\operatorname{Re} H_1(\mu^{-1}e^{-j\omega}) \geq \delta |H_1(\mu^{-1}e^{-j\omega})|^2 + \kappa; \quad 0 < \delta,\ \kappa < \infty \tag{11.241}$$

Adding (11.236) and (11.240) one gets:

$$\left(\delta - \frac{f}{2}\right) \sum_0^t \bar{\varepsilon}^0(t+1)^2 + \kappa \sum_0^t [\hat{\theta}_C^T(t)\bar{\phi}_C(t)]^2 \leq \gamma_1^2 + \gamma_2^2 \tag{11.242}$$

and one concludes that if $\delta > f/2$:

$$\lim_{t\to\infty} \bar{\varepsilon}^0(t+1) = \lim_{t\to\infty} \tilde{\theta}_C^T(t)\bar{\phi}_C^T(t) = 0 \tag{11.243}$$

This analysis can be summarized as follows.

Theorem 11.7 *Assume that there is a* $\delta > f/2$ *for which* (11.241) *holds, then*:

$$\|\bar{\varepsilon}^0(t+1)\|_2, \|\tilde{\theta}_C^T \bar{\phi}_C(t)\|_2 \leq c\|\bar{e}^*(t)\|_2 + \beta \tag{11.244}$$

If, in addition:

$$\|\bar{e}^*(t)\|_2 \leq \alpha < \infty \tag{11.245}$$

then:

$$\lim_{t\to\infty} \bar{\varepsilon}^0(t+1) = \lim_{t\to\infty} \tilde{\theta}_C^T(t)\tilde{\phi}_C^T(t) = 0 \tag{11.246}$$

In Ortega et al. (1985), Ortega (1993), it is shown that $\phi_C(t)$ is bounded and this allows to conclude that the multiplier $m(t)^{-1}$ and its inverse $m(t)$ are bounded, which leads to conclusion that:

$$\|\varepsilon^0(t+1)\|_2, \|\tilde{\theta}_C^T(t)\phi_C(t)\|_2 \leq c\|e^*(t)\|_2 + \beta \tag{11.247}$$

The major implications of this result are:

1. The PAA with dead zone may not be necessary in direct adaptive control in the presence of unmodeled dynamics.
2. It gives a direct relationship between on the one hand, the characteristics of the unmodeled dynamics and of the linear design (see the expression of $H_1(q^{-1})$) and, on the other hand, the value of the adaptation gain. This relationship has been deeply investigated in Cluett et al. (1987) for analysis and design of a robust direct adaptive control scheme.

The condition (11.241) combined with the condition $\delta > f/2$ has a very nice graphical interpretation which is very useful for analysis and design of robust direct adaptive controllers. To do this, we will make first the observation that based on Lemma 11.5 the feedforward block in Fig. 11.3 is very strictly passive provided that the transfer function $H_1(\mu^{-1}z^{-1})$ is strictly positive real and satisfies (11.241). Therefore, Fig. 11.3 takes the form shown in Fig. 11.4a where the feedback pass has a lack of input passivity which can be compensated by adding in parallel a block with a gain $f/2$. The resulting equivalent scheme is shown in Fig. 11.4b. It can be easily checked (see Lemma 11.4) that the new system with input $\bar{\varepsilon}_0(t+1)$ and output $\tilde{\theta}_C^T(t)\bar{\phi}_C(t) + \frac{f}{2}\bar{\varepsilon}^0(t+1)$ is passive.

Using the asymptotic hyperstability theorem (see Appendix C), asymptotic stability (for $\|\bar{e}^*\|_2 < \infty$) is assured if the equivalent new feedforward path is characterized by a strictly positive real transfer function, i.e.:

$$H_1'(\mu^{-1}z^{-1}) = \frac{H_1(\mu^{-1}z^{-1})}{1 - \frac{f}{2}H_1(\mu^{-1}z^{-1})} = SPR \tag{11.248}$$

But using the circle criterion (Zames 1966), this is equivalent to: $H_1(\mu^{-1}z^{-1})$ *should lie inside a circle centered on the real axis and passing through* $(0, \frac{2}{f})$ *with center* $c = 1/f$ *and radius* $r = 1/f$ (the original feedback path lies in a cone

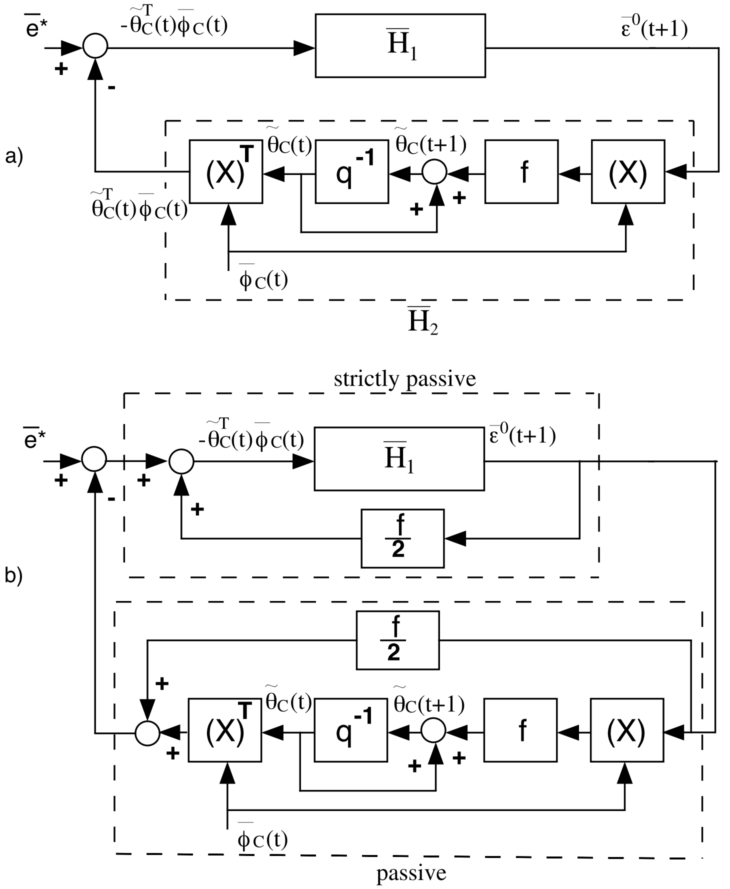
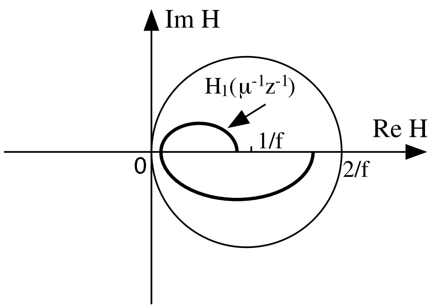


Fig. 11.4 Equivalent representations for the feedback system of Fig. 11.6

Fig. 11.5 Graphical interpretation of the condition (11.248)



$(0, -\frac{f}{2})$). This result is illustrated in Fig. 11.5. Therefore, for assumed nominal reduced order model, desired closed-loop poles and assumed unmodeled dynamics one can:

- compute $H_1'(\mu^{-1}z^{-1})$ given by (11.231);
- find the value of μ for which $H_1(\mu^{-1}z^{-1})$ has its poles inside $|z| < \mu < 1$ (this value is used also for data normalization);
- trade between μ and f in order that $H_1(\mu^{-1}z^{-1})$ lies inside the circle with $c = \frac{1}{f}, j0$ and $r = \frac{1}{f}$.

Note that as f increases, the circle becomes smaller. Note also that in the absence of unmodeled dynamics $H_1(z^{-1}) = 1$ and the stability condition is always satisfied. From Fig. 11.5, one also concludes that either condition (11.248) or (11.241) plus the condition $\delta > f/2$ is equivalent to:

$$\left\| H_1(\mu^{-1}z^{-1}) - \frac{1}{f} \right\|_{\infty} < \frac{1}{f}$$

which can be rewritten as:

$$\| f H_1(\mu^{-1}z^{-1}) - 1 \|_{\infty} < 1$$

Since 1 inside the norm sign can be interpreted as the value of $H_1(z^{-1})$ for the "tuned" case without unmodeled dynamics, one concludes that the image of the unmodeled dynamics multiplied by f should be relatively close to one at all frequencies.

11.6 An Example

In this example, we will illustrate the influence of the regulation dynamics (the polynomial $P(q^{-1})$) on the performance of the adaptive tracking and regulation with independent objectives (Sect. 11.2).

Two different plant models are considered. The plant model before a parameter change occurs is characterized by the discrete transfer operator:

$$G_1(q^{-1}) = \frac{q^{-2}(1 + 0.4q^{-1})}{(1 - 0.5q^{-1})[1 - (0.8 + 0.3j)q^{-1}][1 - (0.8 - 0.3j)q^{-1}]}$$

At time $t = k$, a change of the plant model parameters is made. The new plant model is characterized by the transfer operator:

$$G_2(q^{-1}) = \frac{q^{-2}(0.9 + 0.5q^{-1})}{(1 - 0.5q^{-1})[1 - (0.9 + 0.42j)q^{-1}][1 - (0.9 - 0.42j)q^{-1}]}$$

The simulations have been carried out for two different values of the regulation polynomial:

1. $P_1(q^{-1}) = 1$ (deadbeat control);
2. $P_2(q^{-1}) = 1 - 1.262q^{-1} + 0.4274q^{-2}$.

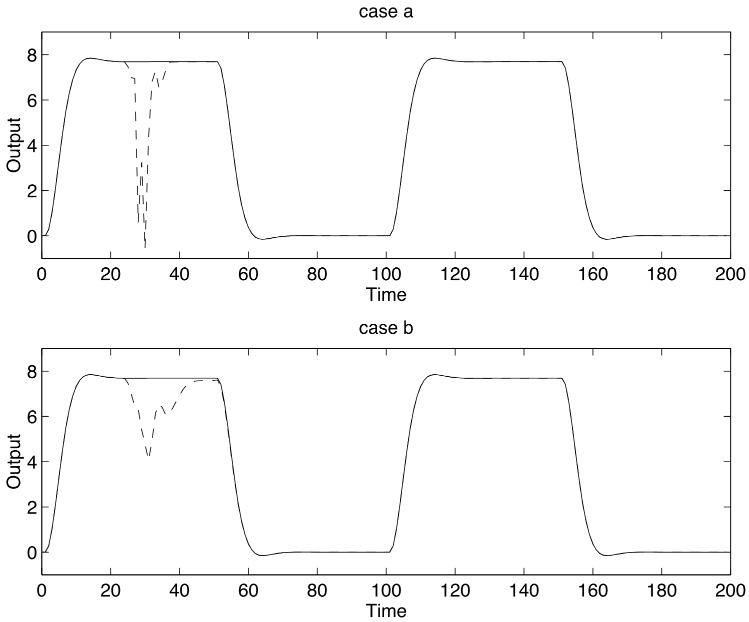


Fig. 11.6 Tracking behavior in the presence of plant model parameters changes at $t = 25$: [–] desired output, [- -] achieved output. (**a**) Regulation dynamics $P(q^{-1}) = 1$, (**b**) regulation dynamics $P(q^{-1}) = 1 - 1.262q^{-1} + 0.4274q^{-2}$

$P_2(q^{-1})$ corresponds to the discretization ($T_S = 1$ s) of a continuous-time second-order system with $\omega_0 = 0.5$ rad/s and $\zeta = 0.85$.

The tracking reference model is characterized by:

$$\frac{B_m(q^{-1})}{A_m(q^{-1})} = \frac{(0.28 + 0.22q^{-1})}{(1 - 0.5q^{-1})[1 - (0.7 + 0.2j)q^{-1}][1 - (0.7 - 0.2j)q^{-1}]}$$

In all simulations a PAA with constant trace adaptation gain has been used with $\operatorname{tr} F(t) = \operatorname{tr} F(0)$, $F_0 = \operatorname{diag}[10]$ and $[\lambda_1(t)/\lambda_2(t)] = 1$.

Figure 11.6 shows the desired trajectory and the achieved trajectory during a sequence of step changes on the reference and in the presence of the parameter variations occurring at $t = 25$. One can observe that the adaptation transient is much smoother using the $P_2(q^{-1})$ regulation polynomial than in the case $P_1(q^{-1})$.

Figure 11.7 shows the behavior in regulation, i.e., the evolution of the output from an initial condition at $t = 0$ in the presence of a change in the parameters at $t = 0$. Same conclusion can be drawn: the poles defined by the regulation polynomial strongly influence the adaptation transient. Two fast dynamics in regulation with respect to the natural response of the system will induce undesirable adaptation transients.

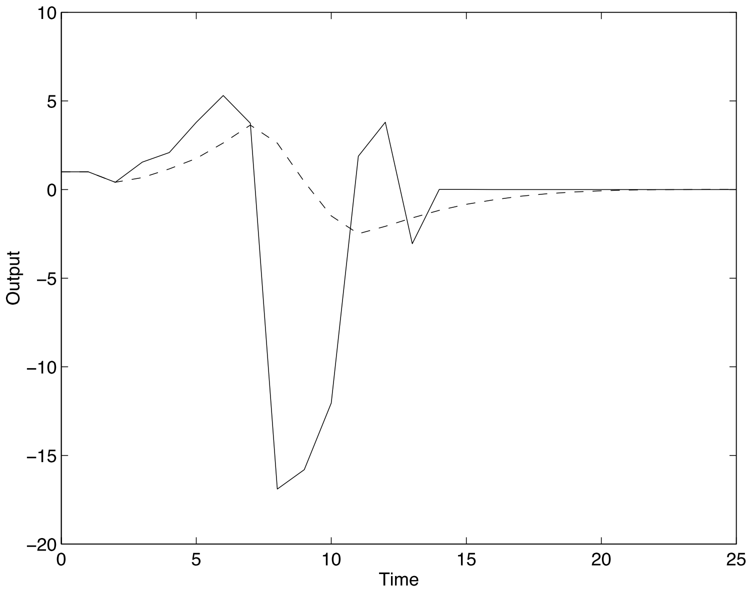


Fig. 11.7 Regulation behavior in the presence of plant model parameters changes at $t = 0$. [–] Regulation dynamics $P(q^{-1}) = 1$, [- -] regulation dynamics $P(q^{-1}) = 1 - 1.262q^{-1} + 0.4274q^{-2}$

11.7 Concluding Remarks

The direct adaptive control algorithms presented in this chapter share a number of features which are summarized below.

1. Direct adaptive control schemes lead to simple implementations since the parameters of the controller are directly updated (however, for large integer delay d it may be more convenient to use an indirect adaptive control scheme).
2. The specification of the desired performance is simple for adaptive tracking and regulation and adaptive minimum-variance tracking and regulation.
3. For adaptive tracking and regulation with weighted input and the adaptive generalized minimum variance, one cannot guarantee the resulting closed-loop performances (the resulting closed-loop poles will depend also upon the current values of $A(q^{-1})$ and $B(q^{-1})$).
4. The weakness of this approach is the fact that it can be used only for a restricted class of plants characterized by stable zeros, for the whole domain of possible variations of the plant model parameters.
5. In general, in the deterministic environment, one uses non-vanishing adaptation gains and in the stochastic environment one uses time-decreasing adaptation gains with resetting from time to time.

6. Robustification of the parameter adaptation algorithms may be necessary in the presence of unmodeled dynamics (i.e., when one uses for design a nominal model of lower order than the true one).
7. Most industrial applications up to 1986 have been accomplished using this approach and several industrial products incorporating these techniques have been developed, see Seborg et al. (1989), Dumont (1992), Åström and Wittenmark (1995).

11.8 Problems

11.1 Give the details of the proof for the Theorem 11.5 using the averaging method (convergence analysis of the adaptive generalized minimum variance and regulation).

11.2 Show that for a recursive least square one step ahead predictor estimating an ARMAX model:

$$A(q^{-1})y(t+1) = B^*(q^{-1})u(t) + C(q^{-1})e(t+1);\ n_A = n_C$$

(a) $\theta^{*^T} = [(c_1 - a_1), \ldots, (c_{n_A} - a_{n_A}), b_1, \ldots, b_{n_B}]$ is a possible convergence point.
(b) the sufficient condition for w.p.1 convergence towards

$$D_C = \{\hat{\theta} | [\theta^* - \hat{\theta}]^T \phi(t, \hat{\theta}) = 0\}$$

is the strict positive realness of $\frac{1}{C(z^{-1})} - \frac{\lambda_2}{2}$ for $2 > \lambda_2 \geq \sup_t \lambda_2(t)$.
(c) assuming that $[\theta^* - \hat{\theta}]^T \phi(t, \hat{\theta}) = 0$ has only one solution $\hat{\theta} = \theta^*$, the estimated parameters correspond to the coefficients of the minimum variance regulator.

Interpret the results and compare with the analysis given in Sect. 11.4.2.

11.3 In the stochastic environment, for an ARMAX plant model with $d = 0$, define an adaptation error as:

$$\varepsilon^0(t+1) = y(t+1) - y^*(t+1)$$

Using the averaging method (Theorem 4.1) derive directly the adaptive minimum variance tracking and regulation scheme assuring:

$$\text{Prob}\left\{\lim_{t\to\infty} \varepsilon^0(t+1) = \lim_{t\to\infty} [y(t+1) - y^*(t+1)] = e(t+1)\right\} = 1$$

11.4 The adaptive tracking and regulation with independent objectives (Sect. 11.2) uses in the observation vector entering in the adaptation algorithm $y(t), y(t-1), \ldots, u(t), u(t-1)$, which are directly corrupted by noise. In order to remove asymptotically the effect of the measurement noise, try to develop a direct adaptive control scheme which uses the ideas of closed-loop output error recursive identification algorithms (Sect. 9.2).

11.5 Extend the results of Theorem 11.5 for robust direct adaptive control in the presence of bounded disturbances for the case of a plant with pure time delay $d \geq 1$.

11.6 Consider the following plant model:

$$(1 + a_1 q^{-1}) y(t+1) = b_1 u(t) + b_2 u(t-1) = b_1 [1 + \beta q^{-1}] u(t)$$

For the case $\beta = \frac{b_2}{b_1} > 1$ ($b_1, b_2 > 0$), the system has an unstable zero. A robust direct adaptive control for the reduced order system with $B^*(q^{-1}) = b_1$ using data normalization is considered. Assuming that the nominal model is characterized by $a_1 = -0.5$, $b_1 = 1$, that the neglected dynamics is characterized by $\beta = 1.5$ or 2 and the desired closed-loop pole is characterized by $P(q^{-1}) = 1 + p_1 q^{-1}$ with $p_1 = -0.5$ or -0.3 find:

1. The value of μ to be used in the dynamic data normalization for the various values of p_1 and β when using the PAA given in (11.234).
2. Discuss the influence of the values of the unmodeled dynamics and of the desired closed-loop values upon μ.

Hint: Use Lemma 11.5.

11.7 Consider the system:

$$A(q^{-1}) y(t+1) = q^{-d} B^*(q^{-1}) u(t)$$

Assume that $A(z^{-1})$ is asymptotically stable and well damped but $B^*(z^{-1})$ does not need to be stable.

Consider the pole placement strategy (internal model control—Sect. 7.3.4) where the polynomials $S(q^{-1})$ and $R(q^{-1})$ of the controller are solutions of:

$$A(q^{-1}) S(q^{-1}) + q^{-d-1} B^*(q^{-1}) R(q^{-1}) = A(q^{-1}) P_0(q^{-1})$$

Show that it is possible to build a direct adaptive control scheme in which one directly estimates the parameters of the controller when $A(q^{-1})$ and $B^*(q^{-1})$ are unknown.

11.8 (Adaptive generalized minimum variance tracking and regulation.) Consider the control objective for $d = 0$:

$$\text{Prob}\left\{ \lim_{t\to\infty} \varepsilon^0(t+1) = \lim_{t\to\infty} \left[y(t+1) + \lambda \frac{Q}{C} u(t) - y^*(t+1) \right] = e(t+1) \right\} = 1$$

1. Develop an adaptive control for the system given in (11.78).
   Hint: define the generalized output:

$$\bar{y}(t+1) = y(t+1) + \lambda \frac{Q}{C} u(t)$$

   and use the methodology presented in Sect. 11.4 (see also Sect. 11.3).
2. Analyse the asymptotic behavior of the scheme using the averaging method given in Sect. 4.2 (see also Sect. 11.4.2).