Chapter 12
Indirect Adaptive Control

12.1 Introduction

As indicated in Chap. 1, *indirect adaptive control* is a very general approach to adaptive control since, in principle, one can combine any parameter estimation scheme with any control strategy. The adaptation of the controller parameters is carried out in two steps:

Step 1: On-line estimation of the plant model parameters.
Step 2: On-line computation of the controller parameters based on the estimated plant model.

The resulting control scheme should guarantee that the input and the output of the plant remain bounded and that some indices of performance are achieved asymptotically. This requires a deep analysis of the indirect adaptive control system which is nonlinear and time-varying.

Fortunately, good properties of the indirect adaptive control schemes will be guaranteed if, separately, the parameter estimation algorithm and the control strategy satisfy a number of properties (largely similar to those required for indirect adaptive prediction) which are summarized next:

1. The a posteriori adaptation error in the parameter estimation algorithm goes to zero asymptotically.
2. The a priori adaptation error does not grow faster than the observation vector (containing the input and output of the plant).
3. The estimated plant model parameters are bounded for all t.
4. The variations of the estimated plant model parameters go to zero asymptotically.
5. The *design equation*(*s*) provide(s) bounded controller parameters for bounded plant parameter estimates.
6. The estimated plant model is *admissible* with respect to the *design equation* which means that assuming that it corresponds to the exact plant model, the resulting controller has bounded parameters, stabilizes the system and achieves the desired performances.

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If these conditions are satisfied, it is then possible to show that the input and the output of the plant remain bounded and that some indices of performances are achieved asymptotically.

Probably, the most difficult problem (at least theoretically) is to guarantee that any estimated model is *admissible* with respect to the control design strategy. For every control strategy, even if the estimated plant parameters are bounded at each time t, the current estimated model may not be admissible in the sense that there is no solution for the controller. For example, if pole placement is used as the control strategy, an estimated model at time t, which features a pole-zero cancellation, will not allow to compute the controller. Similarly, using a LQ control, if the estimated model at time t is not stabilizable the control cannot be computed. Furthermore, getting close to non-admissibility situations will lead to numerical problems resulting in very large and undesirable control actions. The consequence of this fact is twofold:

1. One has to be aware of the admissibility conditions and take appropriate ad hoc action in practice, in order to deal with the singularities which may occur during adaptation.
2. One can make a theoretical analysis and develop *modifications* of the parameter estimates (or of the algorithms) in order to avoid the singularities corresponding to the non-admissibility of an estimated plant model.

As indicated earlier (Property 6), to remove the singularities one has to assume that the unknown plant model is admissible with respect to the control strategy.

While the various *modifications* of the estimated parameters or of the algorithms resulting from a theoretical analysis are in general complex to implement, they have the merit of showing that there are solutions that avoid the *non-admissibility* of the estimated model and give hints for simple ad hoc modifications to be implemented.

The analysis of the indirect adaptive control schemes is carried out in two steps:

1. One assumes that the estimated models are always in the admissibility set.
2. One modifies the parameter estimation algorithm in order to satisfy the admissibility condition.

The objective of the analysis is to guarantee that a stabilizing controller is obtained without any assumption about the presence or the richness of an external excitation. However, the use of an external or internal excitation signal, even for short periods of time, is useful for speeding up the convergence of the adaptive control scheme since it can be shown that exponential stability is achieved under richness conditions (Anderson and Johnstone 1985).

To implement an indirect adaptive control strategy effectively, we have two major options. The choice is related to a certain extent to the ratio between the computation time and the sampling period.

Strategy 1

1. Sample the plant output;
2. Update the plant model parameters;

3. Compute the controller parameters based on the new plant model parameter estimates;
4. Compute the control signal;
5. Send the control signal;
6. Wait for the next sample.

Using this strategy, there is a delay between $u(t)$ and $y(t)$ which will essentially depend upon the time required to achieve (2) and (3). This delay should be small with respect to the sampling period and, of course, smaller than the delay between $u(t)$ and $y(t)$ scheduled in the I/O system (see Chap. 16 for details). In this strategy, a posteriori parameter estimates are used and the a posteriori adaptation error will occur in the stability analysis.

Strategy 2

1. Sample the plant output;
2. Compute the control signal based on the controller parameters computed during the previous sampling period;
3. Send the control signal;
4. Update the plant model parameters;
5. Compute the controller parameters based on the new plant model parameter estimates;
6. Wait for the next sample.

Using this strategy, the delay between $u(t)$ and $y(t)$ is smaller than in the previous case. In fact, this is the strategy used with constant parameters controllers (Steps 4 and 5 are deleted). In this strategy, one uses a priori parameter estimates and the a priori adaptation error will appear in the stability analysis.

The analysis of the resulting schemes is very similar except that as indicated earlier, in Strategy 1 the a posteriori adaptation error will play an important role, while in Strategy 2, the properties of the a priori adaptation error will be used.

One uses a non-vanishing adaptation gain to get an indirect adaptive control scheme which can react to changes in plant model parameters. One uses a time-decreasing adaptation gain to implement indirect adaptive control schemes for the case of plant models with unknown but constant parameters (over a large time horizon). In the latter case, adaptation can be restarted either on demand or automatically, based on the analysis of an index of performance.

As indicated in Chap. 1, nothing stops us updating the estimates of the plant model parameters at each sampling instant, but updating the controller parameters only every N sampling instants. The analysis remains the same as long as N is finite. The use of this approach is related to:

- the possibility of getting better parameter estimates for control design,
- the eventual reinitialization of the plant parameters estimation algorithm after each controller updating,

- the use of more sophisticated control designs requiring complex computations (in particular robust control design),
- the reduction of the risk for getting non-admissible estimated plant models.

If the plant to be controlled has constant parameters over a large time horizon and if one considers a large horizon N for plant model parameters estimation, followed by the updating of the controller based on the results of the plant model identification in closed loop, one gets what is called the *iterative identification in closed loop and controller redesign*. This can be considered as a limit case of indirect adaptive control. The main difference is that the plant model parameters estimation is done in closed loop in the presence of a linear controller with fixed parameters. This suggests that *among all the parameter estimation algorithms which allow us to obtain a stable indirect adaptive control scheme, it may be useful to use those dedicated to identification in closed loop* (see Chap. 9).

For the case of plant models with unknown but constant parameters over a large horizon, one may ask what is the best choice among:

- adaptive control with controller updating each sampling instant;
- adaptive control with controller updating each N sampling instants;
- plant model identification in closed loop followed by the redesign of the controller.

The answer is an "engineering" type answer, since essentially a trade-off between computation power available and performances will, to a large extent, dictate the choice of one or another strategy.

Adaptive pole placement and adaptive generalized predictive control are probably the most popular indirect adaptive control strategies used in applications. We have chosen adaptive pole placement as the prototype for the analysis of the indirect adaptive control schemes (Sect. 12.2). The results can be applied *mutatis mutandis* to the other indirect adaptive control strategies. Robustness aspects related to the violation of the basic hypotheses will be examined in detail in Sect. 12.3. Adaptive generalized predictive control and adaptive linear quadratic control will be presented in Sects. 12.4 and 12.5, respectively.

A non-exhaustive list of references detailing various indirect adaptive control strategies is given below.

Adaptive pole placement (Goodwin and Sin 1984; Anderson and Johnstone 1985; de Larminat 1980, 1984, 1986; Guo 1996; Lozano and Goodwin 1985; Lozano 1992; Giri et al. 1990, 1989; de Larminat and Raynaud 1988; Middleton et al. 1988; Mareels and Polderman 1996)

Adaptive generalized predictive control (Clarke and Mohtadi 1989; Bitmead et al. 1990; Giri et al. 1991; M'Saad et al. 1993a)

Adaptive linear quadratic control (Samson 1982; Lam 1982; Ossman and Kamen 1987; Giri et al. 1991; M'Saad and Sanchez 1992)

12.2 Adaptive Pole Placement

12.2.1 The Basic Algorithm

We will assume that the system operates in a deterministic environment. The basic algorithm combines any of the parameter estimation algorithms presented in Chaps. 3, 5 and 9 with the pole placement control strategy presented in Sect. 7.3. The plant model (with unknown parameters) is assumed to be described by:

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) = q^{-d-1}B^*(q^{-1})u(t) \tag{12.1}$$

where $u(t)$ and $y(t)$ are the input and the output of the plant and:

$$A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_{n_A}q^{-n_A} = 1 + q^{-1}A^*(q^{-1})$$
$$B(q^{-1}) = b_1q^{-1} + \cdots + b_{n_B}q^{-n_B} = q^{-1}B^*(q^{-1})$$

One assumes that:

- the orders of the polynomials $A(q^{-1})$, $B(q^{-1})$ and of the delay d are known (n_A, n_B, d-known);[1]
- $A(q^{-1})$ and $B(q^{-1})$ do not have common factors (admissibility condition).

Step I: Estimation of the Plant Model Parameters

As indicated earlier, one can use any parameter estimation algorithm presented in Chaps. 3, 5 and 9 since we operate in a deterministic environment. However, one has to take into account the following:

1. Convergence conditions (positive real type), which are required for some algorithms, will become also convergence conditions for the adaptive control scheme.
2. It is reasonable to use parameter estimation algorithms inspired from those dedicated to the identification in closed loop.

In a deterministic environment, the parameter estimates and the adaptation errors will satisfy the properties resulting from Theorem 3.2 and they will fulfill the points 1 to 5 mentioned in Sect. 12.1. To simplify the analysis, we will assume that a *recursive least squares* type parameter estimation algorithm will be used. The cases of using recursive least squares on filtered data or appropriate algorithms for closed-loop identification are illustrated in the example presented in Sect. 12.4 and in the applications presented in Sect. 12.7.

The plant output can be expressed as:

$$y(t+1) = \theta^T\phi(t) \tag{12.2}$$

[1] What is in fact needed is the knowledge of $n_B + d$. However, the knowledge of d reduces the number of estimated parameters.

where:

$$\theta^T = [a_1, \ldots, a_{n_A}, b_1, \ldots, b_{n_B}] \tag{12.3}$$

$$\phi^T(t) = [-y(t), \ldots, -y(t-n_A+1), u(t-d), \ldots, u(t-d-n_B+1)] \tag{12.4}$$

The a priori output of the adjustable predictor is given by:

$$\hat{y}^0(t+1) = \hat{\theta}^T(t)\phi(t) \tag{12.5}$$

The a posteriori output of the adjustable predictor is given by:

$$\hat{y}(t+1) = \hat{\theta}^T(t+1)\phi(t) \tag{12.6}$$

where:

$$\hat{\theta}^T(t) = [\hat{a}_1(t), \ldots, \hat{a}_{n_A}(t), \hat{b}_1(t), \ldots, \hat{b}_{n_B}(t)] \tag{12.7}$$

Accordingly, the a priori and the a posteriori prediction (adaptation) errors are given by:

$$\varepsilon^0(t+1) = y(t+1) - \hat{y}^0(t+1) \tag{12.8}$$

$$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) \tag{12.9}$$

The parameter adaptation algorithm is:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t)\phi(t)\varepsilon(t+1) \tag{12.10}$$

$$\begin{aligned} F(t+1)^{-1} &= \lambda_1(t)F(t)^{-1} + \lambda_2(t)\phi(t)\phi^T(t); \\ 0 < \lambda_1(t) &\le 1;\; 0 \le \lambda_2(t) < 2;\; F(0) > 0 \end{aligned} \tag{12.11}$$

$$\varepsilon(t+1) = \frac{\varepsilon^0(t+1)}{1 + \phi^T(t)F(t)\phi(t)} \tag{12.12}$$

It is also assumed that:

$$F(t)^{-1} \ge \alpha F(0)^{-1}; \quad F(0) > 0;\; \alpha > 0,\; \forall t \in [0, \infty] \tag{12.13}$$

Selection of $\lambda_1(t)$ and $\lambda_2(t)$ allows various forgetting profiles to be obtained (see Sect. 3.2.3) and this leads to:

- a time-decreasing adaptation gain;
- a time-decreasing adaptation gain with reinitialization;
- a non vanishing adaptation gain.

It also allows the introduction of the various modifications discussed in Chap. 10 in order to have a robust adaptation algorithm (this will be discussed in Sect. 12.3). Using this algorithm, one will have, according to Theorem 3.2, the following properties:

- The a posteriori adaptation error is bounded and goes to zero asymptotically, i.e.:

$$\lim_{t_1 \to \infty} \sum_{t=1}^{t_1} \varepsilon^2(t+1) \le C < \infty \tag{12.14}$$

$$\lim_{t \to \infty} \varepsilon(t+1) = 0 \tag{12.15}$$

- The a priori adaptation error satisfies:

$$\lim_{t \to \infty} \frac{[\varepsilon^0(t+1)]^2}{1 + \phi^T(t)F(t)\phi(t)} = 0 \tag{12.16}$$

- The parameter estimates are bounded for all t

$$\|\hat{\theta}(t)\| \le C < \infty; \quad \forall t \ge 0 \tag{12.17}$$

- The variations of the parameter estimates go to zero asymptotically:

$$\lim_{t \to \infty} \|\hat{\theta}(t+k) - \hat{\theta}(t)\| = 0; \quad \forall k < \infty \tag{12.18}$$

Step II: Computation of the Controller Parameters and of the Control Law

We will use the Strategy 1 (see Sect. 12.1) for updating the controller parameters. The controller equation generating $u(t)$ is:

$$\boxed{\hat{S}(t, q^{-1})u(t) + \hat{R}(t, q^{-1})y(t) = \hat{\beta}(t)P(q^{-1})y^*(t+d+1)} \tag{12.19}$$

or alternatively:

$$u(t) = -\hat{S}^*(t, q^{-1})u(t-1) - \hat{R}(t, q^{-1})y(t) + \hat{\beta}(t)P(q^{-1})y^*(t+d+1) \tag{12.20}$$

where:

$$\hat{\beta}(t) = 1/\hat{B}(t, 1) \tag{12.21}$$

$$\hat{S}(t, q^{-1}) = 1 + \hat{s}_1(t)q^{-1} + \cdots + \hat{s}_{n_S}(t)q^{-n_S} = 1 + q^{-1}\hat{S}^*(t, q^{-1}) \tag{12.22}$$

$$\begin{aligned} \hat{R}(t, q^{-1}) &= \hat{r}_0(t) + \hat{r}_1(t)q^{-1} + \cdots + \hat{r}_{n_R}(t)q^{-n_R} \\ &= \hat{r}_0(t) + q^{-1}\hat{R}^*(t, q^{-1}) \end{aligned} \tag{12.23}$$

and $\hat{S}(t, q^{-1})$, $\hat{R}(t, q^{-1})$ are solutions of:[2]

$$\boxed{\hat{A}(t, q^{-1})\hat{S}(t, q^{-1}) + q^{-d-1}\hat{B}^*(t, q^{-1})\hat{R}(t, q^{-1}) = P(q^{-1})} \tag{12.24}$$

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[2]In fact $\hat{S}(t, q^{-1}) = \hat{S}'(t, q^{-1})H_S(q^{-1})$, $\hat{R}(t, q^{-1}) = \hat{R}'(t, q^{-1})H_R(q^{-1})$ where $H_S(q^{-1})$ and $H_R(q^{-1})$ are the fixed parts of the controller.

$P(q^{-1})$ in (12.19), (12.20) and (12.24) are the desired closed-loop poles and y^* is the desired tracking trajectory. Equation (12.24) can be reformulated in matrix form:

$$M[\hat{\theta}(t)]\hat{x} = p \tag{12.25}$$

where:

$$\hat{x}^T = [1, \hat{s}_1, \ldots, \hat{s}_{n_S}, \hat{r}_0, \ldots, \hat{r}_{n_R}] \tag{12.26}$$

$$p^T = [1, p_1, \ldots, p_{n_P}, 0, \ldots, 0] \tag{12.27}$$

and M is the Sylvester matrix which has the form (see Sect. 7.3 for details):

$$\left.\begin{array}{c}
\overbrace{\hphantom{1 \quad 0 \quad \ldots \quad 0}}^{n_{B'}} \qquad \overbrace{\hphantom{0 \quad \ldots \quad \ldots \quad 0}}^{n_{\hat{A}}} \\
\underbrace{\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
\hat{a}_1 & 1 & \ddots & \vdots & \hat{b}'_1 & 0 & \ddots & \vdots \\
\vdots & & \ddots & 0 & \vdots & & \ddots & \\
 & & & 1 & & & & 0 \\
\vdots & & & \hat{a}_1 & \vdots & & & \hat{b}'_1 \\
\hat{a}_{n_{\hat{A}}} & & & & \hat{b}'_{n_{\hat{B}}} & & & \\
0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\
\vdots & \ddots & & & \vdots & \ddots & & \\
0 & \ldots & 0 & \hat{a}_{n_{\hat{A}}} & 0 & \ldots & 0 & \hat{b}'_{n_{\hat{B}}}
\end{bmatrix}}_{n_{\hat{A}}+n_{\hat{B}}}
\end{array}\right\} n_{\hat{A}}+n_{\hat{B}} \tag{12.28}$$

where $\hat{a}_i$ are estimated coefficients of the polynomial $A(q^{-1})$ and $\hat{b}'_i$ are the estimated coefficients of $B'(q^{-1}) = q^{-d}B(q^{-1})$ ($b'_i = 0$ for $i = 0, 1, \ldots, d$). Therefore, $\hat{S}(t)$ and $\hat{R}(t)$ are given by:

$$\hat{x} = M^{-1}[\hat{\theta}(t)]p \tag{12.29}$$

The admissibility condition for the estimated model is:

$$|\det M[\hat{\theta}(t)]| \geq \delta > 0 \tag{12.30}$$

which can alternatively be evaluated by the condition number:

$$\frac{\lambda_{\min} M[\hat{\theta}(t)]}{\lambda_{\max} M[\hat{\theta}(t)]} > \delta_0 > 0 \tag{12.31}$$

Remark If Strategy 2 is used, (12.19) through (12.31) remain the same except that the index t becomes $(t-1)$. The analysis of the resulting schemes follows the one for Strategy 1.

12.2.2 Analysis of the Indirect Adaptive Pole Placement

The objective of the analysis will be to show that:

- $y(t)$ and $u(t)$ are bounded;
- some indices of performance go to zero asymptotically.

This implies that the resulting controller stabilizes the system.

The analysis will be carried on without requiring the persistence of excitation. Before embarking in the analysis of the behavior of $y(t), u(t)$ and of certain error signals, it is useful to write first the equations describing the time evolution of the observation vector $\phi(t)$ which contains $y(t)$ and $u(t)$. To simplify the writing, we will furthermore assume that $d = 0$. We start by making the observation that using (12.2), (12.6) and (12.9), the plant output at time t can be expressed as:

$$\begin{aligned} y(t) &= \hat{y}(t) + \varepsilon(t) = \hat{\theta}^T(t)\phi(t-1) + \varepsilon(t) \\ &= -\sum_{i=1}^{n_A} \hat{a}_i(t)y(t-i) + \sum_{i=1}^{n_B} \hat{b}_i(t)u(t-i) + \varepsilon(t) \end{aligned} \tag{12.32}$$

On the other hand, introducing the expression of $y(t)$ given by (12.2) in the controller equation (12.20), one gets:

$$\begin{aligned} u(t) &= -\hat{S}^*(t, q^{-1})u(t-1) - \hat{r}_0(t)[\theta^T\phi(t-1)] \\ &\quad - \hat{R}^*(t, q^{-1})y(t-1) + \hat{\beta}(t)P(q^{-1})y^*(t+1) \\ &= f[y(t-1), \ldots, u(t-1), \ldots] + \bar{y}^*(t) \\ &= f[\phi(t-1)] + \bar{y}^*(t) \end{aligned} \tag{12.33}$$

where:

$$\bar{y}^*(t) = \hat{\beta}(t)P(q^{-1})y^*(t+1) \tag{12.34}$$

is a quantity perfectly known at time t and which is bounded. Defining now:

$$\hat{r}_i' \stackrel{\Delta}{=} \hat{r}_i(t) - \hat{r}_0(t)a_i, \quad i = 1, 2, \ldots, n_A \tag{12.35}$$

$$\hat{s}_i' \stackrel{\Delta}{=} \hat{s}_i(t) + \hat{r}_0(t)b_i, \quad i = 1, 2, \ldots, n_B \tag{12.36}$$

Equation (12.33) becomes:

$$u(t) = -\sum_{i=1}^{n_A} \hat{r}_i'(t)y(t-i) - \sum_{i=1}^{n_B} \hat{s}_i'(t)u(t-i) + \bar{y}^*(t) \tag{12.37}$$

Combining now (12.32) with (12.37), one obtains a state space equation for $\phi(t)$:

$$\underbrace{\begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-n_A+1) \\ u(t) \\ u(t-1) \\ \vdots \\ u(t-n_B+1) \end{bmatrix}}_{\phi(t)} = \underbrace{\begin{bmatrix} -\hat{a}_1(t) & \dots & & -\hat{a}_n(t) & \hat{b}_1(t) & \dots & \hat{b}_{n_B}(t) \\ 1 & & & 0 & 0 & 0 & 0 \\ & \ddots & & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ -\hat{r}'_1(t) & \dots & \dots & -\hat{r}'_{n_A}(t) & -\hat{s}'_1(t) & \dots & -\hat{s}'_{n_B}(t) \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \ddots & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{L(t)}$$

$$\times \underbrace{\begin{bmatrix} y(t-1) \\ y(t-2) \\ \vdots \\ y(t-n_A) \\ u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n_B) \end{bmatrix}}_{\phi(t-1)} + \underbrace{\begin{bmatrix} \varepsilon(t) \\ 0 \\ 0 \\ \\ \bar{y}^*(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{z(t)} \tag{12.38}$$

Moving one step further in time, this equation can be written:

$$\phi(t+1) = L(t+1)\phi(t) + z(t+1) \tag{12.39}$$

where $L(t)$ is a time-varying matrix containing the estimated parameters of the plant model and of the controller and $z(t+1)$ is a vector which contains the a posteriori prediction error $\varepsilon(t+1)$ and the term $\bar{y}^*(t+1)$ which depend on the reference signal. Both $\varepsilon(t+1)$ and $\bar{y}^*(t+1)$ depend on the estimated parameters at time $t+1$. For bounded reference signals and bearing in mind (12.17), it results that $\bar{y}^*(t+1)$ is bounded. If one uses at each time t (12.24) for the computation of the controller parameters, the eigenvalues of the matrix $L(t)$ are the roots of the polynomial equation: $\lambda^{n_A+n_B-1}P(\lambda) = 0$. This can be easily seen if one takes a first order example with $n_A = 1, n_B = 1$.

Remarks

1. In general, the eigenvalues of the matrix $L(t)$ for given $\hat{R}(t,q^{-1})$, $\hat{S}(t,q^{-1})$, $\hat{A}(t,q^{-1})$, $\hat{B}(t,q^{-1})$ are the solution of:

$$\lambda^{n_A+n_B+d-1}[\hat{A}(\lambda^{-1})\hat{S}(\lambda^{-1}) + \lambda^{-d}\hat{B}(\lambda^{-1})\hat{R}(\lambda^{-1})] = 0$$

   This equation is used for computing the eigenvalues of $L(t)$ when $\hat{R}(t,q^{-1})$ and $\hat{S}(t,q^{-1})$ are not obtained as solutions of an equation of the type (12.24). Such a situation occurs in the case of adaptive generalized predictive control or adaptive linear quadratic control where $\hat{R}(t, q^{-1})$ and $\hat{S}(t, q^{-1})$ result as solutions of a quadratic criterion minimization.

2. An interesting interpretation of (12.38) is obtained by rewriting it as:

$$\begin{bmatrix} \hat{y}(t) \\ y(t-1) \\ \vdots \\ y(t-n_A+1) \\ u(t) \\ u(t-1) \\ \vdots \\ u(t-n_B+1) \end{bmatrix} = L(t) \begin{bmatrix} y(t-1) \\ y(t-2) \\ \vdots \\ y(t-n_A) \\ u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n_B) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ \bar{y}^*(t) \\ 0 \\ \\ 0 \end{bmatrix}$$

which is the closed-loop equation of the output of the predictor. It corresponds to a pole placement control of a perfectly known time-varying system (the adjustable predictor) (see also Fig. 1.13, Sect. 1.3).

Returning now to (12.39), one sees that it corresponds to a time-varying system with a driving signal. Roughly speaking, in order that $\phi(t)$ be bounded, the system should be asymptotically stable in the absence of the external excitation z and the external excitation either should be bounded, or should not grow faster than $\phi(t)$. While these properties of $z(t)$ can be easily checked, the difficult part is that even if the eigenvalues of $L(t)$ are inside the unit circle at each time t, this does not necessarily guarantee the stability of the system.

Let us look now to the performance indices characterizing the adaptive pole placement. To do so, let us first observe that the plant output can be written:

$$y(t+1) = \hat{y}(t+1) + \varepsilon(t+1)$$

which leads to (see (12.32)):

$$\hat{A}(t)y(t) = q^{-d}\hat{B}(t)u(t) + \varepsilon(t) \tag{12.40}$$

where:[3]

$$\begin{aligned} \hat{A}(t) &= 1 + \hat{a}_1(t)q^{-1} + \cdots + \hat{a}_{n_A}(t)q^{-n_A} \\ \hat{B}(t) &= \hat{b}_1(t)q^{-1} + \cdots + \hat{b}_{n_B}(t)q^{-n_B} \end{aligned} \tag{12.41}$$

Together with (12.40), one considers the design equation (12.24) at time t:

$$\hat{A}(t)\hat{S}(t) + q^{-d}\hat{B}(t)\hat{R}(t) = P \tag{12.42}$$

[3]The index q^{-1} has been dropped to simplify the notation.

and the controller equation at t:

$$\hat{S}(t)u(t) + \hat{R}(t)y(t) = \hat{\beta}(t)Py^*(t+d+1) \tag{12.43}$$

Combining (12.40) and (12.43) by appropriately passing the generated signals through the operators $\hat{A}(t)$, $\hat{B}(t)$, $\hat{R}(t)$ and $\hat{S}(t)$, and taking into account the non-commutativity of the time-varying operators, i.e.:

$$\hat{A}(t)\hat{S}(t)x(t) \neq \hat{S}(t)\hat{A}(t)x(t)$$

and (12.42), one gets:

$$\begin{aligned} P[y(t) - \hat{B}^*(t)\hat{\beta}(t)y^*(t)] = {} & \hat{S}(t)\varepsilon(t) + \Delta_{11}(t)y(t) \\ & + \Delta_{12}(t)u(t) + \Delta_{13}(t)y^*(t+1) \end{aligned} \tag{12.44}$$

and:

$$\begin{aligned} P[u(t) - \hat{A}(t)\hat{\beta}(t)y^*(t+1)] = {} & -\hat{R}(t)\varepsilon(t) + \Delta_{21}(t)y(t) \\ & + \Delta_{22}(t)u(t) + \Delta_{23}(t)y^*(t+1) \end{aligned} \tag{12.45}$$

where:

$$\begin{aligned} \Delta_{11}(t) &= \hat{A}(t)\hat{S}(t) - \hat{S}(t)\hat{A}(t) \\ \Delta_{12}(t) &= \hat{S}(t)\hat{B}(t) - \hat{B}(t)\hat{S}(t) \\ \Delta_{13}(t) &= \hat{B}(t)P\hat{\beta}(t) - P\hat{B}^*(t)\hat{\beta}(t)q^{-1} \\ \Delta_{21}(t) &= \hat{R}(t)\hat{A}(t) - \hat{A}(t)\hat{R}(t) \\ \Delta_{22}(t) &= \hat{B}(t)\hat{R}(t) - \hat{R}(t)\hat{B}(t) \\ \Delta_{23}(t) &= \hat{A}(t)P\hat{\beta}(t) - P\hat{A}(t)\hat{\beta}(t) \end{aligned}$$

If $y(t)$ and $u(t)$ are bounded and $\varepsilon(t)$ goes to zero, taking into account that $\Delta_{ij}(t)$ are bounded and go to zero, the left hand of (12.44) and (12.45) will go to zero. The left hand sides of (12.44) and (12.45) correspond in fact to the time domain objectives of the pole placement in the known parameter case. These objectives will be achieved asymptotically except that the filtered trajectories which will be followed by $u(t)$ and $y(t)$ may be different if $\hat{B}(\infty) \neq B$ and $\hat{A}(\infty) \neq A$. One has the following result:

Theorem 12.1 (Strategy 1) *Consider the indirect adaptive pole placement for the plant model* (12.1) *where the plant parameter estimates are given by the algorithm of* (12.5) *through* (12.13) *and the controller is given by* (12.20) *through* (12.24). *Assume that*:

(i) *The plant model is admissible for the pole placement control* (*i.e.*, *the polynomials* $A(q^{-1})$ *and* $B(q^{-1})$ *do not have common factors*).

(ii) *The orders of the polynomials* $A(q^{-1})$, $B(q^{-1})$ *and of the delay* d *are known.*
(iii) *The reference signal is bounded.*
(iv) *The estimated models are admissible for each time* t (*i.e.*, *the estimated polynomials* $\hat{A}(t,q^{-1})$ *and* $\hat{B}(t,q^{-1})$ *do not have common factors*).

Then:

1. *The sequences* $\{u(t)\}$ *and* $\{y(t)\}$ *are bounded.*
2. *The a priori prediction error* $\varepsilon^0(t+1)$ *converges to zero*, *i.e.*:

$$\lim_{t\to\infty} \varepsilon^0(t+1) = 0$$

3. $\lim_{t\to\infty} P[y(t+d) - \hat{B}^*(t,q^{-1})\hat{\beta}(t)y^*(t+d)] = 0.$
4. $\lim_{t\to\infty} P[u(t) - \hat{A}(t,q^{-1})\hat{\beta}(t)y^*(t+d+1)] = 0.$

Proof The proof is divided in two stages. In the first stage, one proves that $\{y(t)\}$ and $\{u(t)\}$ are bounded. Then the other properties of the scheme result straightforwardly. The proof of boundedness of $\{y(t)\}$ and $\{u(t)\}$ is based on the following technical lemma. □

Lemma 12.1 *Consider the system*:

$$\phi(t+1) = L(t)\phi(t) + x(t) \tag{12.46}$$

1. $L(t)$ *has finite coefficients for all* t.
2. *The eigenvalues of* $L(t)$ *are inside the unit circle for all* t.
3. $\|L(t) - L(t-1)\| \to 0$ *for* $t \to \infty$.

Then there exists a time t *such that for* $t \geq t^*$:

$$\|\phi(t+1)\|^2 \leq C_1 + C_2 \max_{0\leq\tau\leq t} \|x(\tau)\|^2; \quad 0 \leq C_1, C_2 < \infty \tag{12.47}$$

Proof Since the eigenvalues of $L(t)$ are inside the unit circle, for a given matrix $Q > 0$ there is a matrix $P(t) > 0$ such that:

$$P(t) - L^T(t)P(t)L(t) = Q \tag{12.48}$$

Note also that since $\|L(t) - L(t-1)\| \to 0$ and $\|L(t)\| < \infty$ we have that $\|P(t) - P(t-1)\| \to 0$ and $\|P(t)\| < \infty$. From (12.46) and (12.48) we have:

$$\begin{aligned} V(t+1) &= \phi^T(t+1)P(t)\phi(t+1) \\ &= [\phi^T(t)L^T(t) + x^T(t)]P(t)[L(t)\phi(t) + x(t)] \\ &= \phi^T(t)L^T(t)P(t)L(t)\phi(t) + 2\phi^T(t)P(t)x(t) + x^T(t)P(t)x(t) \end{aligned}$$

$$
\begin{aligned}
&= V(t) - \phi^T(t)Q\phi(t) + 2\phi^T(t)P(t)x(t) \\
&\quad + x^T(t)P(t)x(t) + \phi^T(t)[P(t) - P(t-1)]\phi(t)
\end{aligned} \tag{12.49}
$$

Defining:

$$
\begin{aligned}
&\lambda_1 = \lambda_{\min} Q; \qquad \lambda_2 = \lambda_{\min} P(t); \\
&\lambda_3 = \lambda_{\max} P(t); \qquad \lambda_4 = \max_t \|P(t)\|
\end{aligned} \tag{12.50}
$$

and:

$$
\Delta P(t) = \|P(t) - P(t-1)\| \tag{12.51}
$$

it results:

$$
\begin{aligned}
V(t+1) \le V(t) - \lambda_1\|\phi(t)\|^2 + 2\|\phi(t)\|\|x(t)\|\lambda_4 \\
+ \lambda_3\|x(t)\|^2 + \Delta P(t)\|\phi(t)\|^2
\end{aligned} \tag{12.52}
$$

Note that since $2ab \le a^2 + b^2 \ \forall a, b \in R$:

$$
\begin{aligned}
2\lambda_4\|\phi(t)\|\|x(t)\| &= 2\lambda_4\left(\frac{2}{\lambda_1}\right)^{1/2}\|x(t)\|\left(\frac{\lambda_1}{2}\right)^{1/2}\|\phi(t)\| \\
&\le \lambda_4^2\frac{2}{\lambda_1}\|x(t)\|^2 + \frac{\lambda_1}{2}\|\phi(t)\|^2
\end{aligned} \tag{12.53}
$$

From (12.52) and (12.53) and for a time $t \ge t^*$ such that $\Delta P(t) \le \frac{\lambda_1}{4}$ we have:

$$
V(t+1) \le V(t) - \frac{\lambda_1}{4}\|\phi(t)\|^2 + \lambda_5\|x(t)\|^2 \tag{12.54}
$$

with:

$$
\lambda_5 = 2\frac{\lambda_4^2}{\lambda_1} + \lambda_3
$$

Since $V(t) \le \lambda_3\|\phi(t)\|^2$, one obtains from (12.54):

$$
V(t+1) \le \alpha V(t) + \lambda_5\|x(t)\|^2
$$

with:

$$
0 < \alpha = 1 - \frac{\lambda_1}{4\lambda_3} < 1
$$

Note from (12.48) and (12.50) that $\lambda_3 > \lambda_1$ which implies $\alpha > 0$. Therefore, one immediately gets:

$$
V(t+1) = \alpha^{t+1}V(0) + \frac{(1-\alpha^{t+1})}{1-\alpha}\lambda_5 \max_{0\le\tau\le t}\|x(\tau)\|^2 \tag{12.55}
$$

which implies (12.47) and ends the proof of Lemma 12.1. □

We will use Lemma 12.1 to analyze the state equation for $\phi(t+1)$ given by (12.39). Taking into account the properties of the parameter estimation algorithm and the *admissibility* hypothesis for the estimated model with respect to the control law, all the assumptions for applying Lemma 12.1 are satisfied. Since $L(t+1)$ has bounded coefficients, $\phi(t)$ can become unbounded only asymptotically, but for $t \geq t^*$ one has:

$$\begin{aligned}\|\phi(t+1)\|^2 &\leq C_1' + C_2 \max_{0\leq\tau\leq t} [\varepsilon(t+1)^2 + \bar{y}^*(t)^2] \\ &\leq C_1 + C_2 \max_{0\leq\tau\leq t} \varepsilon(t+1)^2; \quad C_1, C_2 < \infty \end{aligned} \tag{12.56}$$

From (12.14), it results that $\varepsilon(t+1)^2 < \infty$ for all t and one concludes that $\phi(t)$ is bounded. Recalling that $u(t)$ and $y(t)$ are bounded, that the various $\Delta_{ij}(t)$ in (12.44) and (12.45) go to zero asymptotically and that $\varepsilon(t+1)$ goes to zero, one concludes that properties (1) through (4) of Theorem 12.1 are true.

For Strategy 2, one has a similar result summarized in Theorem 12.2.

Theorem 12.2 (Strategy 2) *Consider the indirect adaptive pole placement for the plant model* (12.1) *where the plant parameters estimates are given by the algorithm of* (12.5) *through* (12.13) *and the controller is given by*:

$$\hat{S}(t-1,q^{-1})u(t) + \hat{R}(t-1,q^{-1})y(t) = \hat{\beta}(t-1)P(q^{-1})y^*(t+d+1) \tag{12.57}$$

or alternatively:

$$\begin{aligned}u(t) = &-\hat{S}^*(t-1,q^{-1})u(t-1) - \hat{R}(t-1,q^{-1})y(t) \\ &+ \hat{\beta}(t-1)P(q^{-1})y^*(t+d+1)\end{aligned} \tag{12.58}$$

where:

$$\hat{\beta}(t-1) = 1/\hat{B}(t-1,1) \tag{12.59}$$

$$\hat{S}(t,q^{-1}) = 1 + \hat{s}_1(t)q^{-1} + \cdots + \hat{s}_{n_S}(t)q^{-n_S} = 1 + q^{-1}\hat{S}^*(t,q^{-1}) \tag{12.60}$$

$$\begin{aligned}\hat{R}(t,q^{-1}) &= \hat{r}_0(t) + \hat{r}_1(t)q^{-1} + \cdots + \hat{r}_{n_R}(t)q^{-n_R} \\ &= \hat{r}_0(t) + q^{-1}\hat{R}^*(t,q^{-1})\end{aligned} \tag{12.61}$$

and $\hat{R}(t,q^{-1})$ *and* $\hat{S}(t,q^{-1})$ *are solutions of* (12.24). *Assume that the hypotheses* (i) *through* (iv) *from Theorem* 12.1 *hold. Then*:

1. *The sequences* $\{u(t)\}$ *and* $\{y(t)\}$ *are bounded.*
2. *The a priori prediction error converges to zero, i.e.*: $\lim_{t\to\infty} \varepsilon^0(t+1) = 0$.
3. $\lim_{t\to\infty} P[y(t+d) - \hat{B}^*(t-1,q^{-1})\hat{\beta}(t-1)y^*(t+d)] = 0$.
4. $\lim_{t\to\infty} P[u(t) - \hat{A}(t-1,q^{-1})\hat{\beta}(t-1)y^*(t+d+1)] = 0$.

Proof The proof is similar to that of Theorem 12.1, except that one replaces in the various equations: $\varepsilon(t+1)$ by $\varepsilon^0(t+1)$, $\hat{y}(t+1)$ by $\hat{y}^0(t+1)$ and the time-varying parameters at t by their values at time $t-1$. The major difference occurs in (12.56) which becomes:

$$
\begin{aligned}
\|\phi(t+1)\|^2 &\le C_1' + C_2 \max_{0\le\tau\le t} [\varepsilon^0(t+1)^2 + \bar{y}^*(t)^2] \\
&\le C_1 + C_2 \max_{0\le\tau\le t} \varepsilon^0(t+1)^2; \quad c_1, c_2 < \infty
\end{aligned}
\tag{12.62}
$$

since $\bar{y}^*(t)$ is a bounded signal. On the other hand, from the properties of the parameter estimation algorithm one has (see (12.16)):

$$
\lim_{t\to\infty} \frac{[\varepsilon^0(t+1)]^2}{1+\phi^T(t)F(t)\phi(t)} = 0
$$

and applying the "bounded growth" lemma (Lemma 11.1) one concludes that $\|\phi(t)\|$ is bounded, which implies that $y(t)$ and $u(t)$ are bounded, as well as $\lim_{t\to\infty} \varepsilon^0(t+1) = 0$. □

12.2.3 The "Singularity" Problem

The question of avoiding singularity points in the parameter space during adaptation, i.e., points which correspond to non-admissible models, has received a lot of attention. While most of the techniques proposed are related to the pole placement where the problem is to avoid plant parameter estimates for which the plant model is not controllable (the determinant of the Sylvester matrix is zero), they can be used also for other types of admissibility conditions.

The various techniques can be classified as follows:

- Techniques based on excitation signals internally generated or externally applied, where the singularities are avoided by securing the convergence of the parameter estimates toward the true plant parameters (Anderson and Johnstone 1985; Eliott 1985; Goodwin and Teoh 1985; Kreisselmeier and Smith 1986; Polderman 1989; Giri et al. 1991; M'Saad et al. 1993b).
- Techniques based on the projection of the parameter estimates in a convex region (or a set of convex regions) (Kreisselmeier 1986; Ossman and Kamen 1987; Middleton et al. 1988; Barmish and Ortega 1991).
- Techniques based on the correction of the parameter estimates before using them to compute the controller (de Larminat 1984; Lozano and Goodwin 1985; Lozano 1989, 1992; Lozano et al. 1993; Lozano and Zhao 1994).
- Search for another type of plant model parameterization allowing to define a convex set of admissible models (Barmish and Ortega 1991).

A detailed review of the various techniques can be found in Lozano and Zhao (1994), M'Saad et al. (1993b). We will focus our attention on a technique of correcting the *singular* estimated parameter vector without modifying the asymptotic properties of the adaptation algorithm. When modifying the current parameter estimates, one should provide an algorithm for which:

- The modification is easily constructed.
- The number of possible modifications is finite, and within this finite set, there is at least one modification which gives a nonsingular parameter estimation.
- The modification algorithm stops (it converges) once a satisfactory result is obtained.

The algorithm which will be presented is based on the technique presented in Lozano and Zhao (1994). This technique starts from the observation that for the parameter estimation algorithms covered by Theorem 3.2, one has (see (3.248))

$$\|F(t)^{-1/2}[\hat{\theta}(t)-\theta]\| \leq h_0 \leq \infty; \quad \forall t \tag{12.63}$$

where θ is the true parameter vector characterizing the unknown plant model.

If one now defines a vector:

$$\beta^*(t) = F(t)^{-1/2}[\theta - \hat{\theta}(t)] \tag{12.64}$$

one can write the true parameter vector as:

$$\theta = \hat{\theta}(t) + F(t)^{1/2}\beta^*(t) \tag{12.65}$$

This means that there is a modification of the vector $\hat{\theta}(t)$ such that the new parameter estimation is equal to the true one for which the determinant of the Sylvester matrix is non-null. This of course implies that there is a modification of $\hat{\theta}(t)$ having the structure:

$$\bar{\theta}(t) = \hat{\theta}(t) + F(t)^{1/2}\beta(t) \tag{12.66}$$

such that the determinant of the Sylvester matrix is non-null. The vector $\beta(t)$ should satisfy in addition the following conditions:

1. $\beta(t)$ should converge in finite time;
2. $\beta(t)$ should be such that the absolute value of the determinant of the Sylvester matrix associated with $\bar{\theta}(t)$ is uniformly bounded from below.

The Modification Algorithm

1. Define $\beta(t)$ as follows:

$$\beta^T(t) = [\sigma(t), \sigma(t)^m, \sigma(t)^{m^2}, \ldots, \sigma(t)^{m^{(m-1)}}] \tag{12.67}$$

where:

$$m = n_A + n_B + d; \qquad m^{(m-1)} = l' \tag{12.68}$$

2. $\sigma(t)$ takes values in the set D

$$D = [\sigma_1, \sigma_2, \ldots, \sigma_l]; \qquad l = m^m \tag{12.69}$$

with

$$\sigma_i \in R; \quad \sigma_i \geq \sigma_{i-1} + 1; \quad i = 1, 2, \ldots, l \tag{12.70}$$

The values of $\sigma(t)$ will switch from a value to another in D such that an appropriate $\beta(t)$ can be found. In order to stop the searching procedure, a switching function with a constant hysteresis width $\mu > 0$ ($\mu \ll 1$) is introduced. Denote:

$$x(t, \sigma) = |\det M[\hat{\theta}(t) + F(t)^{-1/2}(\sigma, \sigma^m, \ldots, \sigma^{l'})^T]| \tag{12.71}$$

3. The hysteresis switching function defining $\sigma(t)$ is defined as follows:

$$\sigma(t) = \begin{cases} \sigma(t-1) & \text{if } x(t, \sigma_j) < (1+\gamma)x[t, \sigma(t-1)] \\ & \text{for all } \sigma_j \in D \\ \sigma_j & \text{if } j \text{ is the smallest integer such that} \\ & x(t, \sigma_j) \geq (1+\gamma)x[t, \sigma(t-1)] \\ & \text{and } x(t, \sigma_j) \geq x(t, \sigma_i) \quad \forall \sigma_i \in D \end{cases} \tag{12.72}$$

It is shown in Lozano and Zhao (1994) that using this type of algorithm, one has the following result:

Theorem 12.3 *Subject to the assumptions that the plant model polynomials $A(q^{-1})$ and $B(q^{-1})$ do not have common factors and their orders as well the delay d are known, the parameter estimation algorithm (given by (12.5) through (12.13)) combined with the modification procedure defined by (12.67) through (12.72) assures:*

(i) *a lower bound for the absolute value of the determinant of the Sylvester matrix, i.e.,*

$$|\det M[\bar{\theta}(t)]| \geq \alpha\delta_0; \quad \alpha > 0 \tag{12.73}$$

where δ_0 is a measure of the controllability of the unknown system (related to the condition number)

$$0 < \delta_0 \leq |\det M(\theta)| \tag{12.74}$$

(ii) *convergence in finite time toward the modified value $\bar{\theta}(t)$ assuring the property above.*

Analysis A full proof of this result is beyond our scope. In what follows, we would like to outline the basic steps and the rationale behind the choice proposed for $\beta(t)$. The key ideas are the following:

1. Using the proposed modification it is possible to write:

$$\det M[\bar{\theta}(t)] = g^T(t)v[\beta(t)] \tag{12.75}$$

where $g(t)$ contains combinations of the coefficient of the unmodified parameter vector $\hat{\theta}(t)$ and $v(\beta(t))$ contain combinations of σ_i at various powers.

2. Instead of trying to show that there exists a value σ_i such that (12.75):

$$\|g^T(t)v[\beta(t)]\| > 0$$

one constructs a vector collecting the values of $\det M[\bar{\theta}(t)]$ for all σ_i in D, and one shows that the norm of this vector is bounded from below. This implies that at least one of the determinants is different from zero. An important role in this analysis is played by a Vandermonde matrix (Kailath 1980) and this explains the rationale behind the structure of the modifications defined by (12.67) and (12.69).

To be more specific, let us consider the case of a constant adaptation gain matrix $F(t) = F(0) = I$. In this case, one has:

$$\bar{\theta}(t) = \hat{\theta}(t) + \beta(t)$$

where $\beta(t)$ is given by (12.67). Consider now the case of a plant model with $n_A = 1$ and $n_B = 2$ and furthermore let us assume that there is a common factor for the estimated model at time t. One has therefore:

$$\hat{A}(t, q^{-1}) = 1 + \hat{a}_1(t)q^{-1}$$

$$\hat{B}(t, q^{-1}) = \hat{b}_1(t)q^{-1} + \hat{b}_2(t)q^{-1} \quad \text{with } \hat{a}_1(t) = \frac{\hat{b}_2(t)}{\hat{b}_1(t)}$$

The Sylvester matrix is:

$$M[\hat{\theta}(t)] = \begin{bmatrix} 1 & 0 & 0 \\ \hat{a}_1(t) & 1 & \hat{b}_1(t) \\ 0 & \hat{a}_1(t) & \hat{b}_2(t) \end{bmatrix}$$

and:

$$\det M[\hat{\theta}(t)] = \hat{b}_2(t) - \hat{a}_1(t)\hat{b}_1(t) = 0$$

The modified vector will have the form:

$$\begin{bmatrix} \bar{a}_1(t) \\ \bar{b}_1(t) \\ \bar{b}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{a}_1(t) \\ \hat{b}_1(t) \\ \hat{b}_2(t) \end{bmatrix} + \begin{bmatrix} \sigma \\ \sigma^3 \\ \sigma^9 \end{bmatrix}$$

where $\sigma \in D$

$$D = [\sigma_1, \sigma_2, \ldots, \sigma_{27}]$$

which for this example can be chosen as:

$$[1, 2, 3, \ldots, 27]$$

The determinant of the modified Sylverster matrix takes the form:

$$\det M[\bar{\theta}(t)] = \det \begin{bmatrix} 1 & \hat{b}_1(t) + \sigma^3 \\ \hat{a}_1(t) + \sigma & \hat{b}_2(t) + \sigma^9 \end{bmatrix}$$
$$= \sigma^9 + \sigma^4 - \hat{b}_1(t)\sigma - \hat{a}_1(t)\sigma^3 = g^T(t)v[\beta(t)]$$

where:

$$v[\beta(t)]^T = [1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6, \sigma^7, \sigma^8, \sigma^9]$$
$$g^T(t) = [0 - \hat{b}_1, 0, -\hat{a}_1, 1, 0, 0, 0, 0, 1]$$

For this particular case, the highest power of σ is 9 and therefore we can limit the domain D for $i_{\max} = 10$ (but this is a particular case). Defining now a vector whose components are $\det M[\hat{\theta}(t)]$ obtained for various σ_i, one gets:

$$p^T(t) = [g^T(t)v(\sigma_1), \ldots, g^T(t)v(\sigma_{10})] = g^T(t)N \tag{12.76}$$

where N is a 10×10 Vandermonde matrix:

$$N = \begin{bmatrix} 1 & 1 & \ldots & \ldots & 1 \\ \sigma_1 & \sigma_2 & & & \sigma_{10} \\ \sigma_1^2 & \sigma_2^2 & & & \sigma_{10}^2 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \sigma_1^9 & \sigma_2^9 & & & \sigma_{10}^9 \end{bmatrix} \tag{12.77}$$

Therefore, the objective is to show that indeed $\|p(t)\| > 0$ which will imply that:

$$\exists \sigma_i \in D \quad \text{such that } |\det M[\bar{\theta}(t)]| > 0$$

From (12.76) one has:

$$\|p(t)\|^2 = p^T(t)p(t) = g^T(t)NN^Tg(t) \geq \|g(t)\|^2\lambda_{\min}[NN^T] \tag{12.78}$$

From matrix calculus, one has (Lancaster and Tismenetsky 1985):

$$(\det N)^2 = \det NN^T = \lambda_{\min}[NN^T] \cdots \lambda_{\max}[NN^T]$$
$$\leq \lambda_{\min} NN^T(\mathrm{tr}[NN^T])^{l-1} \tag{12.79}$$

taking into account that:

$$\mathrm{tr}[NN^T] = \sum_{i=1}^{l} \lambda_i[NN^T]$$

Combining (12.78) and (12.79), one gets:

$$\|p(t)\| \geq \|g(t)\| \frac{|\det N|}{(\mathrm{tr}[NN^T])^{(l-1)/2}} \tag{12.80}$$

A lower bound for the vector $\|g(t)\|$ can be obtained taking into account the hypothesis that the true plant model polynomials do not have common factors.

$$\begin{aligned} 0 < \delta_0 \leq |\det M(\theta)| &= |\det M[\hat{\theta}(t) + F(t)^{1/2}\beta^*(t)]| \\ &= |g^T(t)v(\beta^*(t))| \leq \|g(t)\|\|v(\beta^*(t))\| \end{aligned} \tag{12.81}$$

From the above relationship, one gets:

$$\|g(t)\| \geq \frac{\delta_0}{\max \|v(\beta^*(t))\|} \tag{12.82}$$

and from (12.80), it results that:

$$\|p(t)\| \geq \frac{\delta_0 |\det N|}{\max \|v(\beta^*(t))\| (\mathrm{tr}\, NN^T)^{(l-1)/2}} \tag{12.83}$$

which is the desired result. The solution to the "singularity" problem presented above can be extended to cover the case when the system is subject to disturbances (see Sect. 12.3). The extension of this solution to the stochastic case is discussed in Guo (1996).

12.2.4 Adding External Excitation

For constant controllers, it was shown in Chap. 9 in the context of plant model identification in closed loop, that richness conditions on the external excitation can be imposed in order to obtain richness conditions on the observation vector. This in turn will lead to a correct estimation of the parameter vector θ characterizing the plant model.

In the indirect adaptive pole placement control, an additional problem comes from the fact that the controller is time-varying. However, since the variations of the plant parameter estimates tend to zero, the time-varying coefficients of the controller will also exhibit slower and slower coefficients variations.

Let us $\{\bar{\phi}(t)\}$ denote the sequence of $\{\phi(t)\}$ generated by a fixed controller starting at time t^*. Since $\lim_{t\to\infty} \|\hat{\theta}(t+k) - \hat{\theta}(t)\| = 0$ and $\|\hat{\theta}(t)\|$ is bounded for all t, there is a time t^* such that for $t \geq t^*$, one has:

$$\|\hat{\theta}_c(t^*) - \hat{\theta}_c(t)\| < h_0$$

where:

$$\hat{\theta}_c^T(t) = [\hat{r}_0(t), \hat{r}_1(t), \ldots, \hat{r}_{n_R}(t), \hat{s}_1(t), \ldots, \hat{s}_{n_S}(t)]$$

Since $\phi(t)$ is bounded (we are assuming $\hat{\theta}(t)$ to be in a convex region around the correct plant model) it results that:

$$\|\bar{\phi}(t) - \phi(t)\| < \delta$$

from which one concludes that:

$$\|\bar{\phi}(t)\bar{\phi}^T(t) - \phi(t)\phi^T(t)\| < 0(\delta); \quad \forall t > t^*$$

Therefore, if $\bar{\phi}(t)$, corresponding to a fixed controller for $t \geq t^*$, has the desired richness properties, i.e.:

$$\beta_2 I > \sum_{t=t^*}^{n_A+n_B+d-1+L} \bar{\phi}(t)\bar{\phi}^T(t) > \beta_1 I; \quad \beta_1, \beta_2 > 0$$

then, $\phi(t)$ will also have the desired richness property and this implies that:

$$\lim_{t\to\infty} \hat{A}(t, q^{-1}) = A(q^{-1}); \qquad \lim_{t\to\infty} \hat{B}(t, q^{-1}) = B(q^{-1})$$

Therefore, the external excitation (if it is sufficiently rich) will allow, not only to obtain a stabilizing controller, but to converge toward the controller corresponding to the exact knowledge of the plant model. Furthermore, since convergence under rich excitation is related to asymptotic exponential stability, the adaptation transients will be improved (Anderson and Johnstone 1985; Anderson and Johnson 1982).

It should, however, be noted that the technique based on the introduction of external excitation may have several drawbacks. Introducing an external excitation signal all the time is not always feasible or desirable in practice. Furthermore, due to the presence of the external signals, the plant output cannot reach the exact desired value. Often, the external excitation is used for the initialization of the scheme. To avoid the problems caused by the external excitation, it is possible to use the so-called self-excitation. In this case, the external excitation is introduced only when the plant output is far enough from its desired value (Giri et al. 1991; M'Saad et al. 1993b).

12.3 Robust Indirect Adaptive Control

In this section, we will present a design for indirect adaptive control (adaptive pole placement) which is robust with respect to:

- bounded disturbances;
- unmodeled dynamics.

As discussed in Sect. 10.6, the effect of unmodeled dynamics will be represented as a disturbance bounded by a function of $\|\phi(t)\|$, which is not necessarily bounded a priori. Several representations of the unmodeled dynamics are possible. Consider again the plant in (12.1), but with a disturbance $w(t)$ acting as follows:

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + w(t) \tag{12.84}$$

The plant (12.84) can be rewritten as (see also (12.2)):

$$y(t+1) = \theta^T\phi(t) + w(t+1) \tag{12.85}$$

In this section, we will focus on disturbances defined by:

$$|w(t+1)| \le d_1 + d_2\eta(t); \quad 0 < d_1, d_2 < \infty \tag{12.86}$$

$$\eta^2(t) = \mu^2\eta^2(t-1) + \|\phi(t)\|^2 \tag{12.87}$$

where d_1 accounts for a bounded external disturbance and $d_2\eta(t)$ accounts for the equivalent representation of the effect of unmodeled dynamics. More details can be found in Sect. 10.6 (the chosen representation for the unmodeled dynamics corresponds to Assumption B in Sect. 10.6).

It should be noted that even if the plant parameters are perfectly known, there is no fixed linear controller that could stabilize the plant for all possible values of d_2. Nevertheless, we can design a controller that stabilizes the plant provided that $d_2 \le d_2^*$ where d_2^* is a threshold that depends on the system parameters.

The assumptions made upon the system of (12.84) through (12.87) are:

- the orders of the polynomials $A(q^{-1})$, $B(q^{-1})$ and of the delay d are known (n_A, n_B, d-known),
- $A(q^{-1})$ and $B(q^{-1})$ do not have common factors,
- the disturbance upper bounds d_1 and d_2 are known.

We will discuss the robust adaptive pole placement in two situations.

(a) without taking into account the possible singularities in the parameter space during adaptation (standard robust adaptive pole placement),
(b) taking into account the modification of parameter estimates in the singularity points (modified robust adaptive pole placement).

12.3.1 Standard Robust Adaptive Pole Placement

Bearing in mind the type of equivalent disturbance considered in (12.86), we will use a parameter estimation algorithm with data normalization and dead zone. This algorithm and its properties are described in Sect. 10.6, Theorem 10.5.

We will discuss next the direct extension of the design given in Sect. 12.2 for the ideal case using Strategy 1. How to incorporate the modification of the parameter estimates in order to avoid singularities will be discussed afterwards in Sect. 12.3.2. For the analysis of the resulting scheme, one proceeds exactly as in Sect. 12.2.2.

Step I: Estimation of the Plant Model Parameters

Define the normalized input-output variables:

$$\bar{y}(t+1) = \frac{y(t+1)}{m(t)}; \qquad \bar{u}(t) = \frac{u(t)}{m(t)}; \qquad \bar{\phi}(t) = \frac{\phi(t)}{m(t)} \tag{12.88}$$

where:

$$m^2(t) = \mu^2 m^2(t-1) + \max(\|\phi(t)\|^2, 1); \quad m(0) = 1;\ 0 < \mu < 1 \tag{12.89}$$

The a priori output of the adjustable predictor is given by:

$$\hat{y}^0(t+1) = \hat{\theta}^T(t)\bar{\phi}(t) \tag{12.90}$$

The a posteriori output of the adjustable predictor is given by:

$$\hat{y}(t+1) = \hat{\theta}^T(t+1)\bar{\phi}(t) \tag{12.91}$$

where:

$$\hat{\theta}^T(t) = [\hat{a}_1(t), \dots, \hat{a}_{n_A}(t), \hat{b}_1(t), \dots, \hat{b}_{n_A}(t)] \tag{12.92}$$

The a priori and the a posteriori prediction (adaptation) errors are given by:

$$\bar{\varepsilon}^0(t+1) = \bar{y}(t+1) - \hat{y}^0(t+1) \tag{12.93}$$

$$\bar{\varepsilon}(t+1) = \bar{y}(t+1) - \hat{y}(t+1) \tag{12.94}$$

The parameter adaptation algorithm is:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \alpha(t)F(t)\bar{\phi}(t)\bar{\varepsilon}(t+1) \tag{12.95}$$

$$F(t+1)^{-1} = F(t)^{-1} + \alpha(t)\bar{\phi}(t)\bar{\phi}(t)^T; \quad F(0) > 0 \tag{12.96}$$

$$\bar{\varepsilon}(t+1) = \frac{\bar{\varepsilon}^0(t+1)}{1 + \bar{\phi}^T(t)F(t)\bar{\phi}(t)} \tag{12.97}$$

$$\alpha(t) = \begin{cases} 1 & |\bar{\varepsilon}(t+1)| > \bar{\delta}(t+1) \\ 0 & |\bar{\varepsilon}(t+1)| \le \bar{\delta}(t+1) \end{cases} \tag{12.98}$$

where $\bar{\delta}(t)$ is given by:

$$\bar{\delta}^2(t+1) = \delta^2(t+1) + \sigma\delta(t+1); \quad \delta(t) > 0;\ \sigma > 0 \tag{12.99}$$

and (see (12.86) and (12.87)):

$$\delta(t+1) = d_2 + \frac{d_1}{m(t)} \ge |w(t+1)|/m(t) \tag{12.100}$$

Using this algorithm, we will have, according to Theorem 10.5, the following properties:

- The parameter vector $\hat{\theta}(t)$ is bounded and converges.
- The normalized a posteriori adaptation error is bounded:

$$\lim_{t\to\infty} \sup |\bar{\varepsilon}(t+1)| \le \bar{\delta}(t+1) \tag{12.101}$$

- The a posteriori prediction error defined as:

$$\varepsilon(t+1) = y(t+1) - \hat{\theta}^T(t+1)\phi(t) \tag{12.102}$$

  satisfies:

$$\lim_{t\to\infty} \sup \varepsilon(t+1)^2 \le \bar{\delta}^2(t+1)m^2(t) \tag{12.103}$$

Step II: Computation of the Control Law

We will use Strategy 1, as described in Sect. 12.2, using the parameter estimates provided by the robust parameter adaptation algorithm above. One finally obtains (see (12.2.56)):

$$\|\phi(t+1)\|^2 \le C_1 + C_2 \max_{0\le\tau<t} \varepsilon(t+1)^2; \quad 0 \le C_1, C_2 < \infty \tag{12.104}$$

In the presence of the disturbance $w(t+1)$, $\varepsilon(t+1)$ is no more necessarily bounded as in the ideal case. On the other hand, from Theorem 10.5, one has:

$$\lim_{t\to\infty} \sup \varepsilon^2(t+1) \le \bar{\delta}^2(t+1)m^2(t) \tag{12.105}$$

where:

$$\bar{\delta}^2(t+1) = \left(d_2 + \frac{d_1}{m(t)}\right)^2 + \sigma\left(d_2 + \frac{d_1}{m(t)}\right) \tag{12.106}$$

Assume now that $\phi(t+1)$ diverges. It follows that there is a subsequence such that for this subsequence $t_1, t_2, \ldots, t_n$, one has:

$$\|\phi(t_1)\| \le \|\phi(t_2)\| \le \cdots \le \|\phi(t_n)\| \tag{12.107}$$

or, equivalently:

$$\|\phi(t)\| \le \|\phi(t_n)\|; \quad \forall t \le t_n \tag{12.108}$$

In the meantime, as $\phi(t)$ diverges, it follows that $\bar{\delta}(t)^2 \to d_2^2 + \sigma d_2$, therefore in the limit as $t \to \infty$:

$$\begin{aligned}\lim_{t\to\infty} \sup \varepsilon^2(t+1) &\le (d_2^2 + \sigma d_2)m^2(t) \\ &\le (d_2^2 + \sigma d_2)m^2(t+1)\end{aligned} \tag{12.109}$$

Using these results, one gets from (12.104) that:

$$\|\phi(t+1)\|^2 \leq C_1 + C_2(d_2^2 + \sigma d_2)m^2(t+1) \tag{12.110}$$

and thus:

$$\frac{\|\phi(t+1)\|^2 - C_1}{m^2(t+1)} \leq C_2(d_2^2 + \sigma d_2) \tag{12.111}$$

The LHS of (12.111) converges toward a positive number ρ smaller than 1 as $\phi(t+1)$ diverges ($m^2(t+1) \leq \gamma + \|\phi(t+1)\|^2, \gamma > 0$). Therefore, there are values of $d_2 \leq d_2^*$ such that $C_2(d_2^* + \sigma d_2^*) < \rho$ which leads to a contradiction. Therefore, $d_2 \leq d_2^*$, $\|\phi(t+1)\|$ is bounded and the a posteriori and a priori prediction errors are bounded. Since d_2 corresponds to the "μ-scaled infinity norm" of the unmodeled dynamics $H(z^{-1})$, one gets the condition that boundedness of $\phi(t+1)$ is assured for $\|H(\mu^{-1}z^{-1})\|_\infty < d_2^*$.

We will also have:

$$\lim_{t\to\infty} P[y(t+d) - B^*(t,q^{-1})\hat{\beta}(t)y^*(t+d)] = S(t)\varepsilon(t) \tag{12.112}$$

$$\lim_{t\to\infty} P[u(t) - A(t,q^{-1})\hat{\beta}(t)y^*(t+d+1)] = -R(t)\varepsilon(t) \tag{12.113}$$

Taking into account (12.105) and the boundedness of $\phi(t)$, it results that the two indices of performance will be bounded ($\phi(t)$ bounded implies that $m(t)$ is bounded).

12.3.2 Modified Robust Adaptive Pole Placement

In order to use the technique for the modification of the parameter estimates in the eventual singularity points, a slight modification of the parameter estimation algorithm given by Theorem 10.5 should be considered. This algorithm and its properties are summarized in the next theorem (Lozano and Zhao 1994).

Theorem 12.4 *Under the same hypothesis as in Theorem* 10.4 ((10.100) *through* (10.102)) *and Theorem* 10.5 *except that* $w(t+1) = d_1 + d_2\|\phi(t)\|$ *and* $m(t) = 1 + \|\phi(t)\|$ *using a PAA of the form*:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \alpha(t)F(t+1)\bar{\phi}(t)\bar{\varepsilon}^0(t+1) \tag{12.114}$$

$$F(t+1)^{-1} = F(t)^{-1} + \alpha(t)\bar{\phi}(t)\bar{\phi}^T(t) \tag{12.115}$$

$$\bar{\varepsilon}^0(t+1) = \bar{y}(t+1) - \hat{y}^0(t+1) = \bar{y}(t+1) - \hat{\theta}^T(t)\bar{\phi}(t) \tag{12.116}$$

$$\alpha(t) = \begin{cases} 1 & \varepsilon_a^2(t+1) > \bar{\delta}^2(t+1) \\ 0 & \varepsilon_a^2(t+1) \leq \bar{\delta}^2(t+1) \end{cases} \tag{12.117}$$

where $\varepsilon_a(t+1)$ *is given by*:

$$\varepsilon_a(t+1) = [\bar{\varepsilon}^0(t+1)^2 + \bar{\phi}^T(t)F(t)\bar{\phi}(t)]^{1/2} \tag{12.118}$$

and:

$$\bar{\delta}^2(t+1) = [\delta^2(t+1) + \sigma\delta(t)][1 + \operatorname{tr} F(0)]; \quad \sigma > 0 \tag{12.119}$$

where:

$$\delta(t+1) = d_2 + \frac{d_1}{1 + \|\phi(t)\|} \tag{12.120}$$

one has:

$$(1) \quad \tilde{\theta}^T(t)F(t)^{-1}\tilde{\theta}(t) \leq \tilde{\theta}^T(0)F(0)^{-1}\tilde{\theta}(0) < \infty \tag{12.121}$$

$$\textit{where}: \tilde{\theta}(t) = \hat{\theta}(t) - \theta \tag{12.122}$$

$$(2) \quad \lim_{t\to\infty} \sup[\varepsilon_a^2(t+1) - \bar{\delta}^2(t)] = 0 \tag{12.123}$$

(3) $F(t)$ *and* $\tilde{\theta}(t)$ *converge.*

Remark It is worth noting that the properties of the algorithm stated above are independent of the values of d_1 and d_2 which characterize the equivalent disturbance.

Proof From the definition of the normalized variables (10.102) and of the disturbance $w(t)$, it results that:

$$\bar{y}(t+1) = \theta^T\bar{\phi}(t) + \bar{w}(t+1) \tag{12.124}$$

where:

$$\bar{w}(t+1) = w(t+1)/(1 + \|\phi(t)\|) \tag{12.125}$$

and, therefore:

$$\begin{aligned}|\bar{w}(t+1)| &\leq \frac{|d_1 + d_2\|\phi(t)\||}{1 + \|\phi(t)\|} \\ &\leq d_2 + \frac{d_1}{1 + \|\phi(t)\|} = \delta(t+1)\end{aligned} \tag{12.126}$$

Introducing (12.124) in (12.93), one gets:

$$\bar{\varepsilon}^0(t+1) = -\tilde{\theta}^T(t)\bar{\phi}(t) + \bar{w}(t+1) \tag{12.127}$$

From (12.114) and (12.122), it results that the parameter error is described by:

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) + \alpha(t)F(t+1)\bar{\phi}(t)\bar{\varepsilon}^0(t+1) \tag{12.128}$$

Combining (12.115) and (12.128), one has:

$$\begin{aligned}&\tilde{\theta}^T(t+1)F(t+1)^{-1}\tilde{\theta}(t+1) \\ &\quad = [\tilde{\theta}(t) + \alpha(t)F(t+1)\bar{\phi}(t)\bar{\varepsilon}^0(t+1)]F(t+1)^{-1}\end{aligned}$$

$$
\begin{aligned}
&\times [\tilde{\theta}(t) + \alpha(t)F(t+1)\bar{\phi}(t)\bar{\varepsilon}^0(t+1)] \\
&= \tilde{\theta}^T(t)F(t)^{-1}\tilde{\theta}(t) + \alpha(t)[\tilde{\theta}^T(t)\bar{\phi}(t)]^2 \\
&\quad + 2\alpha(t)\tilde{\theta}(t)\bar{\phi}(t)\bar{\varepsilon}^0(t+1) \\
&\quad + \alpha^2(t)\bar{\phi}^T(t)F(t+1)\bar{\phi}(t)[\bar{\varepsilon}^0(t+1)]^2 \\
&= \tilde{\theta}(t)F(t)^{-1}\tilde{\theta}(t) + \alpha(t)[\tilde{\theta}^T(t)\bar{\phi}(t) + \bar{\varepsilon}^0(t+1)]^2 \\
&\quad + \alpha(t)[\varepsilon^0(t+1)]^2[\alpha(t)\bar{\phi}(t)F(t+1)\bar{\phi}(t) - 1]
\end{aligned}
\tag{12.129}
$$

On the other hand, from (12.115) and the matrix inversion lemma it follows:

$$
\bar{\phi}(t)F(t+1)\bar{\phi}(t) = \frac{\bar{\phi}^T(t)F(t)\bar{\phi}(t)}{1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t)} \tag{12.130}
$$

Therefore:

$$
\alpha(t)\bar{\phi}^T(t)F(t+1)\bar{\phi}(t) - 1 = \frac{-1}{1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t)} \tag{12.131}
$$

Introducing (12.127) and (12.131) in (12.129), one obtains:

$$
V'(t+1) = V'(t) + \alpha(t)\bar{w}^2(t+1) - \frac{\alpha(t)[\bar{\varepsilon}^0(t+1)]^2}{1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t)} \tag{12.132}
$$

where:

$$
V'(t) = \tilde{\theta}^T(t)F(t)\tilde{\theta}(t) \tag{12.133}
$$

From (12.115) using the matrix inversion lemma (see Chap. 3) one also has:

$$
\operatorname{tr} F(t+1) = \operatorname{tr} F(t) - \frac{\alpha(t)\bar{\phi}(t)F^2(t)\bar{\phi}(t)}{1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t)} \tag{12.134}
$$

Adding (12.132) to (12.134) and using (12.118) one gets:

$$
V(t+1) = V(t) + \alpha(t)\bar{w}^2(t+1) - \frac{\alpha(t)\varepsilon_a^2(t+1)}{1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t)} \tag{12.135}
$$

with:

$$
V(t) = V'(t) + \operatorname{tr} F(t) = \tilde{\theta}^T(t)F(t)^{-1}\tilde{\theta}(t) + \operatorname{tr} F(t) \tag{12.136}
$$

In view of (10.102), (12.115) and (12.117) one has:

$$
1+\alpha(t)\bar{\phi}(t)F(t)\bar{\phi}(t) \le 1 + \alpha(t)\operatorname{tr} F(t)\|\bar{\phi}(t)\|^2 \le 1 + \operatorname{tr} F(0) \triangleq \gamma \tag{12.137}
$$

($\|\bar{\phi}(t)\| \le 1$ from (10.7)). Introducing (12.126) and (12.137) in (12.135) one obtains:

$$
V(t+1) \le V(t) + \frac{\alpha(t)}{\gamma}[\gamma\delta^2(t+1) - \varepsilon_a^2(t+1)] \tag{12.138}
$$

Using (12.117), (12.118) and (12.119) it follows:

$$\begin{aligned}\frac{\alpha(t)}{\gamma}[\gamma\delta^2(t+1)-\varepsilon_a^2(t+1)] &= \frac{\alpha(t)}{\gamma}[\gamma\bar{\delta}^2(t+1)-\gamma\sigma\delta(t+1)-\varepsilon_a^2(t+1)] \\ &\le -\alpha(t)\sigma\delta(t+1) \end{aligned} \tag{12.139}$$

Introducing (12.139) into (12.138) one finally gets:

$$0 \le V(t+1) \le V(t)-\alpha(t)\sigma\delta(t+1) \le V(0)-\sigma\sum_{i=0}^{t+1}\alpha(i)\delta(i+1) \tag{12.140}$$

$V(t)$ is therefore a nonincreasing positive sequence (see (12.140) and (12.136)) and thus $V(t)$ converges to a constant value. This proves that $\tilde{\theta}^T(t)F(t)^{-1}\tilde{\theta}(t)$ is bounded. The proofs of (12.123) as well as the convergence of $F(t)$ and $\tilde{\theta}(t)$ are similar to those of Theorem 10.5 and are omitted. □

The algorithm given in Theorem 12.4 will be used together with the modification of parameter estimates discussed in Sect. 12.2.3 in order to avoid the singularities in the computation of the controller parameters. The parameter estimates will be modified according to:

$$\bar{\theta}(t) = \hat{\theta}(t) + F(t)^{1/2}\beta(t) \tag{12.141}$$

where:

$$\bar{\theta}(t) = [\bar{a}_1(t),\ldots,\bar{a}_{n_A}(t),\bar{b}_1(t),\ldots,\bar{b}_{n_B}(t)] \tag{12.142}$$

Since $\|F(t)^{-1/2}\tilde{\theta}(t)\|$ is bounded and $\hat{\theta}(t)$ converges, it is possible to compute $\beta(t)$ as in Sect. 12.2.3 in such a way that $\beta(t)$ converges and that the Sylvester matrix associated to $\bar{\theta}(t)$ is nonsingular.

Let us define the modified a posteriori prediction error:

$$\begin{aligned}\varepsilon(t) &= y(t)-\bar{\theta}(t)\phi(t-1) \\ &= y(t)+\sum_{i=1}^{n_A}\bar{a}_i(t)y(t-i)-\sum_{i=1}^{n_B}\bar{b}_i(t)u(t-d-i)\end{aligned} \tag{12.143}$$

Using similar notations as in (12.1) through (12.4), (12.143) can also be written as:

$$\bar{A}(t)y(t) = q^{-d}\bar{B}(t)u(t)+\varepsilon(t) \tag{12.144}$$

where:

$$\bar{A}(t) = 1+\bar{a}_1(t)q^{-1}+\cdots+\bar{a}_{n_A}(t)q^{-n_A} \tag{12.145}$$

$$\bar{B}(t) = \bar{b}_1(t)q^{-1}+\cdots+\bar{b}_{n_B}(t)q^{-n_B} \tag{12.146}$$

are relatively prime polynomials having coefficients that converge. A bound for $\varepsilon(t)$ in (12.143), (12.144) which will be required later is given in the following lemma (Lozano 1992).

Lemma 12.2 *$\varepsilon(t)$ in (12.143), (12.90) satisfies the following inequality*:

$$\lim_{t\to\infty}\sup\left\{\frac{\varepsilon^2(t)}{(1+\|\phi(t-1)\|)^2}-3\bar{\delta}^2(t)\beta_{\max}^2\right\}\leq 0 \tag{12.147}$$

where $\beta_{\max}$ is such that:

$$\beta_{\max}=\max(1,\|\beta(t)\|) \tag{12.148}$$

Proof Introducing (12.141) into (12.143), one gets:

$$\begin{aligned}\varepsilon(t)&=y(t)-\hat{\theta}^T(t)\phi(t-1)-\beta^T(t)F^{1/2}(t)\phi(t-1)\\&=[\bar{y}(t)-\hat{\theta}^T(t)\bar{\phi}(t-1)-\beta^T(t)F^{1/2}(t)\bar{\phi}(t-1)]\\&\quad\times[1+\|\phi(t-1)\|]\end{aligned} \tag{12.149}$$

Taking into account that:

$$\bar{\phi}(t)=\frac{\phi(t)}{1+\|\phi(t)\|} \tag{12.150}$$

and using (12.86) and (12.93) one obtains:

$$\frac{\varepsilon(t)}{1+\|\phi(t-1)\|}=[\bar{\varepsilon}^0(t)-\beta^T(t)F^{1/2}(t-1)\bar{\phi}(t-1)+z(t)] \tag{12.151}$$

where:

$$\begin{aligned}z(t)&=[\hat{\theta}(t-1)-\hat{\theta}(t)]^T\bar{\phi}(t-1)\\&\quad+\beta^T(t)[F^{1/2}(t-1)-F^{1/2}(t)]\bar{\phi}(t-1)\end{aligned} \tag{12.152}$$

From (12.118), (12.148), (12.151) and the fact that $(a+b+c)^2\leq 3(a^2+b^2+c^2)$ for any a, b and c, one has:

$$\begin{aligned}\frac{\varepsilon^2(t)}{(1+\|\phi(t-1)\|)^2}&\leq 3[\bar{\varepsilon}^0(t)^2+\beta_{\max}^2\bar{\phi}^T(t-1)F(t-1)\bar{\phi}(t-1)+z^2(t)]\\&\leq 3\beta_{\max}^2\varepsilon_a^2(t)+3z^2(t)\end{aligned} \tag{12.153}$$

Note that $z(t)$ in (12.152) converges to zero because $F(t)$ and $\hat{\theta}(t)$ converge and $\beta(t)$ and $\bar{\phi}(t-1)$ are bounded. Equation (12.153) can also be written as:

$$\frac{\varepsilon^2(t)}{(1+\|\phi(t-1)\|)^2}-3\beta_{\max}^2\bar{\delta}^2(t)\leq 3\beta_{\max}^2[\varepsilon_a^2(t)-\bar{\delta}^2(t)]+3z^2(t) \tag{12.154}$$

Since $z(t)\to 0$ as $t\to\infty$ and in view of (12.3.40), the rest of the proof follows. □

From this point, the robust adaptive pole placement is obtained by combining Strategy 1 with the modified parameter estimates provided by (12.141), the parameters being estimated using the algorithm given in Theorem 12.4. The analysis of the resulting scheme is similar to the one discussed in Sect. 12.3.1 since $\varepsilon^2(t+1)$ in the limit is bounded by:

$$\lim_{t\to\infty} \sup \varepsilon^2(t+1) \le 3\bar{\delta}^2(t+1)\beta^2_{\max}(1+\|\phi(t)\|)^2 \tag{12.155}$$

Note that this expression is very similar to (12.106). Using similar arguments we can obtain an equation of the type of (12.111). Therefore, there is threshold d_2^* such that if $d_2 \le d_2^*$, one can prove by contradiction that $\|\phi(t)\|$ is bounded. The rest of the analysis follows.

12.3.3 Robust Adaptive Pole Placement: An Example

The example which will be considered here comes from Rohrs et al. (1985). The continuous time plant to be controlled is characterized by the transfer function:

$$G(s) = \frac{2}{s+1} \cdot \frac{229}{(s^2+30s+229)}$$

where the first order part is considered as the dominant dynamics and the second order part as the unmodeled dynamics. The system will be controlled in discrete time with a sampling period $T_S = 0.04$ s. For this sampling period the true discrete time plant model is given by:

$$G(q^{-1}) = \frac{b_1q^{-1}+b_2q^{-2}+b_3q^{-3}}{1-a_1q^{-1}+a_2q^{-2}+a_3q^{-3}}$$

with:

a_1	a_2	a_3
−1.8912	1.1173	−0.21225

b_1	b_2	b_3
0.0065593	0.018035	0.0030215

Since this model has unstable zeros, it is reasonable to use an indirect adaptive control strategy which can handle unstable zeros. Adaptive pole placement will be used. The plant model will be estimated using a filtered recursive least squares with dynamic data normalization. Adaptation freezing will be enforced in the absence of enough rich information (i.e., the scheduling variable for the dead zone will depend on the signal richness measured by $\bar{\phi}_f^T(t)F(t)\bar{\phi}_f(t)$—see Chap. 16 for details). Estimated models of different orders will be used ($n = 1, 2, 3$).

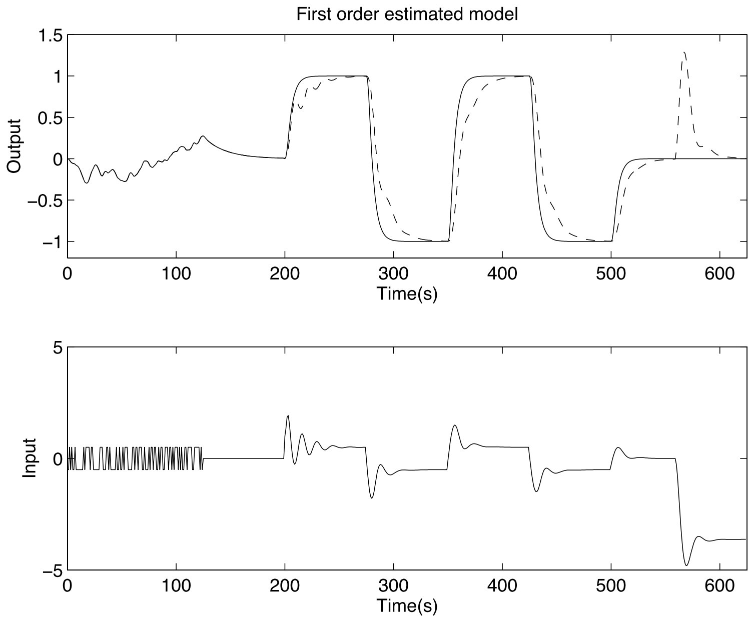

Fig. 12.1 First order estimated model (— reference trajectory, - - - plant output)

Figures 12.1, 12.2 and 12.3 summarize the results obtained with various orders for the estimated model. The controller is first initialized using an open-loop recursive identification, then the loop is closed and a series of step reference changes is applied followed by the application of an output disturbance.

For a first order estimated model, while the system is stable and the signals have acceptable form, one can see that the tracking performances are not very good. Second order and third order estimated models give almost the same results and they are very good.

Figure 12.4 shows the frequency characteristics of the estimated models for $n = 1, 2, 3$. The identified third order model corresponds exactly to the true model. From this figure, one can see that the second and third order models have similar frequency characteristics in the interesting frequency range for control (related to the desired closed-loop poles) while the first order model cannot cope with the frequency characteristics of the third order model in a sufficiently large frequency region. The desired closed-loop poles used are as follows:

$$\text{for } n = 1: \quad P(q^{-1}) = (1 - 0.8q^{-1})(1 - 0.9q^{-1})$$

$$\text{for } n = 2, 3: \quad P(q^{-1}) = (1 - 0.8q^{-1})(1 - 0.4q^{-1})(1 - 0.2q^{-1})(1 - 0.1q^{-1})$$

The controller has an integrator. For the case $n = 2$, a filter $H_R(q^{-1}) = 1 + q^{-1}$ has been introduced in the controller in order to reduce the "activity" of the control action by reducing the magnitude of the input sensitivity function in the high frequencies. Same dynamics has been used in tracking and regulation.

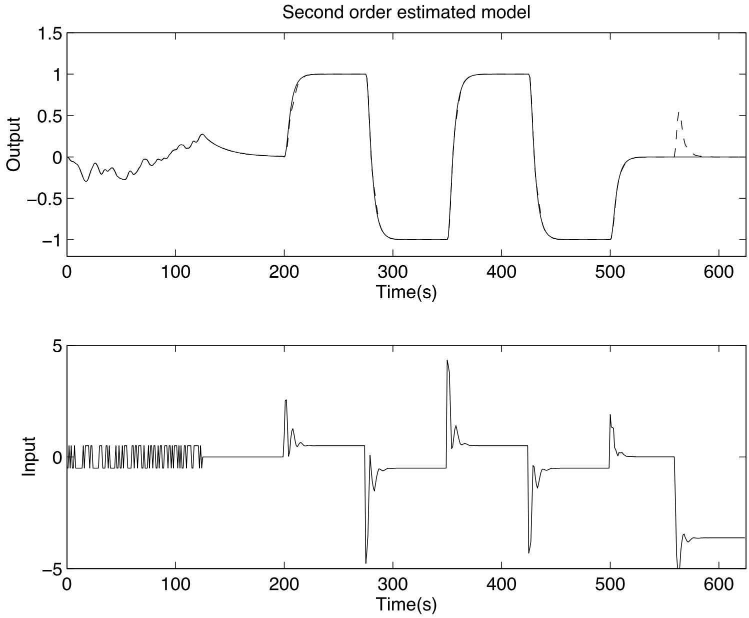

Fig. 12.2 Second order estimated model (— reference trajectory, - - - plant output)

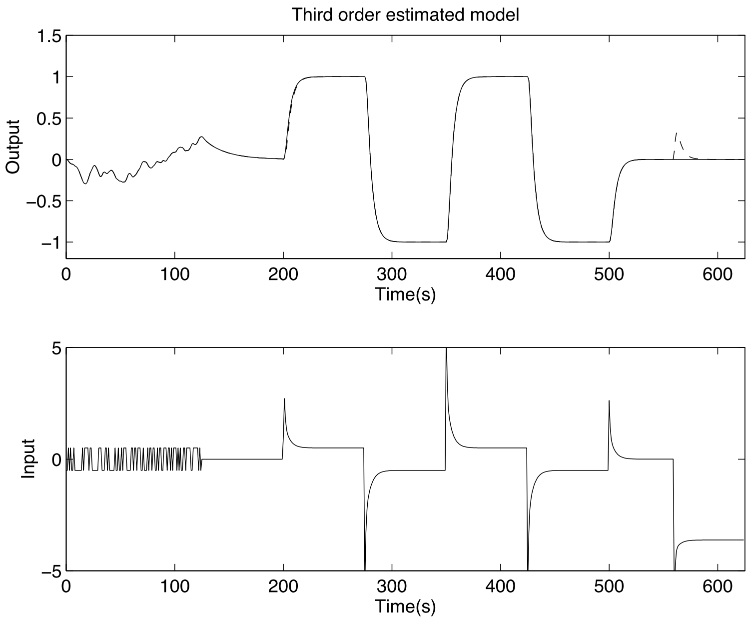

Fig. 12.3 Third order estimated model (— reference trajectory, - - - plant output)

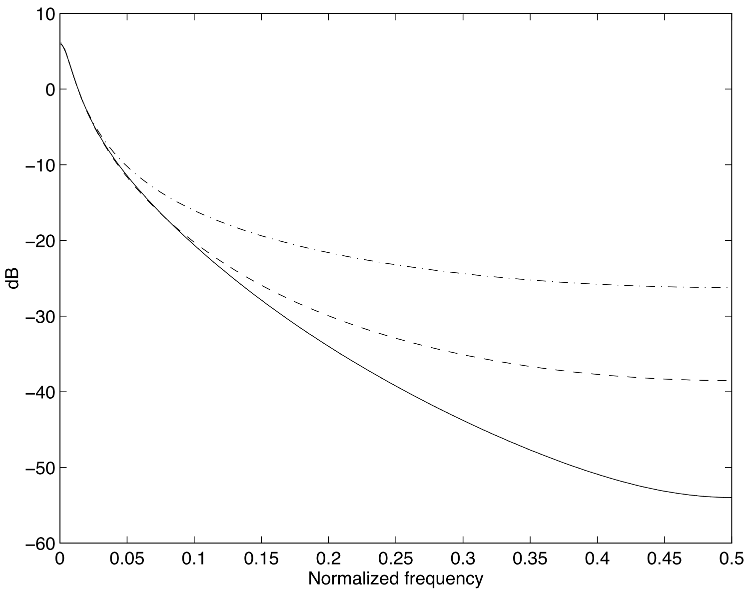

Fig. 12.4 Frequency characteristics for $n = 1(-.-), 2(--), 3(-)$

An adaptation gain with variable forgetting factor combined with a constant trace adaptation gain has been used ($F(0) = \alpha I$; $\alpha = 1000$ desired trace: $\operatorname{tr} F(t) = 6$). The filter used on input/output data is: $L(q^{-1}) = 1/P(q^{-1})$. The normalizing signal $m(t)$ has been generated by $m^2(t) = \mu^2 m^2(t-1) + \max(\|\phi_f(t)\|, 1)$ with $\mu = 0.9$. For $n = 1$, taking into account the unmodeled dynamics, the theoretical value for μ is $0.6 < \mu < 1$ (the results are not very sensitive with respect to the choice of the desired trace and μ).

The conclusion is that for this example, despite the fact that a stable adaptive controller can be obtained with a first order estimated model corresponding to the dominant dynamics of the true plant model, one should use a second order estimated model in order to obtain a good performance.

The rule which can be established is that low-order modeling can be used in adaptive control but good performance requires that this model be able to copy the frequency characteristics of the true plant model in the frequency region relevant for control design.

12.4 Adaptive Generalized Predictive Control

The basic algorithm for implementing Adaptive (PSMR) Generalized Predictive Control is similar to that for adaptive pole placement except for the details of Step II: Computation of the controller parameters and of the control law. In this case, using

Strategy 1, for the updating of the controller parameters (see Sect. 12.1) the controller equation generating $u(t)$ is:

$$\hat{S}(t,q^{-1})u(t)+\hat{R}(t,q^{-1})y(t)=\hat{\beta}(t)\hat{T}(t,q^{-1})y^*(t+d+1) \tag{12.156}$$

where:

$$\hat{\beta}(t)=1/\hat{B}(t,1) \tag{12.157}$$

$$\hat{S}(t,q^{-1})=\hat{S}'(t,q^{-1})H_S(q^{-1}) \tag{12.158}$$

$$\hat{R}(t,q^{-1})=\hat{R}'(t,q^{-1})H_R(q^{-1}) \tag{12.159}$$

$$\hat{T}(t,q^{-1})=\hat{A}(t,q^{-1})\hat{S}(t,q^{-1})+q^{-d-1}\hat{B}^*(t,q^{-1})\hat{R}(t,q^{-1}) \tag{12.160}$$

From (7.222) and (7.223), one gets for a certain value of the estimated plant model parameter vector $\hat{\theta}(t)$:

$$\hat{S}'(t,q^{-1})=P_D(q^{-1})+q^{-1}\sum_{j=h_i}^{h_p}\hat{\gamma}_j(t)\cdot\hat{H}_{j-d}(t,q^{-1}) \tag{12.161}$$

$$\hat{R}'(t,q^{-1})=\sum_{j=h_i}^{h_p}\hat{\gamma}_j(t)\hat{F}_j(t,q^{-1}) \tag{12.162}$$

where $\hat{\gamma}_j(t)$ are the elements of the first row of the matrix $[\hat{G}^T(t)\hat{G}(t)+\lambda I_{h_c}]^{-1}\hat{G}^T(t)$ and $\hat{G}(t)$ is an estimation of the matrix (7.215). To effectively compute $\hat{\gamma}_j(t)$, $\hat{H}_{j-d}(t,q^{-1})$ and $\hat{F}_j(t,q^{-1})$, one has to solve at each sampling instant t the two polynomial divisions:

$$P_D(q^{-1})=\hat{\hat{A}}(t,q^{-1})\hat{E}_j(t,q^{-1})+q^{-j}\hat{F}_j(t,q^{-1}) \tag{12.163}$$

$$\begin{aligned}\hat{\hat{B}}^*(t,q^{-1})\hat{E}_j(t,q^{-1})&=P_D(q^{-1})\hat{G}_{j-d}(t,q^{-1})\\&\quad+q^{-j+d}\hat{H}_{j-d}(t,q^{-1})\end{aligned} \tag{12.164}$$

for the estimated value of the plant model parameter vector $\hat{\theta}(t)$ given in (12.7).

As in the case of the pole placement, one has to check the admissibility of the estimated plant model. For the case of generalized predictive control, one checks if the estimated characteristic polynomial

$$\hat{P}(t,q^{-1})=\hat{A}(t,q^{-1})\hat{S}(t,q^{-1})+q^{-d-1}\hat{B}^*(t,q^{-1})\hat{R}(t,q^{-1}) \tag{12.165}$$

is asymptotically stable and that $\hat{\beta}(t)>0$.

12.5 Adaptive Linear Quadratic Control

The basic algorithm for implementing (PSMR) Linear Quadratic Control is similar to that for adaptive pole placement except for the details of Step II: Computation of the controller parameters and of the control law. In this case, using strategy I for the updating of the controller parameters one has:

$$\hat{x}(t+1) = \hat{A}(t)\hat{x}(t) + \hat{b}(t)\bar{e}_u(t) + \hat{k}(t)[e_y(t) - \hat{e}_y(t)] \tag{12.166}$$

$$\hat{e}_y(t) = c^T\hat{x}(t) \tag{12.167}$$

$$\hat{k}^T(t) = [\hat{k}_1(t), \ldots, \hat{k}_n(t)]; \quad \hat{k}_i(t) = p_i - \hat{a}_i(t) \tag{12.168}$$

$$\bar{e}_u(t) = -\frac{\hat{b}^T(t)\hat{\Gamma}(t)A(t)}{\hat{b}^T(t)\hat{\Gamma}(t)\hat{b}(t) + \lambda}\hat{x}(t) \tag{12.169}$$

where p_i are the coefficients of the polynomial $P_D(q^{-1})$ defining desired observer dynamics and $\Gamma(t)$ is the positive definite solution of the Algebraic Riccati Equation computed for the estimated values $\hat{A}(t)$ and $\hat{b}(t)$:

$$\begin{aligned} &\hat{A}^T(t)\hat{\Gamma}(t)\hat{A}(t) - \hat{\Gamma}(t) \\ &\quad - \hat{A}^T(t)\hat{\Gamma}(t)\hat{b}(t)[\hat{b}^T(t)\hat{\Gamma}(t)\hat{b}(t) + \lambda]^{-1}\hat{b}^T(t)\hat{\Gamma}(t)\hat{A}(t) \\ &\quad + cc^T = 0 \end{aligned} \tag{12.170}$$

As indicated in Samson (1982), instead of solving an ARE at each sampling instant one can use a Time Varying Riccati Equation with an iteration at each sampling instant, i.e.:

$$\begin{aligned} \hat{\Gamma}(t) = {} & \hat{A}^T(t)\hat{\Gamma}(t-1)\hat{A}(t) - \hat{A}^T(t)\hat{\Gamma}(t-1)\hat{b}(t)[\hat{b}^T(t)\Gamma(t-1)\hat{b}(t) + \lambda]^{-1} \\ & \times \hat{b}^T(t)\hat{\Gamma}(t-1)\hat{A}(t) + cc^T = 0 \end{aligned} \tag{12.171}$$

since if asymptotically $\hat{A}(t)$ and $\hat{b}(t)$ tend toward constant values satisfying the existence condition for a positive definite solution of the ARE, $\hat{\Gamma}(t)$ given by (12.171) will tend toward the solution of the ARE. From (12.166) through (12.169), using (7.293) through (7.297), one gets at each instant t the RST controller form.

As in the case of the pole placement, one has to check the admissibility of the estimated plant model at each instant t. For the case of linear quadratic control, one checks if the eventual common factors of $\hat{A}(t, q^{-1})$ and $\hat{B}(t, q^{-1})$ are inside the unit circle.

12.6 Adaptive Tracking and Robust Regulation

The idea in this approach is to:

1. use of a robust linear controller with fixed parameters for regulation ($R(q^{-1})$ and $S(q^{-1})$);

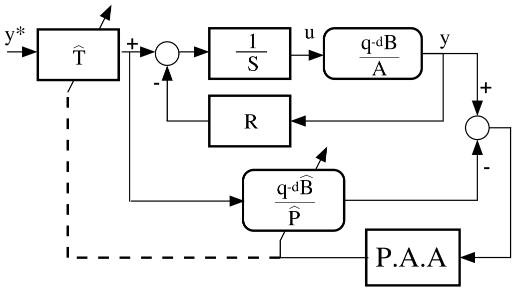

Fig. 12.5 Adaptive tracking and robust regulation

2. identify the closed-loop poles (either directly or indirectly by identifying the plant model in closed-loop operation and computing the closed-loop poles);
3. adapt the parameters of the precompensator $\hat{T}(t, q^{-1})$ based on the current poles and zeros of the closed loop.

This is illustrated in Fig. 12.5.

This technique is compared with a robust controller and with an indirect adaptive controller in Sect. 12.7 (Fig. 12.14) for the case of the control of the flexible transmission.

12.7 Indirect Adaptive Control Applied to the Flexible Transmission

The performance of several indirect adaptive control schemes will be illustrated by their applications to the control of the flexible transmission shown in Fig. 1.19.

12.7.1 Adaptive Pole Placement

In this case, the pole placement will be used as control strategy but two types of parameter estimators will be used:

1. filtered recursive least squares;
2. filtered closed-loop output error (F-CLOE).

As indicated in Chap. 1 in indirect adaptive control the objective of the plant parameter estimation is to provide the best prediction for the behavior of the closed loop system, for given values of the controller parameters. This can be achieved by either using appropriate data filters on plant input-output data or by using adaptive predictors for the closed-loop system parameterized in terms of the controller parameters and plant parameters (Landau and Karimi 1997b). The corresponding parameter estimators are illustrated in Figs. 12.6a and 12.6b and details can be found in Chap. 9.

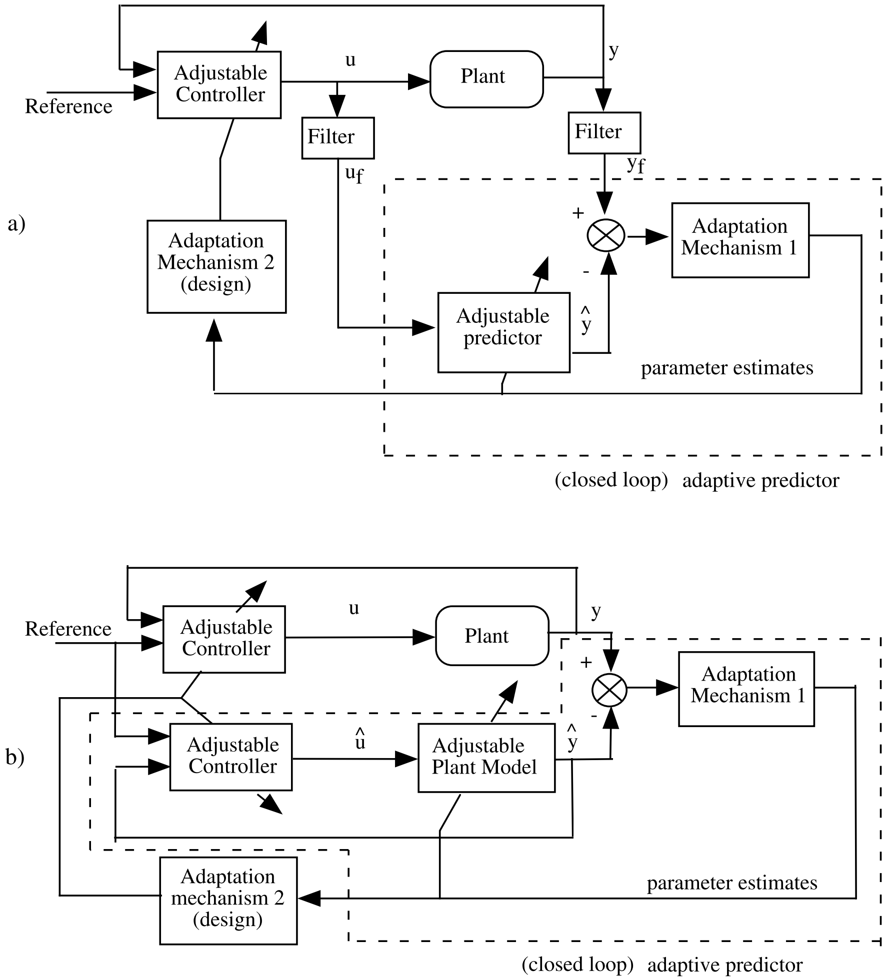

Fig. 12.6 Indirect adaptive control with closed-loop adjustable predictors, (**a**) using input-output data filters, (**b**) using closed-loop predictors

These experiments have been carried out using the real time simulation package Vissim/RT (Visual Solutions 1995). Figs. 12.7, 12.8 and 12.9 show the behavior of the adaptive pole placement using filtered recursive least squares without adaptation freezing for various loads. In each experiment, an open-loop identification for the initialization of the adaptive scheme is carried out during 128 samples, then one closes the loop and one sends a sequence of step reference changes followed by the application of a position disturbance. The upper curves show the reference trajectory and the output of the system, the curves in the middle show the evolution of the input and the lower curves show the evolution of the estimated parameters. One can observe that the system is almost tuned at the end of the initialization period.

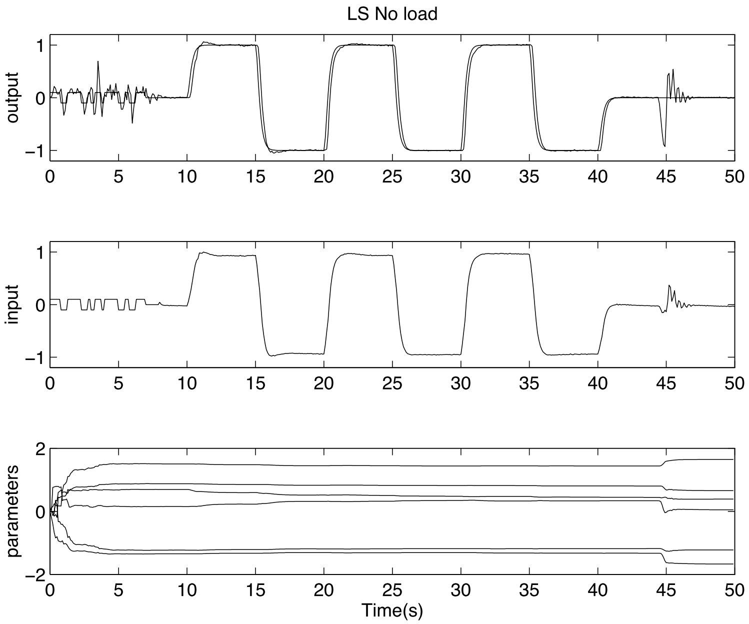

Fig. 12.7 Flexible transmission—adaptive pole placement using filtered RLS. The no load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

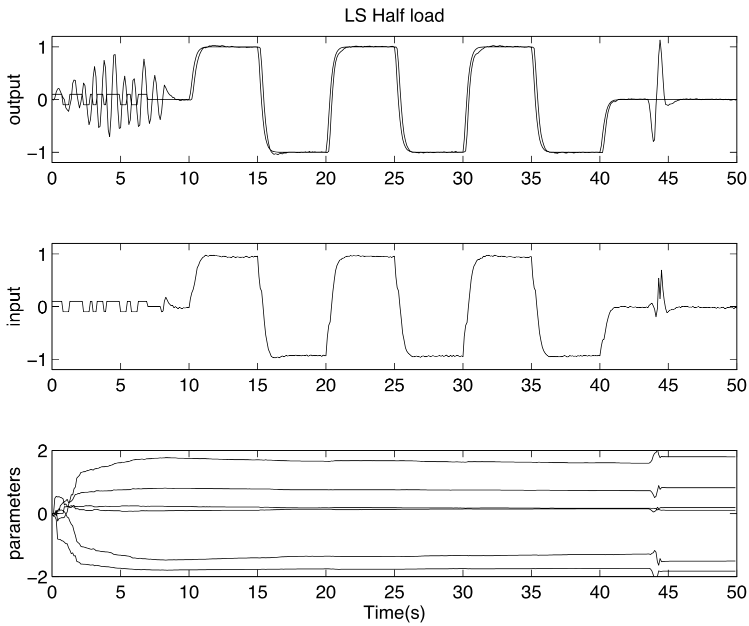

Fig. 12.8 Flexible transmission—adaptive pole placement using filtered RLS. The half load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

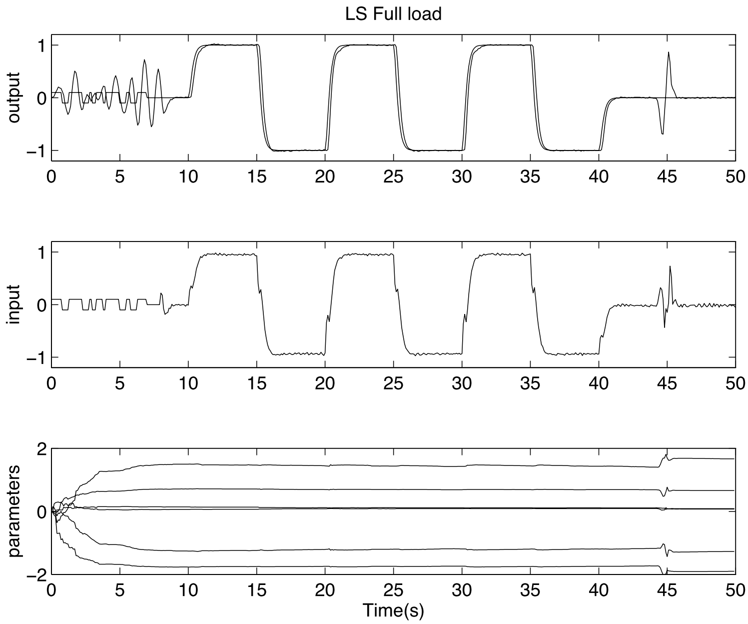

Fig. 12.9 Flexible transmission—adaptive pole placement using filtered RLS. The full load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

Since adaptation freezing is not used, a drift in the estimated parameters occurs in the presence of the disturbance. Successive applications of such disturbances will destabilize the system. Therefore, in the case of recursive least squares (or any other open-loop type identification algorithm) the use of adaptation freezing is mandatory (see Sect. 12.7.2).

For this application the pole placement design is characterized by the following desired closed-loop poles:

1. A pair of dominant complex poles ($P_D(q^{-1})$) with the frequency of the loaded first vibration mode ($\omega_0 = 6$ rad/s) and with a damping $\zeta = 0.9$.
2. Auxiliary poles as follows: a pair of complex poles ($\omega_0 = 33$ rad/s, $\zeta = 0.3$) and the real poles $(1 - 0.5q^{-1})^3(1 - 0.1q^{-1})^3$.

The controller has an integrator ($H_S(q^{-1}) = 1 - q^{-1}$) and contains a filter $H_R(q^{-1}) = (1 + q^{-1})^2$ which reduces the modulus of the input sensitivity function in the high frequencies. The plant model estimator uses a decreasing adaptation gain combined with constant trace adaptation gain with $\operatorname{tr} F(t) = \operatorname{tr}[I_6]$. The input/output data are filtered by a band pass Butterworth filter with two cells (low frequency 0.05 Hz, high frequency 7 Hz). Figures 12.10, 12.11 and 12.12 show the results for the same type of experiment when using F-CLOE identification algorithm instead of the filtered least squares. Almost similar results are obtained in tracking but there is a significant difference in regulation. Despite the absence of adaptation freezing, the parameter are not drifting in the presence of disturbances and therefore

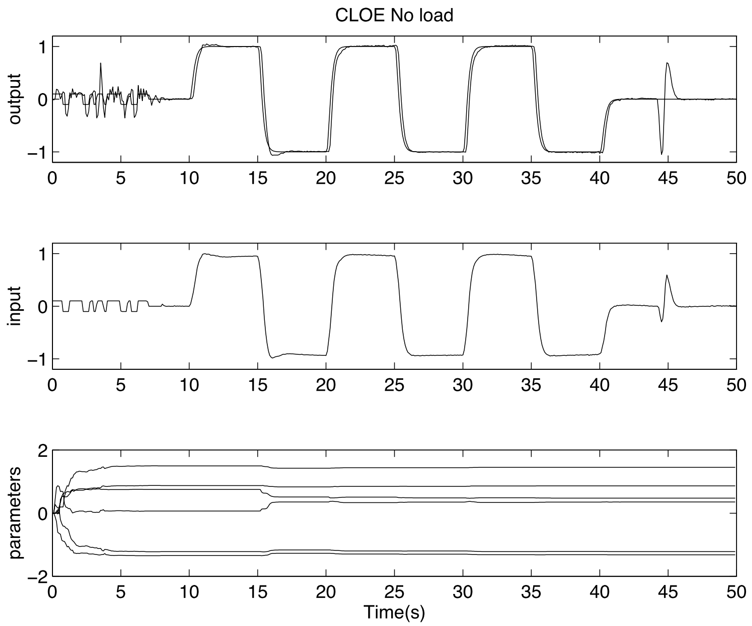

Fig. 12.10 Flexible transmission—adaptive pole placement using F-CLOE. The no load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

there is no risk of instability. The observation vector used in F-CLOE algorithm is filtered through $(1 - q^{-1})/P_D(q^{-1})$ where $P_D(q^{-1})$ corresponds to the desired dominant closed-loop pole. Figure 12.13 shows the influence of the various linear design choices (the F-CLOE algorithm is used). For this case (full load) the auxiliary poles are all set to zero and $H_R(q^{-1}) = 1 + 0.5q^{-1}$. Comparing these results with those shown in Figs. 12.9 and 12.12, one sees an increase of the variance of the input. This is caused by higher values of the modulus of the input sensitivity function in the high frequencies and justify:[4]

1. introduction of auxiliary poles;
2. use of $H_R(q^{-1}) = (1 + \beta q^{-1})$ or $H_R(q^{-1}) = (1 + \beta q^{-1})^2$ with $0.7 \leq \beta \leq 1$ ($\beta = 1$ corresponds to the opening of the loop at $0.5 f_s$).

Figure 12.14 shows a comparison between an adaptive pole placement controller using F-CLOE parameter estimation algorithm (upper curves), a robust controller designed using the method given in Langer and Landau (1999) which meets all the specifications of the robust digital control benchmark for the flexible transmission (Landau et al. 1995a) (middle curves) and an adaptive tracking with robust regulation using the above robust regulator (lower curves).

[4]See Chap. 8, Sect. 8.7 for details.

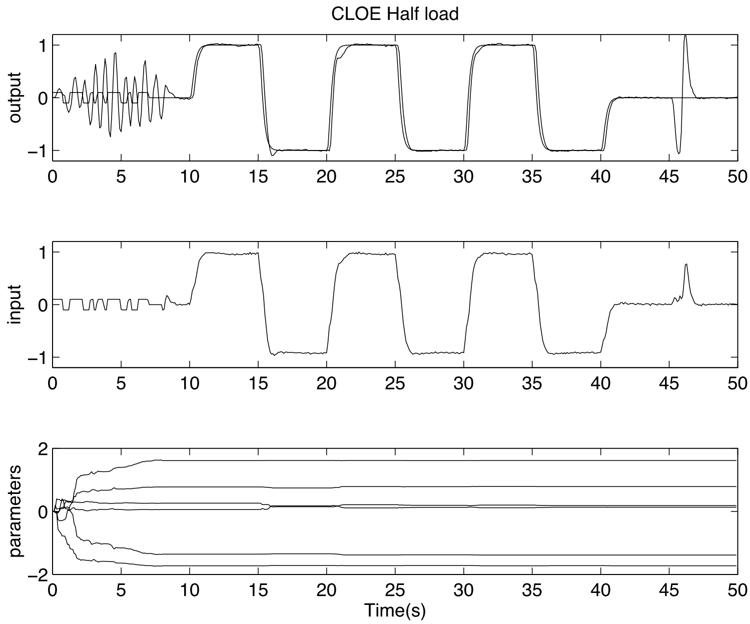

Fig. 12.11 Flexible transmission—adaptive pole placement using F-CLOE. The half load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

The experiment is carried out from "no load" case at $t = 0$, toward the "full load" case and the load is increased after each step reference change. At the end (full load) a position disturbance is applied. The adaptive scheme is tuned for the no load case at $t = 0$. One clearly sees the improvement of performance when using adaptive control. One also can see that adaptive tracking improves the performance in tracking with respect to the robust controller with a fixed polynomial T. The performance can be further improved after several more step reference changes.

12.7.2 Adaptive PSMR Generalized Predictive Control

In this case, the *partial state model reference* generalized predictive control combined with a filtered recursive least squares parameter estimation algorithm will be used. Adaptation freezing in the absence of significant information is incorporated in the algorithm.

The experiments have been carried out using the SIMART package (M'Saad and Chebassier 1997).[5] Figures 12.15, 12.16 and 12.17 show the performance of the

[5]These experiments have been carried out by J. Chebassier (Lab. d'Automatique de Grenoble).

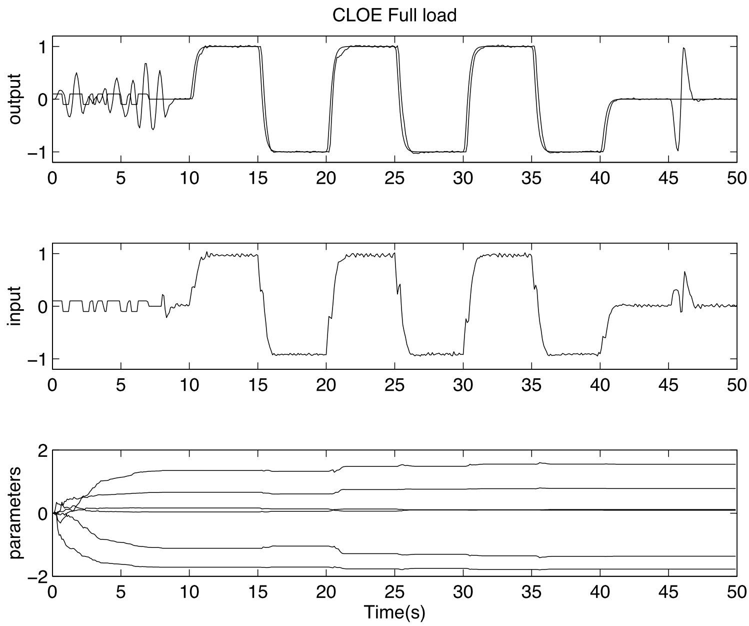

Fig. 12.12 Flexible transmission—adaptive pole placement using F-CLOE. The full load case, (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

adaptive PSMR/GPC for various loads. In each experiment, an open-loop identification for the initialization of the adaptive scheme is carried out during 128 samples, then one closes the loop and one sends a sequence of step reference changes followed by the application of a position disturbance. The upper curve shows the output of the system. The curve below shows the input of the system. The next curves show the evolution of two estimated parameters. The lower curve shows the evolution of the scheduling variable for adaptation freezing (value 1) and of $\phi_f^T(t)F(t)\phi_f(t)$.

For this application the PSMR/GPC is characterized by:

1. Predictor poles: Two pairs of complex poles corresponding to ($\omega_0 = 10.7$ rad/s, $\zeta = 1$) and ($\omega_0 = 30$ rad/s, $\zeta = 1$) and three real poles $z_1 = z_2 = 0.585$ and $z_3 = 0.7$.
2. Initial horizon $h_i = 3$, prediction horizon $h_p = 20$, control horizon $h_c = 3$, input weighting $\lambda = 0.5$, $H_S(q^{-1}) = 1 - q^{-1}$ (integrator behavior).

The plant model estimator uses an adaptation gain with variable forgetting factor combined with constant trace adaptation gain with $\operatorname{tr} F(t) = \operatorname{tr}[I_6]$. The input/output data are filtered through a band pass Butterworth filter with two cells (low frequency: 0.005 Hz, high frequency: 7 Hz). The adaptation freezing is based on the evaluation of the $\phi_f^T(t)F(t)\phi_f(t)$ over a sliding horizon of 10 samples (i.e., if this quantity is below the threshold during 10 samples, the adaptation is switched off). Figure 12.18 shows the performance of the adaptive PSMR/GPC in the presence of

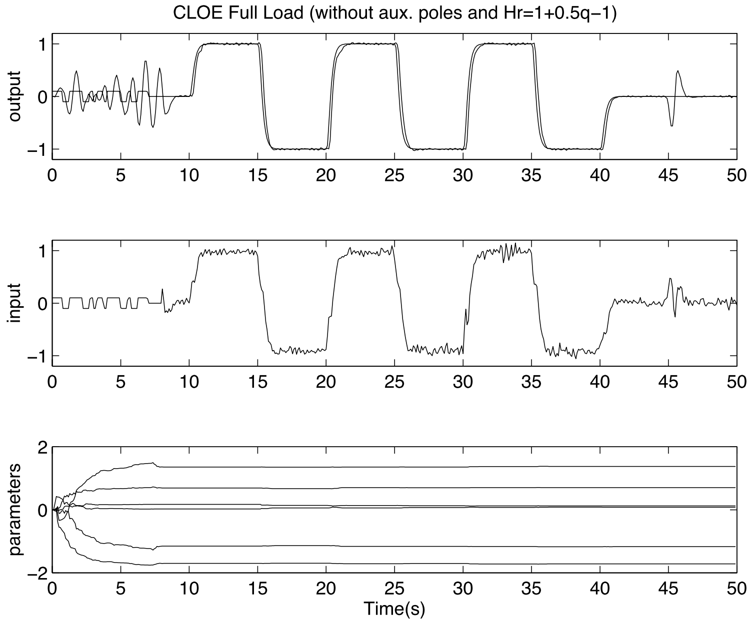

Fig. 12.13 Adaptive pole placement without auxiliary poles and $H_R(q^{-1}) = 1 + 0.5q^{-1}$ (full load), (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters

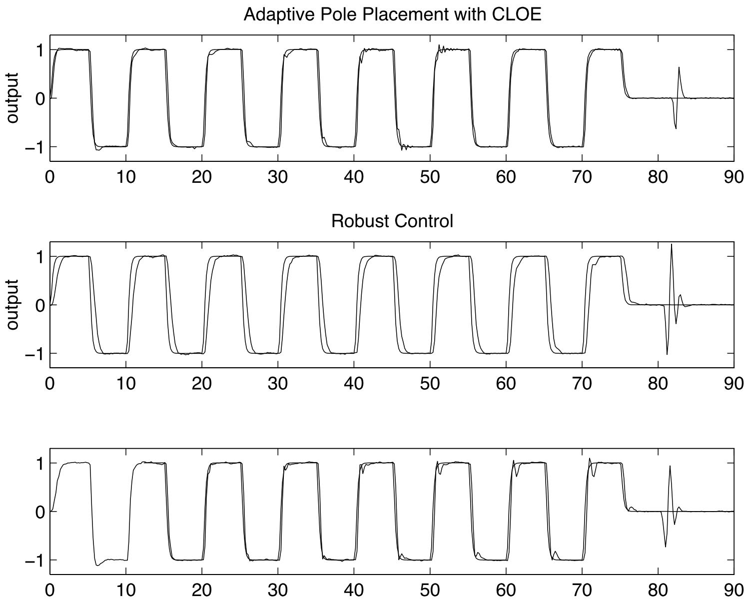

Fig. 12.14 Flexible transmission—comparison of three control strategies, (**a**) adaptive pole placement with CLOE, (**b**) robust control, (**c**) adaptive tracking with robust regulation

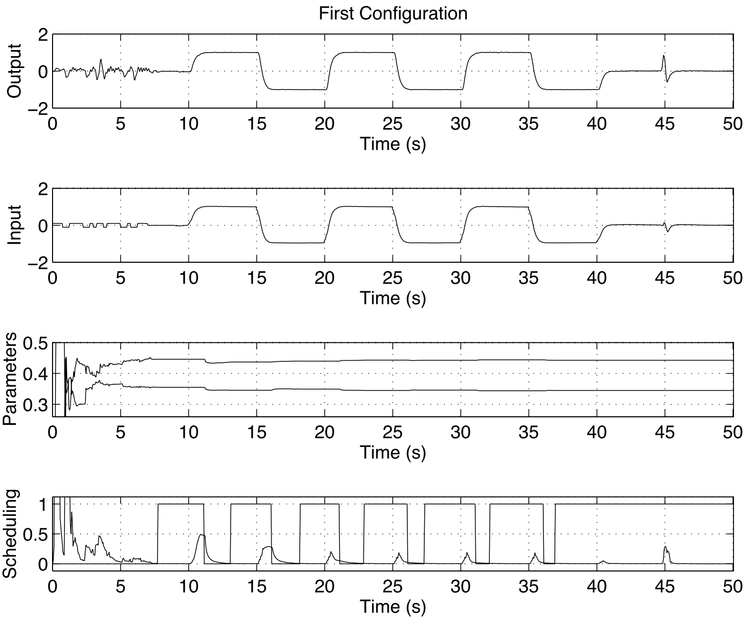

Fig. 12.15 Flexible transmission—adaptive PSMR/GPC. The no load case, (**a**) output, (**b**) input, (**c**) estimated parameters, (**d**) scheduling variable for adaptation freezing

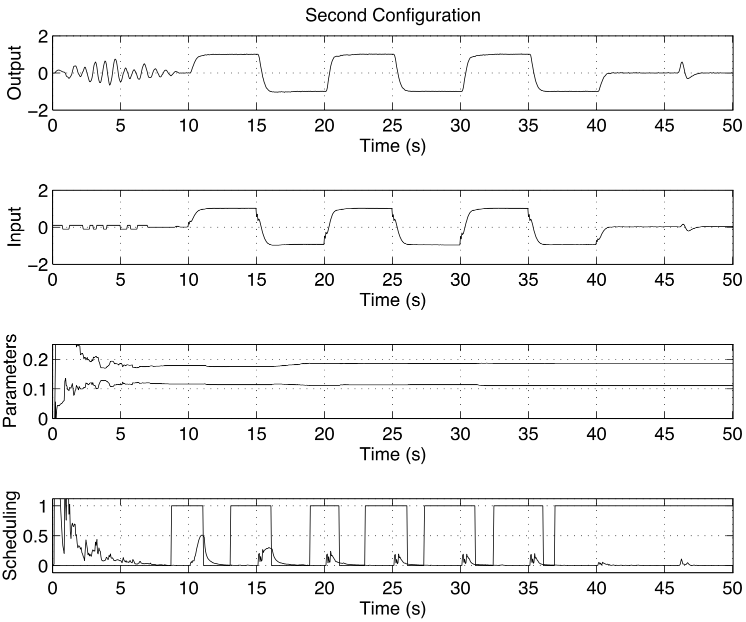

Fig. 12.16 Flexible transmission—adaptive PSMR/GPC. The half load case, (**a**) output, (**b**) input, (**c**) estimated parameters, (**d**) scheduling variable for adaptation freezing

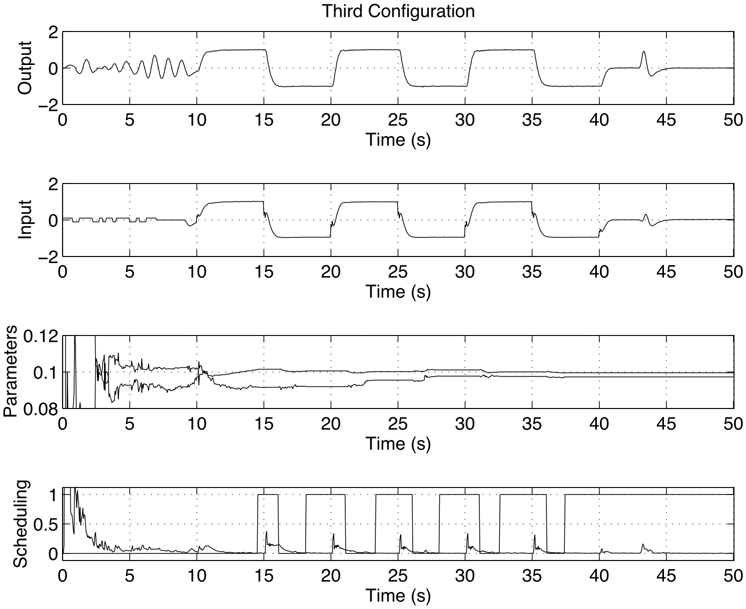

Fig. 12.17 Flexible transmission—adaptive PSMR/GPC. The full load case, (**a**) output, (**b**) input, (**c**) estimated parameters, (**d**) scheduling variable for adaptation freezing

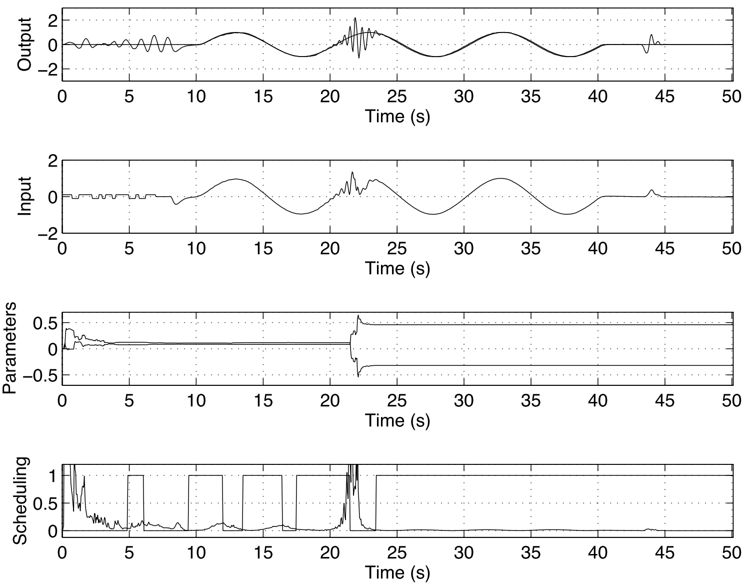

Fig. 12.18 Flexible transmission—adaptive PSMR/GPC. Effect of step load change (100% $\rightarrow$ 0%), (**a**) reference trajectory and output, (**b**) input, (**c**) estimated parameters, (**d**) scheduling variable for adaptation freezing

a step load change from the full load case to the no load case (the most difficult situation). The reference signal is a sinusoid. As a consequence of the drastic change in the parameters of the model, an instability phenomena is ignited, but the adaptation succeeds to quickly stabilize the system.

12.8 Concluding Remarks

1. Indirect adaptive control algorithms emerged as a solution for adaptive control of systems featuring discrete time models with unstable zeros.
2. Indirect adaptive control offers the possibility of combining (in principle) any linear control strategy with a parameter estimation scheme.
3. The plant model parameter estimator should allow a good prediction of the behavior of the closed loop.
4. The design of the controller based on the estimated plant models should be done such that some robustness constraints on the sensitivity functions be satisfied (in particular on the output and on the input sensitivity functions). This can be achieved by a suitable choice of the desired performances and introduction of some fixed filters in the controller.
5. For each type of underlying linear control strategy used in indirect adaptive control a specific admissibility test has to be done on the estimated model prior to the computation of the control.
6. Robustification of the parameter adaptation algorithms used for plant model parameter estimation may be necessary in the presence of unmodeled dynamics and bounded disturbances.
7. Adaptive pole placement and adaptive generalized predictive control are the most used indirect adaptive control strategies.

12.9 Problems

12.1 Under what conditions, (12.16) will imply $\lim_{t\to\infty} \varepsilon^0(t+1) = 0$?

12.2 Show that (12.18) does not imply that $\hat{\theta}(t)$ converges towards a constant θ.

12.3 Using first order polynomials, show that in general

$$\hat{A}(t)\hat{S}(t)y(t) \neq \hat{S}(t)\hat{A}(t)y(t)$$

12.4 Theorem 12.1 shows that

$$\lim_{t\to\infty} P[y(t+d) - \hat{B}^*(t, q^{-1})\hat{\beta}(t)y^*(t+d)] = 0$$

Explain the meaning of this result (hint: see Sect. 7.3.3).

12.5 Motivate the input-output variables normalization (12.88) (hint: See Sect. 10.6).

12.6 Explain what happens if the parameter adaptation is not frozen in the absence of the persistent excitation condition.