Teoría de Control

Root Locus design

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Design criteria

- Steady state performance – exactitud - tracking error – step, ramp
  - Steady state error:

$$e = \lim_{t \to \infty} \left| \frac{r(t) - y_{ss}(t)}{a} \right|$$

  - Position error: step input *r(s)=a*
  - Velocity error: ramp input *r(s)=at*

- Dada

$$G_o(s) = \frac{\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n}{\beta_0 + \beta_1 s + \cdots + \beta_m s^m} \qquad n \geq m$$

Design criteria

- Position error: step input *r(s)=a*

$$e_p = \left|1 - G_o(0)\right| = \left|\frac{\alpha_0 - \beta_0}{\alpha_0}\right|$$

- Velocity error: ramp input *r(s)=at*

$$e_v = \left|\left(1 - G_o(0)\right)t - \frac{d}{ds}G_o(s)\Big|_{s=0}\right|$$

$$e_v = \left|\left(\frac{\alpha_0 - \beta_0}{\alpha_0}\right)t - \frac{\alpha_0\beta_1 - \beta_0\alpha_1}{\alpha_0^2}\right|$$

Design criteria

- The plant output *y(t)* is said to track asymptotically the reference input *r(t)* if

$$\lim_{t\to\infty}\left|r(t)-y(t)\right|=0$$

- step input *r(s)=a*

$$\alpha_0=\beta_0$$

- ramp input *r(s)=at*

$$\alpha_0=\beta_0$$
$$\alpha_1=\beta_1$$

- parable input *r(s)=at^2*

$$\alpha_0=\beta_0$$
$$\alpha_1=\beta_1$$
$$\alpha_2=\beta_2$$

Design criteria

- System type: number of poles at the origen

- Loop transfer function $G_l(s)$

Tipo\\e_{ss}	e_p	e_v	e_a
0	K	∞	∞
1	0	K	∞
2	0	0	K

Design criteria

• Well-posedness:

- All transfer function in the system must be proper and

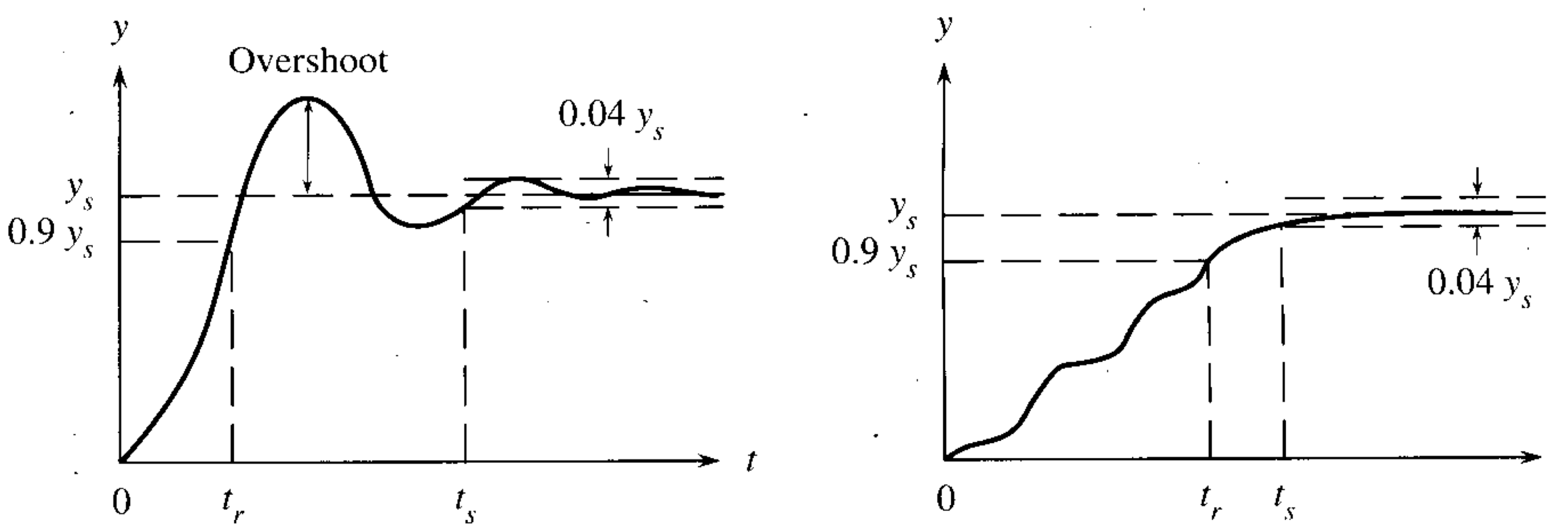
$$\Delta(\infty) = 1 + C(\infty)G(\infty) \neq 0$$

• Total stability

- *Poles of $G_o(s)$ and its missing poles are all stable*
- *Avoid unstable pole-zero cancellations*

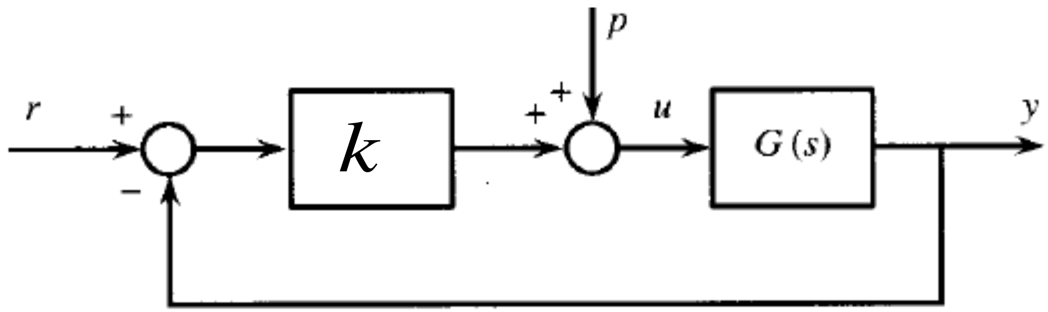
Design criteria

- Transient state – speed of response – step response



Root Locus Method

- Find the closed-loop pole location for any *k* in



Closed-loop transfer function

$$G_o(s) = \frac{kG(s)}{1 + kG(s)}$$

Closed-loop poles:

$$1 + kG(s) = 0$$

Then:

$$G(s) = -\frac{1}{k}$$

$$\left|G(s)\right| = \left|-\frac{1}{k}\right|$$

$$\hat{G}(s) = \arg\left(-\frac{1}{k}\right) = \pm 180^\circ$$

Root Locus Method

- In general

- The roots of the characteristic equation as a function of k_x

$$p(s) + k_x q(s) = 0$$

Then:

$$G_{pq}(s) = \frac{q(s)}{p(s)} = -\frac{1}{k_x}$$

$$\left|G_{pq}(s)\right| = \left|-\frac{1}{k_x}\right|$$

$$\hat{G}_{pq}(s) = \arg\left(-\frac{1}{k_x}\right) = \pm 180^\circ$$

Root Locus Method

- Procedure:
  - Find the design region
  - Plot the roots of the characteristic equation as a function of k

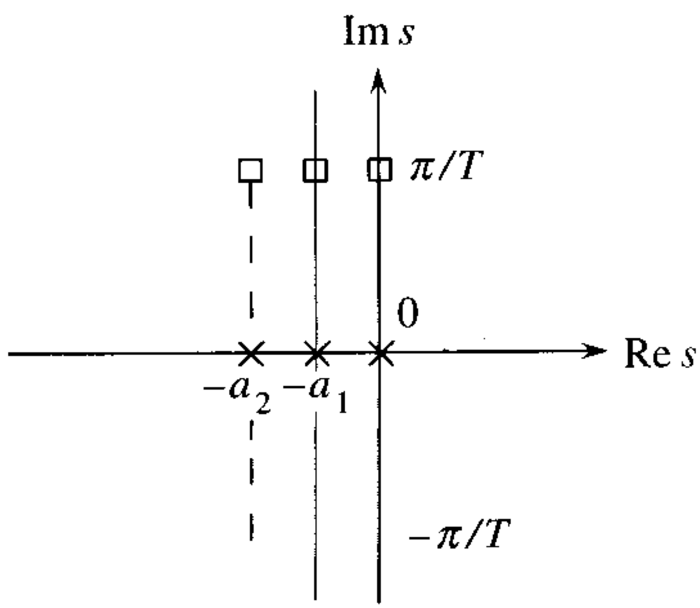
$$p(s)+kq(s)=0$$

- The region is defined by the time domain criteria (step signal response) :
  - t_s settling time
  - Overshoot
  - t_r rise time

Root Locus Method

- t_s settling time

$$-a = -\frac{4.5}{t_s} \longrightarrow t_s = \frac{4.5}{a} < t_{s0}$$



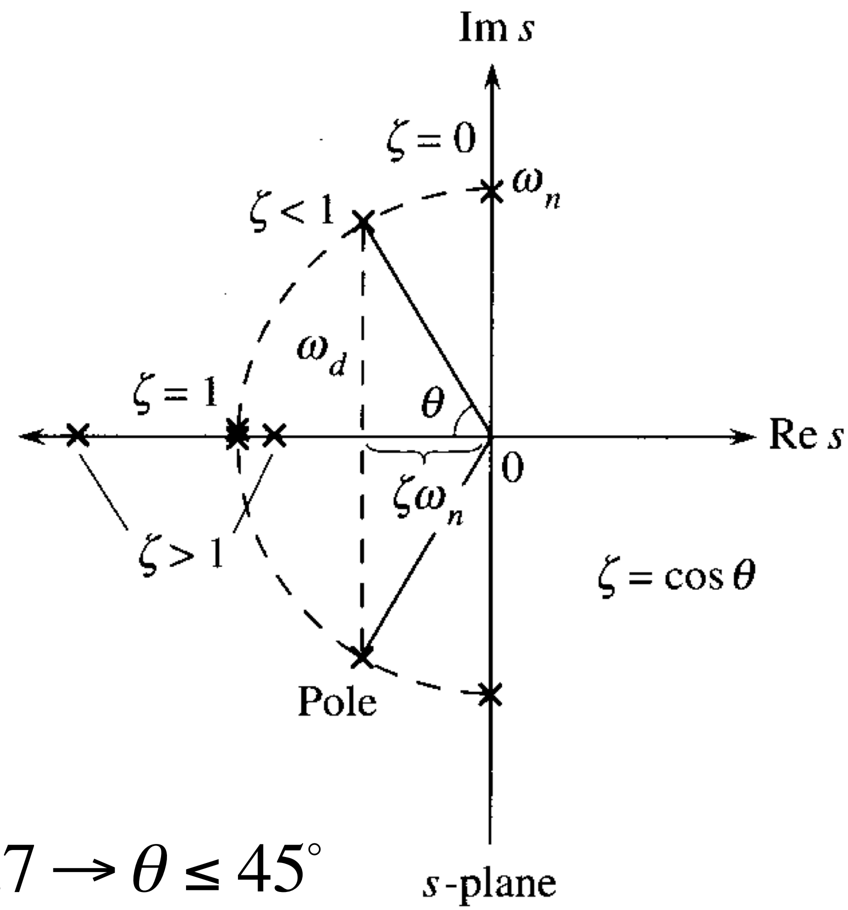
Root Locus Method

- Given

$$G_o(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Its poles are

$$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



Also we have:

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

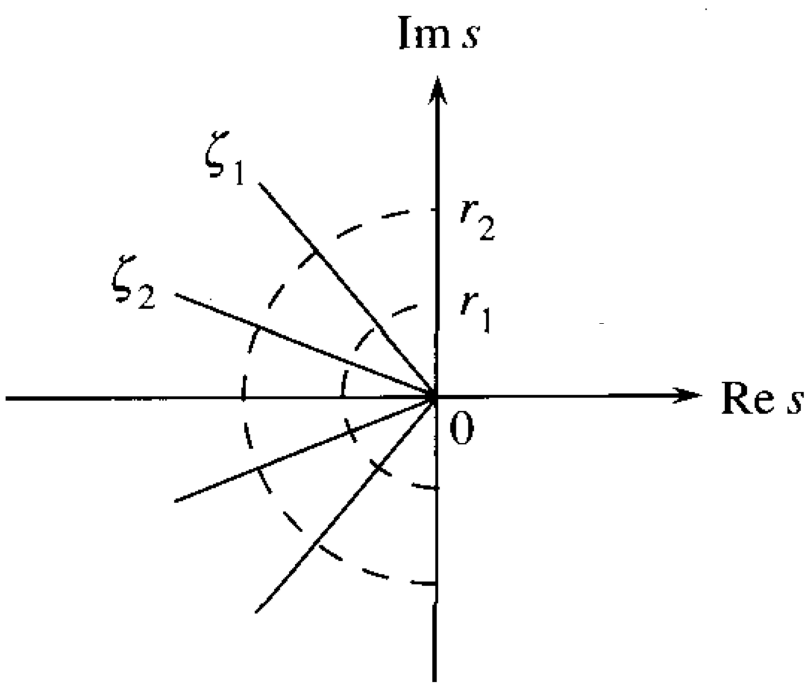
$$\zeta = \cos\theta$$

$$M_p \le 5\% \to \zeta \ge 0.7 \to \theta \le 45^\circ$$

Root Locus Method

• Rise time t_r

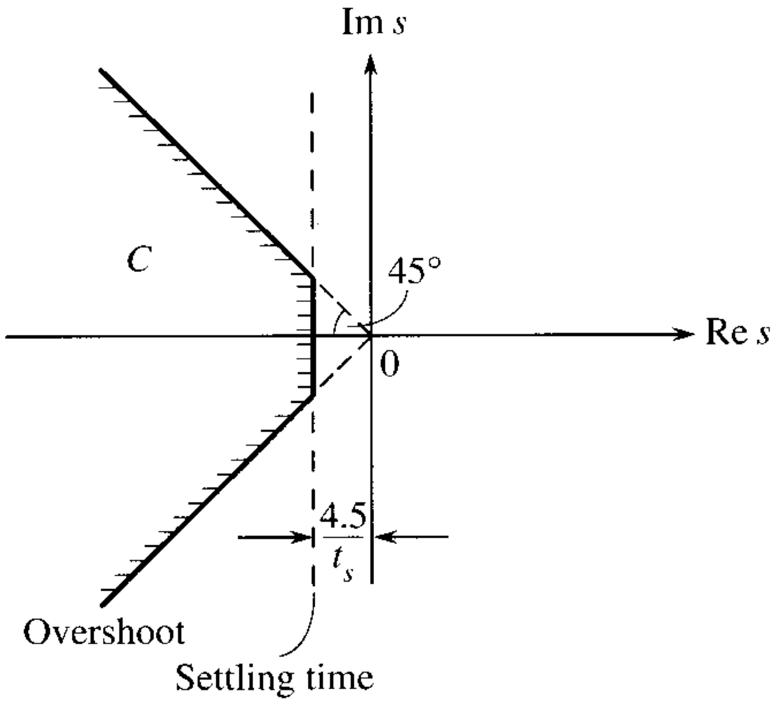
The farther away the closest pole from the origin, the smaller the rise time → ?



$$t_r \approx \frac{1.8}{\omega_n}$$

Root Locus Method

- Desired pole region



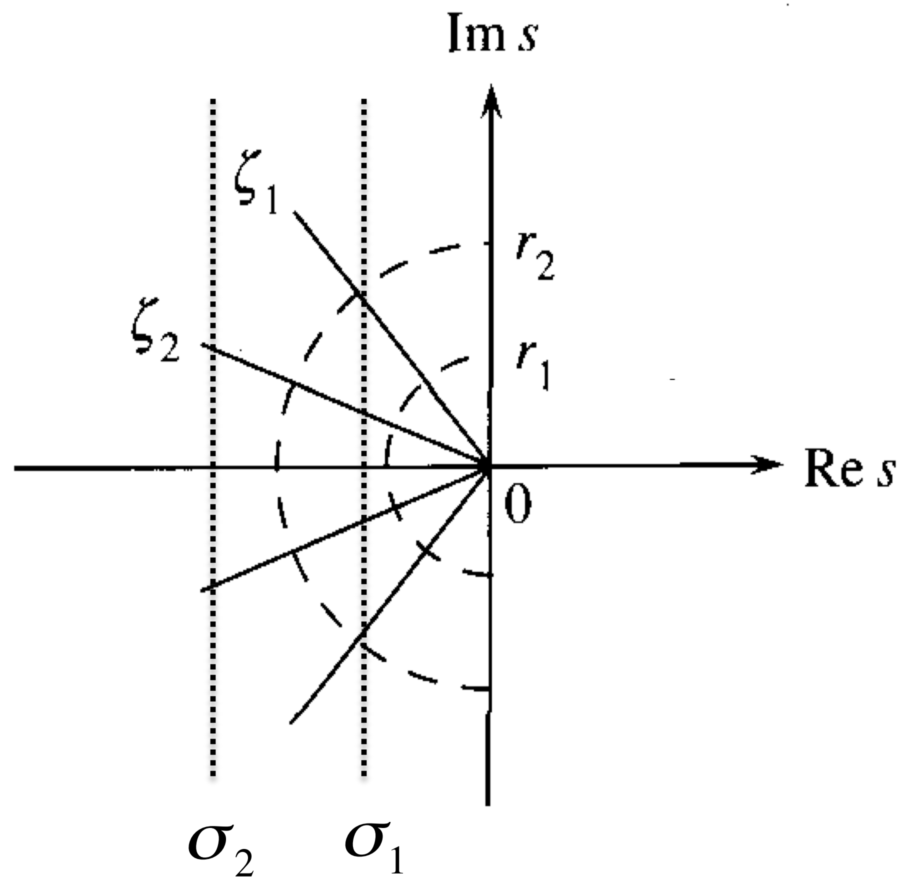
$$\zeta < \zeta_o$$

$$t_s < t_{so}$$

Root Locus Method

- Desired pole region

$$\zeta < \zeta_o$$
$$t_s < t_{so}$$
$$\omega_n < \omega_o$$



Root Locus method

- Ejemplo: $G(s) = \dfrac{1}{s(s+2)}$

- Position error $e_p = 0$

- Overshoot $Mp \leq 5\%$

- Settling time $t_s \leq 9$ s

- Rise time as small as possible

Root Locus method

- Ejemplo: $$G(s) = \frac{1}{(s+1)}$$

- Position error $e_p = 0$

- Overshoot $Mp \leq 5\%$

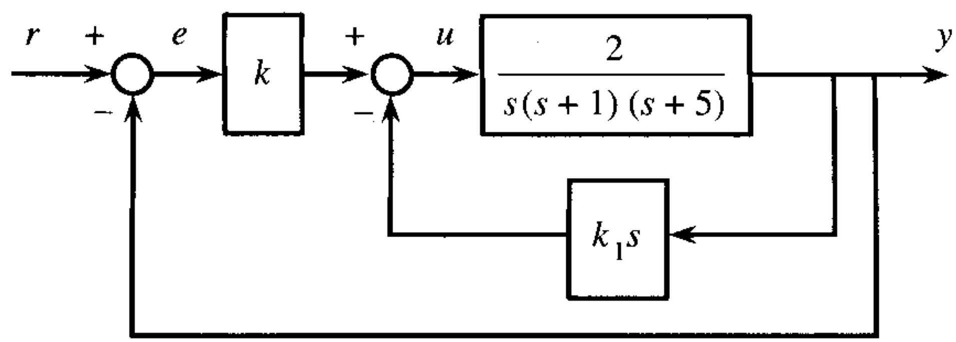
- Settling time $t_s \leq 2$ s

- Rise time as small as possible

Root Locus method

- Ejemplo: $G(s) = \dfrac{2}{s(s+1)(s+5)}$

- Position error $e_p=0$



- Overshoot $Mp \leq 5\%$

- Settling time $t_s \leq 5$ s

- Rise time as small as possible