Leicester University

Fundamentals of Sliding Mode Control

Fundamentals of Sliding Mode Control

Chris Edwards

Control Systems Research
University of Leicester

An overview:

1) Historical Development and Motivation
2) Basic Design Steps
3) Robustness and Performance Issues
4) Design Case Study - automotive actuator

Consider the double integrator

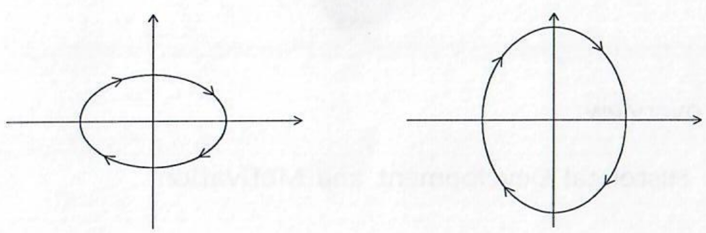
$$\ddot{y}(t) = u(t) \tag{1}$$

and the effect of using the feedback control law

$$u(t) = -ky(t) \tag{2}$$

where k is a strictly positive scalar.

The phase plot of $\dot{y}$ against y is an ellipse



The 'shape' of the ellipse depends on the size of k
$k < 1$ (left) and $k > 1$ (right)

N.B. bounded for all time; but not asymptotically stable

Consider instead the control law

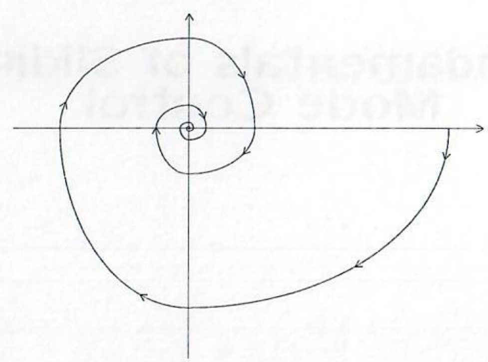
$$u(t) = \begin{cases} -k_2 y(t) & \text{if } y\dot{y} > 0 \\ -k_1 y(t) & \text{otherwise} \end{cases} \tag{3}$$

where $k_2 > 1 > k_1 > 0$.

Note structure varies according to some rule

The phase plane $(y, \dot{y})$ is partitioned by the switching rule into four quadrants separated by the axes.

The resulting phase portrait is



Splicing together the elliptical phase portraits gives one which spirals towards the origin i.e. asymptotically stable.

A more significant example results from

$$u(t) = \begin{cases} -1 & \text{if } s(y,\dot{y}) > 0 \\ 1 & \text{if } s(y,\dot{y}) < 0 \end{cases} \tag{4}$$

where the switching function is defined by

$$s(y,\dot{y}) = my + \dot{y} \tag{5}$$

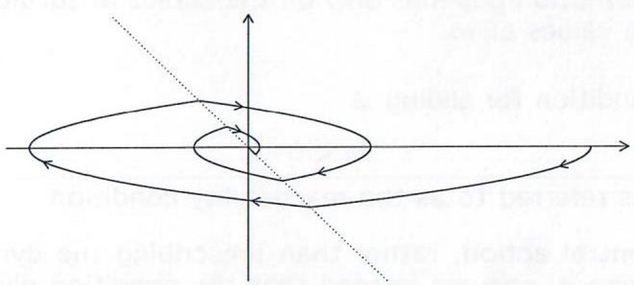
and m is a positive design scalar.

The expression in (4) is usually written as

$$u(t) = -\text{sgn}\,(s(t)) \tag{6}$$

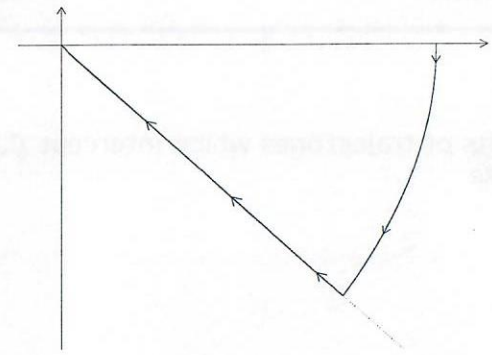
where sgn is the signum or the sign function.

The closed loop phase portrait, obtained from joining the parabolic components of the constituent laws, is



The dotted line in the figure represents the set of points for which $s(y,\dot{y}) = 0$; a straight line through the origin of gradient $-m$.

Look in detail at the area near the origin:



where $|\dot{y}| < \frac{1}{m}$. Consider

$$s\dot{s} = s\,(m\dot{y} + \ddot{y}) = s\,(m\dot{y} - \text{sgn}(s)) < |s|\,(m|\dot{y}| - 1) < 0$$

since

$$\text{sgn}(s)\,s = |s| \quad \text{and} \quad sm\dot{y} \le m|s||\dot{y}|$$

In other words the trajectories on either side of the line

$$\mathcal{L}_s = \{(y,\dot{y}) \ : \ s(y,\dot{y}) = 0\} \tag{7}$$

point towards the line.

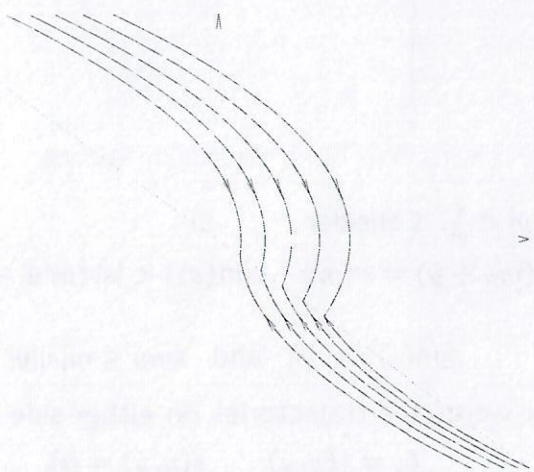
Intuitively high frequency switching between the different control structures will take place as the system trajectories repeatedly cross the line $\mathcal{L}_s$.

If infinite frequency switching were possible the motion would constrained to remain on the line $\mathcal{L}_s$.

Phase Portraits

Leicester University

Phase portraits of trajectories which intercept $\mathcal{L}_s$ near the origin look like



Properties of Sliding Modes

Leicester University

When confined to the line $\mathcal{L}_s$ i.e. $s(y, \dot{y}) = 0$

$$\dot{y}(t) = -my(t) \tag{8}$$

This represents a first order decay and the trajectories will 'slide' along the line $\mathcal{L}_s$ to the origin.

Such dynamical behaviour is described as an ideal sliding mode and the line $\mathcal{L}_s$ is termed the sliding surface.

Key points

During the sliding motion the system behaves as

- a reduced order motion which (apparently) does not depend on the control;
- the motion depends only on the choice of surface i.e. the values of m.

The condition for sliding is

$$\dot{s}s < 0 \tag{9}$$

which is referred to as the reachability condition

The control action, rather than prescribing the dynamic performance, ensures instead that the condition given in (9) is satisfied.

It should be noted that the control action required to bring about such a motion is discontinuous, and on the sliding surface is not even defined.

Discussion

Important issues to address:

- How to design sliding surfaces for general systems ?
- Examples of discontinuous control laws which ensure the reachability condition is satisfied?

(These will be tackled in due course.)

Questions:

What are the advantages of this approach ?

1) the sliding motion is of lower order than the original system (the double integrator became a first order system)

2) sliding mode systems exhibit insensitivity properties to certain kinds of parameter variations ...

The 'Equivalent Control'

Consider the double integrator $\ddot{y}(t) = u(t)$ and the control law

$$u(t) = -\text{sgn}\, s(t)$$

where

$$s(t) = \dot{y} + my$$

for some positive design constant m.

If $s(t) = 0$ for all $t > t_s$ this implies that $\dot{s}(t) = 0$ for all $t > t_s$ and therefore

$$\dot{s} = \frac{d}{dt}(my + \dot{y}) = m\dot{y}(t) + u(t) = 0 \qquad (10)$$

a control law which maintains the motion on $\mathcal{L}_s$ is

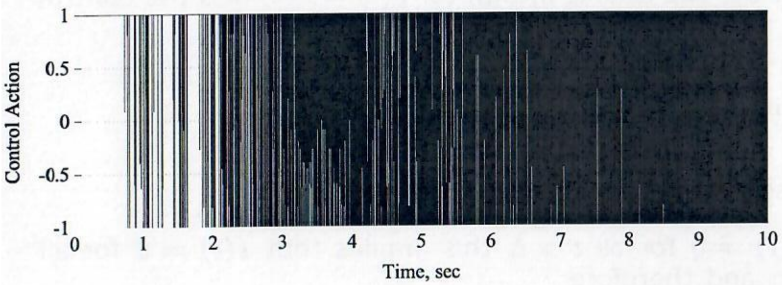
$$u_{eq}(t) = -m\dot{y}(t), \qquad t \geq t_s \qquad (11)$$

This is referred to as the equivalent control action.

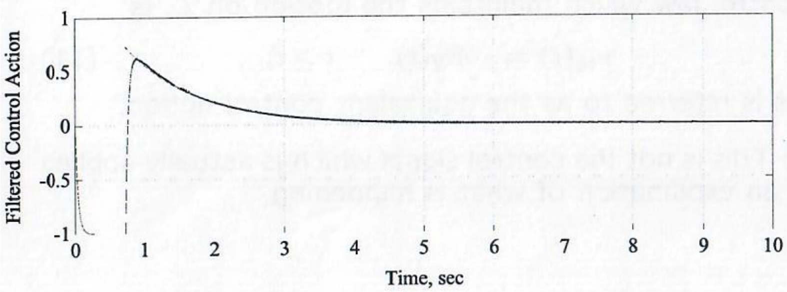
NB: This is not the control signal which is actually applied but an explanation of what is happening.

'Equivalent Control' (ctd)

The actual switched signal for the double integrator looks like



A filtered version together with u_{eq} is



An Uncertain System

Consider the following uncertain system

$$\ddot{y}(t) = bu(t) \tag{12}$$

where b is not known precisely but belongs to a known interval $0 < b_1 < b < b_2$

As before consider the switching function

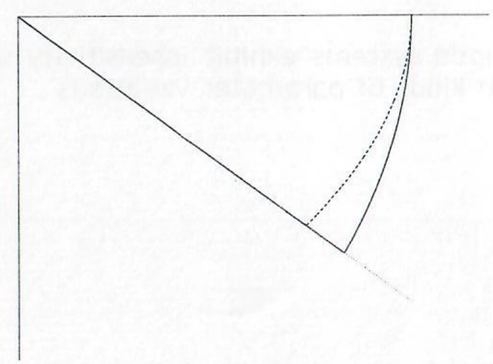
$$s = my + \dot{y} \tag{13}$$

Suppose we can find a control law to guarantee a sliding motion on $\mathcal{L}_s$ i.e. $s = 0$ then once again the reduced order sliding motion is

$$\dot{y}(t) = -my(t) \tag{14}$$

NB This is independent of the parameter b

This robustness property makes sliding mode control a powerful technique for controlling uncertain systems



Guaranteeing Sliding

Leicester University

Recall the reachability condition

$$s\dot{s} < 0$$

needs to be satisfied.

Earlier $\dot{s} = m\dot{y} + \ddot{y} = m\dot{y} + bu$ and so the requirement is

$$s(m\dot{y} + bu) < 0$$

The control law must be designed to ensure that this holds for all values of the uncertain parameter b.

Consider a control law of the form

$$u(t) = -k\dot{y}(t)$$

then we require that

$$s\dot{y}(m - bk) \leq 0$$

Two cases to consider:

$$s\dot{y} > 0 \Rightarrow \text{ we need } (m - bk) < 0 \Rightarrow k > \tfrac{m}{b}$$
$$s\dot{y} < 0 \Rightarrow \text{ we need } (m - bk) > 0 \Rightarrow k < \tfrac{m}{b}$$

Thus an appropriate variable structure control law is

$$u(t) = \begin{cases} -\frac{m}{b_1}\dot{y}(t) & \text{if } s\dot{y} > 0 \\ -\frac{m}{b_2}\dot{y}(t) & \text{if } s\dot{y} < 0 \end{cases} \qquad (15)$$

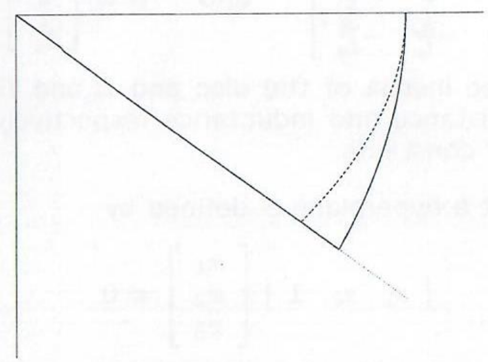
NB other discontinuous strategies could be used - some include linear terms or more complex scaled relays (see case study)

A Design Strategy

Leicester University

A design strategy maybe summarized as:

- design the sliding surface (in this case the gradient m) to provide an appropriate sliding motion to satisfy any specifications (in this just the rate of first order decay)
- design a control law (which will usually depend on the choice of sliding surface) to ensure the reachability condition is satisfied



The two stage nature of the dynamics can be observed: the initial (parabolic) motion towards the sliding surface, followed by a motion along the line $\dot{y} = -y$ towards the origin.

Hyperplane Design

Leicester University

Consider the problem of controlling the angular position of the shaft in a DC motor.

The nominal system equations can be written in state-space form as

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (16)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{K_t}{J_o} \\ 0 & -\frac{K_e}{L_o} & -\frac{R}{L_o} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_o} \end{bmatrix} \qquad (17)$$

Here J_0 is the inertia of the disc and R and L_0 are the armature resistance and inductance respectively, K_e and K_t are motor constants.

Aim to select a hyperplane $\mathcal{S}$ defined by

$$\begin{bmatrix} s_1 & s_2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

to ensure the reduced order sliding motion confined to $\mathcal{S}$ is stable and meets any design specifications.

This is equivalently to

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hyperplane Design (ctd)

Leicester University

Because of the special form of the state-space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_t}{J_o} \end{bmatrix} x_3 \qquad (18)$$

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (19)$$

i.e. a second order system in which x_3 has the role of the full state-feedback matrix

Choosing the design parameters specifying the switching function as

$$s_1 = \frac{J_o}{K_t}\omega_n^2$$

and

$$s_2 = 2\frac{J_o}{K_t}\zeta\omega_n$$

During the sliding motion

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_n^2 & 2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (20)$$

the characteristic equation of the sliding motion is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \qquad (21)$$

Therefore the parameters ζ and ω_n represent the damping ratio and natural frequency respectively.

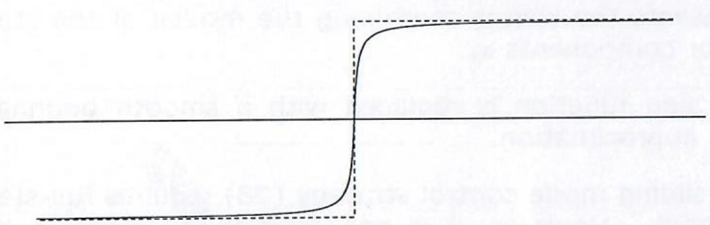
Pseudo-Sliding Modes

One problem is that in many situations a discontinuous control law is unacceptable. (In some electric motors the control law is naturally discontinuous and sliding mode control provides extremely good performance.)

In practise it is not possible to obtain a sliding motion – delays, hysteresis and digital implementation results in chattering whereby the phase portrait repeatedly crosses the sliding surface never remaining on it.

One solution is to smooth the discontinuity by using a sigmoidal function

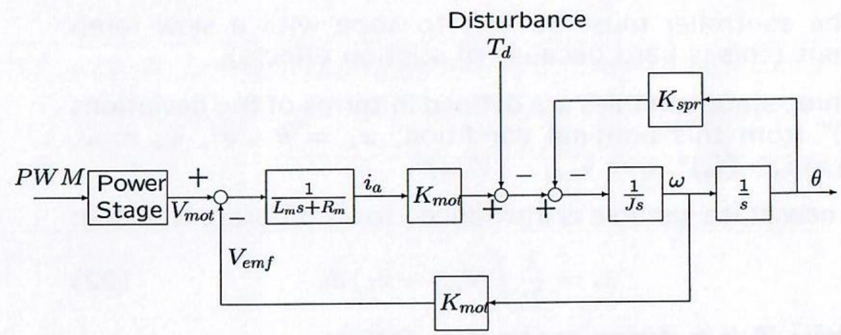
$$u(t) = \frac{s(t)}{|s(t)| + \delta}, \qquad \delta > 0$$



An approximate pseudo-sliding motion is obtained.

Automotive Actuator with Stiction

The actuation system is severely nonlinear and contains significant uncertainty. A critical component is the large stiction torque. If tight performance specifications are required, robust performance across the operation range can be difficult to achieve.



The actuator comprises a DC motor which directly drives a spring-loaded inertia.

The model is developed from the linear representation of a DC motor: R_m is winding resistance; L_m the inductance; K_{mot} the motor gain. These are assumed constant (which is a considerable over-simplification).

The disturbance $T_d(s)$ incorporates stiction effects and the offset torque of the spring.

Actuator Example (ctd)

Leicester University

The behaviour is determined by the following states:

- θ angular position of the rotor (rad)
- ω angular velocity of the rotor (rad/sec)
- i_a current flowing in the motor coil (A)

The design specification requires both large and small step responses (settling time < 0.1 sec, $< 0.1^o$ s.s.error).

The controller must be able to cope with a slow ramp input (this is hard because of stiction effects).

Three state variables are defined in terms of the deviations $(.)^d$ from this nominal condition; $x_1 = \theta - \theta^d$, $x_2 = \omega$, $x_3 = i_a - (i_a)^d$, $u = V_{mot} - (V_{mot})^d$.

A new state variable is introduced to incorporate tracking:

$$x_r = \frac{1}{T_i}\int (\theta_{ref} - x_1)\, dt \tag{22}$$

where T_i is a design parameter. Define

$$\tilde{x} = \begin{bmatrix} x_r \\ x \end{bmatrix} \tag{23}$$

then a state-space description is

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \tilde{B}_w\theta_{ref} \tag{24}$$

with system matrices

$$\tilde{A} = \begin{bmatrix} 0 & -\frac{1}{T_i}e_1 \\ 0 & A \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \quad \tilde{B}_w = \begin{bmatrix} \frac{1}{T_i} \\ 0 \end{bmatrix} \tag{25}$$

where $e_1 = [1 \;\; 0 \;\; 0]$ and (A, B) is the state-space model of the DC motor given earlier.

Actuator Example (ctd)

Leicester University

Consider a switching function of the form

$$s(t) = S\tilde{x}(t) \tag{26}$$

where

$$S = [\; m_1 \;\; m_2 \;\; m_3 \;\; 1 \;] \tag{27}$$

Here a design was performed using robust pole placement to pick $m_1, \ldots m_3$.

Many forms of control are possible here

$$u = -(S\tilde{B})^{-1}S\tilde{B}_w\theta_{ref} + k^T|x|\,\mathrm{sgn}(s) \tag{28}$$

with

$$k_i < -|((S\tilde{B})^{-1}S\tilde{A})_i| \tag{29}$$

where $(.)_i$ indicates the ith element of a vector and $|x|$ represents the vector containing the moduli of the state vector components x_i.

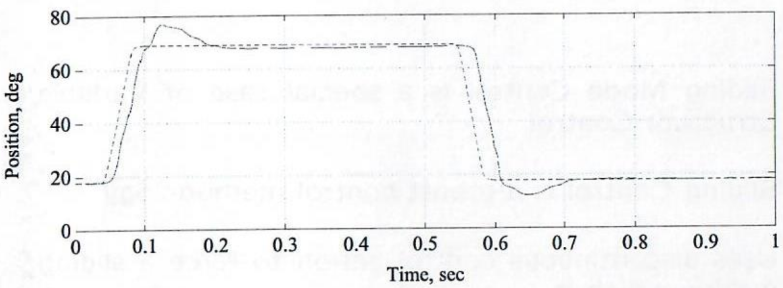
The sign function is replaced with a smooth boundary layer approximation.

The sliding mode control strategy (28) requires full-state feedback. However, it is not practical to measure the angular velocity. Straightforward numerical differentiation of position produced adequate results.

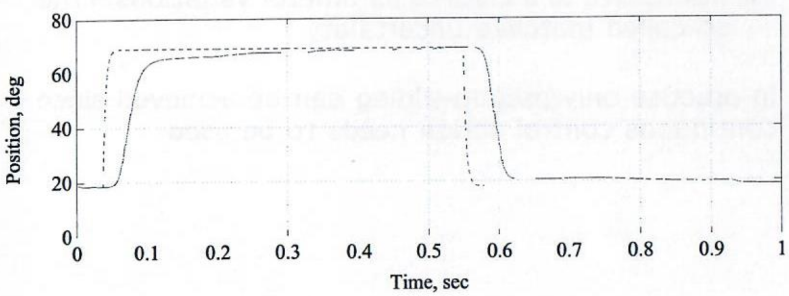
Rig Trial Results

Leicester University

Implementation was carried out on a microprocessor using integer arithmetic.



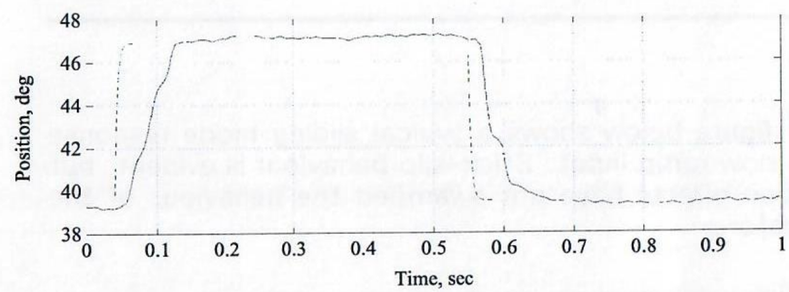
Response of the sliding mode controller to a large step



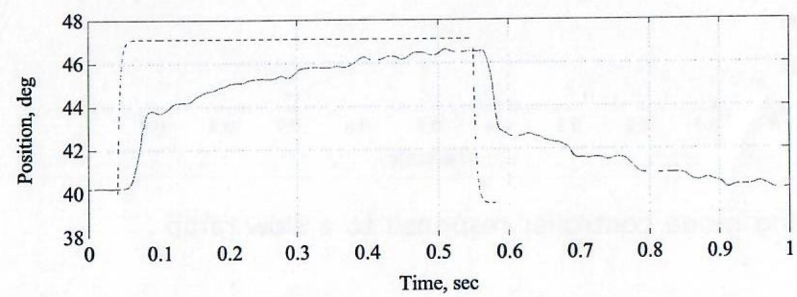
Response of the linear controller to a large step demand

Rig Trial Results (ctd)

Leicester University



Response of the sliding mode controller to a small step

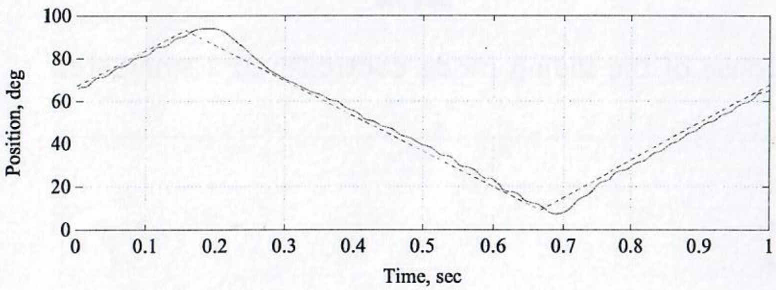


Response of the linear controller to a small step

Rig Trial Results

Leicester University

The figure below shows a typical sliding mode response to a slow ramp input. Stick–slip behaviour is evident, but stiction effects have not swamped the behaviour of the actuator.



Sliding mode controller response to a slow ramp

Concluding Remarks

Leicester University

1) Sliding Mode Control is a special case of Variable Structure Control

2) Sliding Control is a robust control methodology

3) Uses discontinuous control action to force a sliding motion which is

    - of lower order than the original system
    - insensitive to a class of parameter variations – the so-called matched uncertainty

4) In practise only pseudo-sliding can be achieved since continuous control action needs to be used.