State space design

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$

Controllability matrix:

$$\mathbf{U} = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \mathbf{A}^2\mathbf{b} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{b}]$$

Observability matrix:

$$\mathbf{V} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{n-1} \end{bmatrix}$$

State space design

Given:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

Controllable cannonical form

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [b_1 \quad b_2 \quad b_3 \quad b_4]\mathbf{x}$$

always controlable

It is observable if N(s) and D(s) have no common factors

$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$ is the characteristic polynomial

State space design

Given:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

Obervable cannonical form

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ -a_4 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}$$

always observable

It is controllable if N(s) and D(s) have no common factors

$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$ is the characteristic polynomial

State space design

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$

Let $\mathbf{P}$ be an arbitrary nonsingular matrix.

Define $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$. $\quad \longrightarrow \quad \mathbf{x} = \mathbf{P}^{-1}\bar{\mathbf{x}} \qquad \dot{\mathbf{x}} = \mathbf{P}^{-1}\dot{\bar{\mathbf{x}}}$

which yields

$$\mathbf{P}^{-1}\dot{\bar{\mathbf{x}}} = \mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{x}} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{P}^{-1}\bar{\mathbf{x}} + du$$

which become

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} + \bar{d}u$$

with

$$\bar{\mathbf{A}} := \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \qquad \bar{\mathbf{b}} := \mathbf{P}\mathbf{b} \qquad \bar{\mathbf{c}} := \mathbf{c}\mathbf{P}^{-1} \qquad \bar{d} := d$$

State space design

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$

Let $\mathbf{P}$ be an arbitrary nonsingular matrix.

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which yields

$$\mathbf{P}^{-1}\dot{\bar{\mathbf{x}}} = \mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{x}} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{P}^{-1}\bar{\mathbf{x}} + du$$

P is the equivalence transformation

which become

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} + \bar{d}u$$

The transformation:

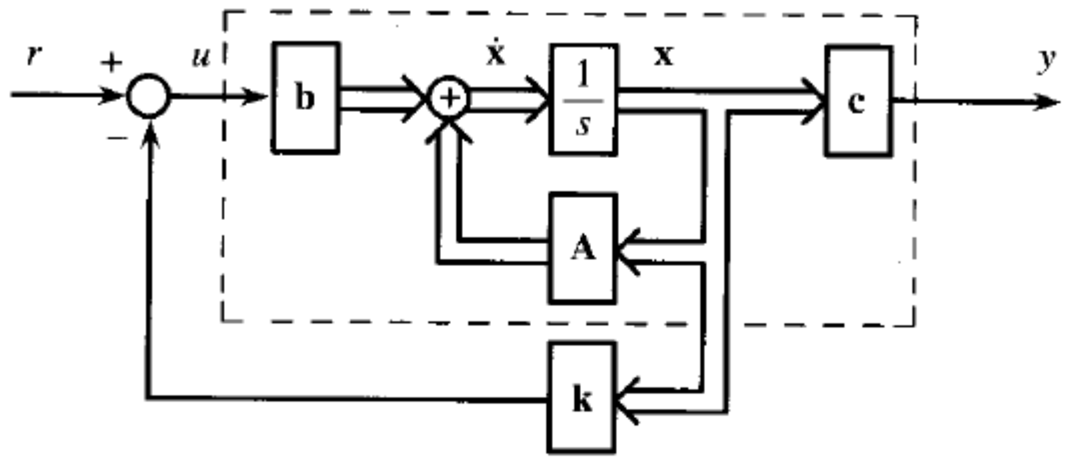
$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$$

is the similarity transformation

with

$$\bar{\mathbf{A}} := \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \qquad \bar{\mathbf{b}} := \mathbf{P}\mathbf{b} \qquad \bar{\mathbf{c}} := \mathbf{c}\mathbf{P}^{-1} \qquad \bar{d} := d$$

State feedback – pole placement



Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

Defining the following control law:

$$u(t) = r(t) - \mathbf{k}\mathbf{x}(t) \qquad \text{where } \mathbf{k} = [k_1 \quad k_2 \quad \cdots \quad k_n]$$

State feedback – pole placement

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

Defining the following control law:

$$u(t) = r(t) - \mathbf{k}\mathbf{x}(t) \qquad \text{where } \mathbf{k} = [k_1 \quad k_2 \quad \cdots \quad k_n]$$

We obtain:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{k}\mathbf{x} + \mathbf{b}r = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r$$

$$y = \mathbf{c}\mathbf{x}$$

wich has the following characteristic equation:

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{k})$$

If the system is controllable we can relocate all the system poles

State feedback – pole placement

Closed -loop characteristic equation:

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{bk})$$

Desired characteristic equation: $\alpha(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n$

$\boxed{\text{Problem:}}$

Find $\mathbf{k} = [k_1 \quad k_2 \quad \cdots \quad k_n]$ such that

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{bk}) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n$$

$\boxed{\text{It is easier from the controllable cannonical form}}$

State feedback – pole placement

From the controllable cannonical form:

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = [b_1 \quad b_2 \quad b_3 \quad b_4]\bar{\mathbf{x}}$$

Using the state feedback: $u = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - [\bar{k}_1 \quad \bar{k}_2 \quad \bar{k}_3 \quad \bar{k}_4]\bar{\mathbf{x}}$

We obtain:

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}) + \bar{\mathbf{b}}r$$

$$= \begin{bmatrix} -a_1 - \bar{k}_1 & -a_2 - \bar{k}_2 & -a_3 - \bar{k}_3 & -a_4 - \bar{k}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}r$$

$$y = [b_1 \quad b_2 \quad b_3 \quad b_4]\bar{\mathbf{x}}$$

State feedback – pole placement

The system:

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}) + \bar{\mathbf{b}}r$$

$$= \begin{bmatrix} -a_1 - \bar{k}_1 & -a_2 - \bar{k}_2 & -a_3 - \bar{k}_3 & -a_4 - \bar{k}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = [b_1 \quad b_2 \quad b_3 \quad b_4]\bar{\mathbf{x}}$$

has the characteristic polynomial:

$$\det(s\mathbf{I} - \bar{\mathbf{A}} + \bar{\mathbf{b}}\bar{\mathbf{k}}) = s^n + (a_1 + k_1)s^{n-1} + (a_2 + k_2)s^{n-2} + \cdots + (a_{n-1} + k_{n-1})s + (a_n + k_n)$$

State feedback – pole placement

Using the desired characteristic equation: $\alpha(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$

Equaling

$$s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n = s^n + (a_1 + k_1) s^{n-1} + (a_2 + k_2) s^{n-2} + \cdots + (a_{n-1} + k_{n-1}) s + (a_n + k_n)$$

Thus, we obtain

$$\bar{\mathbf{k}} = \begin{bmatrix} \alpha_1 - a_1 & \alpha_2 - a_2 & \cdots & \alpha_{n-1} - a_{n-1} & \alpha_n - a_n \end{bmatrix}$$

State feedback – pole placement

Procedure:

equivalence transformation

$$\mathbf{S} := \mathbf{P}^{-1}$$

Original space:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$

System characteristic equation:

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A})$$

$$\Delta(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

Desired characteristic equation:

$$\alpha(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n$$

Controllable cannonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} + \bar{d}u$$

Calculate $\bar{\mathbf{k}}$

$$\bar{\mathbf{k}} = [\alpha_1 - a_1 \quad \alpha_2 - a_2 \quad \cdots \quad \alpha_{n-1} - a_{n-1} \quad \alpha_n - a_n]$$

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \bar{\mathbf{k}}\mathbf{S}^{-1}$$

$$\mathbf{S}^{-1}$$

equivalence transformation

State feedback – pole placement

equivalence transformation to the controllable form

$$\mathbf{S} := \mathbf{P}^{-1}$$

Original system controllability matrix: $\mathbf{U} = [\mathbf{b} \quad \mathbf{Ab} \quad \mathbf{A}^2\mathbf{b} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{b}]$

Controllable form controllability matrix:

$$\overline{\mathbf{U}} = \begin{bmatrix} 1 & -a_1 & e_2 & e_3 \\ 0 & 1 & -a_1 & e_2 \\ 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \overline{\mathbf{U}}^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

wich are related by $\boxed{\overline{\mathbf{U}} = \mathbf{PU}}$

It results:

$$\mathbf{S} := \mathbf{P}^{-1} = \mathbf{U}\overline{\mathbf{U}}^{-1} = [\mathbf{b} \quad \mathbf{Ab} \quad \mathbf{A}^2\mathbf{b} \quad \mathbf{A}^3\mathbf{b}] \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

State feedback – pole placement

equivalence transformation to the controllable form

$$\mathbf{S} := \mathbf{P}^{-1}$$

Original system controllability matrix: $\mathbf{U} = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \mathbf{A}^2\mathbf{b} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{b}]$

Controllable form controllability matrix:

$$\overline{\mathbf{U}} = \begin{bmatrix} 1 & -a_1 & e_2 & e_3 \\ 0 & 1 & -a_1 & e_2 \\ 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \overline{\mathbf{U}}^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

wich are related by $\boxed{\overline{\mathbf{U}} = \mathbf{P}\mathbf{U}}$

It results:

$$\mathbf{S} := \mathbf{P}^{-1} = \mathbf{U}\overline{\mathbf{U}}^{-1} = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \mathbf{A}^2\mathbf{b} \quad \mathbf{A}^3\mathbf{b}] \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Controllability

State feedback – pole placement

Compute the desired characteristic polynomial

$$\bar{\Delta}(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4)$$

$$\bar{\Delta}(s) = s^4 + \bar{a}_1 s^3 + \bar{a}_2 s^2 + \bar{a}_3 s + \bar{a}_4$$

Ackermann formula

$$\begin{aligned}\mathbf{k} &= [0 \quad 0 \quad 0 \quad 1][\mathbf{b} \quad \mathbf{Ab} \quad \mathbf{A}^2\mathbf{b} \quad \mathbf{A}^3\mathbf{b}]^{-1}\bar{\Delta}(\mathbf{A}) \\ &= [0 \quad 0 \quad 0 \quad 1][\mathbf{b} \quad \mathbf{Ab} \quad \mathbf{A}^2\mathbf{b} \quad \mathbf{A}^3\mathbf{b}]^{-1}[\mathbf{A}^4 + \bar{a}_1\mathbf{A}^3 + \bar{a}_2\mathbf{A}^2 + \bar{a}_3\mathbf{A} + \bar{a}_4\mathbf{I}]\end{aligned}$$

Quadratic optimal regulator

Consider

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$ controllable and observable

regulator problem $u = -\mathbf{k}\mathbf{x}$ $r = 0$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}$$

QUADRATIC OPTIMAL REGULATOR

find $\mathbf{k}$ to minimize the quadratic performance index

$$J = \int_0^\infty [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + Ru^2(t)]dt$$

$\mathbf{Q}$ is a symmetric positive semidefinite matrix and R is a positive constant.

Quadratic optimal regulator

THEOREM

A symmetric matrix $\mathbf{Q}$ of order n is positive definite (positive semidefinite) if and only if any one of the following conditions holds:

1. All n eigenvalues of $\mathbf{Q}$ are positive (zero or positive).
2. It is possible to decompose $\mathbf{Q}$ as $\mathbf{Q} = \mathbf{N}'\mathbf{N}$, where $\mathbf{N}$ is a nonsingular square matrix (where $\mathbf{N}$ is an $m \times n$ matrix with $0 < m < n$).
3. All the *leading* principal minors of $\mathbf{Q}$ are positive (all the principal minors of $\mathbf{Q}$ are zero or positive). ■

If $\mathbf{Q}$ is symmetric and of order 3, or

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{21} & q_{22} & q_{32} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

then the *leading* principal minors are

$$q_{11} \qquad \det \begin{bmatrix} q_{11} & q_{21} \\ q_{21} & q_{22} \end{bmatrix} \qquad \det \mathbf{Q}$$

The principal minors of $\mathbf{Q}$ are

$$q_{11} \qquad q_{22} \qquad q_{33} \qquad \det \begin{bmatrix} q_{11} & q_{21} \\ q_{21} & q_{22} \end{bmatrix}$$

$$\det \begin{bmatrix} q_{11} & q_{31} \\ q_{31} & q_{33} \end{bmatrix} \qquad \det \begin{bmatrix} q_{22} & q_{32} \\ q_{32} & q_{33} \end{bmatrix} \qquad \det \mathbf{Q}$$

All eigenvalues of symmetric matrices are real

Quadratic optimal regulator

If $\mathbf{Q}$ is chosen as $\mathbf{c}'\mathbf{c}$, then

$$J = \int_0^\infty [\mathbf{x}'(t)\mathbf{c}'\mathbf{c}\mathbf{x}(t) + Ru^2(t)]dt = \int_0^\infty [y^2(t) + Ru^2(t)]dt$$

with $r(t) = 0$ and $R = 1/q$. $\longrightarrow$ $J = \int_0^\infty [q(y(t) - r(t))^2 + u^2(t)]\,dt$

Quadratic Optimal Systems

$\mathbf{k}$ to minimize this quadratic performance index is given by

$$\mathbf{k} = R^{-1}\mathbf{b}'\mathbf{K}$$

where $\mathbf{K}$ is the symmetric and positive definite matrix meeting

$$-\mathbf{KA} - \mathbf{A}'\mathbf{K} + \mathbf{K}\mathbf{b}R^{-1}\mathbf{b}'\mathbf{K} - \mathbf{c}'\mathbf{c} = \mathbf{0}$$

This is called the *algebraic Riccati equation.*

only one solution is symmetric and positive definite.

Quadratic optimal regulator

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1]\mathbf{x}$$

Find the feedback gain to minimize the performance index

$$J = \int_0^\infty \left[y^2(t) + \frac{1}{9} u^2(t) \right] dt \qquad \mathbf{Q} = \mathbf{c}'\mathbf{c} \text{ and } R = 1/9.$$

Let $\mathbf{K} = \begin{bmatrix} k_{11} & k_{21} \\ k_{21} & k_{22} \end{bmatrix}$

the *algebraic Riccati equation*

$$- \begin{bmatrix} k_{11} & k_{21} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{21} \\ k_{21} & k_{22} \end{bmatrix}$$

$$+ 9 \begin{bmatrix} k_{11} & k_{21} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] \begin{bmatrix} k_{11} & k_{21} \\ k_{21} & k_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Quadratic optimal regulator

Equating the corresponding entries yields

$$4k_{11} - 2k_{21} + 9k_{11}^2 = 0$$

$$2k_{21} - k_{22} + 9k_{11}k_{21} = 0$$

and

$$9k_{21}^2 - 1 = 0$$

Therefore, we have

$$\mathbf{K} = \begin{bmatrix} 0.129 & 0.333 \\ 0.333 & 1.05 \end{bmatrix}$$

the feedback gain is given by

$$\mathbf{k} = R^{-1}\mathbf{b}'\mathbf{K} = 9[1 \quad 0]\begin{bmatrix} 0.129 & 0.333 \\ 0.333 & 1.05 \end{bmatrix} = [1.2 \quad 3]$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{bk})\mathbf{x} = \left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1.2 \quad 3]\right)\mathbf{x}$$

$$= \begin{bmatrix} -3.2 & -3 \\ 1 & 0 \end{bmatrix}\mathbf{x}$$

The characteristic polynomial

$$\det\begin{bmatrix} s + 3.2 & 3 \\ -1 & s \end{bmatrix} = s^2 + 3.2s + 3$$

$D_o(s)$ obtained by spectral factorization equals the characteristic polynomial of $(\mathbf{A} - \mathbf{bk})$.

State estimator

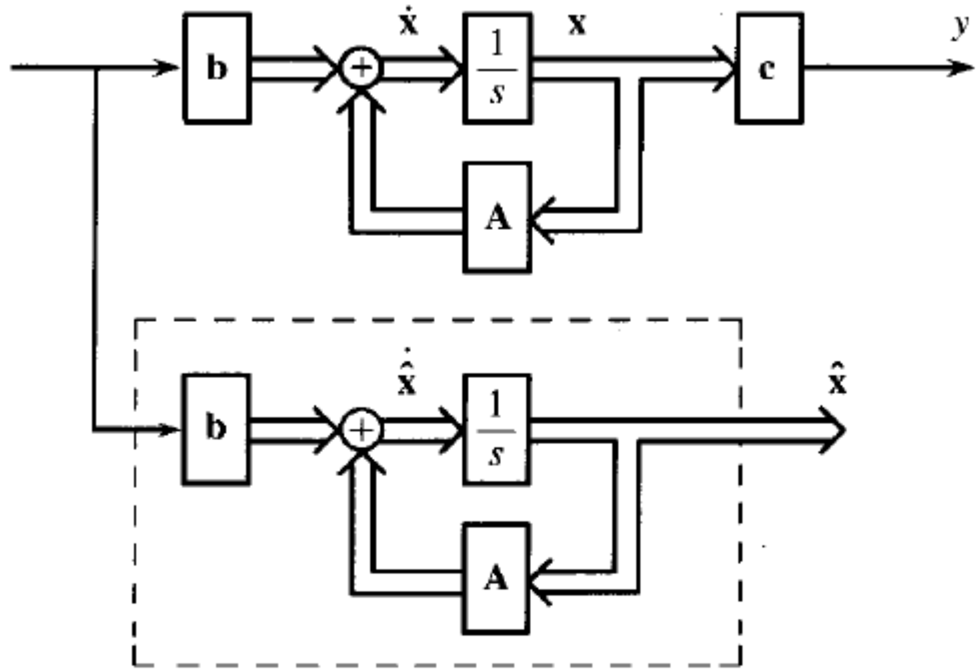
Consider

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

open-loop state estimator

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u$$



Open-loop state estimator:

Estimation error:

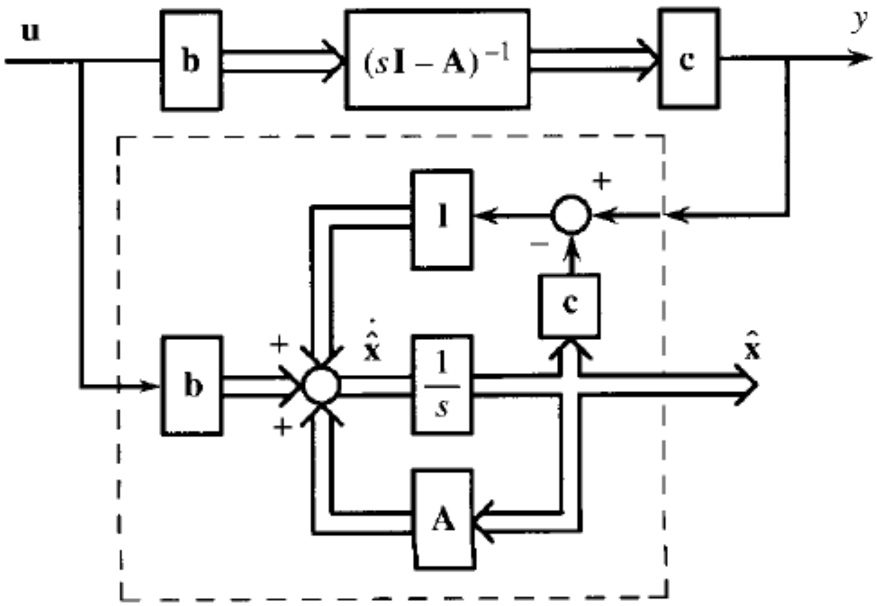
$$\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

We obtain

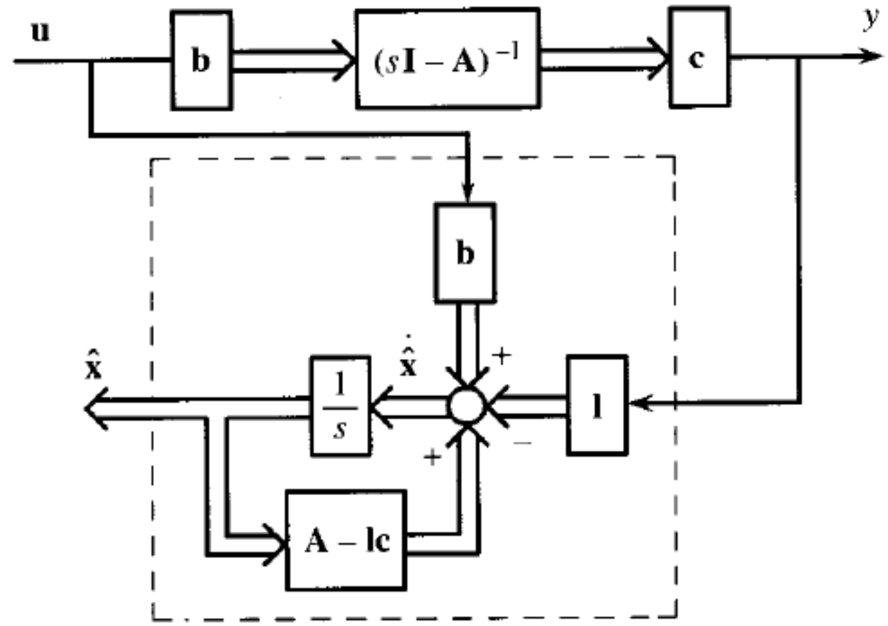
$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e}$$

State estimator



$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}(y - \mathbf{c}\hat{\mathbf{x}})$$

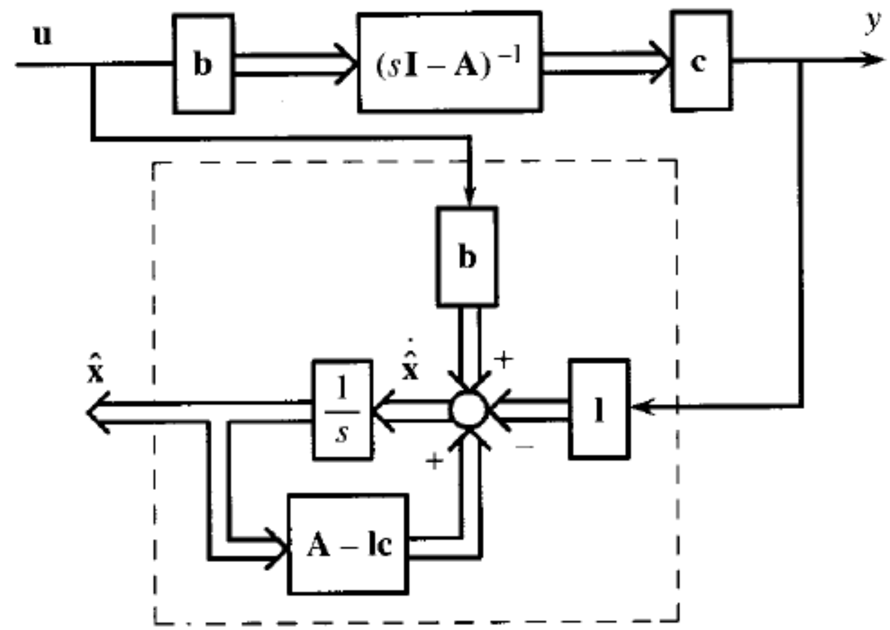


$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$

$$\mathbf{l}' = \begin{bmatrix} l_1 & l_2 & \mathrm{L} & l_{n-1} & l_n \end{bmatrix}$$

If the system is observable the estimator can be designed to estimate the system state as quickly as desired

State estimator



The plant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

The estimator

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$

Estimation error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\begin{aligned}\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u - (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} - \mathbf{b}u - \mathbf{l}\mathbf{c}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} - (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} = (\mathbf{A} - \mathbf{l}\mathbf{c})(\mathbf{x} - \hat{\mathbf{x}})\end{aligned}$$

We obtain

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\dot{\mathbf{e}}$$

State estimator

Estimation error characteristic polynomial:

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{lc})$$

Desired characteristic equation:

$$(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) = s^4 + \bar{a}_1 s^3 + \bar{a}_2 s^2 + \bar{a}_3 s + \bar{a}_4$$

Problem:

Find $\mathbf{l}' = \begin{bmatrix} l_1 & l_2 & \mathrm{L} & l_{n-1} & l_n \end{bmatrix}$ such that

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{lc}) = s^n + \bar{a}_1 s^{n-1} + \bar{a}_2 s^{n-2} + \mathrm{L} + \bar{a}_{n-1} s + \bar{a}_n$$

It is easier from the observable cannonical form

State estimator

Procedure:

equivalence transformation

$$\mathbf{S} := \mathbf{P}^{-1}$$

Original space:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$

Observable cannonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} + \bar{d}u$$

System characteristic equation:

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A})$$

$$\Delta(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

Desired characteristic equation:

$$(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4)$$

$$s^4 + \bar{a}_1 s^3 + \bar{a}_2 s^2 + \bar{a}_3 s + \bar{a}_4$$

Calculate

$$\bar{\mathbf{l}}' = [\bar{a}_1 - a_1 \quad \bar{a}_2 - a_2 \quad \bar{a}_3 - a_3 \quad \bar{a}_4 - a_4]$$

$$\mathbf{l} = \mathbf{P}\bar{\mathbf{l}} = \mathbf{S}^{-1}\bar{\mathbf{l}}$$

$$\mathbf{S}^{-1}$$

equivalence transformation

State estimator

equivalence transformation

$$\mathbf{S} := \mathbf{P}^{-1}$$

$$\mathbf{S} := \mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_1 & 1 & 0 \\ a_3 & a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix}$$

$$\Downarrow$$

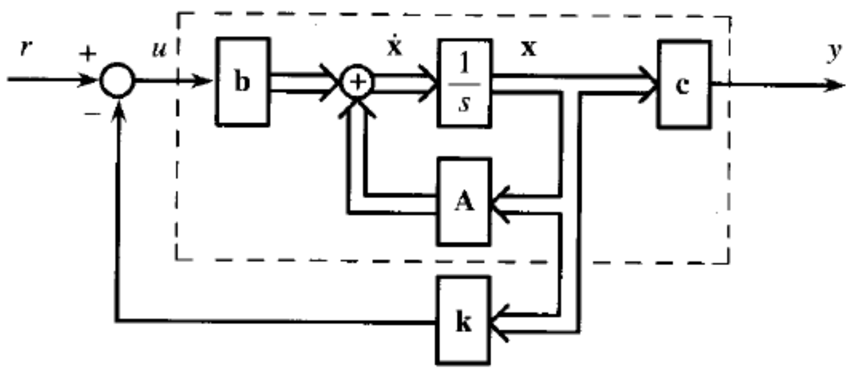
$$\mathbf{l} = \mathbf{P}\bar{\mathbf{l}} = \mathbf{S}^{-1}\bar{\mathbf{l}}$$

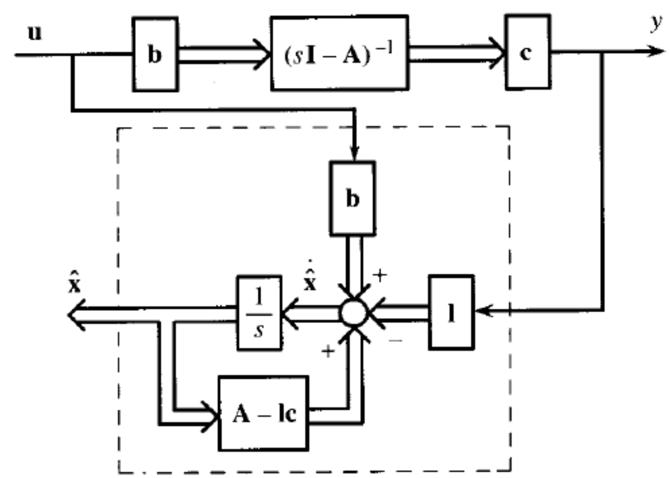
Connection of state feedback and state estimator

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$





The control law:

$$u(t) = r(t) - \mathbf{k}\mathbf{x}(t)$$

where $\mathbf{k} = [k_1 \quad k_2 \quad \cdots \quad k_n]$

The estimator

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$

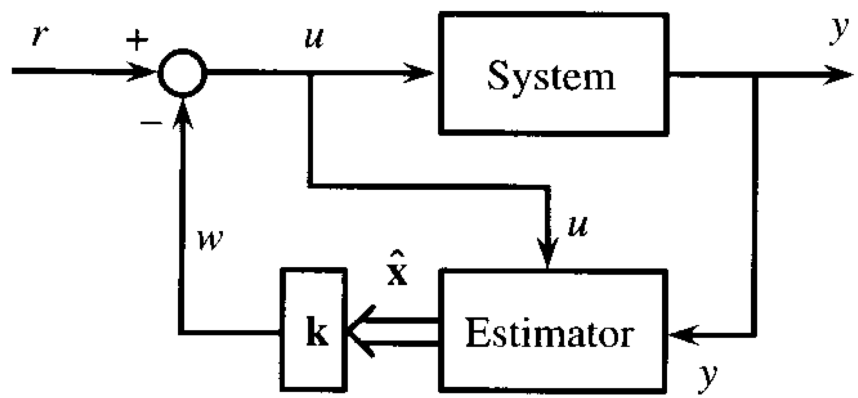
where $\mathbf{l}' = \begin{bmatrix} l_1 & l_2 & \cdots & l_{n-1} & l_n \end{bmatrix}$

Connection of state feedback and state estimator

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$



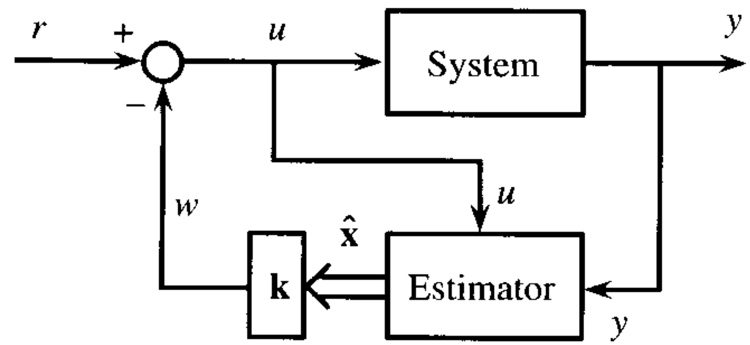
The control law: $u = r - \mathbf{k}\hat{\mathbf{x}}$

Connection of state feedback and state estimator

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x} + du$$



The substitution of the control law $u = r - \mathbf{k}\hat{\mathbf{x}}$ in the system plant and estimator equations yields

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}(r - \mathbf{k}\hat{\mathbf{x}})$$

and

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{l}\mathbf{c}\mathbf{x} + \mathbf{b}(r - \mathbf{k}\hat{\mathbf{x}})$$

They can be combined as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}\mathbf{k} \\ \mathbf{l}\mathbf{c} & \mathbf{A} - \mathbf{l}\mathbf{c} - \mathbf{b}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r$$

$$y = \mathbf{c}\mathbf{x} = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Connection of state feedback and state estimator

Consider

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} = \mathbf{P}^{-1}$$

By applying the equivalence transformation

$$\begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

we obtain

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\hat{\mathbf{x}}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix}$$

any equivalence transformation will not change the characteristic polynomial

Connection of state feedback and state estimator

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\hat{\mathbf{x}}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix}$$

using

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} = \det \mathbf{A} \det \mathbf{B}$$

where $\mathbf{A}$ and $\mathbf{B}$ have the same order

we obtain

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{bk} & -\mathbf{bk} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{lc} \end{bmatrix}$$
$$= \det (s\mathbf{I} - \mathbf{A} + \mathbf{bk}) \det (s\mathbf{I} - \mathbf{A} + \mathbf{lc})$$

separation property

Connection of state feedback and state estimator

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\hat{\mathbf{x}}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix}$$

The transfer function of the overall system is

$$[\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{bk} & -\mathbf{bk} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{lc} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

using

$$\begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \alpha \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \quad \text{where } \alpha = -\mathbf{A}^{-1}\mathbf{D}\mathbf{B}^{-1}$$

Connection of state feedback and state estimator

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\hat{\mathbf{x}}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix}$$

we obtain

$$[\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} (s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1} & \alpha \\ \mathbf{0} & (s\mathbf{I} - \mathbf{A} + \mathbf{lc})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

where $\alpha = (s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1} \mathbf{bk} (s\mathbf{I} - \mathbf{A} + \mathbf{lc})^{-1}$

and

$$[\mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1} \quad \mathbf{c}\alpha] \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1}\mathbf{b}$$

Connection of state feedback and state estimator

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\hat{\mathbf{x}}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\hat{\mathbf{x}}} \end{bmatrix}$$

not minimal

uncontrollable and unobservable

we obtain

$$[\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} (s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1} & \alpha \\ \mathbf{0} & (s\mathbf{I} - \mathbf{A} + \mathbf{lc})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

where $\alpha = (s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1}\mathbf{bk}(s\mathbf{I} - \mathbf{A} + \mathbf{lc})^{-1}$

and

$$[\mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1} \quad \mathbf{c}\alpha] \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1}\mathbf{b}$$

the state estimator is hidden from the input r and output y

Connection of state feedback and state estimator

Consider a minimal state-variable equation with transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

After introducing state feedback and the state estimator

$$G_o(s) = \frac{N(s)}{D_o(s)}$$

where $D_o(s)$ has the same degree and the same leading coefficient as $D(s)$

the numerator of $G_o(s)$ is the same as that of $G(s)$

Lyapunov stability theorem

THEOREM All eigenvalues of $\mathbf{A}$ have negative real parts if for any symmetric positive definite matrix $\mathbf{N}$, the Lyapunov equation

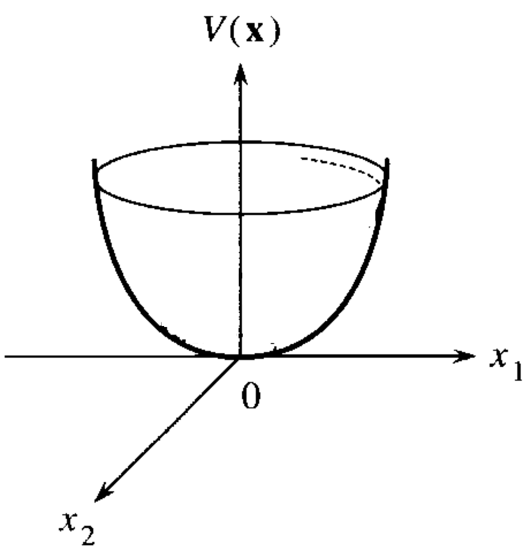
$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

has a symmetric positive definite solution $\mathbf{M}$. ■

Consider

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

We define $V(\mathbf{x}) := \mathbf{x}'\mathbf{M}\mathbf{x}$ a *Lyapunov function.*



If $\mathbf{M}$ is symmetric positive definite

↓

$V(\mathbf{x})$ is positive for any nonzero $\mathbf{x}$

$V(\mathbf{x})$ is zero only at $\mathbf{x} = \mathbf{0}$

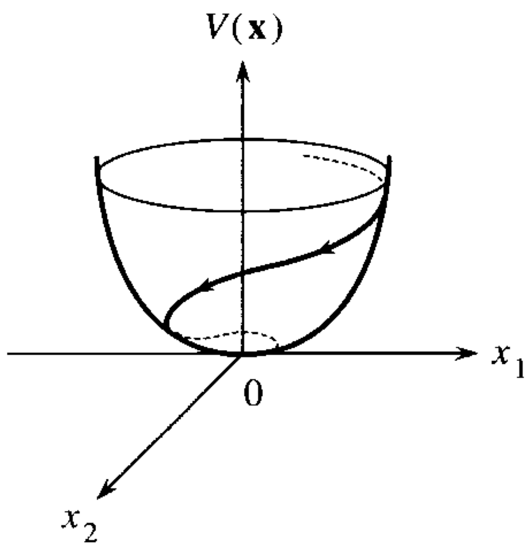
Lyapunov stability theorem

THEOREM All eigenvalues of $\mathbf{A}$ have negative real parts if for any symmetric positive definite matrix $\mathbf{N}$, the Lyapunov equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

has a symmetric positive definite solution $\mathbf{M}$. ■

we compute
$$\frac{d}{dt}V(\mathbf{x}(t)) = \frac{d}{dt}(\mathbf{x}'\mathbf{M}\mathbf{x}) = \dot{\mathbf{x}}'\mathbf{M}\mathbf{x} + \mathbf{x}'\mathbf{M}\dot{\mathbf{x}}$$
$$= \mathbf{x}'\mathbf{A}'\mathbf{M}\mathbf{x} + \mathbf{x}'\mathbf{M}\mathbf{A}\mathbf{x} = \mathbf{x}'(\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A})\mathbf{x}$$



using $\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$

becomes

$$\frac{d}{dt}V(\mathbf{x}(t)) = -\mathbf{x}'\mathbf{N}\mathbf{x}$$

If $\mathbf{N}$ is positive definite, then $dV(\mathbf{x})/dt$ is strictly negative for all nonzero $\mathbf{x}$