Lecture 7 (Weeks 13-14)

Introduction to Multivariable Control (SP - Chapters 3 & 4)

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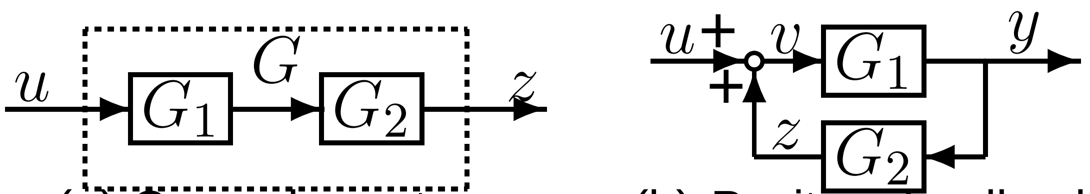
Introduction to Multivariable Control

7.1 Transfer functions for MIMO systems [3.2]



(a) Cascade system (b) Positive feedback system

Figure 1: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule.** (Figure 1(a)) $G = G_2G_1$
2. **Feedback rule.** (Figure 1(b)) $v = (I - L)^{-1}u$ where $L = G_2G_1$
3. **Push-through rule.**

$$G_1(I - G_2G_1)^{-1} \;\; = \;\; (I - G_1G_2)^{-1}G_1$$

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MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term $(I - L)^{-1}$ where L is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example:

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \tag{7.1}$$

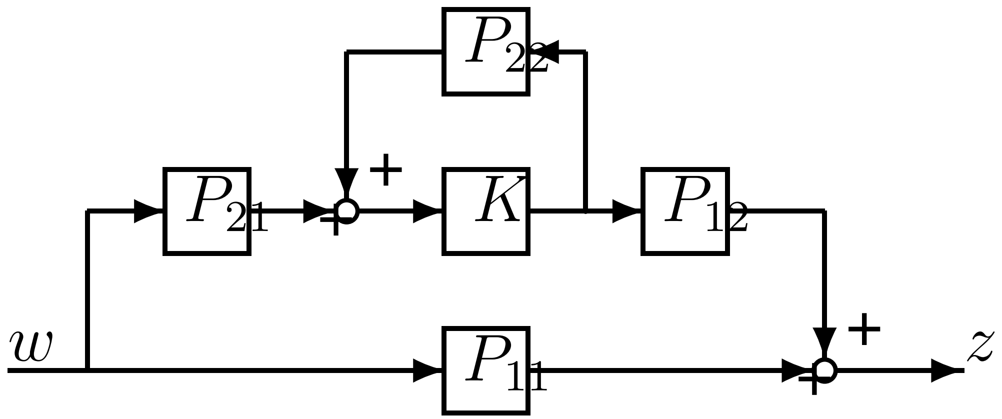

Figure 2: Block diagram corresponding to (7.1)

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Negative feedback control systems

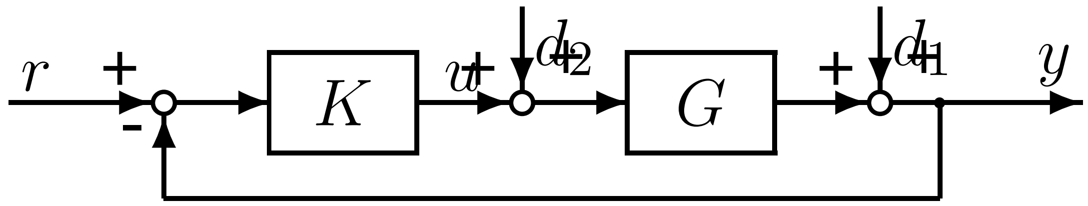

Figure 3: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at the *output* of the plant.

$$L = GK \tag{7.2}$$

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Accordingly

$$S \triangleq (I+L)^{-1}$$

(7.3) *output sensitivity*

$$T \triangleq I - S = (I+L)^{-1}L = L(I+L)^{-1}$$

(7.4) *output complementary sensitivity*

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

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- L_I is the loop transfer function at the *input* to the plant

  (7.5) $$L_I = KG$$

  Input sensitivity:

  $$S_I \triangleq (I + L_I)^{-1}$$

  Input complementary sensitivity:

  $$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

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- Some relationships:

(7.6) $$(I+L)^{-1} + (I+L)^{-1}L = S + T = I$$

(7.7) $$G(I+KG)^{-1} = (I+GK)^{-1}G$$

(7.8) $$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK$$

(7.9) $$T = L(I+L)^{-1} = (I+L^{-1})^{-1} = (I+L)^{-1}L$$

Rule to remember: "G comes first and then G and K alternate in sequence".

7.2 Multivariable frequency response [3.3]

$$
\begin{aligned}
G(s) &= \text{transfer (function) matrix} \\
G(j\omega) &= \text{complex matrix representing response} \\
&\ \text{to sinusoidal signal of frequency } \omega
\end{aligned}
$$

Note: $d \in R^m$ and $y \in R^l$

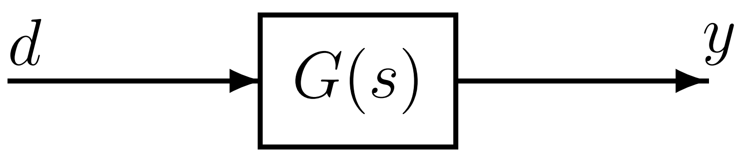

Figure 4: System $G(s)$ with input d and output y

(7.10) $$y(s) = G(s)d(s)$$

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Sinusoidal input to channel j

(7.11) $$d_j(t) = d_{j0}\sin(\omega t + \alpha_j)$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

(7.12) $$y_i(t) = y_{i0}\sin(\omega t + \beta_i)$$

Amplification (gain):

(7.13) $$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)|$$

Phase shift:

(7.14) $$\beta_i - \alpha_j = \angle g_{ij}(j\omega)$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

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Example: 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

(7.15)
$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix}$$

The output signal

(7.16)
$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix}$$

can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega)d(\omega) \\ y(\omega) &= \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \ d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \end{aligned} \tag{7.17}$$

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7.2.1 Directions in multivariable systems [3.3.2]

SISO system ($y = Gd$): gain

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.
MIMO system: input and output are vectors.
$\Rightarrow$ need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \tag{7.18}$$

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(7.19) $$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots}$$

The *gain* of the system $G(s)$ is

(7.20) $$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}}$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the *direction* of the input d.

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Example: Consider the five inputs (all $\|d\|_2 = 1$)

$$\begin{aligned} d_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\ d_4 &= \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, \; d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \end{aligned}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{7.21}$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \; y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \; y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix}, \; y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\|y_1\|_2 = 5.83, \; \|y_2\|_2 = 4.47, \; \|y_3\|_2 = 7.30, \|y_4\|_2 = 1.00, \; \|y_5\|_2 = 0.28$$

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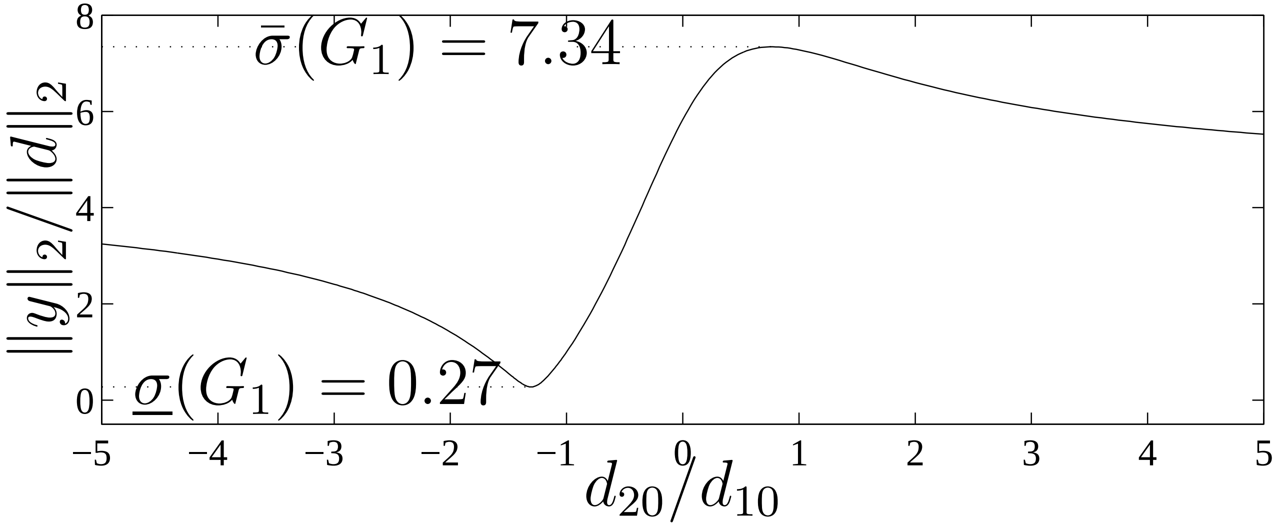

Figure 5: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (7.21)

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The maximum value of the gain in (7.20) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \tag{7.22}$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \tag{7.23}$$

7.2.2 Eigenvalues are a poor measure of gain [3.3.3]

Example:

(7.24) $$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

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For generalizations of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

(7.25) $$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\|$$

and the multiplicative property

(7.26) $$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\|$$

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm

7.2.3 Singular value decomposition [3.3.4]

Any matrix G may be decomposed into its singular value decomposition,

(7.27) $$G = U\Sigma V^H$$

- Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;
- U is an $l \times l$ unitary matrix of output singular vectors, u_i,
- V is an $m \times m$ unitary matrix of input singular vectors, v_i,

(7.28) $$\sigma_i(G) = \sqrt{\lambda_i(G^H G)} = \sqrt{\lambda_i(GG^H)}$$

(7.29) $$(GG^H)U = U\Sigma\Sigma^H, \qquad (G^H G)V = V\Sigma^H\Sigma$$

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Input and output directions. The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

(7.30) $$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1$$

(7.31) $$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

(7.32) $$G = U\Sigma V^H \Rightarrow GV = U\Sigma \quad (V^H V = I) \Rightarrow G v_i = \sigma_i u_i$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

(7.33) $$\sigma_i(G) = \|G v_i\|_2 = \frac{\|G v_i\|_2}{\|v_i\|_2}$$

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Maximum and minimum singular values. The largest gain for *any* input direction is

(7.34) $$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2}$$

The smallest gain for any input direction is

(7.35) $$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2}$$

where $k = \min\{l, m\}$. For any vector d we have

(7.36) $$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G)$$

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Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \tag{7.37}$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the “strongest”, “high-gain” or “most important” directions.

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Example:

(7.38)
$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (7.38) both inputs affect both outputs, we say that the system is *interactive*.

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The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*; $\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example: Shopping cart. Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.

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Example: **Distillation process.** Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{7.39}$$

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{7.40}$$

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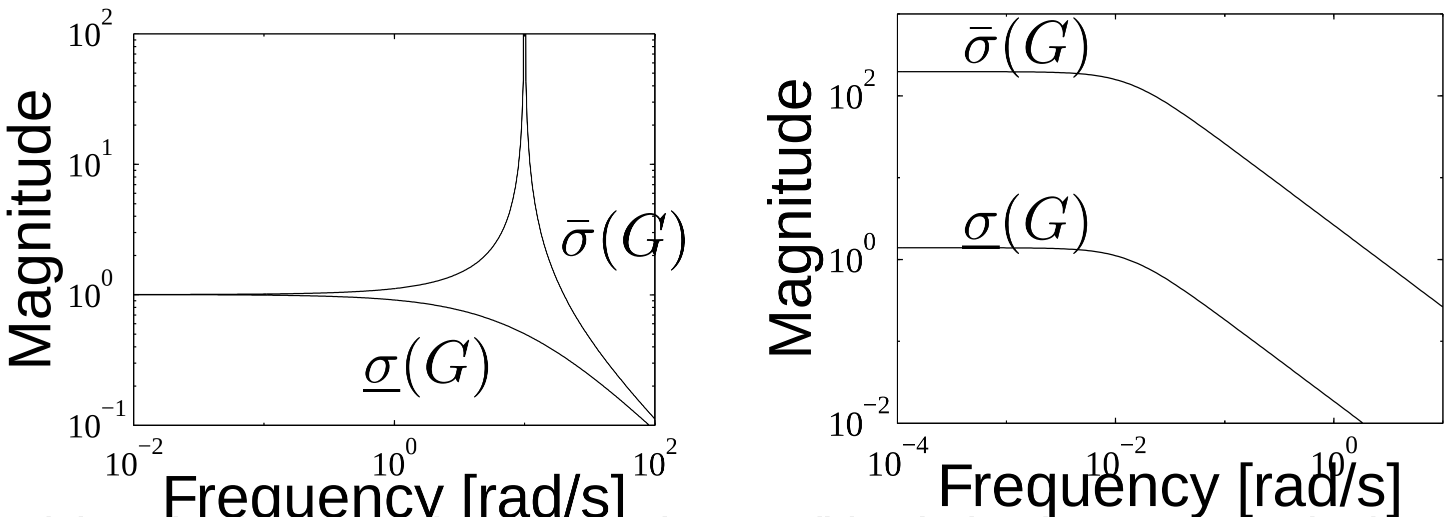
The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure 6).



(a) Spinning satellite in (7.71) (b) Distillation process in (7.76)

Figure 6: Typical plots of singular values

7.2.4 Singular values for performance [3.3.5]

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

.

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)) \tag{7.41}$$

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For *performance* we want the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned} \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \ \forall\omega &= \bar{\sigma}(w_P S) < 1, \forall\omega \\ &= \|w_P S\|_\infty < 1 \end{aligned} \tag{7.42}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \stackrel{\Delta}{=} \max_\omega \bar{\sigma}(M(j\omega)) \tag{7.43}$$

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Typical singular values of $S(j\omega)$ in Figure 7.

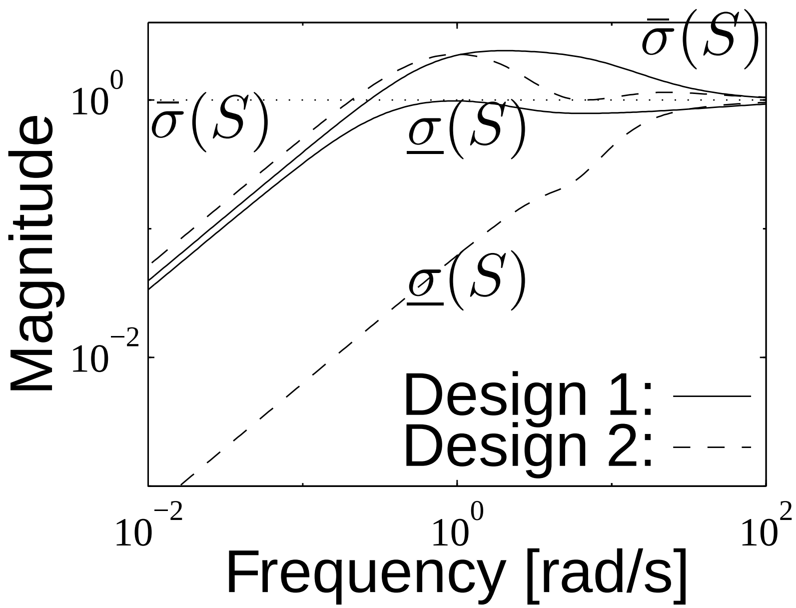

Figure 7: Singular values of S for a 2×2 plant with RHP-zero

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- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

Since $S = (I+L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \le \frac{1}{\bar{\sigma}(I+A)^{-1}} \le \underline{\sigma}(A) + 1$ yields

(7.44) $$\underline{\sigma}(L) - 1 \le \frac{1}{\bar{\sigma}(S)} \le \underline{\sigma}(L) + 1$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

7.3 Poles [4.4]

Definition

Poles. The poles p_i of a system with state-space description

(7.45) $$\dot{x} = Ax + Bu$$

(7.46) $$y = Cx + Du$$

are the eigenvalues $\lambda_i(A), i = 1, \ldots, n$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as $\phi(s) \stackrel{\Delta}{=} \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$. Thus the poles are the roots of the characteristic equation

(7.47) $$\phi(s) \stackrel{\Delta}{=} \det(sI - A) = 0$$

7.3.1 Poles and stability

Theorem 1 *A linear dynamic system $\dot{x} = Ax + Bu$ is stable if and only if all the poles are in the open left-half plane (LHP), that is, $\mathrm{Re}\{\lambda_i(A)\} < 0, \forall i$. A matrix A with such a property is said to be "stable" or Hurwitz.*

7.3.2 Poles from transfer functions

Theorem 2 *The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$, is the least common denominator of all non-identically-zero minors of all orders of $G(s)$.*

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Example:

(7.48) $$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

The minors of order 1 are the four elements all have $(s+1)(s+2)$ in the denominator.
Minor of order 2

(7.49) $$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)}$$

Least common denominator of all the minors:

(7.50) $$\phi(s) = (s+1)(s+2)$$

Minimal realization has two poles: $s = -1$; $s = -2$.

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Example: Consider the 2×3 system, with 3 inputs and 2 outputs,

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} *$$

(7.51) $$* \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}$$

Minors of order 1:

(7.52) $$\frac{1}{s+1}, \frac{s-1}{(s+1)(s+2)}, \frac{-1}{s-1}, \frac{1}{s+2}, \frac{1}{s+2}$$

Minor of order 2 corresponding to the deletion of column 2:

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$$M_2 = \frac{(s-1)(s+2)(s-1)(s+1)+(s+1)(s+2)(s-1)^2}{((s+1)(s+2)(s-1))^2} =$$

$$= \frac{2}{(s+1)(s+2)} \tag{7.53}$$

The other two minors of order two are

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}, \quad M_3 = \frac{1}{(s+1)(s+2)} \tag{7.54}$$

Least common denominator:

$$\phi(s) = (s+1)(s+2)^2(s-1) \tag{7.55}$$

The system therefore has four poles: $s=-1$, $s=1$ and two at $s=-2$. Note MIMO-poles are essentially the poles of the elements. A procedure is needed to determine multiplicity.

7.4 Zeros [4.5]

SISO system: zeros z_i are the solutions to $G(z_i) = 0$.
In general, zeros are values of s at which $G(s)$ loses rank.

Example:

$$\left[Y = \frac{s+2}{s^2+7s+12} U \right]$$

Compute the response when

$$u(t) = e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1$$

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$$\begin{aligned} \mathcal{L}\{u(t)\} &= \frac{1}{s+2} \\ s^2Y &- sy(0) - \dot{y}(0) + 7sY - 7y(0) + 12Y = 1 \\ s^2Y &+ 7sY + 12Y + 1 = 1 \\ &\Rightarrow Y(s) = 0 \end{aligned}$$

Assumption: $g(s)$ has a zero z, $g(z) = 0$.
Then for input $u(t) = u_0 e^{zt}$ the output is $y(t) \equiv 0$, $t > 0$.
(with appropriate initial conditions)

7.4.1 Zeros from state-space realizations [4.5.1]

The state-space equations of a system can be written as:

$$P(s)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}, \qquad P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \tag{7.56}$$

The zeros are then the values $s = z$ for which the polynomial system matrix $P(s)$ loses rank, resulting in zero output for some non-zero input

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0$$

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The zeros are the solutions of

$$\det \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = 0$$

MATLAB

```
zero = tzero(A,B,C,D)
```

7.4.2 Zeros from transfer functions [4.5.2]

Definition

Zeros. z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{n_z}(s - z_i)$ where n_z is the number of finite zeros of $G(s)$.

Theorem 3 *The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.*

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Example:

(7.57)
$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

The normal rank of $G(s)$ is 2.

Minor of order 2: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = 2\frac{s-4}{s+2}$.

Pole polynomial: $\phi(s) = s+2$.

Zero polynomial: $z(s) = s-4$.

Note: Multivariable zeros have no relationship with the zeros of the transfer function elements.

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Example:

(7.58) $$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

Minor of order 2 is the determinant

(7.59) $$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)}$$

$$\phi(s) = 1.25^2(s+1)(s+2)$$

Zero polynomial = numerator of (7.59)
$\Rightarrow$ no multivariable zeros.

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Example:

(7.60) $$G(s) = \begin{bmatrix} \dfrac{s-1}{s+1} & \dfrac{s-2}{s+2} \end{bmatrix}$$

The normal rank of $G(s)$ is 1
No value of s for which $G(s) = 0 \Rightarrow G(s)$ has no zeros.

7.5 More on poles and zeros [4.6]

7.5.1 Directions of poles and zeros

Let $G(s) = C(sI - A)^{-1}B + D$.

Zero directions. Let $G(s)$ have a zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there exist non-zero vectors u_z and y_z such that

$$G(z)u_z = 0, \quad y_z^H G(z) = 0 \tag{7.61}$$

$u_z =$ input zero direction
$y_z =$ output zero direction
y_z gives information about which output (or combination of outputs) may be difficult to control.

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Pole directions. Let $G(s)$ have a pole at $s = p$. Then $G(p)$ is infinite, and we may write

(7.62) $$G(p)u_p = \infty, \quad y_p^H G(p) = \infty$$

$u_p =$ input pole direction
$y_p =$ output pole direction.

SVD:

$$G(z/p) = U \Sigma V^H$$

$u_z =$ last column in V, $y_z =$ last column of U
(corresponding to the zero singular value of $G(z)$)
$u_p =$ first column in V, $y_p =$ first column of U
(corresponding to the infinite singular value of $G(p)$)

Introduction to Multivariable Control

Example:
Plant in (7.57) has a RHP-zero at $z = 4$ and a LHP-pole at $p = -2$.

$$\begin{aligned} G(z) &= G(4) = \frac{1}{6}\begin{bmatrix} 3 & 4 \\ 4.5 & 6 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0.55 & -0.83 \\ 0.83 & 0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}^H \end{aligned}$$

(7.63) $$u_z = \begin{bmatrix} -0.80 \\ 0.60 \end{bmatrix} \quad y_z = \begin{bmatrix} -0.83 \\ 0.55 \end{bmatrix}$$

For pole directions consider

(7.64) $$G(p+\epsilon) = G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -3+\epsilon & 4 \\ 4.5 & 2(-3+\epsilon) \end{bmatrix}$$

Introduction to Multivariable Control

The SVD as $\epsilon \to 0$ yields

$$G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -0.55 & -0.83 \\ 0.83 & -0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}^H$$

(7.65) $$u_p = \begin{bmatrix} 0.60 \\ -0.80 \end{bmatrix} \quad y_p = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix}$$

Note: Locations of poles and zeros are independent of input and output scalings, their directions are *not*.

Introduction to Multivariable Control

7.5.2 Remarks on poles and zeros [4.6.2]

1. For square systems the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det G(s)$.
   This fails when zero and pole in different parts of the system cancel when forming $\det G(s)$.

   $$G(s) = \begin{bmatrix} (s+2)/(s+1) & 0 \\ 0 & (s+1)/(s+2) \end{bmatrix} \tag{7.66}$$

   $\det G(s) = 1$, although the system obviously has poles at -1 and -2 and (multivariable) zeros at -1 and -2.

2. System (7.66) has poles and zeros at the same locations (at -1 and -2). Their directions are different. They do not cancel or otherwise interact.

Introduction to Multivariable Control

3. There are no zeros if the outputs contain direct information about all the states; that is, if from y we can directly obtain x (e.g. $C = I$ and $D = 0$);
4. Zeros usually appear when there are fewer inputs or outputs than states
5. **Moving poles.** (a) feedback control ($G(I + KG)^{-1}$) moves the poles, (b) series compensation (GK, feedforward control) can cancel poles in G by placing zeros in K (but not move them), and (c) parallel compensation ($G + K$) cannot affect the poles in G.

Introduction to Multivariable Control

6. **Moving zeros.** (a) With feedback, the zeros of $G(I + KG)^{-1}$ are the zeros of G plus the poles of K. , i.e. the zeros are unaffected by feedback. (b) Series compensation can counter the effect of zeros in G by placing poles in K to cancel them, but cancellations are not possible for RHP-zeros due to internal stability (see Section 7.7). (c) The only way to move zeros is by parallel compensation, $y = (G + K)u$, which, if y is a physical output, can only be accomplished by adding an extra input (actuator).

7.6 Stability [4.3]

Definition

A system is **(internally) stable** if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded output signals measured anywhere in the system. The word “internal” implies that all the states must be stable not only inputs/outputs.

Definition

State stabilizable, state detectable and hidden unstable modes. A system is state stabilizable if all unstable modes are state controllable. A system is state detectable if all unstable modes are state observable. A system with unstabilizable or undetectable modes is said to contain hidden unstable modes.

7.7 Internal stability of feedback systems [4.7]

Note: Checking the pole of S or T is not sufficient to determine internal stability

Example: (Figure 8). In forming $L = GK$ we cancel the term $(s-1)$ (a RHP pole-zero cancellation) to obtain

$$L = GK = \frac{k}{s}, \text{ and } S = (I+L)^{-1} = \frac{s}{s+k} \tag{7.67}$$

$S(s)$ is stable, i.e. transfer function from d_y to y is stable. However, the transfer function from d_y to u is unstable:

$$u = -K(I+GK)^{-1}d_y = -\frac{k(s+1)}{(s-1)(s+k)}d_y \tag{7.68}$$

Introduction to Multivariable Control

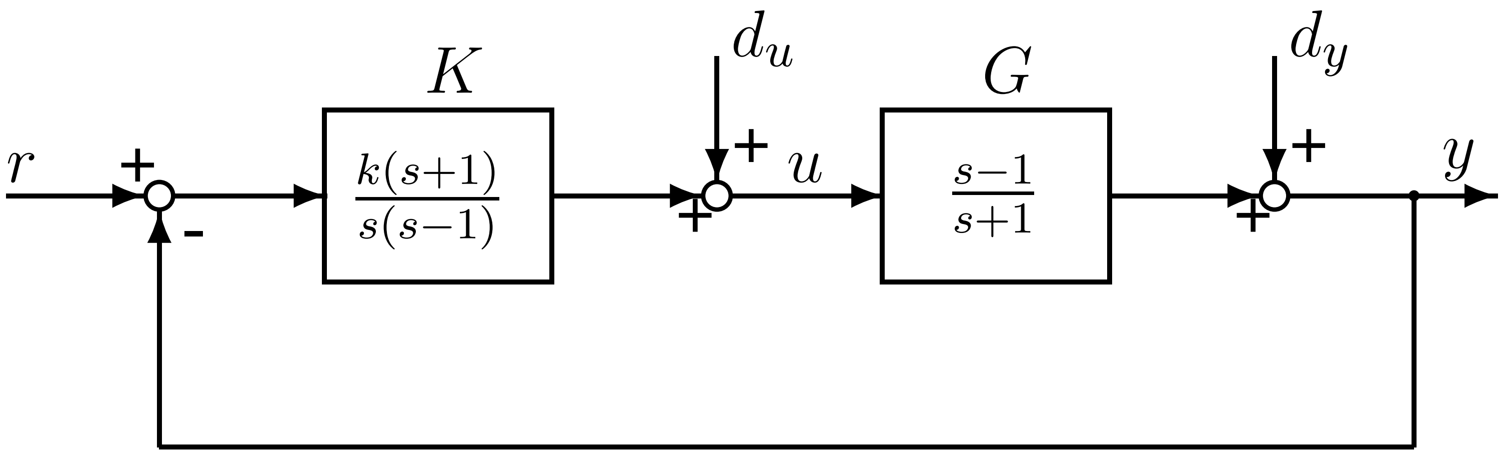


Figure 8: Internally unstable system

Introduction to Multivariable Control

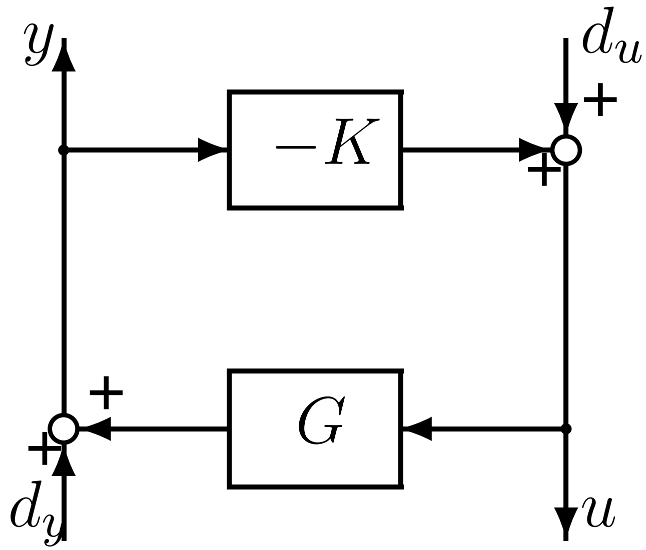


Figure 9: Block diagram used to check internal stability of feedback system

For *internal* stability consider

$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y \tag{7.69}$$

$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y \tag{7.70}$$

Introduction to Multivariable Control

Theorem 4 *The feedback system in Figure 9 is* **internally stable** *if and only if all four closed-loop transfer matrices in (7.69) and (7.70) are stable.*

Theorem 5 *Assume there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then the feedback system in Figure 9 is internally stable if and only if <u>one</u> of the four closed-loop transfer function matrices in (7.69) and (7.70) is stable.*

Introduction to Multivariable Control

Implications of the internal stability requirement

1. *If $G(s)$ has a RHP-zero at z, then $L = GK$, $T = GK(I + GK)^{-1}$, $SG = (I + GK)^{-1}G$, $L_I = KG$ and $T_I = KG(I + KG)^{-1}$ will each have a RHP-zero at z.*
2. *If $G(s)$ has a RHP-pole at p, then $L = GK$ and $L_I = KG$ also have a RHP-pole at p, while $S = (I + GK)^{-1}, KS = K(I + GK)^{-1}$ and $S_I = (I + KG)^{-1}$ have a RHP-zero at p.*

7.8 Introduction to MIMO robustness [3.7]

7.8.1 Motivating robustness example no. 1: Spinning Satellite [3.7.1]

Angular velocity control of a satellite spinning about one of its principal axes:

(7.71) $$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}; \quad a = 10$$

A minimal, state-space realization, $G = C(sI - A)^{-1}B + D$, is

Introduction to Multivariable Control

(7.72)
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right]$$

Poles at $s = \pm ja$ For stabilization:

$$K = I$$

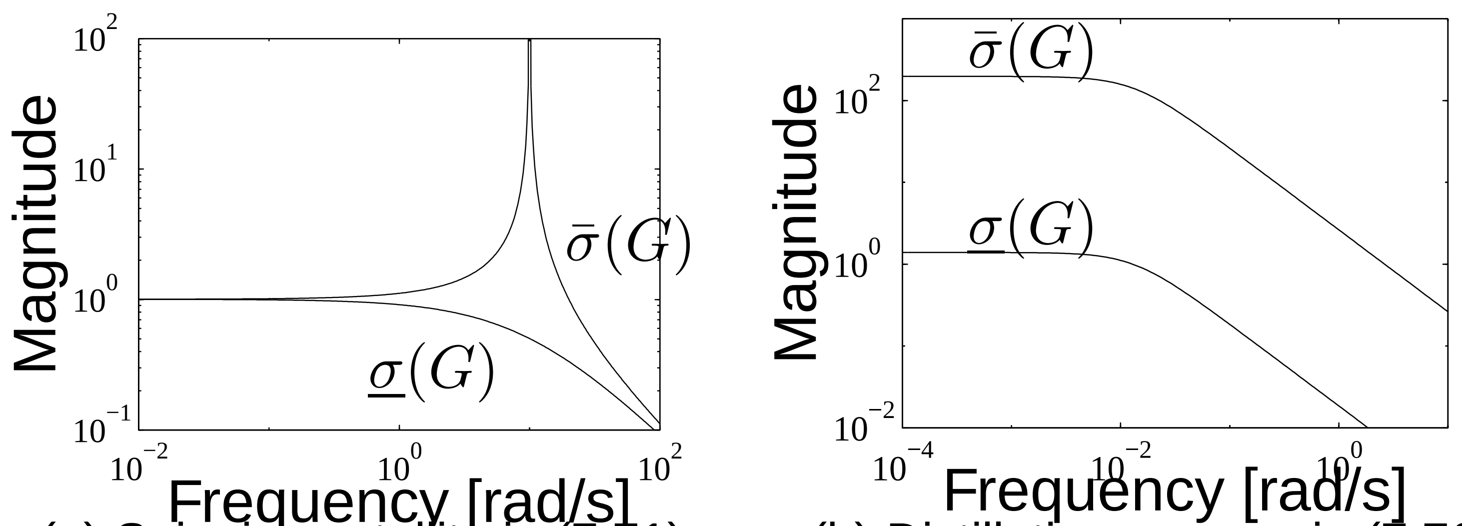
(7.73)
$$T(s) = GK(I + GK)^{-1} = \frac{1}{s+1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Introduction to Multivariable Control

Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nominal performance (NP). Figure 10(a)



(a) Spinning satellite in (7.71) (b) Distillation process in (7.76)

Figure 10: Typical plots of singular values

Introduction to Multivariable Control

- $\underline{\sigma}(L) \leq 1 \;\; \forall \omega$ poor performance in low gain direction
- g_{12}, g_{21} large $\Rightarrow$ strong interaction

Robust stability (RS).
Check stability: one loop at a time.

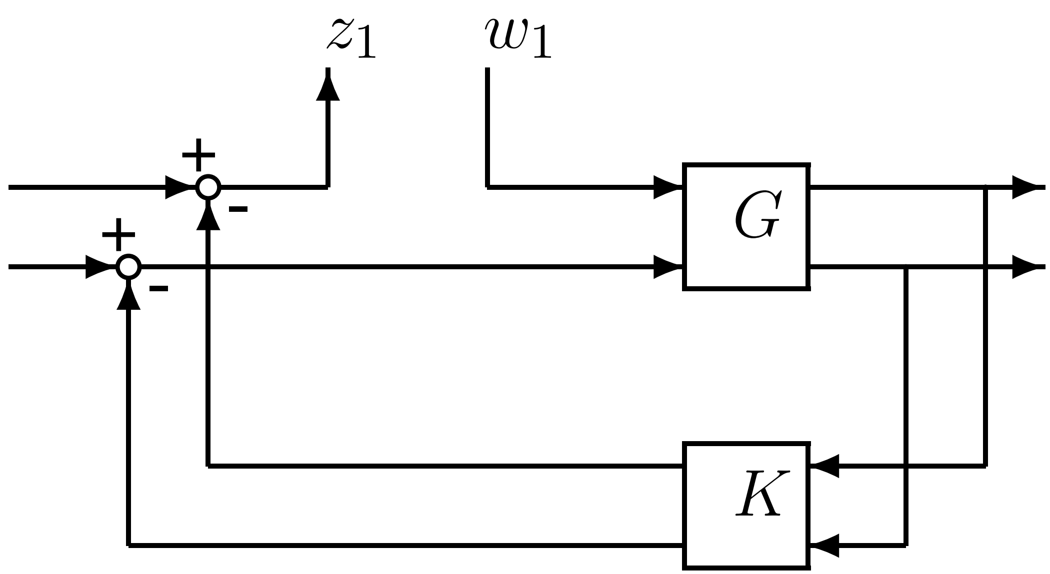


Figure 11: Checking stability margins "one-loop-at-a-time"

Introduction to Multivariable Control

(7.74) $$\frac{z_1}{w_1} \triangleq L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ$$

- Good Robustness? NO
- Consider perturbation in each feedback channel

(7.75) $$u_1' = (1+\epsilon_1)u_1, \quad u_2' = (1+\epsilon_2)u_2$$

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$$

Closed-loop state matrix:

$$A'_{cl} = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Introduction to Multivariable Control

Characteristic polynomial:

$$\det(sI - A'_{cl}) = s^2 + \underbrace{(2 + \epsilon_1 + \epsilon_2)}_{a_1} s + \\ + \underbrace{1 + \epsilon_1 + \epsilon_2 + (a^2 + 1)\epsilon_1\epsilon_2}_{a_0}$$

Stability for $(-1 < \epsilon_1 < \infty, \epsilon_2 = 0)$ and $(\epsilon_1 = 0, -1 < \epsilon_2 < \infty)$ (GM=∞)
But only *small simultaneous changes* in the two channels: for example, let $\epsilon_1 = -\epsilon_2$, then the system is unstable ($a_0 < 0$) for

$$|\epsilon_1| > \frac{1}{\sqrt{a^2 + 1}} \approx 0.1$$

Summary. Checking single-loop margins is inadequate for MIMO problems.

7.8.2 Motivating robustness example no. 2: Distillation Process [3.7.2]

Idealized dynamic model of a distillation column,

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{7.76}$$

(time is in minutes).

Introduction to Multivariable Control

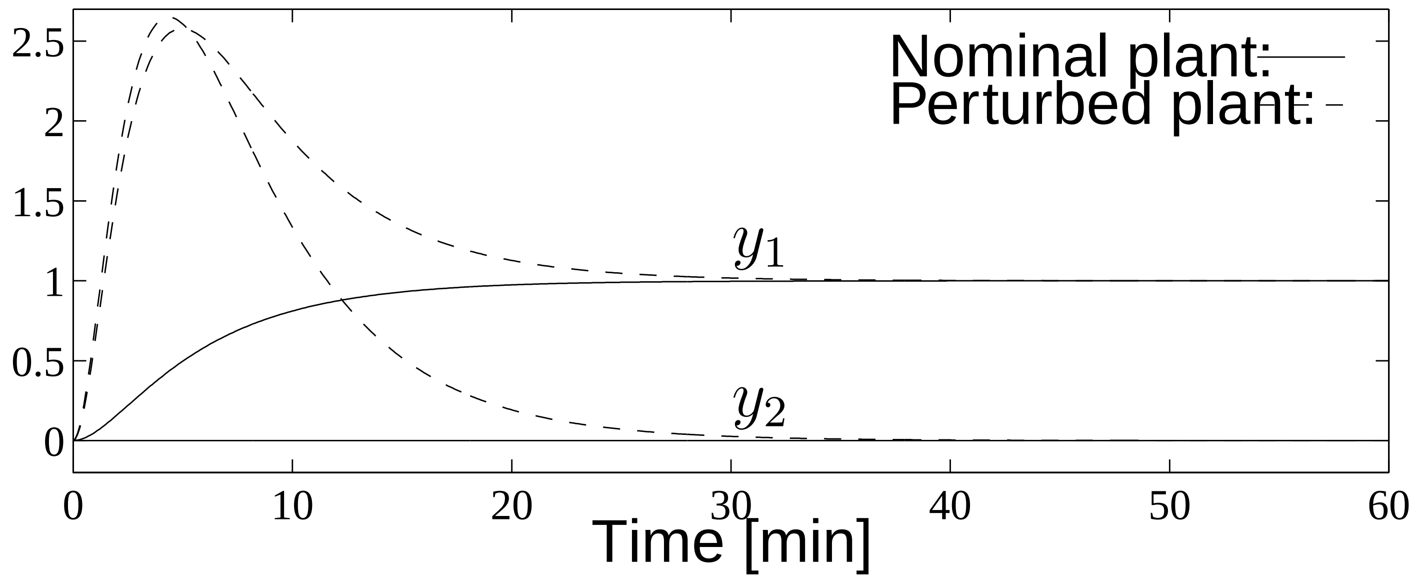


Figure 12: Response with decoupling controller to filtered reference input $r_1 = 1/(5s+1)$. The perturbed plant has 20% gain uncertainty as given by (7.79).

Introduction to Multivariable Control

Inverse-based controller or equivalently steady-state decoupler with a PI controller ($k_1 = 0.7$)

$$K_{\text{inv}}(s) = \frac{k_1}{s}G^{-1}(s) = \frac{k_1(1+75s)}{s}\begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \tag{7.77}$$

Nominal performance (NP).

$$GK_{\text{inv}} = K_{\text{inv}}G = \frac{0.7}{s}I$$

first order response with time constant 1.43 (Fig. 12). *Nominal performance (NP) achieved with decoupling controller*.

Robust stability (RS).

(7.78) $$S = S_I = \frac{s}{s+0.7} I; \quad T = T_I = \frac{1}{1.43s+1} I$$

In each channel: GM=∞, PM=90°.

Input gain uncertainty (7.75) with $\epsilon_1 = 0.2$ and $\epsilon_2 = -0.2$:

(7.79) $$u_1' = 1.2 u_1, \quad u_2' = 0.8 u_2$$

(7.80) $$\begin{aligned} L_I'(s) &= K_{\text{inv}} G' = K_{\text{inv}} G \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} = \\ & \frac{0.7}{s} \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \end{aligned}$$

Introduction to Multivariable Control

Perturbed closed-loop poles are

$$s_1 = -0.7(1+\epsilon_1), \quad s_2 = -0.7(1+\epsilon_2) \tag{7.81}$$

Closed-loop stability as long as the input gains $1+\epsilon_1$ and $1+\epsilon_2$ remain positive
$\Rightarrow$ *Robust stability (RS) achieved with respect to input gain errors for the decoupling controller*.

Robust performance (RP).
Performance with model error poor (Fig. 12)

- SISO: NP+RS $\Rightarrow$ RP
- MIMO: NP+RS $\not\Rightarrow$ RP

RP is not achieved by the decoupling controller.

7.8.3 Robustness conclusions [3.7.3]

Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.