Updated: Monday, September 29, 2008

MULTIVARIABLE ZEROS OF STATE-SPACE SYSTEMS

If a system has more than one input or output, it is called multi-input/multi-output (MIMO) or multivariable. The poles for multivariable systems are defined the same as for single-input/single/output (SISO) systems. On the other hand, for MIMO systems, there are several kinds of zeros and the discussion of zeros is complex. A complete treatment is given in terms of the Smith and McMillan forms. References are the books:

H.H. Rosenbrock, “State-Space and Multivariable Theory,” T. Nelson, London, 1070.
P.J. Antsaklis and A.N. Michel, “A Linear systems Primer,” Birkhauser, Boston, 2007.

The following discussion holds for either continuous-time or discrete-time systems.

Consider the system *(A,B,C,D)* with $x \in R^n, u \in R^m, y \in R^p$. The transfer function of the state-space system *(A,B,C,D)* is given by

$$H(s) = C\Phi(s)B + D = C(sI-A)^{-1}B + D$$
$$= \frac{C[adj(sI-A)]B}{|sI-A|} + D = \frac{C[adj(sI-A)]B + D|sI-A|}{|sI-A|} = \frac{N(s)}{\Delta(s)} = \frac{N_c(s)}{\Delta_c(s)},$$

where *adj(.)* denotes the adjoint of a matrix. $N_c(s), \Delta_c(s)$ are the numerator matrix and denominator after pole/zero cancellation. Thus, one sees that the poles, which are the roots of the denominator of *H(s)*, are given only in terms of *A*. The zeros generally depend on all four matrices *A,B,C,D.*

The system poles are the roots of the system characteristic polynomial $\Delta(s)$. The transfer function poles are the roots of the transfer function denominator AFTER cancellation, $\Delta_c(s)$.

Zeros for Single-Input/Single-Output (SISO) Case

In the SISO case when *m=1, p=1* the numerator *N(s)* is a scalar polynomial. Then, the system zeros are the roots of *N(s).* The transfer function zeros are the roots of

$N_c(s)$. The relative degree of *H(s)* is the degree of the denominator minus the degree of the numerator. The number of finite zeros is equal to the degree of the numerator. There are *n* poles and *n* zeros. Any missing zeros are at infinity. The number of infinite zeros is equal to the relative degree.

A is an *n x n* matrix, so the degree of $|sI - A|$ is *n*, while the degree of $adj(sI - A)$ is at most *n-1* . Therefore, if *D=0* then the relative degree of *H(s)* must be greater than 1. If *D* is not zero, then the transfer function has relative degree of zero. This means there is a direct feed term.

A transfer function is said to be *proper* if its relative degree is greater than or equal to zero, and *strictly proper* if the relative degree is greater than or equal to one.

Transfer Function Zeros for Multivariable (MV) Case

Let the system have $x \in R^n, u \in R^m, y \in R^p$. If the number of inputs *m* or the number of outputs *p* is greater than one, the system is said to be multi-input/multi-output (MIMO) or *multivariable*. Then the transfer function consists of a *p x m* numerator matrix $N_c(s)$ divided by a scalar denominator $\Delta_c(s)$.

If the number of inputs equals the number of outputs, *m=p*, then the system is said to be *square*. Then, the numerator after pole/zero cancellation $N_c(s)$ is a square polynomial matrix. If it has full rank, then the *numerator polynomial* is defined as the determinant of $N_c(s)$

$$p_N(s) = |N_c(s)|.$$

For square systems having a transfer function of full rank, the transmission zeros occur at the roots of the numerator polynomial

$$\textit{transmission zeros at} \quad p_N(s) = |N_c(s)| = 0$$

Note that the transmission zeros only depend on the transfer function. No state space form *(A,B,C,D)* is needed to find the transmission zeros.

If the system is not square or *H(s)* is not of full rank, one may compute the McMillan form of *H(s)* to determine the transmission zeros.

Recall that $Y(s) = H(s)U(s)$ when the initial conditions *x(0)* are zero. It is always possible to find an input *u(t)* with frequency at a transmission zero that yields an output *y(t)* that does not contain that frequency. (See the example below.) This means there is zero transmission at that frequency for correct choice of input. The transmission zeros are also called blocking zeros.

Zeros for MV Systems

In fact, there are several kinds of zeros. These are given in terms of the Rosenbrock system matrix

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$

and its block matrix components.

The Rosenbrock system matrix is important in that it captures the structure of a dynamical system and provides a unifying point of view for analysis and design. In fact, the input *U(s)*, state *X(s)*, and output *Y(s)* satisfy

$$\begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} -X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y(s) \end{bmatrix}$$

which is exactly equivalent

$$(sI - A)X(s) = BU(s)$$

$$CX(s) + DU(s) = Y(s)$$

or

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \left(C(sI - A)^{-1} B + D\right) U(s)$$

the system response for zero initial conditions.

System Poles

The *system poles* are the values of *s* for which the $n \times n$ matrix

$$P_p(s) = [sI - A]$$

loses rank. Since this is a square matrix, this occurs when

$$\Delta(s) = |sI - A|$$

is equal to zero.

The next two tests are known as the Popov-Belevitch-Hautus (PBH) rank tests).

Input-Decoupling Zeros

The *input-decoupling zeros* are those values of *s* for which the $n \times (n + m)$ input-coupling matrix

$$P_I(s) = [sI - A \quad B]$$

loses rank, i.e. has rank less than *n*. Note that this matrix can lose rank only where $(sI - A)$ loses rank, so the input-decoupling zeros must be a subset of the poles.

Output-Decoupling Zeros

The *output-decoupling zeros* are those values of s for which the $(n+p)\times n$ output-coupling matrix

$$P_O(s)=\begin{bmatrix} sI-A \\ -C \end{bmatrix}$$

loses rank, i.e. has rank less than n. Note that this matrix can lose rank only where $(sI-A)$ loses rank, so the output-decoupling zeros must be a subset of the poles.

System Zeros

The system zeros are defined by

System zeros =
transmission zeros(zeros of transfer function) + input-decoupling zeros + output-decoupling zeros – input/output decoupling zeros.

The input/output decoupling zeros are both input-decoupling zeros and output-decoupling zeros.

Thus, the system zeros consist of the transmission zeros which appear in the numerator of the transfer function (after p/z cancellation), and the input and output decoupling zeros. In fact, the poles that cancel out in computing the transfer function are exactly the decoupling zeros. One has,

System poles =
Poles of transfer function + input-decoupling zeros + output-decoupling zeros – input/output decoupling zeros.

Note that transmission zeros depend only on the transfer function, whereas the state space description (A,B,C,D) is needed to find the system zeros, input-decoupling zeros, and output-decoupling zeros.

The Zero Polynomial of *(A,B,C,D)*

To find the system zeros, one can study the Rosenbrock system matrix

$$P(s)=\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}$$

P(s) is an $(n+p)\times(n+m)$ matrix. Let *P(s)* have rank *n*+*r*. Note that $n\le n+r\le n+\min(m,p)$ since *(sI-A)* is nonsingular.

If the system is square, i.e. number of inputs m equals number of outputs p, and *P(s)* is nonsingular, then one can set the determinant of *P(s)* equal to zero to find the system zeros. Otherwise, certain minors of *P(s)* must be considered, as follows.

Define a reduced matrix, which consists of some of the rows and columns of *P(s)*, as follows. Take the *n* first rows and columns of P(s) (i.e. matrix *(sI-A)*), and select any *r* of the columns of $\begin{bmatrix} B \\ D \end{bmatrix}$ and *r* of the rows of $\begin{bmatrix} -C & D \end{bmatrix}$ such that the resulting submatrix is nonsingular. This amounts to selecting *r* of the *m* system inputs and *r* of the *p* system outputs. Find the determinant of this submatrix, which is called a minor of order *(n+r)* of *P(s)*, and denote it by $p_1(s)$. The roots of this polynomial are the zeros of the system with respect to the selected *r* inputs and *r* outputs.

Now repeat this procedure using a different set of r inputs and outputs. Define the determinant of the new submatrix as $p_2(s)$, the minor with respect to the new rows and columns selected. The roots of this polynomial are the zeros of the system with respect to the newly selected *r* inputs and *r* outputs.

Continue until all possible combinations of *r* inputs and outputs have been chosen.

The *zero polynomial* $p_z(s)$ is defined as the monic greatest common divisor of all these *(n+r)*-order minors. The system zeros are the roots of $p_z(s)$. Thus, the system zeros are those zeros that are common to any selection of r inputs and r outputs, with *n+r* the rank pf *P(s)*.

The Smith form of *P(s)* also gives information about the system zeros.

Meaning of Decoupling Zeros

The transfer function is given by

$$H(s) = C\Phi(s)B + D = C(sI - A)^{-1}B + D\,.$$

Input-Decoupling Zeros. Note that the rank of $P_I(s)$ is equal to *n* over the complex numbers *s*, since *(sI-A)* is nonsingular over the complex numbers. If $P_I(s)$ loses rank, i.e. for some value of *s* it has rank less than *n*, then there exist a frequency s_0 (the input-decoupling zero) and a nonzero *n*-vector *w* such that

$$w^T P_I(s_0) = w^T \begin{bmatrix} s_0 I - A & B \end{bmatrix} = 0$$

i.e.

$$w^T (s_0 I - A) = 0$$

$$w^T B = 0$$

Therefore, s_0 is an eigenvalue of *(sI-A)* with a left eigenvector *w* which is in the range perpendicular of B. Therefore, any input decoupling zero is also a system pole.

One can show that this equivalent to

$$w^T (s_0 I - A)^{-1} B = 0$$

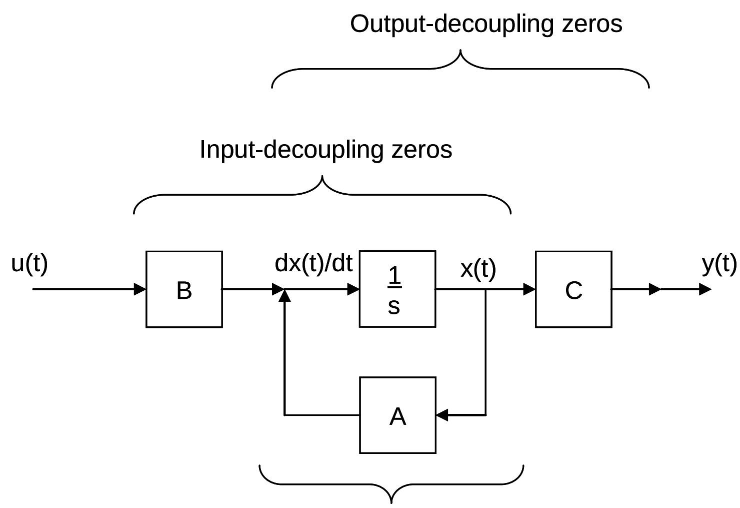
Therefore, *(A,B)* has no input decoupling zeros if and only if

$$rank[(sI - A)^{-1} B] = n$$

over the complex numbers. That is, there exists no nonzero n-vector *w* such that

$$w^T (sI - A)^{-1} B = 0 \quad \textit{for all } s$$

Note that $(sI - A)^{-1} B$ is the right-hand or input portion of the transfer function. We shall see that the input decoupling zeros mean a *loss of control effectiveness at that frequency*, and we cannot fully control the system with the given inputs. We should design systems with no input-decoupling zeros, i.e. with a fully effective set of inputs.



$(sI - A)^{-1}$

Output-Decoupling Zeros. Note that the rank of $P_O(s)$ is equal to *n* over the complex numbers, since *(sI-A)* is nonsingular over the complex numbers. If $P_O(s)$ loses rank, i.e. for some value of *s* it has rank less than *n*, then there exist s_0 (the output-decoupling zero) and an n-vector *v* such that

$$P_O(s_0)v = \begin{bmatrix} s_0 I - A \\ -C \end{bmatrix} v = 0$$

i.e.

$$(s_0 I - A)v = 0$$
$$Cv = 0$$

Therefore, s_0 is an eigenvalue of *(sI-A)* with a right eigenvector *v* which is in the nullspace of *C*. Therefore, any output decoupling zero is also a system pole.

One can show that this equivalent to

$$C(s_0 I - A)^{-1} v = 0$$

Therefore, *(A,C)* has no output decoupling zeros if and only if

$$rank[C(sI - A)^{-1}] = n$$

over the complex numbers. That is, there exists no nonzero n-vector *v* such that

$$C(sI-A)^{-1}v=0 \quad \textit{for all } s$$

Note that $C(sI-A)^{-1}$ is the left-hand or output portion of the transfer function. We shall see later that the output decoupling zeros mean a *loss of measurement effectiveness at that frequency*, and we cannot observe the full state behavior with the given outputs. We should design systems with no output-decoupling zeros.

Meaning of System Zeros

Suppose that $P(s)$ is of full rank, and there is a system zero at s_0. Then, $P(s_0)$ loses rank, and there exist a nonzero n-vector x_0 and m-vector u_0 such that

$$\begin{bmatrix} s_0 I - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} -x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \end{bmatrix} = 0$$

Therefore

$$x_0 = (s_0 I - A)^{-1} B u_0$$

$$0 = y_0 = \left(C(s_0 I - A)^{-1} B + D \right) u_0 = H(s_0) u_0$$

i.e., any input in the direction u_0 with frequency s_0 results in a zero output at that frequency.

Now, the system output is given by

$$Y(s) = C(sI-A)^{-1}x(0) + H(s)U(s)$$

Select the input as $u(t) = u_0 e^{s_0 t}$, i.e. at the frequency of s_0 and in the direction of u_0. Then one has

$$Y(s) = C(sI-A)^{-1}x(0) + H(s)\frac{u_0}{s-s_0}$$

and performing the standard PFE yields

$$Y(s) = C(sI-A)^{-1}x(0) + H(s_0)u_0 + \sum_i \frac{R_i}{s-p_i}\frac{u_0}{p_i - s_0}$$

with p_i the transfer function poles and R_i the corresponding residues.

Note that $H(s_0)u_0 = 0$ since s_0 is a system zero. Select now the initial conditions $x(0)$ so that

$$0 = C(sI-A)^{-1}x(0) + \sum_i \frac{R_i}{s-p_i}\frac{u_0}{p_i - s_0}$$

If this is possible, then the output is equal to zero. Note that for SISO systems, this IC selection is always possible.

Thus, *the transmission can be blocked at the system zeros* with appropriate selection of the input direction and the initial conditions.

Example 1- Transmission Zeros

For classical i/o systems described by transfer functions, the initial condition is normally assumed equal to zero. Recall that for linear systems, the output *y(t)* generally contains frequencies at the poles of the system plus the frequencies of the input *u(t).* Transmission zeros are frequencies at which one may find a suitable input direction such that the input frequency does not appear in the system output. i.e. the frequency of the transmission zero is blocked.

Consider the transfer function

$$H(s) = \frac{1}{s^2+4}\begin{bmatrix} 2 & s \\ -2(s-2) & 1 \end{bmatrix} = \frac{N(s)}{d(s)}$$

where the denominator $d(s) = s^2 + 4$ determines the poles, which are at $s = \pm 2j$. The natural modes are $\sin 2t, \cos 2t$.

The numerator *N(s)* is a 2×2 matrix, so the system is square with two inputs and two outputs. The transmission zeros are defined where the determinant of *N(s)* vanishes. Then,

$$|N(s)| = \left\| \begin{bmatrix} 2 & s \\ -2(s-2) & 1 \end{bmatrix} \right\| = 2s^2 - 4s + 2 = 2(s-1)^2$$

so the MV transmission zeros are at *s= 1, 1*. The relative degree is zero.

Evaluate the numerator at s=1 to obtain

$$N(s=1) = \begin{bmatrix} 2 & 1 \\ -2(1-2) & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

which has rank=1. The vector $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$ is in the nullspace of *N(s=1)* since

$$N(s=1)\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$$

Therefore, select the input as

$$u(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t u_{-1}(t)$$

where $u_{-1}(t)$ is the unit step. This input has frequency of s=1 and is said to be in the direction $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$.

Compute the output as

$$Y(s) = H(s)U(s) = \frac{1}{s^2+4}\begin{bmatrix} 2 & s \\ -2(s-2) & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\frac{1}{s-1}$$

$$Y(s) = H(s)U(s) = \frac{1}{s^2+4}\begin{bmatrix} -2(s-1) \\ -2(s-1) \end{bmatrix}\frac{1}{s-1} = \frac{1}{s^2+4}\begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

where the transmission zero has been cancelled. The inverse Laplace transform yields

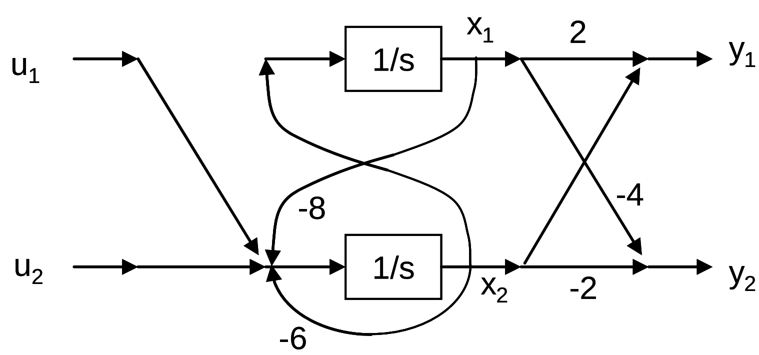
$$y(t) = -\sin 2t\, u_{-1}(t)$$

which does not contain any component of the input signal e^t which occurs at the transmission zero frequency *s=1*.

Example 2- System Zeros, Decoupling Zeros

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u$=A$x$+B$u$ y=$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} x$=C$x$

The block diagram of this system is shown. Note that BD and SV description are exactly equivalent.



1. Poles

$$\Delta(s) = |sI - A| = \left| \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix} \right| = s^2 + 6s + 8 = (s+2)(s+4)$$

The poles are at s = -2 and s = -4.

2. The system zeros are found as follows. The system matrix P(s)= $\begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$ is given by

$$P(s) = \begin{matrix} s & -1 & 0 & 0 \\ 8 & s+6 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{matrix}$$

loses rank where s=-2

and has rank of 3. Find the minor of the upper left *3x3* submatrix. This polynomial has a root at s=-2. Now find the minor obtained by striking out the third row and column of *P(s)*. This polynomial also has a root at s=-2.

Therefore, there is a system zero at s=-2

3. Input decoupling zeros occur where the matrix $P_1(s)$=[sI-A B] loses rank

$$P_1(s)= \begin{matrix} s & -1 & 0 & 0 \\ 8 & s+6 & 1 & 1 \end{matrix}$$

has rank of 2 for all values of *s*. Therefore, there are no input decoupling zeroes.

4. Output decoupling zeros occur where the matrix $P_2(s)=\begin{bmatrix} sI - A \\ -C \end{bmatrix}$

$$P_2(s)= = \begin{bmatrix} s & -1 \\ 8 & s+6 \\ -2 & -1 \\ 4 & 2 \end{bmatrix} \text{ loses rank where s=-2}$$

Therefore, the system has an output decoupling zero at s=-2

5. Transfer function

$$H(s) = C(sI - A)^{-1}B = \frac{1}{(s+2)(s+4)}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)}\begin{bmatrix} s+2 & s+2 \\ -2(s+2) & -2(s+2) \end{bmatrix} = \frac{1}{(s+4)}\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = \frac{N_c(s)}{\Delta_c(s)}$$

There are no transmission zeros, since, after pole/zero cancellation, the rank of *N(s)* is 1 for all values of *s*.

Minimality

A SV system *(A,B,C,D)* is *minimal* if it is a system with the least number of states giving its transfer function. If there is another system *(A1,B1,C1,D1)* with fewer states having the same transfer function, the given system is not minimal.

For SISO systems, a system is minimal iff the transfer function numerator polynomial and denominator polynomial have no common roots.

For MV systems one must use the definition of MV zeros.

Theorem. A system *(A,B,C,D)* is minimal iff it has no input-decoupling zeros and no output decoupling zeros.

In fact, the decoupling zeros are exactly the zeros that cause pole-zero cancellation in computing the transfer function.

Realization theory covers the problem of finding a minimal SV realization (A,B,C,D) for a given transfer function. We will cover this later.

Example 3- Nonminimal System

For Example 2, one has

$$H)(s) = \frac{1}{(s+2)(s+4)}\begin{bmatrix} s+2 & s+2 \\ -2(s+2) & -2(s+2) \end{bmatrix} = \frac{1}{(s+4)}\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = \frac{N(s)}{d(s)}$$

Note that there is a pole/zero cancellation due to the output decoupling zero at *s= -2*. Therefore, the system is not minimal.

In fact, a system of order *n=1* with the same transfer function is

$$\dot{x} = -4x + \begin{bmatrix} 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 \\ -2 \end{bmatrix} x$$

Example 4- PBH Rank Tests

A system has A matrix of $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$.

a. Find output matrix C to make the unstable mode unobservable.

The poles are given by

$$\Delta(s) = |sI - A| = \begin{vmatrix} s-1 & -1 \\ -6 & s \end{vmatrix} = s^2 - s - 6 = (s-3)(s+2) = 0$$

so there is a stable pole at s=-2 and an unstable pole at s=3.

Define $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$. The output-coupling matrix is then

$$P_O(s) = \begin{bmatrix} sI - A \\ -C \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ -6 & s \\ -c_1 & -c_2 \end{bmatrix}$$

Evaluate at the unstable pole to get

$$P_O(3) = \begin{bmatrix} 3I - A \\ -C \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \\ -c_1 & -c_2 \end{bmatrix}$$

This matrix has rank of 1 if one selects

$$C = \begin{bmatrix} c_1 & c_2 \end{bmatrix} = c\begin{bmatrix} 2 & -1 \end{bmatrix}$$

for any value of c.

b. Find input matrix B to make the stable mode uncontrollable.

Define $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. The input-coupling matrix is then

$$P_I(s) = \begin{bmatrix} sI - A & B \end{bmatrix} = \begin{bmatrix} s-1 & -1 & b_1 \\ -6 & s & b_2 \end{bmatrix}$$

Evaluate at the stable pole to get

$$P_I(-2) = \begin{bmatrix} -2I - A & B \end{bmatrix} = \begin{bmatrix} -3 & -1 & b_1 \\ -6 & -2 & b_2 \end{bmatrix}$$

This matrix has rank of 1 if one selects

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for any value of b.

c. Minimality

For these choices of B and/or C, the system is not minimal as there is pole/zero cancellation at the input and output decoupling zeros.