H

H_2 Optimal Control

Ben M. Chen
Department of Mechanical and Automation Engineering, Chinese University of Hong Kong, Shatin/Hong Kong, China

Abstract

An optimization-based approach to linear feedback control system design uses the H_2 norm, or energy of the impulse response, to quantify closed-loop performance. In this entry, an overview of state-space methods for solving H_2 optimal control problems via Riccati equations and matrix inequalities is presented in a continuous-time setting. Both regular and singular problems are considered. Connections to so- called LQR and LQG control problems are also described.

Keywords

Feedback control · H_2 control · Linear matrix inequalities · Linear systems · Riccati equations · State-space methods

Introduction

Modern multivariable control theory based on state-space models is able to handle multi-feedback-loop designs, with the added benefit that design methods derived from it are amenable to computer implementation. Indeed, over the last five decades, a number of multivariable analysis and design have been developed using the state-space description of systems. Of these design tools, H_2 optimal control problems involve minimizing the H_2 norm of the closed-loop transfer function from exogenous disturbance signals to a pertinent controlled output signals of a given plane by appropriate use of an internally stabilizing feedback controller. It was not until the 1990s that a complete solution to the general H_2 optimal control problem began to emerge. To elaborate on this, let us concentrate our discussion on H_2 optimal control for a continuous-time system Σ expressed in the following state-space form:

$$\dot{x} = Ax + Bu + Ew \tag{1}$$

$$y = C_1 x + D_{11} u + D_1 w \tag{2}$$

$$z = C_2 x + D_2 u + D_{22} w \tag{3}$$

where x is the state variable; u is the control input; w is the exogenous disturbance input; y is the measurement output; and z is the controlled output. The system Σ is typically an augmented or generalized plant model including weighting functions that reflect design requirements. The H_2 optimal control problem is to find an appropriate control law, relating the control input u to the measured output y, such that when it is applied to the given plant in Eqs. (1), (2) and (3), the resulting closed-loop system is internally

J. Baillieul, T. Samad (eds.), *Encyclopedia of Systems and Control*,
https://doi.org/10.1007/978-3-030-44184-5

stable and the H_2 norm of the resulting closed-loop transfer matrix from the disturbance input w to the controlled output z, denoted by $T_{zw}(s)$, is minimized. For a stable transfer matrix $T_{zw}(s)$, the H_2 norm is defined as

$$\|T_{zw}\|_2 = \left(\frac{1}{2\pi} \operatorname{trace}\left[\int_{-\infty}^{\infty} T_{zw}(j\omega) T_{zw}^{\mathrm{H}}(j\omega) d\omega\right]\right)^{\frac{1}{2}} \quad (4)$$

where T_{zw}^{H} is the conjugate transpose of T_{zw}. Note that the H_2 norm is equal to the energy of the impulse response associated with $T_{zw}(s)$ and this is finite only if the direct feedthrough term of the transfer matrix is zero.

It is standard to make the following assumptions on the problem data: $D_{11} = 0$; $D_{22} = 0$; (A, B) is stabilizable; (A, C_1) is detectable. The last two assumptions are necessary for the existence of an internally stabilizing control law. The first assumption can be made without loss of generality via a constant loop transformation. Finally, either the assumption $D_{22} = 0$ can be achieved by a pre-static feedback law or the problem does not yield a solution that has finite H_2 closed-loop norm.

There are two main groups into which all H_2 optimal control problems can be divided. The first group, referred to as regular H_2 optimal control problems, consists of those problems for which the given plant satisfies two additional assumptions:

1. the subsystem from the control input to the controlled output, i.e., (A, B, C_2, D_2), has no invariant zeros on the imaginary axis, and its direct feedthrough matrix, D_2, is injective (i.e., it is tall and of full rank); and
2. the subsystem from the exogenous disturbance to the measurement output, i.e., (A, E, C_1, D_1), has no invariant zeros on the imaginary axis, and its direct feedthrough matrix, D_1, is surjective (i.e., it is fat and of full rank).

Assumption 1 implies that (A, B, C_2, D_2) is left invertible with no infinite zero, and Assumption 2 implies that (A, E, C_1, D_1) is right invertible with no infinite zero. The second, referred to as singular H_2 optimal control problems, consists of those which are not regular.

Most of the research in the literature was expended on regular problems. Also, most of the available textbooks and review articles, see, for example, Anderson and Moore (1989), Bryson and Ho (1975), Fleming and Rishel (1975), Kailath (1974), Kwakernaak and Sivan (1972), Lewis (1986), and Zhou et al. (1996), to name a few, cover predominantly only a subset of regular problems. The singular H_2 control problem with state feedback was studied in Geerts (1989) and Willems et al. (1986). Using different classes of state and measurement feedback control laws, Stoorvogel et al. (1993) studied the general H_2 optimal control problems for the first time. In particular, necessary and sufficient conditions are provided therein for the existence of a solution in the case of state feedback control and in the case of measurement feedback control. Following this, Trentelman and Stoorvogel (1995) explored necessary and sufficient conditions for the existence of an H_2 optimal controller within the context of discrete-time and sampled-data systems. At the same time, Chen et al. (1993) and Chen et al. (1994a) provided a thorough treatment of the H_2 optimal control problem with state feedback controllers. This includes a parameterization and construction of the set of all H_2 optimal controllers and the associated sets of H_2 optimal fixed modes and H_2 optimal fixed decoupling zeros. Also, they provided a computationally feasible design algorithm for selecting an H_2 optimal state feedback controller that places the closed-loop poles at desired locations whenever possible. Furthermore, Chen and Saberi (1993) and Chen et al. (1996) developed the necessary and sufficient conditions for the uniqueness of an H_2 optimal controller. Interested readers are referred to the textbook (Saberi et al. 1995) for a detailed treatment of H_2 optimal control problems in their full generality.

Regular Case

Solving regular H_2 optimal control problems is relatively straightforward. In the case that all of the state variables of the given plant are available for feedback, i.e., $y = x$, and Assumption 1 holds, the corresponding H_2 optimal control problem can be solved in terms of the unique positive semi-definite stabilizing solution $P \geq 0$ of the following algebraic Riccati equation:

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 - (PB + C_2^{\mathrm{T}}D_2)(D_2^{\mathrm{T}}D_2)^{-1}(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P) = 0 \quad (5)$$

The H_2 optimal state feedback law is given by

$$u = Fx = -(D_2^{\mathrm{T}}D_2)^{-1}(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P)\, x \quad (6)$$

and the resulting closed-loop transfer matrix from w to z, $T_{zw}(s)$, has the following property:

$$\|T_{zw}\|_2 = \sqrt{\operatorname{trace}(E^{\mathrm{T}}PE)} \quad (7)$$

Note that the H_2 optimal state feedback control law is generally nonunique. A trivial example is the case when $E = 0$, whereby every stabilizing control law is an optimal solution. It is also interesting to note that the closed-loop system comprising the given plant with $y = x$ and the state feedback control law of Eq. (6) has poles at all the stable invariant zeros and all the mirror images of the unstable invariant zeros of (A, B, C_2, D_2) together with some other fixed locations in the left half complex plane. More detailed results about the optimal fixed modes and fixed decoupling zeros for general H_2 optimal control can be found in Chen et al. (1993).

It can be shown that the well-known linear quadratic regulation (LQR) problem can be reformulated as a regular H_2 optimal control problem. For a given plant

$$\dot{x} = Ax + Bu, \quad x(0) = X_0 \quad (8)$$

with (A, B) being stabilizable, the LQR problem is to find a control law $u = Fx$ such that the following performance index is minimized:

$$J = \int_0^\infty (x^{\mathrm{T}}Q_\star x + u^{\mathrm{T}}R_\star u)dt, \quad (9)$$

where $R_\star > 0$ and $Q_\star \geq 0$ with $(A, Q_\star^{\frac{1}{2}})$ being detectable. The LQR problem is equivalent to finding a static state feedback H_2 optimal control law for the following auxiliary plant Σ_{LQR}:

$$\dot{x} = Ax + Bu + X_0 w \quad (10)$$

$$y = x \quad (11)$$

$$z = \begin{pmatrix} 0 \\ Q_\star^{\frac{1}{2}} \end{pmatrix} x + \begin{pmatrix} R_\star^{\frac{1}{2}} \\ 0 \end{pmatrix} u \quad (12)$$

For the measurement feedback case with both Assumptions 1 and 2 being satisfied, the corresponding H_2 optimal control problem can be solved by finding a positive semi-definite stabilizing solution $P \geq 0$ for the Riccati equation given in Eq. (5) and a positive semi-definite stabilizing solution $Q \geq 0$ for the following Riccati equation:

$$QA^{\mathrm{T}} + AQ + EE^{\mathrm{T}} - (QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}})(D_1D_1^{\mathrm{T}})^{-1}(D_1E^{\mathrm{T}} + C_1Q) = 0 \quad (13)$$

The H_2 optimal measurement feedback law is given by

$$\dot{v} = (A + BF + KC_1)v - Ky, \quad u = Fx \quad (14)$$

where F is as given in Eq. (6) and

$$K = -(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}})(D_1D_1^{\mathrm{T}})^{-1} \quad (15)$$

In fact, such an optimal control law is unique, and the resulting closed-loop transfer matrix from w to z, $T_{zw}(s)$, has the following property:

$$\|T_{zw}\|_2 = \left\{\operatorname{trace}(E^{\mathrm{T}}PE) + \operatorname{trace}\left[\left(A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2\right)Q\right]\right\}^{\frac{1}{2}} \quad (16)$$

Similarly, consider the standard LQG problem for the following system

$$\dot{x} = Ax + Bu + G_\star d \quad (17)$$

$$y = Cx + N_\star n, \quad N_\star > 0 \quad (18)$$

$$z = \begin{pmatrix} H_\star x \\ R_\star u \end{pmatrix}, \quad R_\star > 0, \quad w = \begin{pmatrix} d \\ n \end{pmatrix} \tag{19}$$

where x is the state, u is the control, d and n are white noises with identity covariance, and y is the measurement output. It is assumed that (A, B) is stabilizable and (A, C) is detectable. The control objective is to design an appropriate control law that minimizes the expectation of $|z|^2$. Such an LQG problem can be solved via the H_2 optimal control problem for the following auxiliary system Σ_{LQG}, see Doyle (1983):

$$\dot{x} = Ax + Bu + [\, G_\star \quad 0 \,]w \tag{20}$$

$$y = Cx + [\, 0 \quad N_\star \,]w \tag{21}$$

$$z = \begin{pmatrix} H_\star \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ R_\star \end{pmatrix} u \tag{22}$$

H_2 optimal control problem for discrete-time systems can be solved in a similar way via the corresponding discrete-time algebraic Riccati equations. It is worth noting that many works can be found in the literature that deal with solutions to discrete-time algebraic Riccati equations related to optimal control problems; see, for example, Kucera (1972), Pappas et al. (1980), and Silverman (1976), to name a few. It is proven in Chen et al. (1994b) that solutions to the discrete- and continuous-time algebraic Riccati equations for optimal control problems can be unified. More specifically, the solution to a discrete-time Riccati equation can be done through solving an equivalent continuous-time one and vice versa.

Singular Case

As in the previous section, only the key procedure in solving the singular H_2-optimization problem for continuous-time systems is addressed. For the singular problem, it is generally not possible to obtain an optimal solution, except for some situations when the given plant satisfies certain geometric constraints, see, e.g., Chen et al. (1993) and Stoorvogel et al. (1993). It is more feasible to find a suboptimal control law for the singular problem, i.e., to find an appropriate control law such that the H_2 norm of the resulting closed-loop transfer matrix from w to z can be made arbitrarily close to the best possible performance. The procedure given below is to transform the original problem into an H_2 almost disturbance decoupling problem; see Stoorvogel (1992) and Stoorvogel et al. (1993).

Consider the given plant in Eqs. (1), (2) and (3) with Assumption 1 and/or Assumption 2 not satisfied. First, find the largest solution $P \geq 0$ for the following linear matrix inequality

$$F(P) = \begin{pmatrix} A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 & PB + C_2^{\mathrm{T}}D_2 \\ B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{pmatrix} \geq 0 \tag{23}$$

and find the largest solution $Q \geq 0$ for

$$G(Q) = \begin{pmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix} \geq 0 \tag{24}$$

Note that by decomposing the quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1) into various subsystems in accordance with their structural properties, solutions to the above linear matrix inequalities can be obtained by solving a Riccati equation similar to those in Eq. (5) or Eq. (13) for the regular case. In fact, for the regular problem, the largest solution $P \geq 0$ for Eq. (23) and the stabilizing solution $P \geq 0$ for Eq. (5) are identical. Similarly, the largest solution $Q \geq 0$ for Eq. (24) and the stabilizing solution $Q \geq 0$ for Eq. (13) are also the same. Interested readers are referred to Stoorvogel et al. (1993) for more details or to Chen et al. (2004) for a more systematic treatment on the structural decomposition of linear systems and its connection to the solutions of the linear matrix inequalities.

It can be shown that the best achievable H_2 norm of the closed-loop transfer matrix from w to z, i.e., the best possible performance over all internally stabilizing control laws, is given by

$$\gamma_2^\star = \Big\{\text{trace}(E^\mathrm{T} P E) + \text{trace}\Big[\Big(A^\mathrm{T} P + P A + C_2^\mathrm{T} C_2\Big) Q\Big]\Big\}^{\frac{1}{2}} \tag{25}$$

Next, partition

$$F(P) = \begin{pmatrix} C_\mathrm{P}^\mathrm{T} \\ D_\mathrm{P}^\mathrm{T} \end{pmatrix} \begin{pmatrix} C_\mathrm{P} & D_\mathrm{P} \end{pmatrix} \quad \text{and}$$

$$G(Q) = \begin{pmatrix} E_\mathrm{Q} \\ D_\mathrm{Q} \end{pmatrix} \begin{pmatrix} E_\mathrm{Q}^\mathrm{T} & D_\mathrm{Q}^\mathrm{T} \end{pmatrix} \tag{26}$$

where $[C_\mathrm{P} \quad D_\mathrm{P}]$ and $[E_\mathrm{Q}^\mathrm{T} \quad D_\mathrm{Q}^\mathrm{T}]$ are of maximal rank, and then define an auxiliary system Σ_PQ:

$$\dot{x}_\mathrm{PQ} = A x_\mathrm{PQ} + B u + E_\mathrm{Q} w_\mathrm{PQ} \tag{27}$$

$$y = C_1 x_\mathrm{PQ} + D_\mathrm{Q} w_\mathrm{PQ} \tag{28}$$

$$z_\mathrm{PQ} = C_\mathrm{P} x_\mathrm{PQ} + D_\mathrm{P} u \tag{29}$$

It can be shown that the quadruple $(A, B, C_\mathrm{P}, D_\mathrm{P})$ is right invertible and has no invariant zeros in the open right-half complex plane and the quadruple $(A, E_\mathrm{Q}, C_1, D_\mathrm{Q})$ is left invertible and has no invariant zeros in the open right-half complex plane. It can also be shown that there exists an appropriate control law such that when it is applied to Σ_PQ, the resulting closed-loop system is internally stable and the H_2 norm of the closed-loop transfer matrix from w_PQ to z_PQ can be made arbitrarily small. Equivalently, H_2 almost disturbance decoupling problem for Σ_PQ is solvable.

More importantly, it can further be shown that if an appropriate control law that solves the H_2 almost disturbance decoupling problem for Σ_PQ, then it solves the H_2 suboptimal problem for Σ. As such, the solution to the singular H_2 control problem for Σ can be done by finding a solution to the H_2 almost disturbance decoupling problem for Σ_PQ. There are vast results available in the literature dealing with disturbance decoupling problems. More detailed treatments can be found in Saberi et al. (1995).

Conclusion

This entry considers the basic solutions to H_2 optimal control problems for continuous-time systems. Both the regular problem and the general singular problem are presented. Readers interested in more details are referred to Saberi et al. (1995) and the references therein for the complete treatment of H_2 optimal control problems and to Chapter 10 of Chen et al. (2004) for the unification and differentiation of H_2 control, H_∞ control, and disturbance decoupling control problems. H_2 optimal control is a mature area and has a long history. Possible future research includes issues on how to effectively utilize the theory in solving real-life problems.

Cross-References

- ▶ H-Infinity Control
- ▶ Linear Quadratic Optimal Control
- ▶ Optimal Control via Factorization and Model Matching
- ▶ Stochastic Linear-Quadratic Control

Bibliography

Anderson BDO, Moore JB (1989) Optimal control: linear quadratic methods. Prentice Hall, Englewood Cliffs

Bryson AE, Ho YC (1975) Applied optimal control, optimization, estimation, and control. Wiley, New York

Chen BM, Saberi A (1993) Necessary and sufficient conditions under which an H_2-optimal control problem has a unique solution. Int J Control 58:337–348

Chen BM, Saberi A, Sannuti P, Shamash Y (1993) Construction and parameterization of all static and dynamic H_2-optimal state feedback solutions, optimal fixed modes and fixed decoupling zeros. IEEE Trans Autom Control 38:248–261

Chen BM, Saberi A, Shamash Y, Sannuti (1994a) Construction and parameterization of all static and dynamic H_2-optimal state feedback solutions for discrete time systems. Automatica 30:1617–1624

Chen BM, Saberi A, Shamash Y (1994b) A non-recursive method for solving the general discrete time algebraic Riccati equation related to the H_∞ control problem. Int J Robust Nonlinear Control 4:503–519

Chen BM, Saberi A, Shamash Y (1996) Necessary and sufficient conditions under which a discrete time H_2-optimal control problem has a unique solution. J Control Theory Appl 13:745–753

Chen BM, Lin Z, Shamash Y (2004) Linear systems theory: a structural decomposition approach. Birkhäuser, Boston
Doyle JC (1983) Synthesis of robust controller and filters. In: 22nd IEEE Conference on Decision and Control, San Antonio
Fleming WH, Rishel RW (1975) Deterministic and stochastic optimal control. Springer, New York
Geerts T (1989) All optimal controls for the singular linear quadratic problem without stability: a new interpretation of the optimal cost. Linear Algebra Appl 122:65–104
Kailath T (1974) A view of three decades of linear filtering theory. IEEE Trans Inf Theory 20:146–180
Kucera V (1972) The discrete Riccati equation of optimal control. Kybernetika 8:430–447
Kwakernaak H, Sivan R (1972) Linear optimal control systems. Wiley, New York
Lewis FL (1986) Optimal control. Wiley, New York
Pappas T, Laub AJ, Sandell NR (1980) On the numerical solution of the discrete-time algebraic Riccati equation. IEEE Trans Autom Control AC-25:631–641
Saberi A, Sannuti P, Chen BM (1995) H_2 optimal control. Prentice Hall, London
Silverman L (1976) Discrete Riccati equations: alternative algorithms, asymptotic properties, and system theory interpretations. Control Dyn Syst 12:313–386
Stoorvogel AA (1992) The singular H_2 control problem. Automatica 28:627–631
Stoorvogel AA, Saberi A, Chen BM (1993) Full and reduced order observer based controller design for H_2-optimization. Int J Control 58:803–834
Trentelman HL, Stoorvogel AA (1995) Sampled-data and discrete-time H_2 optimal control. SIAM J Control Optim 33:834–862
Willems JC, Kitapci A, Silverman LM (1986) Singular optimal control: a geometric approach. SIAM J Control Optim 24:323–337
Zhou K, Doyle JC, Glover K (1996) Robust and optimal control. Prentice Hall, Upper Saddle River

H-Infinity Control

Keith Glover
Department of Engineering, University of Cambridge, Cambridge, UK

Abstract

The area of robust control, where the performance of a feedback system is designed to be robust to uncertainty in the plant being controlled, has received much attention since the 1980s. System analysis and controller synthesis based on the H-infinity norm has been central to progress in this area. This article outlines how the control law that minimizes the H-infinity norm of the closed-loop system can be derived. Connections to other problems, such as game theory and risk-sensitive control, are discussed and finally appropriate problem formulations to produce "good" controllers using this methodology are outlined.

Keywords

Loop-shaping · Robust control · Robust stability

Introduction

The $\mathcal{H}_\infty$-norm probably first entered the study of robust control with the observations made by Zames (1981) in the considering optimal sensitivity. The so-called $\mathcal{H}_\infty$ methods were subsequently developed and are now routinely available to control engineers. In this entry we consider the $\mathcal{H}_\infty$ methods for control, and for simplicity of exposition, we will restrict our attention to linear, time-invariant, finite dimensional, continuous-time systems. Such systems can be represented by their transfer function matrix, $G(s)$, which will then be a rational function of s. Although the Hardy Space, $\mathcal{H}_\infty$, also includes nonrational functions, a rational $G(s)$ is in $\mathcal{H}_\infty$ if and only if it is proper and all its poles are in the open left half plane, in which case the $\mathcal{H}_\infty$-norm is defined as:

$$\|G(s)\|_\infty = \sup_{\mathrm{Re}s>0} \sigma_{\max}(G(s)) = \sup_{-\infty<\omega<\infty} \sigma_{\max}((j\omega))$$

(where $\sigma_{\max}$ denotes the largest singular value). Hence for a single input/single output system with transfer function, $g(s)$, its $\mathcal{H}_\infty$-norm, $\|g(s)\|_\infty$ gives the maximum value of $|g(j\omega)|$ and hence the maximum amplification of sinusoidal signals by a system with this transfer function. In the multi-input/multi-output case a similar result holds regarding the system amplification of a vector of sinusoids. There is now a good collection of graduate level textbooks that cover the area in some detail from a variety of approaches, and these are listed in the Recommended Reading section and the references in this article are generally to these texts rather than to the original journal papers.

Consider a system with transfer function, $G(s)$, input vector, $u(t) \in \mathcal{L}_2(0,\infty)$ and an output vector, $y(t)$, whose Laplace transforms are given by $\bar{u}(s)$ and $\bar{y}(s)$. Such a system will have a state space realization,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

giving $G(s) = D + C(sI - A)^{-1}B$, which we also denote

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

and hence $\bar{y}(s) = G(s)\bar{u}(s)$ if $x(0) = 0$.

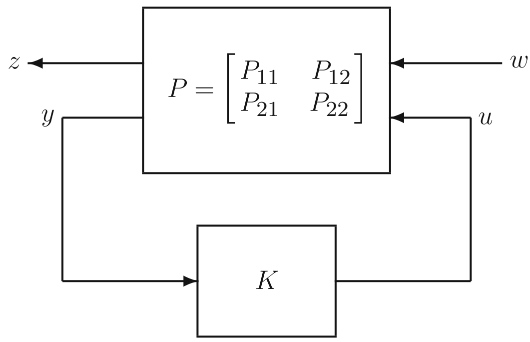
There are two main reasons for using the $\mathcal{H}_\infty$-norm. Firstly in representing the system gain for input signals $u(t) \in \mathcal{L}_2(0,\infty)$ or equivalently $\bar{u}(j\omega) \in \mathcal{L}_2(-\infty,\infty)$, with corresponding norm $\|u\|_2^2 = \int_0^\infty u(t)^* u(t)\,dt$ (where x^* denotes the conjugate transpose of the vector x (or a matrix)). With these input and output spaces the induced norm of the system is easily shown to be the $\mathcal{H}_\infty$-norm of $G(s)$, and in particular,

$$\|y\|_2 \le \|G(s)\|_\infty \|u\|_2$$

Hence in a control context the $\mathcal{H}_\infty$-norm can give a measure of the gain, for example, from disturbances to the resulting errors. In the interconnection of systems, the property that $\|P(s)Q(s)\|_\infty \le \|P(s))\|_\infty \|Q(s)\|_\infty$ is often useful.

The second reason for using the $\mathcal{H}_\infty$-norm is in representing uncertainty in the plant being controlled, e.g., the nominal plant is $P_o(s)$ but the actual plant is $P(s) = P_o(s) + \Delta(s)$ where $\|\Delta(s)\|_\infty \le \delta$.

A typical control design problem is given in Fig. 1, i.e.,



H-Infinity Control, Fig. 1 Lower linear fractional transformation: feedback system

$$\begin{aligned}
\begin{bmatrix} \bar{z} \\ \bar{y} \end{bmatrix} &= P \begin{bmatrix} \bar{w} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} P_{11}\bar{w} + P_{12}\bar{u} \\ P_{21}\bar{w} + P_{22}\bar{u} \end{bmatrix} \\
\bar{u} &= K\bar{y} \\
\Rightarrow \quad \bar{y} &= (I - P_{22}K)^{-1} P_{21}\bar{w}, \\
\bar{u} &= K(I - P_{22}K)^{-1} P_{21}\bar{w} \\
\bar{z} &= \left(P_{11} + P_{12}K(I - P_{22}K)^{-1} P_{21}\right)\bar{w} \\
&=: \mathcal{F}_l(P,K)\bar{w} =: T_{z\leftarrow w}\bar{w}
\end{aligned}$$

where $\mathcal{F}_l(P,K)$ denotes the lower Linear Fractional Transformation (LFT) with connection around the lower terminals of P as in Fig. 1.

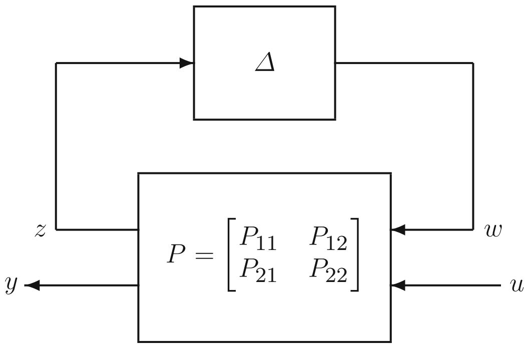
The standard $\mathcal{H}_\infty$-control synthesis problem is to find a controller with transfer function, K, that

> stabilizes the closed-loop system in Fig. 1 and minimizes $\|\mathcal{F}_l(P,K)\|_\infty$.

That is, the controller is designed to minimize the worst-case effect of the disturbance w on the output/error signal z as measured by the $\mathcal{L}_2$ norm of the signals. This article will describe the solution to this problem.

Robust Stability

Before we describe the solution to the synthesis problem, consider the problem of the robust stability of an uncertain plant with a feedback controller. Suppose the plant is given by the upper



H-Infinity Control, Fig. 2 Upper linear fractional transformation

LFT, $\mathcal{F}_u(P, \Delta)$ with $\|\Delta\|_\infty \leq 1/\gamma$ as illustrated in Fig. 2,

$$\bar{y} = \mathcal{F}_u(P, \Delta)\bar{u}, \tag{1}$$

$$\text{where } \mathcal{F}_u(P,K) := P_{22}+P_{21}\Delta(I-P_{11}\Delta)^{-1}P_{12} \tag{2}$$

The *small gain theorem* then states that the feedback system of Fig. 3 will be stable for all such Δ if the feedback connection of P_{22} and K is stable and $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$. This robust stability result is valid if P and Δ are both stable; more care is required when either or both are unstable but with such care a similar result is true.

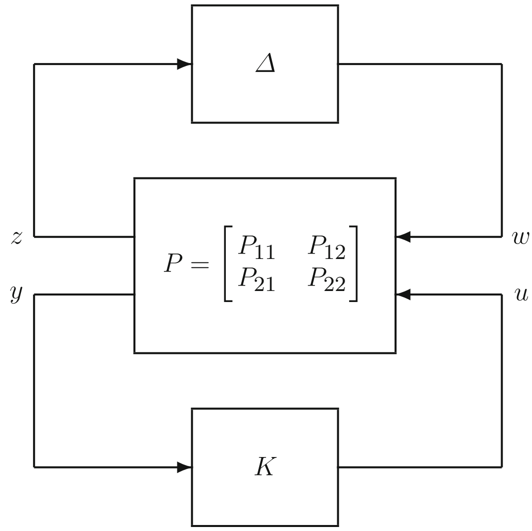
Let us consider a couple of examples. First suppose that the uncertainty is represented as output multiplicative uncertainty,

$$P_\Delta = (I + W_1 \Delta W_2)P_o = \mathcal{F}_u\left(\begin{bmatrix} 0 & W_2 P_o \\ W_1 & P_o \end{bmatrix}, \Delta\right)$$

with robust stability test given by

$$\begin{aligned} &\|\mathcal{F}_l\left(\begin{bmatrix} 0 & W_2 P_o \\ W_1 & P_o \end{bmatrix}, K\right)\|_\infty \\ &= \|W_2 P_o K(I - P_o K)^{-1} W_1\|_\infty < \gamma \end{aligned}$$

As a second example consider the plants $P_\Delta = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$, with $\Delta = \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix}$ and $\|\Delta\|_\infty \leq 1/\gamma$. Here $P_o = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization of the nominal plant and the plants P_Δ are represented by perturbations



H-Infinity Control, Fig. 3 Feedback system with plant uncertainty

to these coprime factors. In this case $P_\Delta = \mathcal{F}_u(P, \Delta)$, where

$$P = \begin{bmatrix} \begin{bmatrix} 0 \\ -\tilde{M}^{-1} \end{bmatrix} & \begin{bmatrix} I \\ -\tilde{M}^{-1}\tilde{N} \end{bmatrix} \\ \tilde{M}^{-1} & \tilde{M}^{-1}\tilde{N} \end{bmatrix}$$

and the robust stability test will be

$$\|\mathcal{F}_l(P,K)\|_\infty = \left\| \begin{bmatrix} K \\ -I \end{bmatrix} (I - P_o K)^{-1} \tilde{M}^{-1} \right\|_\infty < \gamma$$

This is related to plant perturbations in the gap metric (see Vinnicombe 2001). It is therefore observed that the robust stability test for these useful representations of uncertain plants is given by an $\mathcal{H}_\infty$-norm test just as in the controller synthesis problem.

Derivation of the $\mathcal{H}_\infty$-Control Law

In this section we present a solution to the $\mathcal{H}_\infty$-control problem and give some interpretations of the solution. The approach presented is as in by Doyle et al. (1989); see also Zhou et al. (1996). We will make some simplifying structural assumptions to make the formulae less complex and will *not* state the required assumptions on rank, stabilizability, and detectability. Let the system in Fig. 1 be described by the equations:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad (3)$$

$$z(t) = C_1 x(t) + D_{12} u(t) \quad (4)$$

$$y(t) = C_2 x(t) + D_{21} w(t) \quad (5)$$

i.e., in Fig. 1

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

where we also assume, with little loss of generality, that $D_{12}^* D_{12} = I$, $D_{21} D_{21}^* = I$, $D_{12}^* C_1 = 0$ and $B_1 D_{21}^* = 0$. Since we wish to have $\|T_{z\leftarrow w}\|_\infty < \gamma$, we need to find u such that

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \text{ for all } w \neq 0 \in \mathcal{L}_2(0,\infty).$$

We could consider w to be an adversary trying to make this expression positive, while u has to ensure that it always remains negative in spite of the malicious intentions of w, as in a noncooperative game. Suppose that there exists a solution, X_∞, to the Algebraic Riccati Equation (ARE),

$$\begin{aligned} &A^* X_\infty + X_\infty A + C_1^* C_1 \\ &\quad + X_\infty(\gamma^{-2} B_1 B_1^* - B_2 B_2^*) X_\infty = 0 \quad (6) \end{aligned}$$

with $X_\infty \geq 0$ and $A + (\gamma^{-2} B_1 B_1^* - B_2 B_2^*) X_\infty$ a stable "A-matrix." A simple substitution then gives that

$$\begin{aligned} \frac{d}{dt}(x(t)^* X_\infty x(t)) &= -z^* z + \gamma^2 w^* w \\ &\quad + v^* v - \gamma^2 r^* r \end{aligned}$$

where

$$v := u + B_2^* X_\infty x, \quad r := w - \gamma^{-2} B_1^* X_\infty x.$$

Now let $x(0) = 0$ and assuming stability so that $x(\infty) = 0$, then integrating from 0 to ∞ gives

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \|v\|_2^2 - \gamma^2 \|r\|_2^2 \quad (7)$$

If the state is available to u, then the control law $u = -B_2^* X_\infty x$ gives $v = 0$ and $\|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$ for all $w \neq 0$. It can be shown that (6) has a solution if there exists a controller such that $\|\mathcal{F}_l(P,K)\|_\infty < \gamma$. In addition since transposing a system does not change its $\mathcal{H}_\infty$-norm, the following dual ARE will also have a solution, $Y_\infty \geq 0$,

$$\begin{aligned} &A Y_\infty + Y_\infty A^* + B_1 B_1^* \\ &\quad + Y_\infty(\gamma^{-2} C_1^* C_1 - C_2^* C_2) Y_\infty = 0 \quad (8) \end{aligned}$$

To obtain a solution to the output feedback case, note that (7) implies that $\|z\|_2^2 < \gamma^2 \|w\|_2^2$ if and only if $\|v\|_2^2 < \gamma^2 \|r\|_2^2$ and $\bar{v} = \mathcal{F}_l(P_{\text{tmp}}, K)\bar{r}$ where

$$\begin{bmatrix} \bar{v} \\ \bar{y} \end{bmatrix} = P_{\text{tmp}} \begin{bmatrix} \bar{r} \\ \bar{u} \end{bmatrix},$$

and

$$P_{\text{tmp}} = \left[\begin{array}{c|cc} A + \gamma^{-2} B_1 B_1^* X_\infty & B_1 & B_2 \\ \hline B_2^* X_\infty & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

The special structure of this problem enables a solution to be derived in much the same way as the dual of the state feedback problem. The corresponding ARE will have a solution $Y_{\text{tmp}} = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty \geq 0$ if and only if the spectral radius $\rho(Y_\infty X_\infty) < \gamma^2$.

The above outline, supported by significant technical detail and assumptions, will therefore demonstrate that there exists a stabilizing controller, $K(s)$, such that the system described by (3–5) satisfies $\|T_{z\leftarrow w}\|_\infty < \gamma$ if and only if there exist stabilizing solutions to the AREs in (6) and (8) such that

$$X_\infty \geq 0, \quad Y_\infty \geq 0, \quad \rho(Y_\infty X_\infty) < \gamma^2 \quad (9)$$

The state equations for the resulting controller can be written as

$$\dot{\hat{x}} = A\hat{x} + B_1\hat{w}_{\text{worst}} + B_2u + Z_\infty L_\infty(C_2\hat{x} - y)$$
$$u = F_\infty\hat{x}, \quad \hat{w}_{\text{worst}} = \gamma^{-2}B_1^* X_\infty\hat{x}$$
$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*,$$
$$Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$$

giving feedback from a state estimator in the presence of an estimate of the worst-case disturbance.

As $\gamma \to \infty$ the standard LQG controller is obtained with state feedback of a state estimate obtained from a Kalman filter. In contrast to the LQG problem, the controller depends on the value of γ, and if this is chosen to be too small, then one of the conditions in (9) will be violated. In order to determine the minimum achievable value of γ, a bisection search over γ can be performed checking (9) for each candidate value of γ.

In the limit as $\gamma \to \gamma_{\text{opt}}$ (its minimum value), a variety of situations can arise and the formulae given here may become ill-conditioned. Typically achieving γ_{opt} is more of an interesting and sometimes challenging mathematical exercise rather than a control system requirement.

This control problem does not have a unique solution, and all solutions can be characterized by an LFT form such as $K = \mathcal{F}_l(M, Q)$ where $Q \in \mathcal{H}_\infty$ with $\|Q\|_\infty < 1$, the present solution is sometimes referred to as the "central solution" obtained with $Q = 0$.

Relations for Other Solution Methods and Problem Formulations

The $\mathcal{H}_\infty$-control problem has been shown to be related to an extraordinarily wide variety of mathematical techniques and to other problem areas, and investigations of these connections have been most fruitful. Earlier approaches (see Francis 1988) firstly used the characterization of all stabilizing controllers of Youla et al. (see Vidyasagar 1985) which shows that all stable closed-loop systems can be written as

$$\mathcal{F}_l(P, K) = T_1 + T_2QT_3, \text{ where } Q \in \mathcal{H}_\infty$$

and then solved the model matching problem $\inf_{Q\in\mathcal{H}_\infty} \|T_1 + T_2QT_3\|_\infty$. This model matching problem is related to interpolation theory and resulted in a productive interaction with the operator theory. One solution method reduces this problem to J-spectral factorisation problems $\left(\text{where } J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\right)$ and generates state-space solutions to these problems (Kimura 1997).

The derivation above clearly demonstrates relations to noncooperative differential games, and this is fully developed in Başar and Bernhard (1995) and Green and Limebeer (1995).

The model matching problem is clearly a convex optimization problem. The solution of linear matrix inequalities can give effective methods for solving certain convex optimization problems (e.g., calculating the $\mathcal{H}_\infty$ norm using the bounded real lemma) and can be exploited in the $\mathcal{H}_\infty$-control problem. See Boyd and Barratt (1991) for a variety of results on convex optimization and control and Dullerud and Paganini (2000) for this approach in robust control.

As noted above there is a family of solutions to the $\mathcal{H}_\infty$-control problem. The central solution in fact minimizes the entropy integral given by

$$I(T_{z\leftarrow w};\gamma) := -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln \left|\det(I - \gamma^{-2}T_{z\leftarrow w}(j\omega)^* T_{z\leftarrow w}(j\omega))\right| \, d\omega \tag{10}$$

It can be seen that this criterion will penalize the singular values of $T_{z\leftarrow w}(j\omega)$ from being close to γ for a large range of frequencies.

One of the more surprising connections is with the risk-sensitive stochastic control problem (Whittle 1990) where w is assumed to be Gaussian white noise and it is desired to minimize

$$J_T(\gamma) := \frac{\gamma^2}{T} \ln \mathbf{E}\left\{e^{\frac{1}{2}\gamma^{-2}V_T}\right\} \tag{11}$$

$$\text{where } V_T := \int_{-T}^{T} z(t)^* z(t)\, dt \tag{12}$$

The situation with $\gamma^2 > 0$ corresponds to the risk averse controller since large values of V_T

are heavily penalized by the exponential function. It can be shown that if $\|T_{z\leftarrow w}\|_\infty < \gamma$, then

$$\lim_{T\to\infty} J_T(\gamma) = I(T_{z\leftarrow w};\gamma)$$

and hence the central controller minimizes both the entropy integral and the risk-sensitive cost function. When γ is chosen to be too small, Whittle refers to the controller having a "neurotic breakdown" because the cost will be infinite for all possible control laws! If in (11) we set $\gamma^2 = -\theta^{-1}$, then the entropy minimizing controller will have $\theta < 0$ and will be risk-averse. The risk neutral controller is when $\theta \to 0$, $\gamma \to \infty$ and gives the standard LQG case. If $\theta > 0$, then the controller will be risk-seeking, believing that large variance will be in its favor.

Controller Design with $\mathcal{H}_\infty$ Optimization

The above solutions to the $\mathcal{H}_\infty$ mathematical problem do not give guidance on how to set up a problem to give a "good" control system design. The problem formulation typically involves identifying frequency-dependent weighting matrices to characterize the disturbances, w, and the relative importance of the errors, z (see Skogestad and Postlethwaite 1996). The choice of weights should also incorporate system uncertainty to obtain a robust controller.

One approach that combines both closed-loop system gain and system uncertainty is called $\mathcal{H}_\infty$ *loop-shaping* where the desired closed-loop behavior is determined by the design of the loop-shape using pre- and post-compensators and the system uncertainty is represented in the gap metric (see Vinnicombe 2001). This makes classical criteria such as low frequency tracking error, bandwidth, and high-frequency roll-off all easily incorporated. In this framework the performance and robustness measures are very well matched to each other. Such an approach has been successfully exploited in a number of practical examples (e.g., Hyde (1995) for flight control taken through to successful flight tests). Standard control design software packages now routinely have $\mathcal{H}_\infty$-control design modules.

Summary and Future Directions

We have outlined the derivation of $\mathcal{H}_\infty$ controllers with straightforward assumptions that nevertheless exhibit most of the features of linear time-invariant systems without such assumptions and for which routine design software is now available. Connections to a surprisingly large range of other problems are also discussed.

Generalizations to more general cases such as time-varying and nonlinear systems, where the norm is interpreted as the induced norm of the system in $\mathcal{L}_2$, can be derived although the computational aspects are no longer routine. For the problems of robust control, there are necessarily continuing efforts to match the mathematical representation of system uncertainty and system performance to the physical system requirements and to have such representations amenable to analysis and computation.

Cross-References

▶ Fundamental Limitation of Feedback Control
▶ H_2 Optimal Control
▶ Linear Quadratic Optimal Control
▶ LMI Approach to Robust Control
▶ Robust $\mathcal{H}_2$ Performance in Feedback Control
▶ Structured Singular Value and Applications: Analyzing the Effect of Linear Time-Invariant Uncertainty in Linear Systems

Bibliography

Dullerud GE, Paganini F (2000) A course in robust control theory: a convex approach. Springer, New York
Green M, Limebeer D (1995) Linear robust control. Prentice Hall, Englewood Cliffs
Başar T, Bernhard P (1995) H^∞-optimal control and related minimax design problems, 2nd edn. Birkhäuser, Boston
Boyd SP, Barratt CH (1991) Linear controller design: limits of performance. Prentice Hall, Englewood Cliffs

Doyle JC, Glover K, Khargonekar PP, Francis BA (1989) State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. IEEE Trans Autom Control 34(8):831–847
Francis BA (1988) A course in $\mathcal{H}_\infty$ control theory. Lecture notes in control and information sciences, vol 88. Springer, Berlin, Heidelberg
Hyde RA (1995) $\mathcal{H}_\infty$ aerospace control design: a VSTOL flight application. Springer, London
Kimura H (1997) Chain-scattering approach to $\mathcal{H}_\infty$-control. Birkhäuser, Basel
Skogestad S, Postlethwaite I (1996) Multivariable feedback control: analysis and design. Wiley, Chichester
Vidyasagar M (1985) Control system synthesis: a factorization approach. MIT, Cambridge
Vinnicombe G (2001) Uncertainty and feedback: $\mathcal{H}_\infty$ loop-shaping and the ν-gap metric. Imperial College Press, London
Whittle P (1990) Risk-sensitive optimal control. Wiley, Chichester
Zames G (1981) Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. IEEE Trans Automat Control 26:301–320
Zhou K, Doyle JC, Glover K (1996) Robust and optimal control. Prentice Hall, Upper Saddle River, New Jersey

History of Adaptive Control

Karl Åström
Department of Automatic Control, Lund University, Lund, Sweden

Abstract

This entry gives an overview of the development of adaptive control, starting with the early efforts in flight and process control. Two popular schemes, the model reference adaptive controller and the self-tuning regulator, are described with a thumbnail overview of theory and applications. There is currently a resurgence in adaptive flight control as well as in other applications. Some reflections on future development are also given.

Keywords

Adaptive control · Auto-tuning · Flight control · History · Model reference adaptive control · Process control · Robustness · Self-tuning regulators · Stability

Introduction

In everyday language, *to adapt* means to change a behavior to conform to new circumstances, for example, when the pupil area changes to accommodate variations in ambient light. The distinction between adaptation and conventional feedback is subtle because feedback also attempts to reduce the effects of disturbances and plant uncertainty. Typical examples are *adaptive optics* and *adaptive machine tool control* which are conventional feedback systems, with controllers having constant parameters. In this entry we take the pragmatic attitude that an adaptive controller is a controller that can modify its *behavior* in response to changes in the dynamics of the process and the character of the disturbances, by adjusting the controller parameters.

Adaptive control has had a colorful history with many ups and downs and intense debates in the research community. It emerged in the 1950s stimulated by attempts to design autopilots for supersonic aircrafts. Autopilots based on constant-gain, linear feedback worked well in one operating condition but not over the whole flight envelope. In process control there was also a need for automatic tuning of simple controllers.

Much research in the 1950s and early 1960s contributed to conceptual understanding of adaptive control. Bellman showed that *dynamic programming* could capture many aspects of adaptation (Bellman 1961). Feldbaum introduced the notion of *dual control*, meaning that control should be probing as well as directing; the controller should thus inject test signals to obtain better information. Tsypkin showed that schemes for *learning and adaptation* could be captured in a common framework (Tsypkin 1971).

Gabor's work on adaptive filtering (Gabor et al. 1959) inspired Widrow to develop an analogue neural network (Adaline) for adaptive control (Widrow 1962). Widrow's adaptation mechanism was inspired by Hebbian learning in biological systems (Hebb 1949).

There are adaptive control problems in economics and operations research. In these fields the problems are often called *decision making* *under uncertainty*. A simple idea, called the *certainty equivalence principle* proposed by Simon (1956), is to neglect uncertainty and treat estimates as if they are true. Certainty equivalence was commonly used in early work on adaptive control.

A period of intense research and ample funding ended dramatically in 1967 with a crash of the rocket powered X15-3 using Honeywell's MH-96 self-oscillating adaptive controller. The self-oscillating adaptive control system has, however, been successfully used in several missiles.

Research in adaptive control resurged in the 1970s, when the two schemes the model reference adaptive control (MRAC) and the self-tuning regulator (STR) emerged together with successful applications. The research was influenced by stability theory and advances in the field of system identification. There was an intensive period of research from the late 1970s through the 1990s. The insight and understanding of stability, convergence, and robustness increased. Recently there has been renewed interest because of flight control (Hovakimyan and Cao 2010; Lavretsky and Wise 2013) and other applications; there is, for example a need for adaptation in autonomous systems.

The Brave Era

Supersonic flight posed new challenges for flight control. Eager to obtain results, there was a very short path from idea to flight test with very little theoretical analysis in between. A number of research projects were sponsored by the US air force. Adaptive flight control systems were developed by General Electric, Honeywell, MIT, and other groups. The systems are documented in the Self-Adaptive Flight Control Systems Symposium held at the Wright Air Development Center in 1959 (Gregory 1959) and the book (Mishkin and Braun 1961).

Whitaker of the MIT team proposed the model reference adaptive controller system which is based on the idea of specifying the performance of a servo system by a reference. Honeywell proposed a self-oscillating adaptive system (SOAS) which attempted to keep a given gain margin by bringing the system to self-oscillation. The system was flight-tested on several aircrafts. It experienced a disaster in a test on the X-15. Combined with the success of gain scheduling based on air data sensors, the interest in adaptive flight control diminished significantly.

There was also interest of adaptation for process control. Foxboro patented an adaptive process controller with a pneumatic adaptation mechanism in 1950 (Foxboro 1950). DuPont had joint studies with IBM aimed at computerized process control. Kalman worked for a short time at the Engineering Research Laboratory at DuPont, where he started work that led to a paper (Kalman 1958), which is the inspiration of the self-tuning regulator. The abstract of this entry has the statement, *This paper examines the problem of building a machine which adjusts itself automatically to control an arbitrary dynamic process*, which clearly captures the dream of early adaptive control.

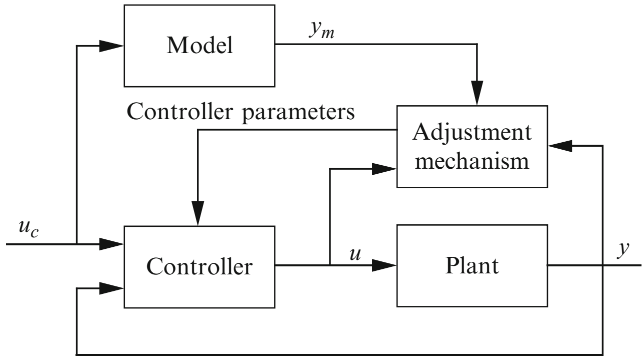
Draper and Li investigated the problem of operating aircraft engines optimally, and they developed a self-optimizing controller that would drive the system towards optimal working conditions. The system was successfully flight-tested (Draper and Li 1966) and initiated the field of *extremal control*.

Many of the ideas that emerged in the brave era inspired future research in adaptive control. The MRAC, the STR, and extremal control are typical examples.

Model Reference Adaptive Control (MRAC)

The MRAC was one idea from the early work on flight control that had a significant impact on adaptive control. A block diagram of a system with model reference adaptive control is shown in Fig. 1. The system has an ordinary feedback loop with a controller, having adjustable parameters, and the process. There is also a reference model which gives the ideal response y_m to the command signal y_m and a mechanism for adjusting the controller parameters θ. The parameter

H



History of Adaptive Control, Fig. 1 Block diagram of a feedback system with a model reference adaptive controller (MRAC)

adjustment is based on the process output y, the control signal u, and the output y_m of the reference model. Whitaker proposed the following rule for adjusting the parameters:

$$\frac{d\theta}{dt} = -\gamma e \frac{\partial e}{\partial \theta}, \tag{1}$$

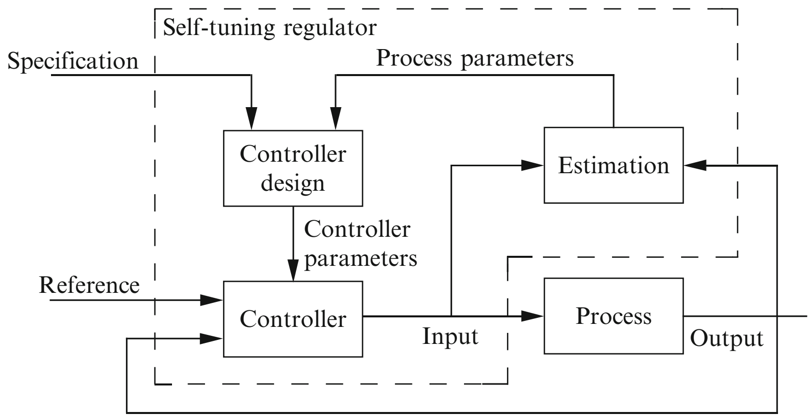
where $e = y - y_m$ and $\partial e/\partial\theta$ is the sensitivity derivative. Efficient ways to compute the sensitivity derivative were already available in sensitivity theory. The adaptation law (1) became known as the *MIT rule*.

Experiments and simulations of the model reference adaptive systems indicated that there could be problems with instability, in particular if the adaptation gain γ in Eq. (1) is large. This observation inspired much theoretical research. The goal was to replace the MIT rule by other parameter adjustment rules with guaranteed stability; the models used were non linear continuous time differential equations. The papers Butchart and Shackcloth (1965) and Parks (1966) demonstrated that control laws could be obtained using Lyapunov theory. When all state variables are measured, the adaptation laws obtained were similar to the MIT rule (1), but the sensitivity function was replaced by linear combinations of states and control variables. The problem was more difficult for systems that only permitted output feedback. Lyapunov theory could still be used if the process transfer function was strictly positive real, establishing a connection with Popov's hyper-stability theory (Landau 1979). The assumption of a positive real process is a severe restriction because such systems can be successfully controlled by high-gain feedback. The difficulty was finally resolved by using a scheme called *error augmentation* (Monopoli 1974; Morse 1980).

There was much research, and by the late 1980s, there was a relatively complete theory for MRAC and a large body of literature (Egardt 1979; Goodwin and Sin 1984; Anderson et al. 1986; Kumar and Varaiya 1986; Åström and Wittenmark 1989; Narendra and Annaswamy 1989; Sastry and Bodson 1989).The problem of flight control was, however, solved by using gain scheduling based on air data sensors and not by adaptive control (Stein 1980). The MRAC was also extended to nonlinear systems using *backstepping* (Krstić et al. 1993); Lyapunov stability and passivity were essential ingredients in developing the algorithm and analyzing its stability.

The Self-Tuning Regulator

The self-tuning regulator was inspired by steady-state regulation in process control. The mathematical setting was discrete time stochastic systems. A block diagram of a system with a self-tuning regulator is shown in Fig. 2. The system has an ordinary feedback loop with a controller and the process. There is an external loop for adjusting the controller parameters based on real-time parameter estimation and control design. There are many ways to estimate the process parameters and many ways to do the control design. Simple schemes do not take parameter uncertainty into account when computing the



History of Adaptive Control, Fig. 2 Block diagram of a feedback system with a self-tuning regulator (STR)

controller parameters invoking *the certainty equivalence principle*.

Single-input, single-output stochastic systems can be modeled by

$$
\begin{aligned}
y(t)+a_1 y(t-h) + \cdots a_n y(t-nh) = \\
b_1 u(t-h) + \cdots + b_n u(t-nh) + \\
c_1 w(t-h) + \cdots c_n w(t-nh) + e(t),
\end{aligned}
\qquad (2)
$$

where u is the control signal, y the process output, w a measured disturbance, and e a stochastic disturbance. Furthermore, h is the sampling period and a_k, b_k and c_k, are the parameters. Parameter estimation is typically done using least squares, and a control design that minimized the variance of the variations was well suited for regulation. A surprising result was that if the estimates converge, the limiting controller is a minimum variance controller even if the disturbance e is colored noise (Åström and Wittenmark 1973). Convergence conditions for the self-tuning regulator were given in Goodwin et al. (1980), and a very detailed analysis was presented in Guo and Chen (1991).

The problem of output feedback does not appear for the model (2) because the sequence of past inputs and outputs $y(t-h), \ldots, y(t-nh), u(t-h), \ldots, u(t-nh)$ is indeed a state, albeit not a minimal state representation. The continuous analogue would be to use derivatives of states and inputs which is not feasible because of measurement noise. The selection of the sampling period is however important.

Early industrial experience indicated that the ability of the STR to adapt feedforward gains was particularly useful, because feedforward control requires good models.

Insight from system identification showed that *excitation* is required to obtain good estimates. In the absence of excitation, a phenomenon of *bursting* could be observed. There could be epochs with small control actions due to insufficient excitation. The estimated parameters then drifted towards values close to or beyond the stability boundary generating large control axions. Good parameter estimates were then obtained and the system quickly recovered stability. The behavior then repeated in an irregular fashion. There are two ways to deal with the problem. One possibility is to detect when there is poor excitation and stop adaptation (Hägglund and Åström 2000). The other is to inject perturbations when there is poor excitation in the spirit of dual control.

Robustness and Unification

The model reference adaptive control and the self-tuning regulator originate from different application domains, flight control and process control. The differences are amplified because they are typically presented in different frameworks, continuous time for MRAC and discrete time for the STR. The schemes are, however, not too different. For a given process model and given design criterion the process model can often be re-parameterized in terms of controller parameters, and the STR is then equivalent to an MRAC. Similarly there are indirect MRAC where the process parameters are estimated (Egardt 1979).

A fundamental assumption made in the early analyses of model reference adaptive controllers was that the process model used for analysis had the same structure as the real process. Rohrs at MIT, which showed that systems with guaranteed convergence could be very sensitive to unmodeled dynamics, generated a good deal of research to explore robustness to unmodeled dynamics. Averaging theory, which is based on the observation that there are two loops in an adaptive system, a fast ordinary feedback and a slow parameter adjustment loop, turned out to be a key tool for understanding the behavior of adaptive systems. A large body of theory was generated and many books were written (Sastry and Bodson 1989; Ioannou and Sun 1995).

The theory resulted in several improvements of the adaptive algorithms. In the MIT rule (1) and similar adaptation laws derived from Lyapunov theory, the rate of change of the adaptation rate is a multiplication of the error e with other signals in the system. The adaptation rate may then become very large when signals are large. The analysis of robustness showed that there were advantages in avoiding large adaptation rates by *normalizing* the signals. The stability analysis also required that parameter estimates had to be bounded. To achieve this, parameters were *projected* on regions given by prior parameter bounds. The projection did, however, require prior process knowledge. The improved insight obtained from the robustness analysis is well described in the books Goodwin and Sin (1984), Egardt (1979), Åström and Wittenmark (1989), Narendra and Annaswamy (1989), Sastry and Bodson (1989), Anderson et al. (1986), and Ioannou and Sun (1995).

Applications

There were severe practical difficulties in implementing the early adaptive controllers using the analogue technology available in the brave era. Kalman used a hybrid computer when he attempted to implement his controller. There were dramatic improvements when mini- and microcomputers appeared in the 1970s. Since computers were still slow at the time, it was natural that most experimentats were executed in process control or ship steering which are slow processes. Advances in computing eliminated the technological barriers rapidly.

Self-oscillating adaptive controllers are used in several missiles. In piloted aircrafts there were complaints about the perturbation signals that were always exciting the system.

Self-tuning regulators have been used industrially since the early 1970s. Adaptive autopilots for ship steering were developed at the same time. They outperformed conventional autopilots based on PID control, because disturbances generated by waves were estimated and compensated for. These autopilots are still on the market (Northrop Grumman 2005). Asea (now ABB) developed a small distributed control system, Novatune, which had blocks for self-tuning regulators based on least-squares estimation, and minimum variance control. The company First Control, formed by members of the Novatune team, has delivered SCADA systems with adaptive control since 1985. The controllers are used for high-performance process control systems for pulp mills, paper machines, rolling mills, and pilot plants for chemical process control. The adaptive controllers are based on recursive estimation of a transfer function model and a control law based on pole placement. The controller also admits feedforward. The algorithm is provided with extensive safety logic, parameters are projected, and adaptation is interrupted when variations in measured signals and control signals are too small.

The most common industrial uses of adaptive techniques are automatic tuning of PID controllers. The techniques are used both in single loop controllers and in DCS systems. Many different techniques are used, pattern recognition as well as parameter estimation. The relay autotuning has proven very useful and has been shown to be very robust because it provides proper excitation of the process automatically. Some of the systems use automatic tuning to automatically generate gain schedules, and they also have adaptation of feedback and feedforward gains (Åström and Hägglund 2005).

Summary and Future Directions

Adaptive control has had turbulent history with alternating periods of optimism and pessimism. This history is reflected in the conferences. When the IEEE Conference on Decision and Control started in 1962, it included a Symposium on Adaptive Processes, which was discontinued after the 20th CDC in 1981. There were two IFAC symposia on the Theory of Self-Adaptive Control Systems, the first in Rome in 1962 and the second in Teddington in 1965 (Hammond 1966). The symposia were discontinued but reappeared when the Theory Committee of IFAC created a working group on adaptive control chaired by Prof. Landau in 1981. The group brought the communities of control and signal processing together, and a workshop on Adaptation and Learning in Signal Processing and Control (ALCOSP) was created. The first symposium was held in San Francisco in 1983 and the 11th in Caen in 2013.

Adaptive control can give significant benefits, it can deliver good performance over wide operating ranges, and commissioning of controllers can be simplified. Automatic tuning of PID controllers is now widely used in the process industry. Auto-tuning of more general controller is clearly of interest. Regulation performance is often characterized by the Harris index which compares actual performance with minimum variance control. Evaluation can be dispensed with by applying a self-tuning regulator.

There are adaptive controllers that have been in operation for more than 30 years, for example, in ship steering and rolling mills. There is a variety of products that use scheduling, MRAC, and STR in different ways. Automatic tuning is widely used; virtually all new single loop controllers have some form of automatic tuning. Automatic tuning is also used to build gain schedules semiautomatically. The techniques appear in tuning devices, in single loop controllers, in distributed systems for process control, and in controllers for special applications. There are strong similarities between adaptive filtering and adaptive control. Noise cancellation and adaptive equalization are widely spread uses of adaptation. The signal processing applications are a little easier to analyze because the systems do not have a feedback controller. New adaptive schemes are appearing. The $\mathcal{L}_1$ adaptive controller is one example. It inherits features of both the STR and the MRAC. The *model-free controller* by Fliess and Join (2013) is another example. It is similar to a continuous time version of the self-tuning regulator.

There is renewed interest in adaptive control in the aerospace industry, both for aircrafts and missiles (Lavretsky and Wise 2013). Good results in flight tests have been reported both using MRAC and the recently developed $\mathcal{L}_1$ adaptive controller (Hovakimyan and Cao 2010).

Adaptive control is a rich field, and to understand it well, it is necessary to know a wide range of techniques: nonlinear, stochastic, and sampled data systems, stability, robust control, and system identification.

In the early development of adaptive control, there was a dream of the universal adaptive controller that could be applied to any process with very little prior process knowledge. The insight gained by the robustness analysis shows that knowledge of bounds on the parameters is essential to ensure robustness. With the knowledge available today, adaptive controllers can be designed for particular applications. Design of proper safety nets is an important practical issue. One useful approach is to start with a basic constant-gain controller and provide adaptation as an add-on. This approach also simplifies design of supervision and safety networks.

There are still many unsolved research problems. Methods to determine the achievable adaptation rates are not known. Finding ways to provide proper excitation is another problem. The dual control formulation is very attractive because it automatically generates proper excitation when it is needed. The computations required to solve the Bellman equations are prohibitive, except in very simple cases. The self-oscillating adaptive system, which has been successfully applied to missiles, does provide excitation. The success of the relay auto-tuner for simple controllers indicates that it may be called in to provide excitation of adaptive controllers. Adaptive control can be an important component of the emerging autonomous system. One may expect that the current upswing in systems biology may provide more inspiration because many biological clearly have adaptive capabilities.

Cross-References

- ▶ Adaptive Control: Overview
- ▶ Autotuning
- ▶ Extremum Seeking Control
- ▶ Model Reference Adaptive Control
- ▶ PID Control

Bibliography

Anderson BDO, Bitmead RR, Johnson CR, Kokotović PV, Kosut RL, Mareels I, Praly L, Riedle B (1986) Stability of adaptive systems. MIT, Cambridge

Åström KJ, Hägglund T (2005) Advanced PID control. ISA – The Instrumentation, Systems, and Automation Society, Research Triangle Park

Åström KJ, Wittenmark B (1973) On self-tuning regulators. Automatica 9:185–199

Åström KJ, Wittenmark B (1989) Adaptive control. Addison-Wesley, Reading. Second 1994 edition reprinted by Dover 2006 edition

Bellman R (1961) Adaptive control processes—a guided tour. Princeton University Press, Princeton

Butchart RL, Shackcloth B (1965) Synthesis of model reference adaptive control systems by Lyapunov's second method. In: Proceedings 1965 IFAC symposium on adaptive control, Teddington

Draper CS, Li YT (1966) Principles of optimalizing control systems and an application to the internal combustion engine. In: Oldenburger R (ed) Optimal and self-optimizing control. MIT, Cambridge

Egardt B (1979) Stability of adaptive controllers. Springer, Berlin

Fliess M, Join C (2013) Model-free control.

Foxboro (1950) Control system with automatic response adjustment. US Patent 2,517,081

Gabor D, Wilby WPL, Woodcock R (1959) A universal non-linear filter, predictor and simulator which optimizes itself by a learning process. Proc Inst Electron Eng 108(Part B):1061–, 1959

Goodwin GC, Sin KS (eds) (1984) Adaptive filtering prediction and control. Prentice Hall, Englewood Cliffs

Goodwin GC, Ramadge PJ, Caines PE (1980) Discrete-time multivariable adaptive control. IEEE Trans Autom Control AC-25:449–456

Gregory PC (ed) (1959) Proceedings of the self adaptive flight control symposium. Wright Air Development Center, Wright-Patterson Air Force Base, Ohio

Guo L, Chen HF (1991) The Åström-Wittenmark's self-tuning regulator revisited and ELS-based adaptive trackers. IEEE Trans Autom Control 30(7):802–812

Hägglund T, Åström KJ (2000) Supervision of adaptive control algorithms. Automatica 36:1171–1180

Hammond PH (ed) (1966) Theory of self-adaptive control systems. In: Proceedings of the second IFAC symposium on the theory of self-adaptive control systems, 14–17 Sept, National Physical Laboratory, Teddington. Plenum, New York

Hebb DO (1949) The organization of behavior. Wiley, New York

Hovakimyan N, Chengyu Cao (2010) $\mathcal{L}_1$ adaptive control theory. SIAM, Philadelphia

Ioannou PA, Sun J (1995) Stable and robust adaptive control. Prentice-Hall, Englewood Cliffs

Kalman RE (1958) Design of a self-optimizing control system. Trans ASME 80:468–478

Krstić M, Kanellakopoulos I, Kokotović PV (1993) Nonlinear and adaptive control design. Prentice Hall, Englewood Cliffs

Kumar PR, Varaiya PP (1986) Stochastic systems: estimation, identification and adaptive control. Prentice-Hall, Englewood Cliffs

Landau ID (1979) Adaptive control—the model reference approach. Marcel Dekker, New York

Lavretsky E, Wise KA (2013) Robust and adaptive control with aerospace applications. Springer, London

Mishkin E, Braun L (1961) Adaptive control systems. McGraw-Hill, New York

Monopoli RV (1974) Model reference adaptive control with an augmented error signal. IEEE Trans Autom Control AC-19:474–484

Morse AS (1980) Global stability of parameter-adaptive control systems. IEEE Trans Autom Control AC-25:433–439

Narendra KS, Annaswamy AM (1989) Stable adaptive systems. Prentice Hall, Englewook Cliffs

Northrop Grumman (2005) SteerMaster. http://www.srhmar.com/brochures/as/SPERRY%20SteerMaster%20Control%20System.pdf

Parks PC (1966) Lyapunov redesign of model reference adaptive control systems. IEEE Trans Autom Control AC-11:362–367

Sastry S, Bodson M (1989) Adaptive control: stability, convergence and robustness. Prentice-Hall, New Jersey

Simon HA (1956) Dynamic programming under uncertainty with a quadratic criterion function. Econometrica 24:74–81

Stein G (1980) Adaptive flight control: a pragmatic view. In: Narendra KS, Monopoli RV (eds) Applications of adaptive control. Academic, New York
Tsypkin YaZ (1971) Adaptation and learning in automatic systems. Academic, New York
Widrow B (1962) Generalization and information storage in network of Adaline neurons. In: Yovits et al. (ed) Self-organizing systems. Spartan Books, Washington

Human Decision-Making in Multi-agent Systems

Ming Cao
Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands

Abstract

In order to avoid suboptimal collective behaviors and resolve social dilemmas, researchers have tried to understand how humans make decisions when interacting with other humans or smart machines and carried out theoretical and experimental studies aimed at influencing decision-making dynamics in large populations. We identify the key challenges and open issues in the related research, list a few popular models with the corresponding results, and point out future research directions.

Keywords

Decision making · Multi-agent systems

Introduction

More and more large-scale social, economic, biological, and technological complex systems have been analyzed using models of complex networks of interacting autonomous agents. Researchers are not only interested in using agent-based models to gain insight into how large dynamical networks evolve over time but also keen to introduce control actions into such networks to influence (and/or simulate) the collective behaviors of large populations. This includes social and economic networks, distributed smart energy grids, intelligent transportation systems, as well as mobile robot and smart sensor networks. However, in practice, such large numbers of interacting autonomous agents making decisions, in particular when humans are involved as participating agents, can result in highly complex, sometimes surprising, and often suboptimal, collective behaviors. It is for this very reason that human decision-making in large multi-agent networks has become a central topic for several research disciplines including economics, sociology, biology, psychology, philosophy, neuroscience, computer science, artificial intelligence, mathematics, robotics, electrical engineering, civil engineering, and last but not least systems and control.

One example to illustrate the complexity is the famous Braess paradox, observed more and more often recently in transportation, communication, and other types of flow networks, in which adding new links can actually worsen the performances of a network and vice versa (Steinberg and Zangwill 1983; Gisches and Rapoport 2012), when each agent decides to optimize its route based on its own local information. Another well-known example is the tragedy of the commons (Ostrom 2008), in which individuals use an excess of a certain shared-resource to maximize their own short-term benefit, leading to the depletion of the resource at a great long-term cost to the group. These considerations impact on a range of emerging engineering applications, e.g., smart transportation and autonomous robots, as well as many of the most pressing current societal concerns, including strategies for energy conservation, improving recycling programs, and reducing carbon emissions.

In order to resolve these types of behavioral and social dilemmas, researchers have tried to gain insight into how humans make decisions when interacting with other autonomous agents, smart machines, or humans and carried out theoretical and experimental studies aimed at influencing decision-making dynamics in large populations in the long run. In fact, control theorists and engineers have been working at the forefront, and a collection of related works

H

have appeared in the literature. For example, *Proceedings of the IEEE* published in 2012 a special issue on the topic of the decision-making interactions between humans and smart machines (Baillieul et al. 2012). After that, a range of new results have been reported in the literature. In what follows we first identify a few key challenges and difficulties of a number of open issues in the related research, then list a few of the most popular models and the corresponding results, and in the end point out some future research directions.

Challenges and Difficulties

Significantly, when smart machines gain increasing autonomy, empowered by recent breakthroughs in data-driven cognitive learning technologies, they interact with humans in the joint decision-making processes. Humans and smart machines, collectively as networks of autonomous agents, give rise to evolutionary dynamics (Sandholm 2010) that cannot be easily modeled, analyzed, and/or controlled using current systems and control theory that has been effective thus far for engineering practices over the past decades. To highlight the evolutionary nature of the collective decision dynamics, four features can be distinguished:

(i) Learning and adaption: Autonomous agents may learn and adapt in response to the collaboration or competition pressure from their local peers or an external changing environment.
(ii) Noise and stochastic effects: Random noise and stochastic deviation are unavoidable in both agents' decisions and the interactions among them.
(iii) Heterogeneity: Agents differ in their perception capabilities; the agents' interaction patterns co-evolve with the agents' dynamics both in space and time.
(iv) Availability for control: Some agents may not be available to be controlled directly, and even for those who are, the control is usually in the form of incentives that may only take effect in the long run or "nudge" the agent in the desired direction.

Therefore, there are still many open problems from the viewpoint of systems and control in developing a general framework for studying human decision-making in multi-agent systems; more generally, there has been an urgent need to develop new theoretical foundations together with computational and experimental tools to tackle the emerging challenging control problems associated with these evolutionary dynamics for networked autonomous agents.

Standard Models and Related Results

Models for human decision-making processes are numerous, and we list a few that are popular and becoming standard especially in the context of multi-agent systems.

Diffusion Model

The standard diffusion model was first proposed by cognitive neural scientists in the 1970s for simple sequential two-choice decision tasks and has been developed into the broader framework known as "evidence accumulation models" (Ratcliff 1978; Ratcliff et al. 2016). The internal process for a human to make a decision is taken to be a process of accumulating evidence from some external stimulus, and once the accumulated evidence is strong enough, a decision is made. More specifically, a one-dimensional drift-diffusion process can be described by the following stochastic differential equation:

$$dz = \alpha dt + \sigma dW, \qquad z(0) = 0, \qquad (1)$$

where z is the accumulated evidence in favor of a choice of interest, α is the drift rate representing the signal intensity of the stimulus acting on z, and σdW is a Wiener process with the standard deviation σ, called the diffusion rate, representing the effect of white noise. Roughly speaking, when $z(t)$ grows to reach a certain boundary level $\bar{z} > 0$ at time $\bar{t} > 0$, a decision is made at time $\bar{t}$ in favor of the choice that z corresponds to; otherwise, when another accumulator of some other choice reaches its boundary level first, then a decision favoring that choice is made. Such descriptions match the experimental data recording neural activities when human subjects make sequential decisions in lab environments (Ratcliff et al. 2016). Neural scientists have also used this model to look into the decision time and thus study the speed-accuracy trade-off in decision-making. So far, the model has been successfully applied to a range of cognitive decision-making tasks and used in clinical research (Ratcliff et al. 2016). Researchers from systems and control have looked into the convergence properties of this model (Cao et al. 2010; Woodruff et al. 2012) and used it to predict group decision-making dynamics (Stewart et al. 2012). New applications to robotic systems have also been reported (Reverdy et al. 2015).

Bayesian Model

The Bayesian model, or more generally Bayesian decision theory, is built upon the concepts of Bayesian statistics (Bernardo 1994); when making decisions, humans constantly update their estimates of the beliefs or preference values of different options using new observed inputs through Bayesian inference. It is particularly suitable for multi-alternative, multi-attribute decision-making where observations on different alternatives are dependent (Broder and Schiffer 2004; Evans et al. 2019). Let $p(A_i|x(0), x(1), \ldots, x(t))$ denote the posterior belief that alternative A_i is preferred given observations $x(j)$, $0 \leq j \leq t$ up to time $t \geq 0$. Then a Bayesian decision refers to the action at time t of choosing that alternative A_i among all i that maximizes the posterior probability p just given. Note that in some cases, the Bayesian model and diffusion model are closely related (Bitzer et al. 2014). The Bayesian model has found broad applications in dealing with neural and behavior data (Broder and Schiffer 2004; Evans et al. 2019).

Threshold Model

The threshold model is a classic model in sociology to study collective behavior (Granovetter 1978). It stipulates that an individual in a large population will engage in one of several possible behaviors only after a sufficiently large proportion of her surrounding individuals or the population at large have done so. Let the binary state $z_i(t)$ denote the decision of agent i in a large population at time t, $t = 0, 1, 2, \ldots$, and $\mathcal{N}_i(t)$ the set of other individuals whose decisions can be observed by agent i at time t. Then the linear threshold model dictates that

$$z_i(t+1) = \begin{cases} 1 \text{ if } \sum_{j \in \mathcal{N}_i(t)} z_j(t) \geq \Theta_i |\mathcal{N}_i(t)| \\ 0 \text{ otherwise,} \end{cases} \tag{2}$$

where $0 < \Theta_i < 1$ is called the *threshold* of agent i and $|\cdot|$ returns the size of the corresponding set. The model or its variants have been widely used to study propagation of beliefs and cascading of social norm violations in social networks (Centola et al. 2016; Mas and Opp 2016). Control theorists have looked into the convergence of dynamics of large populations of individuals whose behaviors follow the threshold model (Ramazi et al. 2016; Rossi et al. 2019).

H

Evolutionary Game Models

Game theory has been linked to decision-making processes from the day of its birth, and there are several classic textbooks covering the topic, e.g., Myerson (1991). As a branch of game theory, evolutionary game theory is relevant to decision-making in large multi-agent networks especially considering the evolution of proportions of populations making certain decisions over time (Sandholm 2010). It is worth mentioning that under certain conditions, the threshold model turns out to be equivalent to some evolutionary game models (Ramazi et al. 2016). Learning strategies, such as imitation and best response, can be naturally built into different evolutionary game models; the same also holds for network structures and stochastic effects. The notions of *evolutionarily stable strategies* and *stochastically stable strategies* play prominent roles in a variety of studies (Sandholm 2010). There have been reviews on the analysis and control of evolutionary game dynamics, and we refer the interested reader to Riehl et al. (2018).

Summary and Future Directions

Human decision-making in multi-agent systems remains a fascinating and challenging topic even after various models have been proposed, tested, and compared both theoretically and experimentally in the past few decades. There are several research directions that keep gaining momentum across different disciplines:

(i) Using dynamical system models to help match human decision-making behavioral data with neural activities in the brain and provides a physiological foundation for decision theories: Although we have listed the diffusion model and Bayesian model and mentioned a few related works, the gap between behavioral and neural findings is still significant, and there is still no widely accepted unified framework that can accommodate both.

(ii) Better embedding of human decision theory into the fast developing field of network science so that the network effects in terms of information flow and adaptive learning can be better utilized to understand how a human makes decisions within a group of autonomous agents: Threshold models and evolutionary game models are developments in this direction.

(iii) Influencing and even controlling the decision-making processes: This is still a new area that requires many more novel ideas and accompanying theoretical and empirical analysis. Behavioral and social dilemmas are common in practice, and difficult to prevent for the betterment of society. Systems and control theory can play a major rule in looking for innovative decision policies and intervention rules (Riehl et al. 2018) to steer a network of autonomous agents away from suboptimal collective behaviors and toward behaviors desirable for society.

Cross-References

- ▸ Controlling Collective Behavior in Complex Systems
- ▸ Cyber-physical-Human Systems
- ▸ Dynamical Social Networks
- ▸ Evolutionary Games
- ▸ Learning in Games
- ▸ Stochastic Games and Learning

Bibliography

Baillieul J, Leonard NE, Morgansen KA (2012) Interaction dynamics: the interface of humans and smart machines. Proc IEEE 100:567–570

Bernardo JM (1994) Bayesian theory. Wiley, Chichester

Bitzer S, Park H, Blankenburg F, Kiebel SJ (2014) Perceptual decision making: drift-diffusion model is equivalent to a Bayesian model. Front Hum Neurosci (102) 8:1–17

Broder A, Schiffer S (2004) Bayesian strategy assessment in multi-attribute decision-making. J Behav Decis Mak 16:193–213

Cao M, Stewart A, Leonard NE (2010) Convergence in human decision-making dynamics. Syst Control Lett 59:87–97

Centola D, Willer R, Macy M (2016) The emperor's dilemma: a computational model of self-enforcing norms. Am J Sociol 110(4):1009–1040

Evans NJ, Holmes WR, Trueblood JS (2019) Response-time data provide critical constraints on dynamic models of multi-alternative, multi-attribute choice. Psychon Bull Rev 26:193–213

Gisches EJ, Rapoport A (2012) Degrading network capacity may improve performance: private versus public monitoring in the Braess paradox. Theor Decis 73: 901–933

Granovetter M (1978) Threshold models of collective behavior. Am J Sociol 83:1420–1443

Mas M, Opp K-D (2016) When is ignorance bliss? Disclosing true information and cascades of norm violation in networks. Soc. Networks 47: 116–129

Myerson RB (1991) Game theory: analysis of conflict. Harvard University Press, Cambridge/London

Ostrom E (2008) Tragedy of the commons. The new palgrave dictionary of economics. Palgrave Macmillan, UK, pp 3573–3576

Ramazi P, Riehl J, Cao M (2016) Networks of conforming or nonconforming individuals tend to reach satisfactory decisions. Proc Natl Acad Sci 113(46):12985–12990

Ratcliff R (1978) A theory of memory retrieval. Psychol Rev 85:59–108

Ratcliff R, Smith PL, Brown SD, McKoon G (2016) Diffusion decision model: current issues and history. Trends Cogn Sci 20(4):260–281
Reverdy P, Llhan BD, Koditschek DE (2015) A drift-diffusion model for robotic obstacle avoidance. In: Proceedings of the 2015 IEEE/RSJ international conference on intelligent robots and systems (IROS), number 15666827. IEEE
Riehl J, Ramazi P, Cao M (2018) A survey on the analysis and control of evolutionary matrix games. Ann Rev Control 45:87–106
Rossi W, Como G, Fagnai F (2019) Threshold models of cascades in large-scale networks. IEEE Trans Netw Sci Eng 6:158–172
Sandholm WH (2010) Population games and evolutionary dynamics. MIT Press, Cambridge
Steinberg R, Zangwill WI (1983) The prevalence of Braess' paradox. Transp Sci 17:301–318
Stewart A, Cao M, Nedic A, Tomlin D, Leonard AE (2012) Towards human-robot teams: model-based analysis of human decision making in two-alternative choice tasks with social feedback. Proc IEEE 100(3):751–775
Woodruff C, Vu L, Morgansen KA, Tomlin D (2012) Deterministic modeling and evaluation of decision-making dynamics in sequential two-alternative forced choice tasks. Proc IEEE 100(3):734–750

Human-Building Interaction (HBI)

Burcin Becerik-Gerber
Sonny Astani Department of Civil and Environmental Engineering, Viterbi School of Engineering, University of Southern California, Los Angeles, CA, USA

Abstract

This entry defines and provides an overview of an emerging field, namely, human-building interaction, and identifies some of the key multidisciplinary research directions for this new field. It also provides examples of human-building interaction applications for providing personalized thermal comfort-driven systems, applications for allowing for adaptive light switch and blind control, and a framework for enabling social interactions between buildings and their occupants.

Keywords

Building automation · Intelligent built environments · Energy and comfort management · Trust in automation · Human in the loop control

Definition of Human-Building Interaction

Human-building interaction (HBI), *an emerging field*, represents the dynamic interplay of embodied human and building intelligence; it studies how humans interact, perceive, and navigate through the physical, spatial, and social aspects of the built environment and the artifacts (objects) within it. The field of HBI not only includes buildings but also other built environments people occupy and use, such as public spaces, trains, etc. A new field at the intersection of building science, human computer interaction (HCI), social science, and computer science, HBI entails reciprocal actions/influences of humans and buildings on each other and as a result the change of both building and user behavior. For example, user interactions with building elements, such as windows, fans, heaters, or blinds, would impact both the comfort of users (e.g., changes in lighting levels) and the building performance (e.g., increased solar gain causing cooling system to activate) (Langevin et al. 2016).

Built environments are a collection of artifacts (e.g., spaces, sensors, appliances, lighting, computers, windows, etc.); however, they also entail spatiotemporal experiences. Therefore, one important differentiator of HBI from HCI is the fact that the users are immersed in the physical environment they are interacting with, causing these interactions to have immediate user comfort, user satisfaction, and user acceptability implications. These interactions are also influenced by human behavior (D'Oca et al. 2018; Langevin et al. 2015) (user habits, preferences, requirements) as well as building behavior (usability, aesthetical or architectural quality, indoor environmental quality, and performance, such as safety, sustainability, and maintainability).

H

HBI Research Directions

Emergence of this new research area requires the collaboration of professionals from multiple disciplines, such as computer scientists, interaction designers, architects, civil engineers, mechanical engineers, and social scientists, to name a few. A summary of essential research directions for the future of the HBI field is provided here.

Intelligent Augmentation. With the unparalleled surge of technological advancements, our spaces (e.g., workplace) will change dramatically in the near future. Considering the variety in building spaces (e.g., offices, schools, post offices, warehouses, homes) and the amount of time we spend in these environments, a natural question is how will our spaces change as a result of intelligent environments? As part of HBI, our built environments should take into account both human-centered and building-centered needs and be "aware" of fatigue, stress levels, psychological states, emotions, and moods of users. Through different modes of HBIs, users can offload some of the tasks to the building, reducing cognitive workload of users and also empowering human users and augmenting their performance where and when needed.

Shared Autonomy. Symbiotic relationships between users and buildings will give birth to new forms of artificial intelligence (AI), which will give buildings an unprecedented autonomy and agency. This will require new kinds of caring relationships and trust between buildings and their users. There are several research questions, for example, how does trust change based on the type of tasks users perform in buildings; how does level of intelligence effect trust between buildings and their users; how should AI be designed for buildings that is easily understandable by building users who might not have any background in AI or similar fields? Moreover, when buildings become intelligent agents that can work for and with human users, the desired level of autonomy (e.g., who does what task and when) should be properly defined and also learned on a continuous basis (Ahmadi-Karvigh et al. 2017).

Mutual Adaptation. Buildings should be capable of continuously sensing the changes in the environment, different states and time, enabling context-aware sensing of human intent. Based on this acquired knowledge, both the building and the user could adapt to achieve shared goals, for example, energy efficiency by allowing the building to change set points (Ghahramani et al. 2014).

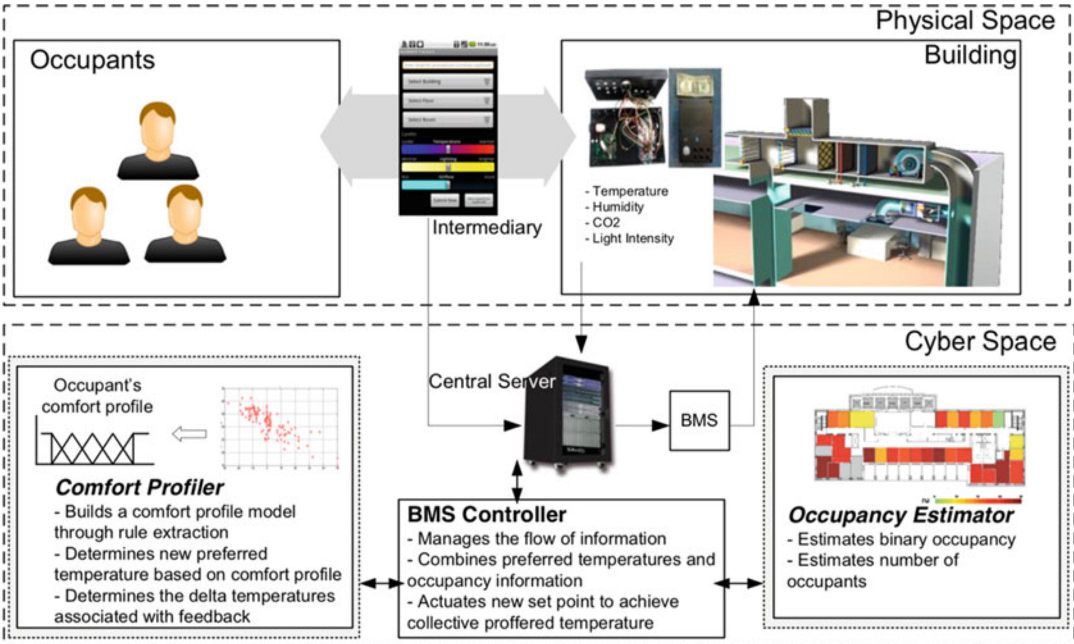
Security and Privacy. With the amount of data collected and stored both about the building and the user and the possibilities to actuate, buildings become hackable. What policies should govern the development and oversight of intelligent environments? What ethical and security measures must be anticipated and addressed? Who should govern the new codes and standards?

New Interfaces. Users and buildings create considerable amounts of data; how to mine and use this information for remembering and reusing of events and activities within a space becomes an interesting question. Moreover, there will exist a need for buildings and users to communicate. How to better design buildings to provide novel interaction opportunities to bi-directionally communicate with their users? What interaction types influence human perception of space and how? How does the style of interaction impact how we perceive our environments? How to enable a building's lifelong interaction with humans? How to adapt interfaces/modalities based on tasks and users?

Education. We need to rethink our curriculum to be able to educate future engineers who can address the complexities of HBIs and broaden our engineering objectives to include human objectives, incorporating the complexity and dynamism of built environments rather than treating them as unmanageable. In addition, we need to incorporate more computational thinking in our traditional engineering curricula. At the same time, we need to educate building users and building managers to ensure sustainable technology adoption.

HBI Applications and Examples

HBI Framework for Personalized Thermal Comfort-Driven Systems. Jazizadeh et al. (2014a) developed an HBI framework for

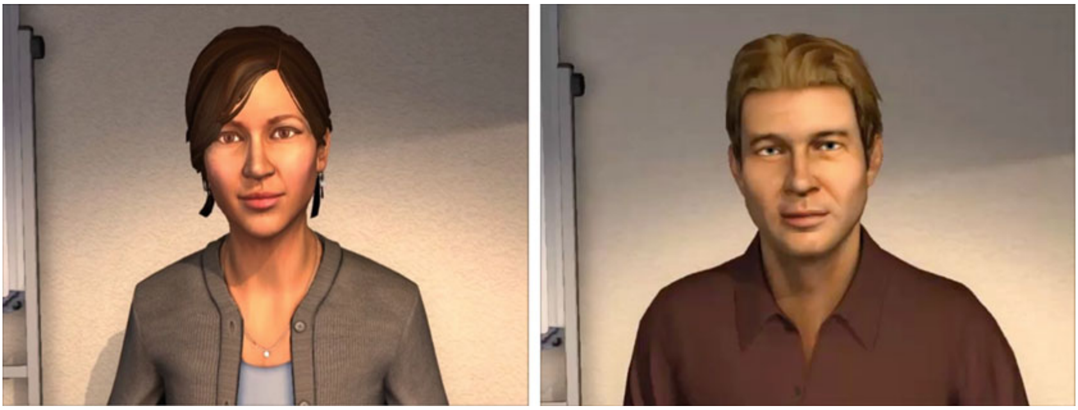


Human-Building Interaction (HBI), Fig. 1 Human-building interaction framework proposed in Jazizadeh et al. (2014a)

integrating building user's thermal comfort preferences into the HVAC control loop. To do so, they have developed an HBI interface, using participatory sensing via smartphones, and collected occupants' comfort perception indices (i.e., comfort votes provided by users and mapped to a numerical value) and ambient temperature data through a sensor network. They computed comfort profiles for each user using a fuzzy rule-based descriptive and predictive model. Implementation of the personalized comfort profiles in the HVAC system was tested through actuation via a proportional controller algorithm in a building. The study proved that the algorithm was capable of keeping the zone temperatures within the range of user's thermal comfort preferences. In a follow-up study (Jazizadeh et al. 2014b), the researchers further tested the framework and demonstrated comfort improvements along with a 39% reduction in daily average airflow when the occupant comfort preferences were used (Fig. 1).

HBI Framework for Adaptive Light Switch and Blind Control. Gunay et al. (2017) analyzed the light switch and blind use behavior of occupants in private offices and developed an adaptive control algorithm for lighting and blinds. Using the observed behavioral data (use of light switches and blinds), the algorithm learns the occupants' illuminance preferences, uses this information to determine photosensor setpoints, and switches off lighting and opens the blinds. The researchers tested the controller in private offices and shared office spaces and found that the solution could substantially reduce the lighting loads (around 25%) in office buildings without compromising the occupant's visual comfort. In another study, also focusing on personalized daylighting control for optimizing visual satisfaction and lighting energy use (Xiong et al. 2019), the researchers developed a personalized shading control framework. This optimization framework used personalized visual satisfaction utility functions and model-predicted lighting energy. Instead of assigning arbitrary weights to objectives, the researchers allowed users to make the final decisions in real time using a slider that balanced personal visual satisfaction and lighting energy use.

HBI Framework for Enabling Social Interactions Between Buildings and Their Users. Khashe et al. (2017) investigated the influence of direct



Human-Building Interaction (HBI), Fig. 2 Female and male avatar representing the building or building manager and communicate with the building occupants (Khashe et al. 2017)

communication between buildings and their users and tested the effectiveness of communicator styles (i.e., avatar, voice, and text) and communicator gender (i.e., female and male) and personification (communicator being either a building or building manager) on building occupant's compliance with pro-environmental requests. While the users were completing an office task (reading a passage), a request was presented to the user delivered through a combination of style, gender, and persona. An example of the request was *"If I open the blinds for you to have natural light, would you please dim or turn off the artificial lights?"* They found that the avatar was more effective than the voice and the voice was more effective than the text requests. Users complied more when the communicator was a female agent and when they thought the communicator was the building manager. In a follow-up study (Khashe et al. 2018), the researchers tested the effectiveness of social dialogue instead of a single request. They again tested two personas, building manager and building, however, this time only with female avatar communicator. The avatar started with a small talk, introduced herself (either as the building or building manager), told the user her name and asked the user's name, and so on, and then followed up with the request. They found that social dialogue persuaded the users to perform more pro-environmental actions. However, differently than the findings in their previous study, when the agent got engaged in a social dialogue with the users, they complied more with the requests if they thought the building was communicating with them. This two-tier study is one of the examples of personification of buildings through digital artifacts with the aim of building a relationship between an inanimate object (i.e., a building) and its users for achieving a shared goal (i.e., pro-environmentalism) (Fig. 2).

Cross-References

- ▶ Building Comfort and Environmental Control
- ▶ Building Control Systems
- ▶ Building Energy Management System
- ▶ Cyber-Physical-Human Systems

Bibliography

Ahmadi-Karvigh S, Becerik-Gerber B, Soibelman L (2017) One size does not fit all: understanding user preferences for building automation systems. J Energ Buildings 145:163–173

D'Oca S, Pisello AL, De Simone M, Bartherlmes V, Hong T, Corgnati S (2018) Human-building interaction at work: findings from an interdisciplinary cross-country survey in Italy. J Building Environ 132: 147–159

Ghahramani A, Jazizadeh F, Becerik-Gerber B (2014) A knowledge based approach for selecting energy- aware and comfort-driven HVAC temperature set points. J Energ Buildings 85:536–548

Gunay B, O-Brien W, Beausoleil-Morrison I, Gilani S (2017) Development and implementation of an adaptive lighting and blinds control algorithm. J Building Environ 113:185–199

Jazizadeh F, Ghahramani A, Becerik-Gerber B, Kichkaylo T, Orosz M (2014a) Human-building interaction framework for personalized thermal comfort driven systems in office buildings. ASCE J Comput Civil Eng 28(1):2–16. Special issue: Computational approaches to understand and reduce energy consumption in the built environment

Jazizadeh F, Ghahramani A, Becerik-Gerber B, Kichkaylo T, Orosz M (2014b) User-led decentralized thermal comfort driven HVAC operations for improved efficiency in office buildings. J Energ Buildings 70:398–410

Khashe S, Lucas G, Becerik-Gerber B, Gratch J (2017) Buildings with persona: towards effective building-occupant communication. J Comput Hum Behav 75:607–618

Khashe S, Lucas G, Becerik-Gerber B, Gratch J (2018) Establishing social dialogue between buildings and their users. Int J Comput Hum Interact. https://doi.org/10.1080/10447318.2018.1555346

Langevin J, Gurian P, Wen J (2015) Tracking the human-building interaction: a longitudinal field study of occupant behavior in air-conditioned offices. J Environ Psychol 42:94–115

Langevin J, Gurian P, Wen J (2016) Quantifying the human–building interaction: considering the active, adaptive occupant in building performance simulation. J Energ Buildings 117:372–386

Xiong J, Tzempelikos A, Bilionis I, Karava P (2019) A personalized daylighting control approach to dynamically optimize visual satisfaction and lighting energy use. J Energ Buildings 193:111–126

Hybrid Dynamical Systems, Feedback Control of

Ricardo G. Sanfelice
Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA, USA

Abstract

The control of systems with hybrid dynamics requires algorithms capable of dealing with the intricate combination of continuous and discrete behavior, which typically emerges from the presence of continuous processes, switching devices, and logic for control. Several analysis and design techniques have been proposed for the control of nonlinear continuous-time plants, but little is known about controlling plants that feature truly hybrid behavior. This short entry focuses on recent advances in the design of feedback control algorithms for hybrid dynamical systems. The focus is on hybrid feedback controllers that are systematically designed employing Lyapunov-based methods. The control design techniques summarized in this entry include control Lyapunov function-based control, passivity-based control, trajectory tracking control, safety, and temporal logic.

This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1150306, ECS-1710621, and CNS-1544396; by the Air Force Office of Scientific Research under Grant no. FA9550-12-1-0366, FA9550-16-1-0015, FA9550-19-1-0053, and FA9550-19-1-0169; and by CITRIS and the Banatao Institute at the University of California.

Keywords

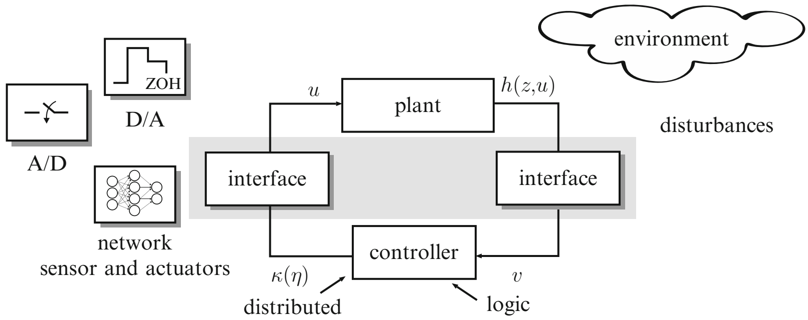
Feedback control · Hybrid control · Hybrid systems · Asymptotic stability

Introduction

A *hybrid control system* is a *feedback system* whose variables may flow and, at times, jump. Such a hybrid behavior can be present in one or more of the subsystems of the feedback system: in the system to control, i.e., *the plant*; in the algorithm used for control, i.e., *the controller*; or in the subsystems needed to interconnect the plant and the controller, i.e., *the interfaces/signal conditioners*. Figure 1 depicts a feedback system in closed-loop configuration with such subsystems under the presence of environmental disturbances. Due to its hybrid dynamics, a hybrid control system is a particular type of *hybrid dynamical system*.

Motivation

Hybrid dynamical systems are ubiquitous in science and engineering as they permit capturing the complex and intertwined continuous/discrete



Hybrid Dynamical Systems, Feedback Control of, Fig. 1 A hybrid control system: a feedback system with a plant, controller, and interfaces/signal conditioners (along with environmental disturbances) as subsystems featuring variables that flow and, at times, jump

behavior of a myriad of systems with variables that flow and jump. The recent popularity of feedback systems combining physical and software components demands tools for stability analysis and control design that can systematically handle such a complex combination. To avoid the issues due to approximating the dynamics of a system, in numerous settings, it is mandatory to keep the system dynamics as pure as possible and to be able to design feedback controllers that can cope with flow and jump behavior in the system.

Modeling Hybrid Control Systems

In this entry, hybrid control systems are represented in the framework of *hybrid equations/inclusions* for the study of hybrid dynamical systems. Within this framework, the continuous dynamics of the system are modeled using a differential equation/inclusion, while the discrete dynamics are captured by a difference equation/inclusion. A solution to such a system can *flow* over nontrivial intervals of time and *jump* at certain time instants. The conditions determining whether a solution to a hybrid system should flow or jump are captured by subsets of the state space and input space of the hybrid control system. In this way, a *plant* with hybrid dynamics can be modeled by the hybrid inclusion (This hybrid inclusion captures the dynamics of (constrained or unconstrained) continuous-time systems when $D_P = \emptyset$ and G_P is arbitrary. Similarly, it captures the dynamics of (constrained or unconstrained) discrete-time systems when $C_P = \emptyset$ and F_P is arbitrary. Note that while the output inclusion does not explicitly include a constraint on (z, u), the output map is only evaluated along solutions.)

$$\mathcal{H}_P \;:\; \begin{cases} \dot{z} \;\in\; F_P(z,u) & (z,u) \in C_P \\ z^+ \in\; G_P(z,u) & (z,u) \in D_P \\ y \;\;= h(z,u) \end{cases} \qquad (1)$$

where z is the *state* of the plant and takes values from the Euclidean space $\mathbb{R}^{n_P}$, u is the *input* and takes values from $\mathbb{R}^{m_P}$, y is the *output* and takes values from the output space $\mathbb{R}^{r_P}$, and (C_P, F_P, D_P, G_P, h) is the *data* of the hybrid system. The set C_P is the *flow set*, the set-valued map F_P is the *flow map*, the set D_P is the *jump set*, the set-valued map G_P is the *jump map*, and the single-valued map h is the *output map*. In (1), $\dot{z}$ denotes time derivative and z^+ denotes an instantaneous change in z.

Example 1 (controlled bouncing ball) Consider the juggling system consisting of a ball moving vertically and bouncing on a fixed horizontal surface. The surface, located at the origin of the line of motion, is equipped with a mechanical actuator that controls the speed of the ball resulting after impacts. From a physical viewpoint, control authority may be obtained varying the viscoelastic properties of the surface and, in turn, the coefficient of restitution of the surface. The position and the velocity of the ball are denoted as z_1 and z_2, respectively. Between bounces, the free motion of the ball is given by

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = -\gamma \tag{2}$$

where $\gamma > 0$ is the gravity constant. The conditions at which impacts occur are modeled as

$$z_1 = 0 \text{ and } z_2 \leq 0 \tag{3}$$

while the new value of the state variables after each impact is described by the difference equations

$$z_1^+ = z_1, \qquad z_2^+ = u \tag{4}$$

where u, which is larger than or equal to zero, is the controlled velocity after impacts, capturing the effect of the mechanism installed on the horizontal surface. In this way, the data (C_P, F_P, D_P, G_P, h) of the bouncing ball model is defined as follows:

$$F_P(z,u) := \begin{bmatrix} z_2 \\ -\gamma \end{bmatrix} \qquad \forall (z,u) \in C_P := \left\{ (z,u) \in \mathbb{R}^2 \times \mathbb{R} \;:\; z_1 \geq 0 \right\}$$

$$G_P(z,u) := \begin{bmatrix} z_1 \\ u \end{bmatrix} \qquad \forall (z,u) \in D_P := \left\{ (z,u) \in \mathbb{R}^2 \times \mathbb{R} \;:\; z_1 = 0, z_2 \leq 0 \right\}$$

and, when assuming that the state z is measured, the output map is $h(z,u) = z$. △

Other examples of hybrid plants whose dynamics can be captured by $\mathcal{H}_P$ in (1) include walking robots, network control systems, and spiking neurons. Systems with different modes of operation can also be modeled as $\mathcal{H}_P$ and, as the following example illustrates, can be captured by the constrained differential equation (or inclusion) part of $\mathcal{H}_P$. Such systems are not necessarily hybrid – at least as the term *hybrid* is used in this short entry – since they can be modeled by a dynamical system with state that only evolves continuously.

Example 2 (Thermostat system) The evolution of the temperature of a room under the effect of a heater can be modeled by a differential equation with constraints on its input. The temperature of the room is denoted by z and takes values from $\mathbb{R}$. The input is given by the pair $u = (u_1, u_2)$, where u_1 denotes whether the heater is turned on ($u_1 = 1$) or turned off ($u_1 = 0$) – these are the constraints on the inputs – while u_2 denotes the temperature outside the room, which can assume any value in $\mathbb{R}$. With these definitions, the evolution of the temperature z is governed by

$$\dot{z} = -z + \begin{bmatrix} z_\Delta & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$(z,u) \in C_P = \left\{ (z,u) \in \mathbb{R} \times \mathbb{R}^2 \;:\; u_1 \in \{0,1\} \right\} \tag{5}$$

where z_Δ is a positive constant representing the heater capacity. Note that C_P captures the constraint on the input u_1, which restricts it to the values 0 and 1. △

Given an input u, a *solution to a hybrid inclusion* is defined by a state trajectory ϕ that satisfies the inclusions. Both the input and the state trajectory are functions of $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} := [0, \infty) \times \{0, 1, 2, \ldots\}$, where t keeps track of the amount of flow while j counting the number of jumps of the solution. These functions are given by *hybrid arcs* and *hybrid inputs*, which are defined on *hybrid time domains*. More precisely, hybrid time domains are subsets E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T', J') \in E$,

$$E \cap \left([0, T'] \times \{0, 1, \ldots J'\} \right)$$

can be written in the form

$$\bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j \right)$$

for some finite sequence of times $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_J$, $J \in \mathbb{N}$. A hybrid arc ϕ is a function on a hybrid time domain. (The set $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ defines a compact hybrid time domain since it is bounded and closed.) The hybrid time domain of ϕ is denoted by $\operatorname{dom}\phi$. A hybrid arc is such that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous on intervals of flow $I^j := \{t : (t, j) \in \operatorname{dom}\phi\}$ with nonzero Lebesgue measure. A hybrid input u is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I^j.

In this way, a solution to the plant $\mathcal{H}_P$ is given by a pair (ϕ, u) with $\operatorname{dom}\phi = \operatorname{dom} u$ $(= \operatorname{dom}(\phi, u))$ satisfying

(S0) $(\phi(0, 0), u(0, 0)) \in \overline{C}_P$ *or* $(\phi(0, 0), u(0, 0)) \in D_P$, *and* $\operatorname{dom}\phi = \operatorname{dom} u$;

(S1) For each $j \in \mathbb{N}$ *such that* I^j *has nonempty interior int*(I^j)*, we have*

$$(\phi(t, j), u(t, j)) \in C_P \qquad \text{for all } t \in \mathrm{int}(I^j)$$

and

$$\frac{d}{dt}\phi(t, j) \in F_P(\phi(t, j), u(t, j))$$

$$\text{for almost all } \ t \in I^j$$

(S2) For each $(t, j) \in \operatorname{dom}(\phi, u)$ *such that* $(t, j + 1) \in \operatorname{dom}(\phi, u)$*, we have*

$$(\phi(t, j), u(t, j)) \in D_P$$

and

$$\phi(t, j + 1) \in G_P(\phi(t, j), u(t, j))$$

A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded and *maximal* if there does not exist another pair $(\phi, u)'$ such that (ϕ, u) is a truncation of $(\phi, u)'$ to some proper subset of $\operatorname{dom}(\phi, u)'$. A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *Zeno* if it is complete and the projection of $\operatorname{dom}(\phi, u)$ onto $\mathbb{R}_{\ge 0}$ is bounded.

On decomposition of inputs and outputs: At times, it is convenient to define inputs $u_c \in \mathbb{R}^{m_{P,c}}$ and $u_d \in \mathbb{R}^{m_{P,d}}$ collecting every component of the input u that affect flows and that affect jumps, respectively (Some of the components of u can be used to define both u_c and u_d, that is, there could be inputs that affect both flows and jumps.). Similarly, one can define y_c and y_d as the components of y that are measured during flows and jumps, respectively.

To control the hybrid plant $\mathcal{H}_P$ in (1), control algorithms that can cope with the nonlinearities introduced by the flow and jump equations/inclusions are required. In general, feedback controllers designed using classical techniques from the continuous-time and discrete-time domain fall short. Due to this limitation, hybrid feedback controllers would be more suitable for the control of plants with hybrid dynamics. Then, following the hybrid plant model above, hybrid controllers for the plant $\mathcal{H}_P$ in (1) will be given by the hybrid inclusion:

$$\mathcal{H}_K \; : \; \begin{cases} \dot{\eta} \;\in F_K(\eta, v) & (\eta, v) \in C_K \\ \eta^+ \in G_K(\eta, v) & (\eta, v) \in D_K \\ \zeta \;\;= \kappa(\eta, v) \end{cases} \tag{6}$$

where η is the *state* of the controller and takes values from the Euclidean space $\mathbb{R}^{n_K}$, v is the *input* and takes values from $\mathbb{R}^{r_P}$, ζ is the *output* and takes values from the output space $\mathbb{R}^{m_P}$, and $(C_K, F_K, D_K, G_K, \kappa)$ is the *data* of the hybrid inclusion defining the hybrid controller.

The control of $\mathcal{H}_P$ via $\mathcal{H}_K$ defines an interconnection through the input/output assignment $u = \zeta$ and $v = y$; the system in Fig. 1 without interfaces represents this interconnection. The resulting closed-loop system is a hybrid dynamical system given in terms of a hybrid inclusion/equation with state $x = (z, \eta)$. We will denote such a closed-loop system by $\mathcal{H}$, with data denoted (C, F, D, G), state $x \in \mathbb{R}^n$, and dynamics:

$$\mathcal{H} \; : \; \begin{cases} \dot{x} \;\in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{7}$$

Its data can be constructed from the data (C_P, F_P, D_P, G_P, h) and $(C_K, F_K, D_K, G_K, \kappa)$ of each of the subsystems. Solutions to both $\mathcal{H}_K$ and $\mathcal{H}$ are understood following the notion introduced above for $\mathcal{H}_P$.

Definitions and Notions

For convenience, we use the equivalent notation $[x^\top\ y^\top]^\top$ and (x, y) for vectors x and y. Also, we denote by $\mathcal{K}_\infty$ the class of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ that are continuous, zero at zero, strictly increasing, and unbounded.

In general, the dynamics of hybrid inclusions have right-hand sides given by set-valued maps. Unlike functions or single-valued maps, set-valued maps may return a set when evaluated at a point. For instance, at points in C_P, the set-valued flow map F_P of the hybrid plant $\mathcal{H}_P$ might return more than one value, allowing for different values of the derivative of z. A particular continuity property of set-valued maps that will be needed later is lower semicontinuity. A set-valued map S from $\mathbb{R}^n$ to $\mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where $\liminf_{x_i \to x} S(x_i) = \{z : \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i)\}$ is the so-called inner limit of S.

A vast majority of control problems consist of designing a feedback algorithm that assures that a function of the solutions to the plant approach a desired set-point condition *(attractivity)* and, when close to it, the solutions remain nearby *(stability)*. In some scenarios, the desired set-point condition is not necessarily an isolated point, but rather a set. The problem of designing a hybrid controller $\mathcal{H}_K$ for a hybrid plant $\mathcal{H}_P$ typically pertains to the stabilization of sets, in particular, due to the hybrid state of the controller including timers that persistently evolve within a bounded time interval and logic variables that take values from discrete sets. Denoting by $\mathcal{A}$ the set of points to stabilize for the closed-loop system $\mathcal{H}$ and $|\cdot|_\mathcal{A}$ as the distance to such set, the following property captures the typically desired properties outlined above. A closed set $\mathcal{A}$ is said to be

(S) *Stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_\mathcal{A} \leq \delta$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;

(pA) *Pre-attractive* if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_\mathcal{A} \leq \mu$ is bounded and if it is complete satisfies $\lim_{(t,j)\in \operatorname{dom}\phi,\ t+j\to\infty} |\phi(t,j)|_\mathcal{A} = 0$;

(FTpA) *Finite time pre-attractive* if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $|x_\circ|_\mathcal{A} \leq \mu$ is such that $|\phi(t,j)|_\mathcal{A} = 0$ for some $(t,j) \in \operatorname{dom}\phi$;

(pAS) *Pre-asymptotically stable* if it is stable and pre-attractive.

The basin of pre-attraction of a pre-asymptotically stable set $\mathcal{A}$ is the set of points from where the pre-attractivity property holds. The set $\mathcal{A}$ is said to be globally pre-asymptotically stable when the basin of pre-attraction is equal to the entire state space. Similarly, notions pertaining to finite-time stability and basin of attraction for finite-time stability can be defined. When every maximal solution is complete, then the prefix "pre" can be dropped since, in that case, the notions resemble those for continuous-time or discrete-time systems with solutions defined for all time.

At times, one is interested in asserting convergence when the state trajectory or the output remains in a set. For instance, a dynamical system (with assigned inputs) is said to be detectable when its output being held to zero implies that its state converges to the origin (or to a particular set of interest). A similar property can be defined for hybrid dynamical systems, for a general set K, which may not necessarily be the set of points at which the output is zero. For the closed-loop system $\mathcal{H}$, given sets $\mathcal{A}$ and K, the distance to $\mathcal{A}$ is said to be

(D) 0-*input detectable relative to* K if every complete solution ϕ to $\mathcal{H}$ is such that

$$\phi(t,j) \in K \qquad \forall (t,j) \in \operatorname{dom}\phi \qquad \Rightarrow \lim_{(t,j)\in \operatorname{dom}\phi,\ t+j\to\infty} |\phi(t,j)|_\mathcal{A} = 0$$

H

Note that "$\phi(t, j) \in K$" captures the "output being held to zero"-like property in the usual detectability notion.

In addition to stability, attractivity, and detectability, in this short note, we are interested in hybrid controllers that guarantee that a set K is forward invariant in the following sense:

(FpI) *Forward pre-invariant* if each maximal solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $x_\circ \in K$, satisfies $\phi(t, j) \in K$ for all $(t, j) \in \operatorname{dom} \phi$.

(FI) *Forward invariant* if each maximal solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x_\circ$, $x_\circ \in K$, is complete and satisfies $\phi(t, j) \in K$ for all $(t, j) \in \operatorname{dom} \phi$.

Feedback Control Design for Hybrid Dynamical Systems

Several methods for the design of a hybrid controller $\mathcal{H}_K$ rendering a given set $\mathcal{A}$ such that the properties defined in section "Definitions and Notions" are given below. At the core of these methods are sufficient conditions in terms of Lyapunov-like functions guaranteeing properties such as asymptotic stability, invariance, and finite-time attractivity of a set. Some of the methods presented below exploit such sufficient conditions when applied to the closed-loop system $\mathcal{H}$, while others exploit the properties of the hybrid plant to design controllers with a particular structure.

CLF-Based Control Design

In simple terms, a control Lyapunov function (CLF) is a regular enough scalar function that decreases along solutions to the system for some values of the unassigned input. When such a function exists, it is very tempting to exploit its properties to construct an asymptotically stabilizing control law. Following the ideas from the literature of continuous-time and discrete-time nonlinear systems, we define control Lyapunov functions for hybrid plants $\mathcal{H}_P$ and present results on CLF-based control design. For simplicity, as mentioned in the *input and output modeling remark* in section "Definitions and Notions", we use inputs u_c and u_d instead of u. Also, for simplicity, we restrict the discussion to sets $\mathcal{A}$ that are compact as well as hybrid plants with F_P, G_P single-valued and such that $h(z,u) = z$. For notational convenience, we use Π to denote the "projection" of C_P and D_P onto $\mathbb{R}^{n_P}$, i.e., $\Pi(C_P) = \{z : \exists u_c \text{ s.t. } (z, u_c) \in C_P\}$ and $\Pi(D_P) = \{z : \exists u_d \text{ s.t. } (z, u_d) \in D_P\}$, and the set-valued maps $\Psi_c(z) = \{u_c : (z, u_c) \in C_P\}$ and $\Psi_d(z) = \{u_d : (z, u_d) \in D_P\}$.

Given a compact set $\mathcal{A}$, a continuously differentiable function $V : \mathbb{R}^{n_P} \to \mathbb{R}$ is a *control Lyapunov function for $\mathcal{H}_P$ with respect to $\mathcal{A}$* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous, positive definite function ρ such that

$$\alpha_1(|z|_\mathcal{A}) \leq V(z) \leq \alpha_2(|z|_\mathcal{A}) \qquad \forall z \in \mathbb{R}^{n_P}$$

$$\inf_{u_c \in \Psi_c(z)} \langle \nabla V(z), F_P(z, u_c)\rangle \leq -\rho(|z|_\mathcal{A}) \qquad \forall z \in \Pi(C_P) \tag{8}$$

$$\inf_{u_d \in \Psi_d(z)} V(G_P(z, u_d)) - V(z) \leq -\rho(|z|_\mathcal{A}) \qquad \forall z \in \Pi(D_P) \tag{9}$$

With the availability of a CLF, the set $\mathcal{A}$ can be asymptotically stabilized if it is possible to synthesize a controller $\mathcal{H}_K$ from inequalities (8) and (9). Such a synthesis is feasible, in particular, for the special case of $\mathcal{H}_K$ being a static state-feedback law $z \mapsto \kappa(z)$. Sufficient conditions guaranteeing the existence of such a controller as well as a particular state-feedback law with point-wise minimum norm are given next.

Given a compact set $\mathcal{A}$ and a control Lyapunov function V (with respect to $\mathcal{A}$), define, for each $r \geq 0$, the set $\mathcal{I}(r) := \{z \in \mathbb{R}^{n_P} : V(z) \geq r\}$. Moreover, for each (z, u_c) and $r \geq 0$, define the function

$$\Gamma_c(z, u_c, r) := \begin{cases} \langle \nabla V(z), F_P(z, u_c) \rangle + \frac{1}{2}\rho(|z|_{\mathcal{A}}) & \text{if } (z, u_c) \in C_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_{P,c}}), \\ -\infty & \text{otherwise} \end{cases}$$

and, for each (z, u_d) and $r \geq 0$, the function

$$\Gamma_d(z, u_d, r) := \begin{cases} V(G_P(z, u_d)) - V(z) + \frac{1}{2}\rho(|z|_{\mathcal{A}}) & \text{if } (z, u_d) \in D_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_{P,d}}), \\ -\infty & \text{otherwise} \end{cases}$$

It can be shown that the following conditions involving V and the data (C_P, F_P, D_P, G_P, h) of $\mathcal{H}_P$ guarantee that, for each $r > 0$, there exists a state-feedback law

$$z \mapsto \kappa(z) = (\kappa_c(z), \kappa_d(z))$$

with κ_c continuous on $\Pi(C_P) \cap \mathcal{I}(r)$ and κ_d continuous on $\Pi(D_P) \cap \mathcal{I}(r)$ rendering the compact set

$$\mathcal{A}_r := \left\{ z \in \mathbb{R}^{n_P} \ : \ V(z) \leq r \right\}$$

pre-asymptotically stable for $\mathcal{H}_P$:

(CLF1) C_P and D_P are closed sets, and F_P and G_P are continuous;

(CLF2) The set-valued maps $\Psi_c(z) = \{u_c \ : \ (z, u_c) \in C_P\}$ and $\Psi_d(z) = \{u_d \ : \ (z, u_d) \in D_P\}$ are lower semicontinuous with convex values;

(CLF3) For every $r > 0$, we have that, for every $z \in \Pi(C_P) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(z, u_c, r)$ is convex on $\Psi_c(z)$ and that, for every $z \in \Pi(D_P) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_c(z, u_d, r)$ is convex on $\Psi_d(z)$;

In addition to guaranteeing the existence of a (continuous) state-feedback law practically pre-asymptotically stabilizing the set $\mathcal{A}$, these conditions also lead to the following natural definition of the feedback: for every $r > 0$, the state-feedback law pair

$$\kappa_c : \Pi(C_P) \to \mathbb{R}^{m_{P,c}}, \quad \kappa_d : \Pi(D_P) \to \mathbb{R}^{m_{P,d}}$$

can be defined on $\Pi(C_P)$ and $\Pi(D_P)$ as

$$\kappa_c(z) := \arg\min \{|u_c| \ : \ u_c \in \mathcal{T}_c(z)\} \qquad \forall z \in \Pi(C_P) \cap \mathcal{I}(r)$$
$$\kappa_d(z) := \arg\min \{|u_d| \ : \ u_d \in \mathcal{T}_d(z)\} \qquad \forall z \in \Pi(D_P) \cap \mathcal{I}(r)$$

where $\mathcal{T}_c(z) = \Psi_c(z) \cap \{u_c \ : \ \Gamma_c(z, u_c, V(z)) \leq 0\}$ and $\mathcal{T}_d(z) = \Psi_d(z) \cap \{u_d \ : \ \Gamma_d(z, u_d, V(z)) \leq 0\}$.

The stability property guaranteed by this feedback is also practical. Under further properties, similar results hold when the input u is not partitioned into u_c and u_d. To achieve asymptotic stability (or stabilizability) of $\mathcal{A}$ with a continuous state-feedback law, extra conditions are required to hold nearby the compact set, which for the case of stabilization of continuous-time systems are the so-called small control properties. Furthermore, the continuity of the feedback law assures that the closed-loop system has closed flow and jump sets as well as continuous flow and jump maps, which, in turn, due to the compactness of $\mathcal{A}$, implies that the asymptotic stability property is robust. Robustness follows from results for hybrid systems without inputs.

Example 3 (controlled bouncing ball revisited) For the juggling system in Example 1, the feedback law $u = \kappa_d \equiv 0$ asymptotically stabilizes its origin – note that for this system, $u_d = u$. In fact, with this feedback, after the first impact (or jump), every solution remains at the origin.

Passivity-Based Control Design

Dissipativity and its special case, passivity, provide a useful physical interpretation of a feedback control system as they characterize the exchange of energy between the plant and its controller. For an open system, passivity (in its very pure form) is the property that the energy stored in the system is no larger than the energy it has absorbed over a period of time. The energy stored in a system is given by the difference between the initial and final energy over a period of time, where the energy function is typically called the storage function. Hence, conveniently, passivity can be expressed in terms of the derivative of a storage function (i.e., the rate of change of the internal energy) and the product between inputs and outputs (i.e., the power flow of the system). Under further observability conditions, this power inequality can be employed as a design tool by selecting a control law that makes the rate of change of the internal energy negative. This method is called *passivity-based control design*.

The passivity-based control design method can be employed in the design of a controller for a "passive" hybrid plant $\mathcal{H}_P$, in which energy might be dissipated during flows, jumps, or both. Passivity notions and a passivity-based control design method for hybrid plants are given next. Since the form of the output of the plant plays a key role in asserting a passivity property, and this property may not necessarily hold both during flows and jumps, as mentioned in the *input and output modeling remark* in section "Definitions and Notions", we define outputs y_c and y_d, which, for simplicity, are assumed to be single-valued: $y_c = h_c(x)$ and $y_d = h_d(x)$. Moreover, we consider the case when the dimension of the space of the inputs u_c and u_d coincides with that of the outputs y_c and y_d, respectively, i.e., a "duality" of the output and input space.

Given a compact set $\mathcal{A}$ and functions h_c, h_d such that $h_c(\mathcal{A}) = h_d(\mathcal{A}) = 0$, a hybrid plant $\mathcal{H}_P$ for which there exists a continuously differentiable function $V : \mathbb{R}^{n_P} \to \mathbb{R}_{\geq 0}$ satisfying for some functions $\omega_c : \mathbb{R}^{m_{P,c}} \times \mathbb{R}^{n_P} \to \mathbb{R}$ and $\omega_d : \mathbb{R}^{m_{P,c}} \times \mathbb{R}^{n_P} \to \mathbb{R}$

$$\langle \nabla V(z), F_P(z, u_c) \rangle \leq \omega_c(u_c, z) \forall (z, u_c) \in C \tag{10}$$

$$V(G_P(z, u_d)) - V(z) \leq \omega_d(u_d, z) \forall (z, u_d) \in D \tag{11}$$

is said to be *passive with respect to a compact set* $\mathcal{A}$ if

$$(u_c, z) \mapsto \omega_c(u_c, z) = u_c^\top y_c \tag{12}$$

$$(u_d, z) \mapsto \omega_d(u_d, z) = u_d^\top y_d \tag{13}$$

The function V is the so-called storage function. If (10) holds with ω_c as in (12), and (11) holds with $\omega_d \equiv 0$, then the system is called *flow-passive*, i.e., the power inequality holds only during flows. If (10) holds with $\omega_c \equiv 0$, and (11) holds with ω_d as in (13), then the system is called *jump-passive*, i.e., the energy of the system decreases only during jumps.

Under additional detectability properties, these passivity notions can be used to design static output feedback controllers. In fact, given a hybrid plant $\mathcal{H}_P = (C_P, F_P, D_P, G_P, h)$ satisfying

(PBC1) C_P and D_P are closed sets; F_P and G_P are continuous; and h_c and h_d are continuous;

and a compact set $\mathcal{A}$, it can be shown that if $\mathcal{H}_P$ is flow-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite with respect to $\mathcal{A}$ and has compact sublevel sets, and if there exists a continuous function $\kappa_c : \mathbb{R}^{m_{P,c}} \to \mathbb{R}^{m_{P,c}}$, $y_c^\top \kappa_c(y_c) > 0$ for all $y_c \neq 0$, such that the resulting closed-loop system with $u_c = -\kappa_c(y_c)$ and $u_d \equiv 0$ has the following properties:

(PBC2) The distance to $\mathcal{A}$ is detectable relative to

$$\Big\{z \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P) : h_c(z)^\top \kappa_c(h_c(z)) = 0,\ (z, -\kappa_c(h_c(z))) \in C_P\Big\};$$

(PBC3) Every complete solution ϕ is such that, for some $\delta > 0$ and some $J \in \mathbb{N}$, we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$;

then the output-feedback law

$$u_c = -\kappa_c(y_c), \qquad u_d \equiv 0$$

renders $\mathcal{A}$ globally pre-asymptotically stable.

In a similar manner, an output-feedback law can be designed when, instead of being flow-passive, $\mathcal{H}_P$ is jump-passive with respect to $\mathcal{A}$. In this case, if the storage function V is positive definite with respect to $\mathcal{A}$ and has compact sublevel sets, and if there exists a continuous function $\kappa_d : \mathbb{R}^{m_{P,d}} \to \mathbb{R}^{m_{P,d}}$, $y_d^\top \kappa_d(y_d) > 0$ for all $y_d \neq 0$, such that the resulting closed-loop system with $u_c \equiv 0$ and $u_d = -\kappa_d(y_d)$ has the following properties:

(PBC4) The distance to $\mathcal{A}$ is detectable relative to

$$\left\{z \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P) : h_d(z)^\top \kappa_d(h_d(z)) = 0, (z, -\kappa_d(h_d(z))) \in D_P\right\};$$

(PBC5) Every complete solution ϕ is Zeno;

then the the output-feedback law

$$u_d = -\kappa_d(y_d), \qquad u_c \equiv 0$$

renders $\mathcal{A}$ globally pre-asymptotically stable. Such a feedback design can be employed to globally asymptotically stabilize the controlled bouncing ball in Example 1.

Strict passivity notions can also be formulated for hybrid plants, including the special cases where the power inequalities hold only during flows or jumps. In particular, strict passivity and output strict passivity can be employed to assert asymptotic stability with zero inputs.

Tracking Control Design

While numerous control problems pertain to the stabilization of a set-point condition, at times, it is desired to stabilize the solutions to the plant to a time-varying trajectory. In this section, we consider the problem of designing a hybrid controller $\mathcal{H}_K$ for a hybrid plant $\mathcal{H}_P$ to *track* a given reference trajectory r (a hybrid arc). The notion of tracking is introduced below. We propose sufficient conditions that general hybrid plants and controllers should satisfy to solve such a problem. For simplicity, we consider tracking of state trajectories and that the hybrid controller can measure both the state of the plant z and the reference trajectory r; hence, $v = (z, r)$.

The particular approach used here consists of recasting the tracking control problem as a set stabilization problem for the closed-loop system $\mathcal{H}$. To do this, we embed the reference trajectory r into an augmented hybrid model for which it is possible to define a set capturing the condition that the plant tracks the given reference trajectory. This set is referred to as *the tracking set*. More precisely, given a reference $r : \operatorname{dom} r \to \mathbb{R}^{n_P}$, we define the set $\mathcal{T}_r$ collecting all of the points (t, j) in the domain of r at which r jumps, that is, every point $(t_j^r, j) \in \operatorname{dom} r$ such that $(t_j^r, j+1) \in \operatorname{dom} r$. Then, the state of the closed loop $\mathcal{H}$ is augmented by the addition of states $\tau \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$. The dynamics of the states τ and k are such that τ counts elapsed flow time, while k counts the number of jumps of $\mathcal{H}$; hence, during flows $\dot{\tau} = 1$ and $\dot{k} = 0$, while at jumps $\tau^+ = \tau$ and $k^+ = k + 1$. These new states are used to parameterize the given reference trajectory r, which is employed in the definition of the tracking set:

$$\mathcal{A} = \left\{(z, \eta, \tau, k) \in \mathbb{R}^{n_P} \times \mathbb{R}^{n_K} \times \mathbb{R}_{\geq 0} \times \mathbb{N} : z = r(\tau, k),\ \eta \in \Phi_K \right\} \tag{14}$$

This set is the target set to be stabilized for $\mathcal{H}$. The set $\Phi_K \subset \mathbb{R}^{n_K}$ in the definition of $\mathcal{A}$ is some closed set capturing the set of points asymptotically approached by the state of the controller η.

Using results to certify pre-asymptotic stability of closed sets for hybrid systems, sufficient conditions guaranteeing that a hybrid controller $\mathcal{H}_K$ stabilizes the tracking set $\mathcal{A}$ in (14) for the hybrid closed-loop system $\mathcal{H}$ can be formulated. In particular, with $\mathcal{H}$ having state $x = (z, \eta, \tau, k)$ and data

H

$$C = \left\{x \ : \ (z, \kappa_c(\eta, z, r(\tau, k))) \in C_P, \tau \in [t_k^r, t_{k+1}^r], (\eta, z, r(\tau, k)) \in C_K \ \right\}$$
$$F(z, \eta, \tau, k) = (F_P(z, \kappa_c(\eta, z, r(\tau, k))), F_K(\eta, z, r(\tau, k)), 1, 0)$$
$$D = \{x \ : \ (z, \kappa_c(\eta, z, r(\tau, k))) \in D_P, (\tau, k) \in \mathcal{T}_r \ \} \cup \left\{x \ : \ \tau \in [t_k^r, t_{k+1}^r), (\eta, z, r(\tau, k)) \in D_K \ \right\}$$
$$G_1(z, \eta, \tau, k) = (G_P(z, \kappa_c(\eta, z, r(\tau, k))), \eta, \tau, k+1),$$
$$G_2(z, \eta, \tau, k) = (z, G_K(\eta, z, r(\tau, k)), \tau, k)$$

given a complete reference trajectory $r : \operatorname{dom} r \to \mathbb{R}^{n_P}$ and associated tracking set $\mathcal{A}$, a hybrid controller $\mathcal{H}_K$ with data $(C_K, F_K, D_K, G_K, \kappa)$ guaranteeing that

(T1) The jumps of r and $\mathcal{H}_P$ occur simultaneously;
(T2) For some continuously differentiable function $V : \mathbb{R}^{n_P} \times \mathbb{R}^{n_K} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$; and continuous, positive definite functions ρ_1, ρ_2, ρ_3, the following hold:

(T2a) For all $(z, \eta, \tau, k) \in C \cup D \cup G_1(D) \cup G_2(D)$

$$\alpha_1(|(z, \eta, \tau, k)|_{\mathcal{A}}) \leq V(z, \eta, \tau, k) \leq \alpha_2(|(z, \eta, \tau, k)|_{\mathcal{A}})$$

(T2b) For all $(z, \eta, \tau, k) \in C$ and all $\zeta \in F(z, \eta, \tau, k)$,

$$\langle \nabla V(z, \eta, \tau, k), \zeta \rangle \leq -\rho_1 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

(T2c) For all $(z, \eta, \tau, k) \in D_1$ and all $\zeta \in G_1(z, \eta, \tau, k)$

$$V(\zeta) - V(z, \eta, \tau, k) \leq -\rho_2 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

(T2d) For all $(z, \eta, \tau, k) \in D_2$ and all $\zeta \in G_2(z, \eta, \tau, k)$

$$V(\zeta) - V(z, \eta, \tau, k) \leq -\rho_3 \left(|(z, \eta, \tau, k)|_{\mathcal{A}}\right)$$

renders $\mathcal{A}$ is globally pre-asymptotically stable for $\mathcal{H}$.

Note that condition (T1) imposes that the jumps of the plant and of the reference trajectory occur simultaneously. Though restrictive, at times, this property can be enforced by proper design of the controller.

Forward Invariance-Based Control Design

As defined in section "Definitions and Notions", a set K is forward invariant if every solution to the system from K stays in K. Also known as *flow-invariance*, *positively invariance*, *viability*, or just *invariance*, this property is very important in feedback control design. In fact, asymptotically stabilizing feedback laws induce forward invariance of the set $\mathcal{A}$ that is asymptotically stabilized. Forward invariance is also key to guarantee safety properties, since safety can be typically recast as forward invariance of the set that excludes every point (and, for robustness, a neighborhood of it) for which the system is considered to be unsafe. In this section, forward invariance (or, equivalently, safety) is guaranteed by infinitesimal conditions that involve functions of the state known as barrier functions.

Given a hybrid closed-loop system $\mathcal{H} = (C, F, D, G)$, a function $B : \mathbb{R}^n \to \mathbb{R}$ is a *barrier function candidate* defining a set $K \subset C \cup D$ if

$$K = \{x \in C \cup D \ : \ B(x) \leq 0\} \qquad (15)$$

In some settings, the set K might be given a priori and then one would seek for a barrier function B such that (15) holds; namely, find a function B that is nonpositive at points in K only. In some other settings, one may generate the set K from the given sets C, D and the given function B.

A function candidate B is a *barrier function* if, in addition, its change along every solution ϕ to $\mathcal{H}$ that starts from K is such that

$$(t, j) \mapsto B(x(t, j))$$

is nonpositive. One way to guarantee such property is as follows. Given a hybrid system $\mathcal{H} = (C, F, D, G)$, suppose the barrier function candidate B defines a closed K as in (15). Furthermore, suppose B is continuously differentiable. Then, B is said to be a barrier function if, for some $\rho > 0$,

$$\langle \nabla B(x), \xi \rangle \leq 0 \quad \forall x \in ((K + \rho\mathbb{B}) \backslash K) \cap C, \quad \forall \xi \in F(x) \cap T_C(x) \tag{16}$$

$$B(\xi) \leq 0 \quad \forall \xi \in G(D \cap K) \tag{17}$$

$$G(D \cap K) \subset C \cup D \tag{18}$$

The barrier function notion introduced above for $\mathcal{H}$ can be formulated for a hybrid plant $\mathcal{H}_P$ given as in (1). In such a setting, since the input to $\mathcal{H}_P$ is not yet assigned, the conditions in (16), (17), and (18) would depend on the input – similar to the conditions that control Lyapunov functions in section "CLF-Based Control Design" have to satisfy. Such an extension is illustrated in the next example for the bouncing ball system, which, since the input only affects the jumps, conditions (17) and (18) become

$$\forall z \in \Pi(D_P)\ \exists u_d \ \text{ such that } \ (z, u_d) \in D_P \text{ and } \begin{cases} B(\xi) \leq 0 & \forall \xi \in G_P(z, u_d) \\ G_P(z, u_d) \subset \Pi(C_P) \cup \Pi(D_P) \end{cases} \tag{19}$$

Example 4 (controlled bouncing ball revisited) Consider the problem of keeping the total energy of the juggling system in Example 1 less than or equal to a constant $V^* \geq 0$. The total energy of the hybrid plant therein is given by

$$V(z) = \gamma z_1 + \frac{1}{2} z_2^2$$

Then, the desired set K to render invariant is defined by the continuously differentiable barrier candidate

$$B(z) := V(z) - V^* \qquad \forall z \in \mathbb{R}^2$$

In fact, for the case of a hybrid plant, the set K in (15) collects all points in $z \in \Pi(C_P) \cup \Pi(D_P)$ such that $V(z) \leq V^*$. It follows that

$$\langle \nabla B(z), F_P(z) \rangle = 0 \qquad \forall z \in \Pi(C_P)$$

since, during flows, the total energy remains constant. At jumps, the following hold – recall that $u_d = u$: for each $z \in \Pi(D_P) = \{z \in \mathbb{R}^2 : z_1 = 0, z_2 \leq 0\}$,

$$G_P(z, u) = \begin{bmatrix} 0 \\ u \end{bmatrix}$$

and

$$B(G_P(z, u)) = V(G_P(z, u)) - V^* = \gamma z_1 + \frac{1}{2} u^2 - V^* = \frac{1}{2} u^2 - V^*$$

Hence, for B to be a barrier function, for each $z \in \Pi(D_P)$, we pick u to satisfy

$$|u| \leq \sqrt{2V^*}$$

In this way, any state feedback $z \mapsto \kappa_d(z)$ such that $|\kappa(z)| \leq \sqrt{2V^*}$ for each $z \in \Pi(D_P)$ leads to a hybrid closed-loop system with K forward invariant (It is easy to show that every maximal solution to such a closed loop is complete.). Note that assigning u to a feedback that is positive (when possible) leads to solutions that, after a jump, flow for some time.

Temporal Logic

Design specifications for control design typically include requirements that go beyond asymptotic stability properties, such as finite-time properties and safety constraints, some of which need to be satisfied at specific times rather than in the limit. A framework suitable for handling such specifications is linear temporal logic (LTL). LTL permits the formulation of desired properties such as *safety*, or equivalently, "something bad never happens," and *liveness*, namely, "something good eventually happens" in finite time.

In LTL, a formula (or sentence) is given in terms of atomic propositions that are combined using Boolean and temporal operators. An atomic proposition is a function of the state that, for each possible value of the state, is either true or false. More precisely, for $\mathcal{H}$ in (7), a proposition $\mathfrak{a}$ is such that $\mathfrak{a}(x)$ is either `True` (1 or $\top$) or `False` (0 or $\bot$). Boolean operators include the following: $\neg$ is the *negation* operator; $\vee$ is the *disjuction* operator; $\wedge$ is the *conjunction* operator; $\Rightarrow$ is the *implication* operator; and $\Leftrightarrow$ is the *equivalence* operator. A way to reason about a solution defined over hybrid time is needed to introduce temporal operators. This is defined by the semantics of LTL, as follows.

Given a solution ϕ to $\mathcal{H}$, a proposition $\mathfrak{a}$ being `True` at $(t, j) \in \operatorname{dom} \phi$ is denoted by

$$\phi(t, j) \Vdash \mathfrak{a}$$

If $\mathfrak{a}$ is `False` at $(t, j) \in \operatorname{dom} \phi$, then we write

$$\phi(t, j) \nVdash \mathfrak{a}$$

Similarly, given an LTL formula $\mathfrak{f}$, we say that it is satisfied by ϕ at (t, j) if

$$(\phi, (t, j)) \vDash \mathfrak{f}$$

while $\mathfrak{f}$ not being satisfied at (t, j) is denoted by

$$(\phi, (t, j)) \nvDash \mathfrak{f}$$

The temporal operators are defined as follows: with $\mathfrak{a}$ and $\mathfrak{b}$ being two atomic propositions

- $\bigcirc$ is the *next* operator: $(\phi, (t, j)) \vDash \bigcirc\mathfrak{a}$ if and only if

$$(t, j+1) \in \operatorname{dom} \phi \text{ and } (\phi, (t, j+1)) \vDash \mathfrak{a}$$

- $\Diamond$ is the *eventually* operator: there exists $(t', j') \in \operatorname{dom} \phi$, $t' + j' \geq t + j$ such that $(\phi, (t', j')) \vDash \mathfrak{a}$
- $\Box$ is the *always* operator: $(\phi, (t, j)) \vDash \Box\mathfrak{a}$ if and only if, for each $t' + j' \geq t + j$, $(t', j') \in \operatorname{dom} \phi$,

$$(\phi, (t', j')) \vDash \mathfrak{a}$$

- $\mathcal{U}_s$ is the strong *until* operator: $(\phi, (t, j)) \vDash \mathfrak{a}\mathcal{U}_s\mathfrak{b}$ if and only if there exists $(t', j') \in \operatorname{dom} \phi$, $t' + j' \geq t + j$, such that

$$(\phi, (t', j')) \vDash \mathfrak{b}$$

and for all $(t'', j'') \in \operatorname{dom} \phi$ such that $t + j \leq t'' + j'' < t' + j'$,

$$(\phi, (t'', j'')) \vDash \mathfrak{a}$$

- $\mathcal{U}_w$ is the weak *until* operator: $(\phi, (t, j)) \vDash \mathfrak{a}\mathcal{U}_w\mathfrak{b}$ if and only if either

$$(\phi, (t', j')) \vDash \mathfrak{a}$$

for all $(t', j') \in \operatorname{dom} \phi$ such that $t' + j' \geq t + j$, or

$$(\phi, (t, j)) \vDash \mathfrak{a}\mathcal{U}_s\mathfrak{b}$$

Similar semantics apply to a formula $\mathfrak{f}$.

Example 5 (Thermostat system revisited) Consider the thermostat system in Example 2. Suppose that the goal is to keep the temperature within the range $[z_{\min}, z_{\max}]$ when the temperature starts in that region, and when it does not start from that range, steer it to that range in finite time and, after that, remain in that range for all time. For simplicity, suppose that the second input is constant and given by $u_2 \equiv z_{\text{out}}$, with $z_{\text{out}} \in (-\infty, z_{\max}]$ and $z_{\text{out}} + z_{\Delta} \in [z_{\min}, \infty)$. It can be shown that the following hybrid controller $\mathcal{H}_K$ accomplishes the state goal: with $\eta \in \{0, 1\}$, and with dynamics

$$\mathcal{H}_K \; : \; \begin{cases} \dot{\eta} \in F_K(\eta, v) := 0 & (\eta, v) \in C_K := (\{0\} \times C_{K,0}) \cup (\{1\} \times C_{K,1}) \\ \eta^+ \in G_K(\eta, v) := 1 - \eta & (\eta, v) \in D_K := (\{0\} \times D_{K,0}) \cup (\{1\} \times D_{K,1}) \\ \zeta \;= \kappa(\eta, v) := \eta \end{cases} \tag{20}$$

where

$$C_{K,0} := \{v : v \geq z_{\min}\},$$
$$C_{K,1} := \{v : v \leq z_{\max}\}$$
$$D_{K,0} := \{v : v \leq z_{\min}\},$$
$$D_{K,1}) := \{v : v \geq z_{\max}\}$$

The input of the controller is assigned via $v = z$ and its output assigned u via $u = \zeta = \eta$. Furthermore, it can be shown that the hybrid closed-loop system satisfies the following LTL formulae: with

$$\mathfrak{a}(z, \eta) = 1 \qquad \text{if } z \in [z_{\min}, z_{\max}],$$
$$\mathfrak{a}(z, \eta) = 0 \qquad \text{if } z \notin [z_{\min}, z_{\max}],$$

the following hold:

- $\square\mathfrak{a}$ for every solution with initial temperature in $[z_{\min}, z_{\max}]$, regardless of the initial value of η.
- $\diamond\mathfrak{a}$, $\diamond\square\mathfrak{a}$, and $\square\diamond\mathfrak{a}$ for every solution.

Sufficient conditions involving Lyapunov functions for finite-time attractivity and barrier functions can be employed to guarantee that certain formulas are satisfied. The following table provides pointers to such results (Table 1).

Hybrid Dynamical Systems, Feedback Control of, Table 1 Sufficient conditions for LTL formulas involving temporal operators $\bigcirc$, $\square$, $\diamond$, $\mathcal{U}_s$, and $\mathcal{U}_w$

$\mathfrak{f}$	Sufficient conditions in the literature
$\bigcirc\mathfrak{a}$	Properties of the data of $\mathcal{H}$ – (Han and Sanfelice 2018, Sections 4.3 and 5.3)
$\square\mathfrak{a}$	Forward invariance – Chai and Sanfelice (2019), Maghenem and Sanfelice (2018), and (Han and Sanfelice 2018, Section 5.1)
$\diamond\mathfrak{a}$	Finite-time attractivity – Li and Sanfelice (2019) and (Han and Sanfelice 2018, Section 5.2)
$\mathfrak{a}\mathcal{U}_s\mathfrak{b}$	Forward invariance and finite-time attractivity – (Han and Sanfelice 2018, Section 5.4)
$\mathfrak{a}\mathcal{U}_w\mathfrak{b}$	Forward invariance or finite-time attractivity – (Han and Sanfelice 2018, Section 5.4)

Summary and Future Directions

Advances over the last decade on modeling and robust stability of hybrid dynamical systems (without control inputs) have paved the road for the development of systematic methods for the design of control algorithms for hybrid plants. The results selected for this short expository entry, along with recent efforts on multi-mode/logic-based control, event-based control, and backstepping, which were not covered here, are scheduled to appear. Future research directions include the development of more powerful tracking control design methods, state observers, and optimal controllers for hybrid plants.

Cross-References

- ▶ Hybrid Model Predictive Control
- ▶ Hybrid Observers
- ▶ Modeling Hybrid Systems
- ▶ Simulation of Hybrid Dynamic Systems
- ▶ Stability Theory for Hybrid Dynamical Systems

Bibliography

Set-Valued Dynamics and Variational Analysis

Aubin J-P, Frankowska H (1990) Set-valued analysis. Birkhauser, Boston

Rockafellar RT, Wets RJ-B (1998) Variational analysis. Springer, Berlin/Heidelberg

Modeling and Stability

Branicky MS (2005) Introduction to hybrid systems. In: Hristu-Varsakelis, Dimitrios, Levine, William S. (Eds.) Handbook of networked and embedded control systems. Springer, Boston, pp 91–116

Goebel R, Sanfelice RG, Teel AR (2012) Hybrid dynamical systems: modeling, stability, and robustness. Princeton University Press, Princeton
Haddad WM, Chellaboina V, Nersesov SG (2006) Impulsive and hybrid dynamical systems: stability, dissipativity, and control. Princeton University Press, Princeton
Lygeros J, Johansson KH, Simić SN, Zhang J, Sastry SS (2003) Dynamical properties of hybrid automata. 48(1):2–17
van der Schaft A, Schumacher H (2000) An introduction to hybrid dynamical systems. Lecture notes in control and information sciences. Springer, Berlin

Control

Biemond JJB, van de Wouw N, Heemels WPMH, Nijmeijer H (2013) Tracking control for hybrid systems with state-triggered jumps. IEEE Trans Autom Control 58(4):876–890
Chai J, Sanfelice RG (2019) Forward invariance of sets for hybrid dynamical systems (part I). IEEE Trans Autom Control 64(6):2426–2441
Forni F, Teel AR, Zaccarian L (2013) Follow the bouncing ball: global results on tracking and state estimation with impacts. IEEE Trans Autom Control 58(6): 1470–1485
Han H, Sanfelice RG (2018) Linear temporal logic for hybrid dynamical systems: characterizations and sufficient conditions. In: Proceedings of the 6th analysis and design of hybrid systems, vol 51, pp 97–102, and ArXiv (https://arxiv.org/abs/1807.02574)
Li Y, Sanfelice RG (2019) Finite time stability of sets for hybrid dynamical systems. Automatica 100: 200–211
Lygeros J (2005) An overview of hybrid systems control. In: Handbook of networked and embedded control systems. Springer, pp 519–538
Maghenem M, Sanfelice RG (2018) Barrier function certificates for invariance in hybrid inclusions. In: Proceedings of the IEEE conference on decision and control, Dec 2018, pp 759–764
Naldi R, Sanfelice RG (2013) Passivity-based control for hybrid systems with applications to mechanical systems exhibiting impacts. Automatica 49(5):1104–1116
Sanfelice RG (2013a) On the existence of control Lyapunov functions and state-feedback laws for hybrid systems. IEEE Trans Autom Control 58(12):3242–3248
Sanfelice RG (2013b) Control of hybrid dynamical systems: an overview of recent advances. In: Daafouz J, Tarbouriech S, Sigalotti M (eds) Hybrid systems with constraints. Wiley, Hoboken, pp 146–177
Sanfelice RG, Biemond JJB, van de Wouw N, Heemels WPMH (2013) An embedding approach for the design of state-feedback tracking controllers for references with jumps. Int J Robust Nonlinear Control 24(11):1585–1608

Hybrid Model Predictive Control

Ricardo G. Sanfelice
Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA, USA

Abstract

Model predictive control is a powerful technique to solve problems with constraints and optimality requirements. Hybrid model predictive control further expands its capabilities by allowing for the combination of continuous-valued with discrete-valued states, and continuous dynamics with discrete dynamics or switching. This short article summarizes techniques available in the literature addressing different aspects of hybrid MPC, including those for discrete-time piecewise affine systems, discrete-time mixed logical dynamical systems, linear systems with periodic impulses, and hybrid dynamical systems.

Keywords

Model predictive control · Receding horizon control · Hybrid systems · Hybrid control · Optimization

Introduction

Model predictive control (MPC) ubiquitously incorporates mathematical models, constraints, and optimization schemes to solve control problems. Through the use of a mathematical model describing the evolution of the system to

This research has been partially supported by the National Science Foundation under Grant no. ECS-1710621 and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, Grant no. FA9550-19-1-0053, and Grant no. FA9550-19-1-0169, and by CITRIS and the Banatao Institute at the University of California.

control and the environment, MPC strategies determine the value of the control inputs to apply by solving an optimization problem over a finite time horizon. The system to control, which is known as the *plant*, is typically given by a state-space model with state, input, and outputs. Most MPC strategies **measure** the output of the plant and, using a mathematical model of the plant, **predict** the possible values of the state and the output in terms of free input variables to be determined. The prediction is performed over a *prediction horizon* which is defined as a finite time interval in the future. With such parameterized values of the state and input over the prediction horizon, MPC **solves** an optimization problem. To formulate the optimization control problem (OCP), a cost functional is to be defined. The OCP can also incorporate constraints, such as those on the state, the input, and the outputs of the plant. Then, MPC **selects** an input signal that solves the OCP and **applies** it to the plant for a finite amount of time, which is defined by the *control horizon*. After the input is applied over the control horizon, the process is repeated.

Over the past two decades, MPC has emerged as one of the preferred planning and control engines in robotics and control applications. Such grown interest in MPC has been propelled by recent advances in computing and optimization, which have led to significantly smaller computation times in the solution to certain optimization problems. However, the complexity of the dynamics, constraints, and control objectives in the MPC has increased so as to accommodate the specifics of the applications of interest. In particular, the dynamics of the system to control might substantially change over the region of operation, the constraints or the control objectives may abruptly change upon certain self-triggered or externally-triggered events, and the optimization problems may involve continuous-valued and discrete-valued variables. MPC problems with such features have been studied in the literature under the name *hybrid model predictive control* (Sanfelice 2018). Within a common framework and notation, this entry summarizes several hybrid MPC approaches available in the literature.

Notation and Preliminaries

Notation

The set of nonnegative integers is denoted as $\mathbb{N} := \{0, 1, 2 \ldots\}$ and the set of positive integers as $\mathbb{N}_{>0} := \{1, 2, \ldots\}$. Given $N \in \mathbb{N}_{>0}$, we define $\mathbb{N}_{<N} := \{0, 1, 2, \ldots, N-1\}$ and $\mathbb{N}_{\leq N} := \{0, 1, 2, \ldots, N\}$. The set of real numbers is denoted $\mathbb{R}$ and, given $n \in \mathbb{N}_{>0}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. The set of nonnegative real numbers is $\mathbb{R}_{\geq 0} := [0, \infty)$ and the set of positive real numbers is denoted as $\mathbb{R}_{>0} := (0, \infty)$. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm and given $p \in [1, \infty]$, $|x|_p$ denotes its p-norm.

Discrete-Time, Continuous-Time, and Hybrid Systems

A general nonlinear discrete-time system takes the form

$$x^+ = f(x, u) \tag{1}$$

$$y = h(x, u) \tag{2}$$

where x is the state, u the input, and y the output. The symbol $^+$ on x is to indicate, when a solution to the system is computed, the new value of the state after each discrete-time step. The parameter $k \in \mathbb{N}$ denotes discrete time. Given an input signal $\mathbb{N} \ni k \mapsto u(k)$, the solution to (1) is given by the function $\mathbb{N} \ni k \mapsto x(k)$ that satisfies

$$x(k+1) = f(x(k), u(k)) \qquad \forall k \in \mathbb{N}$$

Similarly, the output associated to this solution is given by

$$y(k) = h(x(k), u(k)) \qquad \forall k \in \mathbb{N}$$

A general nonlinear continuous-time system takes the form

$$\dot{x} = f(x, u) \tag{3}$$

H

$$y = h(x, u) \tag{4}$$

where x is the state, u the input, and y the output. The dot on x is to indicate the change of x with respect to ordinary time. The parameter t denotes continuous time. Given an input signal $\mathbb{R}_{\geq 0} \ni t \mapsto u(t)$, a solution to (3) is given by a function $\mathbb{R}_{\geq 0} \ni t \mapsto x(t)$ that is regular enough to satisfy

$$\frac{d}{dt}x(t) = f(x(t), u(t))$$

at least for almost all $t \in \mathbb{R}_{\geq 0}$. The output associated to this solution is given by

$$y(t) = h(x(t), u(t)) \qquad \forall t \in \mathbb{R}_{\geq 0}$$

A hybrid dynamical system takes the form

$$\mathcal{H} : \begin{cases} \dot{x} &= F(x, u) \quad (x, u) \in C \\ x^+ &= G(x, u) \quad (x, u) \in D \end{cases} \tag{5}$$

where x is the state and u the input. The set C defines the set of points in the state and input space from which flows are possible according to the differential equation $\dot{x} = F(x, u)$. The function F is called the flow map. The set D defines the set of points in the state and input space from where jumps are possible according to the difference equation $x^+ = G(x, u)$. The function G is called the jump map. Similar to the model for continuous-time systems in (3) and (4), the dot on x is to indicate the change of x with respect to ordinary time. Ordinary time is denoted by the parameter t, which takes values from $\mathbb{R}_{\geq 0}$. As a difference to the model for discrete-time systems in (1) and (2), the symbol $^+$ on x denotes the new value of the state after a jump is triggered. Every time that a jump occurs, the discrete counter j is incremented by one. A solution to the hybrid system can flow or jump, according to the values of the state and of the input relative to C and D, respectively. Solutions to $\mathcal{H}$ are parameterized by t and j, and are defined on hybrid time domains. The hybrid time domain of a solution x is denoted $\operatorname{dom} x$, which is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ and has the following structure: there exists a real nondecreasing sequence $\{t_j\}_{j=0}^{J}$ with $t_0 = 0$, and when J is finite, $t_{J+1} \in [t_J, \infty]$ such that $\operatorname{dom} x = \cup_{j=0}^{J} I_j \times \{j\}$. Here, $I_j = [t_j, t_{j+1}]$ for all $j < J$. If J is finite, I_{J+1} can take the form $[t_J, t_{J+1}]$ (when $t_{J+1} < \infty$), or possibly $[t_J, t_{J+1})$. It is convenient to consider the solution and the input as a pair (x, u) defined on the same hybrid time domain, namely, $\operatorname{dom}(x, u) = \operatorname{dom} x = \operatorname{dom} u$. Given an input signal $(t, j) \mapsto u(t, j)$, a function $(t, j) \mapsto x(t, j)$ is a solution to the hybrid system if, over intervals of flow, it is locally absolutely continuous and satisfies

$$\frac{d}{dt}x(t, j) = F(x(t, j), u(t, j))$$

when

$$(x(t, j), u(t, j)) \in C$$

and, at jump times, it satisfies

$$x(t, j+1) = G(x(t, j), u(t, j))$$

when

$$(x(t, j), u(t, j)) \in D$$

Hybrid Model Predictive Control Strategies

Most MPC strategies in the literature are for plants given in terms of discrete-time models of the form $x^+ = f(x, u)$, where x is the state and u is the input. In particular, the case of f being linear has been widely studied and documented ▶ Model Predictive Control in Practice. The cases in which the function f is discontinuous, or when either x or u have continuous-valued and discrete-value components require special treatment. In the literature, such cases are referred to as *hybrid*: either the system or the MPC strategy are said to be hybrid. Another use of the term hybrid is when the control algorithm obtained from MPC is nonsmooth, e.g., when the MPC strategy selects a control law from a family or when the MPC strategy is implemented using sample and hold. Finally, the term hybrid has been also employed in the literature to indicate that the plant exhibits state jumps at certain events. In such settings, the plant is modeled by the combination of continuous dynamics and discrete dynamics. In this section, after introducing basic definitions and notions, we present hybrid MPC strategies in the literature, starting from those that are for purely discrete-time systems (albeit with nonsmooth right-hand side) and ending at those for systems that are hybrid due to the combination of continuous and discrete dynamics.

MPC for Discrete-Time Piecewise Affine Systems

Piecewise affine (PWA) systems in discrete time take the form

$$x^+ = A_i x + B_i u + f_i \tag{6}$$

$$y = C_i x + D_i u \tag{7}$$

$$\text{subject to } x \in \Omega_i, u \in \mathcal{U}_i(x), i \in S \tag{8}$$

where x is the state, u the input, and y the output. The origin of (6), (7), and (8) is typically assumed to be an equilibrium state for zero input u. The set $S := \{1, 2, \ldots, \overline{s}\}$ with $\overline{s} \in \mathbb{N}_{>0}$ is a finite set to index the different values of the matrices and constraints. For each $i \in S$, the constant matrices $(A_i, B_i, f_i, C_i, D_i)$ define the dynamics of x and the output y. The elements f_i of the collection is such that $f_i = 0$ for all $i \in S$ such that $0 \in \Omega_i$. For each $i \in S$, the state constraints are determined by Ω_i and the state-dependent input constraints by $\mathcal{U}_i(x)$. The collection $\{\Omega_i\}_{i=1}^{\overline{s}}$ is a collection of polyhedra such that

$$\bigcup_{i \in S} \Omega_i = \mathcal{X}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is the region of operation of interest. Moreover, the polyhedra are typically such that their interiors do not intersect, namely,

$$\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset \qquad \forall i, j \in S \,:\, i \neq j$$

For each $x \in \Omega_i$, the set $\mathcal{U}_i(x)$ defines the allowed values for the input u.

For this class of systems, an MPC strategy consists of

1. At the current state of the plant, solve the OCP over a discrete prediction horizon;
2. Apply the optimal control input over a discrete control horizon;
3. Repeat.

The OCP associated to this strategy is as follows: given

- the current state x_0 of (6), (7), and (8),
- a prediction horizon $N \in \mathbb{N}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x, i, u) := \sum_{k=0}^{N-1} \mathcal{L}(x(k), i(k), u(k)) + \mathcal{F}(x(N))$$

over solutions $k \mapsto x(k)$, indices sequence $k \mapsto i(k)$, and inputs $k \mapsto u(k)$ subject to

H

$$x(0) = x_0 \tag{9}$$

$$x(N) \in \mathcal{X}_f \tag{10}$$

$$\forall k \in \mathbb{N}_{<N} : \quad x(k+1) = A_{i(k)} x(k) + B_{i(k)} u(k) + f_{i(k)} \tag{11}$$

$$\forall k \in \mathbb{N}_{\leq N} : \quad y(k) = C_{i(k)} x(k) + D_{i(k)} u(k) \tag{12}$$

$$x(k) \in \Omega_{i(k)} \tag{13}$$

$$u(k) \in \mathcal{U}_{i(k)}(x(k)) \tag{14}$$

$$i(k) \in S \tag{15}$$

A few observations about the OCP above are in order. The solution $k \mapsto x(k)$ is uniquely defined by x_0 and $k \mapsto (i(k), u(k))$. The condition in (9) imposes the initial condition for the solution $k \mapsto x(k)$, while (10) restricts the value of x at the end of the prediction horizon. The condition in (11) is to enforce that $k \mapsto x(k)$ is a solution for the input $k \mapsto u(k)$. Similarly, (12) imposes that an output is generated according to the dynamics of the PWA system in (6), (7), and (8). The conditions in (13) and (14) enforce the state and (state-dependent) input constraints associated with the PWA system. Finally, (15) forces the indexing signal $k \mapsto i(k)$ to belong to the finite set of possible indices S. Typical choices of the functions $\mathcal{L}$ and $\mathcal{F}$ in the cost functional $\mathcal{J}$ are

$$\mathcal{L}(x, i, u) := |Q_i x|_p + |R_i u|_p,$$
$$\mathcal{F}(x) := |Px|_p$$

for some $p \in [1, \infty]$, where, for each $i \in S$, Q_i and R_i and P are matrices of appropriate dimensions.

Further Reading

For key properties of the MPC problem for PWA systems in this section, the reader is referred to Lazar et al. (2006). In particular, sufficient conditions for recursive feasibility and asymptotic stability of the origin of the PWA system appeared in Lazar et al. (2006, Theorem III.2). In that reference, the reader can also find insight on how to select the terminal cost and the terminal constraint set; see Lazar et al. (2006, Section IV and Section V). The same reference also provides insight on how to solve the OCP using off-the-shelf tools. Very importantly, when $p = 1$ or $p = \infty$, the OCP can be rewritten as a mixed integer linear program (MILP). Furthermore, when the stage and terminal costs are quadratic, the OCP can be rewritten as a mixed integer quadratic program (MIQP).

MPC for Discrete-Time Systems with Continuous and Discrete-Valued States

Mixed Logical Dynamical (MLD) systems are discrete-time systems involving states, inputs, and outputs that have continuous-valued and discrete-valued components. MLD systems are given by

$$x^+ = Ax + B_1 u + B_2 \delta + B_3 z + B_4 \tag{16}$$

$$y = Cx + D_1 u + D_2 \delta + D_3 z + D_4 \tag{17}$$

$$\text{subject to } E_2 \delta + E_3 z \leq E_1 u + E_4 x + E_5 \tag{18}$$

where x is the state, u the input, y the output, z continuous-valued auxiliary variables, and δ discrete-valued auxiliary variables. The state, the input, and the output have continuous-valued and discrete-valued components; for example, certain components take values from Euclidean spaces while others from discrete sets, like $\{0, 1\}$. The partition of the state x is typically given by $x = (x_c, x_\ell)$, with x_c being the continuous-valued components and x_ℓ the discrete-valued components of x; similarly for the input $u = (u_c, u_\ell)$ and for the output $y = (y_c, y_\ell)$. The matrices A, $\{B_i\}_{i=1}^3$, B_4, C, $\{D_i\}_{i=1}^3$, and D_4 define the dynamics of x and the output y. The matrices $\{E_i\}_{i=1}^5$ are used in (18) to define constraints coupling the continuous-valued and discrete-valued states. The latter matrices can be properly defined to recast constraints into properties of discrete-valued states. For instance, when $x \in [-x_{\max}, x_{\max}]$ with $x_{\max} \geq 0$, the constraint $x \geq 0$ is equivalent to $\delta = 1$ (and, in turn, $x < 0$ is equivalent to $\delta = 0$) when, for some $\varepsilon > 0$, the following inequalities hold:

$$x_{\max}\delta \leq x + x_{\max}, \qquad -(x_{\max} + \varepsilon)\delta \leq -x - \varepsilon \tag{19}$$

This constraint can be written as (18) with $E_1 = 0$, $E_2 = \begin{bmatrix} x_{\max} & -(x_{\max} + \varepsilon) \end{bmatrix}^\top$, $E_3 = 0$, $E_4 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$, and $E_5 = \begin{bmatrix} x_{\max} & \varepsilon \end{bmatrix}^\top$.

For this class of systems, the MPC strategy in section "MPC for Discrete-Time Piecewise Affine Systems" is typically employed, and the OCP associated to it is as follows: given

- the current state x_0 of (16), (17), and (18),
- a prediction horizon $N \in \mathbb{N}_{>0}$,

- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x, z, \delta, u)$$

over solutions $k \mapsto x(k)$, auxiliary variables $k \mapsto z(k)$ and $k \mapsto \delta(k)$, and inputs $k \mapsto u(k)$ subject to

$$x(0) = x_0 \tag{20}$$

$$x(N) \in \mathcal{X}_f \tag{21}$$

$$\forall k \in \mathbb{N}_{<N}: \quad x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_4 \tag{22}$$

$$\forall k \in \mathbb{N}_{\leq N}: \quad y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_4 \tag{23}$$

$$E_2\delta(k) + E_3z(k) \leq E_1u(k) + E_4x(k) + E_5 \tag{24}$$

H

A particular choice of the cost functional $\mathcal{J}$ used in the MLD literature is

$$\mathcal{J}(x, z, \delta, u) = \sum_{k=0}^{N-1} \mathcal{L}(x(k), z(k), \delta(k), u(k)) + \mathcal{F}(x(N))$$

where the stage and terminal costs are given by

$$\mathcal{L}(x, z, \delta, u) := |Qx|_p + |Q_z z|_p + |Q_\delta \delta|_p + |Ru|_p,$$
$$\mathcal{F}(x) := |Px|_p$$

for some $p \in [1, \infty]$ and constant matrices Q, R, Q_δ, and Q_z of appropriate dimension.

Further Reading

Chapter 18 of Borrelli et al. (2017) provides a detailed presentation of MPC for MLD systems as in (16), (17), and (18). Following the ideas in recasting state constraints using discrete-valued auxiliary states as in (19), the MPC problem for MLD systems is formulated therein as mixed integer (linear and quadratic) problems. Previous articles about MLD systems include Bemporad et al. (2000a, b, 2002), Lazar et al. (2006), and Borrelli et al. (2017). Mixed-integer optimization methods were also used in Karaman et al. (2008) to solve feasibility and optimization problems for MLD systems with temporal logic specifications.

Periodic MPC for Continuous-Time Systems

MPC can be directly applied to plants given by continuous-time nonlinear systems modeled as in (3) and (4), without discretization of the dynamics. The price to pay is that the optimal control input has to be determined over a finite-time window of continuous time. An MPC strategy for such systems consists of:

1. At the current state of the plant, solve the OCP over a continuous-time prediction horizon;
2. Apply the optimal control input over a continuous-time control horizon;
3. Repeat.

The OCP to solve is as follows: given

- the current state x_0 of (3),
- a prediction horizon $T \in \mathbb{R}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$,
- a stage cost $\mathcal{L}$, and
- a terminal cost $\mathcal{F}$

minimize the cost functional

$$\mathcal{J}(x, u) := \int_0^T \mathcal{L}(x(t), u(t))dt + \mathcal{F}(x(T))$$

over solutions $t \mapsto x(t)$ and inputs $t \mapsto u(t)$ subject to

$$x(0) = x_0 \tag{25}$$

$$x(T) \in \mathcal{X}_f \tag{26}$$

$$\forall t \in (0, T): \quad \frac{d}{dt}x(t) = f(x(t), u(t)) \tag{27}$$

$$\forall t \in [0, T]: \quad y(t) = h(x(t), u(t)) \tag{28}$$

$$u(t) \in \mathcal{U} \tag{29}$$

Typically, the right-hand side f is assumed to be twice continuously differentiable to assure uniqueness of solutions. Furthermore, the zero state is typically assumed to be an equilibrium point for (3) with zero input. The input constraint set, which is denoted by $\mathcal{U}$, is commonly assumed to be compact and convex, and to have the property that the origin belongs to its interior.

Further Reading

One of the first articles on MPC for continuous-time systems is Mayne and Michalska (1990). A reference that is more closely related to the MPC algorithm presented in this section is Chen and Allgöwer (1998). The approach in this article is to select the terminal constraint set $\mathcal{X}_f$ as a forward invariant neighborhood of the origin. The invariance-inducing feedback law is taken to be linear. In Chen and Allgöwer (1998), a method to design the feedback, the terminal constraint set, and the terminal cost is provided. Due to their design approach leading to a cost functional that upper bounds the cost of the associated infinite horizon control problem, the MPC strategy in Chen and Allgöwer (1998) is called quasi-infinite horizon nonlinear MPC. While the work in Chen and Allgöwer (1998) allows for general piecewise continuous inputs as candidates for the optimal control, the particular case studied in Magni and Scattolini (2004) considers only piecewise-constant functions that change values periodically. The algorithm in the latter reference results in a type of sample-and-hold MPC strategy. Similar ideas were employed in Nešić and Grüne (2006) for the purposes of finding a sampled version of a continuous-time controller via MPC.

MPC for Linear Systems with Periodic Impulses

Linear systems with periodic impulses on the state are given by

$$\dot{x}(t) = Ax(t) \qquad \forall t \in (k\delta, (k+1)\delta] \tag{30}$$

$$x(t^+) = x(t) + Bu_k \qquad \forall t = k\delta \tag{31}$$

for each $k \in \mathbb{N}$, where $t \mapsto x(t)$ is a left-continuous function and, for each impulse time, namely, each $t \in \{0, \delta, 2\delta, \ldots\}$, $x(t^+)$ is the right limit of $x(t)$. Similarly to the MPC strategies presented in the earlier sections, an output can be attached to the model in (30) and (31). The constant $\delta > 0$ is the sampling period, and u_k is the constant input applied at the impulse time $t = k\delta$. As shown in the literature, the sequence of constant inputs $\{u_k\}_{k\in\mathbb{N}}$ can be determined using the following MPC strategy: for each $k \in \mathbb{N}$,

1. At the current state of the plant at $t = k\delta$, solve the OCP over a prediction horizon including $N-1$ future impulse times;
2. Apply the first entry of the optimal input to the plant until the next impulse time.

The OCP to solve is as follows: given

- the current state x_0 of (30) and (31),
- a prediction horizon $N \in \mathbb{N}_{>0}$,
- a terminal constraint set $\mathcal{X}_f$, and
- a stage cost $\mathcal{L}$

minimize the cost functional

$$\mathcal{J}(x, u) = \sum_{k=0}^{N-1} \mathcal{L}(x(\tau_k), u(\tau_k))$$

over the value of the solutions and inputs at the impulse times, which are denoted as $x(k)$ and $u(k)$, respectively, subject to

$$
\begin{aligned}
& x(0) = x_0 && && (32)\\
& x(N\delta) \in \mathcal{X}_f && && (33)\\
\forall k \in \mathbb{N}_{<N}: \quad & \dot{x}(t) = Ax(t) && \forall t \in (k\delta, (k+1)\delta] && (34)\\
& x(t^+) = x(t) + Bu(t) && \forall t = k\delta && (35)\\
& x(t) \in \mathcal{X} && \forall t \in [k\delta, (k+1)\delta] && (36)\\
& u(\tau_k) \in \mathcal{U} && && (37)
\end{aligned}
$$

The cost functional given above is a particular choice used in the literature, which does not include a terminal cost. The set $\mathcal{X}$ denotes the state constraints and the set $\mathcal{U}$ the input constraints.

Further Reading

The formulation above is based on Sopasakis et al. (2015). The developments therein further exploit the periodicity of the setting and employ the fact that the solutions to the system in (30) and (31) can be directly obtained at the impulse times from the computation of the solutions to the discrete-time system $x^+ = \exp(A\delta)(x + Bu)$. In addition, Sopasakis et al. (2015) employs over approximation techniques with the goal of reducing the number of constraints associated with the OCP above. For this purpose, polytopic over-approximations of the continuous dynamics of the impulsive system are proposed therein so as to arrive at a convex quadratic program. It should be pointed out that the stability concept used in Sopasakis et al. (2015) is weak in the sense that closeness and convergence of the values of the solution are only required to occur at the impulse times. Due to the combination of features of impulsive systems and of sample-data systems, the MPC strategy in Sopasakis et al. (2015) is one of the MPC approaches found in the literature that is closest to hybrid dynamical systems, as introduced in the next section.

MPC for Hybrid Dynamical Systems

Hybrid dynamical systems are systems with state variables that are allowed to evolve continuously and, at times, jump. A mathematical model of a hybrid system, denoted $\mathcal{H}$, was given in (5). This model is general enough to capture the main features of other hybrid system modeling formalisms. For the purposes of MPC, it should be noted that the flow set C and the jump set D in (5) can already accommodate state and input constraints. Furthermore, compared to the MPC strategies for discrete-time systems and for continuous-time systems introduced earlier, the fact that hybrid systems have solution pairs (x, u) defined on a hybrid time domain $\mathrm{dom}(x, u)$, which is parameterized by ordinary time t and jump time j, requires a prediction horizon that includes both prediction of ordinary time and of jump times. One such prediction horizon is given by the set

$$
\mathcal{T} := \left\{ (t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \ : \ \max\{t/\delta, j\} = \tau \right\} \tag{38}
$$

for a positive integer τ and some positive constant δ. These parameters determine the "length" of $\mathcal{T}$ and the step size during flow for prediction. With this construction, the OCP to solve in an MPC strategy for hybrid systems will require that the terminal time, which now will be given by a pair (T, J), satisfies $\max\{T/\delta, J\} = \tau$. A hybrid control horizon, parameterized by a positive integer $\tau_c < \tau$ and positive constant $\delta_c < \delta$, with the same structure as the prediction horizon is defined.

With such a horizon structure, an MPC strategy for a plant modeled as a hybrid dynamical system $\mathcal{H}$ is as follows:

(1) At the current state of the plant, solve the OCP over the hybrid prediction horizon;
(2) Apply the optimal hybrid input to the plant over the hybrid control horizon;
(3) Repeat.

The OCP to solve as part of this MPC strategy is as follows: given

- the current state x_0 of (5),
- prediction horizon parameters $\tau \in \mathbb{N}_{>0}$ and $\delta > 0$,
- control horizon parameters $\tau_c \in \mathbb{N}_{<\tau} \setminus \{0\}$ and $\delta_c \in (0, \delta)$,
- a terminal constraint set $\mathcal{X}_f$,
- stage costs $\mathcal{L}_C$ and $\mathcal{L}_D$, and
- a terminal cost $\mathcal{F}$

the MPC problem consists of minimizing the cost functional

$$\mathcal{J}(x,u) := \left(\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} \mathcal{L}_C(x(t,j), u(t,j))\, dt \right) + \left(\sum_{j=0}^{J-1} \mathcal{L}_D(x(t_{j+1}, j), u(t_{j+1}, j)) \right) + \mathcal{F}(x(T,J))$$

over solution pairs $(t,j) \mapsto (x(t,j), u(t,j))$ with hybrid time domain that is a compact subset of $[0,T] \times \mathbb{N}_{\leq J}$, subject to

$$x(0,0) = x_0 \tag{39}$$

$$(T,J) \in \mathcal{T} \tag{40}$$

$$x(T,J) \in \mathcal{X}_f \tag{41}$$

$$\forall j \in \mathbb{N} \text{ with } \operatorname{int}(I_j) \neq \emptyset : \quad \frac{d}{dt}x(t,j) = F(x(t,j), u(t,j)) \quad (x(t,j), u(t,j)) \in C \quad \text{for almost all } t \in I_j, \tag{42}$$

$$\forall (t,j) \in \operatorname{dom} x \text{ s.t. } (t, j+1) \in \operatorname{dom} x : \quad x(t, j+1) = G(x(t,j), u(t,j)) \quad (x(t,j), u(t,j)) \in D \tag{43}$$

where (T, J) denotes the terminal time of the solution pair (x, u) and $\{t_j\}_{j=0}^{J+1}$ is the sequence defining $\operatorname{dom}(x, u)$, with $t_{J+1} = T$. The function $\mathcal{L}_C$ defines the cost of flowing on the flow set C and the function $\mathcal{L}_D$ defined the cost of jumping from the jump set.

Further Reading

For more details about the MPC strategy for hybrid dynamical systems presented in this section, the reader is referred to Altin et al. (2018) and Altin and Sanfelice (2019). The case when the hybrid dynamics are discretized is treated in Ojaghi et al. (2019). It should be noted that when the flow and jump sets overlap, the OCP may have multiple optimal inputs. Nonuniqueness of optimal inputs already appears in nonlinear MPC (see, e.g., in Grimm et al. 2007; Rawlings and Mayne 2009; Borrelli et al. 2017; Mayne and Michalska 1990; Chen and Allgöwer 1997; Magni and Scattolini 2004), but the source of nonuniqueness of solutions for the case of hybrid dynamical systems is conceptually different. For these systems, conditions guaranteeing that the value function, which at every point x_0 is given by $\mathcal{J}^\star(x_0) := \mathcal{J}(x_\star, u_\star)$ with $(x_\star, u_\star)$ being minimizers of $\mathcal{J}$ from x_0, implies an asymptotic stability property for the plant are given in Altin and Sanfelice (2019). The same reference also highlights key properties of the feasibility set, recursive feasibility, monotonicity properties of the cost functional, and regularity properties of the value function.

Cross-References

▶ Model Predictive Control for Power Networks
▶ Model Predictive Control in Practice
▶ Modeling Hybrid Systems
▶ Robust Model Predictive Control

▶ Simulation of Hybrid Dynamic Systems
▶ Stability Theory for Hybrid Dynamical Systems

Bibliography

Altin B, Sanfelice RG (2019) Asymptotically stabilizing model predictive control for hybrid dynamical systems. In: Proceedings of the American Control Conference

Altin B, Ojaghi P, Sanfelice RG (2018) A model predictive control framework for hybrid systems. In: NMPC, vol 51, pp 128–133, Aug 2018

Bemporad A, Borrelli F, Morari M (2000a) Optimal controllers for hybrid systems: stability and piecewise linear explicit form. In: Proceedings of the 39th IEEE conference on decision and control, vol 2. IEEE, pp 1810–1815

Bemporad A, Borrelli F, Morari M (2000b) Piecewise linear optimal controllers for hybrid systems. In: Proceedings of the 2000 American control conference, vol 2, pp 1190–1194. IEEE

Bemporad A, Heemels WMHP, De Schutter B (2002) On hybrid systems and closed-loop MPC systems. IEEE Trans Autom Control 47(5):863–869

Borrelli F, Bemporad A, Morari M (2017) Predictive control for linear and hybrid systems. Cambridge University Press, Cambridge/New York

Chen H, Allgöwer F (1997) A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. In: European control conference. IEEE, pp 1421–1426

Chen H, Allgöwer F (1998) A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. Automatica 34(10):1205–1217

Grimm G, Messina MJ, Tuna SE, Teel AR (2007) Nominally robust model predictive control with state constraints. IEEE Trans Autom Control 52(10):1856–1870

Karaman S, Sanfelice RG, Frazzoli E (2008) Optimal control of mixed logical dynamical systems with linear temporal logic specifications. In: Proceedings of 47th IEEE conference on decision and control, NULL, pp 2117–2122

Lazar M, Heemels WMPH, Bemporad A (2006) Stabilizing model predictive control of hybrid systems. IEEE Trans. Autom. Control 51(11):1813–1818

Magni L, Scattolini R (2004) Model predictive control of continuous-time nonlinear systems with piecewise constant control. IEEE Trans. Autom. Control 49(6):900–906

Mayne DQ, Michalska H (1990) Receding horizon control of nonlinear systems. IEEE Trans. Autom. Control 35(7):814–824

Nešić D, Grüne L (2006) A receding horizon control approach to sampled-data implementation of continuous-time controllers. Syst. Control Lett. 55(8):660–672

Ojaghi P, Altin B, Sanfelice RG (2019) A model predictive control framework for asymptotic stabilization of discretized hybrid dynamical systems. In: To appear at the IEEE conference on decision and control

Rawlings JB, Mayne DQ (2009) Model predictive control: theory and design. Nob Hill Pub., Madison

Sanfelice RG (2018) Hybrid predictive control, 1 edn. Birkhäuser, Basel, pp 199–220

Sopasakis P, Patrinos P, Sarimveis H, Bemporad A (2015) Model predictive control for linear impulsive systems. IEEE Trans Autom Control 60(8):2277–2282

Hybrid Observers

Daniele Carnevale
Dipartimento di Ing. Civile ed Ing. Informatica, Università di Roma "Tor Vergata", Roma, Italy

Abstract

In first part two hybrid observer designs for non-hybrid systems are presented. In the second part, recently results available in the literature related to the observability and observer design for different classes of hybrid systems are introduced.

Keywords

Hybrid systems · Observer design · Observability · Switching systems

Introduction

Observers' design, which are used to estimate the unmeasured plant state, has received a lot of attention since the late 1960s. One of the first leading contributions to clearly formalize the estimation problem and propose a solution in the linear case has been introduced by Luenberger (1966). The recipe to implement a Luenberger-type observer for linear time-invariant (LTI) continuous-time systems described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times p}$ has three main ingredients: system data (A, B, C, D), the correction term commonly referred to as *output injection*, and the *observability/detectability/determinability* conditions. A Luenberger-type observer for (1) is then a copy of the system dynamics with the output injection term $\mathbf{L}(\mathbf{y} - \hat{\mathbf{y}})$ as

$$\dot{\hat{x}} = A\hat{x} + Bu + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}}), \quad \hat{y} = C\hat{x} + Du, \quad (2)$$

with $\mathbf{L} \in \mathbb{R}^{n\times m}$ and where $\hat{x}$ is the estimated value of x. Certainly if $\hat{x}(0) = x(0)$ and the input to the plant $u(t)$ is known, then $\hat{x}(t) = x(t)$ for all $t \geq 0$. For all nontrivial cases, when $x(0) \neq \hat{x}(0)$, the estimation error defined as $e = x - \hat{x}$ satisfies the differential equation $\dot{e} = (A - LC)e$, with initial condition $e(0) = x(0) - \hat{x}(0)$. The time evolution of the estimation error $e(t)$ depends only on the properties of the matrix $A + \mathbf{L}C$, i.e., if the pair (A, C) is observable (equivalent to determinability for continuous-time systems), there exists a unique $\mathbf{L}$ to assign arbitrarily the eigenvalues of $A + \mathbf{L}C$ with negative real part to render the origin $e = 0$ of the estimation error system globally asymptotically stable (GAS). If the pair (A, C) is detectable, there exists $\mathbf{L}$ such that all the eigenvalues of $A + \mathbf{L}C$ still have negative real parts ensuring that the origin $e = 0$ is GAS, although all the eigenvalues cannot be arbitrarily chosen. The observer (2) exploits only the injection term $\mathbf{L}$ in continuous time to render Hurwitz the matrix $A + \mathbf{L}C$, but one may ask how profitable could be to allow jumps of the observer state designing an observer that has both continuous (flow map) and discrete (jump map) time dynamics, i.e., an *hybrid observer*. On the other way around, there are systems that are intrinsically described by hybrid dynamics and for which it seems natural to directly provide hybrid observers. Designs of hybrid observer is a relatively new area of research, and results are consolidated only for few classes of systems.

In section "Continuous-Time Plants," a hybrid redesign of the observer (2) is discussed first, and then a more general design for nonlinear systems is introduced, whereas in section "Systems with Flows and Jumps," the recent results related to observability and observer designs for hybrid systems are discussed. Conclusions are given in section "Summary and Future Directions."

Hybrid Observers: Different Strategies

The community of researchers working on hybrid observer, which is a quite recent area and is the subject of growing interest, is wide, and a unique formal definition/notation has not been reached yet. This fact is strictly related to the large number of different hybrid system models that are currently adopted by researchers. To render as simple as possible this short presentation, we let the state $x(t)$ of a hybrid system be driven by the flow map (differential equation) when $t \neq t_j$ and by the jump map (difference equation) when $t = t_j$ is the jump time and $x(t)$ is right continuous, i.e., $\lim_{t\to t_j^+} x(t) = x(t_j)$.

Continuous-Time Plants

Linear Case

A simple strategy to improve convergence to zero of the estimation error for (1) has been proposed in Raff and Allgower (2008) and consists in resetting the observer state $\hat{x}$, at predetermined fixed jump times t_j, by means of the linear correction term $\mathbf{K}(\mathbf{t})\left(\mathbf{y}(\mathbf{t}) - \mathbf{C}\hat{\mathbf{x}}(\mathbf{t})\right)$ as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \mathbf{L}\left(\mathbf{y}(\mathbf{t}) - \mathbf{C}\hat{\mathbf{x}}(\mathbf{t})\right), \quad (3a)$$

$$\hat{x}(t_j) = \hat{x}\left(t_j^-\right) + \mathbf{K}\left(\mathbf{t_j^-}\right)\left(\mathbf{y}\left(\mathbf{t_j^-}\right) - \mathbf{C}\hat{\mathbf{x}}\left(\mathbf{t_j^-}\right)\right), \quad (3b)$$

where $t_0 = 0,\ t_{j+1} - t_j = T > 0,\ j \in \mathbb{N}_{\geq 1}$ and T is a parameter that defines the interval times between resets and has to be chosen such that

$$Im\left(\lambda_p - \lambda_r\right) T \neq 2r\pi, \quad r \in \mathbb{Z}\backslash\{0\}, \quad (4)$$

for each pair (λ_p, λ_r) of complex eigenvalues of the matrix $A - LC$. This selection preserves the (continuous time or flow) observability of the system (1) when sampled at time instants t_j. Then, the estimation error $e(t)$ converges to zero in finite time, $t = nT$, if (1) is observable and the matrix $K(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times m}$ is selected such as $K(t) = K_0$ if $t \leq t_n$ and $K(t) = 0$ otherwise where K_0 is such that $(I - K_0 C) \exp((A - LC) T)$ has all its eigenvalues at zero. It is important to note that the state reset (3b) yields a hybrid estimation error system given by

$$\dot{e}(t) = (A - LC) e(t), \tag{5a}$$

$$e(t_j) = \left(I - K\left(t_j^-\right) C\right) e\left(t_j^-\right). \tag{5b}$$

The stability property of the origin can be easily deduced by noting that

$$e(t_j) = \prod_{k=1}^{j} \left(I - K\left(t_j^-\right) C\right) \exp((A - LC) T) e(0),$$

and given that $(I - K_0 C) \exp((A - LC) T)$ is nilpotent, then $e(t_n) = e(nT) = 0$.

Along the same line, to improve convergence performances of continuous-time observer, in Prieur et al. (2012), it is proposed a methodology to limit the well-known *peaking phenomena* affecting the *high-gain observers* (Tornambé 1992; Khalil and Praly 2013) opportunely resetting its (augmented) state. Moreover, when the output of (1) is a nonlinear function of the state, $y = h(x)$, with $h(\cdot)$ not invertible (e.g., the saturation function), it would be possible to rewrite (1) as a hybrid system with linear flow map and augmented state designing a hybrid observer as in Carnevale and Astolfi (2009). In Possieri and Teel (2016), an introduction to structural properties for estimation and output feedback stabilization of linear hybrid systems is given.

Nonlinear Case

When the input of a continuous-time plant is piecewise-constant, the hybrid observer proposed in Moraal and Grizzle (1995), exploiting sampled measurements, can be successfully applied for a class of nonlinear continuous (or discrete-time) systems

$$\dot{x} = f(x(t), u(t)), \; y(t) = h(x(t), u(t)), \tag{6}$$

with sufficiently smooth maps $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ and where

$$x(t_j) = F(x(t_{j-1}), u(t_{j-1})), \tag{7}$$

is the sample data (discrete-time) version of (6) with sampling time $T = t_{j-1} - t_j$. Then, it is possible to define a hybrid observer of the following type:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)), \tag{8a}$$

$$\hat{x}(t_j) = \Gamma\left(y\left(t_j^-\right), \hat{x}\left(t_j^-\right), \xi\left(t_j^-\right)\right), \tag{8b}$$

where the reset map Γ and the dynamics of the new variable $\xi(t)$ have to be properly defined. The main idea in Moraal and Grizzle (1995) is that the Newton method, in continuous and discrete-time, can be used to estimate the value of ξ that renders zero the function

$$W_j^N(\xi) = Y_j^N - H\left(\xi, U_j^N\right), \tag{9}$$

where $U_j^N = \left[u'(t_{j-N+1}), \ldots, u'(t_j)\right]'$ and $Y_j^N = \left[y'(t_{j-N+1}), \ldots, y'(t_j)\right]'$ are the sampled input and output vectors, respectively, and $H : \mathbb{R}^n \times \mathbb{R}^{m \times N} \to \mathbb{R}^N$ maps the state $x(t_j)$ and the N-tuple of control inputs U_j^N into the output vector Y_j^N, i.e., $H\left(x(t_j), U_j^N\right) = Y_j^N$, and is defined as

$$H\left(x, U_j^N\right) \triangleq \begin{bmatrix} h\left(F^{-1}\left(F^{-1}\left(\ldots\right), u\left(t_{j-N+1}\right)\right), u\left(t_{j-N+1}\right)\right) \\ \vdots \\ h\left(F^{-1}\left(x, u\left(t_{j-1}\right)\right), u\left(t_{j-1}\right)\right) \\ h\left(x, u\left(t_j\right)\right) \end{bmatrix}, \tag{10}$$

where F^{-1} shortly represents the inverse of the map F such that $x\left(t_{j-1}\right) = F^{-1}\left(x\left(t_j\right), u\left(t_{j-1}\right)\right)$.

The system (6) and (7) is said to be *N-osbervable,* for some $N \geq 1$ (the generic selection is $N = 2n + 1$), when $W_j^N(\xi) = 0$ hold only if $\xi = x\left(t_j\right)$, uniformly in U_j^N. Then, under certain technical assumptions (see Moraal and Grizzle 1995) related to the derivatives of f and h and the invertibility of the Jacobian matrix $J(x) = \partial H(x) / \partial x$, it is possible to select

$$\dot{\xi}(t) = kJ(\xi(t))^{-1}\left(Y_j^N - H\left(\xi(t), U_j^N\right)\right), \tag{11a}$$

$$\xi\left(t_j\right) = F\left(\xi\left(t_j^-\right), u\left(t_{j-1}\right)\right), \tag{11b}$$

with a sufficiently high-gain $k > 0$ and the reset map $\Gamma(\cdot) = F\left(\xi\left(t_j^-\right), u\left(t_{j-1}\right)\right)$. Note that (11a) is commonly referred to as Newton flow. This approach could be easily extended to other continuous-time minimization algorithms (normalized gradient, line search, etc.) changing the rhs of (11a) or even with discrete-time methods iterated at higher frequency within the sample time T, yielding faster convergence to zero of the estimation error.

The same approach can be used when a continuous-time observer for (6) is considered in place of (8a) and the Newton-based resets can be used to possibly improve the performances. The continuous and discrete-time Newton algorithm require the knowledge of the jump map F to define (7), i.e., the exact discrete-time model of (6), and the Jacobian matrix $J(x) = \partial H(x) / \partial x$. An approach that does not require such knowledge is proposed in Biyik and Arcak (2006), where continuous-time filters and secant method allow to estimate (numerically) the map F and the Jacobian matrix, or in Sassano et al. (2011) where an *extremum-seeking*-based technique is considered.

A different approach to estimate the state of a continuous-time plant, Pursued, for example, in Ahrens and Khalil (2009) and Liu (1997), exploits switching output injections, letting the correction term $l_\sigma(\cdot)$ to switch among opportune values selected by a suitable definition (often derived by a Lyapunov-based proof) of the switching signal $\sigma(t)$. These switching gains allow to improve observer performances and robustness against measurement noise and model uncertainties.

Systems with Flows and Jumps

The classical notion of observability does not hold for hybrid systems. As an example, consider the autonomous linear hybrid system described by $\dot{x}(t) = Ax(t)$ and $x\left(t_j\right) = Jx\left(t_j^-\right)$ with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{12}$$

and $C = [0, 1, 0]$. Evidently the flow is not observable in the classic sense given that $\mathcal{O}_{\text{flow}} = \left[C', (CA)', (CA^2)'\right]'$ is not full rank and the flow-unobservable subspace is $\ker(\mathcal{O}_{\text{flow}}) \triangleq \left\{x \in \mathbb{R}^3 : x_2 = x_3 = 0\right\}$. Nevertheless, in the first flow time interval $\tau = t_1 - t_0$, it is possible to estimate (e.g., in finite time using the observability Gramian matrix) the initial conditions $(x_2(t_0), x_3(t_0))$. Then when the first jump take place at time t_1, thanks to the structure of the jump map J that resets the value of $x_3(t_1)$ with the flow-unobservable $x_1\left(t_1^-\right)$, it is possible to estimate in the next flow time interval the value of $x_1\left(t_1^-\right)$ so that the initial condition $x(t_0)$ can be completely determined. The hybrid observability matrix in this case has the following expression

$$\mathcal{O}_{\text{hybrid}} = \left[\mathcal{O}_{\text{flow}}', (\mathcal{O}_{\text{flow}} J e^{AT_1})', (\mathcal{O}_{\text{flow}} (J e^{AT_2})^2)'\right]'$$

and is full rank for all $T_j = t_j - t_{j-1}$ satisfying (4). Note that from a practical point of view, in this case the time interval that allows to reconstruct the complete state is $[t_0, t_1 + \epsilon)$ since the observer needs at least an ϵ time of the new measurements (after the first jump) to evaluate the full state $\left[\mathcal{O}'_{\text{flow}}, (\mathcal{O}_{\text{flow}} J e^{AT_1})', (\mathcal{O}_{\text{flow}} (J e^{AT_2})^2)'\right]'$. This simple example suggests that (impulsive) hybrid systems might have a reacher notion of observability than the classical ones. These properties have been studied also for mechanical systems subject to non-smooth impacts in Martinelli et al. (2004), where a high-gain-like observer design has been proposed assuming the knowledge of the impact times t_i, no *Zeno* phenomena (no finite accumulation point for t_j's), and a minimum *dwell-time,* $t_{j+1} - t_j \geq \delta > 0$. With the aforementioned assumptions and considering the more general class of hybrid system described by

$$\begin{aligned} \dot{x}(t) &= f(x, u), \\ x(t_j) &= g\left(x\left(t_j^-\right), u\left(t_j^-\right)\right), \end{aligned} \tag{13}$$

with $y = h(x,u)$, a frequent choice is to consider the hybrid observer of the form

$$\dot{\hat{x}}(t) = f(\hat{x}, u) + \mathbf{l}(\mathbf{y}, \mathbf{x}, \mathbf{u}), \tag{14a}$$

$$\hat{x}(t_j) = g\left(\hat{x}\left(t_j^-\right), u\left(t_j^-\right)\right) + \mathbf{m}\left(\hat{\mathbf{x}}\left(\mathbf{t}_j^-\right), \mathbf{u}\left(\mathbf{t}_j^-\right)\right), \tag{14b}$$

with $\mathbf{l}(\cdot)$ and $\mathbf{m}(\cdot)$ that are zero when $\hat{x} = x$ rendering flow and jump-invariant the manifold $\hat{x} = x$ relying only on the correction term l(·) (m $\equiv$ 0) in a high-gain-like design during the flow. The correction during the flow has to recover, within the minimum dwell-time δ, the worst deterioration of the estimation error induced by the jumps (if any) and the transients such that $\|e(t_{j+1})\| < \left\|e\left(t_j^-\right)\right\|$ or $V(e(t_{j+1})) < V\left(e\left(t_j^-\right)\right)$ if a Lyapunov function is available. This type of observer design, with $m = 0$ and the linear choice $l(y, \hat{x}, u) = L(y - M\hat{x})$, has been proposed in Heemels et al. (2011) for *linear complementarity systems* (LCS) in the presence of state jumps induced by impulsive input. Therein, solutions of LCS are characterized by means of piecewise Bohl distributions, and the specially defined *well-posedness* and *low-index* properties, which combined with passivity-based arguments, allow to design a global hybrid observer with exponential convergence. A separation principle to design an output feedback controller is also proposed.

An interesting approach is pursued in Forni et al. (2003) where global output tracking results on a class of linear hybrid systems subject to impacts is introduced. Therein, the key ingredient is the definition of a "mirrored" tracking reference (a change of coordinate) that depends on the sequence of different jumps between the desired trajectory (a virtual bouncing ball) and the plant (the controlled ball). Exploiting this (time-varying) change of coordinates and assuming that the impact times are known, it is possible to define an estimation error that is not discontinuous even when the tracked ball has a bounce (state jump) and the plant does not. A time regularization is included in the model embedding a minimum dwell-time among jumps. In this way, it is possible to design a linear hybrid observer represented by (14b) with a linear (mirrored) term $l(\cdot)$ and $m(\cdot) \equiv 0$, proving (by standard quadratic Lyapunov functions) that the origin of the estimation error system is GES. In this case, the standard observability condition for the couple (A, C) is required.

Switching Systems and Hybrid Automata

Switching systems and hybrid automata have been the subject of intense study of many researchers in the last two decades. For these class of systems, there is a neat separation $x = [z, q]'$ among purely discrete-time state q (*switching signal or system mode*) and rest of the state z that generically can both flow and jump. The observability of the entire system is often divided into the problem of determining the switching signal q first and then z. The switching signal can be divided into two categories: arbitrary (*universal problem*) or specific (*existential problems*) switchings.

In Vidal et al. (2003), the observability of autonomous linear switched systems with no state jump, minimum dwell-time, and unknown switching signal is analyzed. Necessary and sufficient observability conditions based on rank tests and output discontinuities detection strategies are given. Along the same line, the results are extended in Babaali and Pappas (2005) to nonautonomous switched systems with non-Zeno solutions and without the minimum dwell-time requirement, providing state z and mode q observability characterized by linear-algebraic conditions.

Luenberger-type observers with two distinct gain matrices L_1 and L_2 are proposed in the case of bimodal piecewise linear systems in Juloski et al. (2007) (where state jumps are considered), whereas recently in Tanwani et al. (2013), algebraic observability conditions and observer design are proposed for switched linear systems admitting state jumps with known switching signal (although some asynchronism between the observer and the plant switches is allowed). Results related to the observability of hybrid automata, which include switching systems, can be found in Balluchi et al. (2002) and the related references. Therein the *location observer* estimates first the system *current location* q, processing system input and output assuming that it is *current-location observable*, a property that is related to the system *current-location observation tree*. This graph is iteratively explored at each new input to determine the node associated to the current value of $q(t)$. Then, a linear (switched) Luenberger-type observer for the estimation of the state z, assuming minimum dwell-time and observability of each pair (A_q, C_q), is proposed. A local separation principle have been proposed in Teel (2010).

Summary and Future Directions

Observer design and observability properties of general hybrid systems is an active field of research, and a number of different results have been proposed by researchers. Results are based on different notations and definitions for hybrid systems. Efforts to provide a unified approach, in many cases considering the general framework for hybrid systems proposed in Goebel et al. (2009), are pursued by the scientific community to improve consistency and cohesion of the general results. Observer designs, observability properties, and separation principle are still open challenges for the scientific community.

Cross-References

▸ Hybrid Dynamical Systems, Feedback Control of
▸ Observer-Based Control
▸ Observers for Nonlinear Systems
▸ Observers in Linear Systems Theory

Bibliography

Ahrens JH, Khalil HK (2009) High-gain observers in the presence of measurement noise: a switched-gain approach. Automatica 45(5):936–943

Babaali M, Pappas GJ (2005) Observability of switched linear systems in continuous time. In: Morari M, Thiele L (eds) Hybrid systems: computation and control. Volume 3414 of lecture notes in computer science. Springer, Berlin/Heidelberg, pp 103–117

Balluchi A, Benvenuti L, Benedetto MDD, Vincentelli ALS (2002) Design of observers for hybrid systems. In: Hybrid systems: computation and control, vol 2289. Springer, Stanford

Biyik E, Arcak M (2006) A hybrid redesign of Newton observers in the absence of an exact discrete-time model. Syst Control Lett 55(8):429–436

Branicky MS (1998) Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. IEEE Trans Autom Control 43(5):475–482

Carnevale D, Astolfi A (2009) Hybrid observer for global frequency estimation of saturated signals. IEEE Trans Autom Control 54(13):2461–2464

Forni F, Teel A, Zaccarian L (2003) Follow the bouncing ball: global results on tracking and state estimation with impacts. IEEE Trans Autom Control 58(8): 1470–1485

Goebel R, Sanfelice R, Teel AR (2009) Hybrid dynamical systems. IEEE Control Syst Mag 29:28–93

Heemels WPMH, Camlibel MK, Schumacher J, Brogliato B (2011) Observer-based control of linear complementarity systems. Int J Robust Nonlinear Control 21(13):1193–1218. Special issues on hybrid systems

Juloski AL, Heemels WPMH, Weiland S (2007) Observer design for a class of piecewise linear systems. Int J Robust Nonlinear Control 17(15):1387–1404

Khalil HK, Praly L (2013) High-gain observers in nonlinear feedback control. Int J Robust Nonlinear Control 24:993–1015

Liu Y (1997) Switching observer design for uncertain nonlinear systems. IEEE Trans Autom Control 42(12):1699–1703

Luenberger DG (1966) Observers for multivariable systems. IEEE Trans Autom Control 11: 190–197

Martinelli F, Menini L, Tornambè A (2004) Observability, reconstructibility and observer design for linear mechanical systems unobservable in absence of impacts. J Dyn Syst Meas Control 125:549

Moraal P, Grizzle J (1995) Observer design for nonlinear systems with discrete-time measurements. IEEE Trans Autom Control 40(3):395–404

Possieri C, Teel A (2016) Structural properties of a class of linear hybrid systems and output feedback stabilization. IEEE Trans Autom Control 62(6):2704–2719

Prieur C, Tarbouriech S, Zaccarian L (2012) Hybrid high-gain observers without peaking for planar nonlinear systems. In: 2012 IEEE 51st annual conference on decision and control (CDC), Maui, pp 6175–6180

Raff T, Allgower F (2008) An observer that converges in finite time due to measurement-based state updates. In: Proceedings of the 17th IFAC world congress, COEX, South Korea, vol 17, pp 2693–2695

Sassano M, Carnevale D, Astolfi A (2011) Extremum seeking-like observer for nonlinear systems. In: 18th IFAC world congress, Milano, vol 18, pp 1849–1854

Tanwani A, Shim H, Liberzon D (2013) Observability for switched linear systems: characterization and observer design. IEEE Trans Autom Control 58(5): 891–904

Teel A (2010) Observer-based hybrid feedback: a local separation principle. In: American control conference (ACC), 2010, Baltimore, pp 898–903

Vidal R, Chiuso A, Soatto S, Sastry S (2003) Observability of linear hybrid systems. In: Maler O, Pnueli A (eds) Hybrid systems: computation and control. Volume 2623 of lecture notes in computer science. Springer, Berlin/Heidelberg, pp 526–539

Tornambé A (1992) High-gain observers for nonlinear systems. International Journal of Systems Science 23(9): 1475–1489