Fundamentals of Sliding Mode Control

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An overview:

1) Sliding Modes in State-Space

2) An Example

3) A Design Procedure

3) Robustness and Performance Issues

Sliding Modes (in State-Space)

Consider a general linear system with a single input

$$\dot{x} = Ax + bu \tag{1}$$

where $x \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$.

Define a linear function of the states

$$s(x) = Sx \tag{2}$$

where $S \in \mathbf{R}^{1 \times n}$ is a row vector. This will be termed the switching function.

Associated with (2) is a surface (a hyperplane)

$$\mathcal{S} = \{x \in \mathbf{R} \quad Sx = 0\}$$

Based on the above

> If there exists a time $t_s > 0$ so that the solution to (1) represented by $x(t)$ satisfies
>
> $$Sx(t) = 0, \qquad \text{for all } t > t_s$$
>
> then an *ideal sliding motion* is said to be taking place for all $t > t_s$.

Reachability Condition

If $s(t) = Sx(t)$ then the switching function may be thought of as a function of time.

A condition to ensure an ideal sliding motion is

$$\dot{s}s < -\eta|s| \qquad (3)$$

where η is a small positive constant.

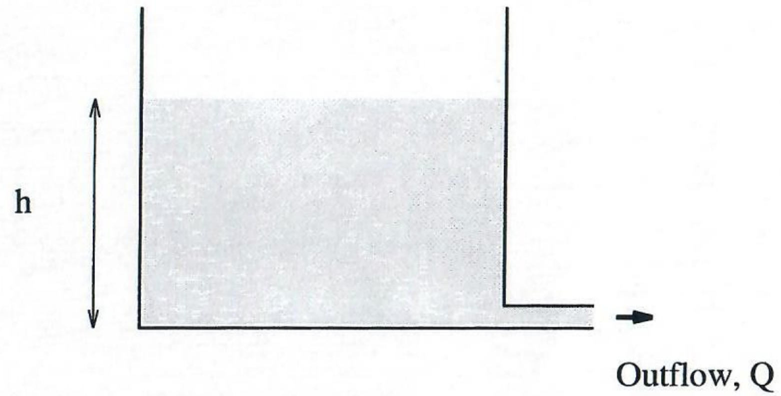
This is referred to as the reachability condition.

The key property of solutions to (3) is that $s(t) = 0$ for some $t = t_s$.

(Aside: Some natural systems possess this property, e.g. water flowing out of a containing vessel.)

Water Tanks ...

Let h represent the height of water in the tank.



The flow rate Q through the outlet

$$Q = K\sqrt{h}$$

where K is a positive coefficient.

The differential equation governing the height of water is

$$A\dot{h} = -K\sqrt{h} \qquad (4)$$

where A is the cross sectional area of the tank.

Writing $h = s^2$ in (4) yields the reachability condition where $\eta = K/A$.

An Example

Consider the state-space model

$$\dot{x}(t) = Ax(t) + bu(t) \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

These can represent the equations of motion of a hot-air balloon where the control input is the fuel flow into the burner and the first component represents the altitude of the balloon.*

The open loop poles are $\{0, -1, -2\}$.

Aim to select a switching function defined by

$$S = \begin{bmatrix} s_1 & s_2 & 1 \end{bmatrix}$$

or equivalently

$$s(x) = s_1x_1 + s_2x_2 + x_3$$

to ensure the reduced order sliding motion confined to $\mathcal{S}$ is stable and meets any design specifications.

*See *Feedback Control of Dynamic Systems*, G.F. Franklin & D. Powell, pp. 726.

Hyperplane Design

Whilst sliding

$$\begin{bmatrix} s_1 & s_2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{6}$$

or equivalently

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Because of the special form of the state-space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3 \tag{7}$$

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

Simplifying

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -s_1 & -2-s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

Remarks:

- The sliding motion will be governed by second order dynamics - i.e. lower than the original system
- Equations (7)-(8) represent a second order system in which x_3 has the role of the control variable and the row vector $\begin{bmatrix} s_1 & s_2 \end{bmatrix}$ is a full state-feedback matrix.

Hyperplane Design (ctd)

The characteristic equation of (9) is

$$\begin{vmatrix} \lambda & -1 \\ s_1 & \lambda + 2 + s_2 \end{vmatrix} = 0$$

in other words

$$\lambda^2 + (2 + s_2)\lambda + s_1 = 0 \qquad (10)$$

Choosing the required sliding mode poles to be $\{-1 \pm j\}$ gives a desired characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

Comparing coefficients with (10) gives

$$s_1 = 2 \quad \text{and} \quad s_2 = 0$$

The switching function

$$s(x) = 2x_1 + x_3 \qquad (11)$$

and the sliding surface is given by

$$\mathcal{S} = \{(x_1, x_2, x_3) \ : \ 2x_1 + x_3 = 0\}$$

Reachability

Now a control law must be developed in order that the reachability condition is satisfied.

It follows (in this case) that

$$\dot{s} = 2\dot{x}_1 + \dot{x}_3$$

Now substituting from the original equations

$$\dot{s} = 2\underbrace{x_2}_{\dot{x}_1} + \underbrace{-x_3 + 10u}_{\dot{x}_3}$$

Now choose

$$u = -\frac{1}{5}x_2 + \frac{1}{10}x_3 - \frac{\eta}{10}\,\text{sign}(s) \qquad (12)$$

where η is a positive scalar.

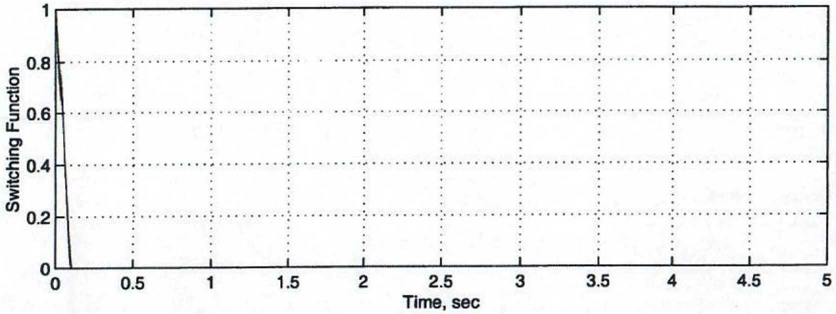
It follows that

$$\dot{s} = -\eta\,\text{sign}(s) \quad \Rightarrow \quad s\dot{s} = -\eta|s|$$

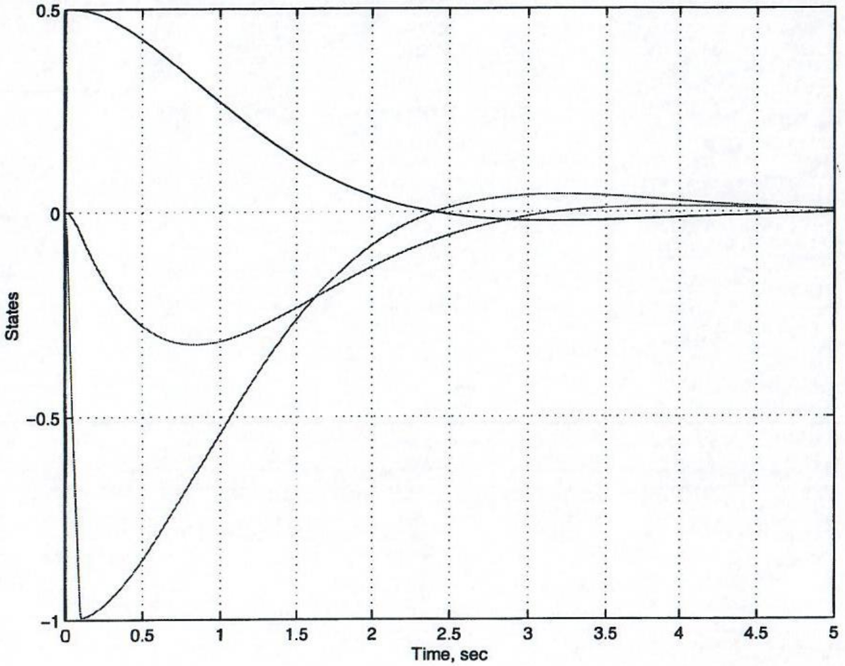
Hence (12) is an appropriate variable structure controller which induces a sliding motion.

Simulations

The switching function plot



The states evolution looks like



A Design Strategy

A design strategy maybe summarized as:

- design the sliding surface to provide an appropriate sliding motion to satisfy any specifications
- design a control law (which will usually depend on the choice of sliding surface) to ensure the reachability condition is satisfied

The resulting system in closed loop will have two distinct characteristics: an initial (fast) motion towards the sliding surface, followed by the sliding motion during which time $s(t) \equiv 0$.

The design earlier exploited the property that the only non-zero term in the input distribution matrix was the last element.

This structure does not always appear naturally in state-space systems. It can always be obtained however through a change of coordinates.*

Any (A, b) can be written in the so-called controllability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -a_1 & -a_2 & \dots & \dots & -a_n \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

where the a_i's are the coefficients of the characteristic equation of the A matrix:

$$\lambda^n + a_n\lambda^{n-1} + \dots + a_2\lambda + a_1 = 0$$

*See the command `tf2ss` in Matlab.

For this general system an appropriate switching function is

$$s(x) = m_1x_1 + m_2x_2 \dots + m_{n-1}x_{n-1} + x_n \tag{14}$$

where the scalars m_i are to be chosen.

The partition of the state-space is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} x_n$$

and so the sliding motion will be governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ -m_1 & \dots & \dots & \dots & -m_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Therefore the characteristic equation of the sliding motion is given by

$$\lambda^{n-1} + m_{n-1}\lambda^{n-2} + \dots + m_2\lambda + m_1 = 0$$

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Consider a general linear system with a single input

$$\dot{x}(t) = Ax(t) + bu(t) \tag{15}$$

where $x \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.

Suppose a switching function has been designed

$$s(x) = Sx \tag{16}$$

where

$$S = \left[\begin{array}{ccccc} s_1 & s_2 & \dots & s_{n-1} & 1 \end{array} \right]$$

Then by definition

$$\dot{s} = S\dot{x} = SAx(t) + \underbrace{Sb}_{1} u(t)$$

Choose

$$u(t) = -SAx(t) - \eta \text{sign}\,(s)$$

where η is a positive scalar.

This guarantees the reachability condition is met.

A slight variation would be the control law

$$u(t) = -SAx(t) - \Phi Sx(t) - \eta \text{sign}\,(s)$$

where Φ is a positive constant.

Now consider

$$\dot{x}(t) = Ax(t) + bu(t) + b\xi(t, x) \tag{17}$$

where $x \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ and (A, b) is in controllability canonical form.

Suppose that $\xi(t, x)$ is an unknown signal. (This could be an external disturbance or parametric variations in the last row of Ax.)

Suppose a switching function has been designed of

$$s(x) = Sx \tag{18}$$

where

$$S = \left[\begin{array}{ccccc} s_1 & s_2 & \dots & s_{n-1} & 1 \end{array} \right]$$

Then providing a sliding motion can be maintained in the presence of $\xi(t, x)$ the dynamics will be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ -m_1 & \dots & \dots & \dots & -m_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$\boxed{\text{This is completely independent of } \xi(t, x)}$

Controller Design

Suppose that $|\xi(t,x)| \leq \rho(t,x)$ where $\rho(\cdot)$ is known.

Then

$$u(t) = -SAx(t) - (\rho(t,x) + \eta)\text{sign}\,(s)$$

where η is a positive scalar guarantees the reachability condition is met.

As before

$$\begin{aligned} \dot{s} = S\dot{x} &= SAx(t) + Sb\xi(t,x) + Sbu(t) \\ &= SAx(t) + \xi(t,x) + u(t) \end{aligned}$$

Substituting for the control law

$$\dot{s} = \xi(t,x) - (\rho(t,x) + \eta)\text{sign}\,(s)$$

and so

$$s\dot{s} = s\,\xi(t,x) - (\rho(t,x) + \eta)|s| \qquad (19)$$

However

$$s\,\xi(t,x) \leq |s||\xi(t,x)| \leq |s|\,\rho(t,x)$$

which implies from (19) that

$$s\dot{s} \leq -\eta|s|$$

$$\dot{s} \leq -\eta \frac{|s|}{s}$$

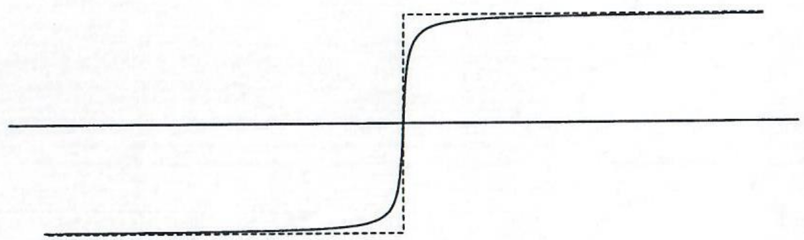
Pseudo-sliding Modes

For most systems a switched control law is not acceptable. (NB in some electric motors the control law is naturally discontinuous and sliding mode control can be used to obtained extremely good performance.)

It is impossible to obtain a sliding motion in reality. Delays, hysteresis and the effects of a digital implementation causes chattering whereby the phase portrait repeatedly crosses the sliding surface never remaining on it.

One solution is to smooth the discontinuity by using a sigmoidal function

$$u(t) = \frac{s(t)}{|s(t)| + \delta}, \qquad \delta > 0$$



An approximate pseudo-sliding motion is obtained (this will be explored in the lab sessions)

In practise a trade-off is made between the conflicting requirements of accuracy (a small δ) and chatter reduction (larger δ).

Concluding Remarks

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1) Sliding Mode Control is a special case of Variable Structure Control

2) Sliding Mode Control is a robust methodology

3) Uses Discontinuous control action forces a sliding motion which is
   - of lower order than the original system
   - and insensitive to a certain class of parameter variations - so-called matched uncertainty

4) In practise often only pseudo-sliding can be achieved because continuous control action needs to be used.