Control Theory

Frequency domain design

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Frequency domain design

- Design based on the system frequency response (bode plot, nyquist plot)

- System robustness can be defined using the gain and phase margins

- Nyquist stability criterion can be applied also to establish gain and phase margins

- System error constants can be read directly from the low frequency asimptote in the magnitude plot

- System speed response is proportional to the crossover frequency

Frequency domain design

In continuous time

- The bode plots are easily plotted and interpreted (by hand)
- Corrections to the frequency response can be easily made using the controller poles and zeros
- $G_s(j\omega)$ is a rational function of $j\omega$ then plot bode plots is simple

In discrete time

- Computer tools
- $G_z(e^{j\omega})$ is an irational function of ω

Frequency domain design

Principio del argumento

From [1]

C_1 → closed curve in the *s*-plane

F(s) → rational function of *s* (no pole nor zero on C_1)

Z, P → numbers of zeros and poles of *F(s)* encircled by C_1

C_2 → maping of C_1 by *F(s)* into the *F*-plane

→ C_2 will encircle the origin of the *F*-plane (*Z*-*P*) times in the same direction as C_1

Frequency domain design

Nyquist stability criterion

Closed-loop transfer function: $G_o(s) = \dfrac{G_l(s)}{1 + G_l(s)}$

Closed-loop characteristic equation: $\Delta(s) = 1 + G_l(s)$

Loop transfer function: $G_l(s)$

Frequency domain design

Nyquist plot

From [5]

Frequency domain design

Nyquist plot

From [1]

$$G_l(s) = \frac{s+1}{s^2(s-2)}$$

$$F(s) = 1 + G_l(s)$$

Frequency domain design

Nyquist plot – polar plot

$G_l(s)$

From [6]

Polar plot

Frequency domain design

Nyquist plot

Distance of the Nyquist curve to the critical point (-1,0):

$$1 + G_l(s)$$

From [7]

Frequency domain design

Nyquist stability criterion

Theorem:

$G_o(s)$ is stable if and only if the Nyquist plot of $G_l(s)$ does not pass through critical point (-1,0) and the number of counterclockwise encirclements of (-1,0) equals the number of open right-half-plane poles of $G_l(s)$

From [7]

Frequency domain design

Nyquist stability criterion

Corollary:

If $G_l(s)$ has no open right-half-plane poles, then $G_o(s)$ is stable if and only if the Nyquist plot of $G_l(s)$ does not encircle nor pass through critical point (-1,0)

From [7]

Frequency domain design

Design criteria

- Gain and phase margins:

From [1]

Frequency domain design

Design criteria

- Gain and phase margins:

From [6]

Frequency domain design

Design criteria

- Gain and phase margins:

From [6]

Frequency domain design

Design criteria

- Gain and phase margins:

From [1]

Frequency domain design

- Gain and phase margins (OL) – transient response (CL)

For a second order continuous time system: $$G_o(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

overshoot

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Resonant peak

From [5]

$$M_r = \begin{cases} \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} & \zeta < 0.707 \\ 1 & \zeta \geq 0.707 \end{cases}$$

Frequency domain design

- Gain and phase margins (OL) – transient response (CL)

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Frequency domain design

- Gain and phase margins (OL) – transient response (CL)

For a second order continuous time system:

$$G_o(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

Gain margin $\geq$ 10 dB, Phase margin $\geq$ 45° $\Rightarrow$ $M_r \approx 1.3$

Gain margin $\geq$ 12 dB, Phase margin $\geq$ 60° $\Rightarrow$ $M_r \approx 1.0$

Frequency domain design

Gain and phase margins (OL) – transient response (CL)

- For the second order continuous time system:

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s} \xrightarrow{\text{closed-loop}} G_o(s) = \frac{G(s)}{1 + G(s)}$$

- Phase margin (*G(s)*) and M_p ($G_o(s)$) are directly related:

$$\phi_m = \tan^{-1}\left[\frac{2\zeta}{\left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)^{1/2}}\right]$$

- Which is usually approximated by: $\boxed{\phi_m = 100\zeta}$

Frequency domain design

Gain and phase margins (OL) – transient response (CL)

- $\boxed{\phi_m = 100\zeta}$

Discrete proportional feedback of

$$G(s) = \frac{1}{s^2 + s}$$

Open loop phase margin and closed-loop damping ratio as the feedback gain was varied

From [6]

Frequency domain design

Gain and phase margins (OL) – transient response (CL)

• The resonant peak (CL) increases as the frequency response (OL) appoaches the point -1

Larger resonance time domain CL	⟷	small stability margins frequency domain CL

Frequency domain design

Gain and phase margins (OL) – transient response (CL)

• Sensitivity to parameter variation: is in general reduced increasing the open loop gain

• Disturbance rejection can be improved using hign loop gain, provided that this gain is not in the direct path between the disturbance input and system output

• Control effort: constraints on the control action need to be considered. High loop gain can cause system saturation

Frequency domain design

Low frequency gains (OL) – steady state error (CL)

- Position error constant:

$$K_p = \lim_{s \to 0} C(s)G(s)$$

- Velocity error constant:

$$K_v = \lim_{s \to 0} sC(z)G(z)$$

Frequency domain design

Given the closed loop transfer function: $E(s) = \dfrac{1}{1+G(s)C(s)} R(s)$

- Position error constant:
  - For a step reference: $R(s) = \dfrac{1}{s}$

$$E(z) = \frac{1}{1+G(s)C(s)} \frac{1}{s}$$

$$e_{ss} = \lim_{s \to 0} \left| sE(s) \right| = \lim_{s \to 0} \left| s \frac{1}{1+G(s)C(s)} \frac{1}{s} \right| = \lim_{s \to 0} \left| \frac{1}{1+G(s)C(s)} \right|$$

Obtaining

$$e_p = \left| \frac{1}{1+K_p} \right| \quad \text{where} \quad K_p = \lim_{s \to 0} C(s)G(s)$$

Frequency domain design

Given the closed loop transfer function: $E(s) = \dfrac{1}{1+G(s)C(s)} R(s)$

- Position error constant:
  - For a step reference: $R(s) = \dfrac{1}{s^2}$

$$E(z) = \frac{1}{1+G(s)C(s)} \frac{1}{s^2}$$

$$e_{ss} = \lim_{s\to 0} \left| sE(s) \right| = \lim_{s\to 0} \left| s \frac{1}{1+G(s)C(s)} \frac{1}{s^2} \right| = \lim_{s\to 0} \left| \frac{1}{s+sG(s)C(s)} \right|$$

Obtaining

$$e_v = \left| \frac{1}{K_v} \right| \qquad \text{where} \qquad K_v = \lim_{s\to 0} sC(z)G(z)$$

Frequency domain design

Syteady state error vs system type

System Type \ Error constant	K_p	K_v	K_a
0	K	0	0
I	∞	K	0
II	∞	∞	K

System Type \ Error	e_p	e_v	e_a
0	k	∞	∞
I	0	k	∞
II	0	0	k

Frequency domain design

Syteady state error in bode plot

From [1]

Frequency domain design

Specifications of control systems

	Time Domain	Frequency Domain	
	Overall System	Overall System	Loop Transfer Function
Steady-State Performance	Position error	$\lvert 1 - G_o(0) \rvert$	$\lvert 1/(1 + K_p) \rvert$
	Velocity error	$\lvert G_o'(0) \rvert$ if $G_o(0) = 1$	$\lvert 1/K_v \rvert$
Transient Performance	Overshoot	Peak resonance	Gain and phase margins
	Rise time	Bandwidth	Gain crossover frequency
	Settling time		
		High-frequency gain	High-frequency gain

Frequency domain design

- Lag phase network

$$C(s) = \frac{1 + aT_1 s}{1 + T_1 s}$$

$$0 < a < 1$$

From [1]

Frequency domain design

Design procedure:

- Compute the error constant from steady state error specification
- Plot bode plot ⇒ gain and phase margins
- Determine the frequency at which the bode plot has the required phase margin plus 6°. This is the new gain crossover frequency ω_g'
- Measure the attenuation α needed to bring the gain plot down to ω_g'

$$\alpha = -20\log(a)$$

- Compute T_1 from:

$$\frac{1}{aT_1} = \frac{\omega_g'}{10}$$

Frequency domain design

- Lead phase network

$$C_2(s) = \frac{1 + bT_2 s}{1 + T_2 s}$$

$$b > 1$$

$$b = \frac{1 + \sin\phi_m}{1 - \sin\phi_m}$$

From [1]

Frequency domain design

Design procedure:

- Compute the error constant from steady state error specification
- Plot bode plot ⇒ gain margin, phase margin ϕ_1, gain crossover frequency ω_g and phase crossover frequency ω_p
- Calculate $\Psi = \phi_{required} - \phi_1 + \theta$
- Compute $$b = \frac{1+\sin\Psi}{1-\sin\Psi}$$
- Draw a horizontal line with gain $-10\log(b)$
- The intersection with the bode plot will be the new gain crossover frequency ω_g'
- Measure the resulting phase margin

Frequency domain design

- Ejemplo: $G(s) = \dfrac{1}{s(s+2)}$

- Velocity error $e_p \leq 10\ \%$

- Phase margin $\geq$ *60*

- Gain margin $\geq$ *12dB*

Frequency domain design

- Ejemplo: $G(s) = \dfrac{1}{s(s+2)}$

- Velocity error $e_p \leq 10\ \%$

- Phase margin $\geq$ *60*

- Gain margin $\geq$ *12dB*

Frequency domain design

Sensitivity function:

The error is given by

$$E(s) = \frac{1}{1 + C(s)G(s)} R(s) = \frac{1}{1 + G_l(s)} R(s)$$

The **sensitivity function** is defined as

$$S(s) = \frac{1}{1 + G_l(s)}$$

Frequency domain design

Sensitivity function:

In the frequency domain

$$S(j\omega) = \frac{1}{1 + G_l(j\omega)}$$

The following can be defined

$$S_\infty = \left\| S(s) \right\|_\infty = \max_\omega \left| S(j\omega) \right|$$

From [2]

Frequency domain design

Sensitivity function:

All points inside the dashed circle have sensitivities greater than 1

From [2]

Distance of the Nyquist curve to the critical point (-1,0):

$$1+G_l(s)=\frac{1}{S(s)}$$

Minimum distance:

$$\frac{1}{\|S(s)\|_\infty}$$

For a larger S_∞ the Nyquist plot comes closer to the point of instability

Frequency domain design

Sensitivity function:

From [6]

Vector gain margin (VGM): $VGM = \dfrac{S_\infty}{S_\infty - 1}$

Distance of the Nyquist curve to the critical point (-1,0):

$$1 + G_l(s) = \frac{1}{S(s)}$$

Minimum distance:

$$\frac{1}{\|S(s)\|_\infty}$$

For a larger S_∞ the Nyquist plot comes closer to the point of instability

Frequency domain design

Sensitivity function – plant variations

From [2]

$$T(s)=\frac{G_l(s)}{1+G_l(s)}=G_o(s)$$

If the process is changed from *G(s)* to *G(s)*+Δ*G(s)*

the loop transfer function changes from *C(s)G(s)* to *C(s)G(s)*+*C(s)* Δ*G(s)*

The Nyquist curve will not reach the critical point −1 provided that

$$\left|C(s)\Delta G(s)\right|<\left|1+G_l(s)\right|$$

This condition can be re-written as

$$\frac{\left|\Delta G(j\omega)\right|}{\left|G(j\omega)\right|}<\frac{1}{\left|T(j\omega)\right|}$$

Frequency domain design

Sensitivity function – plant variations

From [2]

the variations can be large for those frequencies where *T* is small and that smaller variations are allowed for frequencies where *T* is large. A conservative estimate of permissible process variations that will not cause instability is given by

$$\frac{|\Delta G(j\omega)|}{|G(j\omega)|} < \frac{1}{T_\infty}$$

where $T_\infty = \|T(s)\|_\infty = \max_\omega |T(j\omega)|$

If T_∞ = 2 gain variations of 50% and phase variations of 30° are permitted without making the closed loop system unstable

Frequency domain design

Sensitivity function – plant variations

We have also seen from Equations (5.6) and (5.13) that it is advantageous to have a small value of the sensitivity function and it follows from (5.10) that a small value of the complementary sensitivity allows large process uncertainty.

since $$T(s) = \frac{G_l(s)}{1 + G_l(s)} \qquad S(s) = \frac{1}{1 + G_l(s)}$$

then $$T(s) + S(s) = 1$$

This means that *S* and *T* cannot be made small simultaneously. The loop transfer function *L* is typically large for small values of *s* and it goes to zero as *s* goes to infinity. This means that *S* is typically small for small *s* and close to 1 for large. The complementary sensitivity function is close to 1 for small *s* and it goes to 0 as *s* goes to infinity.

Frequency domain design

Sensitivity function – plant variations

Bode has shown that if the loop transfer has poles *pk* in the right half plane and if it goes to zero faster than 1/s for large *s* the sensitivity function satisfies the following integral

$$\int_0^\infty \log|S(i\omega)|d\omega = \int_0^\infty \log\frac{1}{|1+L(i\omega)|}d\omega = \pi\sum \mathrm{Re}\ p_k$$

This equation shows that if the sensitivity function is made smaller for some frequencies it must increase at other frequencies. This means that if disturbance attenuation is improved in one frequency range it will be worse in other. This has been been called the water bed effect.

Frequency domain design

Sensitivity function – plant variations

For a loop transfer function without poles in the right half plane (5.18) reduces to

$$\int_0^\infty \log|S(i\omega)|d\omega = 0$$

This formula can be given a nice geometric interpretation as shown in Figure 5.17 which shows *log|S(jω)|* as a function of *ω* . The area over the horizontal axis must be equal to the area under the axis.

From [2]

Frequency domain design

Sensitivity functions

Form: Stein, G., "Respect the unstable," *Control Systems, IEEE* , vol.23, no.4, pp. 12,25, Aug. 2003

Bibliography

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