151-0591-00 **Control Systems I** (HS 2017) **Exercise Set 5**

Topic: **Transfer functions 1: Definitions and properties**

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Learning objectives: The student can

- Know the equations of the transfer function of an LTI system by heart and can derive them efficiently for systems with up to three states
- Can use the linearity in inputs to calculate output of simple systems to regular inputs
- Can calculate Y(s) based on g(s) and U(s)
- Understand what g(s) represents
- Can place eigenvalues of a closed-loop system using state feedback control

Exercise 1 (Transfer functions. Definitions and derivations)

In this exercise we will derive some basic equations that may come in handy later on. We will also remember what the transfer function is and how it is defined.

a) Given an LTI system in state space form (Equations 1 - 2), derive the equations of the transfer function of the system using the Laplace transform.

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

Hint: You should remember that when deriving transfer function of the system, all initial conditions are set to zero. Remember that the formula for the Laplace transform derivative is: $\mathcal{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$

b) Using your newly derived formula from the previous task, find the transfer functions $g_{x_1} = \frac{x_1(s)}{F(s)}$ and $g_{x_2} = \frac{x_2(s)}{F(s)}$ for the two mass spring-damper system from the exercise 4. Are they similar? Why? (*Use MATLAB.*) Again, the system equations were:

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + F \tag{3}$$
$$m_1\ddot{x}_1 = k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - c_1\dot{x}_1 \tag{4}$$

c) Consider the system given by the equation $\ddot{y}(t) + 5\dot{y}(t) + 10y(t) = \dot{u}(t) + 10u(t)$. Derive the transfer function $g(s) = \frac{y(s)}{u(s)}$ of this system.

d) Use MATLAB to solve this equation in time domain for an input $u(t) = \delta(t)$ (dirac impulse). *You may find the commands 'dsolve' and 'dirac' useful for this*

e) Use MATLAB to calculate the inverse Laplace transform of the transfer function that you calculated in the subtask c). Compare this time response with the one that you got in the previous subtask. What does that tell you about transfer functions (in general)? *You may find the commands 'ilaplace' and 'ezplot' useful*

Exercise 2 (Linearity and complex exponential inputs)

In this exercise we will calculate some steady state responses using solutions for the case

$$u(t) = e^{st}, \quad s \in \mathbb{C}, \quad t \geq 0 \tag{5}$$

and by using linearity. Furthermore, the steady state response can be described by

$$y_{ss} = G(s)e^{st} \tag{6}$$

Some Laplace transforms that you may find useful are:

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2 + a^2} \\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2 + a^2} \\
\mathcal{L}\{1\} &= \frac{1}{s} \\
\mathcal{L}\{\frac{d}{dt}f(t)\} &= sF(s) - f(0)
\end{aligned}$$

a) Calculate the steady state response (in time domain) for the system given with transfer function $g(s) = \frac{1}{s+2}$ and input $u(t) = \sin(\omega t)$.

b) Again calculate the steady state response in time domain for the system $g(s) = \frac{s-1}{s+1}$ when excited with an input: $u(t) = 1 + \cos(\omega t)$.

c) The system from the previous subtask can be represented in time domain with the differential equation: $\dot{y}(t) + y(t) = \dot{u}(t) - u(t)$. Find the solution in time domain for the input $u(t) = 1 + \cos(\omega t)$ with zero initial conditions. Use $\omega = 2$ (it does not really matter which value you use). Did you get the same solution as in the previous subtask? Why? *Hint: Again, MATLAB function 'dsolve' may be helpful.*

Exercise 3 (Basic controller design)

Consider a linear time-invariant system given by:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} \omega & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}$$

where $\omega \in [1, 2]$ is a constant parameter. Define $\mathbf{A}(\omega) = \begin{bmatrix} \omega & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

a) For which values of ω is the system controllable? For which values of ω is it observable?

b) Assume that ω is known. Design a gain matrix $\mathbf{K}(\omega) \in \mathbb{R}^{1 \times 2}$ such that the closed-loop system under the feedback control law $u(t) = \mathbf{K}(\omega)\mathbf{x}(t)$ has eigenvalues equal to -1. *The gain matrix is allowed to depend on ω.*

c) Design a gain matrix $\mathbf{K} = \begin{bmatrix} K_1 & -4 \end{bmatrix} \in \mathbb{R}^{1 \times 2}$ such that the real part of eigenvalues of the closed-loop system under the feedback control law $u(t) = \mathbf{K}\mathbf{x}(t)$ is less than or equal to -1 for every $\omega \in [1, 2]$ (As shown above the term K_2 in the vector $\mathbf{K}$ is assumed to be equal to -4). Derive a condition on K_1 for the above setting. *The gain matrix **is not** allowed to depend on ω.*