Section 3: Random variables – discrete

STA 35C – Statistical Data Science III

Instructor: Akira Horiguchi

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MWF, 12:10 PM – 1:00 PM, Olson 158
University of California, Davis

Overview

Based on Chapter 3 of textbook: `https://www.probabilitycourse.com/`

- Contains problems with solutions, and problems without solutions.

Basic concepts

Random variables

We usually focus on some *numerical aspects* of a random experiment.

- For example, in a soccer game we may be interested in the number of goals, shots, shots on goal, corners kicks, fouls, etc.
- On any given day at UCD, we may be interested in the number of Cheeto sightings.
- These are examples of *random variables*.

Random variables

Definition 1: Random variable

A *random variable* $X\colon \Omega \to \mathbb{R}$ is a function from the sample space Ω to the real numbers.

- E.g., toss a coin three times. Sample space is

$$\Omega = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

  We can define a random variable X whose value is the number of observed heads.
- Usually denote random variables by capital letters such as X, Y, and Z.
- The *range* of a random variable X is the set of possible values for X. For example:
  - I toss a coin 100 times. Let X be the number of heads I observe.
  - I toss a coin until the first heads appears. Let Y be the total number of coin tosses.
  - The random variable T is defined as the time (in hours) from now until the next earthquake occurs in a certain city.

A random variable X is *discrete* if its range is countable.

- Recurring examples:
  1. number of heads after two coin flips,
  2. number of coin flips needed before a heads turns up.
- Here probabilities can be assigned to each realizable value.
  1. For $\{0, 1, 2\}$ (finite), we can assign probabilities $1/4$, $1/2$, and $1/4$.
  2. For $\mathbb{N}$ (countably infinite), we can assign probabilities $(1/2)^k$ to each $k \in \mathbb{N}$.
- For a discrete r.v. X with range $\{x_1, x_2, x_3, \cdots\}$, the function $f_X(\cdot)$ defined as

$$f_X(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3, \cdots,$$

  is called the *probability mass function (PMF)* of X.
  1. $f_X(0) = 1/4$, $f_X(1) = 1/2$, and $f_X(2) = 1/4$.
  2. $f_X(k) = (1/2)^k$ for each $k \in \mathbb{N}$.

  Here $f_X(a)$ is *"the probability that X equals a."*

PMF of the number of heads after two flips of a fair coin.

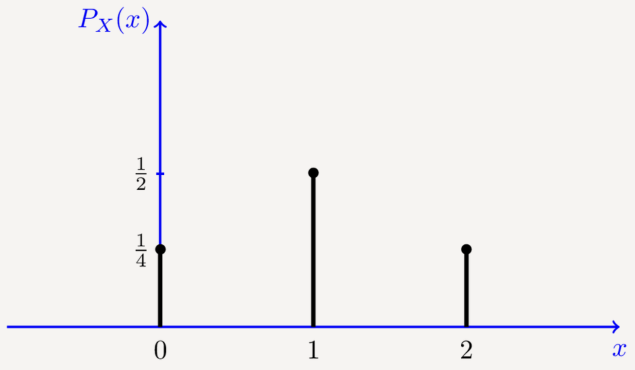


Fig.3.1 - PMF for random Variable X in Example 3.3.

The PMF of a discrete random variable is also called the r.v.'s *probability distribution*.

Discrete random variables: PMF

A PMF is a probabilty measure, so it satisfies Definition 1 from Section 2.

- In particular, it satisfies countable additivity.
- This lets us deduce the probability $P(X \in A)$ that a discrete r.v. X lies in an event A:

$$P(X \in A) = P\left(\bigcup_{a \in A} [X = a]\right) = \sum_{a \in A} f_X(a), \tag{1}$$

Independent random variables

Independent random variables

When dealing with more than one random variable, often need to consider the *dependence/correlation* between them.

- Concept of *independent random variables* is similar to that of independent events.
- Two random variables are independent if knowing the value of one does not change the probabilities for the other.

Definition 2: Two independent random variables

Two discrete random variables X and Y are *independent* if

$$P(X = a, Y = b) = P(X = a)\, P(Y = b) \tag{2}$$

for all a, b.

If two random variables are independent, then we can write

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

for all sets A, B. We can also write

$$P(Y = b \mid X = a) = P(Y = b)$$

for all a, b.

Example

Toss a fair coin four times.

- Let X be the number of heads observed in the first and second coin flips.
- Let Y be the number of heads observed in the third and fourth coin flips.

Find $\underbrace{P\Big((X < 2) \text{ and } (Y > 1)\Big)}$.

$$P(X<2, Y>1) = P(X<2)\,P(Y>1)$$

$$= P(X \in \{0,1\})\,P(Y \in \{2\})$$

$$= \Big[1 - \underbrace{P(X \in \{2\})}_{\frac{1}{2}\cdot\frac{1}{2}}\Big]\underbrace{P(Y \in \{2\})}_{\frac{1}{2}\cdot\frac{1}{2}}$$

$$= \frac{3}{4}\cdot\frac{1}{4} = \boxed{\frac{3}{16}}$$

≥ 2 independent random variables

Definition 3: ≥ 2 independent random variables

Discrete random variables $X_1, X_2, X_3, \cdots, X_n$ are *independent* if

$$\underbrace{P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)} = \underbrace{P(X_1 = x_1)}\,\underbrace{P(X_2 = x_2)} \cdots \underbrace{P(X_n = x_n)} \tag{3}$$

for all $x_1, x_2, \cdots, x_n$.

Special distributions

A random variable X with values in a finite set M is *uniformly* distributed if each element in M has the same probability:

$$P(X = k) = \frac{1}{|M|} \quad \text{for all } k \in M$$

- Such distributions occur when all possible outcomes are equally likely.
- We write $X \sim U(M)$ or $X \sim \mathit{Unif}(M)$.
- Nine random draws in R: `sample(c(1,2,3,4,5,6), size=9, replace=T)`

TRUE

nine rolls of a fair six-sided die

Bernoulli distribution

A random variable X is *Bernoulli* distributed with parameter $p \in (0, 1)$, if $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

- For when our random experiment has only two possible outcomes ("success" and "failure").
- Example: flip a coin with probability p of heads ("success"). Is it heads?
- We write $X \sim Ber_p$ or $X \sim Bern(p)$.
- Nine random draws in R: `rbinom(n=9, size=1, prob=1/3)`

p

Binomial distribution

A random variable X is *Binomial* distributed with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for all } k = 0, \ldots, n.$$

- We think of n as the number of experiments and p the success probability. In the above equation, k is the number of successes.
- For measuring the probability of the number of successes of n independent Bernoulli experiments with parameter p.
- Example: flip a coin n times, each flip with probability p of heads ("success"). How many heads?
- We write $X \sim Bin_{n,p}$ or $X \sim Bin(n, p)$.
- A random draw in R: `rbinom(n=3, size=1, prob=0.25) |> sum()`

n p

Cumulative distribution function

Motivation

The PMF is one way to describe the distribution of a discrete random variable.

- Pro: intuitive.
- Con: it cannot be defined for continuous random variables.

The *cumulative distribution function (CDF)* can characterize the distribution of *any kind* of random variable (discrete, continuous, mixed).

The *CDF* of a random variable X is the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(a) := P(X \le a), \quad a \in \mathbb{R}. \tag{4}$$

This is "the probability that X is less than or equal to a."

- Definition holds regardless of whether X is discrete, continuous, or mixed.
- In the discrete case – recall Eq. (1) – holds for any $a \in \mathbb{R}$,

$A = (-\infty, a)$

$F_x(a) = P(X \in A) = \sum_{s \le a} f_x(s)$

$$F_X(a) = \sum_{s \le a} f_X(s) .$$

$F_x(a) = P(X \le a)$

$= P\left(\bigcup_{s \le a} [X = s] \right)$

countable additivity.

$= \sum_{s \le a} f_x(s)$

- For any $a, b \in \mathbb{R}$ with $b > a$ holds,

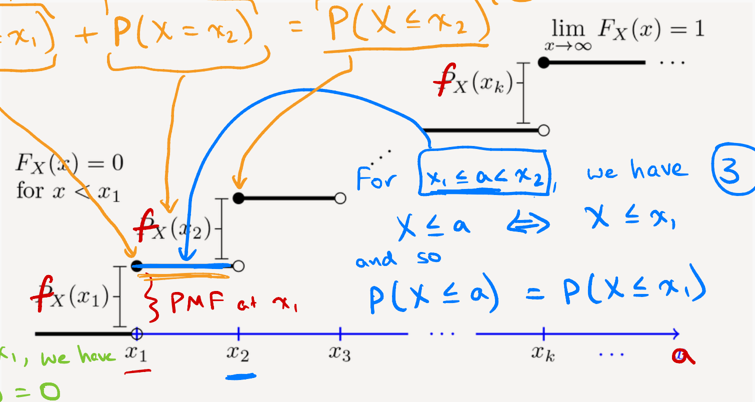
$$P(a < X \le b) = F_X(b) - F_X(a) .$$

From the definition of F_X in Eq. (4) come the following properties:

1. F_X is *right-continuous* and $\underbrace{\text{monotonically increasing}}$,
   "non-decreasing"
2. $\lim_{a\to-\infty} F_X(a) = 0$,
3. $\lim_{a\to+\infty} F_X(a) = 1$.

4

$$\underbrace{P(X \le x_1)}_{\text{CDF at } x_1} + \underbrace{P(X = x_2)}_{\text{PMF at } x_2} = \underbrace{P(X \le x_2)}_{\text{CDF at } x_2}$$



PMF at x_1

3 For $\boxed{x_1 \le a < x_2}$, we have

$X \le a \iff X \le x_1$

and so

$P(X \le a) = P(X \le x_1)$

1 For $a < x_1$, we have

$P(X \le a) = 0$

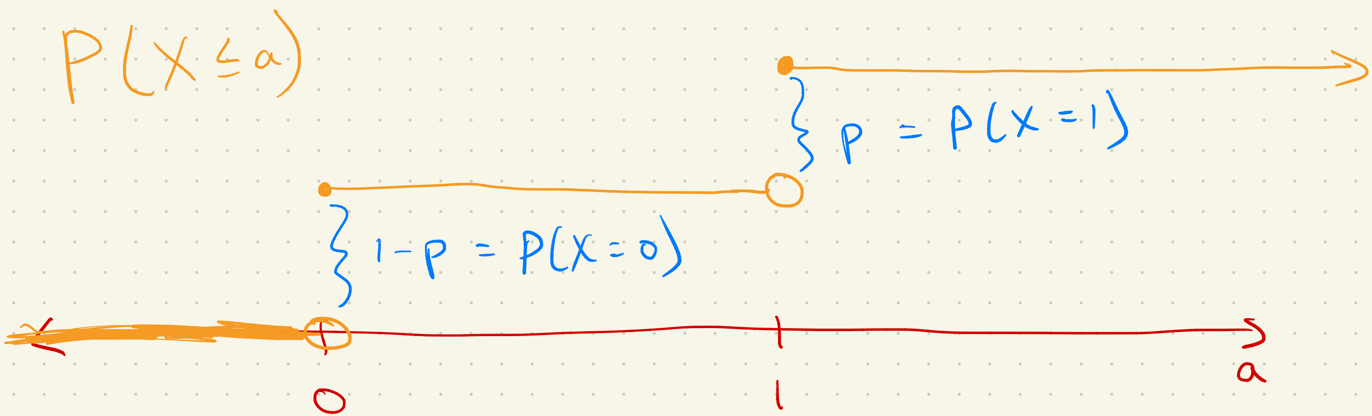
2

$$\underbrace{P(X \le x_1)}_{\text{CDF at } x_1} = \underbrace{P(X = x_1)}_{\text{PMF at } x_1} + \underbrace{P(X < x_1)}_{0}$$

Fig.3.4 - CDF of a discrete random variable.

CDF of Bernuoilli random variable with parameter p

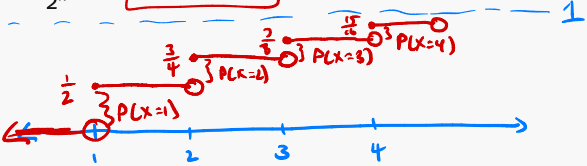
- For $a < 0$, we have $P(X \le a) = 0$.
- For $a = 0$, we have $P(X \le a) = \underbrace{P(X = 0)}_{1-p} + \underbrace{P(X < 0)}_{0} = 1 - p$
- For $0 \le a < 1$, we have $P(X \le a) = P(X \le 0) + \underbrace{P(0 < X \le a)}_{0}$
- For $a = 1$, we have $P(X \le a) = \underbrace{P(X = 1)}_{p} + \underbrace{P(X < 1)}_{P(X \le 0) = 1-p} = 1.$
- For $a > 1$, we have $P(X \le a) = P(X \le 1) = 1.$



Suppose the PMF of a discrete random variable X is given by

$$\boxed{F_X(k) = 1 - \frac{1}{2^k}} \quad \text{CDF}$$

$$P(X=k) = f_X(k) = \frac{1}{2^k} \quad \text{for } \underbrace{k = 1, 2, 3, \dots}$$



1. Find and plot the CDF.
2. Find $P(2 < X \le 5)$.
3. Find $P(X > 4)$.

$a < 1 : \quad P(X \le a) = 0$

$a = 1 : \quad P(X \le 1) = P(X=1) + P(X<1) = \frac{1}{2} + 0 = \frac{1}{2}$

$1 < a < 2 : \quad P(X \le a) = P(X \le 1) = \frac{1}{2}$

$a = 2 : \quad P(X \le 2) = \underbrace{P(X=2)} + \underbrace{P(X<2)}_{P(X \le 1)} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$2 < a < 3 : \quad P(X \le a) = P(X \le 2) + \underbrace{P(2 < X < a)}_{0} = \frac{3}{4} + 0 = \frac{3}{4}$

$a = 3 : \quad P(X \le 3) = \underbrace{P(X=3)}_{\frac{1}{2^3}} + \underbrace{P(X<3)}_{P(X \le 2)} = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$

Example

Suppose the PMF of a discrete random variable X is given by

$$f_X(k) = \frac{1}{2^k} \quad \text{for } k = 1, 2, 3, \ldots$$

1. Find and plot the CDF.
2. Find $P(2 < X \leq 5)$.
3. Find $P(X > 4)$.

Closed form expression for CDF:

$$F_X(a) = P(X \leq a) = 1 - P(X > a)$$

$$= 1 - \sum_{k=a+1}^{\infty} P(X = k)$$

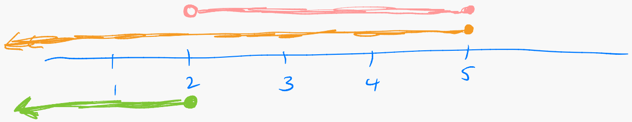
$$= 1 - \sum_{k=a+1}^{\infty} \frac{1}{2^k}$$

$$= \boxed{1 - \frac{1}{2^a}}$$

Suppose the PMF of a discrete random variable X is given by

$$f_X(k) = \frac{1}{2^k} \quad \text{for } k = 1, 2, 3, \ldots$$

1. Find and plot the CDF.
2. Find $P(2 < X \le 5)$.
3. Find $P(X > 4)$.



$$P(2 < X \le 5) = P(X \le 5) - P(X \le 2)$$

$$= F_X(5) - F_X(2)$$

$$= \left[1 - \frac{1}{2^5}\right] - \left[1 - \frac{1}{2^2}\right] = \frac{1}{2^2} - \frac{1}{2^5}$$

$$P(X > 4) = 1 - P(X \le 4)$$

$$= 1 - F_X(4) = 1 - \left\{1 - \frac{1}{2^4}\right\} = \frac{1}{2^4}$$

Expected value

Given some numbers, we often want a descriptive summary of these values.

- Their *average* is a single number that represents/describes the whole collection.
- How might we describe a “representative value” for a random variable?
- With a random variable, some values occur more often than other values.
- We might want to weight the values more if they occur more often.

Example: suppose we have a fair die. How might we summarize the outcomes of this die using a single number? What about for an unfair die?

The *expected value* of a random variable is the weighted average of all of its values, where the weights are the probabilities that these values occur.

Definition 2: Expected value $E(\cdot)$

Let X be a discrete random variable. Then the *expected value* of X is defined as

E[X]   EX

$$E(X) = \sum_{\text{all } k} P(X = k) \cdot k \tag{5}$$

Example: Let $X \sim Bernoulli(p)$. Find $E(X)$.

$$E[X] = \sum_{k=0}^{1} P(X=k) \cdot k$$

$$= \underbrace{P(X=0) \cdot 0}_{0} + \underbrace{P(X=1) \cdot 1}$$

$$= P(X=1)$$

$\boxed{\text{End of } 10/8 \text{ lecture}}$

Linearity

If X is a random variable, then any function of X is also a random variable.

- For example, if $Y = aX + b$, we can talk about $EY = E[aX + b]$.

Theorem 3.2: Expectation is linear

We have

- $E[aX + b] = aEX + b$, for all $a, b \in \mathbb{R}$;
- $E[X_1 + X_2 + \cdots + X_n] = EX_1 + EX_2 + \cdots + EX_n$, for any set of random variables $X_1, X_2, \ldots, X_n$.

Example: Let $X \sim Binomial(n, p)$. Find $E(X)$.

Functions of random variables

If X is a r.v. and any *function $Y = g(X)$ of X* is itself a random variable.

- Range of Y is
$$R_Y := \{g(a) | a \in R_X\}$$
where R_X is the range of X.
- PMF of Y is
$$f_Y(b) \quad = \quad P(Y = b) \quad = \quad P\Big(g(X) = b\Big) \quad = \quad \sum_{a:\, g(a)=b} f_X(a).$$
- Expected value of Y is
$$EY = \sum_{b \in R_Y} b\, f_Y(b).$$

In practice, usually easier to use the *law of the unconscious statistician (LOTUS)*:
$$EY \; = \; E\big[g(X)\big] \; = \; \sum_{a \in R_X} g(a)\, f_X(a).$$

Example

Find $E[\sin(X)]$, where X is a discrete random variable with range

$$R_X = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}$$

and PDF values

$$f_X(0) = f_X\left(\frac{\pi}{4}\right) = f_X\left(\frac{\pi}{2}\right) = f_X\left(\frac{3\pi}{4}\right) = f_X(\pi) = \frac{1}{5}.$$

Variance

Intuition

Often summarize a probability distribution by its center and spread.

- Center: expected value
- Spread: *variance*

Think of variance as "how much a random variable varies about its mean."

Definition 3: Variance $Var(\cdot)$

Let X be a random variable with $E(X^2) < \infty$. Then the *variance* of X is defined as

$$Var(X) := E\left[\{X - E(X)\}^2\right]. \tag{6}$$

- A *large value* of $Var(X)$ means that $\{X - E(X)\}^2$ is often large, so X often takes values far from its mean.
  - Implies that the distribution is very spread out.

- A *small value* of $Var(X)$ means that $\{X - E(X)\}^2$ is often small, so X often takes values close to its mean.
  - Implies that the distribution is concentrated around its average.

Standard deviation

$Var(X)$ has a different unit than X. E.g., if X is a stock price.

- Can instead measure spread using the square root of variance:

Definition 4: Standard deviation $Var(\cdot)$

Let X be a r.v. with $E\left(X^2\right) < \infty$. Then the *standard deviation* of X is defined as

$$SD(X) := \sqrt{Var(X)} \tag{7}$$

- Despite having the same unit of X, the variance is easier to mathematically find the minimum of (i.e., take the derivative of).
- Usually we will describe a distribution's spread using the variance.

Properties and calculation tools

From Definition 3, we can deduce the following properties:

- $Var(X) \geq 0$.
- If $Var(X) = 0$, then X is constant.
- The variance of X can also be calculated as

$$Var(X) = E(X^2) - (E[X])^2. \tag{8}$$

Properties of $Var(\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X be a random variable with $E(X^2) < \infty$. Then

i) $Var(c) = 0$;
ii) $Var(X + c) = Var(X)$;
iii) $Var(cX) = c^2 Var(X)$;

Example: consider $c = 5, Var(X) = 1$.

Properties and calculation tools

Theorem: variance of sum of independent random variables

If $X_1, X_2, \ldots, X_n$ are *independent* random variables, then

$$Var\left(X_1 + X_2 + \cdots + X_n\right) = Var(X_1) + Var(X_2) + \cdots + Var(X_n). \tag{9}$$

Example: if $X \sim Binomial(n, p)$, find $Var(X)$.