

Section 4: Basics in probability theory

STA 141A – Fundamentals of Statistical Data Science

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MWF, 9:00 AM – 9:50 AM, TLC 1215

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Overview

- 1 Probability measure and random variables
- 2 PMF/PDF
- 3 Some distributions
- 4 Expected value
- 5 Variance and covariance
- 6 Conditional probability and independence
 - Bayes' rule

The prerequisite for this class is either STA 108 (regression) or STA 106 (ANOVA), so I expect you have already learned everything in this slide deck.

- If you need a refresher on probability, you can refer to this free textbook:
<https://www.probabilitycourse.com/>

Probability measure and random variables

Probability is a way to quantify randomness and/or uncertainty.

- e.g., coin flips, dice rolls, stocks, weather.
- Rules of probability should be intuitive and self-consistent.
- Self-consistent: the rules shouldn't lead to contradictions.
- Thus these rules must be constructed in a certain way.
- Suppose we want to assign a probability to each event in a set of possible events.
- We would like, at the very least:
 1. each probability to be a value between 0 and 1 (inclusive)
 2. the probability assigned to the full set of events to be 1
 3. the probability assigned to the empty set to be 0
- We need more restrictions to ensure self-consistency.

The following definition will lead to intuitive and self-consistent rules of probability.

Definition 1: Probability measure $P(\cdot)$

For a nonempty set Ω , the set function $P: \Omega \rightarrow [0, 1]$ is a *probability measure*, if

- $P(\Omega) = 1$,
- for any pairwise disjoint sets $A_1, A_2, \dots \subseteq \Omega$ (i.e. $A_i \cap A_j = \emptyset$ for all i, j with $i \neq j$), holds:

$$P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i). \quad (1)$$

This definition fulfills the three properties from the previous slide:

- $P(\Omega) = 1$: the probability of the biggest possible set is equal to 1.
- Property (1) allows us to add probabilities of disjoint sets.
 - ▶ Disjoint means having no shared elements.
 - ▶ (Property (1) is called the *countable additivity* property.)

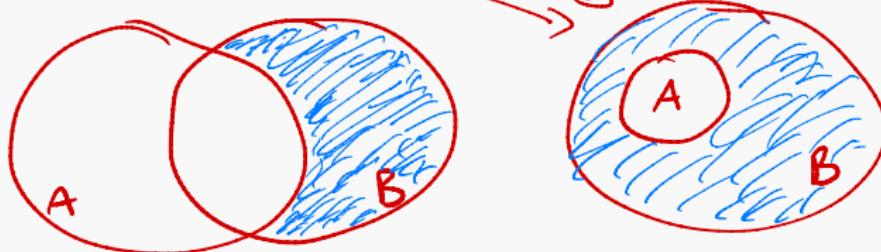
Probability measure - Properties

Definition 1 implies the following additional properties:

Properties of $P(\cdot)$

With \emptyset being the empty set, with some sets $A, B \subset \Omega$, and with $A^c = \Omega \setminus A$ denoting the complement of A , holds,

- i) $P(\emptyset) = 0; 1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) = 1 + P(\emptyset)$
- ii) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$; \rightarrow follows from countable additivity
- iii) $P(A^c) = 1 - P(A); 1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$
- iv) $P(B \setminus A) = P(B) - P(A)$ if $A \subseteq B$; $P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$
- v) $P(A) \leq P(B)$ if $A \subseteq B$.



Probability measures allow us to characterize the "randomness" of events.

- But we are often interested in more than just probabilities. For example:
 - ▶ the number of heads from three (independent) flips of some coin
 - ▶ the sum of the faces after throwing two dice
 - ▶ the lifetime of a battery
- We call each of these a *random variable* because they take on different values based on random events.
- The probability that a random variable is a certain value will depend on the probabilities of individual events.

PMF/PDF

When doing probability calculations, rather than use probability measures (which are functions of sets), it is often easier to describe a probability distribution using functions of single variables

1. PMF/PDF

The idea behind a PMF/PDF is to assign probabilities to the possible values of a random variable.

- The concept is different for discrete and continuous random variables.

PMF/PDF - discrete and continuous case

A random variable X is **discrete** if its range is finite or countably infinite.

- Examples:

1. number of heads after two coin flips,
2. number of coin flips needed before a heads turns up.

e.g. subsets of set of integers

- Here probabilities can be assigned to each realizable value. Examples:

1. For $\{0, 1, 2\}$ (finite), we can assign probabilities $1/4$, $1/2$, and $1/4$.
2. For \mathbb{N} (countably infinite), we can assign probabilities $(1/2)^k$ to each $k \in \mathbb{N}$.

- The **probability mass function** (PMF) f_X of a discrete random variable X assigns probabilities to each realizable value of X . Examples:

1. $f_X(0) = 1/4$, $f_X(1) = 1/2$, and $f_X(2) = 1/4$.
2. $f_X(k) = (1/2)^k$ for each $k \in \mathbb{N}$.

The PMF at a , $f_X(a) := P(X = a)$, is "the probability that X equals a ."

- The probability that X lies in a set A can be calculated by

$$\underline{P(X \in A)} = P\left(\bigcup_{a \in A} [X = a]\right) = \sum_{a \in A} P(X=a) = \sum_{a \in A} f_X(a) \quad (2)$$

Uncountable additivity

- E.g. for example 2, what is the probability that $X < 3$?

$$P(X < 3) = P(X \in \{1, 2\}) = \sum_{k=1}^2 f_X(k) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

PMF/PDF - discrete and continuous case

e.g. \mathbb{R} , intervals of \mathbb{R}

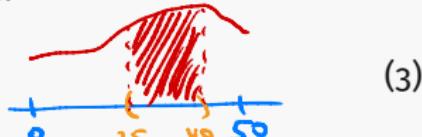
A random variable X is continuous if its range is uncountably infinite.

$[0, \infty)$

$(0, 50]$

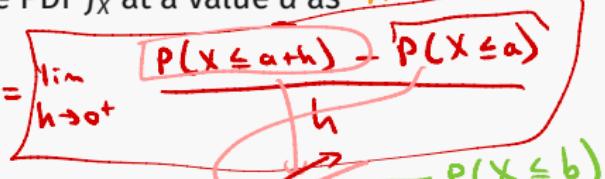
- Examples: lifetime of a person, time it takes you to finish the first midterm exam
- For any value in the range of a continuous random variable X , the probability that X is that value must be zero. Why?
 - If uncountably many values are assigned positive probability, the sum of those values would then be infinity!
- For a continuous random variable X , at any value a we have $P(X = a) = 0$.
- The probability density function (PDF) f_X of a continuous random variable X describes how likely it is for X to lie in a set A of values:

$$P(X \in A) = \int_A f_X(s) ds.$$



- Letting $A = (a, a + h]$, we can think of the PDF f_X at a value a as

$$\lim_{h \rightarrow 0^+} \frac{P(X \in A)}{h} = \lim_{h \rightarrow 0^+} \frac{P(a < X \leq a + h)}{h}$$



Want $f_X(a)$



PMF/PDF - discrete and continuous case

From the properties of probability measures, it follows that any PMF f_X of a discrete random variable X must satisfy both

1. $f_X(a) \geq 0$ for all a , and
2. $\sum_{\text{all } a} f_X(a) = 1$.

Similarly, it follows that any PDF f_X of a continuous random variable X must satisfy both

1. $f_X(a) \geq 0$ for all a , and
2. $\int_{\text{all } a} f_X(a) da = 1$.

Some distributions

Discrete case - Uniform distribution

A random variable X with values in a finite set M is *uniformly* distributed if each element in M has the same probability:

$$P(X = k) = \frac{1}{\#M} \quad \text{for all } k \in M$$

"is distributed as"

- We write $X \sim U(M)$ or $X \sim \text{Unif}(M)$.
- Such distributions occur when all possible outcomes are equally likely.
- Nine random draws in R:

```
sample(c(1,2,3,4,5,6), size=9, replace=TRUE)
```

possible outcomes
of a six-sided die

Discrete case - Bernoulli distribution

A random variable X is *Bernoulli* distributed with parameter $p \in (0, 1)$, if $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

- We write $X \sim \text{Bern}(p)$.
- For when a random experiment has only two possible outcomes ("success" and "failure").
- Example: flip a coin with probability p of heads ("success"). Is it heads?
- Nine random draws in R: `rbinom(n=9, size=1, prob=1/3)`

`rbinom`(`n=9`, `size=1`, `prob=1/3`)

parameter P

End of 10/15

lecture

Discrete case - Binomial distribution

ways we can get k successes and $n-k$ failures

prob of k successes

prob of $n-k$ failures

A random variable X is **Binomial** distributed with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if

$$P(X=0) = (1-p)^n$$
$$P(X=2) = p^2$$
$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for all } k = 0, \dots, n.$$
$$P(X=1) = p(1-p) + (1-p)p \quad \binom{2}{1} = 2$$

- We write $X \sim \text{Bin}(n, p)$.
- For measuring the probability of the number of successes of n **independent** Bernoulli experiments with “success probability” parameter p .
- Example: how many heads in n independent coin flips, each flip with probability p of heads (“success”)?
- A random draw in R: `rbinom(n=3, size=1, prob=0.25) |> sum()`

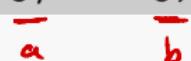
$\binom{n}{k} p^k (1-p)^{n-k}$

e.g. 011
 000

Continuous case - Uniform distribution

A random variable X is *uniformly* distributed on an interval $M = (a, b)$, with $b > a$, if the PDF has the form

$$f_X(c) = \frac{1}{b-a} \quad \text{for all } c \in (a, b).$$

- Here we also write $X \sim U(M)$ or $X \sim \text{Unif}(M)$.
- Such distributions occur when all (uncountably many) possible outcomes are equally likely.
- The interval M can also instead be $[a, b)$, or $(a, b]$, or $[a, b]$.
- Nine random draws in $(3, 5)$ in R: `runif(n=9, min=3, max=5)`
 (a, b) 

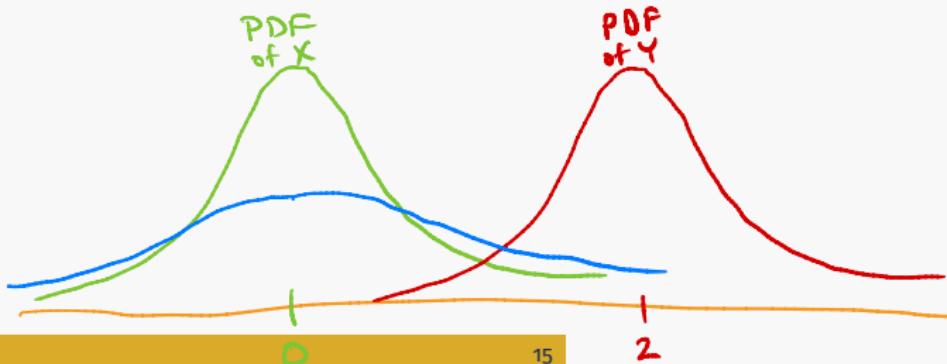
Continuous case - Normal distribution

A random variable X is **normally** distributed with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, if the PDF has the form

a **symmetric function** $f_X(c) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{c-\mu}{\sigma})^2}$ for all $c \in \mathbb{R}$.

- We write $X \sim N(\mu, \sigma^2)$. We also call it **Gaussian** distributed.
- This distribution appears often in this class, in future classes, and in life!
- It can be shown that $E(X) = \mu$ (location parameter) and $Var(X) = \sigma^2$ (squared scale).
- If $X \sim N(0, 1)$, the distribution of X is said to be **standard normal**.
- Nine random draws in R: `rnorm(n=9, mean=2, sd=1)`

PDF of $X \sim N(0, 1)$, $Y \sim N(2, 1)$, $Z \sim N(0, 3)$



Expected value

Expected value - Introduction

The expected value of a random variable is the weighted average of all of its values, where the weights are the probabilities that these values occur.

Definition 2: Expected value $E(\cdot)$

Let X be a random variable. Then, the *expected value* of X is in the discrete case and in the continuous case (given the PDF f_X) is defined as

$$E(X) = \sum_{\text{all } k} P(X = k) \cdot k \quad \text{resp.} \quad E(X) = \int_{\text{all } s} f_X(s) \cdot s \, ds. \quad (4)$$

- The expected value of a random variable sometimes does not exist if, for example, the random variable is continuous and the weights are "large" for large values of the random variable (e.g. $E(X) = \int_1^\infty \frac{1}{s^2} \cdot s \, ds = \infty$).

Expected value - Calculating expected value by hand

Calculate the expected value of a random variable with PDF $f_X(a) = \frac{3}{7}a^2$ where $a \in [1, 2]$

$$E(X) = \int_{-\infty}^{\infty} f_X(a) \cdot a \, da$$

$$= \int_{-\infty}^1 0 \cdot a \, da + \int_1^2 \frac{3}{7}a^2 \cdot a \, da + \int_2^\infty 0 \cdot a \, da$$

$$= \frac{3}{7} \int_1^2 a^3 \, da$$

$$= \frac{3}{7} \left[\frac{a^4}{4} \right]_1^2$$

$$= \frac{3}{28} (2^4 - 1^4)$$

$$= \frac{45}{28}$$

Properties of $E(\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X, Y be random variables for which their expected values $E(X)$ and $E(Y)$ exists. Then, the following rules hold.

- i) $E(c) = c;$
- ii) $E(cX) = cE(X);$
- iii) $E(X + Y) = E(X) + E(Y).$

Example with $c = 2, E(X) = 1, E(Y) = 5$

$$E(cx) = cE(x) = 2 \cdot 1 = 2$$

$$E(x+y) = E(x) + E(y) = 1 + 5 = 6$$

$$E(x+c) = E(x) + E(c) = E(x) + c = 1+2 = 3$$

$$E(cy+x) = E(cy) + E(x) = cE(y) + E(x) = 2 \cdot 5 + 1 = 11$$

Variance and covariance

Heuristics

Variance - Definition and properties

The variance of a random variable is the expected squared deviation of its values to its expected value.

Definition 3: Variance $\text{Var}(\cdot)$

Let X be a random variable with $E(X^2) < \infty$. Then the **variance** of X is defined as

$$\begin{aligned} & E[X - E(X)] \\ &= E[X] - E[E(X)] \\ &= E[X] - E[X] = 0 \quad \text{Var}(cX) = E[\underbrace{\{cX - E(cX)\}}_{\text{centered}}^2] \\ & \text{Var}(X) := E[\{X - E(X)\}^2]. \quad = E[(X^{\text{centered}})^2] \end{aligned} \tag{5}$$

Think of $\text{Var}(X)$ as “how much X varies about its mean.” We can deduce:

- $\text{Var}(X) \geq 0$.
- $\text{Var}(X) = 0 \Rightarrow X$ is constant.
- The variance of X can also be calculated as

$$\text{Var}(X) = E(X^2) - (E(X))^2. \tag{6}$$

Variance - Calculation tools

Properties of $\text{Var}(\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X be a random variable with $E(X^2) < \infty$. Then

- i) $\text{Var}(c) = 0$;
- ii) $\text{Var}(X + c) = \text{Var}(X)$;
- iii) $\text{Var}(cX) = c^2 \text{Var}(X)$;

Recall intuition: $\text{Var}(X)$ is “how much X varies about its mean.”

Example with $c = 5$, $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$.

$$\text{Var}(X + c) = \text{Var}(X) = 1$$

$$\text{Var}(cY) = c^2 \text{Var}(Y) = 5^2 \cdot 2 = 50$$

End of 10/17
lecture

Covariance and correlation - Motivation

Expected value and variance help characterize the distribution of a single random variable X .

Now suppose we want to characterize the relationship between two random variables X and Y .

- A complete characterization requires assigning probabilities to every possible pair of values that (X, Y) could be.
- Simpler characterizations are the *covariance* and *correlation* of X and Y .

Heuristics

Covariance - Definition and properties

$$\text{Var}(x) = E[(x - E(x))^2] = E[(x - E(x))(x - E(x))] = \text{Cov}(X, X)$$

Definition 4: Covariance $\text{Cov}(\cdot, \cdot)$

Let X, Y be random variables with $E(X^2), E(Y^2) < \infty$. Then the **covariance** between X and Y is defined as

$$\text{Cov}(X, Y) := E[(\overset{x \text{ centered}}{(X - E(X))})(\overset{y \text{ centered}}{(Y - E(Y))})]. \quad (7)$$

- The covariance between X and Y can also be calculated as

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y). \quad (8)$$

- We say X and Y are **uncorrelated** if $\text{Cov}(X, Y) = 0$. Then X and Y have no linear relationship, and $E(XY) = E(X)E(Y)$.
- $\text{Cov}(X, Y) > 0$ indicate a positive linear relationship between X and Y .
- $\text{Cov}(X, Y) < 0$ indicate a negative linear relationship between X and Y .
- Covariance is symmetric: $\text{Cov}(X, Y) = \text{Cov}(Y, X)$.

Correlation coefficient

$\sqrt{\text{Var}(X)}$ = standard deviation of X

Definition 5: Correlation coefficient $\rho(\cdot, \cdot)$

Let X, Y be random variables with $E(X^2), E(Y^2) < \infty$. Then, the **correlation coefficient** between X and Y is defined as, provided $\text{Var}(X) > 0$ and $\text{Var}(Y) > 0$,

$$\rho(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}} \in [-1, 1]. \quad (9)$$

Covariance between
two r.v.s w/ unit variance $\leftarrow \text{Cov}\left(\frac{X}{\sqrt{\text{Var}(X)}}, \frac{Y}{\sqrt{\text{Var}(Y)}}\right)$

- $\rho(X, Y) = 0 \Rightarrow$ between X and Y is no linear relationship.
- $\rho(X, Y) = -1 (1) \Rightarrow$ all values of X and Y lie on a line with negative (positive) slope.
- If $\rho(X, Y)$ is close to $-1 (1)$, there is a strong negative (positive) linear relationship between X and Y .

Variance and covariance - More calculation tools

Properties of $\text{Var}(\cdot)$ and $\text{Cov}(\cdot, \cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X, Y, Z be random variables with $E(X^2) < \infty$, $E(Y^2) < \infty$, and $E(Z^2) < \infty$. Then

iv) $\text{Var}(X) = \text{Cov}(X, X)$

v) $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

vi) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

vii) $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$ and $\text{Cov}(cX, Z) = c\text{Cov}(X, Z)$

(Property vii says $\text{Cov}(\cdot, \cdot)$ is linear in its first argument. Because $\text{Cov}(\cdot, \cdot)$ is symmetric, it is also linear in its second argument. Thus we call it **bilinear**.)

Example with $c = 5$, $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$, $\text{Cov}(X, Y) = 1/3$.

$$\begin{aligned}\text{Var}(cx + y) &\stackrel{?}{=} \text{Var}(cx) + \text{Var}(y) + 2\text{Cov}(cx, y) \\ &= c^2\text{Var}(x) + \text{Var}(y) + 2c\text{Cov}(x, y) \\ &= 25 + 2 + \frac{10}{3} = \boxed{\frac{91}{3}}\end{aligned}$$

Conditional probability and independence

Heuristics

Independent \neq disjoint!

If A and B are disjoint,
then knowing B occurred tells
us that A did not occur.



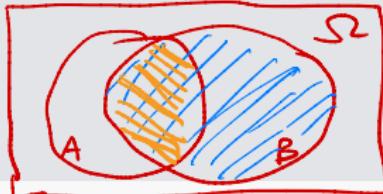
Knowing B occurred tells us
a lot of information about A.

Definition and properties

An **event** is a subset of the sample space Ω .

Definition 6: Conditional probability

For events $A, B \subseteq \Omega$, the **conditional probability** of A given B is defined by

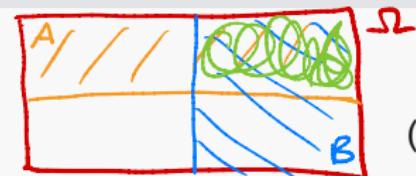


$$P(A|B) = \begin{cases} \frac{P(A \cap B)}{P(B)}, & \text{if } P(B) > 0, \\ 0, & \text{if } P(B) = 0. \end{cases} \quad (10)$$

- Events A and B are called **independent** if

If A and B are disjoint,
then knowing B occurred tells
us that A did not occur

$$P(A \cap B) = P(A)P(B).$$



(11)

Here knowing B provides no information about A , and vice versa.

- Equivalently, events A and B are independent if $P(A|B) = P(A)$.
- Random variables X and Y are called **independent** if for all sets A, B holds,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B). \quad (12)$$

- Independent random variables are uncorrelated.
- But uncorrelated random variables are not necessarily independent!

Conditional probability and independence

Bayes' rule

Introduction

From the definition of conditional probability, we know for any two events A and B that

$$P(B|A) \frac{P(A)}{P(A)} = P(A \cap B) = P(A|B) P(B).$$

Dividing by $P(A)$ (assuming it is not zero), we get **Bayes' rule**:

Theorem 2: Bayes' theorem

Let $\Omega \neq \emptyset$. For any events $A, B \subseteq \Omega$ with $P(A) \neq 0$ holds,



$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}. \quad (13)$$

Often $P(A)$ is unknown and difficult to deduce; can use the **law of total probability** (14).

- Because the sets $A \cap B$ and $A \cap B^c$ partition the set A , we can write $P(A)$ as

$$P(A) = P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c) \stackrel{?}{=} P(A|B)P(B) + P(A|B^c)P(B^c).$$

- More generally, for any partition $\{B_1, B_2, \dots\}$ of Ω , we can write $P(A)$ as

Then $A \cap B_1, A \cap B_2, \dots$ partition A

$$P(A) = \sum_{j=1}^{\infty} P(A|B_j)P(B_j). \quad (14)$$

Example: False positive paradox

$$P(D) \approx \frac{1}{10000} = 0.0001$$

A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. In particular, we know that

- the probability that the test result is positive, given that the person does not have the disease, is 2%;
- the probability that the test result is negative, given that the person has the disease, is 1%. $P(+^c | D) = 0.01 \Rightarrow P(+|D) = 1 - 0.01 = 0.99$

Suppose a random person gets tested for the disease and the test result is positive.

What is the probability that the person has the disease?

"+" - positive test result

"D" - person has disease

$$P(+|D) \quad P(D)$$

$$\text{Bayes Thm: } P(D|+) = \frac{P(+|D) P(D)}{P(+)}$$

End of 10/20
lecture

Example: False positive paradox

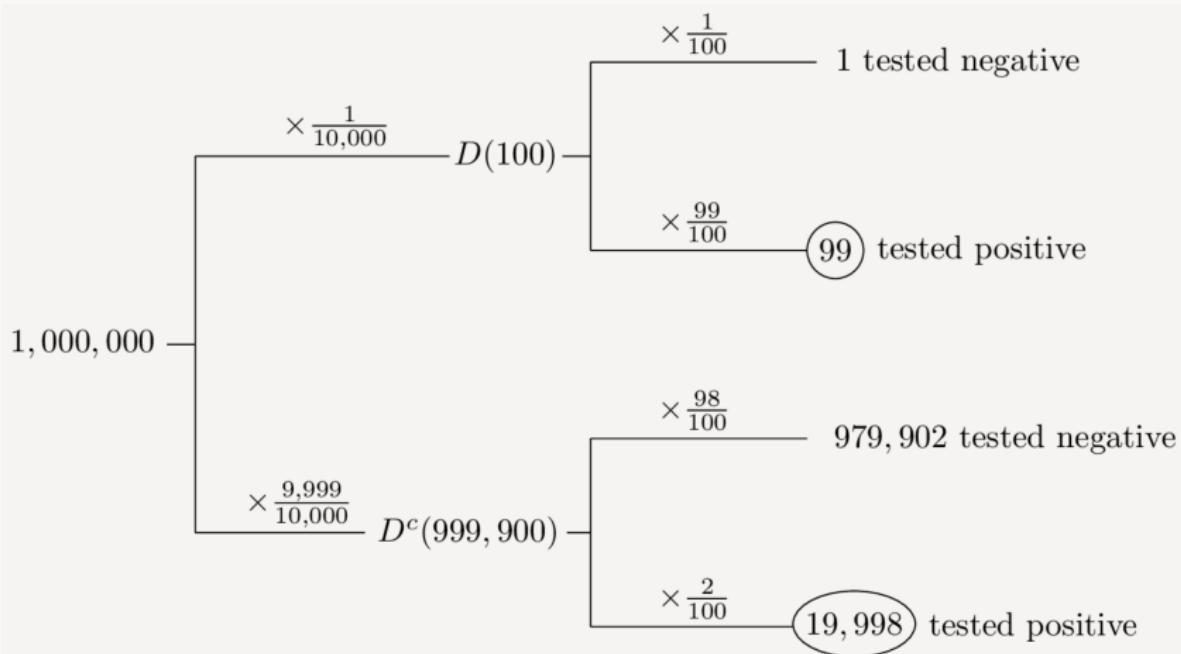


Fig.1.25 - Tree diagram for Example 1.26.

Figure: From textbook Pishro-Nik (2014).

Bayes' rule enables *Bayesian statistics* (STA 145).

- *Bayesian* interpretation: probability expresses a degree of belief in an event.
Use *Bayes' rule* to update degree of belief based on observed data.
- *Frequentist* interpretation: probability is the long-run relative frequency of an event after many trials.
- Don't need to know for this course. More intuition here
<https://www.youtube.com/watch?v=9wCnvr7Xw4E>