Section 4: Random variables – continuous

STA 35C – Statistical Data Science III

Instructor: Akira Horiguchi

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University of California, Davis

Based on Chapter 4 of textbook: `https://www.probabilitycourse.com/`

- Contains problems with solutions, and problems without solutions.

Introduction

- A *discrete* random variable has a countable range.
- A *continuous* random variable has a range in the form of an interval or a union of non-overlapping intervals on the real line (possibly the whole real line).
  For any $a \in \mathbb{R}$, we have $P(X = a) = 0$.
- Need new tools. Overview: replace sums with integrals; replace PMFs with PDFs.

Example

Find the CDF of a real number chosen *uniformly at random* in the interval $[a, b]$.

- *Uniformly at random*: all intervals in $[a, b]$ that have the same length must have the same probability.
- For any interval $[c, d] \subset [a, b]$ with $c < d$, the uniformity assumption implies

$$P\big(X \in [c, d]\big) \propto (d - c). \tag{1}$$

- Since $P\big(X \in [a, b]\big) = 1$, we conclude

$$P\big(X \in [c, d]\big) = \frac{d - c}{b - a}. \tag{2}$$

To get the CDF, note that

- $F_X(w) = 0$ if $w < a$;
- $F_X(w) = 1$ if $w > b$;
- For $w \in [a, b]$, we have

$$F_X(w) = P(X \le w) = P\big(X \in [a, w]\big) = \frac{w - a}{b - a}. \tag{3}$$

Thus the CDF looks like (no jumps)

Definition

In the previous example, the CDF is a continuous function (it has no jumps).

- Recall: jumps in the CDF correspond to values a for which $P(X = a) > 0$.
- "No jumps" is consistent with $P(X = a) = 0$ for all a.

Definition 1: continuous random variable

A random variable X is said to be *continuous* if its CDF $F_X(a)$ is a continuous function for all $a \in \mathbb{R}$.

Probability density function

Motivation

To determine the distribution of a random variable of any type, we can provide its CDF.

- For a *continuous* r.v., we cannot use a PMF because $P(X = a) = 0$ for all $a \in \mathbb{R}$.
- We will instead define its *probability density function (PDF)*.
  Similar to mass density in physics.
- For a continuous r.v. X and a value $a \in \mathbb{R}$, consider the quantity

$$\lim_{h \to 0^+} \frac{P(a < X \leq a + h)}{h}$$

For simplicity, we will assume for this course that the CDF of a continuous random variable is differentiable *except at possibly a few values in $\mathbb{R}$.*

- "almost everywhere" in $\mathbb{R}$

Definition 1: continuous random variable

Consider a continuous random variable X with an "absolutely continuous" CDF F_X. The function f_X defined by

$$f_X(a) = \frac{\mathrm{d}F_X(a)}{\mathrm{d}a} = F_X'(a), \quad \text{if } F_X(a) \text{ is differentiable at } a \tag{4}$$

is called the *probability density function (PDF)* of X.

"absolutely continuous" allows us to obtain the CDF from a PDF (we'll see this later)

Example

Let's find the PDF of the uniform random variable discussed in slide 3.

- This random variable is said to have a *Uniform*(a, b) distribution.
- The CDF of X is given in slide 3. By taking the derivative, we obtain the PDF:

$$f_X(c) = \begin{cases} \frac{1}{b-a} & \text{if } a < c < b, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

This function (the PDF) looks like

Obtain the CDF from a PDF

If we are given a PDF f_X, then the CDF is the function F defined by

$$F(c) = \int_{-\infty}^{c} f_X(u)\,\mathrm{d}u \tag{6}$$

- "absolutely continuous" is a stronger condition than "continuous"; allows us to use the fundamental theorem of calculus.
- For this course, we'll assume that all CDFs of *continuous* random variables are absolutely continuous.

From Eq. (6) we also get

$$P(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f_X(u)\,\mathrm{d}u \tag{7}$$

In particular, since $P(-\infty < X \leq \infty) = 1$, we must also have

$$\int_{-\infty}^{\infty} f_X(u)\,\mathrm{d}u = 1, \tag{8}$$

i.e., *the area under the PDF curve must be equal to one.*

PDF properties

Summary of properties of a PDF

Consider a continuous random variable X with PDF f_X. We have

1. $f_X(c) \geq 0$ for all $c \in \mathbb{R}$.
2. Eq. (8)
3. Eq. (7)
4. More generally, for a set A, we have $P(X \in A) = \int_A f_X(u)\,\mathrm{d}u$.

For item 4, the set A must satisfy some mild conditions which are almost always satisfied in practice. E.g., the set A could be a union of some disjoint intervals:

$$P\Big(X \in [0,2] \cup [3,4]\Big) = \int_0^2 f_X(u)\,\mathrm{d}u + \int_3^4 f_X(u)\,\mathrm{d}u.$$

Example

Let X be a continuous random variable with the following PDF

$$f_X(u) = \begin{cases} c\, e^{-u} & u \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

1. Find c.
2. Find the CDF of X.
3. Find $P(1 < X < 3)$.

Range

If X is a continuous random variable, we can define the *range of X* as

$$R_X = \{c \mid f_X(c) > 0\},$$

i.e., the set of real numbers c for which the PDF f_X is larger than zero.