Section 2: Probability

STA 35C – Statistical Data Science III

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MWF, 12:10 PM – 1:00 PM, Olson 158
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Section 1: Overview

Based on Chapter 1 of textbook: `https://www.probabilitycourse.com/`

1. Probability
   - Finding probabilities
   - Probability models: discrete vs continuous
2. Conditional probability
   - Independence
   - Law of total probability
   - Bayes' rule
   - Conditional independence

Probability

Probability measure: introduction

Probability is a way to quantify randomness and/or uncertainty.

- e.g., coin flips, dice rolls, stocks, weather.
- Rules of probability should be intuitive and self-consistent.
- Self-consistent: the rules shouldn't lead to contradictions.
- Thus these rules must be constructed in a certain way.
- Suppose we want to assign a probability to each event in a set of possible events.
- We would like, at the very least:
  1. each probability to be a value between 0 and 1 (inclusive)
  2. the probability assigned to the full set of events to be 1
     - close to 1 $\Rightarrow$ very likely that A occurs.
  3. the probability assigned to the empty set to be 0
     - close to 0 $\Rightarrow$ very unlikely that A occurs.
- We need more restrictions to ensure self-consistency.

The following definition will lead to intuitive and self-consistent rules of probability.

- We assign a *probability* measure $P(A)$ to an event A.

Probability measure: definition

Definition 1: Probabilty measure $P(\cdot)$

For a nonempty sample space Ω, the set function $P\colon \Omega \to [0, 1]$ is a *probability measure*, if

- $P(\Omega) = 1$,
- for any pairwise disjoints events $A_1, A_2, A_3, \dots \subset \Omega$ (i.e. $A_i \cap A_j = \varnothing$ for all i, j with $i \neq j$), holds:

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots \tag{1}$$

This definition fulfills the three desirable properties:

- $P(\Omega) = 1$: the probability of the biggest possible set is equal to 1.
- Property (1) – called the *countable additivity* property – allows us to add probabilities of disjoint sets.

Probability

Finding probabilities

Finding probabilities

Given a random experiment with a sample space Ω, how do we find the probability of an event of interest? Use:

- the specific information that we have about the random experiment.
- the probability rules induced by Definition 1.

Finding probabilities: example

Example: Roll a fair four-sided die. What is the probability of $E = \{1, 3\}$?

- Information about experiment (fair die): $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\})$.
- Probability rules:

$$\begin{aligned} 1 &= P(S) \\ &= P\left(\{1\} \cup \{2\} \cup \{3\} \cup \{4\}\right) \\ &= P\left(\{1\}\right) + P\left(\{2\}\right) + P\left(\{3\}\right) + P\left(\{4\}\right) \\ &= 4\,P\left(\{1\}\right). \end{aligned}$$

Thus $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = \frac{1}{4}$. Finally,

$$P(E) = P(\{1, 3\}) = P\left(\{1\}\right) + P\left(\{3\}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Finding probabilities: notation

Annoying to write e.g., $P(\{2\})$

- Simplify to $P(2)$
- But always keep in mind that P is a function on sets, not on individual outcomes.

Finding probabilities: more tools

Definition 1 implies the following additional properties:

Properties of $P(\cdot)$

Given a sample space Ω and arbitrary events $A, B \subset \Omega$, Definition 1 implies

1. $P(\varnothing) = 0$
2. $P(A^c) = 1 - P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. $P(B \setminus A) = P(B) - P(A \cap B)$
5. $P(A) \leq P(B)$ if $A \subset B$.

(Pictures for intuition; for formal proofs, see “Example 1.10” in §1.3.3 of textbook)

Finding probabilities: example

Suppose we have the following information:

1. There is a 60 percent chance that it will rain today.
2. There is a 50 percent chance that it will rain tomorrow.
3. There is a 30 percent chance that it does not rain either day.

Find the following probabilities:

a. The probability that it will rain today or tomorrow.
b. The probability that it will rain today and tomorrow.
c. The probability that it will rain today but not tomorrow.
d. The probability that it either will rain today or tomorrow, but not both.

Probability

Probability models: discrete vs continuous

Probability models: discrete vs continuous

Distinguish between two different types of sample spaces: *discrete* and *continuous*.

- Will discuss in more detail in Section 3 of the course.
- Discrete: can compute the probability of an event by adding all outcomes in the event.
- Continuous: need to use integration instead of summation.

Probability models: discrete

If a sample space Ω is a countable set, this refers to a *discrete* probability model.

- Can list all elements: $\Omega = \{s_1, s_2, s_3, \cdots\}$.
- For an event $A \subset \Omega$, by countable additivity (1) we can write

$$P(A) = P\left(\bigcup_{s \in A} \{s\}\right) = \sum_{s \in A} P(s)$$

  Thus, to find probability of an event, just need to sum the probability of individual elements in that event.

Probability models: discrete (example)

Consider a gambling game: win $k - 2$ dollars with probability $\frac{1}{2^k}$ for any $k \in \mathbb{N}$.

- What is the probability of winning at least \$1 and less than \$4?
- What is the probability of winning more than \$1?

Probability models: discrete (equally likely outcomes)

Important special case: finite sample space Ω where each outcome is equally likely.

- Thus for any outcome $s \in \Omega$, we must have

$$P(s) = \frac{1}{|\Omega|}.$$

- In such a case, for any event A, we can write

$$P(A) = \sum_{s \in A} P(s) = \sum_{s \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}.$$

Consider a sample space Ω that is an *uncountable* set.

- E.g., a 50-minute exam (so $\Omega = [0, 50]$), and let T_{Ant} be the time it takes Ant to finish the exam.
- What is the probability of $T_{Ant} \in [42.5, 45)$?

Conditional probability

Introduction

As you obtain *additional information*, how should you update probabilities of events?

- For example, suppose I roll a fair die.
- Let $A = \{1, 3, 5\}$. What is the probability that the outcome is in A? We will write this as $P(A)$.
- Let $B = \{1, 2, 3\}$. What is the probability of A if I know that the outcome is in B? We will write this as $P(A|B)$.

Notation

In the previous example, we call...

- ...$P(A)$ the *prior probability* of A;
- ...$P(A|B)$ the *conditional probability of A given that B has occurred.*
  - Usually shortened to the *conditional probability of A given B.*

The way we obtained $P(A|B)$ in this example can be generalized by the following definition.

Definition 2: Conditional probability

If A and B are two events in a sample space Ω, then the *conditional probability of A given B* is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0. \tag{2}$$

If we know that B has occurred, then we can discard every outcome outside of B.

- In other words, our sample space is reduced to the set B. (Picture)
- We divide $P(A \cap B)$ by $P(B)$ so that the conditional probability of the new sample space B becomes 1.
- $P(A|B)$ is undefined when $P(B) = 0$ (meaning B never occurs).

Conditional probability rules

Conditional probability itself is a probability measure.

- So all probability rules learned so far can be extended to conditional probability.

For example, Definition 1 (slide 3) and other properties (slide 7)

Important special cases

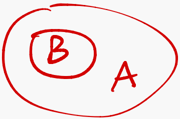
Plug into Definition 2

- When A and B are disjoint:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

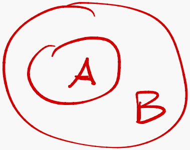
- When B is a subset of A:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$



- When A is a subset of B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$



Example

I roll a fair six-sided die twice. Let

- X_1 be the result of the first roll;
- X_2 be the result of the second roll;

Given that I know $X_1 + X_2 = 7$, what is the probability that $X_1 = 4$ or $X_2 = 4$?

- Let B be the event that $X_1 + X_2 = 7$. Let A be the event that $X_1 = 4$ or $X_2 = 4$.

$$B = \{(1,6), (2,5), \underline{(3,4)}, \underline{(4,3)}, (5,2), (6,1)\}$$

$$A \cap B = \{(3,4), (4,3)\}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B| / 36}{|B| / 36} = \frac{2}{6}$$

Chain rule

We can rearrange the formula in Definition 2 as

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B).$$

We can generalize this to 3 events:

$$P(A \cap B \cap C) = P(A)\, P(B \cap C \mid A)$$

$$P(A)\; P(B \mid A)\; P(C \mid B, A)$$

$$P(B \cap C \mid A) = P(\quad \mid A)\; P(\quad \mid \quad A)$$

We can generalize this to $n \geq 2$ events (chain rule for conditional probability):

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_2, A_1) \cdots P(A_n|A_{n-1}, A_{n-2} \cdots A_1)$$

Example

Of 100 units of a certain product, 5 are defective. If we pick three of the 100 units at random, what is the probability that none of them are defective?

- For $i = 1, 2, 3$, let A_i be the event that the ith picked unit is NOT defective.

$$P(A_1 \cap A_2 \cap A_3) = \underbrace{P(A_1)}_{\frac{95}{100}} \underbrace{P(A_2 \mid A_1)}_{\frac{94}{99}} \underbrace{P(A_3 \mid A_2, A_1)}_{\frac{93}{98}}$$

$$P(A) \; P(B \mid A) \; P(C \mid B, A)$$

Conditional probability

Independence

Introduction

Let A be the event that it rains tomorrow. Let B be the event that the coin I toss (indoors) tomorrow lands heads up.

- Should the result of the coin toss depend on tomorrow's weather?
- Should the probability of A depend on whether or not B happens?
- Two events are *independent* if one does not convey any info about the other.

Definition 3: Independent events

Two events A and B are *independent* if and only if

$$P(A \cap B) = P(A)P(B). \tag{3}$$

If two events A and B are independent and $P(B) \neq 0$, then

$$\underline{P(A|B)} = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = \underline{P(A)},$$

i.e., the conditional probability $P(A|B)$ is the same as the prior probability $P(A)$.

- Sometimes it is obvious if two given events are independent or not.
- Other times, we need to check if they satisfy the independence condition ~~(4)~~.
  (3)

Example

I pick a random number from $\{1, 2, 3, \cdots, 10\}$, and call it N.

- Suppose that all outcomes are equally likely.
- Let A be the event that $N < 7$, and let B be the event that N is even.

Are A and B independent?

$P(A) = \frac{6}{10}$ $\qquad$ $P(B) = \frac{5}{10}$

$A \cap B = \{2, 4, 6\} \implies P(A \cap B) = \frac{3}{10}$

$\therefore$ A and B are independent

Corollary

The following result can now be proven:

Corollary 1

If events A and B are independent, then

- A and B^c are independent,
- A^c and B are independent, and
- A^c and B^c are independent.

Achtung!

Independent $\neq$ disjoint

- Two *independent* events convey no information about the other.
- Two *disjoint* events cannot occur at the same time.

Definition 4: Independent events (≥ 2)

For $n \geq 2$, events $A_1, A_2, \cdots, A_n$ are *independent* if and only if we have

$$P\left(\bigcap_{k \in \mathcal{K}} A_k\right) = \prod_{k \in \mathcal{K}} P(A_k). \qquad (4)$$

for every nonempty subset $\mathcal{K} \subset \{1, 2, \cdots, n\}$.

If $n=3$, need to check

$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$

$P(A_1 \cap A_3) = P(A_1)\,P(A_3)$

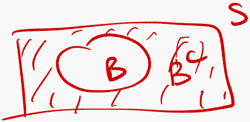
$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$

Conditional probability

Law of total probability

Result



Law of Total Probability:

If events $B_1, B_2, B_3, \cdots$ form a partition of the sample space, then for any event A we have

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)\, P(B_i). \quad (5)$$

countable additivity

definition of conditional probability

Because B and B^c partition the sample space, from (5) we get:

$$P(A) = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)\,.$$

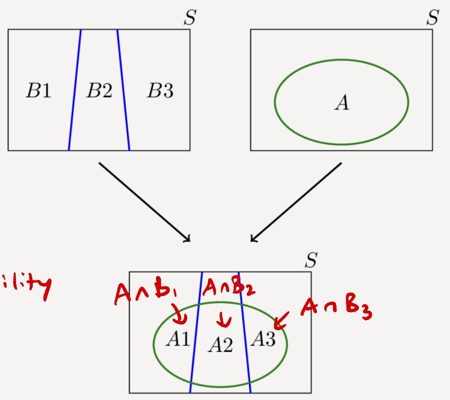


Fig.1.24 - Law of total probability.

$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$$

disjoint

An example

Suppose there is a population of students who are left- or right-handed (assume that no student is ambidextrous). We know that:

- 30% of these students are taller than 6 feet, and of these, 40% are left-handed.
- Of the remaining 70% of students, 20% are left-handed.

Using the law of total probability, calculate the probability that a student chosen uniformly at random from this population is left-handed.

- $P(>6\text{ ft}) = \underline{0.3}$
- $P(L \mid >6\text{ ft}) = \underline{0.4}$
- $P(L \mid \leq 6\text{ ft}) = \underline{0.2}$

$$P(L) = \underbrace{P(L \mid >6\text{ft})}_{0.4}\,\underbrace{P(>6\text{ ft})}_{0.3} + \underbrace{P(L \mid \leq 6\text{ft})}_{0.2}\,\underbrace{P(\leq 6\text{ft})}_{1-0.3}$$

Conditional probability

Bayes’ rule

From the definition of conditional probability, we know for any two events A and B that

$$\underbrace{P(B|A)\,P(A) = P(A \cap B) =}\, P(A|B)\,P(B)\,.$$

Dividing by $P(A)$ (assuming it is not zero), we get *Bayes' rule*:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} \tag{6}$$

Often $P(A)$ is unknown and difficult to deduce.

- Sometimes we can use the law of total probability (5).

Example: False positive paradox

A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. In particular, we know that

- the probability that the test result is positive, given that the person does not have the disease, is 2%;
- the probability that the test result is negative, given that the person has the disease, is 1%. $\leftarrow P(- \mid D) = 0.01 \Rightarrow P(+ \mid D) = 1 - 0.01 = 0.99$

Suppose a random person gets tested for the disease and the test result is positive. What is the probability that the person has the disease?

$$P(D \mid +) = \frac{P(+ \mid D)\, P(D)}{P(+)}$$

$$P(+) = \underbrace{P(+ \mid D)\, P(D)} + \underbrace{P(+ \mid N)\, P(N)}$$

End of 10/1 lecture

Example: False positive paradox

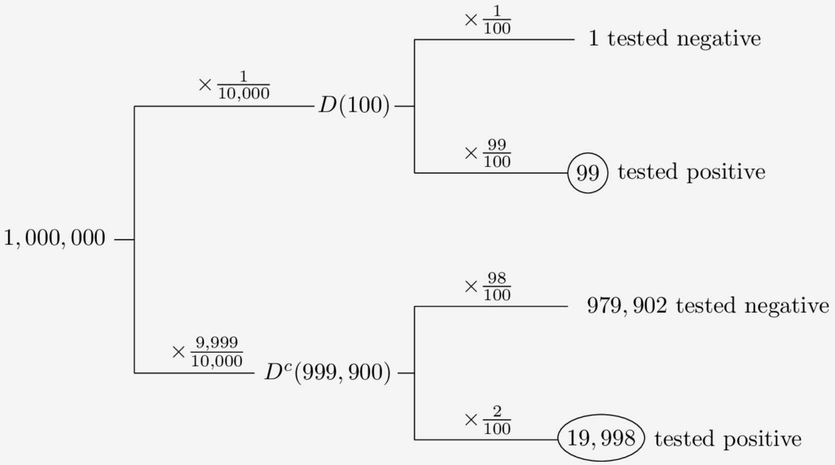


Fig.1.25 - Tree diagram for Example 1.26.

Bayesian paradigm

Bayes' rule leads to *Bayesian statistics*.

- *Bayesian* interpretation: probability expresses a degree of belief in an event. Use *Bayes' rule* to update degree of belief based on observed data.
- *Frequentist* interpretation: probability is the long-run relative frequency of an event after many trials.
- Don't need to know for this course. More intuition here `https://www.youtube.com/watch?v=9wCnvr7Xw4E`

Conditional probability

Conditional independence

Definition

Extend concept of *independence* to conditionally independent events.

Conditional independence

Two events A and B are *conditionally independent* given an event C with $P(C) > 0$ if

$$P(A \cap B|C) = P(A|C)\,P(B|C) \tag{7}$$

Example: Two coins

A box contains two coins: one regular coin and one two-headed coin ($P(H) = 1$). Choose a coin at random and toss it twice. Define the following events.

- A: First coin toss results in an H.
- B: Second coin toss results in an H.
- C: Coin 1 (regular) has been selected.

Note that A and B are not independent, but they are *conditionally independent* given C. Find $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B), P(A \cap B)$.