Online Supplement

"*Identifying Peer Effects on Student Academic Effort*"

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S.1 Additional Notes for the Proofs

S.1.1 Some Basic Properties

In this section, we state and prove some basic properties used throughout the paper.

P.1 Let $[\mathbf{F}_s,\ \bar{\boldsymbol{\ell}}_s/\sqrt{\bar{n}_s},\ \hat{\boldsymbol{\ell}}_s/\sqrt{\hat{n}_s}]$ be the orthonormal matrix of $\mathbf{J}_s$, where the columns in $\mathbf{F}_s$ are eigenvectors of $\mathbf{J}_s$ corresponding to the eigenvalue one. $\|\mathbf{F}_s\|_2 = 1$, where $\|.\|_2$ is the operator norm induced by the ℓ^2-norm.

Proof. $\|\mathbf{F}_s\|_2 = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \sqrt{(\mathbf{F}_s\mathbf{u}_s)'(\mathbf{F}_s\mathbf{u}_s)} = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \sqrt{\mathbf{u}_s'\mathbf{u}_s}$ because $\mathbf{F}_s'\mathbf{F}_s = \mathbf{I}_{n_s-2}$, the identity matrix of dimension $n_s - 2$. Thus, $\|\mathbf{F}_s\|_2 = 1$. □

P.2 For any $n_s \times n_s$ matrix, $\mathbf{B}_s = [b_{s,ij}]$, $|b_{s,ii}| \leqslant \|\mathbf{B}_s\|_2$.

Proof. Let $\mathbf{u}_s$ be the n_s-vector of zeros except for the i-th element, which is one. Note that $\|\mathbf{u}_s\|_2 = 1$. The i-th entry of $\mathbf{B}_s\mathbf{u}$ of $b_{s.ii}$. As a result, $|b_{s,ii}| \leqslant \sqrt{\sum_{j=1}^{n_s} b_{s,ji}^2} = \sqrt{(\mathbf{B}_s\mathbf{u})'(\mathbf{B}_s\mathbf{u})} \leqslant \|\mathbf{B}_s\|_2$. □

P.3 If $\mathbf{B}_s$ is a symmetric matrix of dimension $n_s \times n_s$, then $\|\mathbf{B}_s\|_2 = \pi_{\max}(\mathbf{B}_s)$, where $\pi_{\max}(.)$ is the largest eigenvalue.

Proof. $\|\mathbf{B}_s\|_2 = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \sqrt{(\mathbf{B}_s\mathbf{u}_s)'(\mathbf{B}_s\mathbf{u}_s)} = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \sqrt{\mathbf{u}_s'\mathbf{B}_s^2\mathbf{u}_s} = \sqrt{\pi_{\max}(\mathbf{B}_s^2)} = \pi_{\max}(\mathbf{B}_s)$. □

P.4 If $\mathbf{B}_s$ is a symmetric matrix of dimension $n_s \times n_s$, then $\pi_{\max}(\mathbf{F}_s'\mathbf{B}_s\mathbf{F}_s) \leqslant \pi_{\max}(\mathbf{B}_s)$.

Proof. $\pi_{\max}(\mathbf{F}_s'\mathbf{B}_s\mathbf{F}_s) = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \mathbf{u}_s'\mathbf{F}_s'\mathbf{B}_s\mathbf{F}_s\mathbf{u}_s = \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} (\mathbf{F}_s\mathbf{u}_s)'\mathbf{B}_s(\mathbf{F}_s\mathbf{u}_s)$. As $(\mathbf{F}_s\mathbf{u}_s)'(\mathbf{F}_s\mathbf{u}_s) = 1$, then $\max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} (\mathbf{F}_s\mathbf{u}_s)'\mathbf{B}_s(\mathbf{F}_s\mathbf{u}_s) \leqslant \max\limits_{\mathbf{u}_s'\mathbf{u}_s=1} \mathbf{u}_s'\mathbf{B}_s\mathbf{u}_s = \pi_{\max}(\mathbf{B}_s)$. □

P.5 Let $\mathbf{B}_{s,1}$ and $\mathbf{B}_{s,2}$ be $n_s \times n_s$ matrices. If $\mathbf{B}_{s,1}$ and $\mathbf{B}_{s,2}$ are absolutely bounded in row and column sums, then $\mathbf{B}_{s,1}\mathbf{B}_{s,2}$ is absolutely bounded in row and column sums.

Proof. It is sufficient to show that the entries of $\mathbf{B}_{s,1}\mathbf{B}_{s,2}\mathbf{u}_s$ and $\mathbf{u}_s'\mathbf{B}_{s,1}\mathbf{B}_{s,2}$ are absolutely bounded for all n_s-vector $\mathbf{u}_s$ whose entries take -1 or 1. Assume that $\mathbf{B}_{s,1}$ is absolutely bounded in row sum by $C_{b,1}$ and absolutely bounded in the row sum by $R_{b,1}$. Assume also that $\mathbf{B}_{s,2}$ is absolutely bounded in the row sum by $C_{b,2}$ and absolutely bounded in row sum by $R_{b,2}$. We have $\mathbf{B}_{s,2}\mathbf{u}_s \preceq R_{b,2}\mathbf{1}_{n_s}$ and $\mathbf{B}_{s,1}\mathbf{1}_{n_s} \preceq R_{b,1}\mathbf{1}_{n_s}$, where $\preceq$ is the pointwise inequality $\leqslant$ and $\mathbf{1}_{n_s}$

is an n_s-vector of ones. Thus, $\mathbf{B}_{s,1}\mathbf{B}_{s,2}\mathbf{u}_s \preceq R_{b,2}\mathbf{B}_{s,1}\mathbf{1}_{n_s} \preceq R_{b,1}R_{b,2}\mathbf{1}_{n_s}$. Hence, $\mathbf{B}_{s,1}\mathbf{B}_{s,2}$ is bounded in row sum. Analogously, we have $\mathbf{u}_s'\mathbf{B}_{s,1} \preceq C_{b,1}\mathbf{1}_{n_s}'$ and $\mathbf{1}_{n_s}'\mathbf{B}_{s,2} \preceq C_{b,2}\mathbf{1}_{n_s}'$. Thus, $\mathbf{u}_s'\mathbf{B}_{s,1}\mathbf{B}_{s,2} \preceq C_{b,1}\mathbf{1}_{n_s}'\mathbf{B}_{s,2} \preceq C_{b,1}C_{b,2}\mathbf{1}_{n_s}'$. Hence, $\mathbf{B}_{s,1}\mathbf{B}_{s,2}$ is bounded in column sum. □

P.6 If an $n_s \times n_s$ matrix $\mathbf{B}_s$ is absolutely bounded in both row and column sums, then $|\pi_{\max}(\mathbf{B}_s)| < \infty$ and $\|\mathbf{B}_s\|_2 < \infty$.

Proof. $|\pi_{\max}(\mathbf{B}_s)| < \infty$ is a direct implication of the Gershgorin circle theorem.[1]

Besides, $\|\mathbf{B}_s\|_2 = \sqrt{\pi_{\max}(\mathbf{B}_s'\mathbf{B}_s)} < \infty$ because $\mathbf{B}_s'\mathbf{B}_s$ is absolutely bounded in row and column sums by P.5. □

P.7 Let $\mathbf{B}_s = [b_{ij}]$, $\dot{\mathbf{B}}_s = [\dot{b}_{ij}]$ be $n_s \times n_s$ matrices. Let $\mathbf{G} = \operatorname{diag}(\mathbf{G}_1, \dots, \mathbf{G}_S)$, where diag is the block diagonal operator. Let also $\mu_{4\eta} = \mathbb{E}(\eta_{s,i}^4|\mathbf{G}_s, \mathbf{X}_s)$, $\mu_{4\epsilon} = \mathbb{E}(\varepsilon_{s,i}^4|\mathbf{G}_s, \mathbf{X}_s)$, $\mu_{22} = \mathbb{E}(\eta_{s,i}^2\varepsilon_{s,i}^2|\mathbf{G}_s, \mathbf{X}_s)$, $\mu_{31} = \mathbb{E}(\eta_{s,i}^3\varepsilon_{s,i}|\mathbf{G}_s, \mathbf{X}_s)$, and $\mu_{13} = \mathbb{E}(\eta_{s,i}\varepsilon_{s,i}^3|\mathbf{G}_s, \mathbf{X}_s)$. Under Assumptions 3.1 and A.3,

$\mathbb{V}(\boldsymbol{\eta}_s'\mathbf{B}_s\boldsymbol{\eta}_s|\mathbf{G}) = (\mu_{4\eta} - 3\sigma_{0\epsilon}^4)\sum_{i=1}^{n_s} b_{ii}^2 + \sigma_{0\epsilon}^4(\operatorname{Tr}(\mathbf{B}_s\mathbf{B}_s') + \operatorname{Tr}(\mathbf{B}_s^2))$,

$\mathbb{V}(\boldsymbol{\varepsilon}_s'\mathbf{B}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) = (\mu_{4\epsilon} - 3\sigma_{0\epsilon}^4)\sum_{i=1}^{n_s} b_{ii}^2 + \sigma_{0\epsilon}^4(\operatorname{Tr}(\mathbf{B}_s\mathbf{B}_s') + \operatorname{Tr}(\mathbf{B}_s^2))$,

$\mathbb{V}(\boldsymbol{\varepsilon}_s'\mathbf{B}_s\boldsymbol{\eta}_s|\mathbf{G}) = (\mu_{22} - 3\sigma_{0\eta}^2\sigma_{0\epsilon}^2)\sum_{i=1}^{n_s} b_{ii}^2 + (1-\rho^2)\sigma_{0\eta}^2\sigma_{0\epsilon}^2(\operatorname{Tr}(\mathbf{B}_s))^2 + \sigma_{0\eta}^2\sigma_{0\epsilon}^2\operatorname{Tr}(\mathbf{B}_s\mathbf{B}_s') + \rho^2\sigma_{0\eta}^2\sigma_{0\epsilon}^2\operatorname{Tr}(\mathbf{B}_s^2)$,

$\mathbb{C}\mathbf{ov}(\boldsymbol{\eta}_s'\mathbf{B}_s\boldsymbol{\eta}_s, \boldsymbol{\varepsilon}_s'\dot{\mathbf{B}}_s\boldsymbol{\eta}_s|\mathbf{G}) = (\mu_{31} - 3\rho\sigma_{0\eta}^3\sigma_{0\epsilon})\sum_{i=1}^{n_s} b_{ii}\dot{b}_{ii} + \rho\sigma_{0\eta}^3\sigma_{0\epsilon}(\operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s') + \operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s))$,

$\mathbb{C}\mathbf{ov}(\boldsymbol{\varepsilon}_s'\mathbf{B}_s\boldsymbol{\varepsilon}_s, \boldsymbol{\eta}_s'\dot{\mathbf{B}}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) = (\mu_{13} - 3\rho\sigma_{0\eta}\sigma_{0\epsilon}^3)\sum_{i=1}^{n_s} b_{ii}\dot{b}_{ii} + \rho\sigma_{0\eta}\sigma_{0\epsilon}^3(\operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s') + \operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s))$,

$\mathbb{C}\mathbf{ov}(\boldsymbol{\eta}_s'\mathbf{B}_s\boldsymbol{\eta}_s, \boldsymbol{\varepsilon}_s'\mathbf{B}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) = (\mu_{22} - 2\rho^2\sigma_{0\eta}^2\sigma_{0\epsilon}^2 - \sigma_{0\eta}^2\sigma_{0\epsilon}^2)\sum_{i=1}^{n_s} b_{ii}\dot{b}_{ii} + \rho^2\sigma_{0\eta}^2\sigma_{0\epsilon}^2(\operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s') + \operatorname{Tr}(\mathbf{B}_s\dot{\mathbf{B}}_s))$.

The proof of the lemma is straightforward using the classical definition of variance and covariance.

S.1.2 Identification and Consistent Estimator of $(\sigma_{\epsilon 0}^2,\ \tau_0,\ \rho_0)$

We must show that $\mathbb{V}\left(\hat{\sigma}_\epsilon^2(\tau,\rho)|\mathbf{G}\right) = o_p(1)$.

We have $\hat{\sigma}_\epsilon^2(\tau,\rho) = \displaystyle\sum_{s=1}^{S} \frac{((\mathbf{I}_{n_s} - \lambda_0\mathbf{G}_s)\boldsymbol{\eta}_s + \boldsymbol{\varepsilon}_s)'\mathbf{F}_s\boldsymbol{\Omega}_s^{-1}(\lambda_0,\tau,\rho)\mathbf{F}_s'((\mathbf{I}_{n_s} - \lambda_0\mathbf{G}_s)\boldsymbol{\eta}_s + \boldsymbol{\varepsilon}_s)}{n-2S}$. Thus,

$$\begin{aligned}\mathbb{V}(\hat{\sigma}_\epsilon^2(\tau,\rho)|\mathbf{G}) = &\frac{1}{(n-2S)^2}\sum_{s=1}^{S}\big(\mathbb{V}(\boldsymbol{\eta}_s'\ddot{\mathbf{M}}_s\boldsymbol{\eta}_s|\mathbf{G}) + 4\mathbb{V}(\boldsymbol{\eta}_s'\dot{\mathbf{M}}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) + \mathbb{V}(\boldsymbol{\varepsilon}_s'\mathbf{M}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) + \\ &4\,\mathbb{C}\mathbf{ov}(\boldsymbol{\eta}_s'\ddot{\mathbf{M}}_s\boldsymbol{\eta}_s, \boldsymbol{\eta}_s'\dot{\mathbf{M}}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) + 2\,\mathbb{C}\mathbf{ov}(\boldsymbol{\eta}_s'\ddot{\mathbf{M}}_s\boldsymbol{\eta}_s, \boldsymbol{\varepsilon}_s'\mathbf{M}_s\boldsymbol{\varepsilon}_s|\mathbf{G}) + \\ &4\,\mathbb{C}\mathbf{ov}(\boldsymbol{\varepsilon}_s'\mathbf{M}_s\boldsymbol{\varepsilon}_s, \boldsymbol{\eta}_s'\dot{\mathbf{M}}_s\boldsymbol{\varepsilon}_s|\mathbf{G})\big),\end{aligned} \tag{S.1}$$

where $\mathbf{M}_s = \mathbf{F}_s\boldsymbol{\Omega}_s^{-1}(\lambda_0,\tau,\rho)\mathbf{F}_s'$, $\dot{\mathbf{M}}_s = (\mathbf{I}_{n_s} - \lambda_0\mathbf{G}_s)'\mathbf{M}_s$, and $\ddot{\mathbf{M}}_s = \dot{\mathbf{M}}_s(\mathbf{I}_{n_s} - \lambda_0\mathbf{G}_s)$.

As $\pi_{\min}(\boldsymbol{\Omega}_s(\lambda_0,\tau,\rho)$ is bounded away from zero (Assumption A.2), we have $|\pi_{\max}(\boldsymbol{\Omega}_s^{-1}(\lambda_0,\tau,\rho)| = O_p(1)$. Thus, $\max_s\|\boldsymbol{\Omega}_s^{-1}(\lambda_0,\tau,\rho)\|_2 = O_p(1)$ by P.3. This implies that $\max_s\|\mathbf{M}_s\|_2 = O_p(1)$, $\max_s\|\dot{\mathbf{M}}_s\|_2 = O_p(1)$, and $\max_s\|\ddot{\mathbf{M}}_s\|_2 = O_p(1)$ because $\|\mathbf{F}_s\|_2 = 1$ and $\|\mathbf{I}_{n_s} - \lambda_0\mathbf{G}_s\|_2 = O_p(1)$ by P.6.

[1]See Horn, R. A. and C. R. Johnson (2012): *Matrix analysis*, Cambridge university press.

We now need to show that the sum over s of each term of the variance (S.1) is $o_p((n-2S)^2)$. By P.2, the trace of any product of matrices chosen among $\mathbf{M}_s$, $\dot{\mathbf{M}}_s$, and $\ddot{\mathbf{M}}_s$ is $O_p(n_s)$ and thus, $o_p((n-2S)^2)$. For example, $|\mathrm{Tr}(\mathbf{M}_s\dot{\mathbf{M}}_s)| \leqslant n_s\|\mathbf{M}_s\dot{\mathbf{M}}_s\|_2 \leqslant n_s\|\mathbf{M}_s\|_2\|\dot{\mathbf{M}}_s\|_2 = O_p(n_s) = o_p((n-2S)^2)$. On the other hand, $\sum_{s=1}^{S}(\mathrm{Tr}(\mathbf{M}_s))^2 = O_p(\sum_{s=1}^{S} n_s^2) = o_p((n-2S)^2)$. Moreover, $\sum_{i=1}^{n_s} m_{ii}^2 \leqslant n_s\|\mathbf{M}_s\|_2^2 = O_p(n_s) = o_p((n-2S)^2)$ by P.2. Analogously, $\sum_{i=1}^{n_s} m_{ii}\dot{m}_{ii} = o_p((n-2S)^2)$. As a result, $\mathbb{V}(\hat{\sigma}_\epsilon^2(\tau,\rho)|\mathbf{G}) = o_p(1)$.

The proof implies, by Chebyshev inequality, that $\hat{\sigma}_\epsilon^2(\tau,\rho) - \mathbb{E}\left(\hat{\sigma}_\epsilon^2(\tau,\rho)|\mathbf{G}_1,\ldots,\mathbf{G}_S\right)$ converges in probability to zero. The convergence is uniform in the space of $(\tau,\ \rho)$ because $\hat{\sigma}_\epsilon^2(\tau,\rho)$ and $\mathbb{E}\left(\hat{\sigma}_\epsilon^2(\tau,\rho)|\mathbf{G}_1,\ldots,\mathbf{G}_S\right)$ can be expressed as a polynomial function in $(\tau,\ \rho)$. Thus, $\frac{1}{n}(L_c(\tau,\rho) - L_c^*(\tau,\rho))$ converges uniformly to zero. This proof also implies that $\operatorname{plim}\hat{\sigma}_\epsilon^2(\tau_0,\rho_0) = \sigma_{\epsilon 0}^2$.

S.1.3 Necessary Conditions for the Identification of $(\sigma_{\epsilon 0}^2,\ \tau_0,\ \rho_0)$

As $\lambda_0 \neq 0$ (Condition (i) of Assumption 3.3) and is identified, $\mathbb{E}(\boldsymbol{v}_s\boldsymbol{v}_s'|\mathbf{G}_s)$ implies a unique $(\sigma_{\eta 0},\ \sigma_{\epsilon 0},\ \rho_0)$ if $\mathbf{J}_s$, $\mathbf{J}_s(\mathbf{G}_s+\mathbf{G}_s')\mathbf{J}_s$ and $\mathbf{J}_s\mathbf{G}_s\mathbf{G}_s'\mathbf{J}_s$ are linearly independent. We present a simple subnetwork structure that verifies this condition.

Let $\mathbf{C}_s$ be an arbitrary $n_s \times n_s$ matrix. Unless otherwise stated, we use $\mathbf{C}_{s,ij}$ to denote the $(i,\ j)$-th entry of $\mathbf{C}_s$. Assume that i and j are from the subset of students who have friends in the school s. The (i,j)-th entry of $\mathbf{J}_s\mathbf{C}_s\mathbf{J}_s$ is $\mathbf{C}_{s,ij} - \hat{\mathbf{C}}_{s,\bullet j} - \hat{\mathbf{C}}_{s,i\bullet} + \hat{\mathbf{C}}_{s,\bullet\bullet}$, where $\hat{\mathbf{C}}_{s,\bullet j} = (1/\hat{n}_s)\sum_{k\in\hat{\mathcal{V}}_s}^{n_s}\mathbf{C}_{s,kj}$, $\hat{\mathbf{C}}_{s,i\bullet} = (1/\hat{n}_s)\sum_{l\in\hat{\mathcal{V}}_s}^{n_s}\mathbf{C}_{s,il}$, and $\hat{\mathbf{C}}_{s,\bullet\bullet} = (1/\hat{n}_s^2)\sum_{k,l\in\hat{\mathcal{V}}_s}^{n_s}\mathbf{C}_{s,kl}$.

Let $\tilde{\mathbf{G}}_s = \mathbf{G}_s\mathbf{G}_s'$ and $i_1,\ \ldots,\ i_4$ be four students from $\hat{\mathcal{V}}_s$ who are not directly linked and where only two of them have common friends. Without loss of generality, assume that i_1 and i_3 have common friends. For any $i \in \{i_1,\ i_2\}$ and $j \in \{i_3,\ i_4\}$, $\mathbf{J}_{s,ij} = -1/\hat{n}_s$, $\mathbf{G}_{s,ij} = 0$, and $\mathbf{G}_{s,ij}' = 0$. Moreover, $\tilde{\mathbf{G}}_{s,ij} = 0$ except for the pair $(i_i,\ i_3)$, who have common friends. Let $\mathbf{L}_s = b_1\mathbf{J}_s + b_2\mathbf{J}_s(\mathbf{G}_s+\mathbf{G}_s')\mathbf{J}_s + b_3\mathbf{J}_s\mathbf{G}_s\mathbf{G}_s'\mathbf{J}_s = 0$ for some $b_1,\ b_2,\ b_3 \in \mathbb{R}$. We have $\mathbf{L}_{s,ij} = -b_1/\hat{n}_s - b_2(\mathbf{G}_{s,ij} - \mathbf{G}_{s,\bullet j} - \mathbf{G}_{s,i\bullet} + \mathbf{G}_{s,\bullet\bullet} + \mathbf{G}_{s,ij}' - \mathbf{G}_{s,\bullet j}' - \mathbf{G}_{s,i\bullet}' + \mathbf{G}_{s,\bullet\bullet}') + b_3(\tilde{\mathbf{G}}_{s,ij} - \tilde{\mathbf{G}}_{s,\bullet j} - \tilde{\mathbf{G}}_{s,i\bullet} + \tilde{\mathbf{G}}_{s,\bullet\bullet})$. This implies that $\mathbf{L}_{s,i_1i_3} + \mathbf{L}_{s,i_2i_4} - \mathbf{L}_{s,i_2i_3} - \mathbf{L}_{s,i_1i_4} = b_3\tilde{\mathbf{G}}_{s,i_1i_3}$. Thus, if the combination $\mathbf{L}_s$ is zero, then $b_3 = 0$.

Let $j_1,\ \ldots,\ j_4$ be four students from $\hat{\mathcal{V}}_s$, where only two of them are directly linked (mutually or not), and the others are not directly linked. Without loss of generality, assume that only j_1 to j_3 are linked, i.e., for any $i \in \{j_1,\ j_2\}$ and $j \in \{j_3,\ j_4\}$, $\mathbf{G}_{s,ij} = 0$ and $\mathbf{G}_{s,ij}' = 0$ except for the pairs $(j_1,\ j_3)$ and $(j_3,\ j_1)$. As $b_3 = 0$, we have $\mathbf{L}_{s,j_1j_3} + \mathbf{L}_{s,j_2j_4} - \mathbf{L}_{s,j_2j_3} - \mathbf{L}_{s,j_1j_4} = b_2(\mathbf{G}_{s,j_1j_3} + \mathbf{G}_{s,j_1j_3}')$. Thus if $\mathbf{L}_s$ is zero, then $b_2 = 0$, and it follows that $b_1 = 0$.

As a result, $\mathbf{J}_s$, $\mathbf{J}_s(\mathbf{G}_s+\mathbf{G}_s')\mathbf{J}_s$, and $\mathbf{J}_s\mathbf{G}_s\mathbf{G}_s'\mathbf{J}_s$ are linearly independent if, in some school s, there are four students from $\hat{\mathcal{V}}_s$ who are not directly linked and only two of them have common friends, and if in some school s, there are four students from $\hat{\mathcal{V}}_s$, where only two of them are linked.

We present an example of this condition by adding three nodes to Figure 1 with two additional links (see Figure S.1). There are no links within the nodes i_1, i_4, i_5, and i_6, and only i_5 and i_6 have common a friends (i_7). Besides, only i_5 and i_7 are linked within the nodes i_1, i_2, i_5, and i_7.

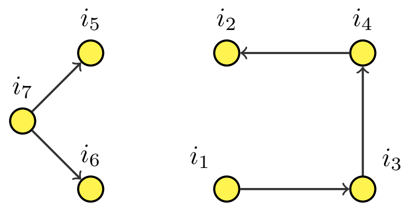

Figure S.1: Illustration of the identification

Note: $\rightarrow$ means that the node on the left side is a friend of the node on the right side.

Many other situations lead to $b_1 = b_2 = b_3 = 0$. In practice, one can easily verify if $\mathbf{J}_s$, $\mathbf{J}_s(\mathbf{G}_s + \mathbf{G}_s')\mathbf{J}_s$ and $\mathbf{J}_s\mathbf{G}_s\mathbf{G}_s'\mathbf{J}_s$ are linearly independent.

S.2 Bayesian Estimation of the Network Formation Model

In the Bayesian approach, we assume that $\mu^{in}_{0,s,i}$ and $\mu^{out}_{0,s,i}$ are random effects following $\mathcal{N}(0,\ \sigma^2_{in})$ and $\mathcal{N}(0,\ \sigma^2_{out})$, respectively, with $\mathbb{E}(\mu^{in}_{0,s,i}\mu^{out}_{0,s,i}) = \rho_\mu$. To simulate the posterior distribution of $\mu^{in}_{0,s,i}$ and $\mu^{out}_{0,s,i}$, we use the data augmentation technique.[2]

Let $a^*_{s,ij} = \ddot{\mathbf{x}}'_{s,ij}\ddot{\boldsymbol{\beta}}_0 + \mu^{in}_{0,s,i} + \mu^{out}_{0,s,j} + u_{s,ij}$, such that $a_{s,ij} = 1$ if $a^*_{s,ij} > 0$ and $a_{s,ij} = 0$ otherwise, where $u_{s,ij} \sim \mathcal{N}(0,\ 1)$. Let $\mathbf{a}_s = (a_{s,ij}; i \neq j)'$ and $\mathbf{a}^*_s = \left(a^*_{s,ij}; i \neq j\right)'$. The density function of $\mathbf{a}^*_s$, conditional on $\mathbf{a}_s$, $\ddot{\mathbf{X}}_s = [\ddot{\mathbf{x}}_{s,ij}; i \neq j]'$, $\ddot{\boldsymbol{\beta}}_0$, $\boldsymbol{\mu}^{in}_s = (\mu^{in}_{0,s,1}, \dots, \mu^{in}_{0,s,i})'$, and $\boldsymbol{\mu}^{out}_s = (\mu^{out}_{0,s,1}, \dots, \mu^{out}_{0,s,i})'$ is proportional to

$$\prod_{i\neq j} \left\{\mathrm{I}\left(a^*_{s,ij} \geqslant 0\right)\mathrm{I}\left(a_{s,ij} = 1\right) + \mathrm{I}\left(a^*_{s,ij} < 0\right)\mathrm{I}\left(a_{s,ij} = 0\right)\right\} \exp\left\{-\tfrac{1}{2}(a^*_{s,ij} - \ddot{\mathbf{x}}'_{s,ij}\ddot{\boldsymbol{\beta}}_0 - \mu^{in}_{0,s,i} - \mu^{out}_{0,s,j})^2\right\},$$

where I(.) is the indicator function. This implies that the distribution of $a^*_{s,ij}|\mathbf{a}_s, \ddot{\mathbf{X}}_s, \ddot{\boldsymbol{\beta}}_0, \boldsymbol{\mu}^{in}_s, \boldsymbol{\mu}^{out}_s$ is $\mathcal{N}(\ddot{\mathbf{x}}'_{s,ij}\ddot{\boldsymbol{\beta}}_0 + \mu^{in}_{0,s,i} + \mu^{out}_{0,s,j},\ 1)$, truncated at the left by 0 if $a_{s,ij} = 1$, and at the right by 0 if $a_{s,ij} = 0$. Given that the number of observations in the network formation model is high, we set a flat prior distribution for $\ddot{\boldsymbol{\beta}}_0$, σ^2_{in}, σ^2_{out}, and ρ_μ. Thus,

$$\ddot{\boldsymbol{\beta}}_0|\mathbf{a}_1, \mathbf{a}^*_1, \ddot{\mathbf{X}}_1, \boldsymbol{\mu}^{in}_1, \boldsymbol{\mu}^{out}_1, \dots, \mathbf{a}_S, \mathbf{a}^*_S, \ddot{\mathbf{X}}_S, \boldsymbol{\mu}^{in}_S, \boldsymbol{\mu}^{out}_S, \sim \mathcal{N}\left(\left(\ddot{\mathbf{X}}'\ddot{\mathbf{X}}\right)^{-1}\sum_{s=1}^S \ddot{\mathbf{X}}'_s\ddot{\mathbf{a}}^*_s,\ \left(\ddot{\mathbf{X}}'\ddot{\mathbf{X}}\right)^{-1}\right),$$

where $\ddot{\mathbf{X}}'\ddot{\mathbf{X}} = \sum_{s=1}^S \ddot{\mathbf{X}}'_s\ddot{\mathbf{X}}_s$ and $\ddot{\mathbf{a}}^*_s = (a^*_{s,ij} - \mu^{in}_{0,s,i} - \mu^{out}_{0,s,j} : i \neq j)'$. For any i,

$$\mu^{in}_{0,s,i}|\ddot{\boldsymbol{\beta}}_0, \mathbf{a}_s, \mathbf{a}^*_s, \ddot{\mathbf{X}}_s, \boldsymbol{\mu}^{in}_{s,-i}, \boldsymbol{\mu}^{out}_s \sim \mathcal{N}\left(\hat{u}_{s,in},\ \hat{\sigma}^2_{s,in}\right),$$

[2]See Albert, J. H., & Chib, S. (1993). Bayesian analysis of binary and polychotomous response data. *Journal of the American statistical Association*, 88(422), 669-679.

where $\hat{u}_{s,in} = \hat{\sigma}^2_{s,in} \sum_{i \neq j} (a^*_{s,ij} - \ddot{\mathbf{x}}'_{s,ij} \ddot{\boldsymbol{\beta}}_0 - \mu^{out}_{0,s,j})$ and $\hat{\sigma}^2_{s,in} = \dfrac{\sigma^2_{in}}{1 + (n_s - 1)\, \sigma^2_{in}}$. Analogously,

$$\mu^{out}_{0,s,i} | \ddot{\boldsymbol{\beta}}_0, \mathbf{a}_s, \mathbf{a}^*_s, \ddot{\mathbf{X}}_s, \boldsymbol{\mu}^{in}, \boldsymbol{\mu}^{out}_{-i} \sim \mathcal{N}\left(\hat{u}_{s,out},\ \hat{\sigma}^2_{s,out}\right),$$

where $\hat{u}_{s,out} = \hat{\sigma}^2_{s,out} \sum_{i \neq j} (a^*_{ji} - \ddot{\mathbf{x}}'_{s,ij} \ddot{\boldsymbol{\beta}}_0 - \mu^{in}_{0,s,j})$, and $\hat{\sigma}^2_{s,out} = \dfrac{\sigma^2_{out}}{1 + (n_s - 1)\, \sigma^2_{out}}$.

For the sake of identification, we normalize $\boldsymbol{\mu}^{in}$ and $\boldsymbol{\mu}^{out}$ to zero mean in each subnetwork for each step in the Gibbs sampling. The means of $\boldsymbol{\mu}^{in}$ and $\boldsymbol{\mu}^{out}$ before this normalization are added to the intercept of the subnetwork for the posterior likelihood not to change.

Finally, let $\boldsymbol{\Sigma}_{\mu,\nu} = \begin{pmatrix} \sigma^2_{in} & \rho_\mu \sigma_{in} \sigma_{out} \\ \rho_\mu \sigma_{in} \sigma_{out} & \sigma^2_{out} \end{pmatrix}$,

$$\boldsymbol{\Sigma}_{\mu,\nu} | \ddot{\boldsymbol{\beta}}_0, \mathbf{a}, \mathbf{a}^*, \ddot{\mathbf{X}}_s, \boldsymbol{\mu}^{in}, \boldsymbol{\mu}^{out} \sim \text{Inverse-Wishart}\left(n, \hat{\mathbf{V}}_{\boldsymbol{\Sigma}_{\mu,\nu}}\right),$$

where $\hat{\mathbf{V}}_{\boldsymbol{\Sigma}_{\mu,\nu}} = \sum_{i=1}^{n} (\mu^{in}_{0,s,i}, \mu^{out}_{0,s,i})$.