Basic Macro Model*

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1 A Two-Period Macro Model

Suppose that Olivia only lives for two periods. She needs to choose how much to consume in each period, denoted by c_0 and c_1 respectively. She values this consumption as follows:

$$U\left(c_0, c_1\right) = u(c_0) + \frac{1}{1+\delta}u(c_1)$$

$u(\cdot)$ denotes Olivia's per period, or **flow**, utility. We assume that u is increasing, concave, and twice-differentiable to ensure we have a well-behaved maximization problem. δ is a parameter governing how much Olivia discounts future utility. For this reason, $\delta > 0$.

Now suppose that Olivia is given ω_0 dollars in wealth in period $t = 0$. She gets no further income, but she can borrow or save at rate r in between period 0 and 1. Lastly, suppose that consumption costs p_0 in period 0 and p_1 in period 1.

We can now construct Olivia's budget constraint. First, note that in period 1:

$$\omega_1 = (1+r)(\omega_0 - p_0 c_0) \tag{1}$$

This should be fairly intuitive. Olivia's wealth tomorrow is her return on what she saved today. We also know, that $\omega_1 \geq p_1 c_1$. Obviously Olivia cannot buy more goods

*These notes draw heavily from notes provided by Maciej Kotowski.

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than she can pay for! Combine these two equations to get:

$$\omega_0 \geq \frac{1}{1+r} p_1 c_1 + p_0 c_0$$

The inequality is mostly a formality here; utility is strictly increasing in consumption, so Olivia will always consume all of her intertemporal wealth.

Before we continue, take another look at equation (1). This equation describes how Olivia gets from her wealth level today to her wealth level tomorrow. We call this a **law of motion**. We call ω_0 a **state** variable because Olivia takes it as a given, or in other words, the current state of the world. We call c_0 a **control** variable because Olivia decides the value of it. Given the state variable and Olivia's choice, we then get tomorrow's state variable (ω_1) from the law of motion.

We can now set-up a Lagrangian to solve this model, assuming $u(c) = \ln(c)$:

$$\mathcal{L} = \ln(c_0) + \frac{1}{1+\delta} \ln(c_1) + \lambda \left(\omega_0 - \frac{1}{1+r} p_1 c_1 - p_0 c_0 \right)$$

Take the first-order conditions with respect to c_0 and c_1:

$$\frac{1}{c_0} = \lambda p_0$$
$$\frac{1}{1+\delta} \frac{1}{c_1} = \frac{\lambda}{1+r} p_1$$

Eliminating λ gives:

$$\frac{1+r}{1+\delta} \frac{1}{p_1 c_1} = \frac{1}{p_0 c_0}$$

We call this equation the **Euler equation**. It relates consumption today to consumption tomorrow. Think about the economics for a moment. The right-hand side is the marginal utility of consumption today. In other words, it is the marginal benefit of consuming one unit today. The left-hand side is the opportunity cost of consuming today. That unit of consumption would have been saved (hence the $1+r$) and consumed tomorrow (hence the marginal utility of consumption tomorrow). We have to discount the future though, explaining the $1+\delta$ term. The Euler equation is the mathematical manifestation of "marginal benefit equals marginal cost."

Moving forward, we can plug the Euler equation into the budget constraint to find

c_1:

$$
\begin{aligned}
\omega_0 - \frac{1}{1+r} p_1 c_1 &= p_0 c_0 \\
\omega_0 - \frac{1}{1+r} p_1 c_1 &= \frac{1+\delta}{1+r} p_1 c_1 \\
(1+r)\frac{\omega_0}{p_1} &= (1+\delta) c_1 + c_1 \\
c_1 &= \frac{1+r}{2+\delta} \frac{\omega_0}{p_1}
\end{aligned}
$$

Plugging c_1 back into the Euler equation gives c_0:

$$
\begin{aligned}
c_0 &= \frac{1+r}{2+\delta} \frac{p_1}{p_0} \frac{\omega_0}{p_1} \frac{1+\delta}{1+r} \\
c_0 &= \frac{1+\delta}{2+\delta} \frac{\omega_0}{p_0}
\end{aligned}
$$

1.1 Eliminating a Choice Variable

Let's resolve the Lagrangian, but by substituting the budget constraint into the maximization problem first. In $t = 1$, we know that Olivia will consume all of her wealth:

$$
\omega_1 = p_1 c_1 \quad \Rightarrow \quad c_1 = \frac{\omega_1}{p_1}
$$

In $t = 0$, we already know the budget constraint:

$$
\omega_0 = \frac{1}{1+r} p_1 c_1 + p_0 c_0 \quad \Rightarrow \quad c_0 = \frac{\omega_0}{p_0} - \frac{1}{1+r} \frac{\omega_1}{p_0}
$$

Plug these into the utility maximization problem:

$$
\mathcal{L} = \ln\left(\frac{\omega_0}{p_0} - \frac{1}{1+r} \frac{\omega_1}{p_0}\right) + \frac{1}{1+\delta} \ln\left(\frac{\omega_1}{p_1}\right)
$$

We only have one choice variable now! We can take the first-order condition to find:

$$
\begin{aligned}
\frac{-1}{\omega_0/p_0 - \frac{1}{1+r}\omega_1/p_0} \frac{1}{(1+r)p_0} + \frac{1}{(1+\delta)p_1} \frac{p_1}{\omega_1} &= 0 \\
(1+\delta)\omega_1 &= (1+r)\omega_0 - \omega_1 \\
\omega_1 &= \frac{1+r}{2+\delta} \omega_0
\end{aligned}
$$

We can now back-out c_0 and c_1.

2 An Infinite-Horizon Macro Model

Much of modern macro uses infinite time models. Essentially, instead of living only two periods, Olivia lives for infinitely many periods. She must choose her consumption in every one of those periods. Her problem is:

$$\max_{(c_0,c_1,\dots)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta = \frac{1}{1+\delta}$. We make this substitution for two reasons: (1) it simplifies the expressions we will derive, and (2) β is used in all the literature. Olivia maximizes given a budget constraint. To derive the budget constraint, we decompose her spending each period into consumption and saving, s:

$$\omega_0 = p_0 c_0 + s_0$$

Next period, Olivia's wealth consists of her savings plus interest:

$$\begin{aligned}(1+r)s_0 &= p_1 c_1 + s_1 \\ \omega_0 - p_0 c_0 &= \frac{p_1 c_1}{1+r} + \frac{s_1}{1+r}\end{aligned}$$

Continue to sub-in for s:

$$\begin{aligned}\omega_0 &= p_0 c_0 + \frac{p_1 c_1}{1+r} + \frac{p_2 c_2}{(1+r)^2} + \frac{s_2}{(1+r)^2} \\ &\vdots \\ \omega_0 &= \sum_{t=0}^{T} \frac{p_t c_t}{(1+r)^t} + \frac{s_T}{(1+r)^T}\end{aligned}$$

This expression looks nice. The present-value of all consumption, plus the present value of savings in period T, must be equal to starting wealth. But recall that Olivia can borrow. Suppose Olivia starts in a world where $\omega_0 = 0$. She has no income, so she cannot increase her wealth. But she can go to the lender, borrow enough to consume one unit, and promise to pay back $(1 + r)$ tomorrow. Next period, Olivia must pay back this loan. But she has no income. So she can go to the lender, borrow enough to consume one unit and pay back her loan, and promise to pay the $2 + r$ the next day. We could carry this thought experiment out to infinity. Clearly, we do not believe the world works like this. It is standard, therefore, to impose a **no Ponzi-game** constraint on Olivia:

$$\lim_{T \to \infty} \frac{s_T}{(1+r)^T} = 0$$

So Olivia's lifetime budget constraint is:

$$\omega_0 = \sum_{t=0}^{\infty} \frac{p_t c_t}{(1+r)^t}$$

Let's now solve Olivia's problem by setting up a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \sum_{t=0}^{\infty} \frac{p_t c_t}{(1+r)^t}$$

and taking the FOCs with respect to c_t and c_{t+1}:

$$\beta^t u'(c_t) = \lambda \frac{p_t}{(1+r)^t} \tag{2}$$

$$\beta^{t+1} u'(c_{t+1}) = \lambda \frac{p_{t+1}}{(1+r)^{t+1}} \tag{3}$$

Divide equation (3) by (2):

$$\begin{aligned}
\frac{\beta^{t+1} u'(c_{t+1})}{\beta^t u'(c_t)} &= \frac{\lambda \frac{p_{t+1}}{(1+r)^{t+1}}}{\lambda \frac{p_t}{(1+r)^t}} \\
\frac{p_{t+1}}{p_t} \frac{1}{1+r} u'(c_t) &= \beta u'(c_{t+1}) \\
u'(c_t) &= \beta u'(c_{t+1})(1+r) \frac{p_t}{p_{t+1}}
\end{aligned}$$

We have our Euler equation! Now, many times we can only get this far analytically.[1] Here, if we assume $p_t = 1$ for all t, are given initial condition ω_0, and impose $u(c) = \ln(c)$,

[1] Technically, we could find the steady-state of the model. But that is for second semester macro.

we can solve for the optimal path of consumption by hand. The Euler equation becomes:

$$\begin{aligned}
\frac{1}{c_t} &= \beta \frac{(1+r)}{c_{t+1}} \\
c_{t+1} &= \beta(1+r)c_t
\end{aligned}$$

Recursively substitute-in for c_t:

$$\begin{aligned}
c_{t+1} &= \beta(1+r)\left(\beta(1+r)\right)c_{t-1} \\
&= \beta^2(1+r)^2 c_{t-1} \\
&\vdots \\
&= \beta^{t+1}(1+r)^{t+1}c_0
\end{aligned}$$

Now plug this into the intertemporal budget constraint:

$$\begin{aligned}
\omega_0 &= \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \\
&= \frac{1}{1-\beta}c_0 \\
\omega_0(1-\beta) &= c_0
\end{aligned}$$

Plugging this back into the Euler equation gives us the optimal path of consumption:

$$c_{t+1} = \beta^t(1+r)^t\omega_0(1-\beta)$$

Now let's calculate Olivia's indirect utility function. An **indirect utility function** is Olivia's maximal attainable utility given prices and income. We denote the indirect utility with $V(\omega_0)$, where the functional arguments always consist of the state variables:

$$\begin{aligned}
V(\omega_0) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\beta^t(1+r)^t\omega_0(1-\beta)\right) \\
&= \sum_{t=0}^{\infty} \beta^t \ln\left(\beta^t(1+r)^t\right) + \sum_{t=0}^{\infty} \beta^t \ln(1-\beta) + \sum_{t=0}^{\infty} \beta^t \ln(\omega_0) \\
&= \sum_{t=0}^{\infty} t\beta^t \ln\left(\beta(1+r)\right) + \frac{\ln(1-\beta)}{1-\beta} + \frac{\ln(\omega_0)}{1-\beta}
\end{aligned}$$

$$V(\omega_0) = \frac{\beta}{(1-\beta)^2}\ln\left(\beta(1+r)\right) + \frac{\ln(1-\beta)}{1-\beta} + \frac{\ln(\omega_0)}{1-\beta}$$

3 Dynamic Programming

Instead of solving Olivia's problem using Lagrangians, we can solve her problem using dynamic programming. **Dynamic programming** is a problem solving technique that takes a large, complex problem and breaks it down into small chunks. For our purposes, this means taking the infinite period model and breaking it into a problem consisting of today's state variable, today's control variable, and a transition function to get us the state variable tomorrow. For simplicity, let's assume that prices are always equal to unity. Write Olivia's problem as:

$$\max_{(c_0, c_1, \dots)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$\omega_{t+1} = (1+r)(\omega_t - c_t)\omega_t \geq 0$$

where we have included the no Ponzi-game constraint. Notice that instead of using a lifetime budget constraint, we are using a **flow** budget constraint, or her constraint from period-to-period. Now let's look at how we can rewrite Olivia's indirect utility function:

$$\begin{aligned}
V(\omega_0) &= \sum_{t=0}^{\infty} \beta^t u(c_t^*) \\
&= u(c_0^*) + \sum_{t=1}^{\infty} \beta^t u(c_t^*) \\
&= u(c_0^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^*) \\
&= u(c_0^*) + \beta V\left((1+r)(\omega_0 - c_0^*)\right) \\
&= u(c_0^*) + \beta V(\omega_1)
\end{aligned}$$

Notice we are claiming that $\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$ is the same function as $V(\omega_0)$. Why? Because it is the original problem shifted one period, with the initial state dependent on Olivia's choice of consumption in period zero, c_0. If the sequence $(c_0^*, c_1^*, \dots)$ solves her problem in period zero, then given optimal c_0, the sequence $(c_1^*, c_2^*, \dots)$ also solves her problem in period one. Another way to think about this structure: Olivia wants to choose optimal consumption today, but also wants to maximize her utility in the future. So $V(\omega_0)$ is her lifetime utility starting in $t = 0$, $V(\omega_1)$ is her lifetime utility starting in period $t = 1$. So $V(\omega_0)$ is just her utility today plus the discounted utility she gets in all future periods.

To choose optimal consumption, Olivia must take into account how her choice impacts her state variable tomorrow:

$$\omega_{t+1} = g(\omega_t, c_t) = (1 + r)(\omega_t - c_t)$$

where $g(\cdot)$ is the transition function from the state and control variables today to the state tomorrow. Let's rewrite her maximization problem again:

$$V(\omega_t) = \max_{c_t} \{u(c_t) + \beta V(g(\omega_t, c_t))\} \tag{4}$$

If we know $V(\omega)$, which we call the **value function**, we can solve Olivia's optimal consumption choice at wealth level ω. Solving for optimal consumption would give us what is called the consumption **policy function**:

$$c^*(\omega) = \operatorname*{argmax}_{c} \{u(c) + \beta V(g(\omega, c))\} \tag{5}$$

When substituted back into the value function:

$$V(\omega) = u(c^*(\omega)) + \beta V(g(\omega, c^*(\omega))) \tag{6}$$

This means that we can compute her consumption at every period! In period 0, $c_0^* = c^*(\omega_0)$. But then we know ω_1 through the transition function and then know that $c_1^* = c^*(\omega_1)$. And so on.

So how do we go about solving for $V(\omega)$ and the policy function? Let's generalize a bit before looking at our specific question. To solve, we first need the **Bellman equation**:

$$V(a) = \max_{a' \in \Gamma(a)} \{F(a, a') + \beta V(a')\} \tag{7}$$

where a is the state today, a' is the state tomorrow, $\Gamma(a)$ is the set of possible states tomorrow given the state today, and $F(a,a')$ is the payoff today. Notice that equation (4) is almost in this form. Given the Bellman equation, the policy function is always:

$$h(a) = \underset{a' \in \Gamma(a)}{\operatorname{argmax}} \ \{F(a,a') + \beta V(a')\} \tag{8}$$

which is close to the form equation (5) takes. Then, putting these together:

$$V(a) = F(a,h(a)) + \beta V(h(a))$$

which is close to equation (6).

By looking at these three equations, we can see three main points:

1. The first-order condition must be satisfied:

$$F_{h(a)}(a,h(a)) + \beta V'(h(a)) = 0$$

   $h(a)$ solves this FOC.

2. Taking the derivative of $V(a)$ with respect to a gives:

$$\begin{aligned} V'(a) &= F_a(a,h(a)) + F_{h(a)}(a,h(a))h'(a) + \beta V'(h(a))h'(a) \\ &= F_a(a,h(a)) + h'(a)\left[F_{h(a)}(a,h(a)) + \beta V'(h(a))\right] \\ &= F_a(a,h(a)) \end{aligned}$$

   The bracketed term is zero because of the first-order condition – this is the envelope theorem. We call:

$$V'(a) = F_a(a,h(a))$$

   the **Benveniste-Scheinkman formula**.

3. Substituting the Benveniste-Scheinkman formula into the FOC gives:

$$F_{h(a)}(a,h(a)) + \beta F_a(h_a, h(h(a))) = 0$$

   The value function is now gone! Not only that, but if you look hard, this equation is the Euler equation!

To solve Olivia's problem, we need to rewrite her value function to maximize with respect to the state variable tomorrow. First, solve for c_t from the transformation equation:

$$\begin{aligned}\omega_{t+1} &= (1+r)(\omega_t - c_t) \\ \omega_t - \frac{\omega_{t+1}}{1+r} &= c_t\end{aligned}$$

Now rewrite the value function to turn it into a Bellman equation:

$$V(\omega) = \max_{\omega' \in \Gamma(\omega)} u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega')$$

This equation is now in the form of equation (7). Let's take the derivatives:

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \omega'} &= u'\left(\omega - \frac{\omega'}{1+r}\right)\frac{-1}{1+r} + \beta V'(\omega') = 0 \\ \frac{\partial V(\omega)}{\partial \omega} &= u'\left(\omega - \frac{\omega'}{1+r}\right)\end{aligned}$$

Get the Euler equation:

$$u'\left(\omega - \frac{\omega'}{1+r}\right) = \beta(1+r)u'\left(\omega' - \frac{\omega''}{1+r}\right)$$

Substitute c back in to see that the Euler equation is indeed the same as before:

$$u'(c) = \beta(1+r)u'(c_{t+1})$$

Of course, when solving dynamic programming problems yourself, you can jump to the Euler equation without the intermediate derivatives. Make sure you understand why the Euler equation comes out the way it does though.

Without solving for $V(\omega)$, we cannot really make any further progress. There are two common ways to do this.

3.1 Guess-and-Verify

First, we can guess a form for the value function and solve that form using the equations above to verify if our guess was correct. Here, let's assume that $u(c) = \ln(c)$. Let's guess that:

$$V(\omega) = A + B \ln(\omega)$$

where A and B are some unknown constants. Let's first take the first-order condition with respect to ω':

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \omega'} &= u'\left(\omega - \frac{\omega'}{1+r}\right)\frac{-1}{1+r} + \beta V'(\omega') = 0\\ &\implies \frac{1}{\omega - \frac{\omega'}{1+r}}\frac{-1}{1+r} + \beta\frac{B}{\omega'} = 0 \qquad (9)\end{aligned}$$

Then let's find the Benveniste-Sheinkman formula:

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \omega} &= u'\left(\omega - \frac{\omega'}{1+r}\right)\\ \frac{B}{\omega} &= \frac{1}{\omega - \frac{\omega'}{1+r}} \qquad (10)\end{aligned}$$

We have two unknowns (ω' and B) in two equations. Solve (9) for B:

$$B = \frac{\omega'}{\beta}\frac{1}{(1+r)\omega - \omega'} \qquad (11)$$

Then plug (11) into (10):

$$\begin{aligned}\frac{\omega'}{\beta}\frac{1}{(1+r)\omega - \omega'} &= \frac{\omega}{\omega - \frac{\omega'}{1+r}}\\ \omega'\frac{\omega - \frac{\omega'}{1+r}}{(1+r)\left(\omega - \frac{\omega'}{1+r}\right)} &= \beta\omega\\ \omega' &= \beta(1+r)\omega \qquad (12)\end{aligned}$$

Plug (12) into (11):

$$\begin{aligned}B &= \frac{\beta(1+r)\omega}{\beta}\frac{1}{(1+r)\omega - \beta(1+r)\omega}\\ B &= \frac{1}{1-\beta}\end{aligned}$$

So far, so good. B has been verified to be a constant. To solve A, we plug everything into the Bellman equation:

$$\begin{aligned}
V(\omega) &= \ln\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') \\
A + B\ln(\omega) &= \ln\left(\omega - \frac{\omega'}{1+r}\right) + \beta(A + B\ln(\omega')) \\
A + \frac{1}{1-\beta}\ln(\omega) &= \ln\left(\omega - \frac{\beta(1+r)\omega}{1+r}\right) + \beta\left(A + \frac{1}{1-\beta}\ln(\beta(1+r)\omega)\right) \\
A + \frac{1}{1-\beta}\ln(\omega) &= \ln(\omega(1-\beta)) + \beta\left(A + \frac{1}{1-\beta}\left(\ln(\beta(1+r)) + \ln(\omega)\right)\right) \\
A(1-\beta) + \frac{1}{1-\beta}\ln(\omega) &= \ln(\omega) + \ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta(1+r)) + \frac{\beta}{1-\beta}\ln(\omega) \\
A &= \frac{1}{1-\beta}\left[\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta(1+r))\right]
\end{aligned}$$

So A is also a constant. The guess is thus verified, and the value function is:

$$V(\omega) = \frac{\beta}{(1-\beta)^2}\ln(\beta(1+r)) + \frac{\ln(1-\beta)}{1-\beta} + \frac{\ln(\omega)}{1-\beta}$$

Which is the same answer we got using the Lagrangian above. Now let's solve for the optimal path of consumption. Plugging the solution for ω' into the transition equation gives:

$$\begin{aligned}
c &= \omega - \frac{\omega'}{1+r} \\
c &= \omega - \frac{\beta(1+r)\omega}{1+r} \\
c &= \omega(1-\beta)
\end{aligned}$$

And recursively substituting in for ω:

$$c_t = (1-\beta)\left(\beta(1+r)\right)^{t-1}\omega_0$$

which also matches the solution from the Lagrangian.

3.2 Value Function Iteration

Sometimes we cannot use guess-and-verify. Either the analytical expressions are too complex or there does not exist an analytical representation of the value function. Instead, we now have to solve numerically using a technique called **value function iteration** (VFI). You'll go over VFI again in Macro I, so I will only briefly cover the theory here. Note that there are technical conditions I have neglected to mention for the sake of intuition.

The idea behind VFI is that $V(\omega)$ is a **fixed point** – a point that does not change upon application of a transformation. In math:

$$Tv = v$$

where v is the fixed point and T is the transformation. We call T a **contraction** if for some distance function d:

$$d(T(x), T(y)) \leq \beta d(x, y)$$

with $\beta \in (0, 1)$. All this inequality says is that T needs to bring x and y closer together. Repeated application of T will bring us to the unique fixed point, where $x = y$. This result is known as the **contraction mapping theorem**.[2]

Theorem 1 (Contraction Mapping Theorem)**.** *Let (S, d) be a complete metric space, where S is some non-empty set and d is a metric, and let $T : S \to S$ be a contraction. Then T has a unique fixed point $x = Tx$ in S.*

But when is T a contraction? **Blackwell's Theorem** provides two sufficient conditions:

Theorem 2 (Blackwell's Theorem)**.** *Let $X \in \mathbb{R}^n$ and let $B(X)$ be the set of bounded functions $f : X \to \mathbb{R}$ equipped with the sup norm. Define $T : B(X) \to B(X)$ such that T satisfies:*

1. *Monotonicity: $f \leq g$ implies $Tf \leq Tg$ for all $f, g \in B(X)$*

2. *Discounting: $\exists\, \beta \in (0, 1)$ such that:*

$$T(f + a) \leq Tf + \beta a$$

[2] Or the **Banach fixed point theorem**.

for all $f \in B(X)$ and where a is a constant ≥ 0.

*These are **sufficient** conditions for T to be a contraction.*

It turns out that the Bellman operator is a contraction. First, write out the Bellman operator:

$$TV(\omega) = \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') \right]$$

Let's check out monotonicity. Let $V \leq G$. Then:

$$u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') \leq u\left(\omega - \frac{\omega'}{1+r}\right) + \beta G(\omega') \qquad \forall\, \omega' \in \Gamma(\omega)$$

Therefore:

$$\max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') \right] \leq \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta G(\omega') \right]$$

But this is just:

$$TV(\omega) \leq TG(\omega)$$

which satisfies monotonicity. Now we need to prove discounting. Let $a \geq 0$. Then:

$$\begin{aligned}
T(V+a)(\omega) &= \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta (V+a)(\omega') \right] \\
&= \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') + \beta a \right] \\
&= \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V(\omega') \right] + \beta a \\
&= TV(\omega) + \beta a
\end{aligned}$$

which satisfies discounting. Therefore the Bellman operator is a contraction.

Now that we know the Bellman operator is a contraction, how do we practically use it to find the value function? The algorithm, broadly speaking, is:

1. Choose an initial starting value $V_0(\omega)$. We often use zero as the initial value.

2. Discretize the state space ω into a grid. For each ω on that grid, define:

$$V_1(\omega) = \max_{\omega' \in \Gamma(\omega)} \left[u\left(\omega - \frac{\omega'}{1+r}\right) + \beta V_0(\omega') \right]$$

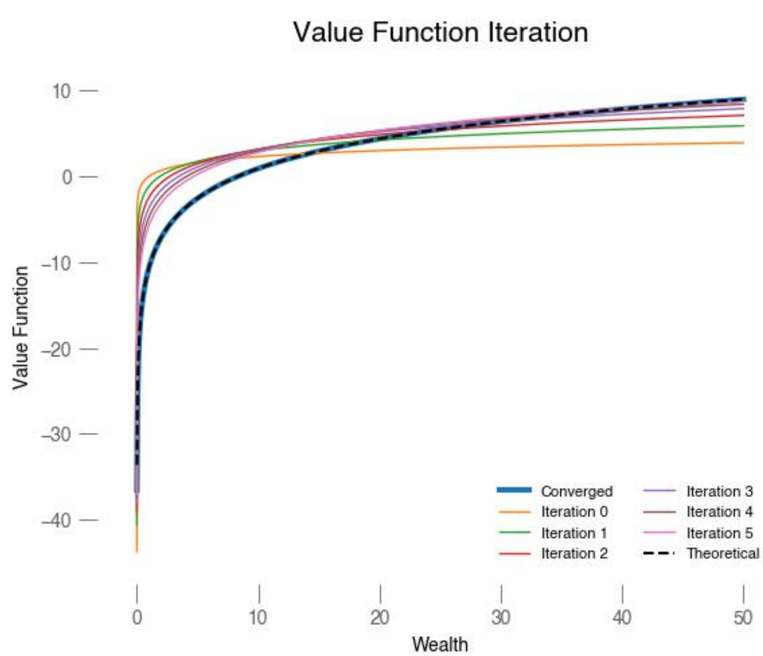
   Find the ω' that maximizes the right-hand side. Store this (ω, ω') pair. Store $V_1(\omega)$.

3. Calculate $|V_1(\omega) - V_0(\omega)|$. Stop if $< \varepsilon$ for some small ε. If not, then redefine $V_1(\omega)$ as $V_0(\omega)$ and repeat step 2.

4. Back out the consumption policy function from the budget constraint:

$$c_t = \omega_t - \frac{\omega_{t+1}}{1+r}$$

And that's it! I wrote a program doing VFI in Python using $r = 0.1$, $\beta = 0.8$, $u(c) = \ln(c)$, $\omega_0 = 50$, and $\varepsilon = 1e^{-8}$. I discretized the state-space into 5001 grid points between 0 and 50. After 112 iterations, which took about 22.5 minutes, the program converged. We can then compare my numerical answers using VFI with the theoretical answers from guess-and-verify:



Optimal Consumption

Period	Theory	Numerical
0	10.000	10.000
1	8.800	8.800
2	7.744	7.738
3	6.815	6.816
4	5.997	5.999
5	5.277	5.281
6	4.644	4.647
7	4.087	4.085
8	3.596	3.598
9	3.165	3.165
10	2.785	2.784