Math Camp Problem Set

Alex Houtz and Jacob Hukill*

August 2023

Day 1

1. **De Morgan's Laws** Let A and B both be events in Ω, the sample space.

   (a) Prove that $(A \cap B)^C = A^C \cup B^C$.

   (b) Prove that $(A \cup B)^C = A^C \cap B^C$.

2. **Hansen 1.3 Adapted** From a 52-card deck of playing cards, draw five cards to make a hand.

   (a) Let A be the event that the hand has exactly two Kings. Find $P(A)$.

   (b) Let B be the event the hand is a straight (not including straight flushes). Find $P(B)$.

   (c) Let C be the event that the hand is a flush (not including straight flushes). Find $P(C)$.

3. **The Monte Hall Problem** You are on the game show "Let's Make a Deal with Monte Hall." There are three doors in front of you: doors A, B, and C. Your goal is to select the door with the prize behind it. Assume, without loss of generality, that you select door A. Monte then opens one of the other two doors, say door B, revealing that there is no prize behind it. He then gives you the option to switch your choice of doors. Should you stick with door A or switch your choice to door

*Math Camp 2023 Instructors | University of Notre Dame

C? Assume that the *ex ante* probability of the prize being behind each door is 1/3.

4. **The St. Petersburg Paradox** Suppose a wealthy billionaire runs a game. Each participant gets to flip a fair coin until the coin comes up as heads for the first time. At that point, the participant wins $\$2^n$, where n denotes the number of flips. The billionaire charges \$1 billion to play.

   (a) Write down the pmf and verify that it is valid.

   (b) What is the probability that you win more than \$4?

   (c) Suppose you are an economic agent that makes decisions based only on your expected payoff. Should you pay the entry fee?

5. **Hansen 2.6** Compute $\mathbb{E}[X]$ and $Var(X)$ for the following distributions:

   (a) $f(x) = ax^{-a-1}$, for $0 < x < 1$ and $a > 0$.

   (b) $f(x) = \frac{1}{n}$, for $x = 1, 2, ..., n$. Hint: show by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and that $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$.

   (c) $f(x) = \frac{3}{2}(x-1)^2$, for $0 < x < 2$.

6. **Hansen 2.7** Let X have density:

$$f_X(x) = \frac{1}{2^{r/2}\Gamma(r/2)} x^{r/2-1} e^{-x/2}$$

   for $x > 0$. Let $Y = 1/X$. Derive the density of Y for $y > 0$.

7. **Hansen 4.3** Let:

$$f(x, y) = \frac{2}{(1 + x + y)^3}$$

   for $x, y \geq 0$.

   (a) Verify that $f(x, y)$ is a valid density.

(b) Find $f_x(x)$

(c) Find $\mathbb{E}[y]$, $Var(y)$, $\mathbb{E}[xy]$, and $\rho_{x,y}$

(d) Find $f(y|x)$

(e) Find $\mathbb{E}[y|x]$

8. **$\varepsilon - \delta$ Continuity** Let $f(x) = \frac{1}{x}$. Prove that for $x \neq 0$, $f(x)$ is continuous (Hint: you may have to assume that $\delta < \frac{1}{2}|x_0|$ to solve for the appropriate δ).

9. **Intersection of open sets** Is the intersection of an infinite collection of open sets open? Either prove the claim, or find a counterexample.

10. **Compactness of a Budget Set** A type of compact set that you will encounter often is called a budget set. Consider a vector of positive prices $p \in \mathbb{R}^N, p \gg 0$ and a wealth level $w \geq 0$. The budget set contains all affordable bundles of goods, and is given by

$$B(p, w) = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } p \cdot x \leq w\}$$

In two dimensions, this looks like a triangle with vertices at $0, w/p_1, w/p_2$. Show that this set is compact.

11. **Taylor Approximation of a Quadratic** Let $f(x) = ax^2 + bx + c$. Write the second order Taylor approximation around 0 for the function. What happens?

Day 2

12. **Positive Semi-Definite Matrices** Suppose we have:

$$C = (x'x)^{-1}(x'\Omega x)(x'x)^{-1} - (x'\Omega^{-1}x)^{-1}$$

Suppose that Ω is positive definite such that a Cholesky decomposition exists. Prove, using the following steps, that C is positive semi-definite. Note that $A - B$ is positive semi-definite iff $B^{-1} - A^{-1}$ is also positive semi-definite.

(a) Simplify the problem to:

$$x' \left[\Omega^{-1} - x(x'\Omega x)^{-1} x\right] x$$

(b) Simplify the problem to the *form* of:

$$W'MW$$

for some matrices W and M using the Cholesky decomposition of Ω.

(c) We call a matrix of the form $z(z'z)^{-1}z'$ the **projection** matrix of z. We call $I - z(z'z)^{-1}z'$ the **annihilator** matrix of z. Denote the annihilator matrix as M_z. M_z is idempotetent, meaning that $M_z M_z = M_z$. It is also symmetric. Rewrite the expression above in a way that looks like a "squared" matrix (i.e. $A^2 = AA'$).

(d) Finish the proof using the definition of positive semi-definite from the quadratic form.

13. Consider the problem of maximizing $f(x, y) = x$ subject to the constraint $x^3+y^2 = 0$.

- Try using the Lagrangian method to find the solution. What happens? (Note: you do not have to verify that the point you find is indeed a maximum)

14. Consider the following maximization problem:

$$\begin{aligned} &Maximize\ \ f(x, y) = x^3 + y^3 \\ &subject\ to\ \ x - y = 0 \end{aligned}$$

- Write down the Lagrangian, and give the FONCs for a maximum
- Solve the problem using the FONCs. What happened? Why might be the problem?

15. Suppose we want to maximize $f(x_1, x_2) = x_1 x_2$ subject to the constraint $x_1+4x_2 = 16$.

- Write down the Lagrangian for this problem and solve it.

16. Consider the following maximization problem:

$$\begin{aligned} &\textit{maximize} \quad f(x_1, x_2) = x_1^2 x_2 \\ &\textit{subject to} \quad 2x_1^2 + x_2^2 = 3 \end{aligned}$$

- Write down the Lagrangian and the FONCs for this problem.
- Without using the second order conditions, how would you find the constrained max? Do so.
- Now use the second order test to find the max.

17. Consider the the problem of maximizing $f(x, y) = xy$ subject to the constraint $x^2 + y^2 \leq 1$.

- Form the Lagrangian and write out the FONCs.
- How many candidates do you need to check?
- Find the point that maximizes the problem

18. Consider a standard utility maximization problem:

$$\begin{aligned} &\max_{x_1, x_2} \; U(x_1, x_2) \\ &s.t. \;\; p_1 x_1 + p_2 x_2 \leq I \end{aligned}$$

where a consumer maximizes utility through two consumption goods, x_1, x_2, subject to their budget constraint. p_1, p_2 represent positive unit prices, and I is the agent's income. We will make the assumption that for each commodity bundle (x_1, x_2),

$$\frac{\partial U}{\partial x_1} > 0 \;\; or \;\; \frac{\partial U}{\partial x_2} > 0$$

which is a common version of what's called a monotonicity assumption.

- Show that the tightness of the budget constraint (that the consumer spends all of their income purchasing goods) is a result of the monotonicity assumption.

19. **Simon and Blume 18.11 Adapted** Consider the following minimization problem:

$$\begin{aligned} &Minimize \;\; f(x,y) = 2y - x^2 \\ &subject \;\; to \;\; x^2 + y^2 \leq 1, \quad x \geq 0 \;\; y \geq 0 \end{aligned}$$

- Check that the NDCQ are satisfied
- Write down the Lagrangian and FONCS
- What is the solution to the minimization problem?