Math Camp: Optimization in $\mathbb{R}$

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We start with simple optimization problems of a single variable function over a real line or its subset. What we mean by optimization is finding a maximum or minimum of a given function over some domain. Why are we interested in such problems? Remember, one of the foundations of economic thinking is studying agents' optimal decision making under limited resources. For example, individuals maximize their well-being under a given budget constraint; firms maximize their profits under some technological constraints; agents decides their optimal allocation of income between saving and consumption. As you might see, optimization problems arise naturally in economics.

Optimization in $\mathbb{R}$

Throughout, we assume that the function $f : \mathcal{D} \implies \mathbb{R}$ is given and that is $\mathcal{D} \subseteq \mathbb{R}$. Before we go through some examples, we need some definitions.

Definition 1. $x^* \in \mathcal{D}$ is called a *local* maximizer (minimizer) if there exists some neighborhood $I_{x^*} = (x^* - \varepsilon, x^* + \varepsilon) \cap \mathcal{D}$ such that $f(x^*) \geq f(x)$ (such that $f(x^*) \leq f(x)$) for all $x \in I_{x^*}$. The value $f(x^*)$ is called the *local* maximum (minimum).

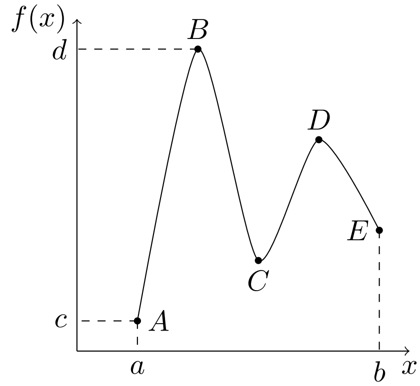
We usually write the maximizer x^* as

$$x^* = \arg\max_{x \in \mathcal{D}} f(x)$$

that is, the value of x that maximizes function f over the domain $\mathcal{D}$.

Definition 2. $x^* \in \mathcal{D}$ is called a *global* maximizer (minimizer) if $f(x^*) \geq f(x)$ (if $f(x^*) \leq f(x)$) for all $x \in \mathcal{D}$. The value $f(x^*)$ is called the *global* maximum (minimum).

We say that x^* is a strict maximizer (minimizer) when the inequalities above are strict. Notice that whenever x^* is a global maximizer, then it is also a local maximizer - this follows immediately from the definitions. The following figure illustrates these concepts.

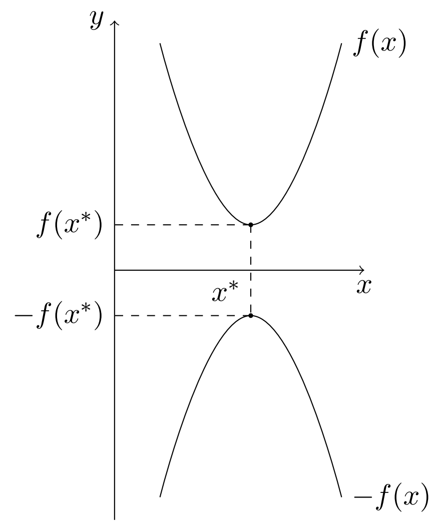


Before we continue, I want to highlight different naming conventions that you may see with regard to the above. You may see that x^* be called a maximizer, maxima, max, maximum points, optimal points, and extreme points. For $f(x^*)$, you might see maximum values, optimal values, or extreme values. You may see other terms as well - we are not very consistent with this.

You will often hear that maximization and minimization are two sides of the same coin. All of the results that we write down in terms of maximization can be translated into minimization. The trick is to remember the following:

$$\max_{x \in \mathcal{D}} -f(x) = \max_{x \in \mathcal{D}} f(x)$$

This follows quite simply. If f is any function with domain $\mathcal{D}$, then $-f$ is defined in $\mathcal{D}$ by $-(f)(x) = -f(x)$. Note that $f(x) \leq f(c)$ for all $x \in \mathcal{D}$ if and only if $-f(x) \geq -f(c)$ for all $x \in \mathcal{D}$. Thus c maximizes f in $\mathcal{D}$ if and only if c minimizes $-f$ in $\mathcal{D}$. The following figure illustrates this idea.



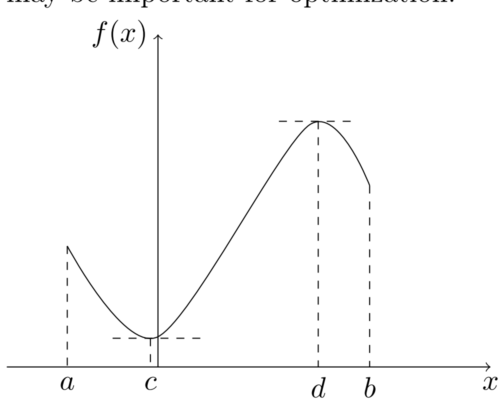
Next, we need to give a couple definitions that will play a crucial role in optimization theory.

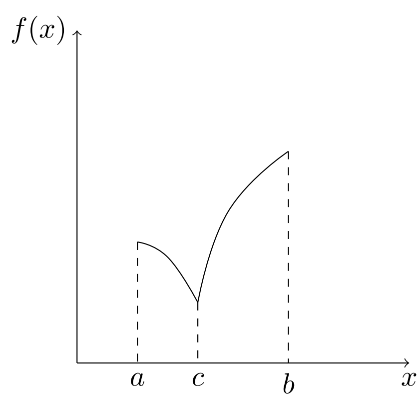
Definition 3. x_0 is a **critical** point for f if $f'(x_0) = 0$, or if $f'(x_0)$ is not defined (not differentiable).

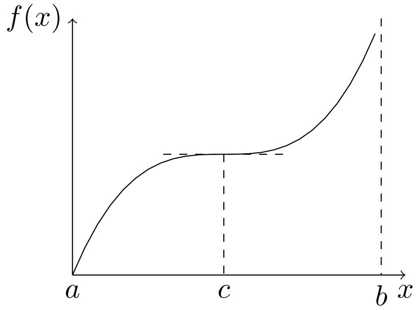
A similar, but slightly different definition:

Definition 4. x_0 is a **stationary** point for f if $f'(x_0) = 0$.

Geometrically, stationary points occur where there the tangent of the function is parallel to the x-axis. The following figures can illustrate this difference, and highlight how these terms may be important for optimization.







In the upper left picture, there are two stationary points, c and d. A c, there is a global minimum, and at d there is a global maximum. In the upper right picture, there is a global maximum at the endpoint b, and a global minimum at c. The point c is a critical point, but it is not differentiable. At b, the left hand derivative is not zero, (and thus we see no tangency), yet it is the maximum. Finally, the function in the bottom figure, we see that there is a minimum at a, but there is no maximum value, as the function approaches infinity as it approaches b. At the critical point c, there is neither a local max, nor a local min - it's a special kind of point called an inflection point. These figures give a broad idea of some of the issues we might end up with when optimizing.

As we study optimization, we want to emphasize the questions that we are interested in. When does a maximizer exist? If it does exist, when is it unique? Given that the solution exists, how do we find a candidate solution? How do we show that the candidate solution is indeed a maximizer? We begin with the first question.

Existence of Solutions: The Weierstrass Theorem

So we begin with the fundamental question of existence. When we ask this question, we are really concerned with the following: Under what conditions on the objective function f and the constraint set $\mathcal{D}$ are we guaranteed that solutions will exist in a maximization or minimization problem? When the domain is a finite set, then this answer is somewhat trivial - it is always guaranteed. We could simply evaluate every possible point, and find the maximum and minimum.

This is not a very good answer, for a variety of reasons. The first, being, that it limits the scope of the question significantly. Furthermore, we are often interested in characterizing solutions to parametric families of problems, such as describing parameter restriction under which a solution exists. Finally, even if our problem falls under this requirement, the finite set could be so large, that it is numerically difficult even if possible. Thus we need a considerably more general method, which we introduce now.

Let's begin by highlighting a particular problem that may arise during optimization. To start, we will add two more definitions to help illustrate.

Definition 5. Let $S \subseteq \mathbb{R}$. A value $b \in \mathbb{R} \cup \{-\infty, \infty\}$ is an upper bound of S if and only if $x \leq b$ for all $x \in S$. The **supremum** of S, denoted as $sup\ S \in \mathbb{R} \cup \{-\infty, \infty\}$ is the least upper bound of S. In other words, $sup\ S \leq b$ for all upper bounds b of S.

Definition 6. Let $S \subseteq \mathbb{R}$. A value $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is a lower bound of S if and only if $x \geq a$ for all $x \in S$. The **infimum** of S, denoted as $inf\ S \in \mathbb{R} \cup \{-\infty, \infty\}$ is the greatest lower bound of S. In other words, $sup\ S \geq a$ for all lower bounds a of S.

The supremum is a natural extension of the maximum, when we consider all values in $\mathbb{R}$. If we allow it to take on the value of ∞ if no upper bound exists, or -∞ if anything is a lower bound, then we are guaranteed that the supremum always exists. Why does this matter?

Example 1. Consider trying to optimize the following function $f(x) = x^2$ on the interval $\mathcal{D} = [1, 2]$. It should be pretty clear to see that the function achieves a maximum at $x^* = 2$. Indeed, $f(2) = 4 \geq f(x)$ for $x \in [1, 2]$. The supremum of $f(x)$ is also 4, as $sup\ f([1, 2]) = max\ f([1, 2]) = 4$.

Now consider the small change, of $\mathcal{D} = (1, 2)$. The supremum of $f(x)$ is still 4, as 4 is the least upper bound of $S = f((1, 2)) = (1, 4)$. The function, however, does not achieve a maximum. At any point $x \in (1, 2)$, there always exists another point $\tilde{x} = \frac{x+2}{2}$ that achieves a higher value of f.

In this example, the problem for maximizaiton lies on the boundary. This does not have to be the case.

Example 2. Consider the following function over the interval $\mathcal{D} = [0, 1]$:

$$f(x) = \begin{cases} x & 0 \leq x \leq 1/2 \\ 2 - x & 1/2 < x \leq 1 \end{cases}$$

Now, at exactly where the maximum 'should' occur, at $f(1/2) = 3/2$, the function instead takes the value $f(1/2) = 1/2$. Indeed, the function does not achieve a maximum at all.

These two examples highlight when a function might not achieve a maximum. In the first example, the domain $\mathcal{D}$ has the 'wrong' features: missing some crucial points. In the second, the function $f(x)$ has the 'wrong' features: it jumps around. What we really want is to describe sufficient features that guarantee $sup_{x \in \mathcal{D}} f(x) = m^*$ is equal to $f(x^*)$ for some $x \in \mathcal{D}$. Under these circumstances, f admits a global maximum.

Theorem 1 (The Weierstrass (Extreme Value) Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is a continuous function, and a and b are finite, then f achieves a global maximum on [a,b].*

Let's first explore each part of this theorem, and then make some general notes about it. The first requirement is that the domain be closed. In the first example above, the open interval created the maximization issue. The second thing to note, is that f must be a continuous function - a problem that was highlighted in the second example. Now let's look at the third major point - that the domain be bounded.

Example 3. Let $\mathcal{D} = \mathbb{R}$, and $f(x) = x^3$. We are maximizing a continuous function over a closed domain. However, because the function is strictly increasing, a global maximum does not exist.

Thus, whenever we are looking to maximize some $f(x)$ on $\mathcal{D} \subset \mathbb{R}$, we should look for ensure that the domain is closed and bounded, and the the function is continuous over this interval.

It is important here to emphasize that this theorem provides the *sufficient* conditions for the existence of optima. **Do Not** mistake it for necessary conditions. The theorem does not show that if the conditions are violated, that there is no maximum. This point is illustrated by the following example.

Example 4. Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{otherwise.} \end{cases}$$

We can see that all three conditions are violated. The domain is neither closed, nor bounded, and the function is heavily discontinuous. Yet, the function attains a maximum at every rational number, and a minimum at every irrational number.

The above theorem can be extended to multidimensional optimization.

Theorem 2 (The Weierstrass (Extreme Value) Theorem)**.** *If K is a compact set of $\mathbb{R}^n$, and $f : K \to \mathbb{R}$ is a continuous function, then f($\boldsymbol{x}$) achieves a global maximum on K.*

These assumptions are usually pretty easy to verify. Compact sets need to be closed and bounded, which usually involve some inequality constraints that we will see later. Furthermore, most of our problems will involve differentiating $f(x)$, which implies continuity. Be aware that when these conditions are not met, you may not have an optimum - you don't want to be the person presenting at a seminar and have someone surprisingly point out that your model doesn't guarantee a maximum.

Locating optima

Now that we have conditions that are sufficient for optima to exist, how do we find them? Though the exact method will differ, the general process will be similar when we extend to multiple dimensions. We will first begin by creating a shortlist of candidates that could possibly be optima. Then, we can look at the local 'shape' of the function near the candidates, and weed out ones that don't match what we are looking for (do they look more like a hill or valley). Finally, we can compare the values of the function for all candidates that remain.

Finding the shortlist of candidates

So how do we find these candidates? Let's begin by examining the first derivative of $f(x)$. Recall the figures of stationary points from earlier. We noticed that the tangent to the function was parallel to the x-axis at the maximum and minimum points. We can utilize this to help characterize local max and min.

Theorem 3 (First Order Necessary Condition (FONC))**.** *Let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function at x_0, and let x_0 be interior to the feasible set. If x_0 is a local max (min), then $f'(x_0) = 0$.*

Let's think about this for a bit. We are now introducing the concept of first order conditions (we will talk about second order conditions is a bit). They are 'necessary', in that in order for x_0 to be a max (min), certain conditions must be true. Recall the propositional logic from earlier.

$$\overbrace{\underbrace{P}_{\text{sufficient}} \implies \underbrace{Q}_{\text{necessary}}}^{\text{Propositional Logic}}$$

We can translate these statements into our theorem from above.

$$\overbrace{\underbrace{\text{If } x_0 \text{ is a local interior max of } f\text{, and } f \text{ is differentiable at } x_0,}_{\text{sufficient}} \implies \underbrace{\text{then } f'(x_0) = 0}_{\text{necessary}}}^{\text{First Order Necessary Condition}}$$

We call these our necessary conditions, because we are 'modifying' or creating conditions for the statement that x_0 is a max. Recall the truth table - the only way that the first statement can be true (the part we care about), is if the second statement is true. So this is a 'necessary' condition for our x_0 to be a max.

The theorem specifies that we are only looking at interior points, because the above theorem may not apply at the boundary.

Example 5. Let's once again consider $f(x) = x^3$, but now under $\mathcal{D} = [-2, 2]$. If we take the first derivative, we will get that $f'(x) = 3x^2$, which is only zero at $x = 0$. This, of course, is not the maximum, nor a minimum, but a particular type of critical point called a **saddle point** or **inflection point.** Because the function is strictly increasing, the maximum is actually attained at the boundary on x=2.

So when we use these tests, we need to be careful to also consider the boundary. When the max occurs at the boundary point of the domain of f, we will call the max a **boundary max**. We will call a max of f that is not an endpoint of the domain of f an **interior max**. If the domain is an open interval, or all of $\mathbb{R}$, then any max will be an interior max.

Finally, we need to ensure that the function is differentiable at x. A differentiable function is continuous, so our extreme-value theorem tells us that a maximum and minimum do exist. If the function is not differentiable, then it will also not be continuous - we will need to separately consider all of these points in our candidate list (if there is still an optima).

We should now have a good idea of what type of candidates for optima we are looking for from our examples. We can construct a candidate list that includes the following:

- all stationary points
- all points where f' does not exist
- all points on the boundary of the feasible set.

As long as we have built our list correctly, then one of these points must be our global maximizer (minimizer). In order to find out which, we could evaluate each point, and the largest function value from these candidates would be our global maximum.

Weeding out candidates - the shape of the function.

Evaluating every point can be exhausting - or impossible. This is especially true if we do not have a numerical example, but want to instead characterize the optimum with respect to our parameters. We need some way to weed out candidates from our list.

Let's suppose that we are considering an interior max. Recall that our Theorem 3 gave a 'necessary' condition. That means that the "Q" from our theorem could be true, but that we still don't have a max. When the derivative of a function is 0, that means that at a particular point, it is neither increasing nor decreasing - it's stationary. So depending on how the derivative behaves on either side of the point, it could either be a (local) maximum or a minimum (or neither). However, we can create a pretty simple test using the first derivative still for finding max and min.

A First Derivative Test for Max/Min
If $f'(x) \geq 0$ for $x \leq c$, and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is a maximum point for f.
If $f'(x) \leq 0$ for $x \leq c$, and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is a minimum point for f.

For a differentiable function, this switch of signs must mean that $f'(c) = 0$. In other words, this is essentially asking if we are looking at a hill (max) or a valley (min). Let's look at an example where we might want to use the above test.

Example 6 (Using Sydsaeter and Hammond 9.3)**.** Let's consider a very basic consumption savings problem. An individual lives for two periods. They have current income y_1, and they expect future income y_2. They are trying to plan current consumption c_1 and future consumption c_2 in order to maximize the utility function

$$ln(c_1) + \frac{1}{1+\delta} ln(c_2)$$

where δ is their discount rate. The individual could potentially borrow money now - but they would have to repay a loan that is charged at interest rate r. They could instead decide to save money, and gain the same interest rate on savings. Together with income from period 2, they have the following budget constraint.

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

Notice that if current consumption is greater than current income, then they have to pay back interest. Otherwise, they can utilize the savings in period 2. Find the optimal consumption plan.

The first thing that we will want to do in approaching this problem is to plug in the budget constraint into the objective function. This way, we are dealing with an "unconstrained" problem where we only need to decide what c_1 is to determine everything else. Thus, we are looking to optimize

$$U = ln(c_1) + \frac{1}{1+\delta} ln(y_2 - (1+r)(c_1 - y_1))$$

The first thing we will do is take the derivative, and then set the function equal to zero.

$$\frac{dU}{dc_1} = \frac{1}{c_1} - \left(\frac{1+r}{1+\delta}\right) \frac{1}{y_2 - (1+r)(c_1 - y_1)} = 0$$

We can rearrange the above into the following:

$$\frac{dU}{dc_1} = \frac{(1+\delta)[(1+r)y_1 + y_2] - (2+\delta)(1+r)c_1}{c_1(1+\delta)[y_2 - (1+r)(c_1 - y_1)]} = 0 \tag{1}$$

and now we can solve for c_1 so that we get

$$c_1^* = y_1 + \frac{(1+\delta)y_2 - (1+r)y_1}{(2+\delta)(1+r)}$$

There's a couple takeaways that we can get from this. The denominator in (1) above has to be positive (consumption is required to be positive here, and the denominator boils down to $c_1(1+\delta)c_2$). Then the only way the sign changes is through the numerator. Now suppose that we change c_1 slightly from c_1^*. Let $\tilde{c_1} = c_1^* + \varepsilon$. If we increase c_1^* just slightly, then the sign of the numerator must deviate from zero, and will do so by $-(2+\delta)(1+r)\varepsilon$. So for $c_1 > c_1^*$, the derivative $\frac{dU}{dc_1}$ is negative, and for $c_1 < c_1^*$, the derivative is positive. By our first derivative test from above, we can verify that c_1^* does indeed maximize our utility. We can get c_2^* from the solution for c_1.

The second takeaway is more of an economic one, than one of optimization. Let's take a look at our solution for c_1^*. The consumption plan of the individual is current income, plus a second term. The sign of this second term determines if the individual borrows ($c_1 > y_1$) or saves ($c_1 < y_1$). The student will borrow if and only if future income is greater than the discounted value that our current income would be worth in the future $y_2 > (\frac{1+r}{1+\delta})y_1$. This is a way for the agent to smooth consumption, something you will learn more about this year.

Now we should have a bit of intuition about how the derivative interacts with optima. Based on how the first derivative behaves around candidate point, we can figure out whether the extreme points are maxima or minima. To analyze this behavior, we turn towards the second derivative.

We can gain some insight by analyzing the pure quadratics, x^2 and $-x^2$. Among functions of one variable, these quadratics are the simplest functions with unique global extrema - a minimum at $x = 0$ for x^2, and a maximum at $x = 0$ for $-x^2$. While we will talk even more about these in a little while, these can help us understand why points are max or min. While the first derivative at 0 is $f'(0) = 0$ in both cases, the second derivative differs. In the case of x^2, the second derivative at zero is $f''(0) = 2 > 0$. On the other hand, the second derivative at zero of $-x^2$ is $f''(0) = -2 < 0$. So the sign of the derivative can help to pinpoint what our candidates are.

Theorem 4 (Second order sufficient conditions (local))**.** *Let f be a twice differentiable function* *on $\mathcal{D}$. If x_0 is a stationary point in the interior of the feasible set and $f''(x_0) < 0$ (and $f''(x_0) > 0$), then x_0 is a local interior max (min) of f.*

Let's analyze this for a bit. Intuitively, if the function at x_0 is not changing locally, and if it looks a bit like $-x^2$ (a hill), then we are dealing with a maximum. If instead it looks a bit like x^2 (a valley) locally, then we are dealing with a minimum. You may notice that the theorem says nothing about if $f''(x_0) = 0$. If this is the case, then the critical point of f is called a **degenerate critical point.** To know more about the point, we would need to know more than just the sign of the second derivative at the candidate - something like the sign of f' over a whole interval around the point.

In contrast to the theorem before, Theorem 4 gives *sufficient* conditions for a local max. If we satisfy these conditions, then we know that our candidate will be a local max (or min). If we don't, such as when $f''(x_0) = 0$, then we cannot use the result of the theorem. We can, however, recover necessary conditions for local optima. Suppose that f is twice differentiable on the interval $\mathcal{D}$, and that x_0 is an interior local max. We know from Theorem 3 that $f'(x_0) = 0$. We also know that it cannot be the case that $f''(x_0) > 0$, by Theorem 4. Then it must be the case that $f''(x_0) \leq 0$.

Theorem 5 (Second order necessary conditions (local))**.** *Let f be a twice differentiable function on $\mathcal{D}$. If x_0 is a local interior max (min) of f, then $f''(x_0) \leq 0$ ($f''(x_0) \geq 0$).*

So now we have a way to narrow down our candidate list. We can then remove all of the interior stationary points that do not meet the above theorem from our candidate list; once done, we can then check each of the remaining candidates to see which is the optimum.

Finding global optima

In our discussion so far, we have mostly talked about local maxima (minima). However, often we are instead more interested in conditions that will guarantee that a stationary point is a global optima. For this, we once again look back to the pure quadratics. As mentioned previously, these are the simplest functions with unique global extrema. We already noted that the second derivative of the functions behaved in a special way around the critical point x_0. However, the function also behaves this way for all x, not just at 0. Because the second derivative of x^2 is positive for all x ($f''(x) = 2$), we say that the function is convex.

Definition 7. Let f be continuous over the interval $\mathcal{D}$, and be twice differentiable in the interior of $\mathcal{D}$, denoted as $\mathcal{D}^0$. Then:
f is **convex** on $\mathcal{D} \iff f''(x) \geq 0$ for all x in $\mathcal{D}^0$
f is **concave** on $\mathcal{D} \iff f''(x) \leq 0$ for all x in $\mathcal{D}^0$

Before we continue, I want to make clear the difference between a convex function and a convex set. Although there is a connection[1], don't be confused when we talk about convexity - we are typically talking about different things. You might note that the above definition requires our function to be twice differentiable - we do not necessarily need this, though this will often be the definition we use. Let's also introduce a more general definition, and one that can be used to extend to functions of several variables later on:

Definition 8. The function f is **concave** in the interval $\mathcal{D}$ if for all $a, b \in \mathcal{D}$ and all $\lambda \in (0, 1)$,

$$f((1-\lambda)a + \lambda b) \geq (1-\lambda)f(a) + \lambda f(b).$$

We can make use of the fact that a function f is convex if $-f$ is concave. Then the following holds:

Definition 9. [2] The function f is **convex** in the interval $\mathcal{D}$ if for all $a, b \in \mathcal{D}$ and all $\lambda \in (0, 1)$,

$$f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b).$$

In Definition 8 (9), if we instead require that the inequality is strict for $a \neq b$, then f is call **strictly concave** (**strictly convex**). A function f is strictly convex if $-f$ is strictly concave.

We are almost done now, and ready to get back to optimization. But first, let's introduce one last thing:

Theorem 6. *Let f be a twice continuously differentiable function over the interval (a, b) :*
$f''(x) < 0$ *for all* $x \in (a, b) \implies f(x)$ *is strictly concave in* (a, b).
$f''(x) > 0$ *for all* $x \in (a, b) \implies f(x)$ *is strictly convex in* (a, b).

We needed to get some definitions out of the way, so now we can finally say a bit more about global optima. Concave functions have a really nice property that relate to optimization:

Theorem 7 (Sufficient Global Conditions)**.** *If f is concave, any stationary point of f is a global maximum. If f is strictly concave, any critical point of f is the unique global maximum.* [3]

This is a really nice property to have, as we often work with concavity in economics. Working with concave functions ensures that our stationary points are global maxima. And, it even has an added property under strictness that gives rise to uniqueness (whuch was one of the questions we asked previously). However, concavity does not *guarantee* the existence

[1] A function is convex iff the set of points on or above the graph of the function is a convex set
[2] Jensen's inequality generalizes this concept.
[3] A similar result occurs for convexity and minimums.

of a maximum. If there are no critical points, there may not be a maximum at all, or if there is one, it may just be at the boundary. The logarithm is a concave function, but it does not admit a maximum, as it is strictly increasing over its domain. Indeed it is often difficult to find the global max, and even more so to prove it. There are certain conditions that can make our lives easier.

The first utilizes our concepts of concavity that we've developed. From Theorem 7, we can guarantee that stationary points for concave or convex functions will be global optima. Of course, there might not be a stationary point, or it might not be the one that we are looking for (min for a max), but it does help. The second is also one we've already talked about, and actually guarantees global optima. Using Weierstrass's theorem from above, if we have a continuous functions who has a closed and bounded domain, then there must be a global maximum and minimum in this domain. We can compute all of the critical points in this interval, evaluate f at these points and at the endpoints of the domain, and then choose the point that gives the largest value of f from these.

Finally, another scenario can arise where there is only one critical point of f in the domain.

Theorem 8. *Suppose that:*

- *The domain of f is an interval I in $\mathbb{R}$,*
- *x_0 is a local maximum of f, and*
- *x_0 is the only critical point of f on I.*

Then x_0 is the global maximum of f on I.

Using these three theorems, we can ensure that we find the global maxima if any of the conditions to the theorems are met.