Sims and Wu (2021) Derivations

DSGE Model

KEY: — Derivation, narrative — Comment to reader — Used for labeling

- In section 2, Sims & Wu (2021) list the principal actors in the model: households, labor union, 4 types of production firms, financial intermediaries, a fiscal authority, and a central bank. This gives 6 major sectors in the model to solve.

① Households (following appendix A.1)

Each household consists of two types of family members - workers and intermediaries (basically a "child"). Intermediaries have a probability of $(1-\sigma)$ of becoming workers. An equal number of workers take their place as intermediaries. New intermediaries are given a fixed amount of net worth to start.

Households' lifetime utility is:

$$E_t\left[\sum_{j=0}^{\infty}\beta^j\left\{\ln(C_{t+j}-bC_{t+j-1})-\frac{\chi L_{t+j}^{1+\eta}}{1+\eta}\right\}\right] \qquad (A.1)$$

Internal Habit

with budget constraint:

Wage from being in a labor union; Nominal Dividends; Paid to new intermediaries; Lump-sum tax; Nominal interest rate on deposits

$$P_tC_t + D_t - D_{t-1} \leq MRS_tL_t + DIV_t - P_tX - P_tT_t + (R^d_{t-1}-1)D_{t-1} \qquad (A.2)$$

Price of goods; Stock of deposits at beginning of period

Dividing by price to make this real:

$$C_t + \frac{D_t}{P_t} - \frac{D_{t-1}}{P_t} \leq mrs_tL_t + div_t - X - T_t + \frac{R^d_{t-1}-1}{P_t}D_{t-1}$$

$$C_t + d_t - \frac{P_{t-1}}{P_t}d_{t-1} \leq mrs_tL_t + div_t - X - T_t + \left(\frac{P_{t-1}}{P_t}\right)(R^d_{t-1}-1)d_{t-1}$$

Putting this in a Lagrangian:

$$\mathcal{L} = E_t\left[\sum_{j=0}^{\infty}\beta^j\left\{\ln(C_{t+j}-bC_{t+j-1})-\frac{\chi L_{t+j}^{1+\eta}}{1+\eta}\right\} + \mu_t\left(mrs_tL_t + div_t - X - T_t + \left(\frac{P_{t-1}}{P_t}\right)(R^d_{t-1}-1)d_{t-1} - C_t - d_t + \frac{P_{t-1}}{P_t}d_{t-1}\right)\right.$$

$\quad\quad\quad\quad\quad\quad\quad\quad$ From this period $\quad\quad\quad\quad$ From next period

FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t - bC_{t-1}} - \beta E_t\left[\frac{b}{C_{t+1} - bC_t}\right] - \mu_t = 0$$

$$\rightarrow \mu_t = \frac{1}{C_t - bC_{t-1}} - \beta E_t\left[\frac{b}{C_{t+1} - bC_t}\right] \qquad (A.3)$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = -\chi L_t^{\eta} + mrs_t \mu_t = 0$$

$$\longrightarrow \chi L_t^{\eta} = mrs_t \mu_t \qquad (A.4)$$

$$\frac{\partial \mathcal{L}}{\partial d_t} = -\mu_t + \beta E_t\left[\mu_{t+1}\left(\Pi_{t+1}^{-1}(R_t^d - 1) + \Pi_{t+1}^{-1}\right)\right] = 0$$

$$\frac{\beta}{\mu_t} E_t\left[\mu_{t+1}\Pi_{t+1}^{-1}(R_t^d - 1 + 1)\right] = 1$$

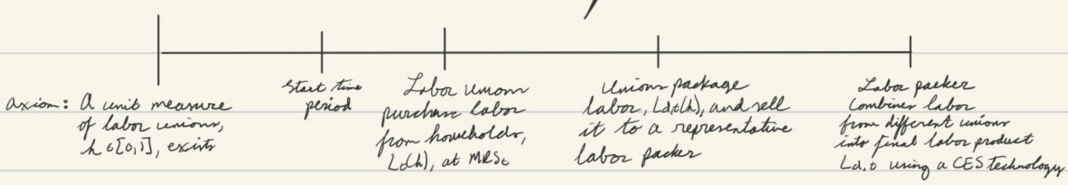
$$E_t\Bigg[\underbrace{\frac{\beta \mu_{t+1}}{\mu_t}}_{\text{Stochastic Discount Factor} \equiv \Lambda_{t,t+1}} \Pi_{t+1}^{-1} R_t^d\Bigg] = 1 \qquad (A.5)$$

$$E_t\left[\Lambda_{t,t+1}\Pi_{t+1}^{-1} R_t^d\right] = 1 \qquad (A.6)$$

This concludes the optimization problem for the household

2 Labor Markets (following Appendix A3)

The labor market works as follows:



So overall demand for labor is:

$$L_{d,t} = \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \quad \text{where } \varepsilon_w > 1 \text{ is the elasticity of substitution}$$

Define aggregate wage expenditure on labor:

$$\int_0^1 w_t(h) L_{d,t}(h)\, dh \equiv Z_t$$

Then optimal demand for labor from each union is:

$$\max_{\{L_{d,t}(h)\}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \quad s.t. \int_0^1 w_t(h) L_{d,t}(h)\, dh \equiv Z_t$$

$$\langle L_{d,t}(h)\rangle \quad \frac{\varepsilon_w}{\cancel{\varepsilon_w - 1}} \cdot \frac{\cancel{\varepsilon_w - 1}}{\cancel{\varepsilon_w}} L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{1}{\varepsilon_w - 1}} - \lambda w_t(h) = 0$$

$$(*) \qquad \lambda w_t(h) = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{1}{\varepsilon_w - 1}}$$

$$\lambda w_t(h) L_{d,t}(h) = L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right]^{\frac{1}{\varepsilon_w - 1}}$$

Integrate both sides:

$$\lambda \int_0^1 w_t(h) L_{d,t}(h)\, dh = \int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w-1}{\varepsilon_w}} dh \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w-1}{\varepsilon_w}} dh\right]^{\frac{1}{\varepsilon_w-1}}$$

$$\lambda Z_t = \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w-1}{\varepsilon_w}} dh\right]^{\frac{\varepsilon_w}{\varepsilon_w-1}}$$

Define W_t as the aggregate wage. Then $W_t L_{d,t} = Z_t$. So:

$$\lambda W_t L_{d,t} = L_{d,t}$$

$$\lambda = \frac{1}{W_t}$$

Sub λ into $(*)$:

$$\frac{W_t(h)}{W_t} = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w-1}{\varepsilon_w}} dh\right]^{\frac{1}{\varepsilon_w-1}}$$

$$\frac{W_t(h)}{W_t} = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} L_{d,t}^{\frac{1}{\varepsilon_w}}$$

$$L_{d,t}(h) = \left(\frac{W_t(h)}{W_t}\right)^{-\varepsilon_w} L_{d,t} \qquad (A.12)$$

Now take the total expenditure on labor:

$$W_t L_{d,t} = \int_0^1 w_t(h) L_{d,t}(h)\, dh \equiv Z_t \quad \text{Sub in } L_{d,t}(h):$$

$$W_t L_{d,t} = \int_0^1 w_t(h) \left(\frac{w_t(h)}{W_t}\right)^{-\varepsilon_w} L_{d,t}\, dh$$

$$W_t = \int_0^1 W_t(h)^{1-\varepsilon_w} dh\, W_t^{\varepsilon_w}$$

$$W_t^{1-\varepsilon_w} = \int_0^1 W_t(h)^{1-\varepsilon_w} dh \qquad (A.13)$$

Labor Unions have dividends of:

$$DIV_{L,t}(h) = w_t(h) L_{d,t}(h) - MRS_t L_t(h)$$

We impose $L_t(h) = L_{d,t}(h)$ (labor supply = labor demand)

$$DIV_{L,t}(h) = (w_t(h) - MRS_t) L_{d,t}(h)$$

Now substituting (A.12):

$$DIV_{L,t}(h) = (w_t(h) - MRS_t)\left(\left(\frac{w_t(h)}{W_t}\right)^{-\varepsilon_w} L_{d,t}\right)$$

$$= w_t(h)^{1-\varepsilon_w} W_t^{\varepsilon_w} L_{d,t} - MRS_t\, w_t(h)^{-\varepsilon_w} W_t^{\varepsilon_w} L_{d,t}$$

We also impose "sticky wages", where unions have a probability of adjusting wages of $(1-\phi_w)$ with $\phi_w \in [0,1]$. For unions that can't update wages in a given period, wages can be written as: $W_t(h)\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{\gamma_w} = W_{t+j}(h)$, with $\gamma_w \in [0,1]$ being an index on inflation

Putting all this together, the union maximization problem is:

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{W_{t+j}(h)^{1-\varepsilon_w} W_{t+j}^{\varepsilon_w} L_{d,t+j} - MRS_{t+j} W_{t+j}(h) W_{t+j}^{-\varepsilon_w}\, W_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

Probability of keeping last period's wage

Discounting by household stochastic discount factor

Then sub in inflation indexing and make terms real:

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{W_t(h)^{1-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} W_{t+j}^{\varepsilon_w} L_{d,t+j} - MRS_{t+j} W_t(h)^{-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} W_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{W_t(h)^{1-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} \frac{W_{t+j}^{\varepsilon_w}}{P_{t+j}^{\varepsilon_w}} L_{d,t+j} P_{t+j}^{\varepsilon_w-1} - P_{t+j}^{\varepsilon_w}\frac{MRS_{t+j}}{P_{t+j}} W_t(h)^{-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w}\frac{W_{t+j}^{\varepsilon_w}}{P_{t+j}^{\varepsilon_w}} L_{d,t+j}\right\}\right]$$

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} W_t(h)^{1-\varepsilon_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j} - mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} W_t(h)^{-\varepsilon_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial W_t(h)} = (1-\varepsilon_w) W_t(h)^{-\varepsilon_w} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right] + \varepsilon_w W_t(h)^{-\varepsilon_w-1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\, mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right] = 0$$

Since all unions are the same, the reset price is the same across all unions, so $W_t(h) = W^*$.

$$(\varepsilon_w - 1) W^{*-\varepsilon_w} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d\,t+j}\right] = \varepsilon_w W^{*-\varepsilon_w-1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\, mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right]$$

$$W^* = \frac{\varepsilon_w}{\varepsilon_w - 1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\, mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right] E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d\,t+j}\right]^{-1}$$

$$\underline{\underline{W}}^* = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{F_{1,t}}{F_{2,t}} \qquad (A.14)$$

where $F_{1,t} = mrs_t P_t^{\varepsilon_w} w_t^{\varepsilon_w} L_{d,t} + \phi_w \Lambda_{t,t+1} \Pi_t^{-\varepsilon_w\gamma_w} F_{1,t+1}$

$F_{2,t} = P_t^{\varepsilon_w - 1} w_t^{\varepsilon_w} L_{d,t} + \phi_w \Lambda_{t,t+1} \Pi_t^{(1-\varepsilon_w)\gamma_w} F_{2,t+1}$

We can divide through by P_t to obtain real wage:

$w^* = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{f_{1,t}}{f_{2,t}}$, where $f_{i,t}$ denotes the real value of the $F_{i,t}$ term

$$f_{1,t} = mrs_t \, w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{\varepsilon_w} \Pi_t^{-\varepsilon_w \gamma_w} f_{1,t+1}\right] \quad (A.15)$$

$$f_{2,t} = w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{\varepsilon_w - 1} \Pi_t^{\gamma_w(1-\varepsilon_w)} f_{2,t+1}\right] \quad (A.16)$$

- Aggregating:

We can now aggregate, integrating A.12 over all unions:

$$\int_0^1 L_{d,t}(h)\,dh = \int_0^1 \left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} L_{d,t}\,dh$$

$$L_t = L_{d,t} \underbrace{\int_0^1 \left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} dh}_{\equiv V_t^w,\ \text{wage dispersion}}$$

$$L_t = L_{d,t} V_t^w \quad (A.17)$$

Looking closer at V_t^w, we can impose wage stickiness:

$$V_t^w = \int_0^{1-\phi_w} \left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} dh + \int_{1-\phi_w}^1 \left(\frac{\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} w_{t-1}(h)}{w_t}\right)^{-\varepsilon_w} dh$$ Using the same reset price:

$$= (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \Pi_{t-1}^{-\varepsilon_w \gamma_w} w_t^{\varepsilon_w} \int_{1-\phi_w}^1 w_{t-1}(h)^{-\varepsilon_w} dh.$$ Multiply right term by $\left(\frac{w_{t-1}}{w_{t-1}}\right)^{-\varepsilon_w}$

$$= (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \Pi_{t-1}^{-\varepsilon_w \gamma_w} w_t^{\varepsilon_w} w_{t-1}^{-\varepsilon_w} \int_{1-\phi_w}^1 \left(\frac{w_{t-1}(h)}{w_{t-1}}\right)^{-\varepsilon_w} dh$$

$$= (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \Pi_{t-1}^{-\varepsilon_w \gamma_w} w_t^{\varepsilon_w} w_{t-1}^{-\varepsilon_w} V_{t-1}^w$$

We need the right term in real values:

The first term is already real, as we just multiply & divide by P_t initially

$$= (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \Pi_{t-1}^{-\varepsilon_w \gamma_w} \frac{W_t^{\varepsilon_w}}{P_t^{\varepsilon_w}} \frac{W_{t-1}^{-\varepsilon_w}}{P_{t-1}^{-\varepsilon_w}} V_{t-1}^w \left(\frac{P_t}{P_{t-1}}\right)^{\varepsilon_w}$$

$$= (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \left(\frac{\Pi_t}{\Pi_{t-1}^{\gamma_w}}\right)^{\varepsilon_w} \left(\frac{w_t}{w_{t-1}}\right)^{\varepsilon_w} V_{t-1}^w \quad (A.18)$$

The last derivation in this section starts from A.13:

$$W_t^{1-\varepsilon_w} = \left[\int_0^1 W_t(h)^{1-\varepsilon_w} dh\right]$$ Use sticky wages:

$$W_t^{1-\varepsilon_w} = \int_0^{1-\phi_w} W_t(h)^{1-\varepsilon_w} dh + \int_{1-\phi_w}^1 \left(\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} W_{t-1}(h)\right)^{1-\varepsilon_w} dh$$ Impose reset wage w^*:

$$W_t^{1-\varepsilon_w} = (1-\phi_w)(W_t^*)^{1-\varepsilon_w} + \int_{1-\phi_w}^1 \left(\Pi_{t-1}^{\gamma_w} W_{t-1}(h)\right)^{1-\varepsilon_w}) dh.$$

If we consider W_{t-1} as the average wage (also as the aggregate wage) in $t-1$, then as $h \to \infty$, by the weak law of large numbers:

$$W_t^{1-\varepsilon_w} = (1-\phi_w)(W_t^*)^{1-\varepsilon_w} + \phi_w\left(\Pi_{t-1}^{\gamma_w} W_{t-1}\right)^{1-\varepsilon_w}$$

Get this into real terms:

$$\frac{W_t^{1-\varepsilon_w}}{P_t^{1-\varepsilon_w}} = (1-\phi_w)\left(\frac{W_t^*}{P_t}\right)^{1-\varepsilon_w} + \phi_w\left(\Pi_{t-1}^{\gamma_w} \frac{P_{t-1}}{P_t} \frac{W_{t-1}}{P_{t-1}}\right)^{1-\varepsilon_w}$$

$$w_t^{1-\varepsilon_w} = (1-\phi_w) w_t^{*\,1-\varepsilon_w} + \phi_w\left(\frac{\Pi_{t-1}^{\gamma_w}}{\Pi_t} w_{t-1}\right)^{1-\varepsilon_w} \quad (A.19)$$

(3) Production

There are 4 firms:

i. Representative Whole-saler - combines capital and labor to make $Y_{m,t}$.

ii. Competitive Capital Producer - Creates new physical capital $\hat{I}_t$.

iii. Retail Firms: Repackage wholesale output using $Y_t(f) = Y_{m,t}(f)$.

iv. Competitive Final Goods Firm: Aggregate $Y_t(f)$ into Y_t using CES aggregation.

Then retailers face demand:

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}}$$

Redefine Z_t as total expenditure on final goods: $Z_t = \int_0^1 P_t(f) Y_t(f)\, df$

Final goods firm face the following problem:

$$\max_{\{Y_t(f)\}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}} \quad \text{s.t.} \quad Z_t = \int_0^1 P_t(f) Y_t(f)\, df$$

$$\langle Y_t(f)\rangle \quad \cancel{\frac{\varepsilon_p}{\varepsilon_p-1}} \cdot \frac{\varepsilon_p-1}{\cancel{\varepsilon_p}} Y_t(f)^{\frac{-1}{\varepsilon_p}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} df\right]^{\frac{1}{\varepsilon_p-1}} - \lambda P_t(f) = 0 \qquad (**)$$

$$Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} df\right]^{\frac{1}{\varepsilon_p-1}} = \lambda P_t(f) Y_t(f)$$

$$\left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p-1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}} = \lambda Z_t$$

$$Y_t = \lambda Z_t \longrightarrow Y_t = \lambda P_t Y_t \longrightarrow \frac{1}{P_t} = \lambda$$

Sub into $(**)$:

$$Y_t(f)^{\frac{-1}{\varepsilon_p}} Y_t^{\frac{1}{\varepsilon_p}} = \frac{P_t(f)}{P_t}$$

$$Y_t(f)^{\frac{1}{\varepsilon_p}} = Y_t^{\frac{1}{\varepsilon_p}} \left(\frac{P_t(f)}{P_t}\right)^{-1}$$

$$Y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t \qquad (A.20)$$

The aggregate price dynamics are:

$$P_t Y_t = \int_0^1 P_t(f) Y_t(f)\, df = Z_t$$

$$P_t Y_t = \int_0^1 P_t(f) \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t \, df$$

$$P_t^{1-\varepsilon_p} = \int_0^1 P_t(f)^{1-\varepsilon_p} df \qquad (A.21)$$

I. Retail Firms (from appendix A.4.1)

First, we define nominal dividends:

$$DIV_{R,t}(f) = P_t(f) Y_t(f) - P_{m,t} Y_{m,t}(f)$$

$$P_t Y_t - P_{m,t} Y_{m,t}$$

Set $Y_t(f) = Y_{m,t}(f)$, imposing demand = supply:

$$DIV_{R,t}(f) = P_t(f)\,Y_t(f) - P_{m,t}\,Y_t(f) \qquad \text{Sub in A.20:}$$

$$= P_t(f)\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t - P_{m,t}\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t$$

$$= P_t(f)^{1-\varepsilon_p} P_t^{\varepsilon_p} Y_t - P_{m,t} P_t(f)^{-\varepsilon_p} P_t^{\varepsilon_p} Y_t$$

Like with wages, we impose sticky prices, where $(1-\phi_p)$ is the probability of adjusting price each period. Also like with wages, we can index to inflation w/ γ_p: $P_{t+j}(f) = P_t(f)\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{\gamma_p}$ for a firm that can't update prices. The retailers problem is thus:

$$\max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_{t+j}(f)^{1-\varepsilon_p} P_{t+j}^{\varepsilon_p} Y_{t+j} - P_{m,t+j} P_{t+j}(f)^{-\varepsilon_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$

$$= \max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_t(f)^{1-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j} - P_{m,t+j} P_t(f)^{-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$

Convert to real values:

$$= \max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_t(f)^{1-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j} - p_{m,t+j} P_t(f)^{-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$

The highlighted parts are what I believe to be corrections to the derivation in the appendix.

$$\langle P_t(f)\rangle \quad (1-\varepsilon_p) P_t(f)^{-\varepsilon_p} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j}\right]$$
$$+ \varepsilon_p P_t(f)^{-\varepsilon_p-1} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\, p_{m,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right] = 0$$

Let $X_{1,t} = \sum_{j=0}^{\infty} (\phi_p\beta)^j \frac{u_{t+j}}{u_t}\, p_{m,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}$

& $X_{2,t} = \sum_{j=0}^{\infty} (\phi_p\beta)^j \frac{u_{t+j}}{u_t}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j}$.

Recursively: $X_{1,t} = p_{m,t} P_t^{\varepsilon_p} Y_t + \phi_p \Lambda_{t,t+1} \Pi_t^{-\varepsilon_p\gamma_p} X_{1,t+1}$

$$X_{2,t} = P_t^{\varepsilon_p-1} Y_t + \phi_p \Lambda_{t,t+1} \Pi_t^{(1-\varepsilon_p)\gamma_p} X_{2,t+1}$$

Because each retailer has the same pricing problem, all reset prices are the same, P_t^*. Rearranging the first-order condition gives:

$$(\varepsilon_p - 1) P_t^{*\,-\varepsilon_p} X_{2,t} = \varepsilon_p P_t^{*\,-\varepsilon_p-1} X_{1,t}$$

$$P_t^* = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{X_{1,t}}{X_{2,t}} \qquad \text{Set in real terms:}$$

$$\frac{P_t^*}{P_t} = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{X_{1,t}}{X_{2,t}}\cdot\frac{P_t^{\varepsilon_p-1}}{P_t^{\varepsilon_p}}$$

$$p_t^* = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{x_{1,t}}{x_{2,t}}, \text{ where} \qquad (A.22)$$

$$\frac{X_{1,t}}{P_t^{\varepsilon_p}} = x_{1,t} = p_{m,t} Y_t + \phi_p E_t\left[\Lambda_{t,t+1} \Pi_t^{-\varepsilon_p\gamma_p}\cdot\frac{P_{t+1}^{\varepsilon_p}}{P_t^{\varepsilon_p}}\, x_{1,t+1}\right] \quad \& \qquad (A.23)$$

$$\frac{x_{2,t}}{P_t^{\varepsilon_p-1}} = x_{2,t} = Y_t + \phi_p E_t\left[\Lambda_{t,t+1}\,\Pi_t^{(1-\varepsilon_p)\gamma_p}\,\frac{P_{t+1}^{\varepsilon_p-1}}{P_t^{\varepsilon_p-1}}\,x_{2,t+1}\right] \qquad (A.24)$$

– Aggregating:

Integrate A.20 over all f:

$$\int_0^1 Y_t(f)\,df = \int_0^1\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t\,df \qquad \text{Imposing eq'm:}$$

$$\int_0^1 Y_{m,t}(f)\,df = Y_t\int_0^1\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} df$$

$$Y_{m,t} = Y_t\,\underbrace{v_t^p}_{\text{Price dispersion}} \qquad (A.25)$$

Looking at just v_t^p and introducing Calvo pricing:

$$v_t^p = \int_0^{1-\phi_p} p_t^{*\,-\varepsilon_p}\,df + \int_{1-\phi_p}^1\left(\frac{\Pi_{t-1}^{\gamma_p}P_{t-1}(f)}{P_t}\right)^{-\varepsilon_p} df$$

$$= (1-\phi_p)\,p_t^{*\,-\varepsilon_p} + \Pi_{t-1}^{-\gamma_p\varepsilon_p}P_t^{\varepsilon_p}P_{t-1}^{-\varepsilon_p}\int_{1-\phi_p}^1\left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\varepsilon_p} df$$

$$\xrightarrow[f\to\infty]{P} (1-\phi_p)\,p_t^{*\,-\varepsilon_p} + \phi_p\left(\frac{\Pi_t}{\Pi_{t-1}^{\gamma_p}}\right)^{\varepsilon_p} v_{t-1}^p \qquad (A.26)$$

Now rewriting A.21 w/ Calvo pricing:

$$P_t^{1-\varepsilon_p} = \int_0^{1-\phi_p} P_t^{*\,1-\varepsilon_p}\,df + \int_{1-\phi_p}^1 \Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}\,P_{t-1}(f)^{1-\varepsilon_p}\,df$$

$$\xrightarrow[f\to\infty]{P} (1-\phi_p)\,P_t^{*\,1-\varepsilon_p} + \phi_p\,\Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}P_{t-1}^{1-\varepsilon_p} \qquad \text{Divide by } P_t^{1-\varepsilon_p}:$$

$$1 = (1-\phi_p)\,p_t^{*\,1-\varepsilon_p} + \phi_p\,\Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}\,\Pi_t^{\varepsilon_p-1} \qquad (A.27)$$

II. Wholesale Firms (from section 2 and appendix A.4.2)

Production technology: $Y_{m,t} = A_t(u_t K_t)^{\alpha} L_{d,t}^{1-\alpha}$ (2.19)

Capital utilization (apparently a choice variable)

Law of Motion: $K_{t+1} = \hat{I}_t + (1-\delta(u_t))K_t$ (2.20)

Loan-in-advance constraint: $\psi P_t^k \hat{I}_t \le Q_t CF_{m,t} = Q_t(F_{m,t} - \kappa F_{m,t-1})$ (2.21)

Dividends: $DIV_{m,t} = P_{m,t}A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha} - W_tL_{d,t} - P_t^k\hat{I}_t - \underbrace{F_{m,t-1}}_{\text{Total coupon liability due in period } t} + Q_t(F_{m,t} - \kappa F_{m,t-1})$ (2.22)

Write dividends in real terms:

$$\frac{DIV_{m,t}}{P_t} = \frac{P_{m,t}}{P_t}A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha} - \frac{W_t}{P_t}L_{d,t} - \frac{P_t^k}{P_t}\hat{I}_t - \frac{P_{t-1}}{P_t}\frac{F_{m,t-1}}{P_{t-1}} + Q_t\left(\frac{F_{m,t}}{P_t} - \frac{\kappa P_{t-1}}{P_t}\frac{F_{m,t-1}}{P_{t-1}}\right)$$

$$div_{m,t} = p_{m,t}A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha} - w_tL_{d,t} - p_t^k\hat{I}_t - \Pi_t^{-1}\frac{F_{m,t-1}}{P_{t-1}} + Q_t\left(\frac{F_{m,t}}{P_t} - \kappa\Pi_t^{-1}\frac{F_{m,t-1}}{P_{t-1}}\right)$$

Write in Lagrangian:

$$\mathcal{L} = E_t\Big[\sum_{j=0}^{\infty}\Lambda_{t,t+j}\Big\{p_{m,t+j}A_{t+j}(u_{t+j}K_{t+j})^{\alpha}L_{d,t+j}^{1-\alpha} - w_{t+j}L_{d,t+j} - p_{t+j}^k\hat{I}_{t+j} - \Pi_{t+j}^{-1}\frac{F_{m,t+j-1}}{P_{t+j-1}} + Q_{t+j}\left(\frac{F_{m,t+j}}{P_{t+j}} - \kappa\Pi_{t+j}^{-1}\frac{F_{m,t+j-1}}{P_{t+j-1}}\right)$$

$$+ \lambda_{1,t+j}\left(\hat{I}_{t+j} + (1-\delta(u_{t+j}))K_{t+j} - K_{t+j+1}\right) + \lambda_{2,t+j}\underbrace{\left(Q_{t+j}\left(\frac{F_{m,t+j}}{P_{t+j}} - \kappa\frac{F_{m,t+j-1}}{P_{t+j-1}}\Pi_{t+j}^{-1}\right) - \psi p_{t+j}^k\hat{I}_{t+j}\right)}_{\text{Loan-in-advance constraint in real terms}}$$

Take FOCs wrt $L_{d,t}$; $\hat{I}_t$; u_t; K_{t+1}; $F_{m,t}$

$$\frac{\partial \mathcal{L}}{\partial L_{d,t}} = (1-\alpha)\, p_{m,t} A_t (u_t K_t)^{\alpha} L_{d,t}^{-\alpha} - w_t = 0 \qquad (A.28)$$

$$\frac{\partial \mathcal{L}}{\partial \hat{I}_t} = -p_t^k + \lambda_{1,t} - \varphi p_t^k \lambda_{2,t} = 0 \longrightarrow \lambda_{1,t} = p_t^k - \varphi p_t^k \lambda_{2,t} \qquad (A.29)$$

$$\frac{\partial \mathcal{L}}{\partial u_t} = p_{m,t} A_t \alpha K_t^{\alpha} u_t^{\alpha-1} L_{d,t}^{1-\alpha} - \lambda_{1,t}\delta'(u_t) K_t = 0 \qquad (A.30)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_{1,t} + E_t\left[\alpha \Lambda_{t,t+1}\, p_{m,t+1} A_{t+1} u_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + \lambda_{1,t+1}\left(1-\delta(u_{t+1})\right)\right] = 0 \qquad (A.31)$$

$$\frac{\partial \mathcal{L}}{\partial F_{m,t}} = \frac{Q_t}{P_t} + \lambda_{2,t}\frac{Q_t}{P_t} - E_t\left[\Lambda_{t,t+1}\left(\Pi_{t+1}^{-1} P_t^{-1} + Q_{t+1}\kappa \Pi_{t+1}^{-1} P_t^{-1} + \lambda_{2,t+1} Q_{t+1}\kappa \Pi_{t+1}^{-1} P_t^{-1}\right)\right] = 0 \qquad (A.32)$$

Let $M_{1,t} = 1+\varphi\lambda_{2,t}$ & $M_{2,t} = 1+\lambda_{2,t}$. Then A.29 becomes $\lambda_{1,t} = p_t^k M_{1,t}$.

Subbing $M_{1,t}$ & $M_{2,t}$ leads to the equations in section 2:

From A.28: $w_t = (1-\alpha)\, p_{m,t} A_t (u_t K_t)^{\alpha} L_{d,t}^{-\alpha}$ (2.23)

From A.30: $p_{m,t} A_t \alpha K_t^{\alpha} u_t^{\alpha-1} L_{d,t}^{1-\alpha} = \lambda_{1,t}\delta'(u_t) K_t$

$$p_{m,t} A_t \alpha K_t^{\alpha} u_t^{\alpha-1} L_{d,t}^{1-\alpha} = p_t^k M_{1,t}\,\delta'(u_t) K_t$$

$$p_{m,t} A_t \alpha K_t^{\alpha-1} u_t^{\alpha-1} L_{d,t}^{1-\alpha} = p_t^k M_{1,t}\,\delta'(u_t) \qquad (2.24)$$

From A.31: $E_t\left[\alpha \Lambda_{t,t+1}\, p_{m,t+1} A_{t+1} u_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + \lambda_{1,t+1}\left(1-\delta(u_{t+1})\right)\right] = \lambda_{1,t}$

$$E_t\left[\alpha \Lambda_{t,t+1}\, p_{m,t+1} A_{t+1} u_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + p_{t+1}^k M_{1,t+1}\left(1-\delta(u_{t+1})\right)\right] = p_t^k M_{1,t} \qquad (2.25)$$

From A.32: $\frac{Q_t}{P_t} + \lambda_{2,t}\frac{Q_t}{P_t} = E_t\left[\Lambda_{t,t+1}\left(\Pi_{t+1}^{-1} P_t^{-1} + Q_{t+1}\kappa \Pi_{t+1}^{-1} P_t^{-1} + \lambda_{2,t+1} Q_{t+1}\kappa \Pi_{t+1}^{-1} P_t^{-1}\right)\right]$

$$\frac{Q_t}{P_t} M_{2,t} = \frac{1}{P_t} E_t\left[\Lambda_{t,t+1}\left(\Pi_{t+1}^{-1} + Q_{t+1}\kappa \Pi_{t+1}^{-1} M_{2,t+1}\right)\right]$$

$$Q_t M_{2,t} = E_t\left[\Lambda_{t,t+1}\Pi_{t+1}^{-1}\left(1 + Q_{t+1}\kappa M_{2,t+1}\right)\right] \qquad (2.26)$$

Divide $M_{1,t}$ by $M_{2,t}$: $\frac{M_{1,t}-1}{M_{2,t}-1} = \frac{\varphi\lambda_{2,t}}{\lambda_{2,t}} = \varphi$ (2.27)

III. Capital Producer

Sims & Wu (2021) give the capital producer investment technology:

$$\hat{I}_t = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \qquad (A.34)$$

where $I_t \equiv$ unconsumed final output, $S(\cdot) \equiv$ adjustment cost function. The dividend function is:

$DIV_{k,t} = P_t^k\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - P_t I_t$ Make into real terms:

$div_{k,t} = p_t^k\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - I_t$

The maximization problem is:

$$\max_{\{I_t\}} E_t\left[\sum_{j=0}^{\infty} \Lambda_{t,t+j}\left\{p_{t+j}^k\left(1 - S\left(\frac{I_{t+j}}{I_{t+j-1}}\right)\right) I_{t+j} - I_{t+j}\right\}\right]$$

$$\langle I_t \rangle \quad 1 = p_t^k\left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) - p_t^k \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right) + E_t\left[\Lambda_{t,t+1}\, p_{t+1}^k \left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\right] \quad (A.35)$$

4 Financial Intermediaries (following Appendix A.2)

Sims & Wu (2021) provide the financial intermediary net worth equation (2.8):

$$N_{i,t} = (R_t^F - R_{t-1}^d) Q_{t-1} F_{1,t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} B_{i,t-1} + (R_{t-1}^{re} - R_{t-1}^d) RE_{i,t-1} + R_{t-1}^d N_{i,t-1}$$

Divide by P_{t-1}:

$$\frac{1}{P_{t-1}} N_{i,t} = (R_t^F - R_{t-1}^d) Q_{t-1} f_{1,t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{i,t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{i,t-1} + R_{t-1}^d n_{i,t-1}$$

$$\Pi_t n_{i,t} = (R_t^F - R_{t-1}^d) Q_{t-1} f_{1,t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{i,t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{i,t-1} + R_{t-1}^d n_{i,t-1} \quad (A.7)$$

$$n_{i,t} = \Pi_t^{-1}\left[(R_t^F - R_{t-1}^d) Q_{t-1} f_{1,t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{i,t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{i,t-1} + R_{t-1}^d n_{i,t-1}\right]$$

Forward this equation, discounting by the stochastic discount factor and multiplying both sides by $\Omega_{t+1} = 1 - \sigma + \sigma \theta_{t+1} \phi_{t+1}$ (2.14)

Probability of financial intermediary surviving to the next period

Portion of bond value financial intermediaries can abscond with.

given in 2.15. Sims & Wu (2021) say that it is an endogenous leverage ratio.

$$\Lambda_{t,t+1} \Omega_{t+1} n_{i,t+1} = \Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1}\left[(R_{t+1}^F - R_t^d) Q_t f_{1,t} + (R_{t+1}^B - R_t^d) Q_{B,t} b_{i,t} + (R_t^{re} - R_t^d) re_{i,t} + R_t^d n_{i,t}\right] \quad (A.8)$$

We can write equation 2.8 recursively:

$$V_{it} = \max_{\{n_{i,t+1}\}} \Big\{ \underbrace{(1-\sigma) E_t\left[\Lambda_{t,t+1} n_{i,t+1}\right]}_{\text{Value of firm today}} + \underbrace{\sigma E_t\left[\Lambda_{t,t+1} V_{i,t+1}\right]}_{\text{Value next period, times the probability of surviving}} \Big\}$$ Plug-in $n_{i,t+1}$:

$$V_{it} = \max_{\{f_{i,t}; b_{i,t}; re_{i,t}\}} \Big\{ (1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1}\left((R_{t+1}^F - R_t^d) Q_t f_{1,t} + (R_{t+1}^B - R_t^d) Q_{B,t} b_{i,t} + (R_t^{re} - R_t^d) re_{i,t} + R_t^d n_{i,t}\right)\right] + \sigma E_t\left[\Lambda_{t,t+1} V_{i,t+1}\right] \Big\}$$

Putting this into a Lagrangian with an incentive compatability constraint (2.9) and a reserve requirement constraint (2.10):

$$\mathcal{L} = V_{i,t} + \lambda_t (V_{i,t} - \theta_t(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t})) + \omega_t (re_{i,t} - \underbrace{\varsigma_t d_{i,t}}_{\text{Sims \& Wu (2021) call this } \overline{re}_{i,t}})$$ Sub-in $V_{i,t}$:

$$\mathcal{L} = \max \Big\{ (1+\lambda_t) E_t\Big[(1-\sigma) \Lambda_{t,t+1} \Big\{ (R_{t+1}^F - R_t^d) \Pi_{t+1}^{-1} Q_t f_{i,t} + (R_{t+1}^B - R_t^d) \Pi_{t+1}^{-1} Q_{B,t} b_{i,t} + (R_t^{re} - R_t^d) \Pi_{t+1}^{-1} re_{i,t} + R_t^d \Pi_{t+1}^{-1} n_{i,t} \Big\} + \sigma \Lambda_{t,t+1} V_{i,t+1} \Big] \Big\} - \lambda_t \theta_t (Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t}) + \omega_t (re_{i,t} - \overline{re}_{i,t})$$

Guess that $V_{i,t}$ is linear in n_{it}. That is: $V_{it} = \theta_t \phi_t n_{i,t}$ (2.16)

FOCs:

$$\frac{\partial \mathcal{L}}{\partial f_{i,t}} = (1+\lambda_t)(1-\sigma) Q_t E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_{t+1}^F - R_t^d)\right] + (1+\lambda_t)\sigma E_t\left[\Lambda_{t,t+1} \frac{\partial v_{i,t+1}}{\partial f_{i,t}}\right]$$

$$= \lambda_t \theta_t Q_t$$

Lump extra terms in here

$$(1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_{t+1}^F - R_t^d)\right] + \sigma E_t\left[\Lambda_{t,t+1} \theta_{t+1} \phi_{t+1} (R_{t+1}^F - R_t^d) \Pi_{t+1}^{-1}\right]$$

$$= \frac{\lambda_t}{1+\lambda_t} \theta_t$$

$$E_t\Big[\Lambda_{t,t+1} \underbrace{(1-\sigma+\sigma\theta_{t+1}\phi_{t+1})}_{\equiv \Omega_{t+1}} \Pi_{t+1}^{-1} (R_{t+1}^F - R_{t+1}^d)\Big] = \frac{\lambda_t}{1+\lambda_t} \theta_t$$

$$E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1} (R_{t+1}^F - R_t^d)\right] = \frac{\lambda_t}{1+\lambda_t} \theta_t \qquad (2.11)$$

$$\frac{\partial \mathcal{L}}{\partial b_{i,t}} = (1+\lambda_t)(1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} Q_{B,t} (R_{t+1}^B - R_t^d)\right] + (1+\lambda_t)\sigma E_t\left[\Lambda_{t,t+1} \frac{\partial v_{i,t+1}}{\partial b_{i,t}}\right] = \lambda_t \theta_t \Delta Q_{B,t}$$

Again, lump everything else in here

$$(1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_{t+1}^B - R_t^d)\right] + \sigma E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} \theta_{t+1} \phi_{t+1} (R_t^B - R_t^d)\right] = \frac{\lambda_t}{1+\lambda_t} \theta_t \Delta$$

$$E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1} (R_{t+1}^B - R_t^d)\right] = \frac{\lambda_t}{1+\lambda_t} \theta_t \Delta \qquad (2.12)$$

$$\frac{\partial \mathcal{L}}{\partial re_{i,t}} = (1+\lambda_t)(1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^d)\right] + (1+\lambda_t)\sigma E_t\left[\Lambda_{t,t+1} \frac{\partial v_{i,t+1}}{\partial re_{i,t}}\right] = -\omega_t$$

$$(1-\sigma) E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^d)\right] + \sigma E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{-1} \theta_{t+1} \phi_{t+1} (R_t^{re} - R_t^d)\right] = \frac{-\omega_t}{1+\lambda_t}$$

$$E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^d)\right] = \frac{-\omega_t}{1+\lambda_t} \qquad (2.13)$$

Now assume that 2.9 binds, that is, that $v_{it} = \theta_t(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t})$. Set this equal to the guess:

$$\theta_t(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t}) = \theta_t \phi_t n_{i,t}$$

$$\phi_t = (Q_t f_{i,t} + \Delta Q_{B,t}) n_{it}^{-1} \qquad (2.17)$$

Now take the expectation of A.8:

$$E_t\left[\Lambda_{t,t+1} \Omega_{t+1} n_{i,t+1}\right] = E_t\Big[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1}\big[(R_{t+1}^F - R_t^d) Q_t f_{i,t} + (R_{t+1}^B - R_t^d) Q_{B,t} b_{i,t} + (R_t^{re} - R_t^d) re_{i,t} + R_t^d n_{i,t}\big]\Big]$$

$$= E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1} (R_{t+1}^F - R_t^d)\right] (Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t})$$

I do not know how this term was derived

$$+ E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1} (R_t^{re} - R_t^d)\right] re_{i,t} + E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1}\right] R_t^d n_{i,t}$$

$$= \frac{\lambda_t}{1+\lambda_t} \theta_t \phi_t n_t - \frac{\omega_t}{1+\lambda_t} + E_t\left[\Lambda_{t,t+1} \Omega_{t+1} \Pi_{t+1}^{-1}\right] R_t^d n_{i,t}$$

We can write A.9 in a similar manner:

$$v_{it} = \theta_t \phi_t n_{it} = \max\left\{E_t\left[\Lambda_{t,t+1} \Omega_{t+1} n_{i,t+1}\right]\right\}$$

$$= \frac{\lambda_t}{1+\lambda_t}\Theta_t \phi_t n_{i,t} - \frac{w_t}{1+\lambda_t} re_{i,t} + E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d n_{i,t}$$

If we assume that $\frac{re_{i,t}}{d_{i,t}} = \frac{re_t}{d_t}$, then the reserve requirement (2.10) implies that $5_t = \frac{re_t}{d_t}$ when the constraint binds. Assuming the same holds for n_{it} ($n_{it}=n_t$):

$$\Theta_t\phi_t = \frac{\lambda_t}{1+\lambda_t}\Theta_t\phi_t - \frac{w_t}{1+\lambda_t}\frac{re_t}{n_t} + E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d$$

$$\left(1-\frac{\lambda_t}{1+\lambda_t}\right)\Theta_t\phi_t = -\frac{w_t}{1+\lambda_t}\frac{re_t}{n_t} + E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d$$

$$\frac{1}{1+\lambda_t}\Theta_t\phi_t = -\frac{w_t}{1+\lambda_t}\frac{re_t}{n_t} + E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d$$

$$\Theta_t\phi_t = (1+\lambda_t)E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d - \frac{w_t re_t}{n_t} \qquad (A.11)$$

$$\phi_t = \frac{(1+\lambda_t)E_t\left[\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\right] R_t^d}{\Theta_t} - \frac{w_t re_t}{\Theta_t n_t} \qquad (2.15)$$

Two other important equations (given):

$$R_t^F = \frac{1+\kappa Q_t}{Q_{t-1}} \quad (2.6) \qquad R_t^B = \frac{1+\chi Q_{B,t}}{Q_{B,t-1}} \quad (2.7)$$

Sims & Wu (2021) go on to derive derivatives of the value function with respect to $n_{i,t}$. I skip this, as I do not see the point of going through this math.

(5) Fiscal Authority (following Appendix A.5)

Sims & Wu (2021) assume that the fiscal authority consumes an exogenously determined, stochastic amount of Y_t, denoted by G_t. G_t is financed by lump-sum taxes on the household and by issuing bonds, $B_{G,t}$ (nominal).

Assume the quantity of real bonds is fixed. Then $b_{G,t} = \bar{b}_G$. Also assume $B_{G,t} = P_t \bar{b}_G$. Then the fiscal authority's budget constraint is:

$$\underbrace{P_t G_t}_{\text{Nominal Expenditure}} + \underbrace{P_{t-1}\bar{b}_g}_{\text{Bond payments this period}} = \underbrace{P_t T_t}_{\text{Taxes collected}} + \underbrace{P_t T_{cb,t}}_{\text{Transfer from Central bank}} + \underbrace{Q_{B,t} P_t \bar{b}_G (1-\kappa \Pi_t^{-1})}_{\text{Value of bonds sold this period}} \qquad (A.36)$$

$$G_t + \Pi_t^{-1}\bar{b}_g = T_t + T_{cb,t} + Q_{B,t}\bar{b}_G(1-\kappa\Pi_t^{-1}) \quad \text{(in real terms)}$$

(6) Central Bank (following section 3)

i. Conventional Policy:

Sims & Wu (2021) define their Taylor rule as:

$$\ln(R_t^{TR}) = (1-\rho_r)\ln(R^{TR}) + \rho_r \ln(R_{t-1}^{TR}) + (1-\rho_r)\Big[\phi_\pi \overbrace{(\ln \Pi_t - \ln \Pi)}^{\text{Deviation from target inflation}} + \phi_y \underbrace{(\ln Y_t - \ln Y_{t-1})}_{\text{Deviation from trend}}\Big] + \underbrace{s_r \varepsilon_{r,t}}_{\text{Interest rate shock}} \qquad (3.1)$$

- Normal times: $R_t^d = R_t^{re} = R_t^{TR}$ (3.2)
- Zero-lower bound: $R_t^{re} = R_t^d = \max\{1, R_t^{TR}\}$ (3.3)

I'm not sure why 1 is used here as the alternative

ii. Quantitative Easing

Central Bank balance sheet: $\overbrace{Q_t F_{cb,t}}^{\text{Value of outstanding gov't bonds}} + \overbrace{Q_{B,t} B_{cb,t}}^{\text{Value of outstanding private bonds}} = \overbrace{RE_t}^{\text{Reserves (interest-bearing)}}$ (3.4)

Created to make equation balance

Remittance: $T_{cb,t} = (1+\kappa Q_t)\Pi_t^{-1} f_{cb,t-1} + (1+\kappa Q_{B,t})\Pi_t^{-1} b_{cb,t-1} - R_{t-1}^{re}\Pi_t^{-1} re_{t-1}$ (Remit)

a) Exogenous QE: holdings follow AR(1) processes:

$$f_{cb,t} = (1-\rho_f) f_{cb} + \rho_f f_{cb,t-1} + s_f \varepsilon_{f,t} \qquad (3.5)$$

$$b_{cb,t} = (1-\rho_b) b_{cb} + \rho_b b_{cb,t-1} + s_b \varepsilon_{b,t} \qquad (3.6)$$

b) Endogenous QE: Private holdings follow a Taylor rule

$$f_{cb,t} = (1-\rho_f) f_{cb} + \rho_f f_{cb,t-1} + (1-\rho_f)\Psi_f\left[\phi_\pi(\ln(\Pi_t) - \ln(\Pi) + \phi_y(\ln(Y_t) - \ln(Y_{t-1})\right] + S_f \varepsilon_{f,t} \quad (5.1)$$

- At zero lower-bound, $\Psi \neq 0$.
- In normal times, $\Psi = 0$.

iii. Forward Guidance

Sims & Wu (2021) model this as a shock to 3.1 with a credibility parameter attached. Therefore, a forward guidance shock is:

$$\gamma S_r \varepsilon_{r,t} \quad \text{with } \gamma \in [0,1]$$

No credibility ⟶ ↑ ↑ ⟵ perfectly credible

iv. Negative Interest Rate Policy

This allows the interest rate on reserves to go negative. So:

$$R_t^d = \max\{1, R_t^{re}\}, \quad R_t^{re} = R_t^{TR} \quad (3.7)$$

Negative rates are introduced by negatively shocking R_t^{TR} when 2.10 binds.

⑦ Aggregation and Exogenous Processes (following Appendix A.6)

There are 3 exogenous processes: A_t, G_t, and Θ_t. Each follow an AR(1) as logs:

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + S_A \varepsilon_{A,t} \quad (A.37)$$

⟵ Mean A = 1, so ln(A) = 0

$$\ln(G_t) = (1-\rho_G)\ln(G) + \rho_G \ln(G_{t-1}) + S_G \varepsilon_{G,t} \quad (A.38)$$

$$\ln(\Theta_t) = (1-\rho_\Theta)\ln(\Theta) + \rho_\Theta \ln(\Theta_{t-1}) + S_\Theta \varepsilon_{\Theta,t} \quad (A.39)$$

Bond market clearing requires:

Wholesaler bonds issued [$f_{m,t} = \underbrace{f_t}_{\text{Bonds held by FIs}} + \underbrace{f_{cb,t}}_{\text{Bonds held by the central bank}}$ (A.40)

Government bonds issued [$\bar{b}_G = b_t + b_{cb,t}$ (A.41)

where $f_t = \sum_i f_{i,t}$ and $b_t = \sum_i b_{i,t}$

Aggregate the financial intermediaries' balance sheets (2.4):

$$\int_0^1 Q_t F_{i,t} + Q_{B,t} B_{i,t} + RE_{i,t}\, di = \int_0^1 D_{i,t} + N_{i,t}\, di$$

$$Q_t F_t + Q_{B,t} B_t + RE_t = D_t + N_t$$ Divide by P_t:

$$Q_t f_t + Q_{B,t} b_t + re_t = d_t + n_t \qquad \text{(A.42)}$$

Aggregate financial intermediaries' net worth (A.7):

$$\int_0^1 n_{i,t}\, di = \int_0^1 \Pi_t^{-1}\Big[(R_t^F - R_{t-1}^d) Q_{t-1} f_{i,t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{i,t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{i,t-1} + R_{t-1}^d n_{i,t-1}\Big] di$$

$$n_t = \Pi_t^{-1}\Big[(R_t^F - R_{t-1}^d) Q_{t-1} f_{t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{t-1} + R_{t-1}^d n_{t-1}\Big]$$

Adjust to reflect σ surviving firms and X start-up funds:

$$n_t = \Pi_t^{-1}\sigma\Big[(R_t^F - R_{t-1}^d) Q_{t-1} f_{t-1} + (R_t^B - R_{t-1}^d) Q_{B,t-1} b_{t-1} + (R_{t-1}^{re} - R_{t-1}^d) re_{t-1} + R_{t-1}^d n_{t-1}\Big] + X \qquad \text{(A.43)}$$

Aggregate the leverage ratio (2.17):

$$\int_0^1 \phi_t n_{i,t}\, di \geq \int_0^1 Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t}\, di$$

$$\phi_t n_t \geq Q_t f_t + \Delta Q_{B,t} b_t \qquad \text{(A.44)}$$

Derive the Aggregate Resource Constraint:

Start with the household budget constraint (A.2):

$$P_t C_t + D_t - D_{t-1} = MRS_t L_t + DIV_t - P_t X - P_t T_t + (R_{t-1}^d - 1) D_{t-1}$$ Put in real terms:

$$C_t + d_t - \frac{P_{t-1}}{P_t} d_{t-1} = mrs_t L_t + div_t - X - T_t + \left(\frac{P_{t-1}}{P_t}\right)(R_{t-1}^d - 1) d_{t-1}$$

$$C_t + d_t = mrs_t L_t + div_t - X - T_t + \Pi_t^{-1} R_{t-1}^d d_{t-1}$$

$$C_t + d_t = mrs_t L_t + div_t - X - G_t - \Pi_t^{-1}\bar{b}_G + T_{cb,t} + Q_{B,t}\bar{b}_G(1 - \kappa\Pi_t^{-1}) + \Pi_t^{-1} R_{t-1}^d d_{t-1}$$

$$C_t + G_t = mrs_t L_t + div_t - X - \Pi_t^{-1}\bar{b}_G + T_{cb,t} + Q_{B,t}\bar{b}_G(1 - \kappa\Pi_t^{-1}) + \Pi_t^{-1} R_{t-1}^d d_{t-1} - d_t$$

$$C_t + G_t = mrs_t L_t + div_{L,t} + div_{m,t} + div_{R,t} + div_{k,t} + n_t^{*} - X - \Pi_t^{-1}\bar{b}_G + T_{cb,t} + Q_{B,t}\bar{b}_G(1 - \kappa\Pi_t^{-1}) + \Pi_t^{-1} R_{t-1}^d d_{t-1} - d_t$$

$$= mrs_t L_t + \cancel{p_{m,t} Y_{m,t}} - w_t L_{d,t} - \cancel{p_t^k \hat{I}_t} - \Pi_t^{-1} f_{m,t-1} + Q_t(f_{m,t} - \kappa\Pi_t^{-1} f_{m,t-1}) + \cancel{p_t^k\left[1 - S\left(\tfrac{I_t}{I_{t-1}}\right)\right]} I_t - I_t$$

$$+ Y_t - \cancel{p_{m,t} Y_{m,t}} + n_t^{*} - X - \Pi_t^{-1}\bar{b}_G + T_{cb,t} + Q_{B,t}\bar{b}_G(1 - \kappa\Pi_t^{-1}) + \Pi_t^{-1} R_{t-1}^d d_{t-1} - d_t$$

$$C_t + I_t + G_t = Y_t + \cancel{div_{L,t} + mrs_t L_t - w_t L_{d,t}} - \Pi_t^{-1} f_{m,t-1} + Q_t(f_{m,t} - \kappa\Pi_t^{-1} f_{m,t-1}) + n_t^{*} - X$$
$$- \Pi_t^{-1}\bar{b}_G + T_{cb,t} + Q_{B,t}\bar{b}_G(1 - \kappa\Pi_t^{-1}) + \Pi_t^{-1} R_{t-1}^d d_{t-1} - d_t$$

$$C_t + I_t + G_t = Y_t - \Pi_t^{-1} f_{m,t-1} + Q_t(f_{m,t} - \kappa \Pi_t^{-1} f_{m,t-1}) + n^*_t - X - \Pi_t^{-1}\bar{b}_G + T_{cb,t}$$
$$+ Q_{B,t}\bar{b}_G(1-\kappa\Pi_t^{-1}) + \Pi_t^{-1} R^d_{t-1} d_{t-1} - d_t$$

$$C_t + I_t + G_t - Y_t = Q_t f_{m,t} - \Pi_t^{-1} f_{m,t-1} - Q_t \kappa \Pi_t^{-1} f_{m,t-1} + Q_{B,t}\bar{b}_G - \Pi_t^{-1}\bar{b}_G - Q_{B,t}\bar{b}_G \kappa \Pi_t^{-1}$$
$$+ \Pi_t^{-1} R^d_{t-1} d_{t-1} - d_t + n^*_t - x + T_{cb,t}$$

$$= Q_t f_{m,t} - \Pi_t^{-1} f_{m,t-1} - Q_t \kappa \Pi_t^{-1} f_{m,t-1} + Q_{B,t}\bar{b}_G - \Pi_t^{-1}\bar{b}_G - Q_{B,t}\bar{b}_G \kappa \Pi_t^{-1}$$
$$+ \Pi_t^{-1} R^d_{t-1}(Q_{t-1} f_{t-1} + Q_{B,t-1} b_{t-1} + re_{t-1} - n_{t-1}) - (Q_t f_t + Q_{B,t} b_t + re_t - n_t) + n^*_t - x$$
$$+ (1+\kappa Q_t)\Pi_t^{-1} f_{cb,t-1} + (1+\kappa Q_{B,t})\Pi_t^{-1} b_{cb,t-1} - R^{re}_{t-1}\Pi_t^{-1} re_{t-1}$$

$$= Q_t f_{m,t} - \Pi_t^{-1} f_{m,t-1} - Q_t \kappa \Pi_t^{-1} f_{m,t-1} + Q_{B,t}\bar{b}_G - \Pi_t^{-1}\bar{b}_G - Q_{B,t}\bar{b}_G \kappa \Pi_t^{-1}$$
$$+ \Pi_t^{-1} R^d_{t-1}\Big(Q_{t-1} f_{t-1} + Q_{B,t-1} b_{t-1} \cancel{+ re_{t-1}} - R^{d^{-1}}_{t-1}\Big(\cancel{\frac{n_t}{\sigma \Pi_t^{-1}}} - (R^F_t - R^d_{t-1})Q_{t-1} f_{t-1} - (R^B_t - R^d_{t-1})Q_{B,t-1} b_{t-1}$$
$$\cancel{-(R^{re}_{t-1} - R^d_{t-1})re_{t-1}}\Big)\Big) \cancel{+X} - Q_t f_t - Q_{B,t} b_t - re_t \cancel{+ n_t} + \cancel{\frac{n^\sigma}{\sigma} n_t} \cancel{- X}$$
$$+ (1+\kappa Q_t)\Pi_t^{-1} f_{cb,t-1} + (1+\kappa Q_{B,t})\Pi_t^{-1} b_{cb,t-1} \cancel{- R^{re}_{t-1}\Pi_t^{-1} re_{t-1}}$$

$$= \cancel{Q_t f_{m,t}} - \Pi_t^{-1} f_{m,t-1} - Q_t \kappa \Pi_t^{-1} f_{m,t-1} + \cancel{Q_{B,t}\bar{b}_G} - \Pi_t^{-1}\bar{b}_G - Q_{B,t}\bar{b}_G \kappa \Pi_t^{-1}$$
$$+ \Pi_t^{-1} R^d_{t-1}\Big(Q_{t-1} f_{t-1} + Q_{B,t-1} b_{t-1} + R^{d^{-1}}_{t-1}\big[(R^F_t - R^d_{t-1})Q_{t-1} f_{t-1} + (R^B_t - R^d_{t-1})Q_{B,t-1} b_{t-1}\big]$$
$$\cancel{- Q_t f_t} \cancel{- Q_{B,t} b_t} \cancel{- Q_t f_{cb,t}} \cancel{- Q_t b_{cb,t}} + (1+\kappa Q_t)\Pi_t^{-1} f_{cb,t-1}$$
$$+ (1+\kappa Q_{B,t})\Pi_t^{-1} b_{cb,t-1}$$

$$= -\Pi_t^{-1} f_{m,t-1} - Q_t \kappa \Pi_t^{-1} f_{m,t-1} - \Pi_t^{-1}\bar{b}_G - Q_{B,t}\kappa\Pi_t^{-1}\bar{b}_G$$
$$\cancel{+ \Pi_t^{-1} R^d_{t-1} Q_{t-1} f_{t-1}} + \cancel{\Pi_t^{-1} R^d_{t-1} Q_{B,t-1} b_{t-1}} + \Pi_t^{-1}\big[(R^F_t \cancel{- R^d_{t-1}})Q_{t-1} f_{t-1}$$
$$+ (R^B_t \cancel{- R^d_{t-1}})Q_{B,t-1} b_{t-1}\big] + \Pi_t^{-1} f_{cb,t-1} + \kappa Q_t \Pi_t^{-1} f_{cb,t-1}$$
$$+ \Pi_t^{-1} b_{cb,t-1} + \kappa Q_{B,t}\Pi_t^{-1} b_{cb,t-1}$$

$$= \underline{-\Pi_t^{-1} f_{m,t-1}} \; \underline{- Q_t \kappa \Pi_t^{-1} f_{m,t-1}} \; \underline{- \Pi_t^{-1}\bar{b}_G} \; \underline{- Q_{B,t}\kappa\Pi_t^{-1}\bar{b}_G}$$
$$+ \Pi_t^{-1} R^F_t Q_{t-1} f_{t-1} + \Pi_t^{-1} R^B_t Q_{B,t-1} b_{t-1} + \Pi_t^{-1} f_{cb,t-1}$$
$$+ \kappa Q_t \Pi_t^{-1} f_{cb,t-1} + \Pi_t^{-1} b_{cb,t-1} + \kappa Q_{B,t}\Pi_t^{-1} b_{cb,t-1}$$

$$= -\Pi_t^{-1} Q_{t-1}\left(\frac{1-\kappa Q_t}{Q_{t-1}}\right) f_{m,t-1} - \Pi_t^{-1} Q_{B,t-1}\left(\frac{1-\kappa Q_{B,t}}{Q_{B,t-1}}\right)\bar{b}_G$$
$$+ \Pi_t^{-1} R^F_t Q_{t-1} f_{t-1} + \Pi_t^{-1} R^B_t Q_{B,t-1} b_{t-1} + \Pi_t^{-1} Q_{t-1}\left(\frac{1-\kappa Q_t}{Q_{t-1}}\right) f_{cb,t-1}$$
$$+ \Pi_t^{-1} Q_{B,t-1}\left(\frac{1-\kappa Q_{B,t}}{Q_{B,t-1}}\right) b_{cb,t-1}$$

Using 2.6 & 2.7 :

$$= \cancel{-\Pi_t^{-1} Q_{t-1} R_t^F f_{m,t-1}} \cancel{-\Pi_t^{-1} Q_{B,t-1} R_t^B \bar{b}_G} + \cancel{\Pi_t^{-1} R_t^F Q_{t-1} f_{t-1}} + \cancel{\Pi_t^{-1} R_t^B Q_{B,t-1} b_{t-1}}$$

$$+ \cancel{\Pi_t^{-1} Q_{t-1} R_t^F f_{cb,t-1}} + \cancel{\Pi_t^{-1} Q_{B,t-1} R_t^B b_{cb,t-1}}$$

$$Y_t = C_t + I_t + G_t \qquad (A.45)$$

(8) Equilibrium (following Appendix A.7)

There are 50 variables in 50 equations:

Equation Number	Labels	Economic Sector
1 - 4	A.3 - A.6	Households
5 - 7	A.14 - A.16	Labor Market
8 - 10	A.22 - A.24	Retail Firms
11 - 18	2.19 - 2.21 & 2.23 - 2.27	Wholesale Firms
19 - 20	A.34 & A.35	Capital Firms
21 - 27	2.6, 2.7, & 2.11 - 2.15	Financial Intermediaries
28	A.36	Fiscal Authority
29 & 30	3.4 & Remit	Central Bank
31 - 35	3.1, 3.2, $\overbrace{3.3 \text{ or } 3.7}^{\text{ZLB or NIRP}}$, 3.6, & 5.1	Monetary Policy
36 - 50	A.37 - A.45, A.17 - A.19, & A.25 - A.27	Aggregation & Exogenous Processes

Variable Category	Variables
Interest Rates	$R_t^F, R_t^B, R_t^{rc}, R_t^d, R_t^{TR}$
Bond Prices	$Q_t, Q_{B,t}$
Stochastic Discounting	$\Lambda_{t,t+1}$
Auxillary	Ω_t, ϕ_t
Inflation	Π_t
Shadow Values	$\lambda_t, \omega_t, \mu_t$ "omega"
Net Worth	n_t, rc_t, f_t, b_t
Household	C_t, L_t, d_t, mrs_t

Labor	$w_t^*, f_{1,t}, f_{2,t}, w_t, \nu_t^w$
Production	$L_{d,t}, p_t^*, x_{1,t}, x_{2,t}, p_{m,t}, Y_t, Y_{m,t}, U_t, K_t, \hat{I}_t, p_t^k, f_{m,t}, M_{1,t}, M_{2,t}, I_t, A_t, \nu_t^p$
Central Banks	$T_{cb,t}, f_{cb,t}, b_{cb,t}$
Fiscal	G_t, T_t
Misc.	Θ_t