Principles of Macroeconomics: The Solow Growth Model Part 1

Class 4

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Overview

- Announcements:
  - You should be able to do LC 7 and GH 7 (due September 12th at 11:59pm)
  - By the end of today we will be halfway through the material for LC 9 and GH 9 (also due September 12th)
- Topics:
  - The production function
  - Simple model of production
- Readings:
  - Chapter 9.2 (Sources of Long-Run Growth), chapters 9.3-9.4

Why Math?

- Remember, I said we'd only use math to clarify ideas
- So why use math here?
  - We want to posit a theory of why countries grow faster than others
  - We could propose all sorts of mechanisms
  - How do we make sure those mechanisms are coherent (as in, will they actually lead to growth given a set of assumptions?)
  - We use math to demonstrate that logic

The Production Function

- Output: Y
  - Think of this as GDP
- Production:

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$$

  - $K \equiv$ capital
  - $L \equiv$ labor hours
  - $A \equiv$ total productivity – scales the inputs
  - $\alpha \equiv$ how important capital is to production ("capital share of production")

Production Functions in General

- $F(K, L)$ can be any function describing how inputs get to outputs
- The particular function on the previous slide is called a Cobb-Douglas production function
- For example, suppose $Y :=$ cars, and $K :=$ factories, and $L :=$ worker work days. Let $A = 2000$ and $\alpha = 1/3$.
  - What is Y if we have 8 factories and 27 worker work days?

$$
\begin{aligned}
Y &= 2000 \times 8^{1/3} \times 27^{2/3} \\
&= 2000 \times 2 \times 9 \\
&= 36,000 \text{ cars}
\end{aligned}
$$

Constant Returns to Scale

- What happens if we increase each input by $x\%$?

$$F(2K, 2L) = A(2K)^{1/3}(2L)^{2/3}$$

- We can simplify this by pulling the 2 out front:

$$\begin{aligned} F(2K, 2L) &= 2AK^{1/3}L^{2/3} \\ &= 2F(K, L) \end{aligned}$$

- So doubling inputs also doubles the output – Constant returns to scale

Even More General

- Let $Y = AK^{\alpha}L^{\beta}$ – more general Cobb-Douglas. Then:
  - Decreasing returns: $\alpha + \beta < 1$
    If we double inputs, we get less than double output
  - Increasing returns: $\alpha + \beta > 1$
    If we double inputs, we get more than double output
  - Constant returns: $\alpha + \beta = 1$
    If we double inputs, we double output
- Note that this is for doubling all inputs

What if we only change one input?

1. Double K, keep L fixed

$$
\begin{aligned}
F(2K, L) &= A(2K)^{1/3}L^{2/3} \\
&= A2^{1/3}K^{1/3}L^{2/3} \\
&< 2AK^{1/3}L^{2/3}
\end{aligned}
$$

2. Covid Labor Shock – L falls by 16% in one quarter

$$Y_t = AK^{1/3}L^{2/3} \qquad Y_{t+1} = AK^{1/3}(0.84L)^{2/3}$$

Output growth is:

$$\frac{Y_{t+1}}{Y_t} = \frac{AK^{1/3}(0.84L)^{2/3}}{AK^{1/3}L^{2/3}} = 0.89$$

So the model predicts output fell by 11% (reality: 10%)

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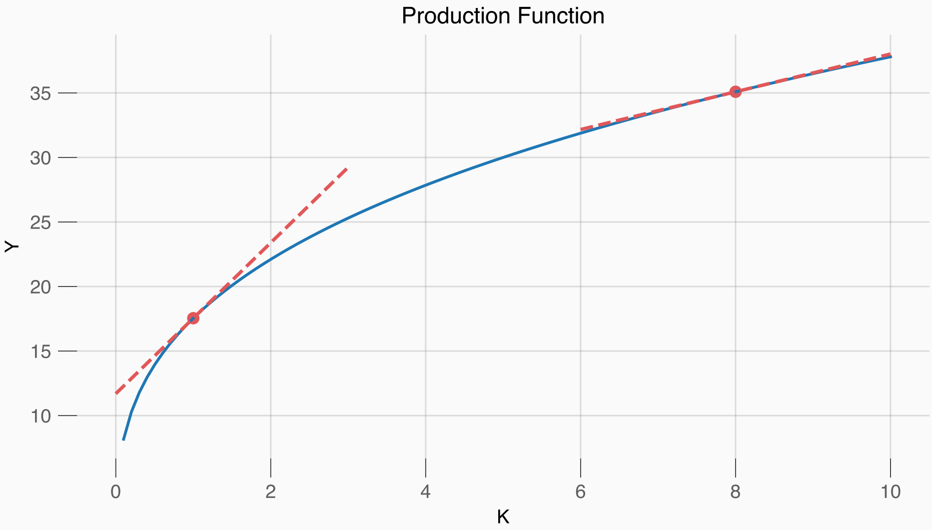
Diminishing Returns – what happens as we keep increasing K?



Marginal Product of Capital (MPK)

- Time for a little bit of calculus: take the partial derivative of the production function with respect to capital:

$$MPK = \frac{\partial Y}{\partial K}$$

  This will tell use how much Y changes when we change K

- In our application, we need the power rule:

$$\text{Let } y = x^a \text{, then } \frac{\partial y}{\partial x} = ax^{a-1}$$

$$\frac{\partial Y}{\partial K} = \frac{1}{3} A K^{-2/3} L^{2/3}$$

- This is also the slope of the production function we just plotted
- As we saw graphically, the slope gets flatter as K increases
  - Check it! I used $A = 6$, $L = 5$, $\alpha = \frac{1}{3}$
- We call this diminishing returns to capital.
- We can do a similar exercise for L and find a similar outcome

Overview

- We just constructed the first piece of the Solow Growth Model

1. The Production Function ✓
2. Factor Supply
   - Fixed endowment of labor: $L = \bar{L}$
   - Fixed endowment of capital: $K = \bar{K}$
3. Producer behavior
   - Competitive producers maximize profits
   - These producers demand capital and labor
4. Equilibrium
   - Set supply $=$ demand
   - Solve for prices and output

Producers

- The producer takes the price of its output (p) and the prices of inputs as given.
- Problem: How much labor and capital should the firm hire?

- To answer, we need to know what the marginal benefit of hiring a worker is.

- What is a firm's revenue?

$$Rev = pY = pF(K, L)$$

- We want to know what the benefit of hiring one additional worker is, so we take the derivative:

$$p\frac{\partial F(K, L)}{\partial L} = \frac{2}{3}AK^{1/3}L^{-1/3}$$

- Recall diminishing returns – first worker adds a lot, each additional worker, less

- Now we need the firm's cost:

$$Cost = wL + rK$$

where $w \equiv$ the wage and $r \equiv$ the price of capital

- We can take the derivative of this with respect to labor and get:

$$\frac{\partial Cost}{\partial L} = w$$

This is marginal cost of labor

- How do people make decisions in economics?
    - Marginal cost = marginal benefit!
- So:

$$\frac{2}{3}AK^{1/3}L^{-1/3} = w$$

- Similarly:

$$\frac{1}{3}AK^{-2/3}L^{2/3} = r$$

The Profit Function

- We could do the calculus all separately, or we could do it all at once.
- We can construct a producer's profit function:

$$\Pi = pY - wL - rK$$

- Given p, r, and w, the producer chooses K and L to maximize profit:

$$\max_{\{K,L\}} \quad pY - wL - rK$$

such that $Y = AK^{1/3}L^{2/3}$

- Take the derivatives and set them equal to zero (also called first-order conditions):

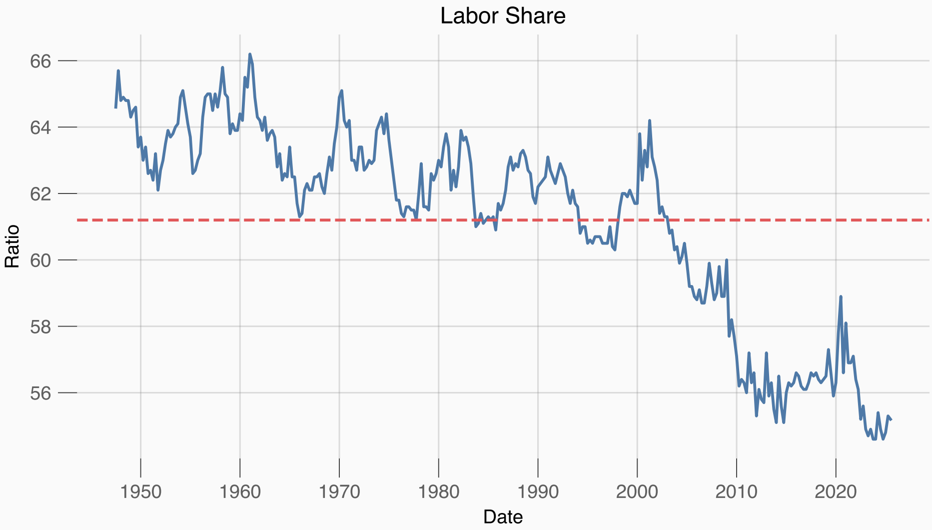
$$\frac{\partial Y}{\partial L}: \quad \frac{2}{3}AK^{1/3}L^{-1/3} = w$$
$$\frac{\partial Y}{\partial K}: \quad \frac{1}{3}AK^{-2/3}L^{2/3} = r$$

- For greater intuition, note that $AK^{1/3}L^{-1/3} = \frac{Y}{L}$, and that $AK^{-2/3}L^{2/3} = \frac{Y}{K}$. Then:

$$\frac{wL}{pY} = \frac{2}{3}$$
$$\frac{rK}{pY} = \frac{1}{3}$$

- So the share of revenue (pY) paid to capital is 1/3
- The share of revenue paid to labor is 2/3

The Data – 2/3 was a decent approximation until recently



Overview

1. The Production Function ✓
2. Factor Supply ✓
   - Fixed endowment of labor: $L = \bar{L}$
   - Fixed endowment of capital: $K = \bar{K}$
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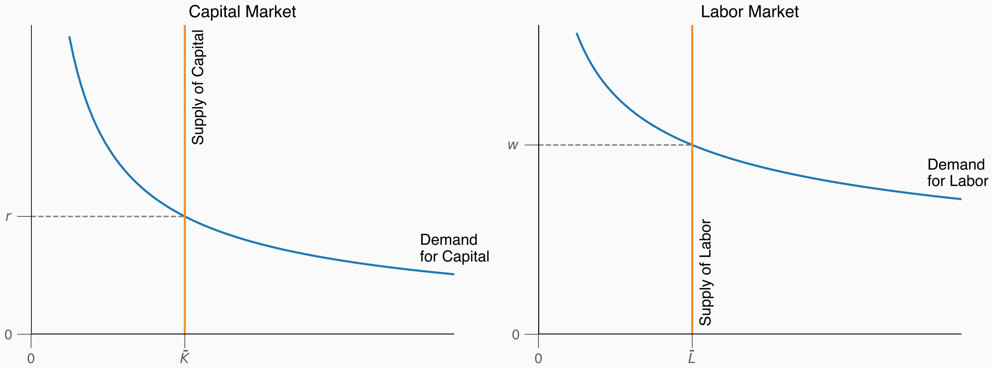
Equilibrium

- Producer decisions are in fact *demand* curves
  - $\downarrow w \longrightarrow \uparrow L$
  - $\downarrow r \longrightarrow \uparrow K$
- Producers have downward-sloping demand for K and L
- We fixed the supply of K and L
- We now equate supply and demand

Graphical Equilibrium



Suppose that $A = 2$, $\alpha = 1/3$, $\bar{K} = 100$, $\bar{L} = 50$, and $p = 1$. Let the production function be a constant returns-to-scale Cobb Douglas function

(1) Compute Y

(2) Derive the marginal product of capital (MPK) and the marginal product of labor (MPL)

(3) Find the factor prices r and w

(4) Verify that capital expenditure relative to output is α. Verify that labor expenditure relative to output is $1 - \alpha$

(5) Suppose A increases by 10%. By what percentage do Y, r, and w change?

Solutions

(1) $Y = 2 \times 100^{1/3} \times 50^{2/3} \approx 126$

(2) The profit function is: $\Pi = pY - wL - rK$. Take the derivatives with respect to K and L:

$$\frac{\partial \Pi}{\partial K} : r = \alpha A K^{\alpha-1} L^{1-\alpha}$$
$$\frac{\partial \Pi}{\partial L} : w = (1-\alpha) A K^{\alpha} L^{-\alpha}$$

(3) Plug-in our specific calibration:

$$r = \frac{1}{3} \times 2 \times 100^{-2/3} \times 50^{2/3} \approx 0.42$$
$$w = \frac{2}{3} \times 2 \times 100^{1/3} \times 50^{-1/3} \approx 1.68$$

(4) Plug-in our solutions:

$$\alpha = \frac{rK}{pY} \qquad\qquad 1-\alpha = \frac{wL}{pY}$$

$$\frac{1}{3} = \frac{0.42 \times 100}{126} \qquad\qquad \frac{2}{3} = \frac{1.68 \times 50}{126}$$

$$\frac{1}{3} = \frac{1}{3} \checkmark \qquad\qquad \frac{2}{3} = \frac{2}{3} \checkmark$$

(5) For Y: $100 \times \frac{(1.1)A_0 K^\alpha L^{1-\alpha} - A_0 K^\alpha L^{1-\alpha}}{A_0 K^\alpha L^{1-\alpha}} = 10\%$

For r: $100 \times \frac{\alpha(1.1)A_0 K^{\alpha-1} L^{1-\alpha} - A_0 K^{\alpha-1} L^{1-\alpha}}{A_0 K^{\alpha-1} L^{1-\alpha}} = 10\%$

For w: $100 \times \frac{(1-\alpha)(1.1)A_0 K^\alpha L^{-\alpha} - (1-\alpha)A_0 K^\alpha L^{-\alpha}}{(1-\alpha)A_0 K^\alpha L^{-\alpha}} = 10\%$

The Solow Model – Summary

- We have three known variables: A, $\bar{K}$, and $\bar{L}$
- We usually set $p = 1$ for convenience
- Five unknowns: K, L, r, w, Y
- Five equations:
    1. Production: $Y = AK^{1/3}L^{2/3}$
    2. Capital demand: $rK = \frac{1}{3}Y$
    3. Labor demand: $wL = \frac{2}{3}Y$
    4. Capital supply: $K = \bar{K}$
    5. Capital demand: $L = \bar{L}$
- Next week: read chapters 9.3-9.4