High-performance defunctionalization in Futhark

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Trends in Functional Programming, 2018

Motivation

- Massively parallel processors, like GPUs, are common but difficult to program.
- Functional programming can make it easier to program GPUs:
  - Referential transparency.
  - Expressing data-parallelism.

Problem Higher-order functions cannot be directly implemented on GPUs.

Can we do higher-order functional GPU programming anyway?

Motivation

Higher-order functions on GPUs?

- Yes!
- Using moderate type restrictions, we can eliminate all higher-order functions at compile-time.
- Gain many benefits of higher-order functions without any run-time performance overhead.

Reynolds's defunctionalization

Defunctionalization (Reynolds, 1972)

John Reynolds: "Definitional interpreters for higher-order programming languages", ACM Annual Conference 1972.

Basic idea:

- Replace each function abstraction by a tagged data value that captures the free variables:

$$\lambda x : \mathbf{int}.\, x + y \quad \Longrightarrow \quad \mathit{LamN}\ y$$

- Replace application by case dispatch over these functions:

$$f\ a \quad \Longrightarrow \quad \begin{array}{l} \mathbf{case}\ f\ \mathbf{of}\ \mathit{Lam1} \ldots \\ \qquad\qquad \mathit{Lam2} \ldots \\ \qquad\qquad \mathit{LamN}\ y \to a + y \\ \qquad\qquad \ldots \end{array}$$

- Branch divergence on GPUs.

Language and type restrictions

Futhark

A purely functional, data-parallel array language with an optimizing compiler that generates GPU code via OpenCL.

- Parallelism expressed through built-in higher-order functions, called *second-order array combinators* (SOACs):

```
map, reduce, scan, ...
```

- No recursion, but sequential loop constructs:

```
loop pat = init for x in arr do body
```

Type-based restrictions on functions

To permit efficient defunctionalization, we introduce type-based restrictions on the use of functions.

Statically determine the form of every applied function.

Transformation is simple and eliminates all higher-order functions.

Instead of allowing unrestricted functions and relying on subsequent analysis, we entirely avoid such analysis.

Type-based restrictions on functions

Conditionals may not produce functions:

```
let f = if b1 then ...
                    if bN then λx → e_n
              else ... λx → e_k
in ... f y
```

Which function `f` is applied?

If our goal is to eliminate higher-order functions without introducing branching, we must restrict conditionals from returning functions.

Require that branches have **order zero** type.

Type-based restrictions on functions

Arrays may not contain functions:

```
let fs = [λy → y+a, λz → z+b, ...]
in ... fs[n] 5
```

Which function `fs[n]` is applied?

Also need to restrict **`map`** to not create array of functions:

```
map (λx → λy → ...) xs
```

Type-based restrictions on functions

Loops may not produce functions:

```
loop f = (λz → z+1) for x in xs
do (λz → x + f z)
```

The shape of `f` depends on the number of iterations of the loop.

Require that loop has **order zero** type.

All other typing rules are standard and do not restrict functions.

Defunctionalization

Defunctionalization

- Type restrictions enable us to track functions precisely.
- Control-flow is restricted so every applied function is known and every application can be specialized.

Defunctionalization

Defunctionalization in a nutshell:

```
let a = 1
let b = 2
let f = λx → x+a
in f b
```

```
let a = 1
let b = 2
let f = {a=a}
in f' f b
```

Create lifted function:

```
let f' env x =
  let a = env.a
  in x+a
```

Defunctionalization

Static values:

$$\begin{aligned} sv ::= &\ \mathit{Dyn}\ \tau \\ \mid &\ \mathit{Lam}\ x\ e_0\ E \\ \mid &\ \mathit{Rcd}\ \{(\ell_i \mapsto sv_i)^{i \in 1..n}\} \end{aligned}$$

- Static approximation of the value of an expression.
- Precisely capture the closures produced by an expression.

Translation environment E maps variables to static values.

Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
                   in twice (λy → y+a)
           in f 1
⇝
```

Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
                   in twice (λy → y+a)
           in f 1
```

$\rightsquigarrow$

```
let twice = {}
```

$Lam\ g\ (\lambda x \rightarrow g\,(g\,x))\ []$

Defunctionalization

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let twice (g: int → int) = λx → g (g x)
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Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
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```

$\leadsto$

```
let twice = {}
let main = let f = let a = 5
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```

$Lam\ g\ (\lambda x \to g\,(g\,x))\ []$

$$
\begin{array}{rcl}
\text{twice} & \leadsto & \text{twice} \\
(\lambda y \to y + a) & \leadsto & \{a = a\}, \quad Lam\ y\ (y + a)\ [a \mapsto Dyn\ \mathbf{int}]
\end{array}
$$

Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
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```

$\rightsquigarrow$

```
let twice = {}                                      Lam g (λx → g (g x)) []
let main = let f = let a = 5
                   in twice' twice {a = a}
           in f 1
let twice' (env: {}) (g: {a: int}) = λx → g (g x)
```

$$\begin{array}{rcl} \text{twice} & \rightsquigarrow & \text{twice} \\ (\lambda y \to y + a) & \rightsquigarrow & \{a = a\}, \quad \mathit{Lam}\ y\ (y + a)\ [a \mapsto \mathit{Dyn}\ \mathbf{int}] \end{array}$$

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let twice' (env: {}) (g: {a: int}) = λx → g (g x)
```

$$\text{twice} \rightsquigarrow \text{twice}$$

$$(\lambda y \to y + a) \rightsquigarrow \{a = a\}, \quad \underbrace{Lam\ y\ (y + a)\ [a \mapsto Dyn\ \mathbf{int}]}_{g}$$

Defunctionalization

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$$
\begin{array}{rcl}
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\lambda x \to g\ (g\ x) & \leadsto & \{g = g\}, \\
 & & \mathit{Lam}\ x\ (g\ (g\ x))\ [g \mapsto \mathit{Lam}\ y\ (y + a)\ \ldots)]
\end{array}
$$

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let twice = {}                                   Lam g (λx → g (g x)) []
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Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
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           in f 1
⇝

let main = let f = let a = 5
                   in {g = {a = a}}
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Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
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$\rightsquigarrow$

```
let main = let f = let a = 5
                   in {g = {a = a}}
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```

$$f \mapsto \mathit{Lam}\ x\ (g\ (g\ x))$$
$$[g \mapsto \mathit{Lam}\ y\ (y + a)\ (a \mapsto \mathit{Dyn}\ \mathbf{int})]$$

Defunctionalization

```
let twice (g: int → int) = λx → g (g x)
let main = let f = let a = 5
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⇝


let main = let f = let a = 5
                   in {g = {a = a}}
           in f' f 1

let f' (env: {g: {a: int}}) (x: int) =
  let g = env.g in g (g x)
```

$$f \mapsto Lam\ x\ (g\ (g\ x))$$
$$[g \mapsto Lam\ y\ (y + a)\ (a \mapsto Dyn\ \mathbf{int})]$$

Defunctionalization

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let twice (g: int → int) = λx → g (g x)
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$$g \mapsto Lam\ y\ (y + a)\ [a \mapsto Dyn\ \textbf{int}]$$

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let g' (env: {a: int}) (y: int) =
  let a = env.a in y+a
```

$$g \mapsto Lam\ y\ (y + a)\ [a \mapsto Dyn\ \mathbf{int}]$$

Defunctionalization

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let twice (g: int → int) = λx → g (g x)
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let main = let f = let a = 5
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let f' (env: {g: {a: int}}) (x: int) =
  let g = env.g in g' g (g' g x)

let g' (env: {a: int}) (y: int) =
  let a = env.a in y+a
```

Correctness

Correctness

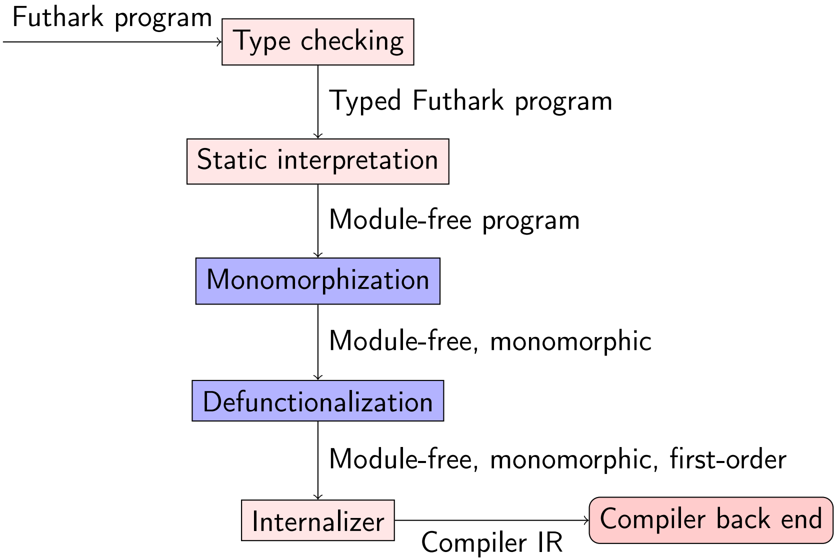
Defunctionalization has been proven correct:

- Defunctionalization terminates and yields a consistently typed residual expression.
  - For order 0, the type is unchanged.
  - Proof using a logical relations argument.
- Meaning is preserved.

More details in the paper.

Implementation

Implementation



Implementation

Polymorphism and defunctionalization

What if type a is instantiated with a function type?

```
let ite 'a (b: bool) (x: a) (y: a) : a =
  if b then x else y
```

Implementation

Polymorphism and defunctionalization

What if type a is instantiated with a function type?

```
let ite 'a (b: bool) (x: a) (y: a) : a =
  if b then x else y
```

Distinguish lifted type variables:

`'a`	regular type variable
`'^a`	*lifted* type variable

Evaluation

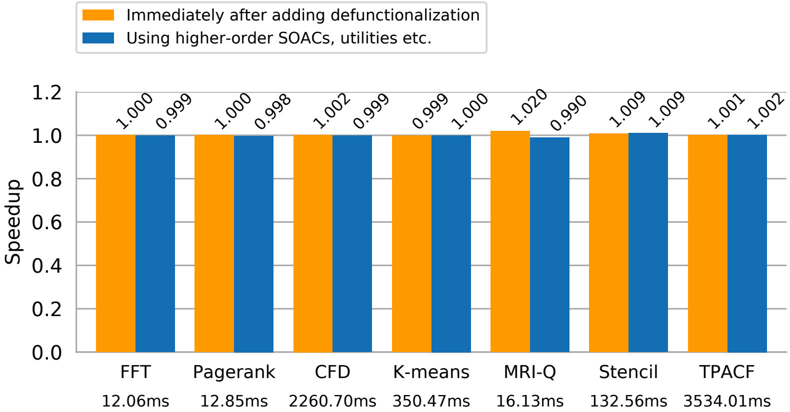
Evaluation

Does defunctionalization yield efficient programs?

Rewrite benchmark programs to use higher-order functions.

- Most SOACs converted to higher-order library functions.
- Higher-order utility functions
  - Function composition, application, `flip`, `curry`, etc.
- Segmented operations and sorting functions in library use higher-order functions instead of parametric modules.

Evaluation



- Run-time performance is unaffected.
- Relies on the optimizations performed by the compiler.

Functional images

Represent images as functions:

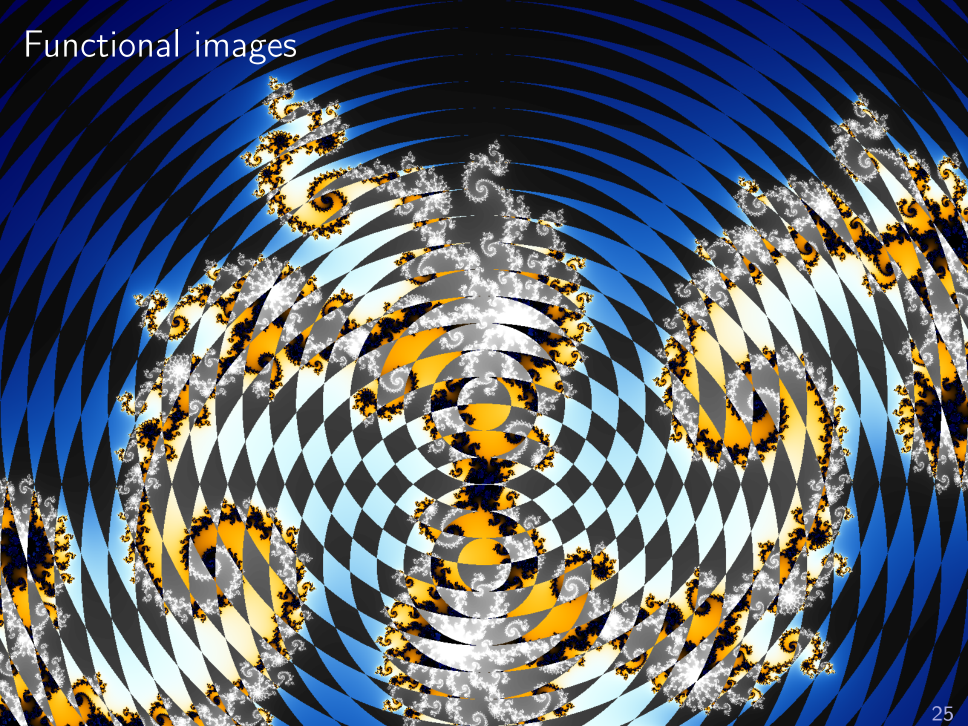
```
type image  'a = point → a

type filter 'a = image a → image a
```

- Due to Conal Elliott.
- Implemented in the Haskell EDSL *Pan*.

The entire Pan library has been translated to Futhark.

Functional images



Function-type conditionals

Support for function-type conditionals

```
let r = if b then {f = λx → x+1, a = 1}
               else {f = λx → x+n, a = 2}
in r.f r.a
```

Support for function-type conditionals

```
let r = if b then {f = λx → x+1, a = 1}
              else {f = λx → x+n, a = 2}
in r.f r.a
```

Introduce new form of static value:

$$Or\ sv_1\ sv_2$$

Static value representation of r:

$$\begin{aligned} Rcd\ \{f \mapsto\ & Or\ (Lam\ x\ (x+1)\ [\,])\\ & \quad\ (Lam\ x\ (x+n)\ [n \mapsto Dyn\ \mathbf{int}])\\ a \mapsto\ & Dyn\ \mathbf{int}\} \end{aligned}$$

Support for function-type conditionals

```
let r = if b then {f = λx → x+1, a = 1}
             else {f = λx → x+n, a = 2}
in r.f r.a
```

Straightforward translation is ill-typed:

```
⇝ if b then {f = {},    a = 1}
        else {f = {n=n}, a = 2}
```

Even worse with nested conditionals.

Binary sum types to complement *Or* static value:

$$\tau_1 + \tau_2$$

Support for function-type conditionals

```
let r = if b then {f = λx → x+1, a = 1}
             else {f = λx → x+n, a = 2}
in r.f r.a
```

$\rightsquigarrow$

```
let r = if b then {f = inl {},    a = 1}
             else {f = inr {n=n}, a = 2}
in let x = r.a
   in case r.f of
        inl e → x+1
        inr e → let n = e.n
                in x+n
```

Conclusion

- General and practical approach to implementing higher-order functions in high-performance functional languages for GPUs.
- Proof of correctness.
- Implementation in Futhark.
- No performance overhead, but gain many of the benefits.

Questions, comments?

Appendix

Typing rules

$$\boxed{\Gamma \vdash e : \tau}$$

$$\text{T-If: } \frac{\Gamma \vdash e_1 : \mathbf{bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau \qquad \tau \ \mathrm{orderZero}}{\Gamma \vdash \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3 : \tau}$$

$$\text{T-Array: } \frac{(\Gamma \vdash e_i : \tau)^{i \in 1..n} \qquad \tau \ \mathrm{orderZero}}{\Gamma \vdash [e_1, \ldots, e_n] : [\,]\tau}$$

$$\text{T-Map: } \frac{\Gamma \vdash e_2 : [\,]\tau_2 \qquad \Gamma, x : \tau_2 \vdash e_1 : \tau \qquad \tau \ \mathrm{orderZero}}{\Gamma \vdash \mathbf{map}\ (\lambda x.\ e_1)\ e_2 : [\,]\tau}$$

$$\text{T-Loop: } \frac{\begin{array}{cc} \Gamma \vdash e_0 : \tau & \tau \ \mathrm{orderZero} \\ \Gamma \vdash e_1 : [\,]\tau' & \Gamma, x : \tau, y : \tau' \vdash e_2 : \tau \end{array}}{\Gamma \vdash \mathbf{loop}\ x = e_0\ \mathbf{for}\ y\ \mathbf{in}\ e_1\ \mathbf{do}\ e_2 : \tau}$$

Defunctionalization rules

$$\boxed{E \vdash e \leadsto \langle e', sv \rangle}$$

D-Var: $$\frac{}{E \vdash x \leadsto \langle x, sv \rangle}\ (E(x) = sv)$$

D-Num: $$\frac{}{E \vdash \overline{n} \leadsto \langle \overline{n}, \mathit{Dyn}\ \mathbf{int} \rangle}$$

D-True: $$\frac{}{E \vdash \mathbf{true} \leadsto \langle \mathbf{true}, \mathit{Dyn}\ \mathbf{bool} \rangle}$$

(equivalent rule D-False)

D-Plus: $$\frac{E \vdash e_1 \leadsto \langle e_1', \mathit{Dyn}\ \mathbf{int} \rangle \qquad E \vdash e_2 \leadsto \langle e_2', \mathit{Dyn}\ \mathbf{int} \rangle}{E \vdash e_1 + e_2 \leadsto \langle e_1' + e_2', \mathit{Dyn}\ \mathbf{int} \rangle}$$

(similar rule D-Leq)

D-If: $$\frac{E \vdash e_1 \leadsto \langle e_1', \mathit{Dyn}\ \mathbf{bool} \rangle \qquad E \vdash e_2 \leadsto \langle e_2', sv \rangle \qquad E \vdash e_3 \leadsto \langle e_3', sv \rangle}{E \vdash \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3 \leadsto \langle \mathbf{if}\ e_1'\ \mathbf{then}\ e_2'\ \mathbf{else}\ e_3', sv \rangle}$$

Defunctionalization rules

$$\boxed{E \vdash e \rightsquigarrow \langle e', sv \rangle}$$

$$\text{D-Lam: } \frac{}{E \vdash \lambda x\colon \tau.\ e_0 \rightsquigarrow \left\langle \{(Lab(y) = y)^{y \in \mathrm{dom}\, E}\}, Lam\ x\ e_0\ E \right\rangle}$$

$$\text{D-App: } \frac{\begin{array}{c} E \vdash e_1 \rightsquigarrow \langle e_1', Lam\ x\ e_0\ E_0 \rangle \\ E \vdash e_2 \rightsquigarrow \langle e_2', sv_2 \rangle \qquad E_0, x \mapsto sv_2 \vdash e_0 \rightsquigarrow \langle e_0', sv \rangle \end{array}}{E \vdash e_1\ e_2 \rightsquigarrow \langle e', sv \rangle}$$

$$\begin{array}{rl} \text{where } e' = & \textbf{let}\ env = e_1'\ \textbf{in}\ (\textbf{let}\ y = env.Lab(y)\ \textbf{in})^{y \in \mathrm{dom}\, E_0} \\ & \textbf{let}\ x = e_2'\ \textbf{in}\ e_0' \end{array}$$

$$\text{D-Let: } \frac{E \vdash e_1 \rightsquigarrow \langle e_1', sv_1 \rangle \qquad E, x \mapsto sv_1 \vdash e_2 \rightsquigarrow \langle e_2', sv \rangle}{E \vdash \textbf{let}\ x = e_1\ \textbf{in}\ e_2 \rightsquigarrow \langle \textbf{let}\ x = e_1'\ \textbf{in}\ e_2', sv \rangle}$$

Defunctionalization rules

$$\boxed{E \vdash e \rightsquigarrow \langle e', sv \rangle}$$

$$\text{D-Rcd: } \frac{\left(E \vdash e_i \rightsquigarrow \langle e'_i, sv_i \rangle\right)^{i \in 1..n}}{\begin{array}{r} E \vdash \{(\ell_i = e_i)^{i \in 1..n}\} \rightsquigarrow \langle \{(\ell_i = e'_i)^{i \in 1..n}\}, \\ Rcd\ \{(\ell_i \mapsto sv_i)^{i \in 1..n}\} \rangle \end{array}}$$

$$\text{D-Proj: } \frac{E \vdash e_0 \rightsquigarrow \left\langle e'_0, Rcd\ \{(\ell_i \mapsto sv_i)^{i \in 1..n}\}\right\rangle}{E \vdash e_0.\ell_k \rightsquigarrow \langle e'_0.\ell_k, sv_k \rangle}\ (1 \leq k \leq n)$$

$$\text{D-Array: } \frac{\left(E \vdash e_i \rightsquigarrow \langle e'_i, sv \rangle\right)^{i \in 1..n}}{E \vdash [e_1, \ldots, e_n] \rightsquigarrow \langle [e'_1, \ldots, e'_n], Arr\ sv \rangle}$$

Defunctionalization rules

$$\boxed{E \vdash e \rightsquigarrow \langle e', sv \rangle}$$

$$\text{D-Update: } \frac{\begin{array}{c} E \vdash e_0 \rightsquigarrow \langle e_0', \mathit{Arr}\ sv \rangle \\ E \vdash e_1 \rightsquigarrow \langle e_1', \mathit{Dyn}\ \mathbf{int} \rangle \qquad E \vdash e_2 \rightsquigarrow \langle e_2', sv \rangle \end{array}}{E \vdash e_0\ \mathbf{with}\ [e_1] \leftarrow e_2 \rightsquigarrow \langle e_0'\ \mathbf{with}\ [e_1'] \leftarrow e_2', \mathit{Arr}\ sv \rangle}$$

$$\text{D-Map: } \frac{E \vdash e_2 \rightsquigarrow \langle e_2', \mathit{Arr}\ sv_2 \rangle \qquad E, x \mapsto sv_2 \vdash e_1 \rightsquigarrow \langle e_1', sv_1 \rangle}{E \vdash \mathbf{map}\ (\lambda x.\ e_1)\ e_2 \rightsquigarrow \langle \mathbf{map}\ (\lambda x.\ e_1')\ e_2', \mathit{Arr}\ sv_1 \rangle}$$

$$\text{D-Loop: } \frac{\begin{array}{c} E \vdash e_0 \rightsquigarrow \langle e_0', sv \rangle \qquad E \vdash e_1 \rightsquigarrow \langle e_1', \mathit{Arr}\ sv_1 \rangle \\ E, x \mapsto sv, y \mapsto sv_1 \vdash e_2 \rightsquigarrow \langle e_2', sv \rangle \end{array}}{\begin{array}{l} E \vdash \mathbf{loop}\ x = e_0\ \mathbf{for}\ y\ \mathbf{in}\ e_1\ \mathbf{do}\ e_2 \\ \qquad \rightsquigarrow \langle \mathbf{loop}\ x = e_0'\ \mathbf{for}\ y\ \mathbf{in}\ e_1'\ \mathbf{do}\ e_2', sv \rangle \end{array}}$$