

The goal of this short note is to provide simple proofs for the “folklore facts” on the sample complexity of learning a discrete probability distribution over a known domain of size  $k$  to various distances  $\varepsilon$ , with error probability  $\delta$ . Thanks to [Gautam Kamath](#) and [John Wright](#) for suggesting “someone should write this up as a note,” and to [Jiantao Jiao](#) for discussions about the Hellinger case.

For a given distance measure  $d$ , we write  $\Phi(d, k, \varepsilon, \delta)$  for the sample complexity of learning discrete distributions over a known domain of size  $k$ , to accuracy  $\varepsilon > 0$ , with error probability  $\delta \in (0, 1]$ . As usual, asymptotics will be taken with regard to  $k$  going to infinity,  $\varepsilon$  going to 0, and  $\delta$  going to 0, in that order. Without loss of generality, we hereafter assume the domain is the set  $[k] \stackrel{\text{def}}{=} \{1, \dots, k\}$ .

## 1 Total variation distance

Recall that  $d_{\text{TV}}(p, q) = \sup_{S \subseteq [k]} (p(S) - q(S)) = \frac{1}{2} \|p - q\|_1 \in [0, 1]$  for any  $p, q \in \Delta([k])$ .

**Theorem 1.**  $\Phi(d_{\text{TV}}, k, \varepsilon, \delta) = \Theta\left(\frac{k + \log(1/\delta)}{\varepsilon^2}\right)$ .

We focus here on the upper bound. The lower bound can be proven, e.g., via Assouad’s lemma (for the  $k/\varepsilon^2$  term), and from the hardness of estimating the bias of a coin ( $k = 2$ ) with high probability (for the  $\log(1/\delta)/\varepsilon^2$  term).

*First proof.* Consider the empirical distribution  $\tilde{p}$  obtained by drawing  $n$  independent samples  $s_1, \dots, s_n$  from the underlying distribution  $p \in \Delta([k])$ :

$$\tilde{p}(i) = \frac{1}{n} \sum_{j=1}^n \mathbb{1}_{\{s_j=i\}}, \quad i \in [k] \quad (1)$$

- First, we bound the *expected* total variation distance between  $\tilde{p}$  and  $p$ , by using  $\ell_2$  distance as a proxy:

$$\mathbb{E}[d_{\text{TV}}(p, \tilde{p})] = \frac{1}{2} \mathbb{E}[\|p - \tilde{p}\|_1] = \frac{1}{2} \sum_{i=1}^k \mathbb{E}[|p(i) - \tilde{p}(i)|] \leq \frac{1}{2} \sum_{i=1}^k \sqrt{\mathbb{E}[(p(i) - \tilde{p}(i))^2]}$$

the last inequality by Jensen. But since, for every  $i \in [k]$ ,  $n\tilde{p}(i)$  follows a  $\text{Bin}(n, p(i))$  distribution, we have  $\mathbb{E}[(p(i) - \tilde{p}(i))^2] = \frac{1}{n^2} \text{Var}[n\tilde{p}(i)] = \frac{1}{n} p(i)(1 - p(i))$ , from which

$$\mathbb{E}[d_{\text{TV}}(p, \tilde{p})] \leq \frac{1}{2\sqrt{n}} \sum_{i=1}^k \sqrt{p(i)} \leq \frac{1}{2} \sqrt{\frac{k}{n}}$$

the last inequality this time by Cauchy–Schwarz. Therefore, for  $n \geq \frac{k}{\varepsilon^2}$  we have  $\mathbb{E}[d_{\text{TV}}(p, \tilde{p})] \leq \frac{\varepsilon}{2}$ .

- Next, to convert this expected result to a *high probability* guarantee, we apply McDiarmid’s inequality to the random variable  $f(s_1, \dots, s_n) \stackrel{\text{def}}{=} d_{\text{TV}}(p, \tilde{p})$ , noting that changing any single sample cannot change its value by more than  $c \stackrel{\text{def}}{=} 1/n$ :

$$\Pr\left[|f(s_1, \dots, s_n) - \mathbb{E}[f(s_1, \dots, s_n)]| \geq \frac{\varepsilon}{2}\right] \leq 2e^{-\frac{2\left(\frac{\varepsilon}{2}\right)^2}{nc^2}} = 2e^{-\frac{1}{2}n\varepsilon^2}$$

and therefore as long as  $n \geq \frac{2}{\varepsilon^2} \ln \frac{2}{\delta}$ , we have  $|f(s_1, \dots, s_n) - \mathbb{E}[f(s_1, \dots, s_n)]| \leq \frac{\varepsilon}{2}$  with probability at least  $1 - \delta$ .

Putting it all together, we obtain that  $d_{\text{TV}}(p, \tilde{p}) \leq \varepsilon$  with probability at least  $1 - \delta$ , as long as  $n \geq \max\left(\frac{k}{\varepsilon^2}, \frac{2}{\varepsilon^2} \ln \frac{2}{\delta}\right)$ .  $\square$

*Second proof – the “fun” one.* Again, we will analyze the behavior of the empirical distribution  $\tilde{p}$  over  $n$  i.i.d. samples from the unknown  $p$  (cf. (1)) – because it is simple, efficiently computable, and *it works*. Recalling the definition of total variation distance, note that  $d_{\text{TV}}(p, \tilde{p}) > \varepsilon$  literally means there exists a subset  $S \subseteq [k]$  such that  $\tilde{p}(S) > p(S) + \varepsilon$ . There are  $2^k$  such subsets, so... let us do a union bound.

Fix any  $S \subseteq [k]$ . We have

$$\tilde{p}(S) = \tilde{p}(i) \stackrel{(1)}{=} \frac{1}{n} \sum_{i \in S} \sum_{j=1}^n \mathbb{1}_{\{s_j=i\}}$$

and so, letting  $X_j \stackrel{\text{def}}{=} \sum_{i \in S} \mathbb{1}_{\{s_j=i\}}$  for  $j \in [n]$ , we have  $\tilde{p}(S) = \frac{1}{n} \sum_{j=1}^n X_j$  where the  $X_j$ 's are i.i.d. Bernoulli random variable with parameter  $p(S)$ . Here comes the Chernoff bound (actually, Hoeffding, the *other* Chernoff):

$$\Pr[\tilde{p}(S) > p(S) + \varepsilon] = \Pr\left[\frac{1}{n} \sum_{j=1}^n X_j > \mathbb{E}\left[\frac{1}{n} \sum_{j=1}^n X_j\right] + \varepsilon\right] \leq e^{-2\varepsilon^2 n}$$

and therefore  $\Pr[\tilde{p}(S) > p(S) + \varepsilon] \leq \frac{\delta}{2^k}$  for any  $n \geq \frac{k \ln 2 + \log(1/\delta)}{2\varepsilon^2}$ . A union bound over these  $2^k$  possible sets  $S$  concludes the proof:

$$\Pr[\exists S \subseteq [k] \text{ s.t. } \tilde{p}(S) > p(S) + \varepsilon] \leq 2^k \cdot \frac{\delta}{2^k} = \delta$$

and we are done. *Badda bing badda boom*, as someone<sup>1</sup> would say.  $\square$

## 2 Hellinger distance

Recall that  $d_{\text{H}}(p, q) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^k (\sqrt{p(i)} - \sqrt{q(i)})^2} = \frac{1}{\sqrt{2}} \|p - q\|_2 \in [0, 1]$  for any  $p, q \in \Delta([k])$ . The Hellinger distance has many nice properties: it is well-suited to manipulating product distributions, its square is subadditive, and is always within a quadratic factor of the total variation distance; see, e.g., [Can15, Appendix C.2].

**Theorem 2.**  $\Phi(d_{\text{H}}, k, \varepsilon, \delta) = \Theta\left(\frac{k + \log(1/\delta)}{\varepsilon^2}\right)$ .

This theorem is “highly non-trivial” to establish, however; for the sake of exposition, we will show increasingly stronger bounds, starting with the easiest to establish.

**Proposition 3** (Easy bound).  $\Phi(d_{\text{H}}, k, \varepsilon, \delta) = O\left(\frac{k + \log(1/\delta)}{\varepsilon^4}\right)$ , and  $\Phi(d_{\text{H}}, k, \varepsilon, \delta) = \Omega\left(\frac{k + \log(1/\delta)}{\varepsilon^2}\right)$ .

*Proof.* This is immediate from Theorem 9, recalling that  $\frac{1}{2} d_{\text{TV}}^2 \leq d_{\text{H}}^2 \leq d_{\text{TV}}$ .  $\square$

**Proposition 4** (More involved bound).  $\Phi(d_{\text{H}}, k, \varepsilon, \delta) = O\left(\frac{k}{\varepsilon^2} + \frac{\log(1/\delta)}{\varepsilon^4}\right)$ .

*Proof.* As for total variation distance, we consider the empirical distribution  $\hat{p}$  (cf. (1)) obtained by drawing  $n$  independent samples  $s_1, \dots, s_n$  from  $p \in \Delta([k])$ .

- First, we bound the *expected* squared Hellinger distance between  $\hat{p}$  and  $p$ : using the simple fact that  $d_{\text{H}}(p, q)^2 = 1 - \sum_{i=1}^k \sqrt{p(i)q(i)}$  for any  $p, q \in \Delta([k])$ ,

$$\mathbb{E}\left[d_{\text{H}}(p, \hat{p})^2\right] = 1 - \sum_{i=1}^k \sqrt{p(i)} \cdot \mathbb{E}\left[\sqrt{\hat{p}(i)}\right].$$

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<sup>1</sup>John Wright.

Now we would like to handle the square root inside the expectation, and *of course* Jensen's inequality is in the wrong direction. However, for every nonnegative r.v.  $X$  with positive expectation, letting  $Y \stackrel{\text{def}}{=} X/\mathbb{E}[X]$ , we have that

$$\begin{aligned}\mathbb{E}[\sqrt{X}] &= \sqrt{\mathbb{E}[X]} \cdot \mathbb{E}[\sqrt{Y}] = \sqrt{\mathbb{E}[X]} \cdot \mathbb{E}[\sqrt{1 + (Y - \mathbb{E}[Y])}] \\ &\geq \sqrt{\mathbb{E}[X]} \left(1 + \frac{1}{2}\mathbb{E}[Y - \mathbb{E}[Y]] - \frac{1}{2}\mathbb{E}[(Y - \mathbb{E}[Y])^2]\right) = \sqrt{\mathbb{E}[X]} \left(1 - \frac{\text{Var } X}{2\mathbb{E}[X]^2}\right)\end{aligned}$$

where we used the inequality  $\sqrt{1+x} \geq 1 + \frac{x}{2} - \frac{x^2}{2}$ , which holds for  $x \geq -1$ .<sup>2</sup> Since, for every  $i \in [k]$ ,  $n\hat{p}(i)$  follows a  $\text{Bin}(n, p(i))$  distribution, we get

$$\mathbb{E}[\text{d}_H(p, \hat{p})^2] \leq 1 - \frac{1}{\sqrt{n}} \sum_{i=1}^k \sqrt{p(i)} \cdot \sqrt{np(i)} \left(1 - \frac{np(i)(1 - np(i))}{2n^2 p(i)^2}\right) \leq 1 - \sum_{i=1}^k p(i) \left(1 - \frac{1}{2np(i)}\right) = \frac{k}{2n}.$$

Therefore, for  $n \geq \frac{k}{\varepsilon^2}$ , we have  $\mathbb{E}[\text{d}_H(p, \hat{p})^2] \leq \frac{\varepsilon^2}{2}$ .

- Next, to convert this expected result to a high probability guarantee, we *would like* to apply McDiarmid's inequality to the random variable  $f(s_1, \dots, s_n) \stackrel{\text{def}}{=} \text{d}_H(p, \hat{p})^2$  as in the (first) proof of [Theorem 9](#); unfortunately, changing a sample can change the value by up to  $c \approx 1/\sqrt{n}$ , and McDiarmid will yield only a vacuous bound.<sup>3</sup> Instead, we will use a stronger, more involved concentration inequality:

**Theorem 5** ([BLM13, Theorem 8.6]). *Let  $f: \mathcal{X}^n \rightarrow \mathbb{R}$  be a measurable function, and let  $X_1, \dots, X_n$  be independent random variables taking values in  $\mathcal{X}$ . Define  $Z \stackrel{\text{def}}{=} f(X_1, \dots, X_n)$ . Assume that there exist measurable functions  $c_i: \mathcal{X}^n \rightarrow [0, \infty)$  such that, for all  $x, y \in \mathcal{X}^n$ ,*

$$f(y) - f(x) \leq \sum_{i=1}^n c_i(x) \mathbb{1}_{\{x_i \neq y_i\}}.$$

Then, setting  $v \stackrel{\text{def}}{=} \mathbb{E} \sum_{i=1}^n c_i(x)^2$  and  $v_\infty \stackrel{\text{def}}{=} \sup_{x \in \mathcal{X}^n} \sum_{i=1}^n c_i(x)^2$ , we have, for all  $t > 0$ ,

$$\Pr[Z \geq \mathbb{E}[Z] + t] \leq e^{-\frac{t^2}{2v}} \quad \Pr[Z \leq \mathbb{E}[Z] - t] \leq e^{-\frac{t^2}{2v_\infty}}.$$

For our  $f$  above, we have, for two any different  $x, y \in [k]^n$ , that

$$\begin{aligned}f(y) - f(x) &= \frac{1}{\sqrt{n}} \sum_{i=1}^k \sqrt{p(i)} \left( \sqrt{\sum_{j=1}^n \mathbb{1}_{\{x_j=i\}}} - \sqrt{\sum_{j=1}^n \mathbb{1}_{\{y_j=i\}}} \right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^k \sqrt{p(i)} \frac{\sum_{j=1}^n (\mathbb{1}_{\{x_j=i\}} - \mathbb{1}_{\{y_j=i\}})}{\sqrt{\sum_{j=1}^n \mathbb{1}_{\{x_j=i\}}} + \sqrt{\sum_{j=1}^n \mathbb{1}_{\{y_j=i\}}}} \\ &\leq \frac{1}{\sqrt{n}} \sum_{i=1}^k \sqrt{p(i)} \frac{\sum_{j=1}^n \mathbb{1}_{\{x_j=i\}} \mathbb{1}_{\{y_j \neq x_j\}}}{\sqrt{\sum_{j=1}^n \mathbb{1}_{\{x_j=i\}}}} = \sum_{j=1}^n \underbrace{\sqrt{\frac{p_{x_j}}{n \sum_{\ell=1}^n \mathbb{1}_{\{x_\ell=x_j\}}}}}_{c_j(x)} \cdot \mathbb{1}_{\{x_j \neq y_j\}}.\end{aligned}$$

In view of [Theorem 5](#), we then must evaluate

$$v \stackrel{\text{def}}{=} \sum_{j=1}^n \mathbb{E}[c_j(X)^2] = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^k p(i)^2 \cdot \mathbb{E} \left[ \frac{1}{1 + \sum_{\ell \neq j} \mathbb{1}_{\{X_\ell=i\}}} \right]$$

<sup>2</sup>And is inspired by the Taylor expansion  $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)$ : there is *some* intuition for it.

<sup>3</sup>Try it: it's a real bummer.

where that last expectation is over  $(x_\ell)_{\ell \neq j}$  drawn from  $p^{\otimes(n-1)}$ . Since  $\sum_{\ell \neq j} \mathbb{1}_{\{X_\ell = i\}}$  is Binomially distributed with parameters  $n-1$  and  $p(i)$ , we can use the simple fact that, for  $N \sim \text{Bin}(r, \rho)$ ,

$$\mathbb{E}\left[\frac{1}{N+1}\right] = \frac{1 - (1-\rho)^{r+1}}{\rho(r+1)} \leq \frac{1}{\rho(r+1)}$$

to conclude that  $v \leq \frac{1}{n^2} \sum_{j=1}^n \sum_{i=1}^k p(i) = \frac{1}{n}$ . By [Theorem 5](#), we obtain

$$\Pr\left[|f(s_1, \dots, s_n) - \mathbb{E}[f(s_1, \dots, s_n)]| \geq \frac{\varepsilon^2}{2}\right] \leq e^{-\frac{1}{8}n\varepsilon^4}$$

and therefore, as long as  $n \geq \frac{8}{\varepsilon^4} \ln \frac{1}{\delta}$ , we have  $|f(s_1, \dots, s_n) - \mathbb{E}[f(s_1, \dots, s_n)]| \leq \frac{\varepsilon^2}{2}$  with probability at least  $1 - \delta$ .

Putting it all together, we obtain that  $d_H(p, \hat{p})^2 \leq \varepsilon^2$  with probability at least  $1 - \delta$ , as long as  $n \geq \max\left(\frac{k}{\varepsilon^2}, \frac{8}{\varepsilon^4} \ln \frac{1}{\delta}\right)$ .  $\square$

We finally get to the final, optimal bound:

*Proof of [Theorem 2](#).* We will rely on a recent – and quite involved – result due to Agrawal [Agr19], analyzing the concentration of the empirical distribution  $\hat{p}$  in terms of its Kullback–Leibler (KL) divergence with regard to the true  $p$ ,

$$\text{KL}(\hat{p} \| p) = \sum_{i=1}^k \hat{p}(i) \ln \frac{\hat{p}(i)}{p(i)} \in [0, \infty].$$

Observing that  $d_H(p, q)^2 \leq \frac{1}{2} \text{KL}(p \| q)$  for any distributions  $p, q$ , the aforementioned result is actually stronger than what we need:

**Theorem 6** ([Agr19, Theorem 1.2]). *Suppose  $n \geq \frac{k-1}{\alpha}$ . Then*

$$\Pr[\text{KL}(\hat{p} \| p) \geq \alpha] \leq e^{-n\alpha} \left(\frac{e\alpha n}{k-1}\right)^{k-1}.$$

In view of the above relation between Hellinger and KL, we will apply this convergence result with  $\alpha \stackrel{\text{def}}{=} 2\varepsilon^2$ , obtaining

$$\Pr[d_H(\hat{p}, p) \geq \varepsilon] \leq e^{-2n\varepsilon^2 + (k-1) \ln \frac{2en\varepsilon^2}{k-1}}.$$

**Fact 7.** *For  $n \geq \frac{15}{2e} \frac{k}{\varepsilon^2}$ , we have  $(k-1) \ln \frac{2en\varepsilon^2}{k-1} \leq n\varepsilon^2$ .*

*Proof.* The conclusion is equivalent to  $2e \cdot \ln \frac{2en\varepsilon^2}{k-1} \leq \frac{2en\varepsilon^2}{k-1}$ , and thus follows from the fact that  $x \geq 2e \ln x$  for  $x \geq 15$ .  $\square$

This fact implies that, for  $n \geq \frac{15k}{2\varepsilon^2}$ ,  $\Pr[d_H(\hat{p}, p) \geq \varepsilon] \leq e^{-n\varepsilon^2}$ . Overall, we obtain that  $d_H(p, \hat{p}) \leq \varepsilon$  with probability at least  $1 - \delta$  as long as  $n \geq \max\left(\frac{15k}{2\varepsilon^2}, \frac{1}{\varepsilon^2} \ln \frac{1}{\delta}\right)$ , as desired.  $\square$

### 3 $\chi^2$ and Kullback—Leibler divergences

In view of the previous section, some remarks on Kullback–Leibler (KL) and chi-squared ( $\chi^2$ ) divergences. Recall their definition, for  $p, q \in \Delta([k])$ ,

$$\text{KL}(p \parallel q) = \sum_{i=1}^k p(i) \ln \frac{p(i)}{q(i)}, \quad \chi^2(p \parallel q) = \sum_{i=1}^k \frac{(p(i) - q(i))^2}{q(i)}$$

both taking values in  $[0, \infty]$ ; as well as the chain of (easily checked) inequalities

$$2d_{\text{TV}}(p, q)^2 \leq \text{KL}(p \parallel q) \leq \chi^2(p \parallel q),$$

where the first one is Pinsker’s. Importantly, KL and  $\chi^2$  divergences are unbounded and asymmetric, so the order of  $p$  and  $q$  matters *a lot*: for instance, it is easy to show that, without strong assumptions on the unknown distribution  $p \in \Delta([k])$ , the empirical estimator  $\hat{p}$  cannot achieve  $\text{KL}(p \parallel \hat{p}) < \infty$  (resp.,  $\chi^2(p \parallel \hat{p}) < \infty$ ) with any finite number of samples.<sup>4</sup> So, that’s uplifting. (On the other hand, *other* estimators than the empirical one, e.g., add-constant estimators, do provide good learning guarantees for those distance measures: see for instance [KOPS15]).

We are going to focus here on getting  $\text{KL}(\hat{p} \parallel p)$  and  $\chi^2(\hat{p} \parallel p)$  down to  $\varepsilon$ . Of course, in view of the inequalities above, the latter is at least as hard as the former, and a lower bound on both follows from that on  $d_{\text{TV}}$ :  $\Omega((k + \log(1/\delta))/\varepsilon^2)$ . And, behold! The result of Agrawal [Agr19] used in the proof of [Theorem 2](#) does provide the optimal upper bound on learning in KL divergence – and it is achieved by the usual suspect, the empirical estimator:

**Theorem 8.**  $\Phi(\text{KL}, k, \varepsilon, \delta) = \Theta\left(\frac{k + \log(1/\delta)}{\varepsilon}\right)$ , where by KL we refer to minimizing  $\text{KL}(\hat{p} \parallel p)$ .

The optimal sample complexity of learning in  $\chi^2$  as a function of  $k, \varepsilon, \delta$ , however, remains open.

### 4 Briefly: Kolmogorov, $\ell_\infty$ , and $\ell_2$ distances

To conclude, let us briefly discuss three other distance measures: Kolmogorov (a.k.a., “ $\ell_\infty$  between cumulative distribution functions”),  $\ell_\infty$ , and  $\ell_2$ :

$$d_K(p, q) = \max_{i \in [k]} \left| \sum_{j=1}^i p(j) - \sum_{j=1}^i q(j) \right|$$

and

$$\ell_2(p, q) = \|p - q\|_2 = \sqrt{\sum_{i=1}^k (p(i) - q(i))^2}, \quad \ell_\infty(p, q) = \|p - q\|_\infty = \max_{i \in [k]} |p(i) - q(i)|.$$

A few remarks first. The Kolmogorov distance is actually defined for any distribution on  $\mathbb{R}$ , not necessarily discrete; one can equivalently define it as  $d_K(p, q) = \sup_i (\mathbb{E}_p[\mathbb{1}_{(-\infty, i]}] - \mathbb{E}_q[\mathbb{1}_{(-\infty, i]}])$ . This has a nice interpretation: recalling the definition of TV distance, both are of the form  $\sup_{f \in \mathcal{C}} (\mathbb{E}_p[f] - \mathbb{E}_q[f])$  where  $\mathcal{C}$  is a class of measurable functions.<sup>5</sup> For TV distance,  $\mathcal{C}$  is the class of indicators of all measurable subsets; for Kolmogorov, this is the (smaller) class of indicators of intervals of the form  $(-\infty, a]$ . (For Wasserstein/EMD distance, this will be the class of continuous, 1-Lipschitz functions.)

<sup>4</sup>You can verify this: intuitively, the issue boils down to having to non-trivially learn even the elements of the support of  $p$  that have arbitrarily small probability.

<sup>5</sup>Such metrics on the space of probability distributions are called *integral probability metrics*.

Second, because of the above, and also monotonicity of  $\ell_p$  norms, Cauchy–Schwarz, the fact that  $\ell_1(p, q) = 2d_{\text{TV}}(p, q)$ , and elementary manipulations, we have

$$\ell_\infty(p, q) \leq \ell_2(p, q) \leq 2d_{\text{TV}}(p, q) \leq \sqrt{k}\ell_2(p, q), \quad \ell_2(p, q) \leq \sqrt{\ell_\infty(p, q)}, \quad \frac{1}{2}\ell_\infty(p, q) \leq d_K(p, q) \leq d_{\text{TV}}(p, q).$$

That can be useful sometimes. Now, I will only briefly sketch the proof of the next theorem: the lower bounds follow from the simple case  $k = 2$  (estimating the bias of a biased coin), the upper bounds are achieved by the empirical estimator (again). Importantly, the result for Kolmogorov distance *still applies to continuous, arbitrary distributions*.

**Theorem 9.**  $\Phi(d_K, k, \varepsilon, \delta), \Phi(\ell_\infty, k, \varepsilon, \delta), \Phi(\ell_2, k, \varepsilon, \delta) = \Theta\left(\frac{\log(1/\delta)}{\varepsilon^2}\right)$ , independent of  $k$ .

*Sketch.* The proof for Kolmogorov distance is the most involved, and follows from a *very* useful and non-elementary theorem due to Dvoretzky, Kiefer, and Wolfowitz from 1956 [DKW56] (with the optimal constant due to Massart, in 1990 [Mas90]):

**Theorem 10** (DKW Inequality). *Let  $\hat{p}$  denote the empirical distribution on  $n$  i.i.d. samples from  $p$  (an arbitrary distribution on  $\mathbb{R}$ ). Then, for every  $\varepsilon > 0$ ,*

$$\Pr[d_K(\hat{p}, p) > \varepsilon] \leq 2e^{-2n\varepsilon^2}.$$

*Note, again, that this holds even if  $p$  is a continuous (or arbitrary) distribution on an unbounded support.*

The proof for  $\ell_\infty$  just follows the Kolmogorov upper bound and the aforementioned inequality  $\ell_\infty(p, q) \leq 2d_K(p, q)$  (which hinges on the fact that  $p(i) = \sum_{j=1}^i p(j) - \sum_{j=1}^{i-1} p(j)$  and the triangle inequality). Finally, the proof for  $\ell_2$  is a nice exercise involving analyzing the expectation of the  $\ell_2^2$  distance achieved by the empirical estimator, and McDiarmid’s inequality.  $\square$

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