A Turing-Complete Resistance-Based Isomorphism for Probabilistic Computation

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Abstract

This paper establishes a rigorous isomorphism between number-theoretic structures and informational resistance. The fundamental objects are **inverse integers** ($1/n$ where $n \in \mathbb{Z}^+$), which inhabit the interval (0,1]. The mapping $\Omega(1/n) = \ln(n)$ transforms multiplication into addition, establishing a monoid isomorphism from $(\{1/n : n \in \mathbb{Z}^+\}, \times)$ to $(\{\ln(n) : n \in \mathbb{Z}^+\}, +)$. This mapping is proven unique up to multiplicative constant (Theorem 2.4).

The correspondence extends across four equivalent representations: natural language, inverse integers, circuit topology, and prime coordinate vectors. For set-theoretic operations on squarefree encodings: **Intersection** corresponds to GCD with resistance Ω_{gcd} (Theorem 9.1); **Union** corresponds to LCM with resistance $\Omega_a + \Omega_b - \Omega_{gcd}$ (Theorem 4.2); **Complement** requires extension to positive rationals via phase interference.

The union formula $\Omega_{OR} = \Omega_a + \Omega_b - \Omega_{gcd}$ is derived from the fundamental identity $\mathrm{lcm}(a,b) \times \gcd(a,b) = ab$.

We validate the framework by deriving the coprimality probability $6/\pi^2$ through all four layers (Theorem 8.1). We demonstrate that relational database operations map exactly to number-theoretic operations, with both domains forming instances of a distributive lattice. We establish Turing completeness by explicit counter machine simulation using Gödel numbering with prime exponents as registers (Theorem 10.4).

Prime numbers are characterized as the irreducible elements—integers $n > 1$ whose resistance $\ln(n)$ cannot be expressed as a sum of smaller resistances (Theorem 11.2).

PART I: FOUNDATIONS

1. Introduction

1.1 The Central Insight

The **inverse integers** are numbers of the form 1/n where n is a positive integer. These objects inhabit the interval (0,1].

The structure of multiplication maps onto addition under the logarithm:

$$\Omega(P_1 \times P_2) = \Omega(P_1) + \Omega(P_2)$$

where $\Omega(1/n) = \ln(n)$. This is an exact algebraic isomorphism.

1.2 Informational Resistance: The Conceptual Framework

The term **resistance** is used in an informational sense throughout this paper. The quantity $\Omega(1/n) = \ln(n)$ represents the **information cost** or **surprisal** of the inverse integer 1/n—the same quantity that appears in information theory as self-information measured in nats.

The framework employs circuit topology (series combination, phase interference) as the algebraic structure governing how these informational costs combine. This should be understood as operating on **informational quantities** rather than physical electrical circuits.

1.3 The Four-Layer Framework

Every computation can be expressed equivalently in four representations:

1. **English**: Natural language description
2. **Inverse Integer**: Values in (0,1] with arithmetic operations
3. **Circuit**: Informational resistance topology with phase
4. **Vector Space**: Prime coordinate vectors

A proof in one layer translates to proofs in the others.

1.4 Domain Specification

Fundamental objects: Inverse integers $I = \{1/n : n \in \mathbb{Z}^+\}$ form a discrete subset of (0,1].

Extended domain: For the NOT operation (complement), the framework extends naturally to all positive rationals in (0,1). The value 1/1 = 1 is included; the limit 0 is excluded.

Boundary behavior: As $P \to 0$, $\Omega \to \infty$. Classical Boolean logic emerges as limiting behavior at the boundaries {0, 1}.

2. The Fundamental Mapping

2.1 Inverse Integers

Definition 2.1. An **inverse integer** is a number of the form 1/n where $n \in \mathbb{Z}^+$. The set of inverse integers is denoted $I = \{1/n : n \in \mathbb{Z}^+\} \subset (0, 1]$.

Every inverse integer inherits unique prime factorization from its denominator:

$$1/n = 1/(p_1\text{^}a_1 \times p_2\text{^}a_2 \times \cdots \times p_k\text{^}a_k)$$

The **inverse primes** {1/2, 1/3, 1/5, 1/7, ...} are those inverse integers whose denominators are prime.

2.2 Informational Resistance

Definition 2.2 (Informational Resistance). For an inverse integer 1/n, the informational resistance is:

$$\Omega(1/n) := \ln(n)$$

Notation convention: We write Ω_a to denote $\Omega(1/a) = \ln(a)$.

Remark. The quantity $\ln(n) = -\ln(1/n)$ is the self-information of an event with probability 1/n, measured in nats. This connection to information theory motivates the "resistance" terminology.

2.3 The Isomorphism Theorem

Theorem 2.3 (Inverse Integer–Resistance Isomorphism). The map Ω: I → $\ln(\mathbb{Z}^+)$ defined by $\Omega(1/n) = \ln(n)$ is a monoid isomorphism from (I, ×) to ($\ln(\mathbb{Z}^+)$, +). That is:

$$\Omega(1/a \times 1/b) = \Omega(1/a) + \Omega(1/b)$$

Proof. $\Omega(1/ab) = \ln(ab) = \ln(a) + \ln(b) = \Omega(1/a) + \Omega(1/b)$. The map is bijective: for each $\ln(n) \in \ln(\mathbb{Z}^+)$, the unique preimage is 1/n. The identity 1/1 maps to 0. □

2.4 Uniqueness

Theorem 2.4 (Uniqueness). Any continuous function $f: (0, 1] \to \mathbb{R}$ satisfying $f(xy) = f(x) + f(y)$ has the form $f(x) = c\cdot\ln(x)$ for some constant c.

Proof. This is Cauchy's logarithmic functional equation. For continuous solutions, $f(x) = c\cdot\ln(x)$ is the unique form. See Aczél (1966), Theorem 1.4.1. □

Corollary 2.5. The resistance mapping $\Omega(P) = -\ln(P)$ is forced (up to scale) by requiring series combination to be additive.

2.5 Series Combination

When two inverse integers combine in series, their resistances add:

$$(1/a) \times (1/b) = 1/ab \quad \text{and} \quad \Omega\_ab = \Omega\_a + \Omega\_b$$

PART II: BOOLEAN LOGIC

3. Series Combination (Multiplication)

3.1 The Series Correspondence

Theorem 3.1 (Series Correspondence). For inverse integers 1/a and 1/b:

$$(1/a) \times (1/b) = 1/ab \quad \text{and} \quad \Omega\_ab = \Omega\_a + \Omega\_b$$

Proof. Direct application of Theorem 2.3. □

3.2 Worked Example

Problem: Probability of drawing a Red Ace from a standard deck.

Since being Red and being an Ace are independent events: P(Ace) = 1/13, P(Red) = 1/2, so P(Red ∧ Ace) = 1/13 × 1/2 = 1/26.

Resistance: Ω = ln(13) + ln(2) = ln(26) ≈ 3.258. Verification: e^(−3.258) ≈ 1/26 ✓

4. Set-Theoretic Union: The LCM Correspondence

4.1 Set Encoding

Definition 4.1. A **squarefree integer** is a positive integer with no repeated prime factors. Sets are encoded as squarefree integers: each element maps to a unique prime; presence in the set means the prime divides the encoding.

4.2 The Union-LCM Formula

Theorem 4.2 (Union Formula). For squarefree integers a, b:

$$\Omega\_lcm = \Omega\_a + \Omega\_b - \Omega\_gcd$$

That is: ln(lcm(a,b)) = ln(a) + ln(b) − ln(gcd(a,b))

Proof. The fundamental identity for positive integers states: lcm(a,b) × gcd(a,b) = a × b. Taking logarithms: ln(lcm) + ln(gcd) = ln(a) + ln(b), therefore ln(lcm) = ln(a) + ln(b) − ln(gcd). □

5. Complement: Phase Interference

5.1 Extension to Positive Rationals

The NOT operation for 1/a yields:

$$NOT(1/a) = 1 - 1/a = (a-1)/a$$

Definition 5.1. For P ∈ (0,1), define NOT(P) = 1 − P.

5.2 Complex Amplitude Representation

Theorem 5.2. The complement NOT(P) = 1 − P can be represented as the interference between a unit reference signal and a phase-shifted input:

$$|e\hat{}(i\theta) - P\cdot e\hat{}(i\theta)| = |1-P|\cdot|e\hat{}(i\theta)| = 1-P \quad for\ P \in (0,1)$$

6. Complete Boolean Logic

6.1 Summary of Operations

Context	AND	OR	NOT
Sets (squarefree)	gcd(a,b)	lcm(a,b)	$1 - 1/a \in \mathbb{Q}$
Resistance	Ω_gcd	Ω_a + Ω_b − Ω_gcd	Phase interference
Probabilities (indep.)	1/a × 1/b	1/a + 1/b − 1/ab	1 − 1/a

6.2 Functional Completeness

Theorem 6.1. {AND, OR, NOT} forms a functionally complete set of Boolean operators.

PART III: THE VECTOR SPACE LAYER

7. Prime Coordinates

7.1 Coordinate Representation

By the Fundamental Theorem of Arithmetic, each n > 1 has unique representation:

$$n = 2^{a_2} \times 3^{a_3} \times 5^{a_5} \times \cdots$$

Definition 7.1. The **prime coordinate vector** of n is: $v(n) = (a_2, a_3, a_5, a_7, \ldots)$ where a_p is the exponent of prime p in n. Define v(1) = (0, 0, 0, ...).

7.2 Linear Independence

Theorem 7.2. The set {ln(p) : p prime} is linearly independent over $\mathbb{Q}$.

Corollary 7.2.1. Prime coordinates form a vector space over $\mathbb{Q}$ with basis {ln(p) : p prime}.

7.3 Operations in Vector Space

Arithmetic	Vector Operation
a × b	v(a) + v(b)
gcd(a, b)	min(v(a), v(b)) componentwise
lcm(a, b)	max(v(a), v(b)) componentwise

PART IV: PROOF OF CONCEPT

8. The Coprimality Theorem

Theorem 8.1. Using natural density:

$$P(\gcd(a,b) = 1) = \lim_{N\to\infty} |\{(a,b) : 1 \le a,b \le N, \gcd(a,b) = 1\}| / N^2 = 6/\pi^2$$

8.1 Layer 1: English

"For every prime p, it is not the case that p divides both a and b."

8.2 Layer 2: Inverse Integer

For each prime p: $P(p \mid a) = 1/p$, $P(p \mid a \text{ and } p \mid b) = 1/p^2$, $P(p \nmid \text{both}) = 1 - 1/p^2$. For coprimality:

$$P(\gcd(a,b) = 1) = \prod_p (1 - 1/p^2)$$

8.3 Layer 3: Circuit

Each prime contributes resistance $\Omega_p = -\ln(1 - 1/p^2)$ in series:

$$\Omega_{total} = \Sigma_p -\ln(1 - 1/p^2)$$

8.4 Layer 4: Vector

Coprime condition: $\min(v(a), v(b)) = \mathbf{0}$ (zero vector).

8.5 Connection to Zeta Function

By the Euler product: $\zeta(s) = \prod_p 1/(1 - p^{-s})$

At s = 2: $\zeta(2) = \prod_p 1/(1 - p^{-2})$. Therefore: $\prod_p (1 - 1/p^2) = 1/\zeta(2) = 6/\pi^2$ (Euler, 1734). □

PART V: DATABASE OPERATIONS

9. Relational Operations as Number Theory

9.1 Set Encoding

Definition 9.1. Sets are encoded as **squarefree** positive integers. Each element maps to a unique prime; the set is represented by the product of its element-primes.

9.2 Fundamental Correspondences

Theorem 9.1 (Intersection = GCD). For squarefree integers a, b encoding sets A, B: $A \cap B$ is encoded by gcd(a, b).

Theorem 9.2 (Union = LCM). For squarefree integers a, b encoding sets A, B: $A \cup B$ is encoded by lcm(a, b).

9.3 Complete Relational Algebra

SQL	Set	Encoding	Vector
INTERSECT	$A \cap B$	gcd(a,b)	min(v_a, v_b)
UNION	$A \cup B$	lcm(a,b)	max(v_a, v_b)
EXCEPT	$A - B$	a/gcd(a,b)	v_a − min

9.4 Binary-Squarefree Bijection

Theorem 9.3. There is a bijection between finite binary strings and squarefree positive integers.

PART VI: COMPUTATIONAL COMPLETENESS

10. Turing Completeness

10.1 Counter Machine Model

A two-counter machine has: Two counters $C_1, C_2 \in \mathbb{Z}{\geq}0$; Operations: INCREMENT($C_i$), DECREMENT($C_i$), JUMP-IF-ZERO($C_i$, L), GOTO(L); A program counter.

Theorem 10.1 (Minsky, 1967). Two-counter machines are Turing complete.

10.2 State Encoding via Gödel Numbering

Definition 10.2. The machine state is encoded as a positive integer:

$$State = 2^{C_1} \times 3^{C_2} \times p\_k$$

where C_1, C_2 are counter values (exponents of primes 2 and 3), and p_k is the k-th prime for instruction k (program counter).

10.3 Operation Implementation

INCREMENT(C_1): Multiply State by 2. Resistance: add ln(2). Vector: v → v + (1, 0, 0, ...).

DECREMENT(C_1): Divide State by 2 (valid iff $C_1 > 0$). Resistance: subtract ln(2). Vector: v → v − (1, 0, 0, ...).

JUMP-IF-ZERO(C_1, L): Test whether $C_1 = 0$ via divisibility gate D_2.

GOTO(L): Replace p_k with p_L: State_new = State × p_L / p_k.

10.4 The Divisibility Gate as Transistor

The divisibility gate functions as a **transistor** in the informational circuit. Like a transistor with base/gate control, the divisibility gate has one input and two outputs: PASS (finite Ω, signal continues) or BLOCK (Ω → ∞, signal redirected).

10.5 Worked Example: Computing 2 + 3

Algorithm: Decrement C_2 and increment C_1 until $C_2 = 0$.

Initial: $C_1 = 2$, $C_2 = 3$, PC at instruction 1 (prime 5). $State_0 = 2^2 \times 3^3 \times 5 = 540$.

Result: $C_1 = 5 = 2 + 3$ ✓. Final state $160 = 2^5 \times 5$.

10.6 Completeness Theorem

Theorem 10.4 (Turing Completeness). The framework is Turing complete.

Corollary 10.5. There is no algorithm to decide, given an arbitrary counter machine encoded in this framework, whether it halts.

PART VII: CONNECTIONS AND IMPLICATIONS

11. Primes as Irreducible Elements

Definition 11.1. A positive integer $n > 1$ is **resistance-irreducible** if there do not exist positive integers $a, b > 1$ such that $\ln(n) = \ln(a) + \ln(b)$.

Theorem 11.2 (Prime Characterization). A positive integer $n > 1$ is prime if and only if it is resistance-irreducible.

12. Connection to the Riemann Zeta Function

12.1 The Euler Product

$$\zeta(s) = \prod_p 1/(1 - p^{-s})$$

At $s = 2$, the reciprocal yields the coprimality probability: $1/\zeta(2) = \prod_p (1 - 1/p^2) = 6/\pi^2$.

12.2 The Möbius Function

The Möbius function $\mu(n)$ equals: 1 if $n = 1$; $(-1)^k$ if n is squarefree with k prime factors; 0 if n has a squared prime factor. The condition $\mu(n) = 0$ identifies non-squarefree integers—exactly those that cannot encode sets in the framework.

13. Lattice-Theoretic Structure

Theorem 13.1. The positive integers under divisibility form a distributive lattice with: Meet ($\wedge$): gcd; Join ($\vee$): lcm.

15. Conclusion

15.1 Summary of Results

Result	Reference
Isomorphism $\Omega(1/n) = \ln(n)$	Theorem 2.3
Uniqueness of logarithm	Theorem 2.4
Union formula: $\Omega_a + \Omega_b - \Omega_{gcd}$	Theorem 4.2
Vector max = Union formula	Theorem 7.3
Linear independence of ln(primes)	Theorem 7.2
Coprimality = $6/\pi^2$	Theorem 8.1
Intersection = GCD (squarefree)	Theorem 9.1
Union = LCM (squarefree)	Theorem 9.2
Binary-squarefree bijection	Theorem 9.3
Turing completeness	Theorem 10.4
Prime = resistance-irreducible	Theorem 11.2
Distributive lattice	Theorem 13.1
Functional completeness	Theorem 6.1

15.2 The Four-Layer Correspondence

Layer	Intersection	Union	Complement
English	$A \cap B$	$A \cup B$	$\bar{A}$
Inverse Integer	$1/\gcd(a,b)$	$1/\mathrm{lcm}(a,b)$	$(a-1)/a \in \mathbb{Q}$
Resistance	Ω_{gcd}	$\Omega_a + \Omega_b - \Omega_{gcd}$	Phase
Vector	$\min(v_a, v_b)$	$\max(v_a, v_b)$	—

15.3 Significance

The framework establishes:

5. An exact correspondence between number theory, set theory, and informational resistance
6. Primes as structurally necessary irreducible elements
7. Boolean logic as boundary behavior of continuous computation
8. Database and number-theoretic operations as instances of a distributive lattice

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Appendix A: Key Formulas

Resistance mapping: $\Omega(1/n) := \ln(n)$

Series (multiplication): $\Omega(1/ab) = \Omega_a + \Omega_b$

Union (set OR via LCM): $\Omega_{lcm} = \Omega_a + \Omega_b - \Omega_{gcd}$

Fundamental identity: $\mathrm{lcm}(a,b) \times \gcd(a,b) = a \times b$

Coprimality: $P(\gcd(a,b) = 1) = 6/\pi^2$

Appendix B: Notation

Symbol	Meaning
I	Set of inverse integers $\{1/n : n \in \mathbb{Z}^+\}$
$\Omega(1/n)$, Ω_n	Informational resistance = $\ln(n)$
$v(n)$	Prime coordinate vector of n
D_p	Divisibility gate (transistor) for prime p
$\zeta(s)$	Riemann zeta function
$\mu(n)$	Möbius function