Ch04 - chapter 4-solution of 13th edition of Elementry Linear Algebra

Linear Algebra (National University of Computer and Emerging Sciences)

CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

1. **(a)** $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$. is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$
$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

2. **(a)** $\mathbf{u}+\mathbf{v}=(0+1+1,4-3+1)=(2,2)$; $k\mathbf{u}=(2\cdot 0,2\cdot 4)=(0,8)$

(b) $(0,0)+(u_1,u_2)=(0+u_1+1,0+u_2+1)=(u_1+1,\ u_2+1)\neq(u_1,u_2)$ therefore $(0,0)$ is not the zero vector $\mathbf{0}$ required by Axiom 4

(c) For all real numbers u_1 and u_2, we have
$(-1,-1)+(u_1,u_2)=(-1+u_1+1,\ -1+u_2+1)=(u_1,\ u_2)$ and
$(u_1,u_2)+(-1,-1)=(u_1-1+1,\ u_2-1+1)=(u_1,\ u_2)$ therefore Axiom 4 holds for
$\mathbf{0}=(-1,-1)$

d) For any pair of real numbers $\mathbf{u}=(u_1,u_2)$, letting $-\mathbf{u}=(-2-u_1,-2-u_2)$ yields
$\mathbf{u}+(-\mathbf{u})=(u_1+(-2-u_1)+1,u_2+(-2-u_2)+1)=(-1,-1)=\mathbf{0}$;
Since $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ holds as well, Axiom 5 holds.

(e) Axiom 7 fails to hold:
$k(\mathbf{u}+\mathbf{v})=k(u_1+v_1+1,u_2+v_2+1)=(ku_1+kv_1+k,ku_2+kv_2+k)$

$k\mathbf{u}+k\mathbf{v}=(ku_1,ku_2)+(kv_1,kv_2)=(ku_1+kv_1+1,ku_2+kv_2+1)$
therefore in general $k(\mathbf{u}+\mathbf{v})\neq k\mathbf{u}+k\mathbf{v}$

Axiom 8 fails to hold:
$(k+m)\mathbf{u}=((k+m)u_1,(k+m)u_2)=(ku_1+mu_1,ku_2+mu_2)$
$k\mathbf{u}+m\mathbf{u}=(ku_1,ku_2)+(mu_1,mu_2)=(ku_1+mu_1+1,ku_2+mu_2+1)$
therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$

3. Let V denote the set of all real numbers.

Axiom 1: $x+y$ is in V for all real x and y;

Axiom 2: $x+y=y+x$ for all real x and y;

Axiom 3: $x+(y+z)=(x+y)+z$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=0$, we have $0+x=x+0=x$ for all real x;

Axiom 5: for each $\mathbf{u}=x$, let $-\mathbf{u}=-x$; then $x+(-x)=(-x)+x=0$

Axiom 6: kx is in V for all real k and x;

Axiom 7: $k(x+y)=kx+ky$ for all real k, x, and y;

Axiom 8: $(k+m)x=kx+mx$ for all real k, m, and x;

Axiom 9: $k(mx)=(km)x$ for all real k, m, and x;

Axiom 10: $1x=x$ for all real x.

This is a vector space – all axioms hold.

4. Let V denote the set of all pairs of real numbers of the form $(x,0)$.

Axiom 1: $(x,0)+(y,0)=(x+y,0)$ is in V for all real x and y;

Axiom 2: $(x,0)+(y,0)=(x+y,0)=(y+x,0)=(y,0)+(x,0)$ for all real x and y;

Axiom 3: $(x,0)+((y,0)+(z,0))=(x,0)+(y+z,0)=(x+y+z,0)=(x+y,0)+(z,0)$
$=((x,0)+(y,0))+(z,0)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(x,0)=(x,0)$ and $(x,0)+(0,0)=(x,0)$ for all real x;

Axiom 5: for each $\mathbf{u}=(x,0)$, let $-\mathbf{u}=(-x,0)$;
then $(x,0)+(-x,0)=(0,0)$ and $(-x,0)+(x,0)=(0,0)$;

Axiom 6: $k(x,0)=(kx,0)$ is in V for all real k and x;

Axiom 7: $k\left((x,0)+(y,0)\right)=k(x+y,0)=(kx+ky,0)=k(x,0)+k(y,0)$
for all real k, x, and y;

Axiom 8: $(k+m)(x,0)=\left((k+m)x,0\right)=(kx+mx,0)=k(x,0)+m(x,0)$
for all real k, m, and x;

Axiom 9: $k\left(m(x,0)\right)=k(mx,0)=(kmx,0)=(km)(x,0)$ for all real k, m, and x;

Axiom 10: $1(x,0)=(x,0)$ for all real x.

This is a vector space – all axioms hold.

5. Axiom 5 fails whenever $x\neq 0$ since it is then impossible to find (x',y') satisfying $x'\geq 0$ for which $(x,y)+(x',y')=(0,0)$. (The zero vector from axiom 4 must be $\mathbf{0}=(0,0)$.)

Axiom 6 fails whenever $k<0$ and $x\neq 0$.

This is not a vector space.

6. Let V denote the set of all n-tuples of real numbers of the form $(x,x,\ldots,x)$.

Axiom 1: $(x,x,\ldots,x)+(y,y,\ldots,y)=(x+y,x+y,\ldots,x+y)$ is in V for all real x and y;

Axiom 2: $(x,x,\ldots,x)+(y,y,\ldots,y)=(x+y,x+y,\ldots,x+y)=(y+x,y+x,\ldots,y+x)$
$=(y,y,\ldots,y)+(x,x,\ldots,x)$ for all real x and y;

Axiom 3: $(x,x,\ldots,x)+\left((y,y,\ldots,y)+(z,z,\ldots,z)\right)=(x,x,\ldots,x)+(y+z,y+z,\ldots,y+z)$
$=(x+y+z,x+y+z,\ldots,x+y+z)=(x+y,x+y,\ldots,x+y)+(z,z,\ldots,z)$
$=\left((x,x,\ldots,x)+(y,y,\ldots,y)\right)+(z,z,\ldots,z)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=(0,0,\ldots,0)$, we have $(0,0,\ldots,0)+(x,x,\ldots,x)=(x,x,\ldots,x)$ and
$(x,x,\ldots,x)+(0,0,\ldots,0)=(x,x,\ldots,x)$ for all real x;

Axiom 5: for each $\mathbf{u}=(x,x,\ldots,x)$, let $-\mathbf{u}=(-x,-x,\ldots,-x)$;
then $(x,x,\ldots,x)+(-x,-x,\ldots,-x)=(0,0,\ldots,0)$ and
$(-x,-x,\ldots,-x)+(x,x,\ldots,x)=(0,0,\ldots,0)$;

Axiom 6: $k(x,x,\ldots,x)=(kx,kx,\ldots,kx)$ is in V for all real k and x;

Axiom 7: $k\left((x,x,\ldots,x)+(y,y,\ldots,y)\right)=k(x+y,x+y,\ldots,x+y)=(kx+ky,kx+ky,\ldots,kx+ky)$
$=k(x,x,\ldots,x)+k(y,y,\ldots,y)$ for all real k, x, and y;

Axiom 8: $(k+m)(x,x,\ldots,x)=((k+m)x,(k+m)x,\ldots,(k+m)x)$
$=(kx+mx,kx+mx,\ldots,kx+mx)=k(x,x,\ldots,x)+m(x,x,\ldots,x)$
for all real k, m, and x;

Axiom 9: $k(m(x,x,\ldots,x))=k(mx,mx,\ldots,mx)=(kmx,kmx,\ldots,kmx)=(km)(x,x,\ldots,x)$
for all real k, m, and x;

Axiom 10: $1(x,x,\ldots,x)=(x,x,\ldots,x)$ for all real x.

This is a vector space – all axioms hold.

7. Axiom 8 fails to hold:

$(k+m)\mathbf{u}=((k+m)^2x,(k+m)^2y,(k+m)^2z)$

$k\mathbf{u}+m\mathbf{u}=(k^2x,k^2y,k^2z)+(m^2x,m^2y,m^2z)=((k^2+m^2)x,(k^2+m^2)y,(k^2+m^2)z)$

therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$.

This is not a vector space.

8. Axiom 1 fails since a sum of two 2×2 invertible matrices may or may not be invertible, e.g. both $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ are invertible, but $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not invertible.

Axiom 6 fails whenever $k=0$.

9. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix}a&0\\0&b\end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix}a&0\\0&b\end{bmatrix}$ be $\begin{bmatrix}-a&0\\0&-b\end{bmatrix}$; follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

10. Let V be the set of all real-valued functions f defined for all real numbers and such that $f(1)=0$.

Axiom 1: If f and g are in V then $f+g$ is a function defined for all real numbers and $(f+g)(1)=f(1)+g(1)=0$ therefore V is closed under the operation of addition defined by Formula (2).

Axiom 6: If k is a scalar and f is in V then kf is a function defined for all real numbers and $(kf)(1)=k(f(1))=0$ therefore V is closed under the operation of scalar multiplication defined by Formula (3).

Verification of the eight remaining axioms proceeds analogously to Example 6.

This is a vector space – all axioms hold.

11. Let V denote the set of all pairs of real numbers of the form $(1,x)$.

Axiom 1: $(1,y)+(1,y')=(1,y+y')$ is in V for all real y and y';

Axiom 2: $(1,y)+(1,y')=(1,y+y')=(1,y'+y)=(1,y')+(1,y)$ for all real y and y';

Axiom 3: $(1,y)+((1,y')+(1,y''))=(1,y)+(1,y'+y'')=(1,y+y'+y'')=(1,y+y')+(1,y'')$
$=((1,y)+(1,y'))+(1,y'')$ for all real y, y', and y'';

Axiom 4: taking $\mathbf{0}=(1,0)$, we have $(1,0)+(1,y)=(1,y)$ and $(1,y)+(1,0)=(1,y)$ for all real y;

Axiom 5: for each $\mathbf{u}=(1,y)$, let $-\mathbf{u}=(1,-y)$;
then $(1,y)+(1,-y)=(1,0)$ and $(1,-y)+(1,y)=(1,0)$;

Axiom 6: $k(1,y)=(1,ky)$ is in V for all real k and y;

Axiom 7: $k((1,y)+(1,y'))=k(1,y+y')=(1,ky+ky')=(1,ky)+(1,ky')=k(1,y)+k(1,y')$ for all real k, y, and y';

Axiom 8: $(k+m)(1,y)=(1,(k+m)y)=(1,ky+my)=(1,ky)+(1,my)=k(1,y)+m(1,y)$ for all real k, m, and y;

Axiom 9: $k(m(1,y))=k(1,my)=(1,kmy)=(km)(1,y)$ for all real k, m, and y;

Axiom 10: $1(1,y)=(1,y)$ for all real y.

This is a vector space – all axioms hold.

12. Let V be the set of polynomials of the form $a+bx$.

Axiom 1: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x$ is in V for all real a_0, a_1, b_0, and b_1;

Axiom 2: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x=(a_1+a_0)+(b_1+b_0)x$
$=(a_1+b_1x)+(a_0+b_0x)$ for all real a_0, a_1, b_0, and b_1;

Axiom 3: $(a_0+b_0x)+((a_1+b_1x)+(a_2+b_2x))=(a_0+a_1+a_2)+(b_0+b_1+b_2)x$
$((a_0+b_0x)+(a_1+b_1x))+(a_2+b_2x)$ for all real a_0, a_1, a_2, b_0, b_1, and b_2;

Axiom 4: taking $\mathbf{0}=0+0x$, we have $(0+0x)+(a+bx)=a+bx$ and
$(a+bx)+(0+0x)=a+bx$ for all real a and b;

Axiom 5: for each $\mathbf{u}=a+bx$, let $-\mathbf{u}=-a-bx$;
then $(a+bx)+(-a-bx)=0+0x=(-a-bx)+(a+bx)$ for all real a and b;

Axiom 6: $k(a+bx)=ka+(kb)x$ is in V for all real a, b, and k;

Axiom 7: $k((a_0+b_0x)+(a_1+b_1x))=k((a_0+a_1)+(b_0+b_1)x)=k(a_0+b_0x)+k(a_1+b_1x)$ for all real a_0, a_1, b_0, b_1, and k;

Axiom 8: $(k+m)(a+bx)=(k+m)a+(k+m)bx=k(a+bx)+m(a+bx)$
for all real a, b, k, and m;

Axiom 9: $k(m(a+bx))=k(ma+mbx)=kma+kmbx=(km)(a+bx)$
for all real a, b, k, and m;

Axiom 10: $1(a+bx)=a+bx$ for all real a and b.

This is a vector space – all axioms hold.

13. Axiom 3: follows from part (b) of Theorem 1.4.1 since

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\left(\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 7: follows from part (h) of Theorem 1.4.1 since

$$k(\mathbf{u}+\mathbf{v})=k\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+k\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=k\mathbf{u}+k\mathbf{v}$$

Axiom 8: follows from part (j) of Theorem 1.4.1 since

$$(k+m)\mathbf{u} = (k+m)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + m\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\mathbf{u} + m\mathbf{u}$$

Axiom 9: follows from part (l) of Theorem 1.4.1 since

$$k(m\mathbf{u}) = k\left(m\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\right) = (km)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (km)\mathbf{u}$$

15. Axiom 1: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$ is in V

Axiom 2: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)=(v_1+u_1,v_2+u_2)=(v_1,v_2)+(u_1,u_2)$

Axiom 3: $(u_1,u_2)+((v_1,v_2)+(w_1,w_2))=(u_1,u_2)+(v_1+w_1,v_2+w_2)$

$=(u_1+v_1+w_1,u_2+v_2+w_2)=(u_1+v_1,u_2+v_2)+(w_1,w_2)$

$=((u_1,u_2)+(v_1,v_2))+(w_1,w_2)$

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(u_1,u_2)=(u_1,u_2)$ and $(u_1,u_2)+(0,0)=(u_1,u_2)$

Axiom 5: for each $\mathbf{u}=(u_1,u_2)$, let $-\mathbf{u}=(-u_1,-u_2)$;

then $(u_1,u_2)+(-u_1,-u_2)=(0,0)$ and $(-u_1,-u_2)+(u_1,u_2)=(0,0)$

Axiom 6: $k(u_1,u_2)=(ku_1,0)$ is in V

Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(ku_1+kv_1,0)=(ku_1,0)+(kv_1,0)=k(u_1,u_2)+k(v_1,v_2)$

Axiom 8: $(k+m)(u_1,u_2)=((k+m)u_1,0)=(ku_1+mu_1,0)=(ku_1,0)+(mu_1,0)$

$=k(u_1,u_2)+m(u_1,u_2)$

Axiom 9: $k(m(u_1,u_2))=k(mu_1,0)=(kmu_1,0)=(km)(u_1,u_2)$

19. $\frac{1}{u}=u^{-1}$

20. For positive real numbers u, $u^k=1$ if and only if $k=0$ or $u=1$.

21.

$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})]=\mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	Axiom 3
$\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$	Axiom 5
$\mathbf{u}=\mathbf{v}$	Axiom 4

22. (1) Axiom 7
(2) Axiom 4
(3) Axiom 5
(4) Axiom 1

(5) Axiom 3
(6) Axiom 5
(7) Axiom 4

True-False Exercises

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty, \infty)$.

4.2 Subspaces

1. **(a)** Let W be the set of all vectors of the form $(a,0,0)$, i.e. all vectors in R^3 with last two components equal to zero.

This set contains at least one vector, e.g. $(0,0,0)$.

Adding two vectors in W results in another vector in W: $(a,0,0)+(b,0,0)=(a+b,0,0)$ since the result has zeros as the last two components.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,0)=(ka,0,0)$ - the result also has zeros as the last two components.

According to Theorem 4.2.1, W is a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a,1,1)$, i.e. all vectors in R^3 with last two components equal to one. The set W is not closed under the operation of vector addition since $(a,1,1)+(b,1,1)=(a+b,2,2)$ does not have ones as its last two components thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of R^3.

(c) Let W be the set of all vectors of the form (a,b,c), where $b = a + c$.

This set contains at least one vector, e.g. $(0,0,0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)

Adding two vectors in W results in another vector in W
$(a, a+c, c)+(a', a'+c', c')=(a+a',\ a+c+a'+c',\ c+c')$ since in this result, the second component is the sum of the first and the third: $a+c+a'+c'=(a+a')+(c+c')$.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,a+c,c)=(ka,k(a+c),kc)$ since in this result, the second component is once again the sum of the first and the third:

$k(a+c) = ka + kc$.

According to Theorem 4.2.1, W is a subspace of R^3 .

2. **(a)** Let W be the set of all vectors of the form (a,b,c), where $b = a + c + 1$. The set W is not closed under the operation of vector addition, since in the result of the following addition of two vectors from W

$(a, a+c+1, c) + (a', a'+c'+1, c') = (a+a',\ a+c+a'+c'+2,\ c+c')$

the second component does not equal to the sum of the first, the third, and 1:

$a+c+a'+c'+2 \neq (a+a')+(c+c')+1$. Consequently, this result is not a vector in W .

According to Theorem 4.2.1, W is not a subspace of R^3 .

(b) Let W be the set of all vectors of the form $(a,b,0)$, i.e. all vectors in R^3 with last component equal to zero.

This set contains at least one vector, e.g. $(0,0,0)$.

Adding two vectors in W results in another vector in W

$(a,b,0) + (a',b',0) = (a+a', b+b', 0)$ since the result has 0 as the last component.

Likewise, a scalar multiple of a vector in W is also in W : $k(a,b,0) = (ka, kb, 0)$ - the result also has 0 as the last component.

According to Theorem 4.2.1, W is a subspace of R^3 .

(c) Let W be the set of all vectors of the form (a,b,c), where $a+b=7$. The set W is not closed under the operation of vector addition, since in the result of the following addition of two vectors from W we obtain

$(a,b,c) + (a',b',c') = (a+a', b+b',\ c+c')$ where

$a+a'+b+b' = a+b+a'+b'$ = 7 + 7 = 14. Consequently, this result is not a vector in W .

According to Theorem 4.2.1, W is not a subspace of R^3 .

3. **(a)** Let W be the set of all $n \times n$ diagonal matrices.

This set contains at least one matrix, e.g. the zero $n \times n$ matrix.

Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W :

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W :

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W. However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are both in W, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\operatorname{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.

Since $\operatorname{tr}(A+B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$

$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.

A scalar multiple of the same matrix A with a scalar k has $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).

This set contains at least one matrix, e.g., I_n is in W.

Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A+B)^T = A^T + B^T = A + B$ therefore W is closed under addition.

From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

4. **(a)** Let W be the set of all $n \times n$ matrices such that $A^T = -A$.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume A and B are both in W, i.e. $A^T = -A$ and $B^T = -B$. By Theorem 1.4.8(b), their sum satisfies $(A+B)^T = A^T + B^T = -A - B = -(A+B)$ therefore W is closed under addition.

From Theorem 1.4.8(d), we have $(kA)^T = kA^T = k(-A) = -kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of $n \times n$ matrices for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. It follows from Theorem 1.5.3 that the set W consists of all $n \times n$ matrices that are invertible. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

(c) Let B be some fixed $n \times n$ matrix, and let W be the set of all $n \times n$ matrices A such that $AB = BA$. This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and C are both in W, i.e. $AB = BA$ and $CB = BC$. By Theorem 1.4.1(d,e), their sum satisfies $(A+C)B = AB + CB = BA + BC = B(A+C)$ therefore W is closed under addition.
From Theorem 1.4.1(m), we have $(kA)B = k(AB) = k(BA) = B(kA)$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all invertible $n \times n$ matrices (i.e., $n \times n$ matrices such that A^{-1} exists). This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

5. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
This set contains at least one polynomial, $0 + 0x + 0x^2 + 0x^3 = 0$.
Adding two polynomials in W results in another polynomial in W:
$(0 + a_1x + a_2x^2 + a_3x^3) + (0 + b_1x + b_2x^2 + b_3x^3)$
$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$.
Likewise, a scalar multiple of a polynomial in W is also in W:
$k(0 + a_1x + a_2x^2 + a_3x^3) = 0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3$.
According to Theorem 4.2.1, W is a subspace of P_3.

(b) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3$.
Adding two polynomials in W results in another polynomial in W
$(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) + (-b_1 - b_2 - b_3 + b_1x + b_2x^2 + b_3x^3)$
$= (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$
since we have $(-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$k(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) = -ka_1 - ka_2 - ka_3 + ka_1x + ka_2x^2 + ka_3x^3$

since it meets the condition $(-ka_1 - ka_2 - ka_3) + (ka_1) + (ka_2) + (ka_3) = 0$.
According to Theorem 4.2.1, W is a subspace of P_3.

6. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers. The set W is not closed under the operation of scalar multiplication, e.g., the scalar product of the polynomial x^3 in W by $k = \pi$ is πx^3, which is not in W.
According to Theorem 4.2.1, W is not a subspace of P_3.

(b) The set of all polynomials of degree ≤ 1 is a subset of P_3. It is also a vector space (called P_1) with same operations of addition and scalar multiplication as those defined in P_3. By Definition 1, we conclude that P_1 is a subspace of P_3.

7. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 0$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(0) = k(f(0)) = 0$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 1$.
We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be two functions in W. Their sum, $f + g$, is not in W since $(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$.
We conclude that W is not a subspace of $F(-\infty,\infty)$.

8. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(-x) = f(x)$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(-x) = f(x)$ and $g(-x) = g(x)$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f+g)(-x) = f(-x) + g(-x) =$ $f(x) + g(x) = (f+g)(x)$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(-x) = k(f(-x)) = k(f(x)) = (kf)(x)$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) A sum of two polynomials of degree 2 may be a polynomial of lower degree, e.g., $(1+x^2)+(x-x^2)=1+x$ therefore the set is not closed under addition, and consequently is not a subspace of $F(-\infty,\infty)$.

9. **(a)** Let W be the set of all sequences in R^∞ of the form $(v,0,v,0,v,0,\ldots)$.

This set contains at least one sequence, e.g. $(0,0,0,\ldots)$.

Adding two sequences in W results in another sequence in W:
$(v,0,v,0,v,0,\ldots)+(w,0,w,0,w,0,\ldots)=(v+w,0,v+w,0,v+w,0,\ldots)$.

Likewise, a scalar multiple of a vector in W is also in W: $k(v,0,v,0,v,0,\ldots)=(kv,0,kv,0,kv,0,\ldots)$.

According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences in R^∞ of the form $(v,1,v,1,v,1,\ldots)$.

This set is not closed under addition since
$(v,1,v,1,v,1,\ldots)+(w,1,w,1,w,1,\ldots)=(v+w,2,v+w,2,v+w,2,\ldots)$ is not in W.

We conclude that W is not a subspace of R^∞.

10. **(a)** Let W be the set of all sequences in R^∞ of the form $(v,2v,4v,8v,16v,\ldots)$.

This set contains at least one sequence, e.g. $(0,0,0,\ldots)$.

Adding two sequences in W results in another sequence in W:
$(v,2v,4v,8v,16v,\ldots)+(w,2w,4w,8w,16w,\ldots)$
$=(v+w,2(v+w),4(v+w),8(v+w),16(v+w),\ldots)$.

Likewise, a scalar multiple of a vector in W is also in W:
$k(v,2v,4v,8v,16v,\ldots)=(kv,2kv,4kv,8kv,16kv,\ldots)$.

According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences in R^∞ whose components are 0 from some point on.

This set contains at least one sequence, e.g. $(0,0,0,\ldots)$.

Let a sequence $\mathbf{u}$ in W have 0 components starting from the i th element; also, let a sequence $\mathbf{v}$ in W have 0 components starting from the j th element. It follows that $\mathbf{u}+\mathbf{v}$ must have 0 component starting no later than from the position corresponding to $\max(i,j)$ - the larger of the two numbers.

Therefore, $\mathbf{u}+\mathbf{v}$ is in W.

The scalar product $k\mathbf{u}$ must have 0 components starting no later than from the i th element, therefore $k\mathbf{u}$ is also in W.

According to Theorem 4.2.1, W is a subspace of R^∞.

11. **(a)** Let W be the set of all matrices of form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix} = \begin{bmatrix} a+a' & 0 \\ b+b' & 0 \end{bmatrix}.$$

Likewise, a scalar multiple of a matrix in W is also in W:

$k\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} ka & 0 \\ kb & 0 \end{bmatrix}$. According to Theorem 4.2.1, W is a subspace of M_{22}.

(b) Let W be the set of all matrices of form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{22}.

(c) Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This set is not closed under addition since if A and B are matrices in W then

$(A+B)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} + B\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Consequently, the matrix $A+B$ is not contained in W. According to Theorem 4.2.1, W is not a subspace of M_{22}.

12. **(a)** Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$(A+B)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} + B\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Likewise, a scalar multiple of a matrix in W is also in W: $(kA)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = k\left(A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = k\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. According to Theorem 4.2.1, W is a subspace of M_{22}.

(b) Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}A$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$(A+B)\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} + B\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}A + \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}B = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}(A+B)$. Likewise, a scalar multiple of a matrix in W is also in W: $(kA)\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = k\left(A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}\right) =$

$k\left(\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}A\right) = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}(kA)$. According to Theorem 4.2.1, W is a subspace of M_{22}.

(c) Let W be the set of all 2×2 matrices A such that $\det(A)=0$. This set is not closed under addition. For example, the matrices $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ are in W because each has determinant zero but $\det\left(\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)=\det\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)=1$. According to Theorem 4.2.1, W is not a subspace of M_{22}.

13. **(a)** Let W be the set of all vectors in R^4 of form (a,a^2,a^3,a^4). This set is not closed under addition. For example, the vector $(1,1,1,1)$ is in W but $(1,1,1,1)+(1,1,1,1)=(2,2,2,2)$ is not. According to Theorem 4.2.1, W is not a subspace of R^4.

(b) Let W be the set of all vectors in R^4 of form $(a,0,b,0)$. This set contains at least one vector, e.g. the zero vector. Adding two vectors in W results in another vector in W:

$(a,0,b,0)+(a',0,b',0)=(a+a',0,b+b',0)$.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,b,0)=(ka,0,kb,0)$. According to Theorem 4.2.1, W is a subspace of R^4.

14. **(a)** Let W be the set of all vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x}=\begin{bmatrix}0\\1\end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of R^4.

(b) Let W be the set of all vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x}=\begin{bmatrix}0\\0\end{bmatrix}$. This set contains at least one vector, e.g. the zero vector. Adding two vectors in W results in another vector in W:

$A(\mathbf{x}+\mathbf{y})-A\mathbf{x}+a\mathbf{y}=\begin{bmatrix}0\\0\end{bmatrix}.+\begin{bmatrix}0\\0\end{bmatrix}.=\begin{bmatrix}0\\0\end{bmatrix}$. Likewise, a scalar multiple of a vector

in W is also in $W: A(k\mathbf{x})=kA(\mathbf{x})=k\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. According to Theorem 4.2.1, W is a subspace of R^4.

15. **(a)** Let W be the set of all polynomials of degree less than or equal to six. This set is not empty. For example, $p(x)=x$ is contained in W. Adding two polynomials in W results in another polynomial in W because the sum of two polynomials of degree at most six is another polynomial of degree at least six. Likewise, a scalar multiple of a polynomial of degree at most six is another polynomial of degree at most six. According to Theorem 4.2.1, W is a subspace of P_∞.

(b) Let W be the set of all polynomials of degree equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

(c) Let W be the set of all polynomials of degree greater than or equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

16. **(a)** Let W be the set of all polynomials with even coefficients. This set is not empty. For example, the polynomial $p(x)=2x^2$ is contained in W. Adding two polynomials in W results in another polynomial in W because the sum of any two corresponding even coefficients is even. Likewise, a scalar multiple of a polynomial with even coefficients is another polynomial with even coefficients.. According to Theorem 4.2.1, W is a subspace of P_∞.

(b) Let W be the set of all polynomials whose coefficients sum to zero. This set is not empty. For example, the polynomial $p(x)=x^2-x$ is contained in W. Adding two polynomials in W results in another polynomial in W:

$\left(a_0+a_1x+a_2x^2+\cdots+a_\mathrm{n}x^\mathrm{n}\right)+\left(b_0+b_1x+b_2x^2+\cdots+b_\mathrm{m}x^\mathrm{m}\right)$
$=\left(a_0+b_0\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2+\cdots+\left(a_\mathrm{m}+b_\mathrm{m}\right)x^\mathrm{m}+a_{m+1}x^{m+1}+\cdots a_nx^n$ where we assume without loss of generality that $n>m$.

The sum of this polynomial's coefficients is:

$(a_0+b_0)+(a_1+b_1)+\cdots(a_\mathrm{m}+b_\mathrm{m})+a_{m+1}+\cdots+a_n=a_0+a_1+\cdots a_\mathrm{n}+b_0+b_1\cdots+b_\mathrm{m}=0+0=0$ which means it is contained in W. Likewise, a scalar multiple of a polynomial whose coefficients sum to zero is another polynomial whose coefficients sum to zero: $k\left(a_0+a_1x+a_2x^2+\cdots+a_\mathrm{n}x^\mathrm{n}\right)=$
$ka_0+\mathrm{k}a_1x+ka_2x^2+\cdots+ka_\mathrm{n}x^\mathrm{n}$ so that $ka_0+ka_1+\cdots ka_\mathrm{n}=k\left(a_0+a_1+\cdots a_\mathrm{n}\right)=k0=0$. According to Theorem 4.2.1, W is a subspace of P_∞.

(c) Let W be the set of all polynomials of even degree. This set is not empty. For example, the polynomial $p(x)=x^2$ is contained in W. Adding two polynomials in W results in another polynomial in W:

$\left(a_0+a_1x+a_2x^2+\cdots+a_\mathrm{n}x^{2\mathrm{n}}\right)+\left(b_0+b_1x+b_2x^2+\cdots+b_\mathrm{m}x^{2\mathrm{m}}\right)$
$=\left(a_0+b_0\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2+\cdots+\left(a_{2\mathrm{m}}+b_{2\mathrm{m}}\right)x^{2\mathrm{m}}+a_{2m+1}x^{2m+1}+\cdots a_nx^{2n}$

where we assume without loss of generality that $n>m$.

This polynomial also has even degree which means it is contained in W. Likewise, a scalar multiple of a polynomial of even degree is another polynomial of even degree:

$k\left(a_0+a_1x+a_2x^2+\cdots+a_\mathrm{n}x^{2\mathrm{n}}\right)=ka_0+\mathrm{k}a_1x+ka_2x^2+\cdots+ka_\mathrm{n}x^{2\mathrm{n}}$. According to Theorem 4.2.1, W is a subspace of P_∞.

17. **(a)** Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\lim_{n\to\infty} v_n = 0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\ldots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W since $\lim_{n\to\infty} v_n + \lim_{n\to\infty} w_n = \lim_{n\to\infty}(v_n + w_n) = 0$. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k(\lim_{n\to\infty} v_n) = \lim_{n\to\infty} kv_n = 0$. (These results both follow because sums and constant multiples of convergent sequences are also convergent.). According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\lim_{n\to\infty} v_n$ exists and is finite. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\ldots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\lim_{n\to\infty} v_n$ and $\lim_{n\to\infty} w_n$ exist and are finite so that $\lim_{n\to\infty} v_n + \lim_{n\to\infty} w_n = \lim_{n\to\infty}(v_n + w_n)$ also exists and is finite. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

(c) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n = 0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\ldots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge to zero so that $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty}(v_n + w_n) = 0$. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n = 0$. According to Theorem 4.2.1, W is a subspace of R^∞.

(d) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n$ converges. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\ldots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge so $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty}(v_n + w_n)$ also converges. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

18. The line L contains at least one point – e.g., the origin.
If the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are both on L, then there must exist real numbers t_1 and t_2 such that $x_1 = at_1$, $y_1 = bt_1$, $z_1 = ct_1$, $x_2 = at_2$, $y_2 = bt_2$, and $z_2 = ct_2$.
L is closed under addition since $(x_1, y_1, z_1) + (x_2, y_2, z_2) = \left((a)(t_1 + t_2), (b)(t_1 + t_2), (c)(t_1 + t_2)\right)$.

It is also closed under scalar multiplication because $k(x_1,y_1,z_1)=((a)(kt_1),(b)(kt_1),(c)(kt_1))$.

It follows from Theorem 4.2.1 that L is a subspace of R^3.

19. **(a)** The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 0\end{bmatrix}$ therefore the solution are $x=-\frac{1}{2}t$, $y=-\frac{3}{2}t$, $z=t$. These are parametric equations of a line through the origin.

(b) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ therefore the only solution is $x=y=z=0$ - the origin.

(c) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix}1 & -3 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ which corresponds to an equation of a plane through the origin $x-3y+z=0$.

(d) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$ therefore the solutions are $x=-3t$, $y=-2t$, $z=t$. These are parametric equations of a line through the origin.

21. Let W denote the set of all continuous functions $f=f(x)$ on $[a,b]$ such that $\int_a^b f(x)\,dx=0$.

This set contains at least one function $f(x)\equiv 0$.

Let us assume $\mathbf{f}=f(x)$ and $\mathbf{g}=g(x)$ are functions in W. From calculus,

$\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx=0$ and $\int_a^b kf(x)\,dx=k\int_a^b f(x)\,dx=0$ therefore both $\mathbf{f}+\mathbf{g}$ and $k\mathbf{f}$ are in W for any scalar k. According to Theorem 4.2.1, W is a subspace of $C[a,b]$.

23. Since $T_A:R^3\to R^m$, it follows from Theorem 4.2.5 that the kernel of T_A must be a subspace of R^3. Hence, according to Table 1 the kernel can be one of the following four geometric obects:

- the origin,
- a line through the origin,
- a plane through the origin,
- R^3.

25. Let W be the set of all function. of the form $x(t)=c_1\cos\omega t+c_2\sin\omega t$ - W is a subset of $C^\infty(-\infty,\infty)$.

This set contains at least one function $x(t)\equiv 0$.

A sum of two functions in W is also in W:

$(c_1 \cos\omega t + c_2 \sin\omega t) + (d_1 \cos\omega t + d_2 \sin\omega t) = (c_1 + d_1)\cos\omega t + (c_2 + d_2)\sin\omega t$.

A scalar product of a function in W by any scalar k is also a function in W:

$k(c_1 \cos\omega t + c_2 \sin\omega t) = (kc_1)\cos\omega t + (kc_2)\sin\omega t$.

According to Theorem 4.2.1, W is a subspace of $C^\infty(-\infty,\infty)$.

26. For example, consider the subsets $U = \{(x,2x) | x \in R\}$ and $V = \{(x,3x) | x \in R\}$ of R^2. Let $W = U \cup V$. Then $(1,2) \in U$ and $(1,3) \in V$ but $(1,2)+(1,3)=(2,5)$ is not contained in W. According to Theorem 4.2.1, W is not a subspace of R^2.

True-False Exercises

(a) True. This follows from Definition 1.

(b) True.

(c) False. The set of all nonnegative real numbers is a subset of the vector space R containing 0, but it is not closed under scalar multiplication.

(d) False. By Theorem 4.2.4, the kernel of $T_A : R^n \to R^m$ is a subspace of R^n.

(e) False. The solution set of a nonhomogeneous system is not closed under addition: $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}$ do not imply $A(\mathbf{x}+\mathbf{y}) = \mathbf{b}$.

(f) True. This follows from Theorem 4.2.2.

(g) False. Consider $W_1 = \text{span}\{(1,0)\}$ and $W_2 = \text{span}\{(0,1)\}$. The union of these sets is not closed under vector addition, e.g. $(1,0)+(0,1)=(1,1)$ is outside the union.

(h) True. This set contains at least one matrix (e.g., I_n). A sum of two upper triangular matrices is also upper triangular, therefore the set is closed under addition. A scalar multiple of an upper triangular matrix is also upper triangular, hence the set is closed under scalar multiplication.

4.3 Spanning Sets

1. **(a)** For $(2,2,2)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2) + b(1,3,-1) = (2,2,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 2 \\ -2a & + & 3b & = & 2 \\ 2a & - & 1b & = & 2 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $(2,2,2)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) For $(0,4,5)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2)+b(1,3,-1)=(0,4,5)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 0 \\ -2a & + & 3b & = & 4 \\ 2a & - & 1b & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0=1$ which is contradictory. We conclude that $(0,4,5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) By inspection, the zero vector $(0,0,0)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ since

$$0(0,-2,2)+0(1,3,-1)=(0,0,0)$$

2. **(a)** For $(-9,-7,-15)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(-9,-7,-15)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a=-2$, $b=1$, $c=-2$, therefore $(-9,-7,-15)=-2\mathbf{u}+1\mathbf{v}-2\mathbf{w}$.

(b) For $(6,11,6)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(6,11,6)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a=4$, $b=-5$, $c=1$, therefore $(6,11,6)=4\mathbf{u}-5\mathbf{v}+1\mathbf{w}$.

(c) For $(0,0,0)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(0,0,0)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. There is only one solution to this system, $a=0$, $b=0$, $c=0$, therefore $(0,0,0)=0\mathbf{u}+0\mathbf{v}+0\mathbf{w}$.

3. **(a)** For $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}+b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}+c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}=\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & 6 \\ 0a & - & 1b & + & 2c & = & -8 \\ -2a & + & 2b & + & 1c & = & -1 \\ -2a & + & 3b & + & 4c & = & -8 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B, and C.

(b) The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, and C since $0A + 0B + 0C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) For $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & -1 \\ 0a & - & 1b & + & 2c & = & 5 \\ -2a & + & 2b & + & 1c & = & 7 \\ -2a & + & 3b & + & 4c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of A, B, and C.

4. **(a)** An arbitrary vector in P_2 has form $\mathbf{p} = a + bx + cx^2$ so it is in the span of $\mathbf{p}_1 = 2 + x + x^2$, $\mathbf{p}_2 = 1 - x^2$, and $\mathbf{p}_3 = 1 + 2x$ if we can solve the equation $k_1\left(2 + x + x^2\right) + k_2\left(1 - x^2\right) + k_3\left(1 + 2x\right) = a + bx + cx^2$. This can be rewritten as

$(2k_1+k_2+k_3)+(k_1+2k_3)x+(k_1-k_2)x^2=a+bx+cx^2$. Equating coefficients yields a linear system with augmented matrix $\begin{bmatrix}2&1&1&a\\1&0&2&b\\1&-1&0&c\end{bmatrix}$. The coefficient matrix $\begin{bmatrix}2&1&1\\1&0&2\\1&-1&0\end{bmatrix}$ has determinant $5\neq 0$ so we can solve the system for all possible choices of $a, b,$ and c. Therefore,

$\mathbf{p}=1+x$ is in the span of $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

(b) From part (a), $\mathbf{p}=1+x^2$ is in the span of $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

(c) From part (a), $\mathbf{p}=1+x+x^2$ is in the span of $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

5. **(a)** We need to solve the equation $k_1\begin{bmatrix}1&-1\\0&2\end{bmatrix}+k_2\begin{bmatrix}0&1\\0&1\end{bmatrix}+k_3\begin{bmatrix}0&1\\0&0\end{bmatrix}+k_4\begin{bmatrix}2&0\\1&-1\end{bmatrix}=\begin{bmatrix}1&2\\2&4\end{bmatrix}$ to express the vector $\begin{bmatrix}1&2\\2&4\end{bmatrix}$ as the desired linear combination. We can rewrite this as

$\begin{bmatrix}k_1+2k_4&-k_1+k_2+k_3\\k_4&2k_1+k_2-k_4\end{bmatrix}=\begin{bmatrix}1&2\\2&4\end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix}1&0&0&2&1\\-1&1&1&0&2\\0&0&0&1&2\\2&1&0&-1&4\end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix}1&0&0&0&-3\\0&1&0&0&12\\0&0&1&0&-13\\0&0&0&1&2\end{bmatrix}$

hence $-3\begin{bmatrix}1&-1\\0&2\end{bmatrix}+12\begin{bmatrix}0&1\\0&1\end{bmatrix}-13\begin{bmatrix}0&1\\0&0\end{bmatrix}+2\begin{bmatrix}2&0\\1&-1\end{bmatrix}=\begin{bmatrix}1&2\\2&4\end{bmatrix}$.

(b) Following part (a) we obtain a linear system whose augmented matrix is $\begin{bmatrix}1&0&0&2&3\\-1&1&1&0&1\\0&0&0&1&1\\2&1&0&-1&2\end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\end{bmatrix}$

hence $\begin{bmatrix}1&-1\\0&2\end{bmatrix}+\begin{bmatrix}0&1\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}2&0\\1&-1\end{bmatrix}=\begin{bmatrix}3&1\\1&2\end{bmatrix}$.

6. **(a)** For $-9-7x-15x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a\left(2+x+4x^2\right)+b\left(1-x+3x^2\right)+c\left(3+2x+5x^2\right)=-9-7x-15x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = -9-7x-15x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a=-2$, $b=1$, $c=-2$, therefore

$$-9-7x-15x^2 = -2\mathbf{p}_1 + 1\mathbf{p}_2 - 2\mathbf{p}_3.$$

(b) For $6+11x+6x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2+x+4x^2)+b(1-x+3x^2)+c(3+2x+5x^2)=6+11x+6x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = 6+11x+6x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a=4$, $b=-5$, $c=1$, therefore $6+11x+6x^2 = 4\mathbf{p}_1 - 5\mathbf{p}_2 + 1\mathbf{p}_3$.

(c) By inspection, $0 = 0\mathbf{p}_1 + 0\mathbf{p}_2 + 0\mathbf{p}_3$.

(d) For $7+8x+9x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2+x+4x^2)+b(1-x+3x^2)+c(3+2x+5x^2)=7+8x+9x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = 7+8x+9x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a=0$, $b=-2$, $c=3$, therefore $7+8x+9x^2 \;= 0\mathbf{p}_1 - 2\mathbf{p}_2 + 3\mathbf{p}_3$.

7. **(a)** The given vectors span R^3 if an arbitrary vector $\mathbf{b}=(b_1,b_2,b_3)$ can be expressed as a linear combination

$$(b_1,b_2,b_3)=k_1(2,2,2)+k_2(0,0,3)+k_3(0,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2k_1 & + & 0k_2 & + & 0k_3 & = & b_1 \\ 2k_1 & + & 0k_2 & + & 1k_3 & = & b_2 \\ 2k_1 & + & 3k_2 & + & 1k_3 & = & b_3 \end{array}$$

By inspection, regardless of the right hand side values b_1, b_2, b_3, the first equation can be solved for k_1, then the second equation can be used to obtain k_3, and the third would yield k_2.
We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3.

(b) The given vectors span R^3 if an arbitrary vector $\mathbf{b}=(b_1,b_2,b_3)$ can be expressed as a linear combination

$$(b_1,b_2,b_3)=k_1(2,-1,3)+k_2(4,1,2)+k_3(8,-1,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

8. **(a)** In order for the vector $(2,3,-7,3)$ to be in $\operatorname{span}\{v_1, v_2, v_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (2,3,-7,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 2 \\ 1a & - & 1b & + & 0c & = & 3 \\ 0a & + & 5b & + & 2c & = & -7 \\ 3a & + & 2b & + & 1c & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This system is consistent (its only solution is $a = 2$, $b = -1$, $c = -1$), therefore $(2,3,-7,3)$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) The vector $(0,0,0,0)$ is obviously in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ since

$$0(2,1,0,3) + 0(3,-1,5,2) + 0(-1,0,2,1) = (0,0,0,0)$$

(c) In order for the vector $(1,1,1,1)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (1,1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 1 \\ 1a & - & 1b & + & 0c & = & 1 \\ 0a & + & 5b & + & 2c & = & 1 \\ 3a & + & 2b & + & 1c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This system is inconsistent therefore $(1,1,1,1)$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d) In order for the vector $(-4,6,-13,4)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(-4,6,-13,4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & -4 \\ 1a & - & 1b & + & 0c & = & 6 \\ 0a & + & 5b & + & 2c & = & -13 \\ 3a & + & 2b & + & 1c & = & 4 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This system is consistent (its only solution is $a=3$, $b=-3$, $c=1$), therefore $(-4,6,-13,4)$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

9. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p}=a_0+a_1x+a_2x^2$ can be expressed as a linear combination

$$a_0+a_1x+a_2x^2=k_1\left(1-x+2x^2\right)+k_2\left(3+x\right)+k_3\left(5-x+4x^2\right)+k_4\left(-2-2x+2x^2\right)$$

Grouping the terms according to the powers of x yields

$$a_0+a_1x+a_2x^2=\left(k_1+3k_2+5k_3-2k_4\right)+\left(-k_1+k_2-k_3-2k_4\right)x+\left(2k_1+4k_3+2k_4\right)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcrcr} 1k_1 & + & 3k_2 & + & 5k_3 & - & 2k_4 & = & a_0 \\ -1k_1 & + & 1k_2 & - & 1k_3 & - & 2k_4 & = & a_1 \\ 2k_1 & + & 0k_2 & + & 4k_3 & + & 2k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$ therefore the system has no solution if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$.

Since polynomials $\mathbf{p} = a_0 + a_1x + a_2x^2$ for which $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$ cannot be expressed as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, we conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ do not span P_2.

10. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1(1+\mathrm{x}) + k_2(1-x) + k_3(1+x+x^2) + k_4(2-x^2)$$

Grouping the terms according to the powers of x yields

$$a_0 + a_1x + a_2x^2 = (k_1 + k_2 + k_3 + 2k_4) + (k_1 - k_2 + k_3)x + (k_3 - k_4)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 1k_2 & + & 1k_3 & + & 2k_4 & = & a_0 \\ 1k_1 & - & 1k_2 & + & 1k_3 & + & 0k_4 & = & a_1 \\ 0k_1 & + & 0k_2 & + & 1k_3 & - & 1k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 2 & a_0 \\ 1 & -1 & 1 & 0 & a_1 \\ 0 & 0 & 1 & -1 & a_2 \end{bmatrix}$ reduces

to $\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{1}{2}a_0 + \frac{1}{2}a_1 - a_2 \\ 0 & 1 & 0 & 1 & \frac{1}{2}a_0 - \frac{1}{2}a_1 \\ 0 & 0 & 1 & -1 & a_3 \end{bmatrix}$ therefore the system has a solution for every choice of a_1, a_2,

and a_3. We conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ span P_2.

11. **(a)** The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination

$k_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as

$\begin{bmatrix} k_1 + k_2 & k_2 + k_3 \\ k_1 + k_4 & k_3 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 1 & 0 & 0 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = 0$ which means the system is not consistent. We conclude that the given matrices do not span M_{22}.

(b) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_3 + k_4 & -k_1 + k_2 + k_3 \\ k_3 & k_1 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 & a \\ -1 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 1 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

(c) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_2 + k_3 + k_4 & k_2 + k_3 + k_4 \\ k_3 + k_4 & k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

12. **(a)** The vector $\mathbf{u} = (1,2)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ since $T_A(\mathbf{e}_1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T_A(\mathbf{e}_2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Observe that $-3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We conclude that $\mathbf{u} = (1,2)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

(b) The vector $\mathbf{u}=(1,2)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ since $T_A(\mathbf{e}_1)=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ and $T_A(\mathbf{e}_2)=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$. Observe that this means every vector in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ is a scalar multiple of the vector $(1,1)$. Since $\mathbf{u}=(1,2)$ not a scalar multiple of $(1,1)$ we conclude that it is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

13. **(a)** The vector $\mathbf{u}=(1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A=\begin{bmatrix}0&2\\1&-2\\1&0\end{bmatrix}$: $T_A(\mathbf{e}_1)=\begin{bmatrix}0&2\\1&-2\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $T_A(\mathbf{e}_2)=\begin{bmatrix}0&2\\1&-2\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\-2\\0\end{bmatrix}$. So there must exist scalars a and b such that $a(0,1,1)+b(2,-2,0)=(1,1,1)$.

Equating corresponding components on both sides leads to the linear system
$$\begin{aligned}0 + 2b &= 1\\ a - 2b &= 1\\ a + 0 &= 1\end{aligned}$$
which is inconsistent since subtracting the last equation from the second yields $-2b=0$ while the first equation is $2b=1$. We conclude that the vector $\mathbf{u}=(1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

(b) The vector $\mathbf{u}=(1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A=\begin{bmatrix}0&2\\1&1\\2&0\end{bmatrix}$: $T_A(\mathbf{e}_1)=\begin{bmatrix}0&2\\1&1\\2&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $T_A(\mathbf{e}_2)=\begin{bmatrix}0&2\\1&1\\2&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix}$. . So there must exist scalars a and b such that $a(0,1,2)+b(2,1,0)=(1,1,1)$. Observe that

$\frac{1}{2}(0,1,2)+\frac{1}{2}(2,1,0)=(1,1,1)$. We conclude that the vector $\mathbf{u}=(1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

14. **(a)** It follows from the trigonometric identity $\cos 2x=\cos^2 x-\sin^2 x$ that $\cos 2x$ is in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$.

(b) In order for $3+x^2$ to be in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$, there must exist scalars a and b such that
$$a\cos^2 x+b\sin^2 x=3+x^2$$
holds for all real x values. When $x=0$ the equation becomes $a=3$, however if $x=\pi$ then it yields $a=3+\pi^2$ - a contradiction. We conclude that $3+x^2$ is not in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$.

(c) It follows from the trigonometric identity $\cos^2 x+\sin^2 x=1$ that 1 is in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$.

(d) In order for $\sin x$ to be in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$, there must exist scalars a and b such that
$$a\cos^2 x+b\sin^2 x=\sin x$$

holds for all real x values. When $x=\frac{\pi}{2}$ the equation becomes $b=1$, however if $x=-\frac{\pi}{2}$ then it yields $b=-1$ - a contradiction. We conclude that $\sin x$ is not in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$.

(e) Since $0\cos^2 x+0\sin^2 x=0$ holds for all real x values, we conclude that 0 is in $\operatorname{span}\{\mathbf{f},\mathbf{g}\}$.

15. **(a)** The solution space W to the homogenous system $A\mathbf{x}=\mathbf{0}$ where $A=\begin{bmatrix}1&1&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\end{bmatrix}$. The general solution in vector form is $(x,y,z,w)=(s,t,-s,-t)=s(1,0,-1,0)+t(0,1,0,-1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1=(1,0,-1,0)$ and $\mathbf{v}_2=(0,1,0,-1)$. We conclude that the vectors $\mathbf{u}=(1,0,-1,0)$ and $\mathbf{v}=(0,1,0,-1)$ span the solution space W.

(b) From part (a) and Theorem 4.3.2 we need to show that the vectors $\mathbf{u}=(1,0,-1,0)$ and $\mathbf{v}=(1,1,-1,-1)$ are contained in the span of the vectors $\mathbf{v}_1=(1,0,-1,0)$ and $\mathbf{v}_2=(0,1,0,-1)$. Observe that $\mathbf{u}=\mathbf{v}_1$ and $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$. We conclude that the vectors $\mathbf{u}=(1,0,-1,0)$ and $\mathbf{v}=(1,1,-1,-1)$ span the solution space W.

16. **(a)** The solution space W to the homogenous system $A\mathbf{x}=\mathbf{0}$ where $A=\begin{bmatrix}0&1&-1&1\\0&2&-2&2\\0&3&-3&3\end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix}0&1&-1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution in vector form is $(x,y,z,w)=(r,s-t,s,t)=r(1,0,0,0)+s(0,1,1,0)+t(0,-1,0,1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1=(1,0,0,0),\mathbf{v}_2=(0,1,1,0)$ and $\mathbf{v}_3=(0,-1,0,1)$. Observe that $(1,0,0,0)=a(1,1,1,0)+b(0,-1,0,1)$ has no solution so that $\mathbf{v}_1$ is not in the span of the vectors $\mathbf{u}=(1,1,1,0)$ and $\mathbf{v}=(0,-1,0,1)$. By Theorem 4.3.2, they do not span the solution space W.

(b) Using part (a), observe that $(1,0,0,0)=a(0,1,1,0)+b(1,0,1,1)$ has no solution so that $\mathbf{v}_1$ is not in the span of the vectors $\mathbf{u}=(0,1,1,0)$ and $\mathbf{v}=(1,0,1,1)$. By Theorem 4.3.2, they do not span the solution space W.

17. **(a)** The vectors $T_A(1,2)=(-1,4)$ and $T_A(-1,1)=(-2,2)$ span R^2 if an arbitrary vector $\mathbf{b}=(b_1,b_2)$ can be expressed as a linear combination

$$(b_1,b_2)=k_1(-1,4)+k_2(-2,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 4k_1 & + & 2k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system is consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ span R^2.

(b) The vectors $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1, 2) + k_2(-2, 4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 2k_1 & + & 4k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ do not span R^2.

18. **(a)** The vectors $T_A(0,1,1) = (1,0)$, $T_A(2,-1,1) = (1,-2)$, and, $T_A(1,1,-2) = (2,3)$ span R^2 if an arbitrary vector

$\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(1, 0) + k_2(1, -2) + k_3(2, 3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 1k_1 & + & 1k_2 & + & 2k_3 & = & b_1 \\ 0k_1 & - & 2k_2 & + & 3k_3 & = & b_2 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$, therefore the system is consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$, $T_A(\mathbf{u}_2)$, and, $T_A(\mathbf{u}_3)$ span R^2.

(b) The vectors $T_A(0,1,1) = (1,4)$, $T_A(2,-1,1) = (-1,4)$, and, $T_A(1,1,-2) = (1,-4)$ span R^2 if an arbitrary vector

$\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(1,4) + k_2(-1,4) + k_3(1,-4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 1k_1 & - & 1k_2 & + & 1k_3 & = & b_1 \\ 4k_1 & + & 4k_2 & - & 4k_3 & = & b_2 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, therefore the system is consistent for all right hand side vectors $\mathbf{b}$. We conclude that $T_A(\mathbf{u}_1)$, $T_A(\mathbf{u}_2)$, and $T_A(\mathbf{u}_3)$ span R^2.

19. Using Theorem 4.3.2, we need to show that the each of the polynomials $\mathbf{q_1} = 2x$ and $\mathbf{q_2} = 1 + x^2$ is in the span of the polynomials $\mathbf{p_1} = 1 + x^2$ and $\mathbf{p}_2 = 1 + x + x^2$. Clearly, $\mathbf{q_2} = \mathbf{p_1}$. Observe that $2x = (-2)(1 + x^2) + 2(1 + x + x^2)$ so that $\mathbf{q_1} = (-2)\mathbf{p_1} + 2\mathbf{p_2}$.

20. We begin by showing that the vector $\mathbf{w}_1$ is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, i.e., that there exist scalars a, b, and c such that

$$a(1,6,4) + b(2,4,-1) + c(-1,2,5) = (1,-2,-5)$$

Equating corresponding components on both sides leads to the linear system

$$\begin{array}{rcrcrcr} 1a & + & 2b & - & 1c & = & 1 \\ 6a & + & 4b & + & 2c & = & -2 \\ 4a & - & 1b & + & 5c & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system is $a = -1 - t$, $b = 1 + t$, $c = t$. E.g., letting $t = 0$ yields a solution $a = -1$, $b = 1$, $c = 0$.

Applying the same procedure repeatedly to each of the remaining four vectors, we can show that

$$\mathbf{w}_1 = 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3$$

$$\mathbf{w}_2 = 2\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3$$

$$\mathbf{v}_1 = 1\mathbf{w}_1 + 1\mathbf{w}_2$$

$$\mathbf{v}_2 = 2\mathbf{w}_1 + 1\mathbf{w}_2$$

$$\mathbf{v}_3 = -1\mathbf{w}_1 + 0\mathbf{w}_2$$

It follows from Theorem 4.3.2 that the sets $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ span the same subspace of R^3.

21. For the vector $(3,5)$ to be expressed as $\mathbf{v}+\mathbf{w}$ where $\mathbf{v}$ is in the subspace spanned by $(3,1)$ and $\mathbf{w}$ is in the subspace spanned by $(2,1)$, we must produce scalars a and b such that

$a(3,1)+b(2,1)=(3,5)$. Equating corresponding components yields a linear system with augmented matrix $\begin{bmatrix} 3 & 2 & 3 \\ 1 & 1 & 5 \end{bmatrix}$ which has reduced row echelon form $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 12 \end{bmatrix}$.

Therefore $\mathbf{v} = -7(3,1) = (-21,-7)$ and $\mathbf{w} = 12(2,1) = (24,12)$.

22. For the vector $(1,0,1)$ to be expressed as $\mathbf{v}+\mathbf{w}$ where $\mathbf{v}$ is in the solution space V of

$4x - y + 2x = 0$ and $\mathbf{w}$ is in the subspace spanned by $(1,1,1)$, we first must find vectors that span V. This do this we write the augmented matrix $\begin{bmatrix} 4 & -1 & 2 \end{bmatrix}$ which has reduced row echelon form $\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$. A general solution is then $(x,y,z) = s\left(\frac{1}{4},1,0\right) + t\left(-\frac{1}{2},0,1\right)$ so that the vectors $\left(\frac{1}{4},1,0\right)$ and $\left(-\frac{1}{2},0,1\right)$ span V. We must produce scalars a,b, and c such that $a\left(\frac{1}{4},1,0\right) + b\left(-\frac{1}{2},0,1\right) + c(1,1,1) = (1,0,1)$. Equating corresponding components yields a linear system with augmented matrix $\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ which has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{6}{5} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{6}{5} \end{bmatrix}$. Therefore, $-\frac{6}{5}\left(\frac{1}{4},1,0\right) - \frac{1}{5}\left(\frac{-1}{2},0,1\right) + \frac{6}{5}(1,1,1) = (1,0,1)$.

Therefore $\mathbf{v} = -\frac{6}{5}\left(\frac{1}{4},1,0\right) - \frac{1}{5}\left(\frac{-1}{2},1,0\right) = \left(-\frac{1}{5},-\frac{6}{5},-\frac{1}{5}\right)$ and $\mathbf{w} = \left(\frac{6}{5},\frac{6}{5},\frac{6}{5}\right)$.

True False Exercises

(a) True.

(b) False. The span of the zero vector is just the zero vector.

(c) False. For example the vectors $(1,1,1)$ and $(2,2,2)$ span a line.

(d) True.

(e) True. This follows from part (a) of Theorem 4.2.1.

(f) False. For any nonzero vector $\mathbf{v}$ in a vector space V, both $\{\mathbf{v}\}$ and $\{2\mathbf{v}\}$ span the same subspace of V.

(g) False. The constant polynomial $p(x) = 1$ cannot be represented as a linear combination of these, since at $x = 1$ all three are zero, whereas $p(1) = 1$.

4.4 Linear Independence

1. **(a)** Since $\mathbf{u}_2 = -5\mathbf{u}_1$, linear dependence follows from Definition 1.

(b) A set of 3 vectors in R^2 must be linearly dependent by Theorem 4.4.3.

(c) Since $\mathbf{p}_2 = 2\mathbf{p}_1$, linear dependence follows from Definition 1.

(d) Since $A = (-1)B$, linear dependence follows from Definition 1.

2. **(a)** The vector equation $a(-3,0,4)+b(5,-1,2)+c(1,1,3)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} -3a & + & 5b & + & 1c & = & 0 \\ 0a & - & 1b & + & 1c & = & 0 \\ 4a & + & 2b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given set of vectors is linearly independent.

(b) A set of 4 vectors in R^3 must be linearly dependent by Theorem 4.4.3.

3. **(a)** The vector equation $a(3,8,7,-3)+b(1,5,3,-1)+c(2,-1,2,6)+d(4,2,6,4)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} 3a & + & 1b & + & 2c & + & 4d & = & 0 \\ 8a & + & 5b & - & 1c & + & 2d & = & 0 \\ 7a & + & 3b & + & 2c & + & 6d & = & 0 \\ -3a & - & 1b & + & 6c & + & 4d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a=-t$, $b=t$, $c=-t$, $d=t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) The vector equation $a(3,0,-3,6)+b(0,2,3,1)+c(0,-2,-2,0)+d(-2,1,2,1)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} 3a & + & 0b & + & 0c & - & 2d & = & 0 \\ 0a & + & 2b & - & 2c & + & 1d & = & 0 \\ -3a & + & 3b & - & 2c & + & 2d & = & 0 \\ 6a & + & 1b & + & 0c & + & 1d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = d = 0$. We conclude that the given set of vectors is linearly independent.

4. **(a)** The terms in the equation

$$a\left(2 - x + 4x^2\right) + b\left(3 + 6x + 2x^2\right) + c\left(2 + 10x - 4x^2\right) = 0$$

can be grouped according to the powers of x

$$\left(2a + 3b + 2c\right) + \left(-a + 6b + 10c\right)x + \left(4a + 2b - 4c\right)x^2 = 0 + 0x + 0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & 2c & = & 0 \\ -a & + & 6b & + & 10c & = & 0 \\ 4a & + & 2b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given set of vectors in P_2 is linearly independent.

(b) The terms in the equation

$$a\left(1 + 3x + 3x^2\right) + b\left(x + 4x^2\right) + c\left(5 + 6x + 3x^2\right) + d\left(7 + 2x - x^2\right) = 0$$

can be grouped according to the powers of x

$$\left(a + 5c + 7d\right) + \left(3a + b + 6c + 2d\right)x + \left(3a + 4b + 3c - d\right)x^2 = 0 + 0x + 0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcrcl} a & & & + & 5c & + & 7d & = & 0 \\ 3a & + & b & + & 6c & + & 2d & = & 0 \\ 3a & + & 4b & + & 3c & - & d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{17}{4} & 0 \\ 0 & 1 & 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 & \frac{9}{4} & 0 \end{bmatrix}$

therefore a general solution of the system is $a = \frac{17}{4}t$, $b = -\frac{5}{4}t$, $c = -\frac{9}{4}t$, $d = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

5. **(a)** The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1a & + & 1b & + & 0c & = & 0 \\ 0a & + & 2b & + & 1c & = & 0 \\ 1a & + & 2b & + & 2c & = & 0 \\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

(b) By inspection, the matrix equation $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

6. The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1a & - & 1b & + & 2c & = & 0 \\ 0a & + & 0b & + & 0c & = & 0 \\ 1a & + & kb & + & 1c & = & 0 \\ ka & + & 1b & + & 3c & = & 0 \end{array}$$

Omitting the second equation (which imposes no restrictions on the unknowns), we obtain the coefficient matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$. Performing elementary row operations

- add -1 times the first row to the second row,
- add $-k$ times the first row to the third row, and
- add -1 times the second row to the third row

yields $B=\begin{bmatrix}1 & -1 & 2\\ 0 & 1+k & -1\\ 0 & 0 & 4-2k\end{bmatrix}$. We have $\det(A)=\det(B)=(1+k)(4-2k)$ therefore by Theorem 2.3.8, the system has only the trivial solution, whenever $(1+k)(4-2k)\neq 0$.

Consequently, the given matrices are linearly independent for all k values except -1 and 2.

7. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) The vector equation $a(2,-2,0)+b(6,1,4)+c(2,0,-4)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 2a & + & 6b & + & 2c & = & 0\\ -2a & + & 1b & + & 0c & = & 0\\ 0a & + & 4b & - & 4c & = & 0\end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given vectors are linearly independent, hence they do not lie in a plane.

(b) The vector equation $a(-6,7,2)+b(3,2,4)+c(4,-1,2)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -6a & + & 3b & + & 4c & = & 0\\ 7a & + & 2b & - & 1c & = & 0\\ 2a & + & 4b & + & 2c & = & 0\end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix}1 & 0 & -\frac{1}{3} & 0\\ 0 & 1 & \frac{2}{3} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ therefore a general solution of the system is $a=\frac{1}{3}t$, $b=-\frac{2}{3}t$, $c=t$.

Since the system has nontrivial solutions, the given vectors are linearly dependent, hence they lie in a plane.

8. **(a)** The set $\{\mathbf{v}_1,\mathbf{v}_3\}$ can be shown to be linearly independent since $a(-1,2,3)+b(-3,6,0)=(0,0,0)$ has only the trivial solution $a=b=0$. Therefore the three vectors do not lie on the same line (even though the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *are* collinear).

(b) Any subset of two vectors chosen from these three vectors can be shown to be linearly independent (e.g., $a(2,-1,4)+b(4,2,3)=(0,0,0)$ has only the trivial solution $a=b=0$). Therefore the three vectors do not lie on the same line.
(An alternate way to show this would be to demonstrate that the three vectors form a linearly independent set, therefore they do not even lie on the same plane, so that they cannot possibly lie on the same line.)

(c) Each subset of two vectors chosen from these three vectors can be shown to be linearly dependent since $-1\mathbf{v}_1+2\mathbf{v}_2=\mathbf{0}$, $1\mathbf{v}_1+2\mathbf{v}_3=\mathbf{0}$, and $1\mathbf{v}_2+1\mathbf{v}_3=\mathbf{0}$. Therefore all three vectors lie on the same line.

9. **(a)** The vector equation $a(0,3,1,-1)+b(6,0,5,1)+c(4,-7,1,3)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} 0a & + & 6b & + & 4c & = & 0 \\ 3a & + & 0b & - & 7c & = & 0 \\ 1a & + & 5b & + & 1c & = & 0 \\ -1a & + & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a=\frac{7}{3}t$, $b=-\frac{2}{3}t$, $c=t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) From part (a), we have $\frac{7}{3}t\mathbf{v}_1-\frac{2}{3}t\mathbf{v}_2+t\mathbf{v}_3=0$.
Letting $t=\frac{3}{7}$, we obtain $\mathbf{v}_1=\frac{2}{7}\mathbf{v}_2-\frac{3}{7}\mathbf{v}_3$.
Letting $t=-\frac{3}{2}$, we obtain $\mathbf{v}_2=\frac{7}{2}\mathbf{v}_1+\frac{3}{2}\mathbf{v}_3$.
Letting $t=1$, we obtain $\mathbf{v}_3=-\frac{7}{3}\mathbf{v}_1+\frac{2}{3}\mathbf{v}_2$.

10. **(a)** The vector equation $a(1,2,3,4)+b(0,1,0,-1)+c(1,3,3,3)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} 1a & + & 0b & + & 1c & = & 0 \\ 2a & + & 1b & + & 3c & = & 0 \\ 3a & + & 0b & + & 3c & = & 0 \\ 4a & - & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is

$$a = -t,\ b = -t,\ c = t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) In the general solution we obtained in part (a), let the parameter t have a nonzero value, e.g., $t = 1$. Then $a = -1$, $b = -1$, and $c = 1$ so that $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. This can be solved for each of the three vectors: $\mathbf{v}_1 = -\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_2 = -\mathbf{v}_1 + \mathbf{v}_3$, and $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

11. By inspection, when $\lambda = -\frac{1}{2}$, the vectors become linearly dependent (since they all become equal). We proceed to find the remaining values of λ.

The vector equation $a\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + b\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + c\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} \lambda a & - & \frac{1}{2}b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & + & \lambda b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & - & \frac{1}{2}b & + & \lambda c & = & 0 \end{array}$$

The determinant of the coefficient matrix is $\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}$. This determinant equals zero for all λ values for which the vectors are linearly dependent. Since we already know that $\lambda = -\frac{1}{2}$ is one of those values, we can divide $\lambda + \frac{1}{2}$ into $\lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}$ to obtain

$\lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = \left(\lambda + \frac{1}{2}\right)\left(\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}\right) = \left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)(\lambda - 1)$.

We conclude that the vectors form a linearly dependent set for $\lambda = -\frac{1}{2}$ and for $\lambda = 1$.

12. By part (b) of Theorem 4.4.2, a set with one vector is linearly independent if that vector is not $\mathbf{0}$.

13. **(a)** We calculate $T_A(1,2) = (-1,4)$ and $T_A(-1,1) = (-2,2)$. The vector equation

$$k_1(-1,4) + k_2(-2,2) = (0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & 0 \\ 4k_1 & + & 2k_2 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly independent set.

(b) We calculate $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$. Since $(-2,4) = 2(-1,2)$, it follows by Definition 1 that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly dependent set.

14. (a) We calculate $T_A(1,0,0) = (1,1,2)$, $T_A(2,-1,1) = (3,-1,2)$, and $T_A(0,1,1) = (3,-3,2)$. The vector equation

$$k_1(1,1,2) + k_2(3,-1,2) + k_3(3,-3,2) = (0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1k_1 & + & 3k_2 & + & 3k_3 & = & 0 \\ 1k_1 & - & 1k_2 & - & 3k_3 & = & 0 \\ 2k_1 & + & 2k_2 & + & 2k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 3 & 3 \\ 1 & -1 & -3 \\ 2 & 2 & 2 \end{vmatrix} = -8 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent.

(b) We calculate $T_A(1,0,0) = (1,1,2)$, $T_A(2,-1,1) = (2,-2,2)$, and $T_A(0,1,1) = (2,-2,2)$. Since $T_A(\mathbf{u}_2) = 1T_A(\mathbf{u}_3)$, it follows that the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly dependent.

15. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) After the three vectors are moved so that their initial points are at the origin, the resulting vectors do not lie on the same plane. Hence these vectors are linearly independent.

(b) After the three vectors are moved so that their initial points are at the origin, the resulting vectors lie on the same plane. Hence these vectors are linearly dependent.

16. (a) From the identity $\sin^2 x + \cos^2 x = 1$ we have $(-1)(6) + (2)(3\sin^2 x) + (3)(2\cos^2 x) = 0$ for all real x. Therefore, the set is linearly dependent.

(b) The equality $ax + b\cos x = 0$ is to hold for all real x. Taking $x = 0$ yields $b = 0$, whereas taking $x = \frac{\pi}{2}$ implies $a = 0$. The set is linearly independent.

(c) The equality $(a)(1) + b\sin x + c\sin 2x = 0$ is to hold for all real x. Taking $x = 0$ yields $a = 0$. When $x = \frac{\pi}{2}$, we obtain $b = 0$. Finally, substituting $x = \frac{\pi}{4}$ results in $c = 0$. The set is linearly independent.

(d) From the identity $\cos^2 x - \sin^2 x = \cos 2x$ we have $(1)(\cos 2x) + (1)(\sin^2 x) + (-1)(\cos^2 x) = 0$ for all real x. Therefore, the set is linearly dependent.

(e) Since $(3-x)^2 = 9 - 6x + x^2$ we can write $(3-x)^2 - (x^2 - 6x) - 9 = 0$ or $(1)(3-x)^2 + (-1)(x^2 - 6x) + (-\frac{9}{5})(5) = 0$. The set is linearly dependent.

(f) From Theorem 4.4.2(a), this set is linearly dependent.

17. The Wronskian is $W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0) = -1 \neq 0$), the functions x and $\cos x$ are linearly independent.

18. The Wronskian is $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $\sin x$ and $\cos x$ are linearly independent.

19. **(a)** The Wronskian is $W(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0) = 1 \neq 0$), the functions 1, x and e^x are linearly independent.

(b) The Wronskian is $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, the functions 1, x and x^2 are linearly independent.

20.

$$W(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix}$$

← The Wronskian

$$= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix}$$

← A common factor of e^x from each row was taken through the determinant sign.

$$= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2+4x \end{vmatrix}$$

← -1 times the first row was added to the second row and to the third row.

$$= (e^{3x})(1)\begin{vmatrix} 1 & 2x \\ 2 & 2+4x \end{vmatrix}$$

← Cofactor expansion along the first column

$$=\left(e^{3x}\right)(1)(2+4x-4x)=2e^{3x}$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

21.

$$W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}.$$ ← The Wronskian

$$=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix}$$ ← The first row was added to the third.

$$=-2\sin x\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$ ← Cofactor expansion along the third row

$$=-2\sin x\left(-\sin^2 x-\cos^2 x\right)$$

$$=(-2\sin x)(-1)=2\sin x$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

True-False Exercises

(a) False. By part (b) of Theorem 4.4.2, a set containing a single *nonzero* vector is linearly independent.

(b) True. This follows directly from Definition 1.

(c) False. For instance $\{(1,1),(2,2)\}$ is a linearly dependent set that does not contain $(0,0)$.

(d) True. If $a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3=\mathbf{0}$ has only one solution $a=b=c=0$ then $a(k\mathbf{v}_1)+b(k\mathbf{v}_2)+c(k\mathbf{v}_3)=k(a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3)$ can only equal $\mathbf{0}$ when $a=b=c=0$ as well.

(e) True. Since the vectors must be nonzero, $\{\mathbf{v}_1\}$ must be linearly independent.
Let us begin adding vectors to the set until the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ becomes linearly dependent, therefore, by construction, $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$ is linearly independent. The equation $c_1\mathbf{v}_1+\cdots+c_{k-1}\mathbf{v}_{k-1}+c_k\mathbf{v}_k=\mathbf{0}$ must have a solution with $c_k\neq 0$, therefore $\mathbf{v}_k=-\frac{c_1}{c_k}\mathbf{v}_1-\cdots-\frac{c_{k-1}}{c_k}\mathbf{v}_{k-1}$. Let us assume there exists another representation $\mathbf{v}_k=d_1\mathbf{v}_1+\cdots+d_{k-1}\mathbf{v}_{k-1}$. Subtracting both sides yields $0=\left(d_1+\frac{c_1}{c_k}\right)\mathbf{v}_1+\cdots+\left(d_{k-1}+\frac{c_{k-1}}{c_k}\right)\mathbf{v}_{k-1}$. By linear independence of $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$, we must have $d_1=-\frac{c_1}{c_k}$, …, $d_{k-1}=-\frac{c_{k-1}}{c_k}$, which shows that $\mathbf{v}_k$ is a *unique* linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}$.

(f) False. The set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is linearly dependent since $\begin{bmatrix}1&1\\0&0\end{bmatrix}=(-1)\begin{bmatrix}0&0\\1&1\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\0&1\end{bmatrix}$.

(g) True. Requiring that for all x values $a(x-1)(x+2)+bx(x+2)+cx(x-1)=0$ holds true implies that the equality must be true for any specific x value. Setting $x=0$ yields $a=0$. Likewise, $x=1$ implies $b=0$, and $x=-2$ implies $c=0$. Since $a=b=c=0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, not both equal 0, such that $k_1f_1(x)+k_2f_2(x)=0$ for all real numbers x.

4.5 Coordinates and Basis

1. Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1)+c_2(3,0)=(0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b}=(b_1,b_2)$ in R^2 as

$$c_1(2,1)+c_2(3,0)=(b_1,b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & & & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & b_1 \\ c_1 & & & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix}2&3\\1&0\end{vmatrix}=-3\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independen and span R^2 so that they form a basis for R^2.

2. Vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent if the vector equation

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(0,0,0)$$

has only the trivial solution. For these vectors to span R^3, it must be possible to express every vector $\mathbf{b}=(b_1,b_2,b_3)$ in R^3 as.

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(b_1,b_2,b_3)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & 0 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & 0 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & b_1 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & b_2 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & b_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1, b_2, and b_3. Therefore the vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent and span R^3 so that they form a basis for R^3.

3. Polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent if the equation

$$c_1\left(x^2+1\right)+c_2\left(x^2-1\right)+c_3\left(2x-1\right)=0$$

has only the trivial solution. For these polynomials to span P_2, it must be possible to express every polynomial $a_0+a_1x+a_2x^2$. as

$$c_1\left(x^2+1\right)+c_2\left(x^2-1\right)+c_3\left(2x-1\right)=a_0+a_1x+a_2x^2$$

Grouping the terms on the left hand side of both equations as $\left(c_1-c_2-c_3\right)+\left(2c_3\right)x+\left(c_1+c_2\right)x^2$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & 0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & a_0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & a_1 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -4 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent and span P_2 so that they form a basis for P_2.

4. Polynomials $1+x$, $1-x$, $1-x^2$, and $1-x^3$ are linearly independent if the equation

$$c_1\left(1+x\right)+c_2\left(1-x\right)+c_3\left(1-x^2\right)+c_4\left(1-x^3\right)=0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0+a_1x+a_2x^2+a_3x^3$ as

$$c_1(1+x)+c_2(1-x)+c_3(1-x^2)+c_4(1-x^3)=a_0+a_1x+a_2x^2+a_3x^3$$

Grouping the terms on the left hand side of both equations as $(c_1+c_2+c_3+c_4)+(c_1-c_2)x-c_3x^2-c_4x^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 &+& 1c_2 &+& 1c_3 &+& 1c_4 &=& 0 \\ 1c_1 &-& 1c_2 &+& 0c_3 &+& 0c_4 &=& 0 \\ 0c_1 &+& 0c_2 &-& 1c_3 &+& 0c_4 &=& 0 \\ 0c_1 &+& 0c_2 &+& 0c_3 &-& 1c_4 &=& 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 &+& 1c_2 &+& 1c_3 &+& 1c_4 &=& a_0 \\ 1c_1 &-& 1c_2 &+& 0c_3 &+& 0c_4 &=& a_1 \\ 0c_1 &+& 0c_2 &-& 1c_3 &+& 0c_4 &=& a_2 \\ 0c_1 &+& 0c_2 &+& 0c_3 &-& 1c_4 &=& a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -2 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials $1+x$, $1-x$, $1-x^2$ and $1-x^3$ are linearly independent and span P_3 so that they form a basis for P_3.

5. Matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides yields linear systems

$$\begin{array}{rcrcrcrcl} 3c_1 &+& 0c_2 &+& 0c_3 &+& 1c_4 &=& 0 \\ 6c_1 &-& 1c_2 &-& 8c_3 &+& 0c_4 &=& 0 \\ 3c_1 &-& 1c_2 &-& 12c_3 &-& 1c_4 &=& 0 \\ -6c_1 &+& 0c_2 &-& 4c_3 &+& 2c_4 &=& 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 3c_1 &+& 0c_2 &+& 0c_3 &+& 1c_4 &=& a_{11} \\ 6c_1 &-& 1c_2 &-& 8c_3 &+& 0c_4 &=& a_{12} \\ 3c_1 &-& 1c_2 &-& 12c_3 &-& 1c_4 &=& a_{21} \\ -6c_1 &+& 0c_2 &-& 4c_3 &+& 2c_4 &=& a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix} = 48 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

6. Matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides in each equation yields linear systems

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcr} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & a_{12} \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & a_{21} \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}=1\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

7. **(a)** Vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly independent if the vector equation

$$c_1(2,-3,1)+c_2(4,1,1)+c_3(0,-7,1)=(0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 4c_2 & + & 0c_3 & = & 0 \\ -3c_1 & + & 1c_2 & - & 7c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly dependent, they do not form a basis for R^3.

(b) Vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly independent if the vector equation

$$c_1(1,6,4)+c_2(2,4,-1)+c_3(-1,2,5)=(0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 2c_2 & - & 1c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ 4c_1 & - & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly dependent, they do not form a basis for R^3.

8. Vectors $\mathbf{p}_1 = 1-3x+2x^2$, $\mathbf{p}_2 = 1+x+4x^2$, and $\mathbf{p}_3 = 1-7x$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.

By grouping the terms on the left hand side as $c_1\left(1-3x+2x^2\right)+c_2\left(1+x+4x^2\right)+c_3\left(1-7x\right)=$ $\left(c_1+c_2+c_3\right)+\left(-3c_1+c_2-7c_3\right)x+\left(2c_1+4c_2\right)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & c_3 & = & 0 \\ -3c_1 & + & c_2 & - & 7c_3 & = & 0 \\ 2c_1 & + & 4c_2 & & & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

9. Matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}+c_3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. Equating corresponding entries on both sides yields a linear system

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 2c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & - & 2c_2 & - & 1c_3 & - & 1c_4 & = & 0 \\ 1c_1 & + & 3c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & + & 2c_2 & + & 0c_3 & + & 1c_4 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent, we conclude that they do not form a basis for M_{22}.

10. **(a)** The identity $\cos^2 x - \sin^2 x = \cos 2x$ implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, therefore it is not a basis for V.

(b) For the equation $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0$ (required when $x = 0$) and $c_2 = 0$ (required when $x = \frac{\pi}{2}$). Therefore the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ are linearly independent.

Any vector $\mathbf{v}$ in V can be expressed as $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$. However, from the identity $\cos^2 x - \sin^2 x = \cos 2x$ it follows that we can express $\mathbf{v}$ as a linear combination of $\cos^2 x$ and $\sin^2 x$ alone: $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 (\cos^2 x - \sin^2 x) = (k_1 + k_3)\cos^2 x + (k_2 - k_3)\sin^2 x$. This proves that the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ span V.

We conclude that $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ form a basis for V.

(Note that $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are also bases for V.)

11. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,1) = c_1(2,-4) + c_2(3,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 1 \\ -4c_1 & + & 8c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 1c_1 & + & 0c_2 & = & a \\ 1c_1 & + & 2c_2 & = & b \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

12. (a) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ -c_1 & + & c_2 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(0,1) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -c_1 & + & c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

13. (a) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(2,-1,3) = c_1(1,0,0) + c_2(2,2,0) + c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3 \end{array}$$

which can be solved by back-substitution to obtain $c_3 = 1$, $c_2 = -2$, and $c_1 = 3$. The coordinate vector is $(\mathbf{v})_S = (3,-2,1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(5,-12,3) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & - & 4c_2 & + & 7c_3 & = & 5 \\ 2c_1 & + & 5c_2 & - & 8c_3 & = & -12 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$. The coordinate vector is $(\mathbf{v})_S = (-2,0,1)$.

14. **(a)** Since $\mathbf{p} = 4\mathbf{p}_1 + (-3)\mathbf{p}_2 + 1\mathbf{p}_3$ we conclude that the coordinate vector is $(\mathbf{p})_S = (4,-3,1)$.

(b) Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0$, $c_2 = 2$, $c_3 = -1$, therefore the coordinate vector is $(\mathbf{p})_S = (0,2,-1)$.

15. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 & k_1 + k_2 \\ k_1 + k_2 + k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{ccccccccc} k_1 & & & & & & & = & 0 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} k_1 & & & & & & & = & a \\ k_1 & + & k_2 & & & & & = & b \\ k_1 & + & k_2 & + & k_3 & & & = & c \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & & & & & & & = & 1 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 1 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array}$$

which can be solved by forward-substitution to obtain $k_1 = 1$, $k_2 = -1$, $k_3 = 1$, $k_4 = -1$.

This allows us to express $A = 1A_1 - 1A_2 + 1A_3 - 1A_4$.

The coordinate vector is $(A)_S = (1, -1, 1, -1)$.

16. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1+k_2+k_3 & k_2 \\ k_1+k_4 & k_3 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 0 \\ & & k_2 & & & & & = & 0 \\ k_1 & & & & & + & k_4 & = & 0 \\ & & & & k_3 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & a \\ & & k_2 & & & & & = & b \\ k_1 & & & & & + & k_4 & = & c \\ & & & & k_3 & & & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 6 \\ & & k_2 & & & & & = & 2 \\ k_1 & & & & & + & k_4 & = & 5 \\ & & & & k_3 & & & = & 3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ therefore the solution is $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$.

This allows us to express $A = 1A_1 + 2A_2 + 3A_3 + 4A_4$. The coordinate vector is $(A)_S = (1,2,3,4)$.

17. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1\left(1+x+x^2\right)+c_2\left(x+x^2\right)+c_3x^2 = c_1+\left(c_1+c_2\right)x+\left(c_1+c_2+c_3\right)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{ccccccc} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccc} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 7 - x + 2x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccc} c_1 & & & & & = & 7 \\ c_1 & + & c_2 & & & = & -1 \\ c_1 & + & c_2 & + & c_3 & = & 2 \end{array}$$

which can be solved by forward-substitution to obtain $c_1 = 7$, $c_2 = -8$, $c_3 = 3$.

This allows us to express $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$. The coordinate vector is $(\mathbf{p})_S = (7,-8,3)$.

18. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1\left(1+2x+x^2\right)+c_2\left(2+9x\right)+c_3\left(3+3x+4x^2\right)=\left(c_1+2c_2+3c_3\right)+\left(2c_1+9c_2+3c_3\right)x+\left(c_1+4c_3\right)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 0 \\ c_1 & & & + & 4c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & a_1 \\ c_1 & & & + & 4c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 2 + 17x - 3x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 17 \\ c_1 & & & + & 4c_3 & = & -3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ therefore the solution is $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. This allows us to express $\mathbf{p} = 1\mathbf{p}_1 + 2\mathbf{p}_2 + (-1)\mathbf{p}_3$.

The coordinate vector is $(\mathbf{p})_S = (1, 2, -1)$.

19. **(a)** The third vector is a sum of the first two. This makes the set linearly dependent, hence it cannot be a basis for R^2.

(b) The two vectors generate a plane in R^3, but they do not span all of R^3. Consequently, the set is not a basis for R^3.

(c) For instance, the polynomial $\mathbf{p} = 1$ cannot be expressed as a linear combination of the given two polynomials. This means these two polynomials do not span P_2, hence they do not form a basis for P_2.

(d) For instance, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be expressed as a linear combination of the given four matrices. This means these four matrices do not span M_{22}, hence they do not form a basis for M_{22}.

20. If the set contains at least two vectors, then the zero vector can be expressed as a scalar product of any other vector in the set and zero scalar. According to Definition 1 in Section 4.3, this makes the set linearly dependent.

A set with only one vector is linearly dependent if and only if the vector is a zero vector (see the margin note next to Definition 1 in Section 4.3).

21. **(a)** We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(1,-3,0)$. The vector equation

$$k_1(1,0,-1)+k_2(1,1,2)+k_3(1,-3,0)=(0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1k_1 & + & 1k_2 & + & 1k_3 & = & 0 \\ 0k_1 & + & 1k_2 & - & 3k_3 & = & 0 \\ -1k_1 & + & 2k_2 & + & 0k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\det(A)=10\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly independent.

(b) We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(2,1,1)$. By inspection,

$$(2,1,1)=(1,0,-1)+(1,1,2)$$

We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly dependent.

22. **(a)** Expressing $T_A(\mathbf{u})=(4,-2,0)$ as a linear combination of the vectors in S we obtain

$$(4,-2,0)=c_1(1,1,0)+c_2(0,1,1)+c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & 4 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & -2 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1=-2$, $c_2=-6$, and $c_3=6$.
The coordinate vector is $(T_A(\mathbf{u}))_S=(-2,-6,6)$.

(b) Expressing $T_A(\mathbf{u})=(-2,0,-1)$ as a linear combination of the vectors in S we obtain

$$(-2,0,-1)=c_1(1,1,0)+c_2(0,1,1)+c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & -2 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & -1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.

The solution of the linear system is $c_1 = 1$, $c_2 = 2$, and $c_3 = -3$.

The coordinate vector is $(T_A(\mathbf{u}))_S = (1,2,-3)$.

23. We have $\mathbf{u}_1 = (\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\mathbf{u}_2 = (0,1)$.

(a) By inspection, we can express $\mathbf{w} = \left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\left(\sqrt{3},1\right) = 2\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + 0(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (2,0)$.

(b) Expressing $\mathbf{w} = \left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} \tfrac{\sqrt{3}}{2}c_1 \phantom{{}+c_2} &= 1 \\ \tfrac{1}{2}c_1 + c_2 &= 0 \end{aligned}$$

The first equation yields $c_1 = \frac{2}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = -\frac{1}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

(c) By inspection, we can express $\mathbf{w} = (0,1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(0,1) = 0\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + 1(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (0,1)$.

(d) Expressing $\mathbf{w} = (a, b)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a, b) = c_1\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} \tfrac{\sqrt{3}}{2}c_1 \phantom{{}+c_2} &= a \\ \tfrac{1}{2}c_1 + c_2 &= b \end{aligned}$$

The first equation yields $c_1 = \frac{2a}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = b - \frac{a}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2a}{\sqrt{3}}, b - \frac{a}{\sqrt{3}}\right)$.

24. **(a)** $\left(0, \sqrt{2}\right)$; **(b)** $(1,0)$; **(c)** $\left(-1, \sqrt{2}\right)$; **(d)** $\left(a-b, \sqrt{2}b\right)$

25. **(a)** Polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent if the equation

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = 0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0 + a_1t + a_2t^2 + a_3t^3$ as

$$c_1(1) + c_2(2t) + c_3\left(-2+4t^2\right) + c_4\left(-12t+8t^3\right) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Grouping the terms on the left hand side of both equations as
$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3t^2 + 8c_4t^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & a_0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 64 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent and span P_3 so that they form a basis for P_3.

(b) To express $\mathbf{p} = -1 - 4t + 8t^2 + 8t^3$ as a linear combination of the four vectors in B, we form the nonhomogeneous system as was done in part (a), with the appropriate right hand side values

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & -1 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & -4 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 8 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 8 \end{array}$$

Back-substitution yields $c_4 = 1$, $c_3 = 2$, $c_2 = 4$, and $c_1 = 3$.
The coordinate vector is $(\mathbf{p})_B = (3,4,2,1)$.

26. **(b)** $(\mathbf{p})_B = (2,-8,0,1)$

27. **(a)** $\mathbf{w} = 6(3,1,-4) - 1(2,5,6) + 4(1,4,8) = (20,17,2)$

(b) $\mathbf{q} = 3(x^2+1) + 0(x^2-1) + 4(2x-1) = 3x^2 + 8x - 1$

(c) $B = -8\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -21 & -103 \\ -106 & 30 \end{bmatrix}$

True-False Exercises

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.5.1.

(d) True. For any vector $\mathbf{v} = (a_1, \ldots, a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1, \ldots, a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4.6 Dimension

1. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = t$, $x_2 = 0$, $x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (t, 0, t) = t(1,0,1)$$

therefore the solution space is spanned by a vector $\mathbf{v}_1 = (1,0,1)$. This vector is nonzero, therefore it forms a linearly independent set (Theorem 4.4.2(b)). We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

2. The augmented matrix of the linear system $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s$, $x_2 = -\frac{1}{4}s - t$, $x_3 = s$, $x_4 = t$. In vector form

$$(x_1,x_2,x_3,x_4) = \left(-\tfrac{1}{4}s,-\tfrac{1}{4}s-t,s,t\right) = s\left(-\tfrac{1}{4},-\tfrac{1}{4},1,0\right)+t(0,-1,0,1)$$

therefore the solution space is spanned by vectors $\mathrm{v}_1 = \left(-\tfrac{1}{4},-\tfrac{1}{4},1,0\right)$ and $\mathrm{v}_2 = (0,-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

3. The augmented matrix of the linear system $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The only solution is $x_1 = x_2 = x_3 = 0$.

The solution space has no basis - its dimension is 0.

4. The augmented matrix of the linear system $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 4r-3s+t$, $x_2 = r$, $x_3 = s$, $x_4 = t$. In vector form

$$(x_1,x_2,x_3,x_4) = (4r-3s+t,r,s,t) = r(4,1,0,0)+s(-3,0,1,0)+t(1,0,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (4,1,0,0)$, $\mathbf{v}_2 = (-3,0,1,0)$, and $\mathbf{v}_3 = (1,0,0,1)$. By inspection, these vectors are linearly independent since $r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$ implies $r = s = t = 0$. We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for the solution space and that the dimension of the solution space is 3.

5. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 3s-t$, $x_2 = s$, $x_3 = t$. In vector form

$$(x_1,x_2,x_3) = (3s-t,s,t) = s(3,1,0)+t(-1,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (3,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

6. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x = 4t$, $y = -5t$, $z = t$. In vector form

$$(x, y, z) = (4t, -5t, t) = t(4, -5, 1)$$

therefore the solution space is spanned by vector $\mathbf{v}_1 = (4, -5, 1)$. By Theorem 4.4.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

7. **(a)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = \frac{2}{3}s - \frac{5}{3}t$. Expressing the solution in vector form $(x, y, z) = \left(\frac{2}{3}s - \frac{5}{3}t, s, t\right) = s\left(\frac{2}{3}, 1, 0\right) + t\left(-\frac{5}{3}, 0, 1\right)$. By Theorem 4.4.2(c), $\left\{\left(\frac{2}{3}, 1, 0\right), \left(-\frac{5}{3}, 0, 1\right)\right\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\left\{\left(\frac{2}{3}, 1, 0\right), \left(-\frac{5}{3}, 0, 1\right)\right\}$. The dimension of the subspace is 2.

(b) If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = s$. Expressing the solution in vector form $(x, y, z) = (s, s, t) = s(1, 1, 0) + t(0, 0, 1)$. By Theorem 4.4.2(c), $\{(1, 1, 0), (0, 0, 1)\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\{(1, 1, 0), (0, 0, 1)\}$. The dimension of the subspace is 2.

(c) In vector form, $(x, y, z) = (2t, -t, 4t) = t(2, -1, 4)$. By Theorem 4.4.2(b), the vector $(2, -1, 4)$ forms a linearly independent set since it is not the zero vector. A basis for the subspace is $\{(2, -1, 4)\}$. The dimension of the subspace is 1.

(d) The subspace contains all vectors $(a, a + c, c) = a(1, 1, 0) + c(0, 1, 1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1, 1, 0), (0, 1, 1)\}$. By Theorem 4.4.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

8. **(a)** The given subspace can be expressed as $\text{span}(S)$ where $S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a, b, a + b, a - b) = a(1, 0, 1, 1) + b(0, 1, 1, -1)$ thus we can express it as $\text{span}(S)$ where $S = \{(1, 0, 1, 1), (0, 1, 1, -1)\}$. By Theorem 4.4.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a,a,a,a) = a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,1,1,1)\}$. By Theorem 4.4.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

9. **(a)** Let W be the space of all diagonal $n \times n$ matrices. We can write

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_2} + \cdots + d_n \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A_n}$$

The matrices $A_1,\dots,A_n$ are linearly independent and they span W; hence, $A_1,\dots,A_n$ form a basis for W. Consequently, the dimension of W is n.

(b) A basis for this space can be constructed by including the n matrices $A_1,\dots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1 = \frac{n(n-1)}{2}$ matrices B_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) and (j,i) entries, which are both 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_{12}}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B_{13}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_{23}}$$

The dimension is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

(c) A basis for this space can be constructed by including the n matrices $A_1,\dots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1 = \frac{n(n-1)}{2}$ matrices C_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) entry, which is 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{12}}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{13}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{23}}$$

The dimension is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

10. The given subspace can be expressed as $\text{span}(S)$ where $S=\{x,x^2,x^3\}$ is a set of linearly independent vectors in P_3. Therefore S forms a basis for the subspace. The dimension of the subspace is 3.

11. **(a)** W is the set of all polynomials $a_0+a_1x+a_2x^2$ for which $a_0+a_1+a_2=0$, i.e. all polynomials that can be expressed in the form $-a_1-a_2+a_1x+a_2x^2$.

Adding two polynomials in W results in another polynomial in W

$\left(-a_1-a_2+a_1x+a_2x^2\right)+\left(-b_1-b_2+b_1x+b_2x^2\right)$

$=\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2$

since we have $\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)+\left(a_2+b_2\right)=0$.

Likewise, a scalar multiple of a polynomial in W is also in W

$k\left(-a_1-a_2+a_1x+a_2x^2\right)=-ka_1-ka_2+ka_1x+ka_2x^2$

since it meets the condition $\left(-ka_1-ka_2\right)+\left(ka_1\right)+\left(ka_2\right)=0$.

According to Theorem 4.2.1, W is a subspace of P_2.

(c) From part (a), an arbitrary polynomial in W can be expressed in the form

$$-a_1-a_2+a_1x+a_2x^2=a_1\left(-1+x\right)+a_2\left(-1+x^2\right)$$

therefore, the polynomials $-1+x$ and $-1+x^2$ span W. Also, $a_1\left(-1+x\right)+a_2\left(-1+x^2\right)=0$ implies $a_1=a_2=0$, so $-1+x$ and $-1+x^2$ are linearly independent, hence they form a basis for W. The dimension of W is 2.

12. **(a)** Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix}-1 & 1 & 1\\ 2 & -2 & 0\\ 3 & -2 & 0\end{vmatrix}=2\neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix}1 & 3 & 1\\ -1 & 1 & 0\\ 0 & -2 & 0\end{vmatrix}=2\neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

13. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{e}_1+k_4\mathbf{e}_2+k_5\mathbf{e}_3+k_6\mathbf{e}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcccccccccc} k_1 & - & 3k_2 & + & k_3 & & & & & & & = & 0\\ -4k_1 & + & 8k_2 & & & + & k_4 & & & & & = & 0\\ 2k_1 & - & 4k_2 & & & & & + & k_5 & & & = & 0\\ -3k_1 & + & 6k_2 & & & & & & & + & k_6 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$.

Based on the leading entries in the first, second, fourth, and fifth columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent. Since $\dim\left(R^4\right) = 4$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^4. (The answer is not unique.)

14. The equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0$ implies $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 0$, i.e., $(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$, which by linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ requires that

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Solving this system by back-substitution yields $c_1 = c_2 = c_3 = 0$ therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Since the dimension of V is 3 (as its basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains three vectors), by Theorem 4.6.4 $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must also be a basis for V.

15. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcl} k_1 & & & + & k_3 & & & & & = & 0 \\ -2k_1 & + & 5k_2 & & & + & k_4 & & & = & 0 \\ 3k_1 & - & 3k_2 & & & & & + & k_5 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} & 0 \end{bmatrix}$.

Based on the leading entries in the first three columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$).

Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ are linearly independent. Since $\dim(R^3)=3$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ form a basis for R^3. (The answer is not unique.)

16. One of the infinitely many ways to enlarge the given set to a basis for R^4 is by adding the vectors $(0,0,1,0)$ and $(0,0,0,1)$ to the set. Since the resulting set contains $\dim(R^4)=4$ vectors, by Theorem 4.6.4 we only need to establish the linear independence of the set to be able to conclude that it forms a basis for R^4. The homogeneous equation $k_1(1,0,0,0)+k_2(1,1,0,0)+k_3(0,0,1,0)+k_4(0,0,0,1)=(0,0,0,0)$ can be rewritten as a linear system whose coefficient matrix $\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ has determinant 1. Using parts (b) and (g) of Theorem 2.3.8, we conclude that there is only the trivial solution, therefore the enlarged set of four vectors is linearly independent (and, consequently, forms a basis for R^4).

17. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcl}1k_1&+&1k_2&+&2k_3&+&0k_4&=&0\\0k_1&+&0k_2&+&0k_3&+&0k_4&=&0\\0k_1&+&1k_2&+&1k_3&-&1k_4&=&0\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1&0&1&1&0\\0&1&1&-1&0\\0&0&0&0&0\end{bmatrix}$.

For arbitrary values of s and t, we have $k_1=-s-t$, $k_2=-s+t$, $x_3=s$, $k_4=t$.
Letting $s=1$ and $t=0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3=\mathbf{v}_1+\mathbf{v}_2$.
Letting $s=0$ and $t=1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4=\mathbf{v}_1-\mathbf{v}_2$.
By part (b) of Theorem 4.6.3, $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}=\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2=\mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. (The answer is not unique.)

18. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 0k_2 & + & 3k_3 & + & 4k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -3s-4t$, $k_2 = \frac{3}{2}s+\frac{1}{2}t$, $k_3 = s$, $k_4 = t$.
Letting $s=1$ and $t=0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = 3\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2$.
Letting $s=0$ and $t=1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = 4\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$.
By part (b) of Theorem 4.6.3, $\text{span}\{\mathbf{v}_1,\mathbf{v}_2\} = \text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. (The answer is not unique.)

19. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -t$, $x_2 = t$, $x_3 = t$. In vector form, $(x_1,x_2,x_3) = (-t,t,t) = t(-1,1,1)$. Since $\{(-1,1,1)\}$ is a basis for the space, the dimension is 1.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -2s$, $x_2 = s$, $x_3 = t$. In vector form, $(x_1,x_2,x_3) = (-2s,s,t) = s(-2,1,0) + t(0,0,1)$. Since $\{(-2,1,0),(0,0,1)\}$ is a basis for the space, the dimension is 2.

(c) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = 0$, $x_2 = -t$, $x_3 = t$. In vector form, $(x_1,x_2,x_3) = (0,-t,t) = t(0,-1,1)$. Since $\{(0,-1,1)\}$ is a basis for the space, the dimension is 1.

20. The space of all vectors $\mathbf{x}=(x_1,x_2,x_3,x_4)$ for which $T_A(\mathbf{x})=\mathbf{0}$ is the solution space of $A\mathbf{x}=\mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix}1&0&2&-1\\0&1&\frac{1}{2}&-\frac{1}{4}\end{bmatrix}$ so $x_1=-2s+t$, $x_2=-\frac{1}{2}s+\frac{1}{4}t$, $x_3=s$, $x_4=t$. In vector form,

$(x_1,x_2,x_3,x_4)=(-2s+t,-\frac{1}{2}s+\frac{1}{4}t,s,t)=s(-2,-\frac{1}{2},1,0)+t(1,\frac{1}{4},0,1)$.

Since $\{(-2,-\frac{1}{2},1,0),(1,\frac{1}{4},0,1)\}$ is a basis for the space, the dimension is 2.

(b) The reduced row echelon form of A is $\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{bmatrix}$ so $x_1=x_2=x_3=-t$, $x_4=t$. In vector form,

$(x_1,x_2,x_3,x_4)=(-t,-t,-t,t)=t(-1,-1,-1,1)$.

Since $\{(-1,-1,-1,1)\}$ is a basis for the space, the dimension is 1.

27. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S=\{1,x,x^2\}$.

(a) Denote $\mathbf{v}_1=-1+x-2x^2$, $\mathbf{v}_2=3+3x+6x^2$, $\mathbf{v}_3=9$.

Then $(\mathbf{v}_1)_S=(-1,1,-2)$, $(\mathbf{v}_2)_S=(3,3,6)$, $(\mathbf{v}_3)_S=(9,0,0)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix

$\begin{bmatrix}-1&3&9&0\\1&3&0&0\\-2&6&0&0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 22, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2)=3$, from Theorem 4.6.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1=1+x$, $\mathbf{v}_2=x^2$, $\mathbf{v}_3=2+2x+3x^2$.

Then $(\mathbf{v}_1)_S=(1,1,0)$, $(\mathbf{v}_2)_S=(0,0,1)$, $(\mathbf{v}_3)_S=(2,2,3)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix

$\begin{bmatrix}1&0&2&0\\1&0&2&0\\0&1&3&0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1&0&2&0\\0&1&3&0\\0&0&0&0\end{bmatrix}$.

This yields solutions $k_1=-2t$, $k_2=-3t$, $k_3=t$. Taking $t=1$, we can express $(\mathbf{v}_3)_S$ as a linear combination of $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$: $(\mathbf{v}_3)_S=2(\mathbf{v}_1)_S+3(\mathbf{v}_2)_S$ - the same relationship holds true for the vectors themselves: $\mathbf{v}_3=2\mathbf{v}_1+3\mathbf{v}_2$. By part (b) of Theorem 4.6.3,

$\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}=\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

Based on the leading entries in the first two columns, the vector equation $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) Clearly, $1 + x - 3x^2 = \frac{1}{2}(2 + 2x - 6x^2) = \frac{1}{3}(3 + 3x - 9x^2)$ therefore from Theorem 4.6.3(b), the subspace is spanned by $1 + x - 3x^2$. By Theorem 4.4.2(b), a set containing a single nonzero vector is linearly independent.

We conclude that $1 + x - 3x^2$ forms a basis for this subspace of P_2.

True-False Exercises

(a) True.

(b) True. For instance, $\mathbf{e}_1, \ldots, \mathbf{e}_{17}$.

(c) False. This follows from Theorem 4.6.2(b).

(d) True. This follows from Theorem 4.6.4.

(e) True. This follows from Theorem 4.6.4.

(f) True. This follows from Theorem 4.6.5(a).

(g) True. This follows from Theorem 4.6.5(b).

(h) True. For instance, invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for M_{22}.

(i) True. The set has $n^2 + 1$ matrices, which exceeds $\dim(M_{nn}) = n^2$.

(j) False. This follows from Theorem 4.6.6(c).

(k) False. For instance, for any constant c, $\text{span}\{x - c, x^2 - c^2\}$ is a two-dimensional subspace of P_2 consisting of all polynomials in P_2 for which $p(c) = 0$. Clearly, there are infinitely many different subspaces of this type.

4.7 Change of Basis

1. **(a)** In this part, B' is the start basis and B is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{10} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$$

The transition matrix is $P_{B'\to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2c_1 + 4c_2 &= 3 \\ 2c_1 - c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{17}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{17}{10}$, $c_2 = \frac{8}{5}$, therefore the coordinate vector is $\left[\mathbf{w}\right]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$.

Using Formula (12), $\left[\mathbf{w}\right]_{B'} = P_{B\to B'}\left[\mathbf{w}\right]_B = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} c_1 - c_2 &= 3 \\ 3c_1 - c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$. The solution of the linear system is $c_1 = -4$, $c_2 = -7$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$. This matches the result obtained in part (c).

2. **(a)** In this part, B' is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right] = [I \mid \text{transition from start to end}]$$

No row operations were necessary to obtain the transition matrix $P_{B' \to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Clearly, $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Using Formula (12), $[\mathbf{w}]_{B'} = P_{B \to B'} [\mathbf{w}]_B = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$ and $\mathbf{u}_2'$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2c_1 - 3c_2 &= 3 \\ c_1 + 4c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{3}{11} \\ 0 & 1 & -\frac{13}{11} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{3}{11}$, $c_2 = -\frac{13}{11}$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

This matches the result obtained in part (c).

3. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 2c_2 & + & c_3 & = & -5 \\ c_1 & - & c_2 & + & 2c_3 & = & 8 \\ c_1 & + & c_2 & + & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{bmatrix}$. The solution of the linear system is $c_1 = 9$, $c_2 = -9$, $c_3 = -5$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ we obtain

$$\begin{bmatrix}-5\\8\\-5\end{bmatrix} = c_1\begin{bmatrix}3\\1\\-5\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-3\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\2\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 3c_1 + c_2 - c_3 &= -5 \\ c_1 + c_2 \phantom{{}-c_3} &= 8 \\ -5c_1 - 3c_2 + 2c_3 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & -\frac{7}{2}\\ 0 & 1 & 0 & \frac{23}{2}\\ 0 & 0 & 1 & 6\end{bmatrix}$.

The solution of the linear system is $c_1 = -\frac{7}{2}$, $c_2 = \frac{23}{2}$, $c_3 = 6$ therefore the coordinate vector is

$[\mathbf{w}]_{B'} = \begin{bmatrix}-\frac{7}{2}\\ \frac{23}{2}\\ 6\end{bmatrix}$, which matches the result we obtained in part (b).

4. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{ccc|ccc}-6 & -2 & -2 & -3 & -3 & 1\\ -6 & -6 & -3 & 0 & 2 & 6\\ 0 & 4 & 7 & -3 & -1 & -1\end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{12}\\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12}\\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3}\end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix}\frac{3}{4} & \frac{3}{4} & \frac{1}{12}\\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12}\\ 0 & \frac{2}{3} & \frac{2}{3}\end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix}-5\\8\\-5\end{bmatrix} = c_1\begin{bmatrix}-3\\0\\-3\end{bmatrix} + c_2\begin{bmatrix}-3\\2\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\6\\-1\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -3c_1 & - & 3c_2 & + & c_3 & = & -5 \\ & & 2c_2 & + & 6c_3 & = & 8 \\ -3c_1 & - & c_2 & - & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = 1$, $c_2 = 1$, $c_3 = 1$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -6c_1 & - & 2c_2 & - & 2c_3 & = & -5 \\ -6c_1 & - & 6c_2 & - & 3c_3 & = & 8 \\ & & 4c_2 & + & 7c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{19}{12} \\ 0 & 1 & 0 & -\frac{43}{12} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$.

The solution of the linear system is $c_1 = \frac{19}{12}$, $c_2 = -\frac{43}{12}$, $c_3 = \frac{4}{3}$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$, which matches the result we obtained in part (b).

5. **(a)** The set $\{\mathbf{f}_1, \mathbf{f}_2\}$ is linearly independent since neither vector is a scalar multiple of the other. Thus $\{\mathbf{f}_1, \mathbf{f}_2\}$ is a basis for V and $\dim(V) = 2$.

Likewise, the set $\{\mathbf{g}_1, \mathbf{g}_2\}$ of vectors in V is linearly independent since neither vector is a scalar multiple of the other. By Theorem 4.6.4, $\{\mathbf{g}_1, \mathbf{g}_2\}$ is a basis for V.

(b) Clearly, $[\mathbf{g}_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $[\mathbf{g}_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ hence $P_{B'\to B} = \left[[\mathbf{g}_1]_B \mid [\mathbf{g}_2]_B\right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

(c) We find the two columns of the transitions matrix $P_{B \to B'} = \left[[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'} \right]$

$$\mathbf{f}_1 = a_1\mathbf{g}_1 + a_2\mathbf{g}_2 \qquad \mathbf{f}_2 = b_1\mathbf{g}_1 + b_2\mathbf{g}_2$$

$$\sin x = a_1(2\sin x + \cos x) + a_2(3\cos x) \qquad \cos x = b_1(2\sin x + \cos x) + b_2(3\cos x)$$

equate the coefficients corresponding to the same function on both sides of each equation

$$\begin{array}{rcrcl} 2a_1 & & & = & 1 \\ a_1 & + & 3a_2 & = & 0 \end{array} \qquad \begin{array}{rcrcl} 2b_1 & & & = & 0 \\ b_1 & + & 3b_2 & = & 1 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

We obtain the transition matrix $P_{B \to B'} = \left[[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'} \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

(An alternate way to solve this part is to use Theorem 4.7.1 to yield

$$P_{B \to B'} = P_{B' \to B}^{-1} = \left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \right)^{-1} = \frac{1}{(2)(3)-(0)(1)} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.)$$

(d) Clearly, the coordinate vector is $[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{h}]_{B'} = P_{B \to B'}[\mathbf{h}]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(e) By inspection, $2\sin x - 5\cos x = (2\sin x + \cos x) - 2(3\cos x)$, hence the coordinate vector is $[\mathbf{p}]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, which matches the result obtained in part (d).

6. **(a)** We find the two columns of the transitions matrix $P_{B' \to B} = \left[[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B \right]$

$$\mathbf{q}_1 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2 \qquad \mathbf{q}_2 = b_1\mathbf{p}_1 + b_2\mathbf{p}_2$$

$$2 = a_1(6+3x) + a_2(10+2x) \qquad 3+2x = b_1(6+3x) + b_2(10+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{array}{rcrcl} 6a_1 & + & 10a_2 & = & 2 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \qquad \begin{array}{rcrcl} 6b_1 & + & 10b_2 & = & 3 \\ 3b_1 & + & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\left[\begin{array}{ccc}1 & 0 & -\frac{2}{9} \\ 0 & 1 & \frac{1}{3}\end{array}\right] \qquad \left[\begin{array}{ccc}1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{1}{6}\end{array}\right]$$

We obtain the transition matrix $P_{B'\to B} = \left[[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B\right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$.

(b) We find the two columns of the transitions matrix $P_{B\to B'} = \left[[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}\right]$

$$\mathbf{p}_1 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 \qquad \mathbf{p}_2 = b_1\mathbf{q}_1 + b_2\mathbf{q}_2$$

$$6+3x = a_1(2) + a_2(3+2x) \qquad 10+2x = b_1(2) + b_2(3+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{array}{rcrcl} 2a_1 & + & 3a_2 & = & 6 \\ & & 2a_2 & = & 3 \end{array} \qquad \begin{array}{rcrcl} 2b_1 & + & 3b_2 & = & 10 \\ & & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\left[\begin{array}{ccc}1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{3}{2}\end{array}\right] \qquad \left[\begin{array}{ccc}1 & 0 & \frac{7}{2} \\ 0 & 1 & 1\end{array}\right]$$

We obtain the transition matrix $P_{B\to B'} = \left[[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}\right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$.

(c) Since $-4+x = (6+3x)-(10+2x)$, the coordinate vector is $[\mathbf{p}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{p}]_{B'} = P_{B\to B'}[\mathbf{p}]_B = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$.

(d) We are looking for the coordinate vector $[\mathbf{p}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ with c_1 and c_2 satisfying the equality

$$-4+x = c_1(2) + c_2(3+2x)$$

for all real values x. Equating the coefficients associated with like powers of x on both sides yields the linear system

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & -4 \\ & & 2c_2 & = & 1 \end{array}$$

which can easily be solved by back-substitution: $c_2 = \frac{1}{2}$, $c_1 = \frac{-4-3\left(\frac{1}{2}\right)}{2} = -\frac{11}{4}$. We conclude that $[\mathbf{p}]_{B'} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$, which matches the result obtained in part (c).

7. **(a)** In this part, B_2 is the start basis and B_1 is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right].$$

The transition matrix is $P_{B_2 \to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$.

(b) In this part, B_1 is the start basis and B_2 is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right].$$

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(c) Since $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & 2c_2 & = & 0 \\ 2c_1 & + & 3c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 2$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_2} = P_{B_1 \to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain

$$\begin{bmatrix}2\\5\end{bmatrix}=c_1\begin{bmatrix}1\\3\end{bmatrix}+c_2\begin{bmatrix}1\\4\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 1c_1 & + & 1c_2 & = & 2 \\ 3c_1 & + & 4c_2 & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 3\\0 & 1 & -1\end{bmatrix}$. The solution of the linear system is $c_1=3$, $c_2=-1$, therefore the coordinate vector is $[\mathbf{w}]_{B_2}=\begin{bmatrix}3\\-1\end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_1}=P_{B_2\to B_1}[\mathbf{w}]_{B_2}=\begin{bmatrix}3 & 5\\-1 & -2\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix}$.

8. **(a)** By Theorem 4.7.2, $P_{B\to S}=\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis: $[\text{end basis} \mid \text{start basis}]=\left[\begin{array}{rr|rr}2 & -3 & 1 & 0\\1 & 4 & 0 & 1\end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}]=\left[\begin{array}{rr|rr}1 & 0 & \frac{4}{11} & \frac{3}{11}\\0 & 1 & -\frac{1}{11} & \frac{2}{11}\end{array}\right]$.

The transition matrix is $P_{S\to B}=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\-\frac{1}{11} & \frac{2}{11}\end{bmatrix}$.

(c) Since $\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\-\frac{1}{11} & \frac{2}{11}\end{bmatrix}\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix}\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\-\frac{1}{11} & \frac{2}{11}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Since $(5,-3)=(2,1)-(-3,4)$ the coordinate vector is $[\mathbf{w}]_B=\begin{bmatrix}1\\-1\end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_S=P_{B\to S}[\mathbf{w}]_B=\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}5\\-3\end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S=\begin{bmatrix}3\\-5\end{bmatrix}$. From Formula (12), $[\mathbf{w}]_B=P_{S\to B}[\mathbf{w}]_S=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\-\frac{1}{11} & \frac{2}{11}\end{bmatrix}\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}-\frac{3}{11}\\-\frac{13}{11}\end{bmatrix}$.

9. **(a)** By Theorem 4.7.2, $P_{B\to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis:

$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$.

The transition matrix is $P_{S\to B} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

(c) Since $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 5c_2 + 3c_3 &= -3 \\ c_1 + 8c_3 &= 1 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -239 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{bmatrix}$. The solution of the linear system is $c_1 = -239$, $c_2 = 77$, $c_3 = 30$ therefore the coordinate vector is

$[\mathbf{w}]_B = \begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix}$. From Formula (12), $[\mathbf{w}]_S = P_{B\to S}[\mathbf{w}]_B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -200 \\ 64 \\ 25 \end{bmatrix}$.

10. Reflecting $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ about the line $y = x$ results in $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Likewise for $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we obtain $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

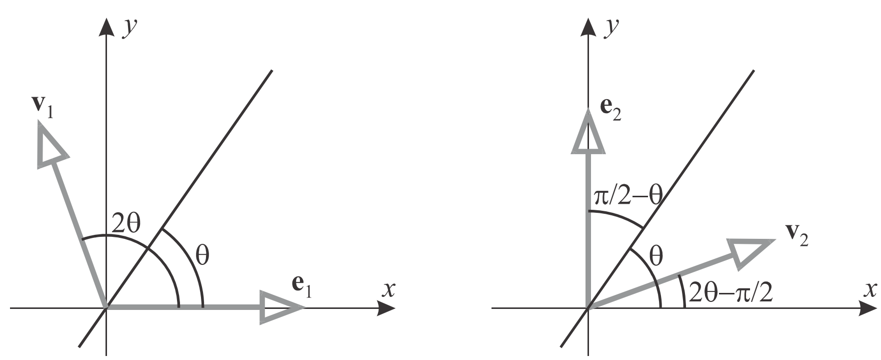
(a) From Theorem 4.7.5, $P_{B\to S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Denoting $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it follows from Theorem 4.7.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

11. **(a)** Clearly, $\mathbf{v}_1 = (\cos(2\theta), \sin(2\theta))$. Referring to the figure on the right, we see that the angle between the positive x-axis and $\mathbf{v}_2$ is $\frac{\pi}{2} - 2\left(\frac{\pi}{2} - \theta\right) = 2\theta - \frac{\pi}{2}$. Hence,

$$\mathbf{v}_2 = \left(\cos\left(2\theta - \tfrac{\pi}{2}\right), \sin\left(2\theta - \tfrac{\pi}{2}\right)\right) = (\sin(2\theta), -\cos(2\theta))$$

From Theorem 4.7.5, $P_{B\to S} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.



(b) Denoting $P = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$, it follows from Theorem 4.7.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

12. Since for every vector $\mathbf{v}$ in R^2 we have $[\mathbf{v}]_{B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ and $[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}[\mathbf{v}]_{B_2}$, it follows that

$[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ so that $P_{B_1 \to B_3} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}$.

From Theorem 4.7.1, $P_{B_3 \to B_1}$ is the inverse of this matrix: $\begin{bmatrix} -\frac{2}{15} & \frac{11}{15} \\ \frac{7}{15} & -\frac{31}{15} \end{bmatrix}$.

13. Since for every vector $\mathbf{v}$ we have $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_C = Q[\mathbf{v}]_B$, it follows that $[\mathbf{v}]_C = QP[\mathbf{v}]_{B'}$ so that $P_{B' \to C} = QP$. From Theorem 4.7.1, $P_{C \to B'} = (QP)^{-1} = P^{-1}Q^{-1}$.

15. **(a)** By Theorem 4.7.2, P is the transition matrix from $B = \{(1,1,0),(1,0,2),(0,2,1)\}$ to S.

(b) By Theorem 4.7.1, $P^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ is the transition matrix from B to S, hence by Theorem 4.7.2, $B = \left\{\left(\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}\right), \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}\right), \left(-\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right\}$.

16. Let the given basis be denoted as $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{v}_3 = (1,0,0)$ and denote the unknown basis as $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We have $P_{B \to B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \left[[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid [\mathbf{u}_3]_{B'} \right]$. Equating the respective columns yields

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = (1,1,1)$$

$$[\mathbf{u}_2]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_2 = 0\mathbf{v}_1 + 3\mathbf{v}_2 + 1\mathbf{v}_3 = (4,3,0)$$

$$[\mathbf{u}_3]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_3 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = (3,2,0)$$

Thus the given matrix is the transition matrix from the basis $\{(1,1,1),(4,3,0),(3,2,0)\}$.

17. From $T(1,0) = (2,5)$, $T(0,1) = (3,-1)$, and Theorem 4.7.2 we obtain $P_{B \to S} = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$.

18. From $T(1,0,0)=(1,2,0)$, $T(0,1,0)=(1,-1,1)$, $T(0,0,1)=(0,4,3)$, and Theorem 4.7.2 we obtain $P_{B\to S}=\begin{bmatrix}1 & 1 & 0\\ 2 & -1 & 4\\ 0 & 1 & 3\end{bmatrix}$.

19. By Formula (10), the transition matrix from the standard basis $S=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ to B is $P_{S\to B}=\left[[\mathbf{e}_1]_B|\ldots|[\mathbf{e}_n]_B\right]=\left[\mathbf{e}_1|\ldots|\mathbf{e}_n\right]=I_n$ therefore B must be the standard basis.

True-False Exercises

(a) True. The matrix can be constructed according to Formula (10).

(b) True. This follows from Theorem 4.7.1.

(c) True.

(d) True.

(e) False. For instance, $B_1=\{(0,2),(3,0)\}$ is a basis for R^2 made up of scalar multiples of vectors in the standard basis $B_2=\{(1,0),(0,1)\}$. However, $P_{B_1\to B_2}=\begin{bmatrix}0 & 3\\ 2 & 0\end{bmatrix}$ (obtained by Theorem 4.7.2) is not a diagonal matrix.

(f) False. A must be invertible.

4.8 Row Space, Column Space, and Null Space

1. **(a)** $\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=1\begin{bmatrix}2\\ -1\end{bmatrix}+2\begin{bmatrix}3\\ 4\end{bmatrix}$

(b) $\begin{bmatrix}4 & 0 & -1\\ 3 & 6 & 2\\ 0 & -1 & 4\end{bmatrix}\begin{bmatrix}-2\\ 3\\ 5\end{bmatrix}=-2\begin{bmatrix}4\\ 3\\ 0\end{bmatrix}+3\begin{bmatrix}0\\ 6\\ -1\end{bmatrix}+5\begin{bmatrix}-1\\ 2\\ 4\end{bmatrix}$

2. **(a)** $\begin{bmatrix}-3 & 6 & 2\\ 5 & -4 & 0\\ 2 & 3 & -1\\ 1 & 8 & 3\end{bmatrix}\begin{bmatrix}-1\\ 2\\ 5\end{bmatrix}=-1\begin{bmatrix}-3\\ 5\\ 2\\ 1\end{bmatrix}+2\begin{bmatrix}6\\ -4\\ 3\\ 8\end{bmatrix}+5\begin{bmatrix}2\\ 0\\ -1\\ 3\end{bmatrix}$

(b) $\begin{bmatrix}2 & 1 & 5\\ 6 & 3 & -8\end{bmatrix}\begin{bmatrix}3\\ 0\\ -5\end{bmatrix}=3\begin{bmatrix}2\\ 6\end{bmatrix}+0\begin{bmatrix}1\\ 3\end{bmatrix}-5\begin{bmatrix}5\\ -8\end{bmatrix}$

3. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$, thus $A\mathbf{x}=\mathbf{b}$ is inconsistent. By Theorem 4.8.1, $\mathbf{b}$ is not in the column space of A.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&0&1\\0&1&0&-3\\0&0&1&1\end{bmatrix}$, so the system has a unique solution $x_1=1$, $x_2=-3$, $x_3=1$. By Theorem 4.8.1, $\mathbf{b}$ is in the column space of A. By Formula (2), we can write $\begin{bmatrix}1\\9\\1\end{bmatrix}-3\begin{bmatrix}-1\\3\\1\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}5\\1\\-1\end{bmatrix}$.

4. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&0&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}$, thus $A\mathbf{x}=\mathbf{b}$ is inconsistent.
By Theorem 4.8.1, $\mathbf{b}$ is not in the column space of A.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&0&0&-26\\0&1&0&0&13\\0&0&1&0&-7\\0&0&0&1&4\end{bmatrix}$, so the system has a unique solution $x_1=-26$, $x_2=13$, $x_3=-7$, $x_4=4$. By Theorem 4.8.1, $\mathbf{b}$ is in the column space of A.
By Formula (2), we can write $-26\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+13\begin{bmatrix}2\\1\\2\\1\end{bmatrix}-7\begin{bmatrix}0\\2\\1\\2\end{bmatrix}+4\begin{bmatrix}1\\1\\3\\2\end{bmatrix}=\begin{bmatrix}4\\3\\5\\7\end{bmatrix}$.

5. **(a)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=r\begin{bmatrix}5\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

(b) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3\\0\\-1\\5\end{bmatrix}+r\begin{bmatrix}5\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

6. **(a)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

(b) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-1\\2\\4\\-3\end{bmatrix}+r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

7. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of this system is $x_1 = 1+3t$, $x_2 = t$; in vector form, $(x_1, x_2) = (1+3t, t) = (1,0) + t(3,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1, x_2) = t(3,1)$.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
The general solution of this system is $x_1 = -2-t$, $x_2 = 7-t$, $x_3 = t$; in vector form, $(x_1, x_2, x_3) = (-2-t, 7-t, t) = (-2,7,0) + t(-1,-1,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1, x_2, x_3) = t(-1,-1,1)$.

8. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of this system is $x_1 = -1+2r-s-2t$, $x_2 = r$, $x_3 = s$, $x_4 = t$; in vector form, $(x_1, x_2, x_3, x_4) = (-1+2r-s-2t, r, s, t) = (-1,0,0,0) + r(2,1,0,0) + s(-1,0,1,0) + t(-2,0,0,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1, x_2, x_3, x_4) = r(2,1,0,0) + s(-1,0,1,0) + t(-2,0,0,1)$.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of this system is $x_1 = \frac{6}{5} + \frac{7}{5}s + \frac{1}{5}t$, $x_2 = \frac{7}{5} + \frac{4}{5}s - \frac{3}{5}t$, $x_3 = s$, $x_4 = t$; in vector form, $(x_1, x_2, x_3, x_4) = \left(\frac{6}{5} + \frac{7}{5}s + \frac{1}{5}t, \frac{7}{5} + \frac{4}{5}s - \frac{3}{5}t, s, t\right) = \left(\frac{6}{5}, \frac{7}{5}, 0, 0\right) + s\left(\frac{7}{5}, \frac{4}{5}, 1, 0\right) + t\left(\frac{1}{5}, -\frac{3}{5}, 0, 1\right)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1, x_2, x_3, x_4) = s\left(\frac{7}{5}, \frac{4}{5}, 1, 0\right) + t\left(\frac{1}{5}, -\frac{3}{5}, 0, 1\right)$.

9. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x}=\mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = 16t$, $x_2 = 19t$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & -16 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -19 \end{bmatrix}$.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = \frac{1}{2}t$, $x_2 = s$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero row of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}$.

10. (a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -s + \frac{2}{7}t$, $x_2 = -s - \frac{4}{7}t$, $x_3 = s$, $x_4 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system

$x_1 = -r - 2s - t$, $x_2 = -r - s - 2t$, $x_3 = r$, $x_4 = s$, $x_5 = t$ can be written in the vector form
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & 1 & 2 \end{bmatrix}$.

11. We use Theorem 4.8.4 to obtain the following answers.

(a) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Nonzero rows form a basis for the row space: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

(b) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Nonzero rows form a basis for the row space: $\begin{bmatrix} 1 & -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$.

12. We use Theorem 4.8.4 to obtain the following answers.

(a) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Nonzero rows form a basis for the row space:
$\begin{bmatrix} 1 & 2 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix}$.

Nonzero rows form a basis for the row space:
$\begin{bmatrix} 1 & 2 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$.

13. **(a)** The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By Theorems 4.8.3 and 4.8.4, the nonzero rows of B form a basis for the row space of A:
$\mathbf{r}_1 = \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

By Theorem 4.8.4, columns of B containing leading 1's form a basis for the column space of B:
$\mathbf{c}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c}'_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A is formed by the corresponding columns of A: $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 5 \\ 3 \\ 8 \end{bmatrix}$, and $\mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) We begin by transposing the matrix A.

We obtain $A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$, whose reduced row echelon form is $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.8.4, columns of C containing leading 1's form a basis for the column space of C:
$\mathbf{c}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c}'_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A^T is formed by the corresponding columns of A^T: $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 5 \\ -7 \\ 0 \\ -6 \end{bmatrix}$, and $\mathbf{c}_3 = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$.

Since columns of A^T are rows of A, a basis for the row space of A is formed by
$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} -2 & 5 & -7 & 0 & -6 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} -1 & 3 & -2 & 1 & -3 \end{bmatrix}$.

14. We construct a matrix whose columns are the given vectors: $A=\begin{bmatrix}1 & 2 & 2\\ 1 & 0 & -1\\ -4 & 2 & 3\\ -3 & -2 & 2\end{bmatrix}$. The reduced row echelon form of A is $B=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$. By Theorem 4.8.4, the three columns of B form a basis for the column space of B. By Theorem 4.8.5(b), the three columns of A form a basis for the column space of A. We conclude that $\{(1,1,-4,-3),(2,0,2,-2),(2,-1,3,2)\}$ is a basis for the subspace of R^4 spanned by these vectors.

15. We construct a matrix whose columns are the given vectors: $A=\begin{bmatrix}1 & 0 & -2 & 0\\ 1 & 0 & 0 & -3\\ 0 & 1 & 2 & 0\\ 0 & 1 & 2 & 3\end{bmatrix}$. The reduced row echelon form of A is $B=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$. By Theorem 4.8.4, the four columns of B form a basis for the column space of B. By Theorem 4.8.5(b), the four columns of A form a basis for the column space of A. We conclude that $\{(1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)\}$ is a basis for the subspace of R^4 spanned by these vectors.

16. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:
$A=\begin{bmatrix}1 & -3 & -1 & -5\\ 0 & 3 & 3 & 3\\ 1 & 7 & 9 & 5\\ 1 & 1 & 3 & -1\end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{c}\begin{bmatrix}1 & 0 & 2 & -2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{array}{cccc}\uparrow & \uparrow & \uparrow & \uparrow\\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4\end{array}\end{array}$$

contains leading 1's in the first two columns, by Theorems 4.8.4 and 4.8.5(b), the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the column space of A, and for $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_4 = -2\mathbf{w}_1 + \mathbf{w}_2$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$.

17. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$:

$A = \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array} \end{array}$$

contains leading 1's in columns 1, 2, and 4, by Theorems 4.8.4 and 4.8.5(b), the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ form a basis for the column space of A, and for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{w}_5 = -\mathbf{w}_1 + 3\mathbf{w}_2 + 2\mathbf{w}_4$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4$.

18. We are employing the procedure developed in Example 9.

The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.8.4 and 4.8.5(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $\begin{bmatrix} 1 & 4 & 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 3 & 0 \end{bmatrix}$, form a basis for the row space of A.

19. We are employing the procedure developed in Example 9.

The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \frac{13}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.8.4 and 4.8.5(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $\begin{bmatrix}1 & 4 & 5 & 6 & 9\end{bmatrix}$ and $\begin{bmatrix}3 & -2 & 1 & 4 & -1\end{bmatrix}$, form a basis for the row space of A.

20. Let $B = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{matrix}1 & 2 \\ -1 & 0 \\ 3 & -2 \\ 2 & 4\end{matrix}$. We are looking for a matrix A such that $AB = O$. Taking a transpose on both sides results in $B^T A^T = \mathbf{0}^T$. We proceed to solve the homogeneous linear system $B^T\mathbf{u} = \mathbf{0}$. The reduced row echelon form of its augmented matrix $\begin{bmatrix}1 & -1 & 3 & 2 & 0 \\ 2 & 0 & -2 & 4 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0\end{bmatrix}$ therefore the general solution in the vector form is $s\begin{bmatrix}1 \\ 4 \\ 1 \\ 0\end{bmatrix} + t\begin{bmatrix}-2 \\ 0 \\ 0 \\ 1\end{bmatrix}$. We can take $A^T = \begin{bmatrix}1 & -2 \\ 4 & 0 \\ 1 & 0 \\ 0 & 1\end{bmatrix}$ thus $A = \begin{bmatrix}1 & 4 & 1 & 0 \\ -2 & 0 & 0 & 1\end{bmatrix}$.

21. Since $T_A(\mathbf{x}) = A\mathbf{x}$, we are seeking the general solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(a) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & 0 \\ 1 & -1 & 4 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & 0\end{bmatrix}$. The general solution is $x_1 = -\frac{8}{3}t$, $x_2 = \frac{4}{3}t$, $x_3 = t$. In vector form, $\mathbf{x} = t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(b) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & 1 \\ 1 & -1 & 4 & 3\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3}\end{bmatrix}$. The general solution is $x_1 = \frac{7}{3} - \frac{8}{3}t$, $x_2 = -\frac{2}{3} + \frac{4}{3}t$, $x_3 = t$.
In vector form, $\mathbf{x} = \left(\frac{7}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(c) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & -1 \\ 1 & -1 & 4 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3}\end{bmatrix}$.
The general solution is $x_1 = \frac{1}{3} - \frac{8}{3}t$, $x_2 = -\frac{2}{3} + \frac{4}{3}t$, $x_3 = t$.
In vector form, $\mathbf{x} = \left(\frac{1}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

22. Since $T_A(\mathbf{x}) = A\mathbf{x}$, we are seeking the general solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(a) The reduced row echelon form of the augmented matrix $\begin{bmatrix}2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$.

The only solution is $x_1 = x_2 = 0$. In vector form, $\mathbf{x} = (0,0)$.

(b) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The system has no solution; no vector $\mathbf{x}$ exists for which $T_A(\mathbf{x}) = \mathbf{b}$.

(c) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The system has no solution; no vector $\mathbf{x}$ exists for which $T_A(\mathbf{x}) = \mathbf{b}$.

23. **(a)** The associated homogeneous system $x + y + z = 0$ has a general solution $x = -s - t$, $y = s$, $z = t$. The original nonhomogeneous system has a general solution $x = 1 - s - t$, $y = s$, $z = t$, which can be expressed in vector form as

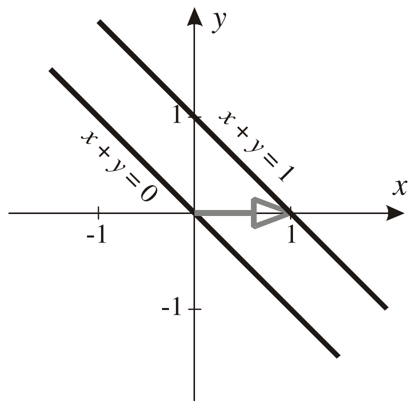
$$(x, y, z) = (1 - s - t, s, t) = \underbrace{(1,0,0)}_{\substack{\text{particular} \\ \text{solution} \\ \text{of the} \\ \text{nonhomogeneous} \\ \text{system}}} + \underbrace{(-s - t, s, t)}_{\substack{\text{general} \\ \text{solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{system}}}$$

(b) Geometrically, the points (x, y, z) corresponding to solutions of $x + y + z = 1$ form a plane passing through the point $(1,0,0)$ and parallel to the vectors $(-1,1,0)$ and $(-1,0,1)$.

24. **(a)** The associated homogeneous system $x + y = 0$ has a general solution $x = -t$, $y = t$.
The original nonhomogeneous system has a general solution $x = 1 - t$, $y = t$, which can be expressed in vector form as

$$(x, y) = (1 - t, t) = \underbrace{(1,0)}_{\substack{\text{particular} \\ \text{solution} \\ \text{of the} \\ \text{nonhomogeneous} \\ \text{system}}} + \underbrace{(-t, t)}_{\substack{\text{general} \\ \text{solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{system}}}$$

(b)



Geometrically, the points (x, y, z) corresponding to solutions of $x + y = 1$ form a line passing through the point $(1,0,)$ and parallel to the vector $(-1,1)$.

25. **(a)** The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{2}{3}s + \frac{1}{3}t$, $x_2 = s$, $x_3 = t$.

(b) Multiplying $\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ therefore $x_1 = 1,\ x_2 = 0,\ x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,0,1)}_{\substack{\text{particular}\\\text{solution}\\\text{of the}\\\text{nonhomogeneous}\\\text{system}}} + \underbrace{\left(-\tfrac{2}{3}s + \tfrac{1}{3}t, s, t\right)}_{\substack{\text{general}\\\text{solution}\\\text{of the}\\\text{homogeneous}\\\text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{2}{3} - \frac{2}{3}p + \frac{1}{3}q$, $x_2 = p$, $x_3 = q$.

If we let $p = s$ and $q = t + 1$ then this agrees with the solution we obtained in part (c).

26. **(a)** The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{11}{5}t$, $x_2 = \frac{2}{5}t$, $x_3 = t$.

(b) Multiplying $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$ therefore $x_1 = x_2 = x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,1,1)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{\left(-\tfrac{11}{5}t, \tfrac{2}{5}t, t\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{16}{5} - \frac{11}{5}s$, $x_2 = \frac{3}{5} + \frac{2}{5}s$, $x_3 = s$.

If we let $s = 1 + t$ then this agrees with the solution we obtained in part (c).

27. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system

$x_1 = \frac{1}{3} - \frac{4}{3}r - \frac{1}{3}s$, $x_2 = r$, $x_3 = s$, $x_4 = 1$ can be expressed in vector form as

$$(x_1, x_2, x_3, x_4) = \underbrace{\left(\tfrac{1}{3}, 0, 0, 1\right)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{\left(-\tfrac{4}{3}r - \tfrac{1}{3}s, r, s, 0\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{associated}\\ \text{homogeneous}\\ \text{system}}}$$

28. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{13}{3} & \frac{13}{3} \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system

$x_1 = \frac{13}{3} + \frac{1}{3}s + \frac{13}{3}t$, $\quad x_2 = s$, $\quad x_3 = -7 - 9t$, $\quad x_4 = t$

can be expressed in vector form as

$$(x_1, x_2, x_3, x_4) = \underbrace{\left(\tfrac{13}{3}, 0, -7, 0\right)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{\left(\tfrac{1}{3}s + \tfrac{13}{3}t, s, -9t, t\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{associated}\\ \text{homogeneous}\\ \text{system}}}$$

29. **(a)** The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution $\mathbf{x} = (x, y, z)$ of $A\mathbf{x} = \mathbf{0}$ is $x = 0$, $y = 0$, $z = t$; in vector form, $\mathbf{x} = t(0,0,1)$. This shows that the null space of A consists of all points on the z-axis.

The column space of A, $\operatorname{span}\{(1,0,0),(0,1,0)\}$ clearly consists of all points in the xy-plane.

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an example of such a matrix.

30. **(a)** e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ null space is the origin

(b) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the z-axis

(c) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the yz-plane

31. **(a)** By inspection, $\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}$ has the desired null space. In general, this will hold true for all matrices of the form $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$ where a and b are not both zero (if $a = b = 0$ then the null space is the entire plane).

(b) Only the zero vector forms the null space for both A and B (their determinants are nonzero, therefore in each case the corresponding homogeneous system has only the trivial solution).
The line $3x + y = 0$ forms the null space for C.
The entire plane forms the null space for D.

True-False Exercises

(a) True.

(b) False. The column space of A is the space spanned by all column vectors of A.

(c) False. Those column vectors form a basis for the column space of R.

(d) False. This would be true if A were in row echelon form.

(e) False. For instance $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ have the same row space, but different column spaces.

(f) True. This follows from Theorem 4.8.3.

(g) True. This follows from Theorem 4.8.3.

(h) False. Elementary row operations generally can change the column space of a matrix.

(i) True. This follows from Theorem 4.8.1.

(j) False. Let both A and B be $n \times n$ matrices. By Theorem 4.8.3, row operations do not change the row space of a matrix. An invertible matrix can be reduced to I thus its row space is always R^n. On the other hand, a singular matrix cannot be reduced to identity matrix - at least one row in its reduced row echelon form is made up of zeros. Consequently, its row space is spanned by fewer than n vectors, therefore the dimension of this space is less than n.

4.9 Rank, Nullity, and the Fundamental Matrix Spaces

1. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A) = 1$ (the number of leading 1's)
- $\operatorname{nullity}(A) = 3$ (by Theorem 4.9.2).

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A) = 2$ (the number of leading 1's)
- $\operatorname{nullity}(A) = 3$ (by Theorem 4.9.2).

2. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A) = 3$ (the number of leading 1's)
- $\operatorname{nullity}(A) = 2$ (by Theorem 4.9.2).

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A) = 3$ (the number of leading 1's)
- $\operatorname{nullity}(A) = 1$ (by Theorem 4.9.2).

3. **(a)** $\operatorname{rank}(A) = 3$; $\operatorname{nullity}(A) = 0$

(b) $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 0 = 3 = n \leftarrow$ number of columns of A

(c) 3 leading variables; 0 parameters in the general solution (the solution is unique)

4. **(a)** $\operatorname{rank}(A) = 2$; $\operatorname{nullity}(A) = 1$;

(b) $\operatorname{rank}(A) + \operatorname{nullity}(A) = 2 + 1 = 3 = n \leftarrow$ number of columns of A

(c) 2 leading variables; 1 parameter in the general solution

5. **(a)** $\operatorname{rank}(A) = 1$; $\operatorname{nullity}(A) = 2$

(b) $\operatorname{rank}(A) + \operatorname{nullity}(A) = 1 + 2 = 3 = n \leftarrow$ number of columns of A

(c) 1 leading variable; 2 parameters in the general solution

6. **(a)** $\operatorname{rank}(A) = 3$; $\operatorname{nullity}(A) = 1$;

(b) $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 1 = 4 = n \leftarrow$ number of columns of A

(c) 3 leading variables; 1 parameter in the general solution

7. **(a)** If every column of the reduced row echelon form of a 4×4 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 4
- the nullity of A has the smallest possible value: 0

(b) If every row of the reduced row echelon form of a 3×5 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 2

(c) If every column of the reduced row echelon form of a 5×3 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 0

8. The largest possible value for the rank of an $m \times n$ matrix A is the smaller of the two dimensions of A:

- n if $m \ge n$ (when every column of the reduced row echelon form of A contains a leading 1),
- m if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

The smallest possible value for the nullity of an $m \times n$ matrix A is

- 0 if $m \ge n$ (when every column of the reduced row echelon form of A contains a leading 1),
- $n - m$ if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

9.

			(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
	Size of A:	$m \times n$	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	$\text{rank}(A)$	$= r$	3	2	1	2	2	0	2
	$\text{rank}(A\,\|\,\mathbf{b})$	$= s$	3	3	1	2	3	0	2
(i)	dimension of the row space of A	$= r$	3	2	1	2	2	0	2
	dimension of the column space of A	$= r$	3	2	1	2	2	0	2
	dimension of the null space of A	$= n - r$	0	1	2	7	7	4	0
	dimension of the null space of A^T	$= m - r$	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x} = \mathbf{b}$ consistent?	Is $r = s$?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x} = \mathbf{b}$	$= n - r$ if consistent	0	-	2	7	-	4	0

10. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ whereas the reduced row echelon form of A^T is

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We conclude that $\text{rank}(A) = \text{rank}(A^T) = 2$.

11. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore $\text{rank}(A) = \text{rank}(A^T) = 2$. Applying

Formula (4) to both A and its transpose yields $2 + \text{nullity}(A) = 2$ and $2 + \text{nullity}(A^T) = 3$.

It follows that bases for $\text{row}(A)$ and $\text{row}(A^T)$ are $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ respectively. Since

$\text{nullity}(A) = 0$ the nullspace of A contains only the zero vector. $A^T = \begin{bmatrix} 1 & 0 & -9 \\ 4 & 3 & 0 \end{bmatrix}$ has reduced row echelon form $\begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 12 \end{bmatrix}$. The general solution is

$x_1 = 9t$, $x_2 = -12t$, $x_3 = t$. In vector form, $\mathbf{x} = t(9,-12,1)$ where t

is arbitrary. So a basis for the left null space (of A^T) is $\left\{\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}\right\}$.

12. The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\text{rank}(A) = \text{rank}(A^T) = 1$.

It follows that bases for $\text{row}(A)$ and $\text{row}(A^T)$ are $\left\{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ respectively and a

basis for the null space of A is $\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}\right\}$.

Applying Formula (4) to A^T yields $2 + \text{nullity}(A^T) = 3$.

Since A^T has reduced row echelon form $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, a basis for the left null space (of A^T) is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

13. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\text{rank}(A) = \text{rank}(A^T) = 2$.

It follows that bases for $\text{row}(A)$ and $\text{row}(A^T)$ are $\left\{\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}\right\}$ respectively

and a basis for the null space of A is $\left\{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right\}$.

Applying Formula (4) to A^T yields $2 + \text{nullity}(A^T) = 3$

Since A^T has reduced row echelon form $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, a basis for the left null space (of A^T) is $\left\{\begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}\right\}$.

14. Since $\det\left(\begin{bmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & -3 \\ 1 & -1 & 2 & 2 \end{bmatrix}\right) = -144$, the nullspaces of A and A^T only contain the zero vector.

The columns and rows of A form bases for $\text{row}(A)$ and $\text{row}(A^T)$ respectively.

15. From Exercise 11, a basis for $\text{row}(A)$ is $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}$ whereas the null space of A contains only the zero vector. These subspaces clearly satisfy Theorem 4.9.7(a). Bases for $\text{row}(A^T)$ and the left null space are given by $\left\{\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}\right\}$ respectively. Since $\det\left(\begin{bmatrix} 1 & 4 & 9 \\ 0 & 3 & -12 \\ -9 & 0 & 1 \end{bmatrix}\right) = 678 \neq 0,$ these three vectors are a basis for R^3. Moreover $\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -12 \\ 0 \end{bmatrix} = 0$. Therefore the left null space is the set of all vectors in R^3 that are orthogonal to all vectors in $\text{row}(A^T)$.

16. From Exercise 12, a basis for $\text{row}(A)$ is $\left\{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right\}$ and a basis for the null space of A is $\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}\right\}$. Since $\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -4 \end{bmatrix}\right) = 24 \neq 0,$ these three vectors are a basis for R^3. Moreover $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = 0$. Therefore, the null space of A is set of all vectors in R^3 that are orthogonal to all vectors in $\text{row}(A)$. Similarly, bases for $\text{row}(A^T)$ and the left null space are given by $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ respectively. Since $\det\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\right) = 5 \neq 0,$ these vectors are a basis for R^2. Moreover, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0.$ Therefore the left null space is the set of all vectors in R^2 that are orthogonal to all vectors in $\text{row}(A^T)$.

17. From Exercise 13, a basis for $\text{row}(A)$ is $\left\{\begin{bmatrix}0\\-1\\-2\end{bmatrix}, \begin{bmatrix}-1\\0\\3\end{bmatrix}\right\}$ and a basis for the null space of A is $\left\{\begin{bmatrix}3\\-2\\1\end{bmatrix}\right\}$.

Since $\det\left(\begin{bmatrix}0 & -1 & 3\\-1 & 0 & -2\\-2 & 3 & 1\end{bmatrix}\right) = -14 \neq 0,$ these three vectors are a basis for R^3. Moreover

$\begin{bmatrix}0\\-1\\-2\end{bmatrix} \cdot \begin{bmatrix}3\\-2\\1\end{bmatrix} = 0$ and $\begin{bmatrix}-1\\0\\3\end{bmatrix} \cdot \begin{bmatrix}3\\-2\\1\end{bmatrix} = 0$. Therefore, the null space of A is the set of all vectors in R^3

that are orthogonal to all vectors in $\text{row}(A)$. Similarly, bases for $\text{row}(A^T)$ and the left null space are

given by $\left\{\begin{bmatrix}0\\-1\\-4\end{bmatrix}, \begin{bmatrix}-1\\0\\-4\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}-4\\-4\\1\end{bmatrix}\right\}$ respectively. Since $\det\left(\begin{bmatrix}0 & -1 & -4\\-1 & 0 & -4\\-4 & -4 & 1\end{bmatrix}\right) = -33 \neq 0,$ these vectors

are a basis for R^3. Moreover, $\begin{bmatrix}0\\-1\\-4\end{bmatrix} \cdot \begin{bmatrix}-4\\-4\\1\end{bmatrix} = 0$ and $\begin{bmatrix}-1\\0\\4\end{bmatrix} \cdot \begin{bmatrix}-4\\-4\\1\end{bmatrix} = 0$. Therefore, the left null space is the

set of all vectors in R^2 that are orthogonal to all vectors in $\text{row}(A^T)$.

18. From Exercise 14, the bases for $\text{row}(A)$ and $\text{row}(A^T)$ are given by the rows and columns of A respectively. These are each bases for R^4. Also, both the null space and the left null space contain only the zero vector so that both parts of Theorem 4.9.7 are clearly satisfied.

19. Following Example 5, we find that the reduced row echelon form of the

augmented matrix $\begin{bmatrix}0 & 2 & 8 & -7 & 1 & 0 & 0\\2 & -2 & 4 & 0 & 0 & 1 & 0\\-3 & 4 & -2 & 5 & 0 & 0 & 1\end{bmatrix}$

is $\begin{bmatrix}1 & 0 & 6 & 0 & \frac{5}{17} & \frac{19}{17} & \frac{7}{17}\\0 & 1 & 4 & 0 & \frac{5}{17} & \frac{21}{34} & \frac{7}{17}\\0 & 0 & 0 & 1 & -\frac{1}{17} & \frac{3}{17} & \frac{2}{17}\end{bmatrix}$. Hence, the column space basis is $\left\{\begin{bmatrix}0\\2\\-3\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}-7\\0\\5\end{bmatrix}\right\}$, the row space basis is $\left\{\begin{bmatrix}0\\2\\8\\-7\end{bmatrix},\begin{bmatrix}2\\-2\\4\\0\end{bmatrix},\begin{bmatrix}-3\\4\\-2\\5\end{bmatrix}\right\}$,the null space basis is $\left\{\begin{bmatrix}-6\\-4\\1\\0\end{bmatrix}\right\}$, and the left null space contains only the zero vector.

20. Following Example 5, we find that the reduced row echelon form of the

Augmented matrix $\begin{bmatrix}1&2&3&1&1&1&0&0&0\\2&8&0&0&2&0&1&0&0\\0&4&-6&0&1&0&0&1&0\\1&0&0&0&0&0&0&0&1\end{bmatrix}$ is $\begin{bmatrix}1&0&0&0&0&0&0&0&1\\0&1&0&0&\frac{1}{4}&0&\frac{1}{8}&0&-\frac{1}{4}\\0&0&1&0&0&0&\frac{1}{12}&-\frac{1}{6}&-\frac{1}{6}\\0&0&0&1&\frac{1}{2}&1&-\frac{1}{2}&\frac{1}{2}&0\end{bmatrix}$

Hence, a basis for

the column space basis is $\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}2\\8\\4\\0\end{bmatrix},\begin{bmatrix}3\\0\\-6\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$, the row space basis is $\left\{\begin{bmatrix}1\\2\\3\\1\\1\end{bmatrix},\begin{bmatrix}2\\8\\0\\0\\2\end{bmatrix},\begin{bmatrix}0\\4\\-6\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right\}$,the null

space basis is $\left\{\begin{bmatrix}0\\-\frac{1}{4}\\0\\-\frac{1}{2}\\1\end{bmatrix}\right\}$, and the left null space contains only the zero vector.

21. **(a)** Applying Formula (4) to both A and its transpose yields

$$2+\text{nullity}(A)=4 \text{ and } 2+\text{nullity}(A^T)=3$$

therefore

$$\text{nullity}(A)-\text{nullity}(A^T)=1$$

(b) Applying Formula (4) to both A and its transpose yields

$$\text{rank}(A)+\text{nullity}(A)=n \text{ and } \text{rank}(A^T)+\text{nullity}(A^T)=m$$

By Theorem 4.9.4, $\text{rank}(A^T)=\text{rank}(A)$ therefore

$$\text{nullity}(A)-\text{nullity}(A^T)=n-m$$

22. $T(x_1, x_2) = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) $\operatorname{rank}(A) = 2$ **(b)** $\operatorname{nullity}(A) = 0$

23. $T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$; the standard matrix is

$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) $\operatorname{rank}(A) = 3$ **(b)** $\operatorname{nullity}(A) = 2$

24. (a) The determinant of A is

$$\begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ t-1 & 0 & 1-t \end{vmatrix}$$

-1 times the first row was added to the second row and to the third row.

$$= \begin{vmatrix} 1 & 1+t & t \\ 0 & 0 & 1-t \\ t-1 & 1-t & 1-t \end{vmatrix}$$

Last column was added to the second column.

$$= -(1-\mathrm{t})\begin{vmatrix} 1 & 1+t \\ t-1 & 1-t \end{vmatrix}$$

Cofactor expansion along the second row.

$$= -(1-t)\left((1-t) - (1+t)(t-1)\right)$$

$$= -(1-t)^2(2+t)$$

From parts (g) and (n) of Theorem 4.9.8, $\operatorname{rank}(A) = 3$ when $\det(A) \neq 0$, i.e. for all t values other than 1 or -2.

If $t=1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 1.

If $t=-2$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

(b) The determinant of A is

$$\begin{vmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{vmatrix} = \begin{vmatrix} t & 3 & -1 \\ 3-2t & 0 & 0 \\ -1 & -3 & t \end{vmatrix}$$

-2 times the first row was added to the second row.

$$= -(3-2t)\begin{vmatrix} 3 & -1 \\ -3 & t \end{vmatrix}$$

Cofactor expansion along the second row.

$$= -(3-2t)(3t-3)$$

$$= 3(2t-3)(t-1)$$

From parts (g) and (n) of Theorem 4.9.8, $\operatorname{rank}(A)=3$ when $\det(A)\neq 0$, i.e. for all t values other than 1 or $\frac{3}{2}$.

If $t=1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

If $t=\frac{3}{2}$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

25. By inspection, there must be leading 1's in the first column (because of the first row) and in the third column (because of the fourth row) regardless of the values of r and s, therefore the matrix cannot have rank 1.

It has rank 2 if $r=2$ and $s=1$, since there is no leading 1 in the second column in that case.

26. **(a)** e.g., $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ - the column space is the xy-plane in R^3

(b) The general solution of $A\mathbf{x}=\mathbf{0}$ is $\mathbf{x}=(0,0,t)$. The null space is the z-axis.

(c) The row space of A is the xy-plane in R^3

27. No, both row and column spaces of A must be planes through the origin since from $\text{nullity}(A)=1$, it follows by Formula (4) that $\text{rank}(A)=3-1=2$.

28. **(a)** 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x}=\mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A^T can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A^T\mathbf{x}=\mathbf{0}$ has three parameters;

29. **(a)** 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x}=\mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A\mathbf{x}=\mathbf{0}$ has three parameters;

30. By part (b) of Theorem 4.9.3, the nullity of A is 0. By Formula(4), $\text{rank}(A)=6-0=6$.

31. **(a)** By Formula (4), $\text{nullity}(A)=7-4=3$ thus the dimension of the solution space of $A\mathbf{x}=\mathbf{0}$ is 3.

(b) No, the column space of A is a subspace of R^5 of dimension 4, therefore there exist vectors $\mathbf{b}$ in R^5 that are outside this column space. For any such vector, the system $A\mathbf{x}=\mathbf{b}$ is inconsistent.

32. The rank of A is 2 if and only if the two row vectors of A are not scalar multiples of one another, i.e. they are nonparallel nonzero vectors. This is equivalent to the cross product of these vectors being nonzero, i.e.

$$\left(\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right) \neq (0,0,0)$$

33. From the result of Exercise 32, the rank of the matrix being less than 2 implies that

$$\begin{vmatrix} x & y \\ 1 & x \end{vmatrix} = x^2 - y = 0, \qquad \begin{vmatrix} x & z \\ 1 & y \end{vmatrix} = xy - z = 0, \qquad \begin{vmatrix} y & z \\ x & y \end{vmatrix} = y^2 - xz = 0$$

therefore $y=x^2$ and $z=xy=x^3$. Letting $x=t$, we obtain $y=t^2$ and $z=t^3$.

34. For instance, let $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A^2=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, therefore $\text{rank}(A^2)=1\neq 0=\text{rank}(B^2)$ even though $\text{rank}(A)=1=\text{rank}(B)$.

35. The reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of $A^T\mathbf{x} = 0$ has components $x_1 = -10t$, $x_2 = 5t$, $x_3 = t$, $x_4 = 0$, so in vector form $\mathbf{x} = t(-10,5,1,0)$. Evaluating dot products of columns of A and $\mathbf{v} = (-10,5,1,0)$, which forms a basis for the null space of A^T we obtain

$\mathbf{c}_1 \cdot \mathbf{v} = (1,2,0,2)\cdot(-10,5,1,0) = (1)(-10)+(2)(5)+(0)(1)+(2)(0) = 0$

$\mathbf{c}_2 \cdot \mathbf{v} = (3,6,0,6)\cdot(-10,5,1,0) = (3)(-10)+(6)(5)+(0)(1)+(6)(0) = 0$

$\mathbf{c}_3 \cdot \mathbf{v} = (-2,-5,5,0)\cdot(-10,5,1,0) = (-2)(-10)+(-5)(5)+(5)(1)+(2)(0) = 0$

$\mathbf{c}_4 \cdot \mathbf{v} = (0,-2,10,8)\cdot(-10,5,1,0) = (0)(-10)+(-2)(5)+(10)(1)+(8)(0) = 0$

$\mathbf{c}_5 \cdot \mathbf{v} = (2,4,0,4)\cdot(-10,5,1,0) = (2)(-10)+(4)(5)+(0)(1)+(4)(0) = 0$

$\mathbf{c}_6 \cdot \mathbf{v} = (0,-3,15,18)\cdot(-10,5,1,0) = (0)(-10)+(-3)(5)+(15)(1)+(18)(0) = 0$

Since the column space of A is $\text{span}\{\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3,\mathbf{c}_4,\mathbf{c}_5,\mathbf{c}_6\}$ and the null space of A^T is $\text{span}\{\mathbf{v}\}$, we conclude that the two spaces are orthogonal complements in R^4.

37. **(a)** $m = 3 > 2 = n$ so the system is overdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & b_1 + b_3 \\ 0 & 1 & b_3 \\ 0 & 0 & 3b_1 + b_2 + 2b_3 \end{bmatrix}$ hence the system is inconsistent for all b's that satisfy $3b_1 + b_2 + 2b_3 \neq 0$.

(b) $m = 2 < 3 = n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 - \frac{1}{4}b_2 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{6}b_1 - \frac{1}{12}b_2 \end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

(c) $m = 2 < 3 = n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & -\frac{3}{2} & -\frac{1}{2}b_1 - \frac{3}{2}b_2 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2}b_1 - \frac{1}{2}b_2 \end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

38. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & -2b_1+3b_2 \\ 0 & 1 & -b_1+b_2 \\ 0 & 0 & 3b_1-4b_2+b_3 \\ 0 & 0 & -2b_1+b_2+b_4 \\ 0 & 0 & 7b_1-8b_2+b_5 \end{bmatrix}$. For the system to be consistent, we must have $3b_1-4b_2+b_3=0$, $-2b_1+b_2+b_4=0$, and $7b_1-8b_2+b_5=0$.

For arbitrary s and t, the b's must satisfy $b_1=s$, $b_2=t$, $b_3=-3s+4t$, $b_4=2s-t$, $b_5=-7s+8t$.

True-False Exercises

(a) False. For instance, in, neither row vectors nor column $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ vectors are linearly independent.

(b) True. In an $m\times n$ matrix, if $m<n$ then by Theorem 4.6.2(a), the n columns in R^m must be linearly dependent. If $m>n$, then by the same theorem, the m rows in R^n must be linearly dependent. We conclude that $m=n$.

(c) False. The nullity in an $m\times n$ matrix is at most n.

(d) False. For instance, if the column contains all zeros, adding it to a matrix does not change the rank.

(e) True. In an $n\times n$ matrix A with linearly dependent rows, $\operatorname{rank}(A)\le n-1$.
By Formula (4), $\operatorname{nullity}(A)=n-\operatorname{rank}(A)\ge 1$.

(f) False. By Theorem 4.9.7, the nullity must be nonzero.

(g) False. This follows from Theorem 4.9.1.

(h) False. By Theorem 4.9.4, $\operatorname{rank}(A^T)=\operatorname{rank}(A)$ for any matrix A.

(i) True. Since each of the two spaces has dimension 1, these dimensions would add up to 2 instead of 3 as required by Formula (4).

(j) False. For instance, if $n=3$, $V=\operatorname{span}\{\mathbf{i},\mathbf{j}\}$ (the xy-plane), and $W=\operatorname{span}\{\mathbf{i}\}$ (the x-axis) then $W^\perp=\operatorname{span}\{\mathbf{j},\mathbf{k}\}$ (the yz-plane) is not a subspace of $V^\perp=\operatorname{span}\{\mathbf{k}\}$ (the z-axis).
(Note that it is true that $V^\perp$ is a subspace of $W^\perp$.)

Chapter 4 Supplementary Exercises

1. **(a)** $\mathbf{u}+\mathbf{v}=(3+1,-2+5,4-2)=(4,3,2)$; $k\mathbf{u}=(-1\cdot 3,0,0)=(-3,0,0)$

(b) For any $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,u_3+v_3)$ is an ordered triple of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u} = (u_1, u_2, u_3)$ in V and for any scalar k, $k\mathbf{u} = (ku_1, 0, 0)$ is an ordered triple of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^3.

(d) Axiom 7:
$k((u_1,u_2,u_3)+(v_1,v_2,v_3)) = k(u_1+v_1, u_2+v_2, u_3+v_3) = (k(u_1+v_1), 0, 0) = k(u_1,u_2,u_3)+k(v_1,v_2,v_3)$
for all real k, u_1, u_2, u_3, v_1, v_2, and v_3.

Axiom 8: $(k+m)(u_1,u_2,u_3) = ((k+m)u_1, 0, 0) = (ku_1+mu_1, 0, 0) = k(u_1,u_2,u_3)+m(u_1,u_2,u_3)$ for all real k, m, u_1, u_2, and u_3;

Axiom 9: $k(m(u_1,u_2,u_3)) = k(mu_1, 0, 0) = (kmu_1, 0, 0) = (km)(u_1,u_2,u_3)$ for all real k, m, u_1, u_2, and u_3;

(e) Axiom 10 fails to hold: $1(u_1,u_2,u_3) = (u_1,0,0)$ does not generally equal (u_1,u_2,u_3). Consequently, V is not a vector space.

2. **(a)** The solution space is R^3 since all vectors (x, y, z) satisfy the system.

(b) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

therefore the general solution is $x = \frac{3}{2}s - \frac{1}{2}t$, $y = s$, $z = t$. The solution space is a plane in R^3; its equation is $2x - 3y + z = 0$, the first equation in our system (the other two equations were its multiples).

(c) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

therefore the general solution is $x = 2t$, $y = t$, $z = 0$ - these form parametric equations for a line in R^3.

(d) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

therefore the homogeneous system has only the trivial solution $(0,0,0)$ - the origin.

3. $$A = \begin{bmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{bmatrix} \longleftarrow \text{The coefficient matrix of the system}$$

$$\begin{bmatrix} 1 & 1 & s \\ 0 & s-1 & 1-s \\ 0 & 1-s & 1-s^2 \end{bmatrix}$$ -1 times the first row was added to the second row and $-s$ times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & s \\ 0 & s-1 & 1-s \\ 0 & 0 & 2-s-s^2 \end{bmatrix}$$ The second row was added to the third row.

After factoring $2-s-s^2=(2+s)(1-s)$, we conclude that

- the solution space is a plane through the origin if $s=1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\text{nullity}(A)=2$),
- the solution space is a line through the origin if $s=-2$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so $\text{nullity}(A)=1$),
- the solution space is the origin if $s \neq -2$ and $s \neq 1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $\text{nullity}(A)=0$),
- there are no values of s for which the solution space is R^3.

4. **(a)** $(4a, a-b, a+2b) = a(4,1,1)+b(0,-1,2)$

(b) $(3a+b+3c, -a+4b-c, 2a+b+2c) = a(3,-1,2)+b(1,4,1)+c(3,-1,2)$
$= (a+c)(3,-1,2)+b(1,4,1)$

(c) $(2a-b+4c, 3a-c, 4b+c) = a(2,3,0)+b(-1,0,4)+c(4,-1,1)$

5. **(a)** Using trigonometric identities we can write

$$\mathbf{f}_1 = \sin(x+\theta) = \sin x\cos\theta + \cos x\sin\theta = (\cos\theta)\mathbf{f} + (\sin\theta)\mathbf{g}$$
$$\mathbf{g}_1 = \cos(x+\theta) = \cos x\cos\theta - \sin x\sin\theta = (-\sin\theta)\mathbf{f} + (\cos\theta)\mathbf{g}$$

which shows that $\mathbf{f}_1$ and $\mathbf{g}_1$ are both in $W = \text{span}\{\mathbf{f},\mathbf{g}\}$.

(b) The functions $\mathbf{f}_1 = \sin(x+\theta)$ and $\mathbf{f}_2 = \cos(x+\theta)$ are linearly independent since neither function is a scalar multiple of the other. By Theorem 4.6.4, these functions form a basis for W.

6. **(a)** We are looking for scalars c_1, c_2, and c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$, i.e.,

$$\begin{array}{rcrcrcl} 1c_1 & + & 3c_2 & + & 2c_3 & = & 1 \\ -1c_1 & & & + & c_3 & = & 1 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & \frac{2}{3} \end{bmatrix}$ so that the general solution is $c_1 = -1 + t$, $c_2 = \frac{2}{3} - t$, $c_3 = t$.

E.g., letting $t = 0$ yields $-1\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{v}$, whereas with $t = 1$ we obtain $0\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{v}$.

(b) The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not form a basis for R^2 therefore Theorem 4.5.1 does not apply here.

7. Denoting $B = \left[\ \mathbf{v}_1 \ \middle|\ \cdots \ \middle|\ \mathbf{v}_n\ \right]$ we can write $AB = \left[\ A\mathbf{v}_1 \ \middle|\ \cdots \ \middle|\ A\mathbf{v}_n\ \right]$.

By parts (g) and (h) of Theorem 4.9.7, the columns of AB are linearly independent if and only if $\det(AB) \neq 0$. This implies that $\det(A) \neq 0$, i.e., the matrix A must be invertible.

8. No, e.g., $x+1$ and $x-1$ form a basis for P_1 even though both are of degree 1.

9. **(a)** The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 1.

(b) The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 2.

(c) For $n = 1$, the rank is 1 and the nullity is 0.
For $n \geq 2$, the reduced row echelon form will always have two nonzero rows; the rank is 2 and the nullity is $n - 2$.

10. **(a)** Adding -1 times the first row to the third row yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; we conclude that the matrix has rank 2 and nullity 1.

(b) Adding -1 times the first row to the fifth row and adding -1 times the second row to the fourth row yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the matrix has rank 3 and nullity 2.

(c) After performing n elementary row operations which follow the same pattern as in parts (a) and (b):

- add -1 times row 1 to row $2n+1$,
- add -1 times row 2 to row $2n$,
- add -1 times row 3 to row $2n-1$,
- …
- add -1 times row n to row $n+2$,

the reduced row echelon form will be obtained: its top $n+1$ rows are identical to those in the original X-matrix, whereas the bottom n rows are completely filled with zeros.
We conclude that the matrix has rank $n+1$ and nullity n.

11. **(a)** Let W be the set of all polynomials p in P_n for which $p(-x)=p(x)$. In order for a polynomial $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ to be in W, we must have $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n=a_0+a_1(-x)+a_2(-x)^2+\cdots+a_n(-x)^n=p(-x)$ which implies that for all x, $2a_1x+2a_3x^3+\cdots=0$ so $a_1=a_3=\cdots=0$.
Any polynomial of the form $p(x)=a_0+a_2x^2+a_4x^4+\cdots+a_{2\lfloor n/2\rfloor}x^{\lfloor 2n/2\rfloor}$ satisfies $p(-x)=p(x)$ (the notation t represents the largest integer less than or equal to t).
This means $W=\text{span}\{1,\ x^2,\ x^4,\ ...,\ x^{2\lfloor n/2\rfloor}\}$, so W is a subspace of P_n by Theorem 4.3.1(a).
The polynomials in $\{1,\ x^2,\ x^4,\ ...,\ x^{2\lfloor n/2\rfloor}\}$ are linearly independent (since they form a subset of the standard basis for P_n), consequently they form a basis for W.

(b) Let W be the set of all polynomials p in P_n for which $p(0)=p(1)$.
In order for a polynomial $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ to be in W, we must have $p(0)=a_0=a_0+a_1+a_2+\cdots+a_n=p(1)$ which implies that $a_1+a_2+\cdots+a_n=0$.
Therefore any polynomial in W can be expressed as

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+(-a_1-a_2-\cdots-a_{n-1})x^n$$
$$=a_0+a_1(x-x^n)+a_2(x^2-x^n)+\cdots+a_{n-1}(x^{n-1}-x^n).$$

This means $W = \text{span}\left\{1, x - x^n, x^2 - x^n, \ldots, x^{n-1} - x^n\right\}$, so W is a subspace of P_n by Theorem 4.3.1(a). Since $a_0 + a_1\left(x - x^n\right) + a_2\left(x^2 - x^n\right) + \cdots + a_{n-1}\left(x^{n-1} - x^n\right) = 0$ implies $a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$, it follows that $\left\{1, x - x^n, x^2 - x^n, \ldots, x^{n-1} - x^n\right\}$ is linearly independent, hence it is a basis for W.

12. For $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ to have a horizontal tangent at $x = 0$, we must have $p'(0) = 0$.

Since $p'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$ it follows that $p'(0) = a_1 = 0$. The set of all polynomials $p(x)$ for which $a_1 = 0$ is $\text{span}\left\{1, x^2, x^3, \ldots, x^n\right\}$ and therefore a subspace of P_n.

Since the set $\left\{1, x^2, x^3, \ldots, x^n\right\}$ is clearly linearly independent and spans the subspace, it forms a basis for the subspace.

13. **(a)** A general 3×3 symmetric matrix can be expressed as $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

$$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ span the space of all 3×3 symmetric matrices. Also, these matrices are linearly indpendent, since $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ requires that all six coefficients in the linear combination above must be zero. We conclude that the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ form a basis for the space of all 3×3 symmetric matrices.

(b) A general 3×3 skew-symmetric matrix can be expressed as

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Clearly the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ span the space of all 3×3 skew-symmetric matrices. Also, these matrices are linearly indpendent, since

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

requires that all three coefficients in the linear combination above must be zero. We conclude that the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ form a basis for the space of all 3×3 skew-symmetric matrices.

14. **(a)** A submatrix $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ has nonzero determinant -1 therefore the rank of the original matrix is 2.

(b) All three 2×2 submatrices have zero determinant: $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$. Since determinant of any 1×1 submatrix of the original matrix is nonzero, the original matrix has rank 1.

(c) The original 3×3 matrix has zero determinant. A submatrix $\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$ has nonzero determinant -1 therefore the rank of the original matrix is 2.

(d) The original 3×3 matrix has nonzero determinant 30 therefore the rank of the original matrix is 3.

15. All submatrices of size 3×3 or larger contain at least two rows that are scalar multiples of each other, so their determinants are 0. Therefore the rank cannot exceed 2. The possible values are:

- $\operatorname{rank}(A)=2$, e.g., if $a_{51}=a_{16}=1$ regardless of the other values,
- $\operatorname{rank}(A)=1$, e.g., if $a_{16}=a_{26}=a_{36}=a_{46}=0$ and $a_{56}=1$ regardless of the other values, and
- $\operatorname{rank}(A)=0$ if all entries are 0.

17. The standard matrices for D_k, R_θ, and S_k are $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (assuming a shear in the x-direction).

(a) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and R_θ commute.

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & k\cos\theta - \sin\theta \\ \sin\theta & k\sin\theta + \cos\theta \end{bmatrix}$ does not generally equal

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + k\sin\theta & -\sin\theta + k\cos\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

therefore R_θ and S_k do not commute (same result is obtained if a shear in the y-direction is taken instead)

(c) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and S_k commute (same result is obtained if a shear in the y-direction is taken instead)

18. **(b)** Every vector (x, y, z) in R^3 can be expressed in exactly one way as a sum of a vector $(x, y, 0)$ in U and $(0, 0, z)$ in W. Consequently, $R^3 = U \oplus W$.

(c) Every vector (x, y, z) in R^3 can be expressed as a sum of a vector in U and a vector in V. However, in this case, this representation is not unique, for instance,

$$(1,2,3) = \underbrace{(1,1,0)}_{\text{vector in } U} + \underbrace{(0,1,3)}_{\text{vector in } V} = \underbrace{(1,3,0)}_{\text{vector in } U} + \underbrace{(0,-1,3)}_{\text{vector in } V}$$

We conclude that R^3 is not a direct sum of the xy-plane and the yz-plane.