Data Science prerequisite

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Linear Algebra

The very basic: Adding two Vectors.

$$c\mathbf{v} + d\mathbf{w} = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}2\\3\end{bmatrix}$$

Definition

The sum of cv and dw is a linear combination of v and w.

- How would cv look like in space?
- How would $cv + dw$ look like in space?

We will come back to this later...

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Dot Product

- Length of a vector. $\|v\| = \mathbf{v.v}$
- Unit vector : $\frac{v}{\|v\|}$
- Dot Product zero when vectors are perpendicular.

$$\mathbf{v.w} = v_1 w_1 + v_2 w_2$$

Matrices

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Linear Combination:

$$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

We can re-write it as:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

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Alternatively:

$$\mathbf{A}x = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \mathbf{b}$$

Here, consider x as input, A as the system model and b as the output.

Digression-Matrices: Dot Product picture

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0).(x_1,x_2,x_3) \\ (-1,1,0).(x_1,x_2,x_3) \\ (1,-1,1).(x_1,x_2,x_3) \end{bmatrix}$$

Ax=b

Think of b as known and look for x that solves it.

- *Old Question:* Compute linear combination of $x_1 u + x_2 v + x_3 w$ to find b.
- *New Question:* Which combination of u, v, w produce a particular vector b.

Independence and Dependence

- Matrix has no inverse.
- One of the vectors is the linear combination of other vectors.

Column Space

In $\mathbf{Ax} = \mathbf{b}$, what happens if we take all linear combinations of the columns of $\mathbf{A}$?

- Equivalent of multiplying A with every possible x.

Definition

The **column space** consists of all **linear combinations of the columns.** The combinations are all possible vectors Ax.

- To solve $Ax = b$ is to express b as the linear combination of the columns of A.
- The system $Ax = b$ is solvable only if b is in the column space of A $(C(A))$.

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Dimension, Basis

Definition

Dimension is the space that the columns of A can span.

Definition

Basis is the first n vectors of the matrix A that span the whole space.

- The columns of A are **independent** if $x = 0$ is the only solution of $Ax = 0$.
- The vectors $v_1, v_2, \cdots, v_n$ span the space if their combinations fill the space.
- Any n independent vectors in R^n must span R^n so they are basis.

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Projections

- What are the projections of $b = (2, 3, 4)$ onto the z-axis and the xy plane?
- What matrices produce those projections onto a line and a plane?

The projection matrix $\mathbf{P}$ multiplies $\mathbf{b}$ to give $\mathbf{p}$. Onto the z-axis:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Onto the xy plane:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Continuation: Projection

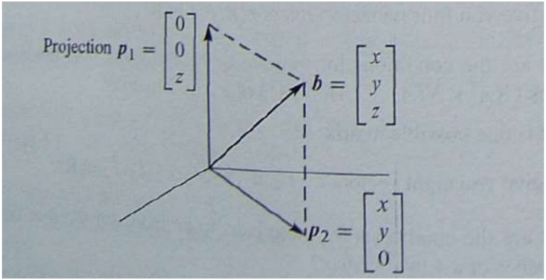

Figure: Projection $\mathbf{p_1} = \mathbf{P_1 b}$ and $\mathbf{p_2} = \mathbf{P_2 b}$ onto the z-axis and xy-plane

What does $p_1 + p_2$ gives???

Projection onto subspaces

- Projecting the given vector onto the column space of A.
- Problem is to project any b onto the column space of any $mbyn$ matrix.

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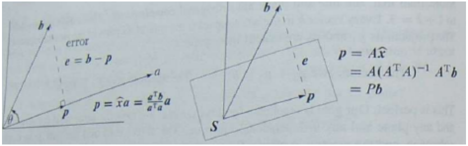
Projection Onto a Line

Problem Statement: Find the point p closest to b on a line in the direction of a.

- Let $p = \hat{x}a$.
- error $e = b - \hat{x}a$
- $a.(b - \hat{x}a) = 0$
- or $a.b - \hat{x}a.a = 0$
- $\hat{x} = \frac{a.b}{a.a} = \frac{a^T b}{a^T a}$



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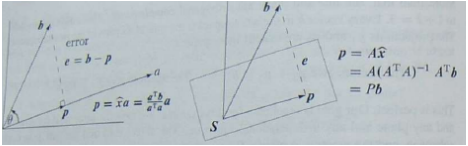

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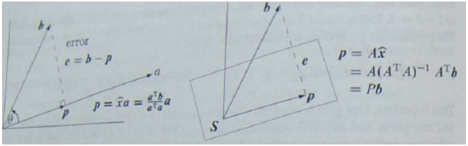

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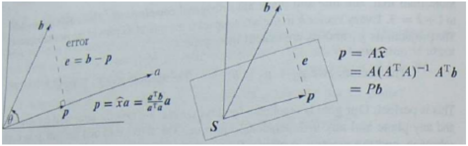

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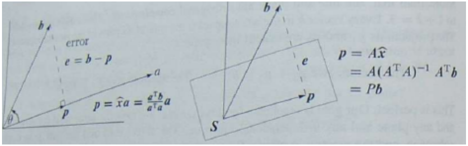

Projection Example

Project $b = (1, 1, 1)$ onto $a = (1, 2, 2)$ to find $p = \hat{x}a$.

- $a^T b = 5$
- $a^t a = 9$
- $p = \frac{5}{9}a$
- $e = b - p$
- $p = (\frac{5}{9}, \frac{10}{9}, \frac{10}{10})$ and $e = (\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9})$

What should be $e^T a$??

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Alternate Projection

- $\|p\| = \|b\| \cos\theta$
- $\|e\| = \|b\| \sin\theta$

Definition

Projection Matrix **P**:

$$p = a\hat{x} = a\frac{a^T b}{a^T a} = Pb$$

where the matrix is $\mathbf{P} = \frac{aa^T}{a^T a}$

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Projection: Example 2

Find Projection matrix onto the line through $a = (1, 2, 2)$.

- $$p = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{bmatrix}$$
- This matrix projects any vector b onto a.
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Projection properties

- What would $\mathbf{P}^2$ give??
- What would be P if we use $2a$??
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Projection Multi-dimension

- $A^T(b - A\hat{x}) = 0$ or $A^T A\hat{x} = A^T b$
- $p = A\hat{x} = A(A^T A)^{-1}A^T b$
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Orthonormal Basis

Definition

The vectors are orthonormal if:

$$q_i^T q_j = \begin{cases} 0, & when \quad i \neq j \quad \text{orthogonal vectors} \\ 1, & when \quad i = j \quad \text{orthonormal} \end{cases}$$

A matrix with orthonormal columns is assigned the letter **Q**.

- $\mathbf{Q^T Q = I}$
- $Q^T = Q^{-1}$

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- $\hat{x} = \frac{q.b}{q.q} = \frac{q^T b}{q^T q}$
- $\hat{x} = q^T b$
- $q^T b = |b| \cos\theta$
- $(q^T b)q =$ Projection of b in the direction of q
- $P = Q(Q^T Q)^{-1} Q^T = QQ^T$

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Projection using Orthonormal basis

Projection onto $q's$

Definition

$$P = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1^T \\ . \\ . \\ . \\ q_N^T \end{bmatrix} b = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1^T b \\ . \\ . \\ . \\ q_N^T b \end{bmatrix}$$

So b can be written as:

$$\mathbf{b} = \mathbf{q_1}(\mathbf{q_1^T b}) + \mathbf{q_2}(\mathbf{q_2^T b}) + \cdots + \mathbf{q_n}(\mathbf{q_n^T b})$$

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Conclusive Equation:

$$\mathbf{b} = \mathbf{q_1}(\mathbf{q_1^T b}) + \mathbf{q_2}(\mathbf{q_2^T b}) + \cdots + \mathbf{q_n}(\mathbf{q_n^T b})$$

The vector b has been broken down into perpendicular pieces. Then, by adding the pieces, we can put back the vector again.

Breaking down: Transform

Figure: Transform

Putting back together: Inverse Transform (King's men didn't know how to.)

Dot Product Continuous space

The standard inner product for complex valued functions is:

$$f.g = \int_{-\infty}^{\infty} f(t)g^*(t)dt$$

Compare it with vector space:

$$a.b = a^T b = \sum_{i=0}^{n} a_i b_i^{*}$$[1]

[1] *: Remember Hermitian matrix? Transpose of complex matrix is taken by inverting signs of complex values and changing rows and columns afterwards.

Fourier Series

Calculating coefficients: (breaking down:)

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{2\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{2\pi} f(x)\cos nx dx \\ b_n &= \frac{1}{\pi}\int_{2\pi} f(x)\sin nx dx \end{aligned}$$

(putting back together:)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Fourier Series

$$f(x) = \sum_{n=0}^{\infty}\left[\left(\frac{1}{\pi}\int_{2\pi} f(x)\cos nx dx\right)\cos nx + \left(\frac{1}{\pi}\int_{2\pi} f(x)\sin nx dx\right)\sin nx\right]$$

$$b = (q_1^T b)q_1 + (q_2^T b)q_2 + \cdots + (q_n^T b)q_n$$

Function/Vector/Signal/Control Input,...
Orthonormal Basis Function, Orthonormal Vectors,...
Brackets (): Dot Product/Inner Product/Convolution (special case),...

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$$f(x) = \sum_{n=0}^{\infty} \left[\left(\frac{1}{\pi} \int_{2\pi} f(x) \cos nx dx \right) \cos nx + \left(\frac{1}{\pi} \int_{2\pi} f(x) \sin nx dx \right) \sin nx \right]$$

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Other Transforms

Laplace Transform:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Power Series:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

Taylor series, Wavelet transforms, Maclaurin Series ...

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