Math-1st Paper

Integration (যোগজীকরণ)

Calculus (study of change)

- Rate of Change → Differenciation (অন্তরকলন)
- Result of Change → Integration (যোগজীকরণ)

Integration (সমাকলন বা যোগজীকরণ)

→ Differenciation এর বিপরীত প্রক্রিয়া

in Differenciation, $\frac{d}{dx}$, $\frac{\partial}{\partial x}$, $\frac{D}{Dx}$

in Integration, $\int$ → Summation

$$\frac{d}{dx}\{f(x)\} = f'(x)$$

$$\Rightarrow d\{f(x)\} = f'(x)\,dx$$

$$\Rightarrow \int d\{f(x)\} = \int f'(x)\,dx$$

$$\Rightarrow f(x) + c = \int f'(x)\,dx$$

$$\Rightarrow \int f'(x)\,dx = f(x) + c$$

→ constant of Integration
(সমাকলন ধ্রুবক)

Now, $\int d(\text{Rudro}) = \text{Rudro} + \boxed{c}$

→ why constant?

সমাকলন ধ্রুবক (Constant of Integration):

Now, $\frac{d}{dx}(\sin x) = \cos x$

$$\therefore \int \cos x\,dx = \sin x + c$$

So, $\frac{d}{dx}(\sin x) = \cos x$

$$\frac{d}{dx}(\sin x + 5) = \cos x$$

$$\frac{d}{dx}(\sin x - 10) = \cos x$$

$$\frac{d}{dx}(\sin x + 10^{1000}) = \cos x$$

Then. In Reverse process (Integration) we don't know the value. That's why we have to use c.

we get
This constant ~~is got~~ based upon on different condition or boundary constant.

Integration

Indefinite Integral (অনির্দিষ্ট সমাকল)

Definite Integral (নির্দিষ্ট সমাকল)

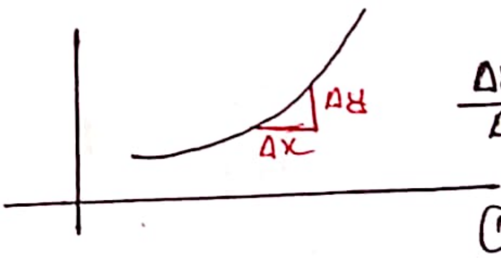
Indefinite Integral

Now, How can we mathematically define Integration?

Let, In differenciation

→ মূল নিয়ম → $\frac{d}{dx}\{f(x)\} = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$

→ ঢাল (Slope)



$\frac{\Delta y}{\Delta x}$ = slope

↓

একটি ভাগ প্রক্রিয়া

↓

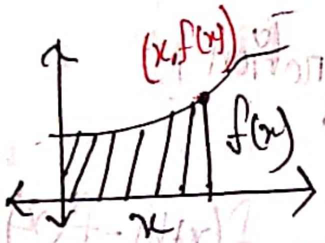
বিপরীত হলো গুন

Now.

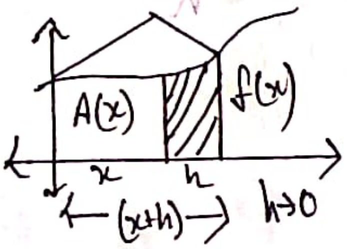
$y \cdot \Delta x = A$

∴ তাদের যোগের মান ক্ষেত্রফল (A)

Again Newton ও Libnitz ⟶



$A(x) = x f(x)$



Now, $A(x+h) - A(x) = h f(x)$

$$\therefore f(x) = \frac{A(x+h) - A(x)}{h}$$

$$\Rightarrow f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$\Rightarrow f(x) = \frac{d}{dx}\{A(x)\}$$

Area এর function কে differenciation করলে Normal function পাওয়া যায়,

যেহেতু Integration ; Differenciation এর বিপরীত—

∴ Normal function ~~এর Area~~ কে integrate করি তাহলে Area এর মানগুলো পাওয়া যাবে।

Laws of Integration

i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

Now, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1} \frac{d}{dx}\left(x^{n+1}\right)$

$\Rightarrow \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}(n+1)\, x^{n+1-1}$

$\Rightarrow \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$

$\Rightarrow \int d\left(\frac{x^{n+1}}{n+1}\right) = \int x^n dx$

$\Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c$ here, $n \neq -1$

$\therefore \int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$

$\therefore \int x dx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

$\therefore \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + c = \frac{0}{0} + c = ?$

So, $\int x^{-1} dx = \int \frac{1}{x} dx$

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ii) $\int \frac{1}{x} dx = \ln x + c$

Now, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\Rightarrow d(\ln x) = \frac{1}{x} dx$

$\Rightarrow \int d(\ln x) = \int \frac{1}{x} dx$

$\Rightarrow \ln x + c = \int \frac{1}{x} dx$

$\Rightarrow \int \frac{1}{x} dx = \ln x + c$

iii) $\int dx = x + c$

Now, $\int dx = \int 1 . dx$

$= \int x^0 . dx$

$= \frac{x^{0+1}}{0+1} + c$

$\Rightarrow \int dx = x + c$

$\int d(\sin\theta) = \sin\theta + c$

$\int d(\text{Math}) = \text{Math} + c$

iv) $\int e^x dx = e^x + c$

Now, $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow d(e^x) = e^x dx$

$\Rightarrow \int d(e^x) = \int e^x dx$

$\Rightarrow \int e^x dx = e^x + c$

VI $\int a^x dx = \frac{a^x}{\ln a} + c$

Now, $\frac{d}{dx}(a^x) = a^x \ln a$

$\Rightarrow d(a^x) = a^x \ln a \, dx$

$\Rightarrow \frac{d(a^x)}{\ln a} = a^x dx$

$\Rightarrow \int \frac{d(a^x)}{\ln a} = \int a^x dx$

$\Rightarrow \frac{1}{\ln a} \int d(a^x) = \int a^x dx$

$\Rightarrow \frac{1}{\ln a} a^x + c = \int a^x dx$

$\Rightarrow \int a^x dx = \frac{a^x}{\ln a} + c$

VII $\int \sin x \, dx = -\cos x + c$

Now, $\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow d(\cos x) = -\sin x \, dx$

$\Rightarrow \int d(\cos x) = -\int \sin x \, dx$

$\Rightarrow -\int d(\cos x) = \int \sin x \, dx$

$\Rightarrow \int \sin x \, dx = -\cos x + c$

Similarly,

vii] $\int \cos x\,dx = \sin x + c$

viii] $\int \sec^2 x\,dx = \tan x + c$

Now, $\frac{d}{dx}(\tan x) = \sec^2 x$

$\Rightarrow d(\tan x) = \sec^2 x\,dx$

$\Rightarrow \int d(\tan x) = \int \sec^2 x\,dx$

$\Rightarrow \int \sec^2 x\,dx = \tan x + c$

Similarly, ix] $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

x] $\int \sec x \tan x\,dx = \sec x + c$

Now, $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\Rightarrow \int d(\sec x) = \int \sec x \cdot \tan x\,dx$

$\Rightarrow \int \sec x \cdot \tan x\,dx = \sec x + c$

Similarly xi] $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + c$

~~xii]~~ $\cancel{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|} + c$

$\underline{x^{i}_{I}|}$ $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

Let, $f(x) = z$

Direct jump to this Line

$\Rightarrow \frac{d}{dx}\{f(x)\} = \frac{d}{dx} z$

$\Rightarrow f'(x) = dz/dx$

$\Rightarrow f'(x)dx = dz$

$\therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{dz}{z} = \int \frac{1}{z} dz = \ln|z| + C$

$\therefore \int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$

$\therefore \int \frac{d(2?)}{2?} = \ln|2?| + C$

$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$

P.1 $\int \sin\theta \, d(\sin\theta) = ?$

Soln: $\int \sin\theta \, d(\sin\theta) = \frac{\sin\theta^{1+1}}{1+1} + C = \frac{\sin^2\theta}{2} + C$ (Ans.)

$\underline{x^{ii}_{II}|}$ $\int \tan x dx = \ln|\sec x| + C$

Now, $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{-\sin x}{\cos x} dx$

$$= -\ln|\cos x| + c \qquad \left[\frac{d}{dx}(\cos x) = -\sin x\right]$$

$$= -\ln\left|\frac{1}{\sec x}\right| + c$$

$$= -\cancel{\ln 1}^{0} + \ln|\sec x| + c$$

$$\therefore \int \tan x\,dx = \ln|\sec x| + c$$

xiii) $\int \cot x\,dx = \ln|\sin x| + c$

Now, $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$$\therefore \int \cot x\,dx = \ln|\sin x| + c$$

xiv) ~~$\int \operatorname{cosec} x\,d$~~ $\int \operatorname{cosec} x\,dx = \ln|\tan x/2| + c$

Now, ~~M.I.~~ $\int \operatorname{cosec} x\,dx = \int \frac{1}{\sin x}\,dx$

$$= \int \frac{1}{2\sin x/2 \cos x/2}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 x/2}}{\frac{2\sin x/2 \cos x/2}{\cos^2 x/2}}\,dx$$

$$= \int \frac{\sec^2 x/2}{2\tan x/2}\,dx$$

$$= \int \frac{\frac{1}{2}\sec^2 x/2}{\tan x/2}\,dx \qquad \left[\frac{d}{dx}\left(\tan x/2\right) = \frac{1}{2}\sec^2 x/2\right]$$

$$\therefore \int \operatorname{cosec} x\,dx = \ln|\tan x/2| + c$$

M.2 $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} dx$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x . \cot x}{\operatorname{cosec} x - \cot x} dx$$

$$\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + C$$

finally, $\int \operatorname{cosec} x\, dx = \ln|\tan x/2| + C = \ln|\operatorname{cosec} x - \cot x| + C$

XVI $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

M.1 Now, $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \ln|\sec x + \tan x| + C \qquad \left[\frac{d}{dx}(\sec x + \tan x) = \sec^2 x + \sec x \tan x\right]$$

$$\therefore \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

M.2 $\int \sec x\, dx = \int \frac{1}{\cos x} dx$

$$= \int \frac{1}{\sin(\pi/2 + x)} dx$$

$$= \int \frac{1}{\sin 2(\pi/4 + x/2)} dx$$

$$= \int \frac{1}{2\sin(\pi/4 + x/2)\cos(\pi/4 + x/2)} dx$$

$$= \int \frac{\frac{1}{\cos^2(\pi/4 + x/2)}}{\frac{2\sin(\pi/4 + x/2)\cos(\pi/4 + x/2)}{\cos^2(\pi/4 + x/2)}} dx$$

$$= \int \frac{\frac{1}{2}\sec^2(\pi/4 + x/2)}{\tan(\pi/4 + x/2)} dx$$

$$\therefore = \ln|\tan(\pi/4 + x/2)| + c$$

$$\int \sec x\, dx = \ln|\tan(\pi/4 + x/2)| + c$$

$$\therefore \int \sec x\, dx = \ln|\sec x + \tan x| + c = \ln|\tan(\pi/4 + x/2)| + c$$

xvii) $\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx$

$\therefore \int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx \rightarrow$ Linear Property

xviii) $\int c f(x)\, dx = c \int f(x)\, dx$ [where, c = const.]

xviiii) $\int 1/\sqrt{x}\, dx = 2\sqrt{x} + c$

Now, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\Rightarrow$ ~~$d/dx\ d/dx(\sqrt{x})$~~ $d(\sqrt{x}) = \frac{1}{2\sqrt{x}} dx$

$\Rightarrow \int d\sqrt{x} = \frac{1}{2} \int \frac{1}{\sqrt{x}} dx$

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$\Rightarrow 2\int d\sqrt{x} = \int \frac{1}{\sqrt{x}} dx$

$\Rightarrow \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

Again, $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

Let, $f(x) = x \quad \therefore f'(x) = 1$

$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

P.1 $\int \frac{\cos x}{\sqrt{\sin x}} dx = 2\sqrt{\sin x} + c$

xix| $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

$\Rightarrow$ Method of substitution:

Let, $ax+b = z$

$\Rightarrow a\,dx = dz$

$\Rightarrow dx = \frac{dz}{a}$

$\therefore \int (ax+b)^n dx = \int z^n \frac{dz}{a}$

$= \frac{1}{a}\int z^n dz = \frac{1}{a}\frac{z^{n+1}}{n+1} + c$

$= \frac{z^{n+1}}{a(n+1)} + c$

$$\therefore \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

xx) $\int \sin(ax+b)\, dx$

Let, $ax+b = z$

$\Rightarrow a\,dx = dz$

$\Rightarrow dx = dz/a$

$$\therefore \int \sin z \frac{dz}{a} = \frac{1}{a}\int \sin z\, dz = \frac{1}{a}(-\cos z) + c$$

$$\therefore \int \sin(ax+b)\, dx = -\frac{\cos(ax+b)}{a} + c$$

$$\therefore \int \sin 5x\, dx = -\frac{\cos 5x}{5} + c$$

Similarly,

xxi) $\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + c$

xxii) $uv \rightarrow$ configauration

$$\int (uv)\, dx = u\int v\, dx - \int \left\{ \frac{d}{dx}(u) \int v\, dx \right\} dx$$

Let, $u, v, w \rightarrow x$ ao ~~...~~

we know, $\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx}$ —(i)

Let, $\frac{d}{dx}(\omega) = v$

$\Rightarrow \int d\omega = \int v\,dx$

$\Rightarrow \omega = \int v\,dx$ — (ii) [এ ক্ষেত্রে ধ্রুবক নাই]

Now, $\frac{d}{dx}(u\omega) = u\frac{d}{dx}(\omega) + \omega\frac{d}{dx}(u)$

$\Rightarrow \frac{d}{dx}\left\{u\int v\,dx\right\} = uv + \frac{d}{dx}(u)\int v\,dx$

$\Rightarrow d\left\{u\int v\,dx\right\} = uv\,dx + \left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

$\Rightarrow \int d\left\{u\int v\,dx\right\} = \int uv\,dx + \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

$\Rightarrow u\int v\,dx = \int uv\,dx + \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

$\Rightarrow \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$ ***

XXIII) $\int \ln x\,dx$

$= \int \underset{v}{1}\cdot\underset{u}{\ln x}\,dx$

$= \ln x\int dx - \int\left\{\frac{d}{dx}(\ln x)\int dx\right\}dx$

$= x\ln x - \int\left(\frac{1}{x}\cdot x\right)dx$

$= x\ln x - \int dx$

$\therefore \int \ln x dx = x\ln x - x + c$

xxiv) $\int \ln(ax)\, dx$

$= \int \underbrace{\ln(ax)}_{u} \cdot \underbrace{1}_{v}\, dx$

$= \ln(ax)\int dx - \int\left\{\frac{d}{dx}(\ln ax)\int dx\right\}dx$

$= x\ln ax - \int\left\{\frac{1}{ax}\cdot a\cdot x\right\}dx$

$= x\ln ax - \int dx$

$\int \ln(ax)dx = x\ln ax - x + c$

p. ~~$\int \ln(ax)dx$~~ $\int e^{mx}dx = \frac{e^{mx}}{m} + c$

xxv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}}$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

Let, $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

$\sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}(x/a)$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}}$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + C$$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}x/a + C$$

if, $a=1$, $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$

xxvii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$

Let, $x = a\tan\theta$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

Now, $\tan\theta = x/a$

$\therefore \theta = \tan^{-1}x/a$

Now, $\int \frac{dx}{a^2+x^2}$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \frac{1}{a}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}(x/a) + C$$

$$\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$$

if, $a=1$, $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

xxvii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$

Now, $\int \frac{dx}{(a^2-x^2)} = \int \frac{dx}{(a+x)(a-x)}$

Again, $\frac{1}{(a+x)(a-x)} = \frac{1}{(a+x)2a} + \frac{1}{(a-x)(2a)}$ [using thumbs Rule]

$= \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right]$

$\therefore \int \frac{dx}{(a+x)(a-x)} = \int \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right]dx$

$= \frac{1}{2a}\int\left[\frac{1}{a+x} + \frac{1}{a-x}\right]dx$

$= \frac{1}{2a}\left[\int \frac{dx}{a+x} + \int \frac{dx}{a-x}\right]$

$= \frac{1}{2a}\left[\ln(a+x) - \ln(a-x)\right] + c$

$\therefore \int \frac{dx}{a^2-x^2} = \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a}\left[\ln(a+x) - \ln(a-x)\right] + c = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

xxviii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$

Now, $\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x+a)(x-a)}$

$= \int \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right] dx$ [By Partial fraction]

$= \frac{1}{2a}\left[\int \frac{dx}{(x-a)} - \int \frac{dx}{(x+a)}\right]$

$= \frac{1}{2a}\left[\ln(x-a) - \ln(x+a)\right] + c$

$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$

xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$

Now, $\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta \tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{a\sec\theta \tan\theta\, d\theta}{\sqrt{a^2(\sec^2\theta-1)}}$

$= \int \frac{a\sec\theta \tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}}$

$= \int \frac{a\sec\theta \tan\theta\, d\theta}{a\tan\theta}$

$= \int \sec\theta\, d\theta$

Let, $x = a\sec\theta$

$\Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$\therefore \sec\theta = \frac{x}{a}$

$\Rightarrow \theta = \sec^{-1}(x/a)$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{x^2/a^2 - 1}$

$= \sqrt{\frac{x^2-a^2}{a^2}}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a^2}$

$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a^2}\right)$

$= \ln|\sec\theta + \tan\theta| + c'$

$= \ln\left|x/a + \frac{\sqrt{x^2-a^2}}{a}\right| + c'$

$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c'$

$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c'$

$= \ln|x+\sqrt{x^2-a^2}| - \ln a + c' \to c$

$= \ln|x+\sqrt{x^2-a^2}| + c \quad [c = c' - \ln a]$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + c$

XXX | $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}| + c$

Now, $\int \frac{dx}{\sqrt{x^2+a^2}}$

$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2(\tan^2\theta+1)}}$

$= \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta}$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + c'$

Let, $x = a\tan\theta$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

$\therefore \tan\theta = x/a$

$\Rightarrow \theta = \tan^{-1}(x/a)$

$\sec\theta = \sqrt{1+\tan^2\theta}$

$= \sqrt{1+x^2/a^2}$

$= \sqrt{\frac{x^2+a^2}{a^2}}$

$= \frac{\sqrt{x^2+a^2}}{a}$

$\therefore \theta = \sec^{-1}\left(\frac{\sqrt{x^2+a^2}}{a}\right)$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right| + c'$$

$$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c'$$

$$= \ln\left|x+\sqrt{x^2+a^2}\right| - \ln a + c'$$

$$= \ln\left|x+\sqrt{x^2+a^2}\right| + c \quad \boxed{c' - \ln a = c}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

xxxii) $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + c$$

Let, $I = \int \sqrt{x^2+a^2}\,dx \quad \boxed{(uv)}$

$$= \sqrt{x^2+a^2}\int dx - \int \left\{\frac{d}{dx}\left(\sqrt{x^2+a^2}\right)\int dx\right\}$$

$$= x\sqrt{x^2+a^2} - \int \left(\frac{2x}{2\sqrt{x^2+a^2}} \cdot x\right) dx$$

$$= x\sqrt{x^2+a^2} - \int \left(\frac{2x^2}{2\sqrt{x^2+a^2}}\,dx\right)$$

$$= x\sqrt{x^2+a^2} - \int \left(\frac{x^2}{\sqrt{x^2+a^2}}\right) dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{\left(\sqrt{a^2+x^2}\right)^2}{\sqrt{a^2+x^2}}\,dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - I + \int \frac{a^2}{\sqrt{x^2+a^2}}\,dx \quad \boxed{I = \int \sqrt{a^2+x^2}\,dx}$$

$$\Rightarrow 2I = x\sqrt{a^2+x^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\therefore I = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

xxxiii) $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \ln\left|x+\sqrt{x^2-a^2}\right| + c$

Let, $I = \int \sqrt{x^2-a^2}\,dx$ (uv)

$$= \sqrt{x^2-a^2}\int dx - \int \left\{\frac{d}{dx}\left(\sqrt{x^2-a^2}\right)\int dx\right\} dx$$

$$= x\sqrt{x^2-a^2} - \int \left(\frac{2x}{2\sqrt{x^2-a^2}} \cdot x\right) dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$\Rightarrow I = x\sqrt{x^2-a^2} - I - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow 2I = x\sqrt{x^2-a^2} - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

XXXIII) $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

Let, $I = \int \sqrt{a^2-x^2}\,dx$

$$= \sqrt{a^2-x^2}\int dx - \int \left\{\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)\int dx\right\} dx$$

$$= x\sqrt{a^2-x^2} - \int \left(\frac{-2x}{2\sqrt{a^2-x^2}} \cdot x\right) dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + \int \frac{a^2}{\sqrt{a^2-x^2}}\,dx$$

$$\therefore I = x\sqrt{a^2-x^2} - I + a^2 \sin^{-1}\frac{x}{a} + c$$

$$\therefore I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

Formulae of Integration

1. $\int dx = x + c$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
3. $\int e^x dx = e^x + c$
4. $\int a^x dx = \frac{a^x}{\ln a} + c$
5. $\int \frac{1}{x} dx = \ln x + c$
6. $\int \sin x \, dx = -\cos x + c$
7. $\int \cos x \, dx = \sin x + c$
8. $\int \sec^2 x \, dx = \tan x + c$
9. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
10. $\int \sec x \cdot \tan x \, dx = \sec x + c$
11. $\int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x + c$
12. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
13. $\int \frac{\frac{d}{dx}(\text{Denominator})}{\text{Denominator}} dx = \ln|\text{Denominator}| + c$
14. $\int \tan x \, dx = \ln|\sec x| + c$
15. $\int \cot x \, dx = \ln|\sin x| + c$
16. $\int \operatorname{cosec} x \, dx = \ln|\tan x/2| + c = \ln|\operatorname{cosec} x - \cot x| + c$

17. $\int \sec x \, dx = \ln|\sec x + \tan x| + c = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

18. $\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx$

19. $\int c f(x)\, dx = c \int f(x)\, dx$

20. $\int \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} + c$

21. $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$

22. $\int \frac{\frac{d}{dx}(\text{Denominator})}{\sqrt{\text{Denominator}}}\, dx = 2\sqrt{\text{Denominator}} + c$

23. $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

24. $\int \sin(ax+b)\, dx = \frac{-\cos(ax+b)}{a} + c$

25. $\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + c$

26. $\int uv\, dx = u\int v\, dx - \int \left\{ \frac{d}{dx}(u) \int v\, dx \right\} dx$

27. $\int \ln x\, dx = x \ln x - x + c$

28. $\int \ln(ax)\, dx = x \ln ax - x + c$

29. $\int e^{mx}\, dx = \frac{e^{mx}}{m} + c$

30. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

31. $a = 1$, $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$

32. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

33. if $a=1$, $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

34. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

35. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

36. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

37. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

38. $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

or, $\int \sqrt{a^2+x^2}\,dx$

39. $\int \sqrt{x^2-a^2} = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x-\sqrt{x^2-a^2}\right| + C$

40. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

Type: 4

Method of Substitution

P.1 $\int \sec^2 x\, e^{\tan x} dx$

Let, $\tan x = z$
$\Rightarrow \sec^2 x\, dx = dz$

$= \int e^z dz$

$= e^z + c$

$= e^{\tan x} + c$

P.2 $\int \frac{dx}{x(1+\ln x)}$

Let, $1+\ln x = z$
$\Rightarrow dx/x = dz$

$= \int dz/z$

$= \ln|z| + c$

$= \ln|1+\ln x| + c$.

P.3 $\int \frac{\sec^2 x\, dx}{\tan^2 x + 4\tan x + 5}$

Let, $\tan x + 2 = z$
$\Rightarrow \sec^2 x\, dx = dz$

$= \int \frac{\sec^2 x\, dx}{(\tan x)^2 + 2\cdot\tan x\cdot 2 + 2^2 + 1}$

$= \int \frac{\sec^2 x\, dx}{(\tan x + 2)^2 + 1} = \int \frac{dz}{(z)^2 + 1^2}$

$= \tan^{-1} z + c$

$= \tan^{-1}(\tan x + 2) + c$

P.4 $\int \frac{x^2 \tan^{-1}x^3}{1+x^6}dx$

$= \frac{1}{3}\int \frac{3x^2 \tan^{-1}x^3}{1+x^6}dx$

Let, $\tan^{-1}x^3 = z$

$\Rightarrow \frac{1}{1+x^6}3x^2dx = dz$

$\Rightarrow \frac{3x^2}{1+x^6}dx = dz$

$= \frac{1}{3}\int z\,dz$

$= \frac{1}{3}\frac{z^2}{2}+c$

$= \frac{(\tan^{-1}x^3)^2}{6}+c$

P.5 $\int \frac{e^x dx}{e^{2x}+4e^x+5}$

$= \int \frac{e^x dx}{(e^x)^2+2.e^x.2+2^2+1}$

Let, $e^x+2 = z$

$\Rightarrow e^x dx = dz$

$= \int \frac{e^x dx}{(e^x+2)^2+1}$

$= \int \frac{dz}{z^2+1} = \tan^{-1}z+c$

$= \tan^{-1}|e^x+z|+c$

P.6 $\int e^x\sqrt{e^{2x}+4e^x+5}\,dx$

$= \int e^x\sqrt{(e^x)^2+2.e^x.2+2^2+1}\,dx$

$= \int e^x\sqrt{(e^x+2)^2+1}\,dx$

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$$= \int \sqrt{z^2+1^2}\, dz$$

Let, $e^x+2 = z$
$\Rightarrow e^x dx = dz$

$$= \frac{z\sqrt{z^2+1}}{2} + \frac{1^2}{2}\ln\left|z+\sqrt{z^2+1^2}\right| + c$$

$$= \frac{(e^x+2)\sqrt{(e^x+2)^2+1}}{2} + \frac{1}{2}\ln\left|(e^x+2)+\sqrt{(e^x+2)^2+1}\right| + c$$

P.6 $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

$= \ln|e^x+e^{-x}| + c$ (using 13)

P.7 $\int \frac{x^4 dx}{1+x^{10}}$

Let, $x^5 = z$
$\Rightarrow 5x^4 dx = dz$
$\Rightarrow x^4 dx = \frac{1}{5} dz$

$$= \int \frac{x^4 dx}{1+(x^5)^2}$$

$$= \int \frac{\frac{1}{5} dz}{1+z^2}$$

$$= \frac{1}{5}\int \frac{dz}{1+z^2}$$

$$= \frac{1}{5}\tan^{-1}(z) + c$$

$$= \frac{1}{5}\tan^{-1}(x^5) + c$$

P.8 $\int \left(1-\frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$

Let, $x+\frac{1}{x} = z$

$\Rightarrow \left(1-\frac{1}{x^2}\right) dx = dz$

$= \int e^z dz$

$= e^z + c$

$= e^{x+\frac{1}{x}} + c$

Type:02

$$\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)} + c$$

P.1 $\int \frac{1+\cos x}{\sqrt{x+\sin x}} dx = 2\sqrt{x+\sin x} + c$

Type:03

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)} = \int \frac{dx}{a\left\{(x+f)^2+g^2\right\}}$$

$\rightarrow$ use desired formulae

P.1 $\int \frac{dx}{x^2+4x+5}$

$= \int \frac{dx}{x^2+2\cdot x\cdot 2+2^2+1}$

$= \int \frac{dx}{(x+2)^2+1}$

Let, $x+2 = z$

$\Rightarrow dx = dz$

$= \int \frac{dz}{z^2+1} = \tan^{-1} z + c = \tan^{-1}(x+2) + c$

Type:04

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \int \frac{dx}{\sqrt{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}} = \int \frac{dx}{\sqrt{a}\sqrt{(x\pm f)^2+g^2}} \longrightarrow \text{use desired formulae}$$

P.1 $\displaystyle\int \frac{dx}{\sqrt{4x^2+16x+20}}$

$$= \int \frac{dx}{\sqrt{4}\sqrt{x^2+4x+5}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2.x.2+2^2+1}}$$

Let, $x+2=z$
$\Rightarrow dx = dz$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{(x+2)^2+1}} = \frac{1}{2}\int \frac{dz}{\sqrt{z^2+1}} = \frac{1}{2}\ln\left|z+\sqrt{z^2+1}\right|+c$$

$$= \frac{1}{2}\ln\left|(x+2)+\sqrt{(x+2)^2+1^2}\right|+c$$

(Ans.)

Type:05

$$\int \frac{(cx+d)dx}{ax^2+bx+c} = \frac{m\frac{d}{dx}(ax^2+bx+c)\pm n}{ax^2+bx+c}$$

P.1 $\displaystyle\int \frac{(3x+5)dx}{x^2+8x+17}$

$$= \int \frac{\frac{3}{2}(2x+8)-7}{x^2+8x+17}dx$$

$$= \frac{3}{2}\int \frac{(2x+8)dx}{x^2+8x+17} - 7\int \frac{dx}{x^2+8x+17}$$

$$= \frac{3}{2}\ln\left|x^2+8x+17\right| - 7\int \frac{dx}{x^2+2.x.4+4^2+1}$$

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$$= \frac{3}{2} \ln|x^2+8x+17| - 7\int \frac{dx}{(x+4)^2+1}$$

Let, $(x+4) = z$

$\Rightarrow dx = dz$

$$= \frac{3}{2} \ln|x^2+8x+17| - 7\int \frac{dz}{z^2+1}$$

$$= \frac{3}{2} \ln|x^2+8x+17| - 7\left(\tan^{-1} z + c\right)$$

$$= \frac{3}{2} \ln|x^2+8x+17| - 7\tan^{-1}(x+4) + C$$

P.2 $\int \frac{(3x+5)}{x^2+4x+5}\,dx$

$$= \int \frac{\frac{3}{2}(2x+4)-1}{x^2+4x+5}\,dx$$

$$= \frac{3}{2}\int \frac{(2x+4)\,dx}{x^2+4x+5} - 1\int \frac{dx}{x^2+4x+5}$$

$$= \frac{3}{2} \ln|x^2+4x+5| - 1\int \frac{dx}{x^2+2.x.2+2^2+1}$$

$$= \frac{3}{2} \ln|x^2+4x+5| - 1\int \frac{dx}{(x+2)^2+1}$$

Let, $(x+2) = z$

$\Rightarrow dx = dz$

$$= \frac{3}{2} \ln|x^2+4x+5| - 1 \cancel{\tan^{-1}} \int \frac{dz}{z^2+1}$$

$$= \frac{3}{2} \ln|x^2+4x+5| - 1\left(\tan^{-1} z + C\right)$$

$$= \frac{3}{2} \ln|x^2+4x+5| - \tan^{-1}(x+2) + C$$

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Type: 06

$$\int \frac{(dx^2+ex+f)}{(ax^2+bx+c)}dx = \int \frac{m(ax^2+bx+c)+n\frac{d}{dx}(ax^2+bx+c)+l}{(ax^2+bx+c)}dx$$

P.1 $\int \frac{2x^2+5x+9}{x^2+4x+5}dx$

$$= \int \frac{2(x^2+4x+5) - \frac{3}{2}(2x+4)+5}{(x^2+4x+5)}dx$$

$$= \int \frac{2(x^2+4x+5)\,dx}{(x^2+4x+5)} - \frac{3}{2}\int \frac{(2x+4)}{x^2+4x+5}dx + 5\int \frac{dx}{x^2+4x+5}$$

$$= 2\int dx - \frac{3}{2}\int \frac{(2x+4)}{\cancel{x^2+4x+5\,(x+2)^2+1}\ x^2+4x+5} + 5\int \frac{dx}{(x+2)^2+1}$$

$$= 2x - \frac{3}{2}\ln|x^2+4x+5| + 5\int \frac{dz}{z^2+1}$$

Let, $(x+2) = z$
$\Rightarrow dx = dz$

$$= 2x - \frac{3}{2}\ln|x^2+4x+5| + 5(\tan^{-1}z + c)$$

$$= 2x - \frac{3}{2}\ln|x^2+4x+5| + 5\tan^{-1}(x+2) + c$$

19

Type: 07

$$\int \frac{(ex+f)}{\sqrt{ax^2+bx+c}}dx = \int \frac{m\frac{d}{dx}(ax^2+bx+c)+l}{\sqrt{ax^2+bx+c}}$$

P.1 $\int \frac{(3x+9)}{\sqrt{4x^2+16x+20}}dx$

$$= \int \frac{\frac{3}{8}(8x+16)+3}{\sqrt{4x^2+16x+20}}\,dx$$

$$= \frac{3}{8}\int \frac{(8x+16)\,dx}{\sqrt{(2x)^2+2\cdot 2x\cdot 4+4^2+4}} + \frac{3}{2}\int \frac{dx}{\sqrt{(x)^2+2\cdot x\cdot 4+2^2+1}}$$

Let, $x+2 = z$
$\Rightarrow dx = dz$

$$= \frac{3}{8}\times 2\sqrt{4x^2+16x+20} + \frac{3}{2}\int \frac{dx}{\sqrt{(x+2)^2+1}}$$

$$= \frac{3}{4}\sqrt{4x^2+16x+20} + \frac{3}{2}\int \frac{dz}{\sqrt{z^2+1}}$$

$$= \frac{3}{4}\sqrt{4x^2+16x+20} + \frac{3}{2}\ln\left|z+\sqrt{z^2+1^2}\right| + c$$

$$= \frac{3}{4}\sqrt{4x^2+16x+20} + \frac{3}{2}\ln\left|(x+2)+\sqrt{(x+2)^2+1}\right| + C$$

Type: 08

$$\int\sqrt{ax^2+bx+c}\,dx = \int\sqrt{a}\,\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\,dx = \sqrt{a}\int\sqrt{(x\pm f)^2+g^2}\,dx \rightarrow \text{use desired formulae}$$

P.1 $\int\sqrt{x^2+4x+5}\,dx$

Let, $x+2=z$
$\Rightarrow dx = dz$

$$= \int\sqrt{(x+2)^2+1^2}\,dx$$

$$= \int\sqrt{z^2+1^2}\,dz$$

$$= \frac{z\sqrt{z^2+1^2}}{2} + \frac{1^2}{2}\ln\left|z+\sqrt{z^2+1^2}\right| + c$$

$$= \frac{(x+2)\sqrt{(x+)^2+1}}{2} + \frac{1}{2}\ln\left|(x+2)+\sqrt{(x+2)^2+1^2}\right| + c$$

Type: 09

$$\int \frac{(ex^2+fx+g)}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{m(ax^2+bx+c)+n\frac{d}{dx}(ax^2+bx+c)+l}{\sqrt{ax^2+bx+c}}\,dx$$

P.1 $\displaystyle\int \frac{(3x^2+5x+9)}{\sqrt{4x^2+16x+20}}\,dx$

$$= \int \frac{\frac{3}{4}(4x^2+16x+20)-\frac{7}{8}(8x+16)+8}{\sqrt{4x^2+16x+20}}$$

$$= \frac{3}{4}\int \frac{(4x^2+16x+20)dx}{\sqrt{4x^2+16x+20}} - \frac{7}{8}\int \frac{(8x+16)\,dx}{\sqrt{4x^2+16x+20}} + 8\int \frac{dx}{\sqrt{4x^2+16x+20}}$$

$$= \frac{3}{4}\int\left(\sqrt{4x^2+16x+20}\right)dx - \frac{7}{8}\,2\sqrt{4x^2+16x+20} + \frac{8}{2}\int \frac{dx}{\sqrt{(x+2)^2+1}}$$

$$= \frac{3}{4}\int 2\sqrt{(x+2)^2+1}\,dx - \frac{7}{16}\sqrt{4x^2+16x+20} + \frac{8}{2}\int \frac{dz}{\sqrt{z^2+1}}$$

Let, $x+2=z$
$\Rightarrow dx = dz$

$$= \frac{3}{4}\times 2\int \sqrt{z^2+1}\,dx - \frac{7}{16}\sqrt{4x^2+16x+20} + \frac{8}{2}\int \frac{dz}{\sqrt{z^2+1}}$$

$$= \frac{3\times 2}{4}\left(\frac{z\sqrt{z^2+1}}{2} + \frac{1}{2}\ln\left|z+\sqrt{z^2+1}\right| + c\right) - \frac{7}{16}\sqrt{4x^2+16x+20} + \frac{8}{2}\left(\cancel{\tan^{-1}z+c}\right)$$
$$\left(\ln\left|z+\sqrt{z^2+1}\right|+c\right)$$

$$= \frac{3}{2}\left(\frac{(x+2)\sqrt{(x+2)^2+1}}{2} + \frac{1}{2}\ln\left|(x+2)+\sqrt{(x+2)^2+1}\right|\right) - \frac{7}{16}\sqrt{4x^2+16x+20} + \frac{8}{2}$$
$$\left(\ln\left|(x+2)+\sqrt{(x+2)+1}\right|\right)+c$$

Type: 10

$$\int \frac{dx}{(ax\pm b)\sqrt{(cx\pm d)}}$$ $\to$ Let, $cx\pm d = z^2$

P.1 $$\int \frac{dx}{(x-1)\sqrt{x+2}}$$

Let, $x+2 = z^2 \therefore x = z^2-2$

$\Rightarrow dx = 2z\,dz$

$$= \int \frac{2z\,dz}{(x-1)\sqrt{z^2}}$$

$$= \int \frac{2z\,dz}{(x-1)z}$$

$$= 2\int \frac{dz}{(z^2-2-1)}$$

$$= 2\int \frac{dz}{z^2-(\sqrt{3})^2}$$

$$= 2\,\frac{1}{2\sqrt{3}}\ln\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| + c$$

$$= 2\,\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + c$$

Type: 11

$$\int \frac{dx}{(cx\pm d)\sqrt{ax^2+bx+c}}$$ $\to$ $cx\pm d = 1/z$

~~P.1~~ $\int \frac{dx}{(cx \pm d)\sqrt{ax}}$

P.1 $\int \frac{dx}{(x-2)\sqrt{x^2-3}}$

Let, $x-2 = \frac{1}{z}$ $\therefore x = \frac{1}{z}+2$

$\Rightarrow dx = -\frac{dz}{z^2}$

$= \int \frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{(\frac{1}{z}+2)^2-3}}$

$= -\int \frac{dz}{z\sqrt{\frac{1}{z^2}+\frac{4}{z}+4-3}}$

$= -\int \frac{dz}{z\sqrt{\frac{1}{z^2}+\frac{4}{z}+1}}$

$= -\int \frac{dz}{z\sqrt{\frac{1+4z+z^2}{z^2}}}$

$= -\int \frac{dz}{z\,\frac{\sqrt{1+4z+z^2}}{z}}$

$= -\int \frac{dz}{\sqrt{1+4z+z^2}}$

$= -\int \frac{dz}{\sqrt{(z+2)^2-(\sqrt{3})^2}} = -\ln\left|(z+2)+\sqrt{(z+2)^2-(\sqrt{3})^2}\right| + C$

$= -\ln\left|\left(\frac{1}{x-2}+2\right)+\sqrt{\left(\frac{1}{x-2}+2\right)^2-(\sqrt{3})^2}\right| + C$

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Type: 12

$$\int \frac{dx}{(ax^2+cx+d)\sqrt{ax^2+bx+c}} \rightarrow \text{Let, } x = 1/z$$

P.1 $$\int \frac{dx}{(x^2-2)\sqrt{x^2+3}}$$

Let, $x = 1/z$

$\Rightarrow dx = -dz/z^2$

$$= \int \frac{-dz/z^2}{\left(\frac{1}{z^2}-2\right)\sqrt{\frac{1}{z^2}+3}}$$

$$= -\int \frac{dz/z^2}{\frac{1-2z^2}{z^2}\sqrt{\frac{1+3z^2}{z^2}}}$$

$$= -\int \frac{dz}{(1-2z^2)\frac{\sqrt{1+3z^2}}{z}}$$

$$= -\int \frac{z\,dz}{(1-2z^2)(\sqrt{1+3z^2})} \longrightarrow$$ square root এর ভেতরে u^2 ধরতে হবে।

Let, $1+3z^2 = u^2$

$\Rightarrow z^2 = \frac{u^2-1}{3}$

$\Rightarrow 6z\,dz = 2u\,du$

$\Rightarrow 3z\,dz = u\,du$

$\Rightarrow z\,dz = \frac{1}{3}u\,du$

$$= -\int \frac{\frac{1}{3}u\,du}{\left\{1-2\frac{(u^2-1)}{3}\right\}\sqrt{u^2}}$$

$$= -\int \frac{\frac{1}{3}u\,du}{\frac{3-2u^2+2}{3}\,u}$$

$$= -\int \frac{du}{5-2u^2}$$

$$= \int \frac{du}{2\left(u^2 - \frac{5}{2}\right)}$$

$$= \frac{1}{2}\int \frac{du}{u^2 - \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{\frac{5}{2}}} \ln\left|\frac{u - \sqrt{\frac{5}{2}}}{u + \sqrt{\frac{5}{2}}}\right| + c$$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{\frac{5}{2}}} \ln\left|\frac{\sqrt{3z^2+1} - \sqrt{\frac{5}{2}}}{\sqrt{3z^2+1} + \sqrt{\frac{5}{2}}}\right| + c \qquad \because \; 1+3z^2 = u^2 \Rightarrow u = \sqrt{1+3z^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{10}} \ln\left|\frac{\sqrt{\frac{3}{x^2}+1} - \sqrt{\frac{5}{2}}}{\sqrt{\frac{3}{x^2}+1} + \sqrt{\frac{5}{2}}}\right| + c \qquad x = \frac{1}{z} \Rightarrow z = \frac{1}{x}$$

Type: 13

$\sqrt{\frac{ax+b}{cx+d}} \rightarrow$ লবকে root free করতে হবে

P.1 $\int \sqrt{\frac{x-5}{x+3}}\, dx$

$$= \int \sqrt{\frac{(x-5)(x-5)}{(x+3)(x-5)}}\, dx$$

$$= \int \sqrt{\frac{(x-5)^2}{x^2-2x+15}}\, dx$$

$$= \int \frac{x-5}{\sqrt{x^2-2x-15}}\,dx$$

$$= \int \frac{\frac{1}{2}(2x-2) - 4}{\sqrt{x^2-2x-15}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x-2)\,dx}{\sqrt{x^2-2x-15}} - 4\int \frac{dx}{\sqrt{(x-1)^2-4^2}}$$

$$= \frac{1}{2} \times 2\sqrt{x^2-2x-15} - 4\ln\left|(x-1)+\sqrt{(x-1)^2-4^2}\right| + c$$

$$= \sqrt{x^2-2x-15} - 4\ln\left|(x-1)+\sqrt{x^2-2x-15}\right| + c$$

Type: 14

$$\int \frac{dx}{\sqrt{ax \pm b} \pm \sqrt{gx \pm c}}$$ → হরকে root free করতে হবে

P.1 $$\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}}$$

$$= \int \frac{(\sqrt{x-1}-\sqrt{x+1})\,dx}{(\sqrt{x-1}+\sqrt{x+1})(\sqrt{x-1}-\sqrt{x+1})}$$

$$= \int \frac{(\sqrt{x-1}-\sqrt{x+1})\,dx}{(\sqrt{x-1})^2-(\sqrt{x+1})^2}$$

$$=\int \frac{(\sqrt{x-1}-\sqrt{x+1})}{x-1-x-1}\,dx$$

$$=\int \frac{(\sqrt{x-1})-\sqrt{x+1})}{-2}\,dx$$

$$=\frac{1}{-2}\int\sqrt{x-1}\,dx+\frac{1}{2}\int\sqrt{x+1}\,dx$$

$$=-\frac{1}{2}\int(x-1)^{1/2}\,dx+\frac{1}{2}\int(x+1)^{1/2}\,dx$$

$$=-\frac{1}{2}\,\frac{(x-1)^{1/2+1}}{\frac{1}{2}+1}+\frac{1}{2}\,\frac{(x+1)^{1/2}}{\frac{1}{2}+1}+C$$

$$=-\frac{1}{2}\sqrt{(x-1)^3}+\frac{1}{2}\sqrt{(x+1)^3}+C$$

Type: 15

$$\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\quad[\beta>\alpha]$$

Let, $x=z^2+\alpha \Rightarrow x-\alpha=z^2$
$\Rightarrow dx=2z\,dz$

$$=\int\frac{2z\,dz}{\sqrt{z^2(\beta-\alpha-z^2)}}=\int\frac{2z\,dz}{z\sqrt{\beta-\alpha-z^2}}=2\int\frac{dz}{\sqrt{\beta-\alpha-z^2}}$$

P.1 $\displaystyle\int\frac{dx}{\sqrt{(\underset{\alpha}{x-1})(\underset{\beta}{2-x})}}$; $\beta>\alpha$

Let, $x-1=z^2$
$\Rightarrow x=z^2+1$
$\Rightarrow dx=2z\,dz$

$$=\int\frac{2z\,dz}{\sqrt{z^2(2-1-z^2)}}$$

$$= \int \frac{2z\,dz}{z\sqrt{1-z^2}}$$

$$= 2\int \frac{dz}{\sqrt{1-z^2}}$$

$$= 2\sin^{-1}z + C = 2\sin^{-1}\left(\sqrt{x-1}\right) + C$$

Type:16

$\int \frac{x^m}{(a\pm bx)^n}$ where. $m \in \mathbb{N} \longrightarrow (a\pm bx) = z$

P.1 $\int \frac{x^2}{(2+x)^{3/2}}\,dx$

Let, $2+x = z \Rightarrow x = z-2$

$\Rightarrow dx = dz$

$$= \int \frac{(z-2)^2}{z^{3/2}}\,dz$$

$$= \int \frac{z^2-4z+4}{z^{3/2}}\,dz$$

$$= \int \frac{z^2 dz}{z^{3/2}} - 4\int \frac{z}{z^{3/2}}\,dz + 4\int \frac{dz}{z^{3/2}}$$

$$= \int z^{1/2} dz - 4\int z^{-1/2} dz + 4\int z^{-3/2} dz$$

$$= \frac{z^{3/2}}{3/2} - 4\,\frac{z^{1/2}}{1/2} + 4\,\frac{z^{-1/2}}{-1/2} + C$$

$$= \frac{2}{3}(x+2)^{3/2} - 8(x+2)^{1/2} - 8(x+2)^{-1/2} + c$$

$$= \frac{2}{3}\sqrt{(x+2)^3} - 8\sqrt{(x+2)} - \frac{8}{\sqrt{x+2}} + c$$

#Note: যদি $m>3$ হয় তবে দ্বিপদী বিস্তৃতির সাহায্যে বিস্তৃতি করে সমাধান করতে হবে, যেমন:

$$(z-5)^5 = 5C_0 z^5(-5)^0 + 5C_1 z^{5-1}(-5)^1 + 5C_2 z^{5-2}(-5)^2 + \dots$$

Type: 17

$$\int \frac{dx}{x^m(a+bx)^n}$$

$m+n \rightarrow$ Positive integer, $m+n>0$

Let, $a+bx = ux$

P.1 $\int \frac{dx}{x^3(3+5x)^2}$

Let, $3+5x = ux \longrightarrow x = \frac{3}{u-5}$

$\Rightarrow \frac{3}{x} + 5 = u$

$\Rightarrow -\frac{3}{x^2}dx = du$

$\Rightarrow \frac{1}{x^2}dx = -\frac{du}{3}$

$$= \int \frac{dx}{x^2 \cdot x\,(ux)^2}$$

$$= \int \frac{-\frac{du}{3}}{u^2x^2 \cdot x}$$

$$= -\frac{1}{3}\int \frac{du}{u^2x^3}$$

$$= -\frac{1}{3}\int \frac{du}{u^2\left(\frac{3}{u-5}\right)^3}$$

$$= -\frac{1}{3}\int \frac{(u-5)^3}{27u^2}\,du$$

$$= -\frac{1}{81}\int \frac{(u-5)^3}{u^2}\,du$$

$\longrightarrow$ Type-16

$$= -\frac{1}{81}\int \frac{u^3-15u^2+75u-125}{u^2}\,du$$

$$= -\frac{1}{81}\int\left(u-15+\frac{75}{u}-\frac{125}{u^2}\right)du$$

$$= -\frac{1}{81}\left[\int u\,du - 15\int du + 75\int \frac{1}{u}\,du - 125\int \frac{1}{u^2}\,du\right]$$

$$= -\frac{1}{81}\left(\frac{u^2}{2} - 15u + 75\ln u + \frac{125}{u} + c'\right)$$

$$= -\frac{u^2}{162} - \frac{15u}{81} - \frac{75}{81}\ln u - \frac{125}{81u} + c$$

$$= -\frac{(3/x+5)^2}{162} - \frac{15(3/x+4)}{81} - \frac{75}{81}\ln\left(\frac{3}{x}+5\right) - \frac{125}{81\left(\frac{3}{x}+5\right)} + c$$

Type: 18

1) $\int \frac{dx}{x(a+bx^n)} \rightarrow x^n = \frac{1}{u}$

2) $\int \frac{dx}{x\sqrt{a+bx^n}} \rightarrow x^n = \frac{1}{u^2}$

P.1 $\int \frac{dx}{x(2+5x^{10})}$

Let, $x^{10} = \frac{1}{u}$

$\Rightarrow x^{10} = u^{-1}$

$\Rightarrow x^9\,dx = -\frac{1}{10u^2}\,du$

$$= \int \frac{x^9\,dx}{x^{10}(2+5x^{10})}$$

$$= \int \frac{-\frac{1}{10u^2}\,du}{\frac{1}{u}\left(2+\frac{5}{u}\right)}$$

$$= -\frac{1}{10}\int \frac{\frac{1}{u^2}\,du}{\frac{1}{u}\cdot\frac{1}{u}(2u+5)}$$

$$= -\frac{1}{10}\int \frac{\frac{1}{u^2}\,du}{\frac{1}{u^2}(2u+5)}$$

$$= -\frac{1}{10}\int \frac{du}{2u+5}$$

$$= -\frac{1}{20}\int \frac{2\,du}{2u+5}$$

$$= -\frac{1}{20}\ln|2u+5| + C$$

$$= -\frac{1}{20}\ln|2x^{-10}+5| + C$$

P2 $\displaystyle\int \frac{dx}{x\sqrt{2+\sqrt[3]{x}}}$

Let, $\sqrt[3]{x} = \frac{1}{u^2}$

$\Rightarrow x = \frac{1}{u^6}$

$\Rightarrow dx = -6\,\frac{1}{u^7}\,du$

$$= \int \frac{-6\frac{1}{u^7}\,du}{\frac{1}{u^6}\sqrt{2+\frac{1}{u^2}}}$$

$$= -6\int \frac{\frac{1}{u^7}\,du}{\frac{1}{u^6}\sqrt{\frac{2u^2+1}{u^2}}}$$

$$= -6\int \frac{\frac{1}{u^7}\,du}{\frac{1}{u^6}\,\frac{1}{u}\sqrt{2u^2+1}}$$

$$= -6\int \frac{dv}{\sqrt{2}\sqrt{v^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -3\sqrt{2}\int \frac{dv}{\sqrt{v^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -3\sqrt{2}\ \ln\left|v+\sqrt{v^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right| + c$$

$$= -3\sqrt{2}\ \ln\left|x^{-1/6}+\sqrt{\left(x^{-1/6}\right)^2+\frac{1}{2}}\right| + c$$

Type: 19

$$\int \frac{dx}{ax^4+bx^2+1}$$

P.1 $\int \sqrt{\tan x}\, dx$

~~Let, $\tan x = z^2 \Rightarrow \sqrt{\tan x} = z$~~

Let, $\sqrt{\tan x} = z$

$\Rightarrow \tan x = z^2$

$\Rightarrow \sec^2 x\, dx = 2z\, dz$

$\Rightarrow (1+\tan^2 x)\, dx = 2z\, dz$

$\Rightarrow (1+z^4)\, dx = 2z\, dz$

$\Rightarrow dx = \frac{2z\, dz}{1+z^4}$

$$= \int \sqrt{z^2}\, \frac{2z\, dz}{1+z^4}$$

$$= \int z\, \frac{2z\, dz}{1+z^4}$$

$$= \int \frac{2z^2\, dz}{1+z^4}\ \cancel{dt}$$

$$= \int \frac{(1+z^2)-(1-z^2)}{1+z^4}\, dz$$

$$= \int \frac{(1+z^2)}{1+z^4}\, dz - \int \frac{1-z^2}{1+z^4}\, dz$$

$$= \int \frac{\frac{1}{z^2}+1}{\frac{1}{z^2}+z^2}\,dz - \int \frac{\frac{1}{z^2}-1}{\frac{1}{z^2}+z^2}\,dz$$ [z^4 থাকলে লব ও হরকে z^2 দ্বারা ভাগ করতে হবে]

$$= \int \frac{\frac{1}{z^2}+1}{z^2+\frac{1}{z^2}}\,dz + \int \frac{1-\frac{1}{z^2}}{z^2+\frac{1}{z^2}}\,dz$$

$$= \int \frac{\frac{1}{z^2}+1}{\left(z-\frac{1}{z}\right)^2+2z\cdot\frac{1}{z}}\,dz + \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2-2z\cdot\frac{1}{z}}\,dz$$

$$= \int \frac{dv}{v^2+(\sqrt{2})^2} + \int \frac{dw}{w^2-(\sqrt{2})^2}$$

Let, $z-\frac{1}{z}=v$

$\Rightarrow \left(1+\frac{1}{z^2}\right)dz = dv$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{w-\sqrt{2}}{w+\sqrt{2}}\right| + c$$

Again, $z+\frac{1}{z}=w$

$\Rightarrow \left(1-\frac{1}{z^2}\right)dz = dw$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right| + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right| + c$$

Type: 20

$$\int \frac{dx}{x^{1/n} - x^{1/m}} \longrightarrow \text{Let, } x^{1/m} = z$$

$m > n \longrightarrow m$ এর মান

$\longrightarrow m, n \rightarrow$ L.C.M (ল.সা.গু)

P.1 $\int \frac{dx}{x^{1/3} - x^{1/6}}$

$6 > 3 \longrightarrow (3, 6) \longrightarrow$ L.C.M $\rightarrow 3 \underline{| 3, 6}$, $1, 2$ $\rightarrow 6$

Let, $x^{1/6} = z$

$\Rightarrow x = z^6$

$\Rightarrow dx = 6z^5 dz$

$$= \int \frac{6z^5 dz}{(z^6)^{1/3} - z}$$

$$= \int \frac{6z^5 dz}{z^2 - z}$$

$$= 6\int \frac{z^5 dz}{z(z-1)}$$

$$= 6\int \frac{z^4 dz}{(z-1)}$$

M.1

$$= 6\int \frac{(z^4 - 1) + 1}{(z-1)} dz$$

$$= 6\int \frac{(z^2-1)(z^2+1)}{(z-1)} dz + 6\int \frac{1}{z-1} dz$$

$$= 6\int \frac{(z-1)(z+1)(z^2+1)}{(z-1)} dz + 6\int \frac{1}{z-1} dz$$

$$= 6\int (z^2+1)(z+1)\,dz + 6\ln|z-1|$$

$$= 6\int (z^3+z^2+z+1)\,dz + 6\ln|z-1|$$

$$= 6\cdot\frac{z^4}{4} + 6\cdot\frac{z^3}{3} + 6\,\frac{z^2}{2} + 6z + 6\ln|z-1| + C$$

$$= \frac{3}{2}(x^{1/6})^4 + 2(x^{1/6})^3 + 3(x^{1/6})^2 + 6\cdot x^{1/6} + 6\ln|x^{1/6}-1| + C$$

$$= \frac{3}{2}x^{2/3} + 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6}-1| + C$$

Method:2

$$6\int \frac{z^4}{z-1}\,dz$$

$$= 6\int \frac{z^4 - z^3 + z^3 - z^2 + z^2 - z + z - 1 + 1}{z-1}\,dz$$

$$= 6\int \frac{z^3(z-1) + z^2(z-1) + z(z-1) + 1(z-1) + 1}{(z-1)}\,dz$$

$$= 6\int z^3\,dz + 6\int z^2\,dz + 6\int z\,dz + 6\int dz + 6\int \frac{dz}{z-1}$$

$$= 6\cdot\frac{z^4}{4} + 6\cdot\frac{z^3}{3} + 6\,\frac{z^2}{2} + 6z + 6\ln|z-1| + C$$

$$= \frac{3}{2}x^{2/3} + 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6}-1| + C$$

Type: 21

- e^x [exponential based Equation]

P.1 $\int \frac{dx}{e^x - e^{-x}}$

$= \int \frac{dx}{e^x - 1/e^x}$

$= \int \frac{e^x dx}{(e^x)^2 - 1}$

Let, $e^x = z$

$\Rightarrow e^x dx = dz$

$= \int \frac{dz}{z^2 - 1}$

$= \frac{1}{2} \ln \left| \frac{z-1}{z+1} \right| + c$

$= \frac{1}{2} \ln \left| \frac{e^x - 1}{e^x + 1} \right| + c$

Type: 22

$\int \frac{dx}{a\sin x + b\cos x + c}$, $\int \frac{dx}{\sin x + \cos x}$, $\int \frac{dx}{a\cos x + b\sin x + c}$, $\int \frac{dx}{a\sin x + b}$, $\int \frac{dx}{a\cos x + b}$

formulae we have to use,

* $\tan x = \frac{2\tan^2 x/2}{1 - \tan^2 x/2}$

* $\sin x = \frac{2\tan^2 x/2}{1 + \tan^2 x/2}$ ***

* $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$ ***

P.1 $\int \frac{dx}{a\sin x + b\cos x + c}$

$$= \int \frac{dx}{a\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + c}$$

$$= \int \frac{dx}{\frac{2a\tan\frac{x}{2} + b - b\tan^2\frac{x}{2} + c + c\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}\right)}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{2a\tan\frac{x}{2} + (c-b)\tan^2\frac{x}{2} + (b+c)}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{(c-b)\tan^2\frac{x}{2} + 2a\tan\frac{x}{2} + (b+c)}$$

$$= \int \frac{2dz}{(c-b)z^2 + 2az + (b+c)}$$

Let, $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

$$= \int \frac{2dz}{(c-b)\left\{z^2 + \left(\frac{2a}{c-b}\right)z + \left(\frac{b+c}{c-b}\right)\right\}}$$

$$= \frac{2}{c-b}\int \frac{dz}{z^2 + \frac{2a}{c-b} + \frac{b+c}{c-b}} \longrightarrow \text{use Type-03}$$

$$= \frac{2}{c-b} \int \frac{dz}{z^2 + \left(\sqrt{\frac{2a+b+c}{c-b}}\right)^2}$$

$$= \frac{2}{c-b} \cdot \frac{1}{\sqrt{\frac{2a+b+c}{c-b}}} \tan^{-1} \frac{z}{\sqrt{\frac{2a+b+c}{c-b}}} + C$$

$$= \frac{2}{c-b} \frac{\sqrt{c-b}}{\sqrt{2a+b+c}} \tan^{-1} \frac{\tan\frac{x}{2}\sqrt{c-b}}{\sqrt{2a+b+c}} + C$$

P.2 $\int \frac{dx}{6\cos x + 2\sin x + 9}$

$$= \int \frac{dx}{6\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 2\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) + 9}$$

$$= \int \frac{dx\,(1+\tan^2 x/2)}{6 - 6\tan^2 x/2 + 4\tan x/2 + 9 + 9\tan^2 x/2}$$

$$= \int \frac{(1+\tan^2 x/2)\,dx}{7\tan^2 x/2 + 15}$$

$$= \frac{1}{7}\int \frac{\sec^2 x/2\,dx}{\tan^2 x/2 + 15/7}$$

$$= \frac{1}{7}\int \frac{2dz}{z^2 + \left(\sqrt{15/7}\right)^2}$$

$$= \frac{2}{7}\int \frac{dz}{z^2 + \left(\sqrt{15/7}\right)^2}$$

$$= \frac{2}{7}\,\frac{1}{\sqrt{15/7}}\,\tan^{-1}\frac{z}{\left(\sqrt{15/7}\right)} + c$$

$$= \frac{2}{7} \times \frac{\sqrt{7}}{\sqrt{15}}\,\tan^{-1}\frac{\sqrt{7}}{\sqrt{15}} + c$$

Let, $\tan x/2 = z$

$\Rightarrow \frac{1}{2}\sec^2 x/2\,dx = dz$

$\Rightarrow \sec^2 x/2\,dx = 2dz$

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Type: 23

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x + c},\ \int \frac{dx}{a\sin^2 x + b},\ \int \frac{dx}{a\cos^2 x + b}$$

↳ প্রথমে লব ও হরকে $\cos^2 x$ দিয়ে ভাগ করে $\tan x = z$ ধরতে হবে।

P.1 $\displaystyle\int \frac{dx}{5\sin^2 x + 6\cos^2 x + 3}$

$$= \int \frac{\sec^2 x\, dx}{5\tan^2 x + 6 + 3\sec^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{5\tan^2 x + 6 + 3(1+\tan^2 x)}$$

$$= \int \frac{\sec^2 x\, dx}{8\tan^2 x + 9}$$

Let, $\tan x = z$
$\Rightarrow \sec^2 x\, dx = dz$

$$= \int \frac{dz}{8z^2 + 9}$$

$$= \int \frac{dz}{8\left(z^2 + \frac{9}{8}\right)}$$

$$= \frac{1}{8} \int \frac{dz}{z^2 + \left(\frac{3}{2\sqrt{2}}\right)^2}$$

$$= \frac{1}{8} \cdot \frac{1}{\frac{3}{2\sqrt{2}}} \tan^{-1}\left(\frac{z}{\frac{3}{2\sqrt{2}}}\right) + c$$

$$= \frac{1}{6\sqrt{2}} \tan^{-1}\left(\frac{2\sqrt{2}z}{3}\right) + c$$

$$= \frac{1}{6\sqrt{2}} \tan^{-1}\left(\frac{2\sqrt{2}\cdot\tan x}{3}\right) + c$$

P.2

$$\int \frac{dx}{6\cos^2 x + 2}$$

$$= \int \frac{\sec^2 x\, dx}{6 + 2\sec^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{6 + 2(1 + \tan^2 x)}$$

$$= \int \frac{\sec^2 x\, dx}{2\tan^2 x + 8}$$

Let, $\tan x = z$

$\Rightarrow \sec^2 x\, dx = dz$

$$= \int \frac{dz}{2z^2 + 8}$$

$$= \frac{1}{2} \int \frac{dz}{z^2 + 4}$$

$$= \frac{1}{2} \int \frac{dz}{z^2 + 2^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\left(\frac{z}{2}\right) + c$$

$$= \frac{1}{4} \tan^{-1}\left(\frac{\tan x}{2}\right) + c$$

Type: 24

$\int \sin^n x\,dx,\ \int \cos^n dx$

Case: 01 $n \rightarrow$ odd (বিজোড়)

i) $\int \cos^n x\,dx$ এর $\cos x\,dx$ বাদ রেখে $\cos^{n-1}x$ কে $\sin x$ এ convert করতে হবে এবং $\sin x = z$ ধরতে হবে।

ii) $\int \sin^n x\,dx$ এর $\sin x\,dx$ বাদ রেখে $\sin^{n-1}x$ কে $\cos x$ এ convert করতে হবে এবং $\cos x = z$ ধরতে হবে।

P.1 $\int \sin^5 x\,dx$

$$= \int \sin x \sin^4 x\,dx$$

$$= \int \sin x (\sin^2 x)^2\,dx$$

$$= \int \sin x (1-\cos^2 x)^2\,dx$$

$$= \int \sin x (1-\cos^2 x)^2\,dx$$

$$= -\int (1-z^2)^2\,dz$$

$$= -\int (1-2z+z^4)\,dz$$

$$= -z + \frac{2}{3}z^3 - \frac{1}{5}z^5 + C$$

Let, $\cos x = z$

$\Rightarrow -\sin x\,dx = dz$

$\Rightarrow \sin x\,dx = -dz$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

P.2 $\int \cos^5 x \, dx$

$= \int \cos x \cos^4 x \, dx$

Let, $\sin x = z$

$\cancel{\Rightarrow \sin x \, dx = dz}$

$\Rightarrow \cos x \, dx = dz$

$= \int \cos x (\cos^2 x)^2 dx$

$= \int \cos x (1 - \sin^2 x)^2 dx$

$= \int (1 - z^2)^2 dz$

$= \int (1 - \cancel{2z^4} \, 2z^2 + z^4) \, dz$

$= \int dz - \int (2z^2) dz + \int z^4 dz$

$= z - 2\frac{z^3}{3} + \frac{z^4}{5} + C$

$= \cancel{z} \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^4 x}{5} + C$

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Case:02 $n \rightarrow$ even (জোড়) হলে,

গুণিতক বা উপগুণিতক কোণে convert করতে হবে, যেমন:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin^3 x = \frac{3\sin x - \sin x}{4}$$

$$\cos^3 x = \frac{\cos 3x - 3\cos x}{4}$$

P.1 $\int \sin^4 x \, dx$

$$= \frac{1}{4}\int (2\sin^2 x)^2 dx$$

$$= \frac{1}{4}\int (1-\cos 2x)^2 dx$$

$$= \frac{1}{4}\int \{1 - 2\cos 2x + \cos^2 2x\} dx$$

$$= \frac{1}{4}x - \frac{1}{4}\times 2\frac{\sin 2x}{2} + \frac{1}{4}\int \cos^2 2x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\frac{\sin 4x}{4} + c$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + c$$

12 $\int \cos^4 x\,dx$

$= \frac{1}{4}\int (2\cos^2 x)^2\,dx$

$= \frac{1}{4}\int (1+\cos 2x)^2\,dx$

$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\,dx$

$= \frac{1}{4}x + \frac{1}{2}\,\frac{\sin 2x}{2} + \frac{1}{4}\int \cos^2 2x\,dx$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx$

$\cancel{= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{4}\int \left(\frac{1}{2} + \frac{\cos 4x}{2}\right) dx}$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\int (1+\cos 4x)\,dx$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\,dx$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\cos 4x}{4 \times 8} + C$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\cos 4x}{32} + C$

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Type: 25

sinx ও cosx এর কোণের মানের যোগ বা বিয়োগফলের আকারে প্রকাশ করতে হবে

$$2\sin A.\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A.\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A.\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\,\sin B = \cos(A-B) - \cos(A+B)$$

P.1 $\int \sin 4x \cos x \, dx$

$= \frac{1}{2}\int (2\sin 4x \cos x)\, dx$

$= \frac{1}{2}\int (\sin 5x + \sin 3x)\, dx$

$= \frac{1}{2}\frac{(-\cos 5x)}{5} + \frac{1}{2}\left(\frac{-\cos 3x}{3}\right) + C$

$= -\frac{1}{10}\cos 5x - \frac{1}{6}\cos 3x + C$

Type: 26

$\int \sin^m x \cos^n x \, dx$

Case: 01 $\left.\begin{array}{l} m \to \text{even} \\ n \to \text{odd} \end{array}\right\} \sin x = z$ Again, $\left.\begin{array}{l} m \to \text{odd} \\ n \to \text{even} \end{array}\right\} \cos x = z$

~~P.1 $\int \sin^4 x$~~

P.1 $\int \sin^4 x \cos^3 x\, dx$

Let,
$\sin x = z$
$\cos x\, dx = dz$

$= \int z^4 \cos^2 x\, dz$

$= \int z^4 (1 - \sin^2 x)\, dz$

$= \int z^4 (1 - z^2)\, dz$

$= \int (z^4 - z^6)\, dz$

$= \frac{z^5}{5} - \frac{z^7}{7} + c$

$= \frac{(\sin x)^5}{5} - \frac{(\sin x)^7}{7} + c$

Case:02 $m \to$ odd $\quad n \to$ odd

$m > n \to \sin x = z$

$m < n \to \cos x = z$

P.1 *** $\int \sin^{99} x \cos^3 x\, dx$

Let,
$\sin x = z$
$\Rightarrow \cos x\, dx = dz$

$= \int \sin^{99} x \cos^2 x \cos x\, dx$

$= \int \sin^{99} x (1 - \sin^2 x) \cos x\, dx$

$= \int z^{99} (1 - z^2)\, dz$

$$= \int (z^{99} - z^{101})\, dz$$

$$= \frac{z^{100}}{100} - \frac{z^{102}}{102} + C$$

$$= \frac{(\sin x)^{100}}{100} - \frac{(\sin x)^{102}}{102} + C$$

Case:03 $m \to$ even $\quad n \to$ even

সংযুক্ত ত্রিকোণমিতিক কোণের সংশ্লেষণ আকারে যেতে হবে.

P.1 $\int \sin^2 x \cos^2 x \, dx$

$$= \frac{1}{4}\int (2\sin x \cos x)^2 dx$$

$$= \frac{1}{4}\int (\sin 2x)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \frac{1}{2}(1 - \cos 4x)\, dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

P.2 $\int \sin^4 x \cdot \cos^2 x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \, dx$

$= \int \sin^4 x \, dx - \int \sin^2 x \, dx$ [type:24 case:02]

$= \frac{1}{4}\int (2\sin^2 x)^2 dx - \frac{1}{2}\int 2\sin^2 x \, dx$

$= \frac{1}{4}\int (1-\cos 2x)^2 dx - \frac{1}{2}\int (1-\cos 2x) \, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int (1-\cos 2x)^2 dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx + \frac{1}{2}x - \frac{1}{2}\frac{\cos 2x}{2}$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x + \frac{1}{4}\int \cos^2 2x \, dx + \frac{x}{2} - \frac{\cos 2x}{4}$

$= \frac{x}{4} - \frac{1}{2}\frac{\cos 2x}{2} + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x) \, dx + \frac{x}{2} - \frac{\cos 2x}{4}$

$= \frac{x}{4} - \frac{\cos 2x}{4} + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x \, dx + \frac{x}{2} - \frac{\cos 2x}{4}$

$= \frac{x}{4} - \frac{\cos 2x}{4} + \frac{x}{8} + \frac{1}{8}\frac{\cos 4x}{4} + \frac{x}{2} - \frac{\cos 2x}{4}$

$= \frac{x}{4} - \frac{\cos 2x}{4} + \frac{x}{8} + \frac{\cos 4x}{32} + \frac{x}{2} - \frac{\cos 2x}{4}$

Case:04 $m+n=-2$ এর, $\tan x = z$

P.1 *** $\int \sin^{-5/3}x \cdot \cos^{-1/3}x \, dx$ $\qquad \left(-\frac{5}{3}-\frac{1}{3}\right)=-2$

$$=\int \frac{dx}{\sin^{5/3}x \cos^{1/3}x}$$

$$=\int \frac{\frac{1}{\cos^{5/3}}\,dx}{\tan^{5/3}x \cos^{1/3}x}$$

$$=\int \frac{\cos^{-5/3}x \cdot \cos^{-1/3}x}{\tan^{5/3}x}\,dx$$

$$=\int \frac{\cos^{-2}x\,dx}{\tan^{5/3}x}$$

$$=\int \frac{\sec^2 x\,dx}{\tan^{5/3}x}$$

Let, $\tan x = z$
$\Rightarrow \sec^2 x\,dx = dz$

$$=\int \frac{dz}{z^{5/3}}$$

$$=\int z^{-5/3}\,dz = \frac{z^{-5/3+1}}{\left(-\frac{5}{3}+1\right)}+C = -\frac{3}{2}z^{-2/3}+C$$

$$=-\frac{3}{2}(\tan x)^{-2/3}+C$$

Type: 27

$\int \sec^m x\,dx,\ \int \operatorname{cosec}^m x\,dx$

Case: 01 : if, m = Positive even (ধনাত্মক জোড়), $\sec^2 x\,dx$ বের করতে হবে ও $\sec^{m-2}x$ কে $\tan x$ এ convert করতে হবে এবং $\tan x = z$ ধরতে হবে।

আবার, $\operatorname{cosec}^2 x\,dx$ বের করে $\operatorname{cosec}^{m-2}x$ কে $\cot x$ এ convert করতে হবে এবং $\cot x = z$ ধরতে হবে।

P.1 $\int \sec^6 x\,dx$

Let, $\tan x = z$

$\Rightarrow \sec^2 x\,dx = dz$

$= \int \sec^4 x \sec^2 x\,dx$

$= \int (1+\tan^2 x)^2 \sec^2 x\,dx$

$= \int (1+z^2)^2\,dz$

$= \int (1+2z^2+z^4)\,dz$

$= z + \frac{2}{3}z^3 + \frac{z^5}{5} + c$

$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$

Case: 02 $m \rightarrow$ Positive odd হলে,

apply Just Simple $\int (uv)dx$ method.

$u \rightarrow$ differenciation করা যাবে

$v \rightarrow$ Integration করা যাবে

P.1 $\int \operatorname{cosec}^3 x\,dx$

Let, $I = \int \operatorname{cosec}^3 x\,dx$

$= \int \operatorname{cosec} x \operatorname{cosec}^2 x\,dx$

$= \operatorname{cosec} x \int \operatorname{cosec}^2 x\,dx - \int \left\{ \frac{d}{dx}(\operatorname{cosec} x) \int \operatorname{cosec}^2 x\,dx \right\} dx$

$= -\operatorname{cosec} x \cdot \cot x - \left\{ \int (-\operatorname{cosec} x \cdot \cot x)(-\cot x)\,dx \right\}$

$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec} x \cdot \cot^2 x\,dx$

$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1)\,dx$

$= -\operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec}^3 x\,dx + \int \operatorname{cosec} x\,dx$

$= -\operatorname{cosec} x \cdot \cot x - I + \ln\left|\tan \frac{x}{2}\right| + c'$

$\Rightarrow 2I = -\operatorname{cosec} x \cdot \cot x + \ln\left|\tan \frac{x}{2}\right| + c'$

$\Rightarrow I = -\frac{1}{2} \operatorname{cosec} x \cdot \cot x + \frac{1}{2} \ln\left|\tan \frac{x}{2}\right| + c$

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P.2 $\int \sec^3 x\,dx$

Let, $I = \int \sec^3 x\,dx$

$= \int \sec x \cdot \sec^2 x\,dx$

$= \sec x \int \sec^2 x\,dx - \int \left\{ \left(\frac{d}{dx} \sec x \right) \int \sec^2 x\,dx \right\} dx$

$= \sec x \cdot \tan x - \int (\sec x \cdot \tan x \cdot \tan x)\,dx$

$= \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x\,dx$

$= \sec x \cdot \tan x - \int \sec x (\cancel{1}\sec^2 x - 1)\,dx$

$= \sec x \cdot \tan x - \int \sec^3 x\,dx + \int \sec x \cdot dx$

$\Rightarrow I = \sec x \cdot \tan x - I + \ln|\sec x + \tan x| + c'$

$\Rightarrow 2I = \sec x \cdot \tan x + \ln|\sec x + \tan x| + c'$

$\Rightarrow I = \cancel{s}\frac{1}{2} \sec x \cdot \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$

Type: 28

$\int \tan^n x\,dx$, $\int \cot^n x\,dx$; $n \to$ odd or even

$\int \tan^n x\,dx \to$ $\tan^2 x$ বের করতে হবে এবং $\tan^2 x$ কে $\sec^2 x$ এ convert করতে হবে তারপর বাকি থাকবে $\tan^{n-2} x$, $\tan x = z$ ধরতে হবে।

$\int \cot^n x\,dx \rightarrow$ $\cot^2 x$ বের করতে হবে এবং $\cot^2 x$ কে $\operatorname{cosec}^2 x$ এ convert করতে হবে। এবং বাকী থাকবে $\cot^{n-2} x$। $\cot x = z$ ধরতে হবে।

P.1 $\int \tan^5 x\,dx$

$= \int \tan^2 x \tan^3 x\,dx$

$= \int \tan^3 x (\sec^2 x - 1)\,dx$

$= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$ | Let

$= \int \tan^3 x \sec^2 x\,dx - \int \tan x(\sec^2 x - 1)\,dx$

$= \int \tan^3 x \sec^2 x\,dx - \int \tan x \sec^2 x\,dx + \int \tan x\,dx$

Let, $\tan x = z$
$\Rightarrow \sec^2 x\,dx = dz$

$= \int z^3\,dz - \int z\,dz + \int \tan x\,dx$

$= \frac{z^4}{4} - \frac{z^2}{2} + \ln|\sec x| + C$

$= \frac{(\tan x)^4}{4} - \frac{(\tan x)^2}{2} + \ln|\sec x| + C$

P.2 $\int \cot^5 x\,dx$

$= \int \cot^3 x \cot^2 x\,dx$

$$= \int \cot^3 x \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= \int \cot^3 x \cdot \operatorname{cosec}^2 x \, dx - \int \cot^3 x \, dx$$

$$= \int \cot^3 x \cdot \operatorname{cosec}^2 x \, dx - \int \cot x \cot^2 x \, dx$$

$$= \int \cot^3 x \cdot \operatorname{cosec}^2 x \, dx - \int \cot x \cdot \operatorname{cosec}^2 x \, dx + \int \cot x \, dx$$

Let, $\cot x = z$

$\Rightarrow -\operatorname{cosec}^2 x \, dx = dz$

$\Rightarrow \operatorname{cosec}^2 x \, dx = -dz$

$$= \int z^3 (-dz) - \int z(-dz) + \int \cot x \, dx$$

$$= -\frac{z^4}{4} + \frac{z^2}{2} + \ln|\sin x| + c$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln|\sin x| + c$$

Type: 29

(i) $\displaystyle\int \frac{a\sin x \pm b\cos x}{e\sin x \pm f\cos x} dx$ (ii) $\displaystyle\int \frac{a\sin x \pm b\cos x \pm c}{e\sin x \pm f\cos x \pm g}$

for, (i) & (ii)

$(a\sin x \pm b\cos x)$ or $(a\sin x \pm b\cos x \pm c)$

$$= M \times (\text{হর}) + N\left\{\frac{d}{dx}(\text{হর})\right\} + P$$

M & N → এর মান নির্ণয় করতে হবে।

P.1 $\int \frac{3\sin x+4\cos x}{7\sin x+5\cos x}\,dx$

Now. $3\sin x+4\cos x = M(7\sin x+5\cos x)+N(7\cos x-5\sin x)$

$= 7M\sin x+5M\cos x+7N\cos x-5N\sin x$

$\Rightarrow 3\sin x+4\cos x = \sin x(7M-5N)+\cos x(7N+5M)$

$\sin x$ ও $\cos x$ এর সহগ সমান করে।

$7M-5N = 3 \quad -(i)$

$5M+7N = 4 \quad -(ii)$

(i) ও (ii) সমাধান করে, $M = 41/74$ এবং $N = 13/74$

Now, $\int \frac{3\sin x+4\cos x}{7\sin x+5\cos x} = \int \frac{41/74(7\sin x+5\cos x)+13/74(7\cos x-5\sin x)}{7\sin x+5\cos x}\,dx$

$= \int 41/74 \frac{(7\sin x+5\cos x)dx}{7\sin x+5\cos x} + \int 13/74 \frac{7\cos x-5\sin x}{7\sin x+5\cos x}\,dx$

$= 41/74 \int dx + 13/74 \int \frac{7\cos x-5\sin x}{7\sin x+5\cos x}\,dx$

$= 41/74\, x + 13/74 \ln|7\sin x+5\cos x| + C$

Type:30

(uv) method & Modified (uv) method;

$$\int (uv)\,dx = u\int v\,dx - \int \left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$$

Now, $\int uv\,dx$

$= u\int v\,dx - \int \left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

$= uv_1 - \int u' v_1\,dx$

$= uv_1 - u'\int v_1\,dx + \int \left\{\frac{d}{dx}(u')\int v_1\,dx\right\}dx$

$= uv_1 - u'v_2 + \int u'' v_2\,dx$

$= uv_1 - u'v_2 + u''\int v_2\,dx - \int \left\{\frac{d}{dx}(u'')\int v_2\,dx\right\}dx$

$= uv_1 - u'v_2 + u''v_3 - \int u''' v_3\,dx$

differenciation $= u', u'', u''' \ldots$

integration $= v_1, v_2, v_3 \ldots$

- - - - - - - - - - - -

$$\boxed{\int (uv)\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + u''''v_5 + u^{n-1} \pm \int (u^n v_n)\,dx}$$

→ Chain Rule of Integration

$u \leftarrow \boxed{\text{LIATE}} \rightarrow v$

- E → Exponential (e^x)
- T → Trigonometry ($\sin x, \cos x$)
- A → Algebric (x^n)
- I → Inverse Trigonometry ($\sin^{-1}x, \cos^{-1}x$)
- L → Logarithamic ($\ln x, \log_{10} x$)

$u \rightarrow$ যাকে Differenciate করতে হবে

$v \rightarrow$ " Integration " "

P.1 $\int \ln x \cdot dx$

$= \int (\ln x) \cdot (x^0)\, dx$

$= \ln x \int dx - \int \left\{ \frac{d}{dx}(\ln x) \int dx \right\} dx$

$= x \ln x - \int \left(\frac{1}{x} \cdot x\right) dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

P.2 $\int \sin^{-1}x \, dx$

$= \int (\underset{u}{\sin^{-1}x}) \cdot \underset{v}{1} \, dx$

$= \sin^{-1}x \int dx - \int \left\{ \frac{d}{dx}(\sin^{-1}x) \int dx \right\} dx$

$= x\sin^{-1}x - \int \left(\frac{1}{\sqrt{1-x^2}} x \right) dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x + \frac{1}{2} 2\sqrt{1-x^2} + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$

P.3 $\int \cos^{-1}x \, dx$

~~$= \int (\sin^{-1}x) \cdot 1 \, dx$~~

$= \int (\underset{u}{\cos^{-1}x}) \underset{v}{1} \, dx$

$= \cos^{-1}x \int dx - \int \left\{ \frac{d}{dx}(\cos^{-1}x) \int dx \right\} dx$

$= x\cos^{-1}x - \int \left(-\frac{1}{\sqrt{1-x^2}} \times x \right) dx$

$$= x\cos^{-1}x - \int\left(-\frac{x}{\sqrt{1-x^2}}\right)dx$$

$$= x\cos^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= x\cos^{-1}x - \frac{1}{2}\,2\sqrt{1-x^2} + c$$

$$= x\cos^{-1}x - \sqrt{1-x^2} + c$$

P.4 $\int \tan^{-1}x\,dx$

$$= \int \underset{u}{\tan^{-1}x} \cdot \underset{v}{1}\,dx$$

$$= \tan^{-1}x\int dx - \int\left\{\frac{d}{dx}(\tan^{-1}x)\int dx\right\}dx$$

$$= x\tan^{-1}x - \int\left(\frac{x}{1+x^2}\right)dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + c$$

P.5 $\int \operatorname{cosec}^{-1}x\,.dx$

$$= \int \underset{u}{\operatorname{cosec}^{-1}x} \cdot \underset{v}{1}\,dx$$

$$= \operatorname{cosec}^{-1}x\int dx - \int\left\{\frac{d}{dx}(\operatorname{cosec}^{-1}x)\int dx\right\}$$

$$= x\operatorname{cosec}^{-1}x - \int -\frac{x}{x\sqrt{x^2-1}}\,dx$$

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$= x\operatorname{cosec}^{-1}x + \frac{1}{2}\int \frac{\cdot 2x}{x\sqrt{x^2-1}}\,dx$

$= x\operatorname{cosec}^{-1}x + \frac{1}{2}2\sqrt{x^2-1} + C$

$= x\operatorname{cosec}^{-1}x + \sqrt{x^2-1} + C$

P.6 $\int \sec^{-1}x\,dx$

$= \int \underset{u}{\sec^{-1}x} \cdot \underset{v}{1}\,dx$

$= \sec^{-1}x\int dx - \int\left\{\frac{d}{dx}(\sec^{-1}x)\cdot\int dx\right\}dx$

$= x\sec^{-1}x - \int \frac{x}{x\sqrt{x^2-1}}\,dx$

$= x\sec^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{x^2-1}}\,dx$

$= x\sec^{-1}x + \frac{1}{2}2\sqrt{x^2-1} + C$

$= x\sec^{-1}x + \sqrt{x^2-1} + C$

P.7 $\int \cot^{-1}x\,dx$

$= \cot^{-1}x\int dx - \int\left\{\frac{d}{dx}(\cot^{-1}x)\int dx\right\}$

$= x\cot^{-1}x - \int -\frac{x}{1+x^2}\,dx$

$= x\cot^{-1}x + \frac{1}{2}\int \frac{\cdot 2x}{1+x^2}\,dx$

$= x\cot^{-1}x + \frac{1}{2}\ln|1+x^2| + C$

P.8 $\int x^5 e^x \, dx$

$= x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + c$

P.9 $\int x^4 \sin x \, dx$

$= x^4(-\cos x) - 4x^3(-\sin x) + 12x^2(\cos x) - 24x(\sin x) + 24(-\cos x) + c$

$= -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c$

P.10 $\int x^7 e^{x^2} dx$

Let, $x^2 = z$
$\Rightarrow 2x\,dx = dz$
$\Rightarrow x\,dx = dz/2$

$= \int (x^2)^3 e^{x^2} x \, dx$

$= \int z^3 e^z \, dz/2$

$= \frac{1}{2}\int z^3 e^z \, dz$

$= \frac{1}{2}\left[z^3e^z - 3z^2e^z + 6ze^z - 6e^z + c'\right]$

$= \frac{1}{2}\left[x^6e^{x^2} - 3x^4e^{x^2} + 6x^2e^{x^2} - 6e^{x^2} + c\right]$

Type: 31

$\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$

P.1 $\int e^x \sin x\,dx$

Let, $I = \int e^x \sin x\,dx$

Ans. $= \sin x\int e^x dx - \int\left\{\frac{d}{dx}(\sin x)\int e^x dx\right\}dx$

$= e^x\sin x - \int \cos x\, e^x dx$

$= e^x\sin x - \cos x\int e^x dx + \int\left\{\frac{d}{dx}(\cos x)\int e^x dx\right\}dx$

$= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx + c'$

$\Rightarrow I = e^x\sin x - e^x\cos x - I + c'$

$\Rightarrow 2I = e^x\sin x - e^x\cos x + c'$

$\Rightarrow I = \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x + c$

P.2 $\int e^x\cos x\,dx$

Let, $I = \int e^x\cos x\,dx$

$= \cos x\int e^x dx - \int\left\{\frac{d}{dx}(\cos x)\int e^x dx\right\}dx$

$= e^x\cos x + \int e^x\sin x\,dx$

$= e^x\cos x + \sin x \int e^x dx - \int \{ \frac{d}{dx}(\sin x) \int e^x dx \} dx$

$= e^x\cos x + e^x \sin x - \int e^x \cos x dx + c'$

$\Rightarrow I = e^x\cos x + e^x\sin x - I + c'$

$\Rightarrow I + I = e^x\cos x + e^x \sin x + c'$

$\Rightarrow 2I = e^x\cos x + e^x\sin x + c'$

$\Rightarrow I = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + c$

Type: 32

$\int e^{ax}[af(x) + f'(x)]dx = e^{ax}f(x) + c$

Proof: $\int e^{ax}[af(x) + f'(x)]dx$

$= \int e^{ax}[af(x)]dx + \int e^{ax} f'(x) dx$

$= a\int e^{ax} f(x) dx + \int e^{ax} f'(x) dx$

$= a\left[f(x)\int e^{ax}dx - \int\{\frac{d}{dx}[f(x)]\int e^{ax}dx\}dx\right] + \int e^{ax} f'(x) dx$

$= a\frac{1}{a}e^{ax}f(x) - a\int f'(a)\cdot\frac{1}{a}e^{ax}dx + \int e^{ax} f'(x) dx + c$

$= e^{ax}f(x) - \int e^{ax}f'(x)dx + \int e^{ax}f'(x)dx + c$

$= e^{ax}f(x) + c$

$\int e^{ax}[af(x) + f'(x)]dx = e^{ax}f(x) + c$

P.1 $\int e^x[\sin x+\cos x]dx = e^x\sin x + c$

P.2 $\int e^x[\sec x+\sec x\tan x]dx = e^x\sec x + c$

P.3 $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{e^x}{x}+c$

P.4 $\int \frac{xe^x}{(x+1)^2}dx = \int \frac{\{(x+1)-1\}e^x}{(x+1)^2}dx$

$$= \int e^x\left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}dx$$

$$= \frac{e^x}{x+1}+c$$

P.5 $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$

$$= \int \frac{(x^2+2x+1-2x)e^x}{(x+1)^2}dx$$

$$= \int e^x dx - 2\int \frac{xe^x}{(x+1)^2}dx$$

$$= e^x - 2\int \frac{\{(x+1)-1\}e^x}{(x+1)^2}dx$$

$$= e^x - 2\int e^x\left\{\frac{1}{(x+1)}-\frac{1}{(x+1)^2}\right\}dx$$

$$= e^x - 2e^x\frac{1}{x+1}+c = e^x - 2\frac{e^x}{x+1}+c$$

P.6 $\int \left\{ \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \right\} dx$

Let, $\ln x = z$

$\Rightarrow x = e^z$

$\Rightarrow dx = e^z dz$

$= \int \left(\frac{1}{z} - \frac{1}{z^2} \right) e^z dz$

$= \int e^z \left(\frac{1}{z} - \frac{1}{z^2} \right) dz$

$= e^z \frac{1}{z} + c$

$= x \frac{1}{\ln x} + c = \frac{x}{\ln x} + c$

Type: 33

Integration of Partial function (আংশিক ভগ্নাংশে প্রকাশ করে যোগজীকরণ)

Case: 01 (P.1) $\int \frac{(x+1)}{(x+2)(x+3)} dx$

M-1 $\frac{x+1}{(x+2)(x+3)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+3)}$

M-2 $\frac{x+1}{(x+2)(x+3)} = \frac{(-2+1)}{(x+2)(-2+3)} + \frac{(-3+1)}{(x+3)(-3+2)}$

$= -\frac{1}{x+2} + \frac{2}{x+3}$

$\therefore \int \frac{x+1}{(x+2)(x+3)} dx = \int \left[-\frac{1}{x+2} + \frac{2}{x+3} \right] dx$

$$= -\int \frac{dx}{x+2} + 2\int \frac{dx}{x+3}$$

$$= -\ln|x+2| + 2\ln|x+3| + C$$

P.2 $\displaystyle\int \frac{2x-1}{x(x-1)(x-2)}\,dx$

$$\frac{2x-1}{x(x-1)(x-2)} = \frac{0-1}{x(0-1)(0-2)} + \frac{(2-1)}{(x-1)(1)(1-2)} + \frac{(2\cdot 2-1)}{(x-2)(2-1)(2)}$$

$$= \frac{-1/2}{x} - \frac{1}{x-1} + \frac{3/2}{x-2}$$

$$\therefore \int \frac{2x-1}{x(x-1)(x-2)}\,dx = \int \left[\frac{-1/2}{x} - \frac{1}{x-1} + \frac{3/2}{x-2}\right] dx$$

$$= \int \frac{-1/2}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{3/2}{x-2}\,dx$$

$$= -\frac{1}{2}\int \frac{dx}{x} - \int \frac{dx}{x-1} + \frac{3}{2}\int \frac{dx}{x-2}$$

$$= -\frac{1}{2}\ln x - \ln|x-1| + \frac{3}{2}\ln|x-2|$$

Case:02 **P.1** $\displaystyle\int \frac{x\,dx}{(x+2)(x-1)^2}$

$$\frac{x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow x = A(x-1)^2 + B(x-1)(x+2) + C(x+2) \text{ --- (i)}$$

$x = 1$	$x = -2$
$1 = C(x+2)$	$-2 = (A)(-3)^2$
$\Rightarrow C = \frac{1}{3}$	$\Rightarrow A = -\frac{2}{9}$

এখন সরলীকরণ করে পাই,

$$x = A(x^2 - 2x + 1) + B(x^2 + 2x - x - 2) + C(x+2)$$

$$\Rightarrow x = (A+B)x^2 - 2Ax + A + Bx - 2B + Cx + C$$

$$\Rightarrow x = (A+B)x^2 + (C - 2A + B)x + (A - 2B + C)$$

x^2- এর সহগ সমান করে পাই,

$$0 = A + B$$

$$\Rightarrow B = -A = \left(-\frac{-2}{9}\right) = \frac{2}{9}$$

$$\therefore \int \frac{x}{(x+2)(x-1)^2} dx = \int \left[\frac{-2/9}{x+2} + \frac{2/9}{x-1} + \frac{1/3}{(x-1)^2}\right] dx$$

$$= -\frac{2}{9}\int \frac{dx}{x+2} + \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2}$$

$$= -\frac{2}{9} \ln|x+2| + \frac{2}{9}|x-1| \ \frac{1}{3} \frac{1}{x-1} + c$$

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P.2 $\int \frac{dx}{x^2(x-1)}$

Let, $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$\Rightarrow 1 = Ax(x-1) + B(x-1) + Cx^2$ —(i)

$x=0$	$x=1$	সহগ সমীকরণ
$1 = B(0-1)$	$1 = C\cdot 1^2$	$0 = A + C$
$\Rightarrow B = -1$	$\Rightarrow C = 1$	$\Rightarrow A = -C = -(1)$
		$\Rightarrow A = -1$

$$\therefore \int \frac{dx}{x^2(x-1)} = \int \left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right] dx$$

$$= -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1}$$

$$= -\ln x + \frac{1}{x} + \ln|x-1| + c$$

* $\frac{x+1}{x(x-1)^2(x^2+4)}$

Case: 03 **P.1** $\int \frac{dx}{x(x^2+1)}$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)x \quad \cdots (i)$

$x=0$	$x^2 \rightarrow A+B=0$	$x \rightarrow 0 = C$
$1 = A(0+1)$	$\Rightarrow B = -A = -1$	$\Rightarrow C = 0$
$\Rightarrow A = 1$	$\Rightarrow B = -1$	

$$\therefore \int \frac{dx}{x(x^2+1)} = \int \left[\frac{1}{x} + \frac{(-x)}{x^2+1}\right] dx$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x\,dx}{x^2+1}$$

$$= \ln x - \frac{1}{2}\ln|x^2+1| + C$$

$$* \quad \frac{x+1}{x(x-1)^2(x^2+4)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+4}$$

Type: 34
Special Problems

P.1 $\displaystyle\int \frac{(2+\tan^2\theta)\sec^2\theta\, d\theta}{1+\tan^3\theta}$

Let, $u = \tan\theta$

$\Rightarrow du = \sec^2\theta\, d\theta$

$$= \int \frac{(2+u^2)\,du}{1+u^3}$$

$$= \int \frac{u^2+2}{u^3+1}\,du$$

$$= \int \frac{(u^2+2)}{(u+1)(u^2-u+1)}\,du$$

Now, $$\frac{u^2+2}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{(u^2-u+1)}$$

$$\Rightarrow u^2+2 = A(u^2-u+1) + (Bu+C)(u+1)$$

$$\Rightarrow u^2+2 = Au^2 - Au + A + Bu^2 + Bu + Cu + C = u^2(A+B) + u(-A+B+C) + (A+C)$$

~~Now~~ $u = -1$	$u^2 \rightarrow$ এর সহগ সমীকৃত.	ধ্রুবক সমান করে
$\Rightarrow 1+2 = A(1+1+1)$	$A+B = 1$	$A+C = 2$
$\Rightarrow A = 1$	$\Rightarrow B = 0$	$\Rightarrow C = 1$

$$\therefore \int \frac{u^2+2}{(u+1)(u^2-u+1)}\,du = \int \left[\frac{1}{u+1} + \frac{C}{(u^2-u+1)}\right] du$$

$$= \int \frac{du}{u+1} + \int \frac{C}{u^2-u+1}\,du$$

$$= \ln|u+1| + \int \frac{du}{\left(u^2 - 2u\frac{1}{2} + \left(\frac{1}{2}\right)^2\right) + \left(\frac{3}{4}\right)}$$

$$= \ln|u+1| + \int \frac{du}{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \ln|u+1| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(u-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$= \ln|u+1| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\left(u-\frac{1}{2}\right)}{\sqrt{3}} + c$$

$$= \ln|\tan\theta + 1| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\tan\theta - 1}{\sqrt{3}} + c$$

Type: 35

$\int \frac{dx}{(x\pm a)^p (x\pm b)^q}$, p & q are positive Integer also $a \neq b \longrightarrow z = \frac{x \pm a}{x \pm b}$

P.1 $\int \frac{dx}{(x-1)^2 (x-2)^3}$

Let, $z = \frac{x-1}{x-2}$ $\therefore (x-1) = z(x-2)$

$\Rightarrow xz - 2z = x - 1$

$\Rightarrow xz - x = 2z - 1$

$\Rightarrow x(z-1) = 2z - 1$

$$\Rightarrow x = \frac{2z-1}{z-1}$$

$$\Rightarrow dx = \frac{(z-1)d(2z-1)-(2z-1)d(z-1)}{(z-1)^2}dz$$

$$\Rightarrow dx = \frac{(z-1)(2)-(2z-1)(1)}{(z-1)^2}dz$$

$$\Rightarrow dx = \frac{2z-2-2z+1}{(z-1)^2}dz$$

$$\Rightarrow dx = -\frac{dz}{(z-1)^2}$$

Now, $$\int \frac{dx}{(x-1)^2(x-2)^3} = \int \frac{-\frac{dz}{(z-1)^2}}{\left\{\frac{2z-1}{z-1}-1\right\}^2\left\{\frac{2z-1}{z-1}-2\right\}^3}$$

$$= -\int \frac{1}{(z-1)^2} \cdot \frac{dz}{\left\{\frac{2z-1-z+1}{z-1}\right\}^2\left\{\frac{2z-1-2z+2}{z-1}\right\}^3}$$

$$= -\int \frac{1}{(z-1)^2} \frac{dz}{\frac{z^2}{(z-1)^2}\frac{1}{(z-1)^3}}$$

$$= -\int \frac{1}{(z-1)^2} \frac{(z-1)^5 dz}{z^2}$$

$$= -\int \frac{(z-1)^3}{z^2}dz$$

$$= -\int \frac{(z^3 - 3z^2 + 3z - 1)}{z^2}\, dz$$

$$= -\left[\int z\,dz - 3\int dz + 3\int \frac{dz}{z} - \int \frac{dz}{z^2}\right]$$

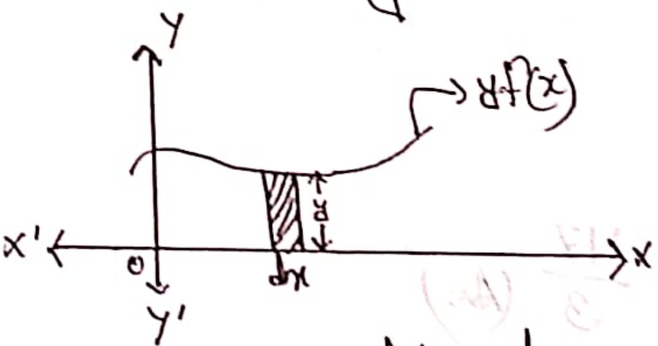
$$= -\left[\frac{z^2}{2} - 3z + 3\ln z + \frac{1}{z}\right] + C$$

$$= -\left[\frac{1}{2}\left(\frac{x-1}{x-2}\right) - 3\left(\frac{x-1}{x-2}\right) + 3\ln\left(\frac{x-1}{x-2}\right) + \frac{x-2}{x-1}\right] + C$$

Definite Integral

At the beginning, ⇒ Normal function কে integrate করলে Area function পাওয়া যায়।

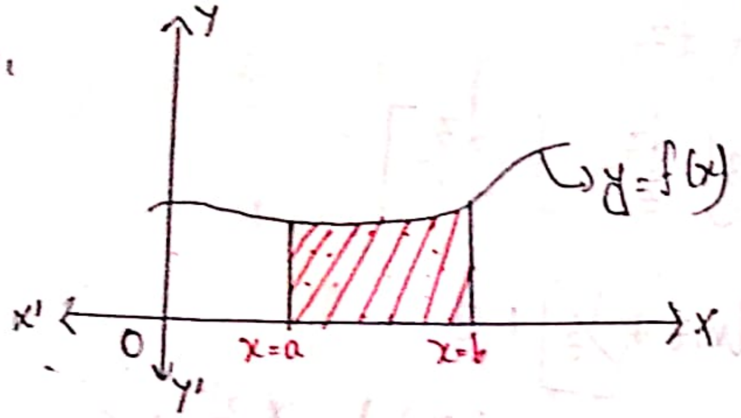
⇒ Integration এর মান এর Area এর সমান হয়েছে।

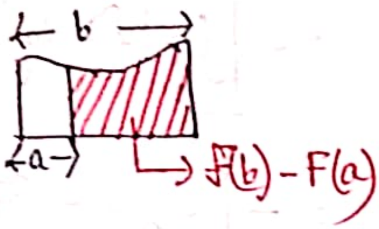


$$dA = y\,dx$$

$$\int y\,dx = \int f(x)\,dx$$

Now,





$$\int_a^b f(x)\,dx = \left[F(x)+C\right]_a^b$$

$$= \left[F(b)+C\right] - \left[F(a)+C\right]$$

$$= F(b)+C-F(a)-C$$

$$\therefore \int_a^b f(x)\,dx = F(b)-F(a)$$

P.1 $\int_2^5 x^2\,dx = \left[\frac{x^3}{3}\right]_2^5$

$$= \left[\frac{5^3}{3} - \frac{2^3}{3}\right]$$

$$= \frac{125}{3} - \frac{8}{3} = \frac{117}{3} \text{ (Ans.)}$$

P.2 $\int_0^1 e^{4x}\,dx = \left[\frac{e^{4x}}{4}\right]_0^1 = \frac{1}{4}\left[e^{4x}\right]_0^2$

$$= \frac{1}{4}\left(e^4 - e^0\right) = \frac{1}{4}\left(e^4 - 1\right)$$

P.3 $\displaystyle\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Let, $\sin^{-1}x = z \;\therefore x = \sin^{-1}z$

$\Rightarrow \dfrac{1}{\sqrt{1-x^2}}\,dx = dz$

x	0	1
z	0	$\pi/2$

$$= \int_0^{\pi/2} z\,dz$$

$$= \left[\frac{z^2}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right]$$

$$= \frac{1}{2}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{8} \text{ (Ans.)}$$

Properties of Definite Integral (নির্দিষ্ট যোগজের বৈশিষ্ট্য)

i| $\displaystyle\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$

ii| $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

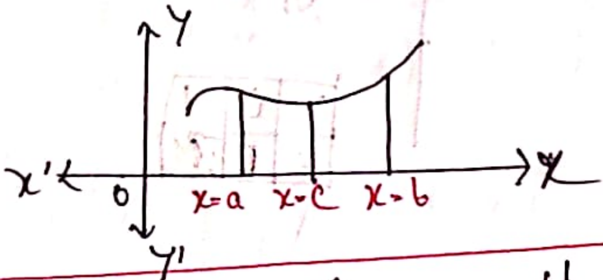
Now, $\displaystyle\int_a^b f(x)\,dx = \left[\varphi(x)\right]_a^b = \varphi(b) - \varphi(a)$

Again, $\displaystyle\int_a^b f(z)\,dz = \left[\varphi(z)\right]_a^b = \varphi(b) - \varphi(a)$

$$\boxed{\therefore \int_a^b f(x)\,dx = \int_a^b f(z)\,dz}$$

iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad ; [a<c<b]$

Now,



$$\boxed{\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$$

iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Now, $\int_b^a f(x)dx = [\varphi(x)]_b^a = \varphi(a) - \varphi(b)$

$= -[\varphi(b) - \varphi(a)]$

$$\boxed{\Rightarrow \int_b^a f(x)dx = -\int_a^b f(x)dx}$$

v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Now, Let, $a - x = z$

$\Rightarrow -dx = dz$

$\Rightarrow dx = -dz$

x	0	a
z	a	0

$$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(z)(-dz)$$

$$= -\int_a^0 f(z)\,dz$$

$$= \int_0^a f(z)\,dz = \int_0^a f(x)\,dx$$

$$\boxed{\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

vi) C.1 $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$; যখন, $f(-x) = f(x)$ [even function]

C.2 $\int_{-a}^{a} f(x)\,dx = 0$; যখন, $f(-x) = -f(x)$ [odd function]

vii) $\int_a^a f(x)\,dx = 0$

viii) $\int_0^{na} f(x) = n\int_0^a f(x)\,dx$

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Proof of (vi)

from (vi) $\rightarrow \displaystyle\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \,; & \text{when } f(-x) = f(x) \text{ [even function]} \\ 0 \,; & \text{" } f(-x) = -f(x) \text{ [odd function]} \end{cases}$

Now, $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x) + f(-x)\}\,dx$

So, $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad \text{—(i)}$

Let, $\displaystyle\int_0^a f(-x)\,dx = -\int_0^{-a} f(z)\,dz$

$\displaystyle= \int_{-a}^{0} f(z)\,dz$

Let,
$-x = z$
$\Rightarrow -dx = dz$
$\Rightarrow dx = -dz$

x	0	a
z	0	$-a$

$\displaystyle\therefore \int_0^a f(-x)\,dx = \int_{-a}^{0} f(x)\,dx \quad \text{—(ii)} \quad \left[\int_{-a}^{0} f(x)\,dx\right]$

from (i) $\rightarrow \displaystyle\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

$\displaystyle= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad$ [from (ii)]

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$$

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$$

Odd function: $f(-x) = -f(x)$

$$\therefore \int_{-a}^{a} f(x) = 0$$

even function: $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = \int_0^a 2f(x)dx$$

$$= 2\int_0^a f(x)dx$$

P.1 $\int_{-1}^{1} x^3 \cos x \, dx$

Now, $f(x) = x^3 \cos x$

$\therefore f(-x) = (-x)^3 \cos(-x)$

$= -x^3 \cos x$

$\therefore f(x) = -f(x)$

$$\therefore \int_{-1}^{1} x^3 \cos x \, dx = 0$$
(Ans)

P.2 $\int_{-1}^{1} x^4 \sin x \, dx$

Soln: $f(x) = x^4 \sin x$

$\therefore f(-x) = x^4(-\sin x)$

$\therefore f(-x) = -x^4 \sin x$

$\therefore \int_{-1}^{1} x^4 \sin x \, dx = 0$ (Ans.)

P.3 $\int_{-1}^{1} \sin x$

Soln: $f(x) = \sin x$

$\therefore f(-x) = \sin(-x)$

$\therefore f(-x) = -\sin x$

$\therefore f(x) = -f(x)$

$\therefore \int_{-1}^{1} \sin x = 0$ (Ans)

Special Problems

***Q. i) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Soln: Let.

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{— (i)}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{— (ii)}$$

$$(i)+(ii) \Rightarrow I+I = \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sqrt{\sin x}+\sqrt{\cos x})}{(\sqrt{\sin x}+\sqrt{\cos x})}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = [\pi/2 - 0]$$

$$\Rightarrow 2I = \pi/2$$

$$\Rightarrow I = \pi/4 \text{ (Ans.)}$$

Q. iii) $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Sol^n: Let,

$$I = \int_0^{\pi} \frac{x}{1+\sin x} dx \text{ ---(i)}$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \text{ ---(ii)}$$

$$(i)+(ii) \Rightarrow 2I = \int_0^{\pi} \frac{x}{1+\sin x} dx + \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{x+\pi-x}{1+\sin x} dx$$

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$$= \int_0^\pi \frac{\pi}{1+\sin x}\,dx$$

$$= \pi \int_0^\pi \frac{1}{1+\sin x}\,dx$$

$$= \pi \int_0^\pi \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \pi \int_0^\pi \frac{1-\sin x}{(1-\sin x)}\,dx$$

$$= \pi \int_0^\pi dx = [x]_0^\pi = [\pi - 0]$$

$\Rightarrow 2I = \pi$

$\Rightarrow I = \pi/2$

(Ans.)

***

Q.111 Show that, $\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

Soln: Let, $I = \int_0^\pi x\, f(\sin x)\,dx$

$= \int_0^\pi (\pi - x)\, f\{\sin(\pi - x)\}\,dx \quad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^\pi (\pi - x)\, f(\sin x)\,dx$

$= \int_0^\pi \pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx$

$\Rightarrow I = \int_0^\pi \pi f(\sin x)\,dx - I$

$\Rightarrow 2I = \pi \int_0^\pi f(\sin x)\,dx$

$\Rightarrow I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

$\Rightarrow \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$

(Proved)

Definite Integral of Modulas Function

Example.01 $\int_{-2}^{7} |x|\,dx$

$$|x| = \begin{cases} x\,;\ x \geq 0 \\ -x\,;\ x < 0 \end{cases}$$

$$= \int_{-2}^{0} (-x)\,dx + \int_{0}^{7} x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{7}$$

$$= -\frac{1}{2}\left[(0)^2 - (2)^2\right] + \frac{1}{2}\left[(7)^2 - (0)^2\right]$$

$$= 2 + \frac{49}{2} = \frac{53}{2} \text{ (Ans.)}$$

Example:02 $\int_{-1}^{5} \frac{|x|}{x}\,dx$

$$|x| = \begin{cases} x\,;\ x \geq 0 \\ -x\,;\ x < 0 \end{cases}$$

$$= \int_{-1}^{0} \frac{(-x)}{x}\,dx + \int_{0}^{5} \frac{x}{x}\,dx$$

$$= -\int_{-1}^{0} dx + \int_{0}^{5} dx$$

$$= -\left[x\right]_{-1}^{0} + \left[x\right]_{0}^{5}$$

$$= -\left[0-(-1)\right] + \left[5-0\right] = 4 \text{ (Ans.)}$$

Example.03 $\displaystyle\int_{-8}^{9} |x-7|\, dx$

$$|x-7| = \begin{cases} +(x-7) \; ; x \geq 7 \\ -(x-7) \; ; x < 7 \end{cases}$$

$$= \int_{-8}^{7} -(x-7)\, dx + \int_{7}^{9} (x-7)\, dx$$

$$= -\int_{-8}^{7} x\, dx + 7\int_{-8}^{7} dx + \int_{7}^{9} x\, dx - 7\int_{7}^{9} dx$$

$$= -\left[\frac{x^2}{2}\right]_{-8}^{7} + 7\left[x\right]_{-8}^{7} + \left[\frac{x^2}{2}\right]_{7}^{9} - 7\left[x\right]_{7}^{9}$$

$$= -\frac{1}{2}\left[(7)^2 - (-8)^2\right] + 7\left[(7)-(-8)\right] + \frac{1}{2}\left[(9)^2 - (7)^2\right] - 7\left[9-7\right]$$

$$= \frac{169}{2} \text{ (Ans.)}$$

Wallis Theorem:

$$\boxed{\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx}$$

Case:01 n is even natural (জোড় স্বাভাবিক)

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)(n-5)\cdots 3.1}{n(n-2)(n-4)\cdots 4.2} \cdot \frac{\pi}{2}$$

P.1 $\int_0^{\pi/2} \sin^8 x\,dx = \int_0^{\pi/2} \cos^8 x\,dx$

$$= \frac{7.5.3.1}{8.6.4.2} \cdot \pi/2$$

$$= \frac{35}{128} \cdot \pi/2$$

Case: 02 n is odd natural (বিজোড় স্বাভাবিক)

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}$$

P.1 $\int_0^{\pi/2} \sin^9 x\,dx = \int_0^{\pi/2} \cos^9 x\,dx$

$$= \frac{8.6.4.2}{9.7.5.3} = \quad \text{(Ans.)}$$

Practice

P.1 $\int_0^{\pi/2} \sin^{10} x\,dx$

$$= \frac{9.7.5.3.1}{10.8.6.4.2} \cdot \pi/2$$

P.2 $\int_0^{\pi/2} \cos^{17} x\,dx$

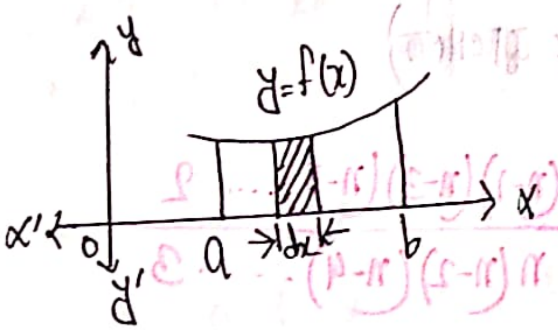
$$\neq \frac{\cancel{16.14.12.10.8.7.6.5}.5.2}{16.14.12.10.8.6.4.2}$$

$$= \frac{16.14.12.10.8.6.4.2}{17.15.13.11.9.7.5.3.}$$

Application of Integration

↳ Area using definite Integral (নির্দিষ্ট যোগজ-সমূহের সাহায্যে ক্ষেত্রফল)

⇒ আমরা জানি, Normal function কে Integration করলে— Area function পাওয়া যায়,

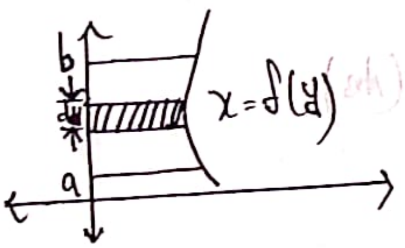


$$dA = y\,dx$$

$$\therefore A = \int_a^b y\,dx$$

$$\Rightarrow A = \int_a^b f(x)\,dx$$

আবার,



$$dA = x\,dy$$

$$\therefore A = \int_a^b x\,dy$$

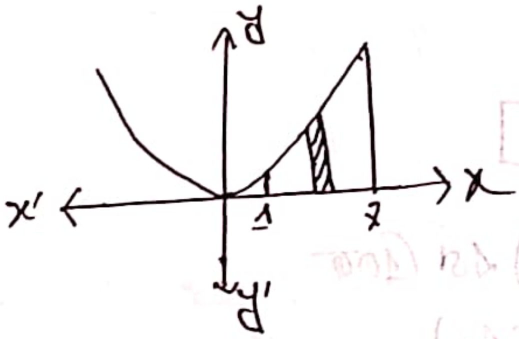
$$\Rightarrow A = \int_a^b f(y)\,dy$$

Type:01

বক্ররেখার অক্ষের সাথে সীমাবদ্ধ ক্ষেত্রফল

P.1 $y=x^2$ বক্ররেখা x-অক্ষ এবং $x=1$, $x=7$ দ্বারা আবদ্ধ-ক্ষেত্র অংশের ক্ষেত্রফল।

Soln:



$$A=\int_1^7 y\,dx=\int_1^7 f(x)\,dx=\int_1^7 x^2\,dx=\left[\frac{x^3}{3}\right]_1^7$$

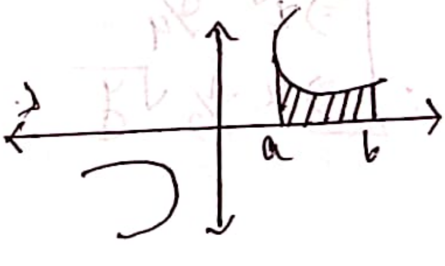
$$=\frac{1}{3}\left[7^3-1^3\right]$$

$$=\frac{342}{3}=114 \text{ বর্গ একক}$$

(Ans.)

P.2 Rectangular hyperbola $xy=c^2$, x অক্ষ এবং $x=a$, $x=b$ দ্বারা আবদ্ধ অংশের ক্ষেত্রফল।

Soln:



$$A = \int_a^b f(x)\,dx$$

$$= \int_a^b y\,dx \qquad \left[\begin{array}{l} xy = c^2 \\ \Rightarrow y = \dfrac{c^2}{x} \end{array}\right]$$

$$= \int_a^b \frac{c^2}{x}\,dx$$

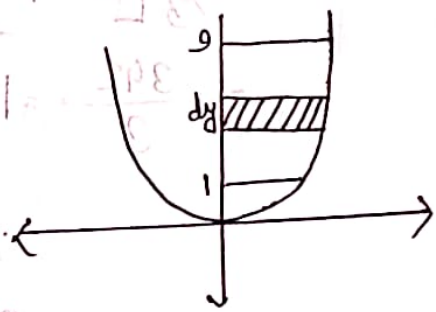
$$= c^2\left[\ln x\right]_a^b = c^2\left[\ln b - \ln a\right]$$

$$= c^2 \ln\left(\frac{b}{a}\right) \text{ বর্গ একক}$$

(Ans.)

P.3 $y = 4x^2$ বক্ররেখা, y-অক্ষ এবং $y = 1$, $y = 9$ দ্বারা আবদ্ধ ক্ষেত্রের ক্ষেত্রফল

Soln:



$$A = \int_1^9 x\,dy \qquad \left[\begin{array}{l} y = 4x^2 \\ \Rightarrow x^2 = y/4 \\ \Rightarrow x = \frac{1}{2}\sqrt{y} \end{array}\right]$$

$$= \int_1^9 \frac{1}{2}\sqrt{y}\,dy$$

$$= \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_1^9$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[(9)^{3/2}-(1)^{3/2}\right]$$

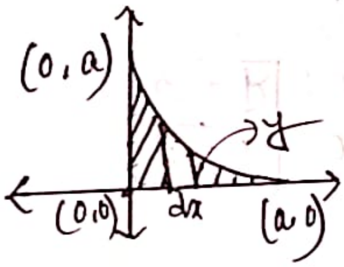
$$= \frac{1}{9}\cdot\left[27-1\right] = \frac{26}{3}$$ বর্গ একক

(Ans.)

P.4 $\sqrt{x}+\sqrt{y}=\sqrt{a}$ বক্ররেখা অক্ষদ্বয়ের সাথে কি পরিমাণ ক্ষেত্রফল উৎপন্ন করে।

Soln:

$$\left[\begin{array}{l} x=0;\ \sqrt{y}=\sqrt{a}\ \therefore \boxed{y=a} \\ y=0;\ \sqrt{x}=\sqrt{a}\ \therefore \boxed{x=a}\end{array}\right.$$



$$A=\int_0^a y\,dx$$

$$\left[\begin{array}{l}\sqrt{y}=\sqrt{a}-\sqrt{x} \\ \Rightarrow y=(\sqrt{a}-\sqrt{x})^2\end{array}\right.$$

$$=\int_0^a(\sqrt{a}-\sqrt{x})^2\,dx$$

$$=\int_0^a(a+x-2\sqrt{a}\cdot\sqrt{x})\,dx$$

$$=\left[ax+\frac{x^2}{2}-2\sqrt{a}\,\frac{x^{3/2}}{3/2}\right]_0^a$$

$$=\left[\left(a^2+\frac{a^2}{2}-2\sqrt{a}\,\frac{a^{3/2}}{3/2}\right)-0\right]$$

$$= a^2+\frac{a^2}{2}-2\cdot\frac{2}{3}\,a^{1/2}\cdot a^{3/2}$$

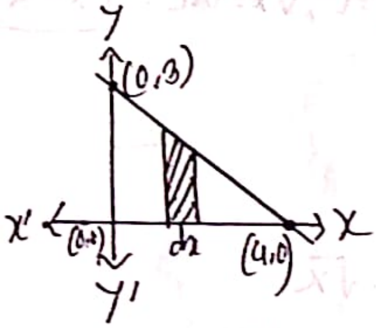
$$= a^2 + \frac{a^2}{2} - \frac{4}{3}a^2$$

$$= \frac{6a^2 + 3a^2 - 8a^2}{6}$$

$$= \frac{a^2}{6} = \frac{1}{6}a^2$$ বর্গ একক (Ans.)

P.5 $3x+4y=12$ সরলরেখা অক্ষদ্বয়ের সাথে কি পরিমাণ ক্ষেত্রফল আবদ্ধ।

Soln:



$$\left[\begin{array}{l} x=0\,;\ \boxed{y=3} \\ y=0\,;\ \boxed{x=4} \end{array}\right.$$

$$\left[\begin{array}{l} 4y = 12 - 3x \\ \Rightarrow y = 3 - \frac{3}{4}x \end{array}\right.$$

$$A = \int_0^4 y\,dx$$

$$= \int_0^4 \left(3 - \frac{3}{4}x\right) dx$$

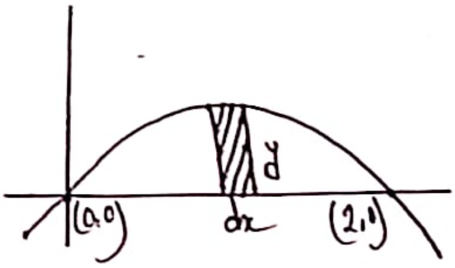
$$= \left[3x - \frac{3}{4}\frac{x^2}{2}\right]_0^4$$

$$= \left[\left\{(3\times4) - \frac{3}{8}\cdot 4^2\right\} - 0\right]$$

$$= 12 - 6 = 6$$ বর্গ একক (Ans.)

P.6 $y = 2x - x^2$ এবং x অক্ষ দ্বারা সীমিত ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Sol^n:



$y = 0$;

$2x - x^2 = 0$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow \cancel{x=0}\ x(x-2) = 0$

$\Rightarrow x = 0, 2$

$A = \int_0^2 y\,dx$

$= \int_0^2 (2x - x^2)\,dx \quad [y = 2x - x^2$

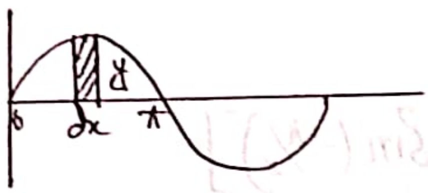
$= \left[2\frac{x^2}{2} - \cancel{x^2}\frac{x^3}{3}\right]_0^2$

$= \left[\left(2^2 - \frac{2^3}{3}\right) - 0\right]$

$= 4 - \frac{8}{3} = \frac{4}{3}$ বর্গ একক (Ans)

P.7 $y = \sin x$ এর একটি চাপ দ্বারা আবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Sol^n:



$A = \int_0^\pi y\,dx$

$$= \int_0^{\pi} \sin x \, dx \qquad [y = \sin x]$$

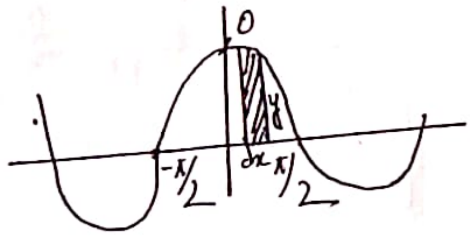
$$= \left[-\cos x\right]_0^{\pi}$$

$$= \left[-\cos\pi + \cos 0\right]$$

$$= \left[-(-1) + 1\right] = 2$$ বর্গ একক (Ans.)

P.8 $y = \cos x$ বক্ররেখা, x অক্ষ এবং $x = -\pi/2$ এবং $x = \pi/2$ দ্বারা আবদ্ধ অংশের ক্ষেত্রফল নির্ণয় কর।

Solⁿ:



$$A = \int_{-\pi/2}^{\pi/2} y \, dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos x \, dx \qquad [y = \cos x]$$

$$= \left[\sin x\right]_{-\pi/2}^{\pi/2} = \left[\sin \pi/2 - \sin(-\pi/2)\right]$$

$$= \sin\pi/2 + \sin\pi/2$$

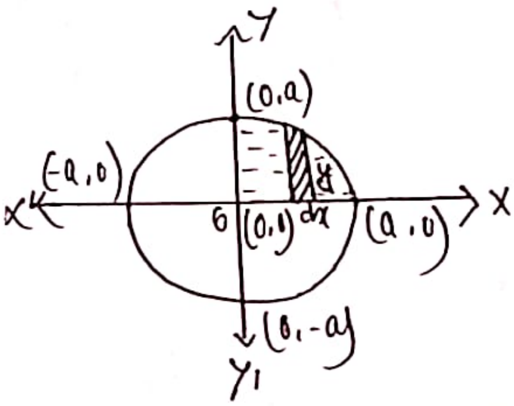
$$= 1 + 1 = 2 \text{ (Ans.)}$$

Type: 02

নির্দিষ্ট ক্ষেত্রফল নির্ণয় করা

P.1 $x^2+y^2=a^2$ দ্বারা সীমিত বৃত্তের ক্ষেত্রফল নির্ণয় কর।

Sol^n:



M.1 Total Area, $A = 4\int_0^a y\,dx$

$$= 4\int_0^a \sqrt{a^2-x^2} \qquad \left|\; \begin{aligned} & x^2+y^2=a^2 \\ \Rightarrow\ & y=\sqrt{a^2-x^2} \end{aligned}\right.$$

$$= 4\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a\sqrt{a^2-a^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right)-0\right]$$

$$= 4\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right)$$

$$= 4\cdot\frac{\pi a^2}{4} = \pi a^2 \text{ বর্গ একক}$$

(Ans.)

M.2 $A = 4\int_0^a \sqrt{a^2 - x^2}\,dx$

$= 4\int_0^{\pi/2} (a\cos\theta)(a\cos\theta)\,d\theta$

$= 4\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$

$= 4a^2 \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$= 2a^2\int_0^{\pi/2} (1+\cos 2\theta)\,d\theta$

$= 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= 2a^2\left[\left(\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2}\right) - 0\right]$

$= 2a^2\,\frac{\pi}{2} = \pi a^2$ (Ans.)

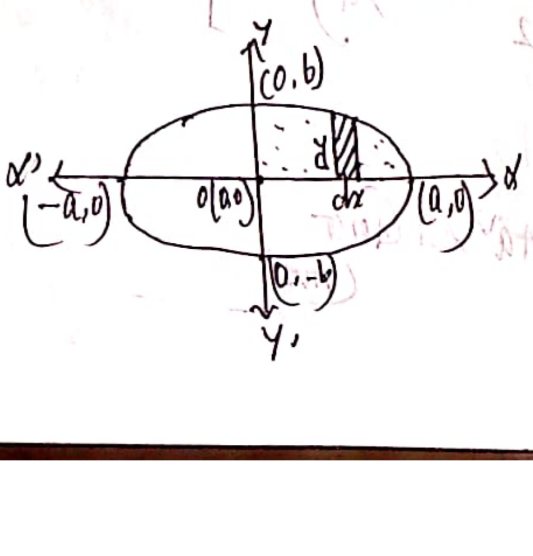
Let $x = a\sin\theta \Rightarrow \sin\theta = x/a$

$\Rightarrow dx = a\cos\theta\,d\theta \quad \therefore \theta = \sin^{-1} x/a$

x	0	a
θ	0	$\pi/2$

P.2 $x^2/a^2 + y^2/b^2 = 1$ উপবৃত্তের ক্ষেত্রফল নির্ণয় কর

Soln:



Total Area, $A = 4 \times \int_0^a y\,dx$

$= 4 \times \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{4b}{a}\left[\left(\frac{a\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - 0\right]$

$= \frac{4b}{a}\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \pi ab$ বর্গ একক (Ans.)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\Rightarrow \left(\frac{y^2}{b^2}\right) = \frac{a^2 - x^2}{a^2}$

$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$

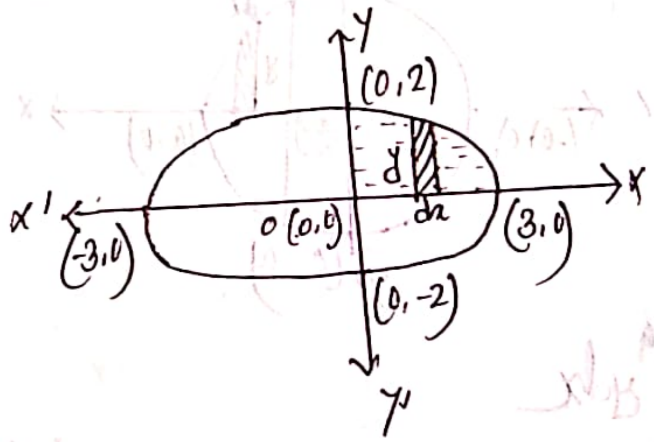
P.3 $4x^2 + 9y^2 = 36$ উপবৃত্ত দ্বারা ~~আবদ্ধ~~ আবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর,

Soln: $4x^2 + 9y^2 = 36$

$\Rightarrow \frac{4x^2}{36} + \frac{9y^2}{36} = 1$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$



Total Area, $A = 4 \times \int_0^3 y\,dx$

$= 4\int_0^3 \frac{2}{3}\sqrt{3^2 - x^2}\,dx$

$\frac{y}{2} = 1 - \frac{x^2}{3^2}$

$\Rightarrow y = \frac{2}{3}\sqrt{3^2 - x^2}$

$$= \frac{8}{3}\int_0^3 \sqrt{3^2-x^2}\,dx$$

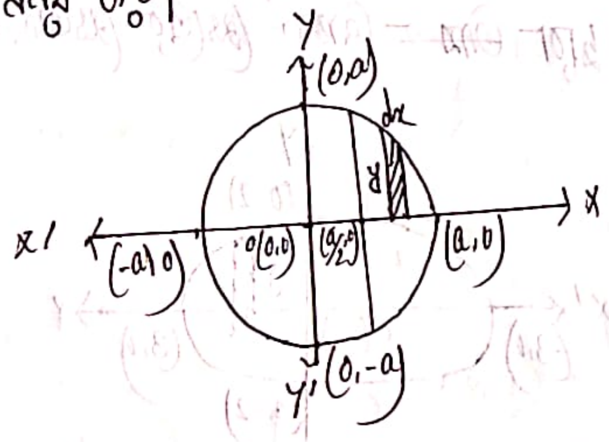
$$= \frac{8}{3}\left[\frac{x\sqrt{3^2-x^2}}{2} + \frac{3^2}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{8}{3}\left[\left(\frac{3\sqrt{3^2-3^2}}{2} + \frac{3^2}{2}\sin^{-1}\frac{3}{3}\right) + 0\right]$$

$$= \frac{8}{3}\left(\frac{9}{2}\cdot\frac{\pi}{2}\right) = 6\pi \text{ বর্গ একক}$$ (Ans.)

P.4 $x^2+y^2=a^2$ বৃত্তটির ক্ষেত্রের যে অংশ $x=\frac{a}{2}$ দ্বারা ~~আবদ্ধ~~ খণ্ডিত হয় তার ক্ষেত্রফল নির্ণয় কর।

Solⁿ:



$$\text{Area, } A = 2\int_{a/2}^{a} y\,dx$$

$$= 2\int_{a/2}^{a}\sqrt{a^2-x^2}\,dx$$

$$x^2+y^2=a^2$$
$$\Rightarrow y=\sqrt{a^2-x^2}$$

$$= 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/2}^{a}$$

$$= 2\left[\left(\frac{a\sqrt{a^2-a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left(\frac{\frac{a}{2}\sqrt{a^2-\frac{a^2}{4}}}{2} + \frac{a^2}{2}\sin^{-1}\frac{a}{2a}\right)\right]$$

$$= 2\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a}{4}\sqrt{\frac{4a^2-a^2}{4}} - \frac{a^2}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\frac{\pi a^2}{4} - \frac{a}{4}\,\frac{\sqrt{3}a}{2} - \frac{a^2}{2}\cdot\frac{\pi}{6}\right]$$

$$= 2\left[\frac{\pi a^2}{4} - \frac{\sqrt{3}a^2}{8} - \frac{\pi a^2}{12}\right]$$

$$= \frac{\pi a^2}{2} - \frac{\sqrt{3}a^2}{4} - \frac{\pi a^2}{6}$$

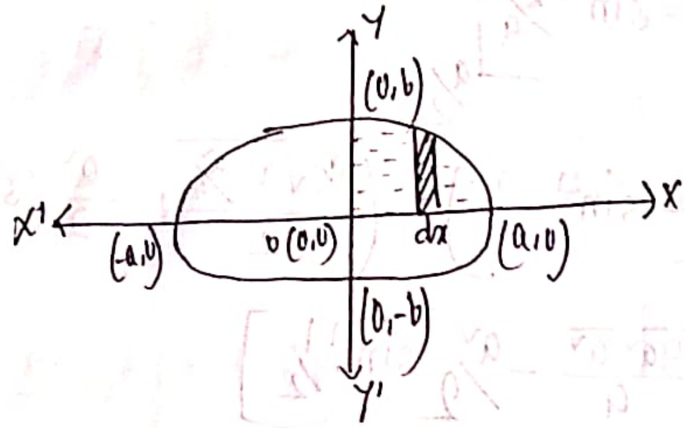
$$= \frac{3\pi a^2 - \pi a^2}{6} - \frac{\sqrt{3}a^2}{4}$$

$$= \frac{2\pi a^2}{6} - \frac{\sqrt{3}a^2}{4}$$

$$= a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$ বর্গ একক (Ans.)

P.5 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ উপবৃত্তের ক্ষেত্রের যে অংশ ধনাত্মক বৃহৎ অক্ষ ও ক্ষুদ্র অক্ষ দ্বারা সীমিত তার ক্ষেত্রফল নির্ণয় কর।

Solⁿ:



$$A = \int_0^a y\,dx$$

$$\boxed{y = \frac{b}{a}\sqrt{a^2 - x^2}}$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\left(\frac{a\sqrt{a^2-a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - 0\right]$$

$$= \frac{b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi ab}{4}$$ বর্গ একক (Ans.)

Special Problem: Q. $y = \frac{1}{2}x^2 + 1$ পরাবৃত্ত এবং তার উপকেন্দ্রিক লম্ব দ্বারা সীমিত ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Solⁿ: $y = \frac{1}{2}x^2 + 1$

$\Rightarrow y = \frac{x^2 + 2}{2}$

$\Rightarrow x^2 = 2y - 2$

$\Rightarrow x^2 = 2(y-1)$

$\Rightarrow x^2 = 4 \cdot \frac{1}{2}(y-1)$

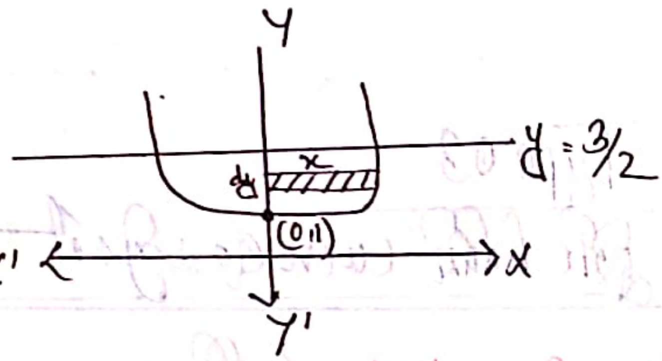
ইহা পরাবৃত্তের সমীকরণ এবং এর কেন্দ্র $(0,1)$ এবং উপকেন্দ্রিক লম্বের দৈর্ঘ্য,

$= 4 \cdot \frac{1}{2} = 2$

এর উপকেন্দ্রিক লম্বের সমীকরণ,

$y - 1 = \frac{1}{2}$

$\Rightarrow y = \frac{3}{2}$



Area $= 2\int_1^{3/2} x\,dy$

$x^2 = 2(y-1)$

$\Rightarrow x = \sqrt{2(y-1)}$

$= 2\int_1^{3/2} \sqrt{2(y-1)}\,dy$

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$$= 2\int_1^{3/2} \sqrt{2}\sqrt{y-1}\, dy$$

$$= 2\sqrt{2}\int_1^{3/2} \sqrt{y-1}\, dy$$

$$= 2\sqrt{2}\int_1^{3/2} (y-1)^{1/2}\, dy$$

$$= 2\sqrt{2}\left[\frac{(y-1)^{3/2}}{3/2}\right]_1^{3/2}$$

$$= 2\sqrt{2}\times\frac{2}{3}\left[\left(\frac{3}{2}-1\right)^{3/2} - 0\right]$$

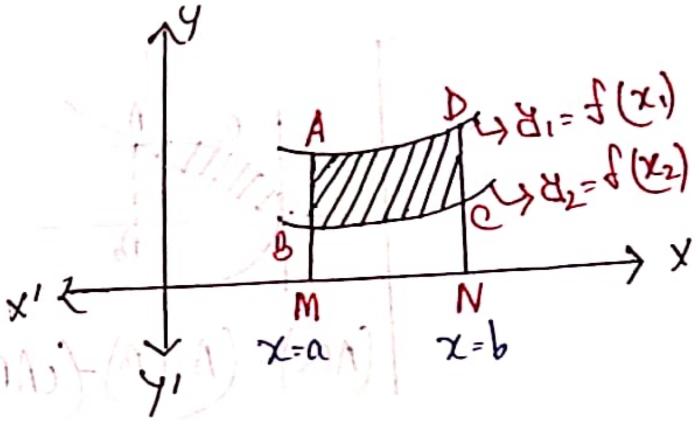
$$= \frac{4\sqrt{2}}{3}\times\left(\frac{1}{2}\right)^{3/2} = \frac{2}{3}$$ বর্গ একক (Ans.)

Type:03

দুইটি নির্দিষ্ট curve এর মধ্যবর্তী অংশের ক্ষেত্রফল

Concept:01 দুইটি সরলরেখা ও দুইটি কোটি দ্বারা আবদ্ধ সমতল ক্ষেত্রের ক্ষেত্রফল

P.T.O→



$$(ABCD) = (AMND) - (BMNC)$$

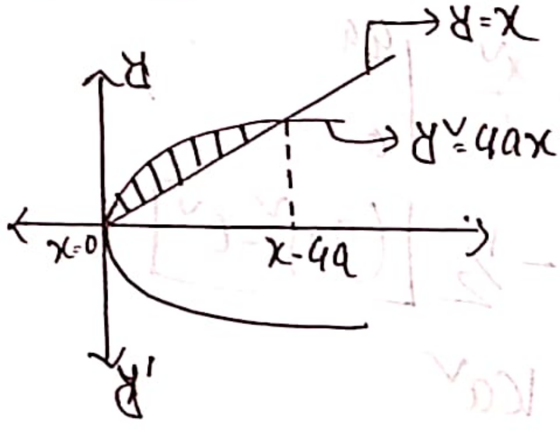
$$= \int_a^b y_1\,dx - \int_a^b y_2\,dx$$

$$(ABCD) = \int_a^b (y_1 - y_2)\,dx$$ অথবা, $$(ABCD) = \int_a^b \{f(x_1) - f(x_2)\}\,dx$$

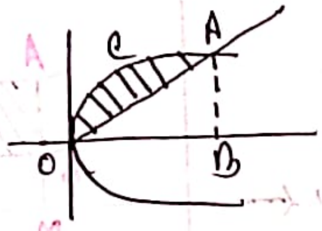
Note: (a,b) দেওয়া না থাকলে Eqn solve করতে হবে।

P.1 $y^2 = 4ax$ ও $y = x$ রেখা দুটি দ্বারা আবদ্ধ অংশের ক্ষেত্রফল নির্ণয় কর।

Soln:



উদা। $y=x$, $y^2=4ax$ এ ক্ষেত্র।



$(AOC) = (ACOB) - (OAB)$

$x^2 = 4ax$

$\Rightarrow x^2 - 4ax = 0$

$\Rightarrow x(x-4a) = 0$

$\Rightarrow \boxed{x = 0, 4a}$

$$\therefore \text{Area} = \int_0^{4a} (y_1 - y_2)\,dx$$

$$= \int_0^{4a} (2\sqrt{a}\sqrt{x} - x)\,dx$$

$y^2 = 4ax$

$\Rightarrow y = 2\sqrt{a}\sqrt{x}$

$$= \int_0^{4a} 2\sqrt{a}\sqrt{x}\,dx - \int_0^{4a} x\,dx$$

$$= 2\sqrt{a}\int_0^{4a} \sqrt{x}\,dx - \int_0^{4a} x\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \left[\frac{x^2}{2}\right]_0^{4a}$$

$$= 2\sqrt{a}\,\tfrac{2}{3}\left[(4a)^{3/2} - 0\right] - \tfrac{1}{2}\left[(4a)^2 - 0^2\right]$$

$$= \frac{4\sqrt{a}}{3}\left\{(4a)^3\right\}^{1/2} - \tfrac{1}{2}\,16a^2$$

$$= \frac{4\sqrt{a}}{3}\sqrt{64a^3} - 8a^2$$

$$= \frac{4\sqrt{a}}{3}\, 8(\sqrt{a})^3 - 8a^2$$

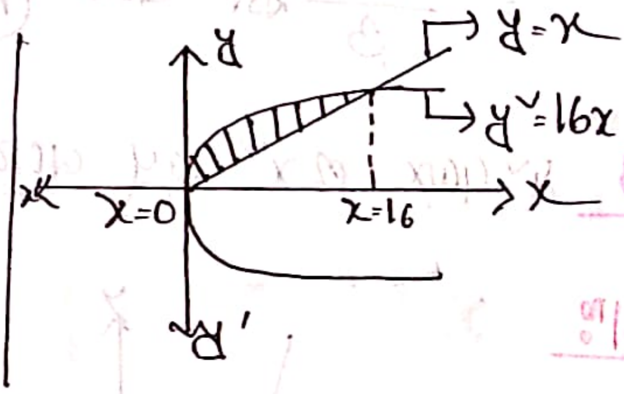
$$= \frac{32(\sqrt{a})^4}{3} - 8a^2$$

$$= \frac{32a^2}{3} - 8a^2$$

$$= \frac{8}{3}a^2 \text{ (বর্গ একক)}$$ (Ans.)

P.2 $y^2 = 16x$ অধিবৃত্ত এবং $y = x$ সরলরেখা দ্বারা সীমাবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Sol^n: $y = x$, $y^2 = 16x$ এ বসিয়ে,



$$x^2 = 16x$$

$$\Rightarrow x^2 - 16x = 0$$

$$\Rightarrow x(x - 16) = 0$$

$$\Rightarrow \boxed{x = 0, 16}$$

$y_1 = 4\sqrt{x}$

$y_2 = x$

$$\text{Area} = \int_0^{16} (y_1 - y_2)\,dx$$

$$= \int_0^{16} (4\sqrt{x} - x)\,dx$$

$$= \int_0^{16} 4\sqrt{x}\, dx - \int_0^{16} x\, dx$$

$$= 4\int_0^{16} \sqrt{x} - \int_0^{16} x\, dx$$

$$= 4\left[\frac{x^{3/2}}{3/2}\right]_0^{16} - \left[\frac{x^2}{2}\right]_0^{16}$$

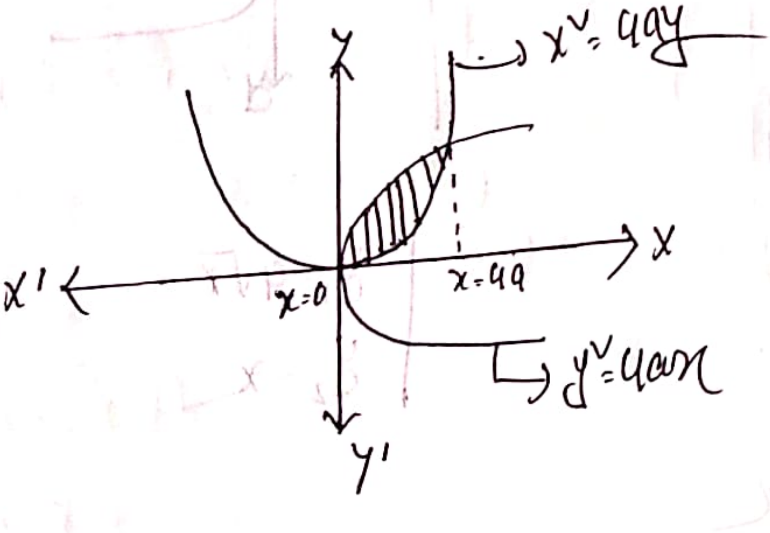
$$= 4\cdot\frac{2}{3}\left[(16)^{3/2} - 0\right] - \frac{1}{2}\left[(16)^2 - 0\right]$$

$$= \left(\frac{8}{3}\times 64\right) - 128$$

$= \frac{128}{3}$ বর্গ একক (Ans.)

P.3 $y^2 = 4ax$ ও $x^2 = 4ay$ বক্ররেখাদ্বয় দ্বারা আবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Solⁿ:



So, $x^2 = 4ay \Rightarrow \boxed{y = x^2/4a}$

আবার, $y^2 = 4ax$

$\Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$

$\Rightarrow \frac{x^4}{16a^2} = 4ax$

$\Rightarrow x^4 = 64a^3x$

$\Rightarrow x^4 - 64a^3x = 0$

$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0, 4a$

$$\text{Area} = \int_0^{4a} (y_1 - y_2)\, dx$$

$$= \int_0^{4a} \left(2\sqrt{a}\sqrt{x} - x^2/4a\right) dx$$

$y^2 = 4ax$

$y_1 = 2\sqrt{a}\sqrt{x}$

$y_2 = x^2/4a$

$$= 2\sqrt{a}\int_0^{4a} \sqrt{x}\, dx - \frac{1}{4a}\int_0^{4a} x^2\, dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left[(4a)^{3/2}-0\right]-\frac{1}{12a}\left[(4a)^3-0\right]$$

$$= \frac{4\sqrt{a}}{3}\times 8\times a^{3/2}-\frac{1}{12a}\times 64a^3$$

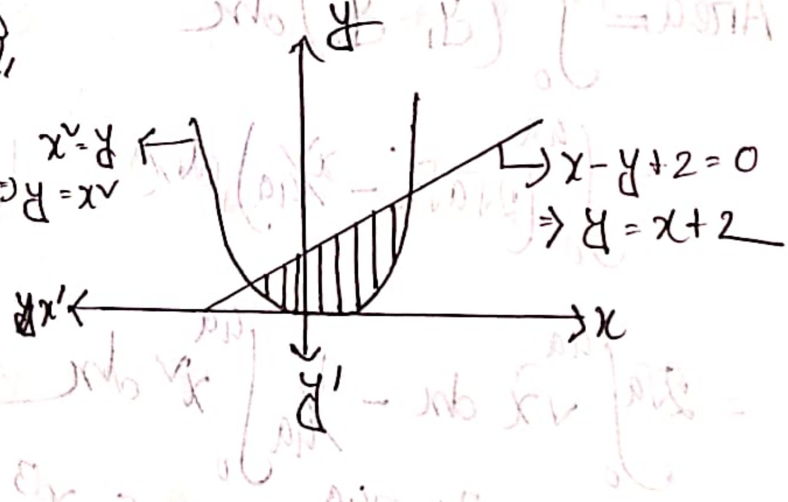
$$= \frac{32a^2}{3}-\frac{16a^2}{3}$$

$$= \frac{32a^2-16a^2}{3}=\frac{16a^2}{3}$$ বর্গ একক

(Ans)

P.4 $y=x^2$ বক্ররেখা এবং $x-y+2=0$ সরলরেখা দ্বারা সীমাবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় কর।

Solⁿ: $y=x^2$, $x-y+2=0$ তে পাই,



$\therefore\ x-x^2+2=0$

$\Rightarrow x^2-x-2=0$

$\Rightarrow x^2-2x+x-2=0$

$\Rightarrow x(x-2)+1(x-2)=0$

$\Rightarrow (x+1)(x-2)=0$

$\Rightarrow \boxed{x=-1,\ 2}$

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$\therefore$ Area $= \int_{-1}^{2} (y_1 - y_2)\, dx$

$$\boxed{\begin{aligned} y_1 &= x+2 \\ y_2 &= x^2 \end{aligned}}$$

$= \int_{-1}^{2} (x+2-x^2)\, dx$

$= \int_{-1}^{2} x\, dx + \int_{-1}^{2} 2\, dx + \int_{-1}^{2} x^2\, dx$

$= \left[\frac{x^2}{2}\right]_{-1}^{2} + \left[2x\right]_{-1}^{2} + \left[\frac{x^3}{3}\right]_{-1}^{2}$

$= \frac{1}{2}\left[2^2 - (-1)^2\right] + \left[(2\cdot 2) - (-1\times 2)\right] + \frac{1}{3}\left[(2)^3 - (-1)^3\right]$

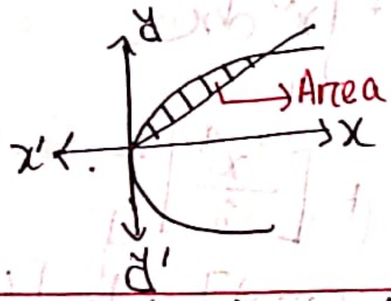
$= \frac{1}{2}(4-1) + (4+2) + \frac{1}{3}(8+1)$

$= \frac{1}{2}(3) + 6 + \frac{1}{3}(9)$

$= \frac{3}{2} + 6 + 3 = \frac{21}{2}$ বর্গ একক (Ans.)

কালিকরণ (Quadrature):

কোন বক্ররেখা এবং সরলরেখা দ্বারা আবদ্ধ ক্ষেত্রের ক্ষেত্রফল নির্ণয় করার পদ্ধতিকে কালিকরণ (Quadrature) বলে।



This concept is used in Astronomy

Also in space engineering

THE END