Deterministic Online Bi-partite Edge Coloring

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Problem Setting

- We have a bipartite graph (two sides: “offline” nodes known up front, “online” nodes arriving one by one).

- - Each time a new online node arrives, all its edges must be *colored immediately and permanently*, with no two edges sharing a node getting the same color.

Performance Measure

- Offline optimal uses **Δ** colors (maximum degree) by König's theorem.

- An online algorithm is **α-competitive** if it never uses more than **α·Δ** colors.

Greedy Baseline

- The naïve ***"use the smallest available color"*** rule is $(2 - 1/\Delta)$-competitive ($\approx 2 \times \Delta$ colors).
- It was previously believed that, to outperform greedy randomization is necessary.

Main Breakthrough

A *deterministic* online algorithm achieving a competitive ratio

$(e/(e-1))+o(1)\approx 1.58$

for any $\Delta \gg \log n$—strictly better than 2.

Partial-Coloring Subroutine

1. **Palettes**: Give each offline node a set of about **(1 + ε)Δ** colors, where

$$\varepsilon = \Theta\left(\sqrt{\frac{\ln n}{\Delta}}\right).$$

2. **Random Pick**: When an online node arrives, each incident edge independently samples one color from its server's palette and then removes that color from the palette.

3. **Contention Resolution**: For each color, among all edges that sampled it, run a CRS (contention resolution scheme) to select exactly one edge to paint with that color—others remain uncolored.

4. **Outcome**: Each round colors about **$1 - e^{-1}$≈63%** of the remaining edges, using only **(1+ε)Δ** colors.

Contention Resolution Scheme CRS

Ensures that if element **i** shows up with probability **xi,** then

$$\Pr[i \text{ wins}] \;\geq\; x_i \times \frac{1 - \prod_j (1 - x_j)}{\sum_j x_j}$$

so every contender keeps a fair fraction of its sampling chance.

Symbol	Meaning
i	Edge (u, v_t) trying to get a color
x_i	Probability that edge i picks the color
x_j	Probability of other edges picking the color
$\sum_j x_j$	Expected number of edges picking the color
$\prod_j (1 - x_j)$	Probability no edge picks the color
$1 - \prod_j (1 - x_j)$	Probability at least one edge tries it
$\Pr[i \in O]$	Probability edge i is chosen (wins)

Final Outcome of Algorithm

- ~63% of edges are successfully colored in each round
- Martingales were used to track how load builds up on color groups as edges choose colors.
- Freedman's Inequality guarantees load stays within safe limits.
- Together, they ensured that the randomized coloring step behaves reliably, allowing the overall algorithm to be both efficient and derandomizable.

Total Color Count

Summing the geometric reduction gives

$$\Delta\,(1 + e^{-1} + e^{-2} + \cdots) \;=\; \Delta\,\frac{e}{e-1} \;+\; o(\Delta),$$

hence competitive ratio $\frac{e}{e-1} + o(1)$. ≈1.58

Robustness to Adaptive Adversary

- Standard random-order analyses fail when the adversary sees your coin flips.

- Here, we prove **strong concentration** of color-loads using a **martingale** construction and **Freedman's inequality**, obtaining failure probabilities like $\Omega(\Delta^2)$.

- This is sharp enough to union-bound over all adversarial choices.

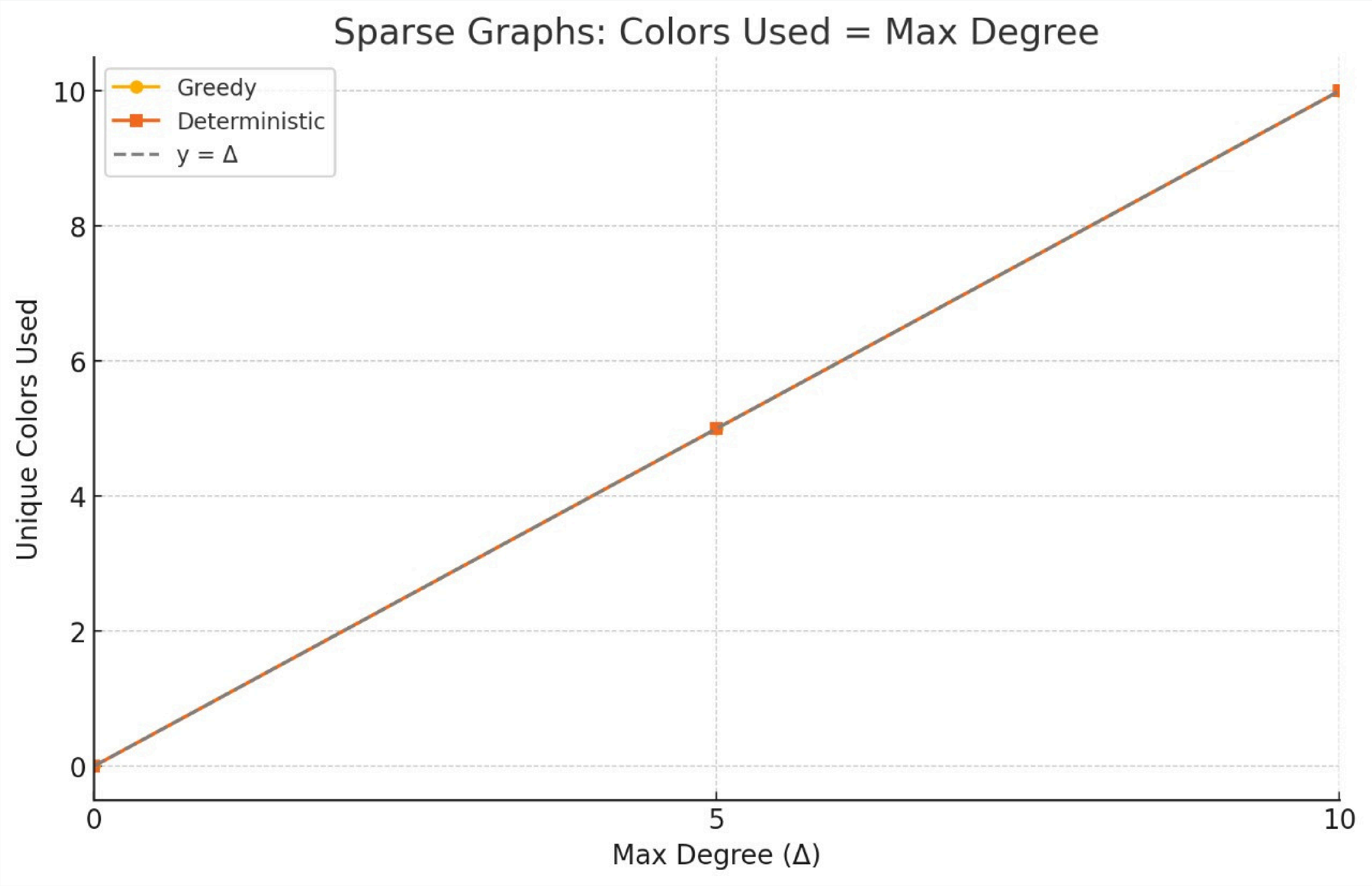
Derandomization

Lemma 2.1 : If there exists a *competitive randomized online edge-coloring algorithm against adaptive adversaries*, then there exists a *deterministic competitive online edge-coloring algorithm*.

Why It Matters

- **First** deterministic algorithm beating the long-standing greedy ratio of **2** for large Δ.
- Introduces a novel blend of **contention resolution** and **martingale concentration** against adaptive adversaries.
- Offers a blueprint for derandomizing other online covering/matching problems.

Visualization



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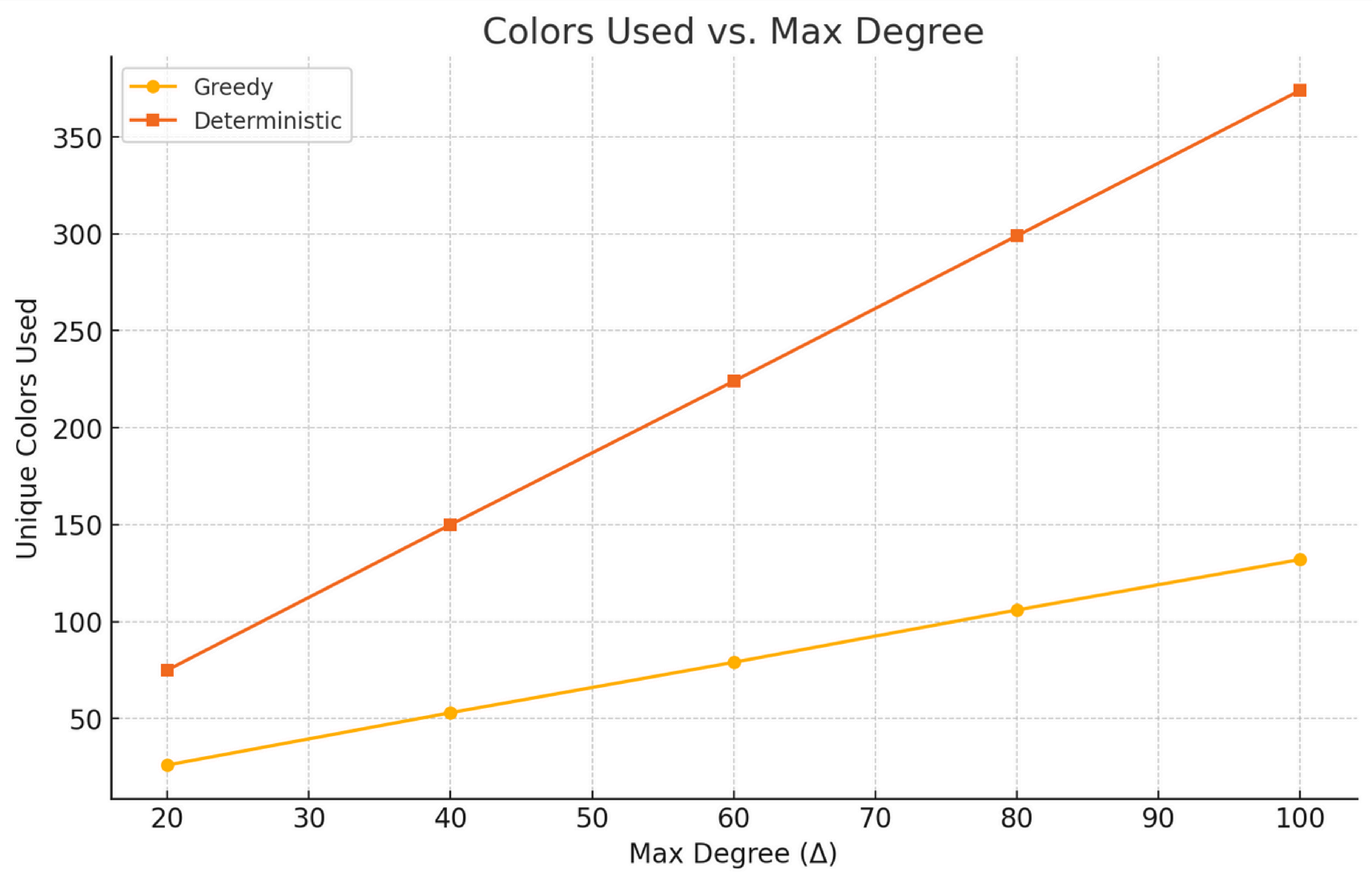

Thank you!