A. S. M. Ahsan-Ul-Haque
ahsanhaquetarique@gmail.com
+8801521496399

1 Problem Statement

User A needs to go point P. On the middle of the road, the road is divided and go to two different direction (R1 & R2) with one checkpoint at the beginning of each road after division. Let us think the checkpoints are C1 & C2. Only one of R1 & R2 will take user A to point P. C1 & C2 have all the information regarding C1, C2 & P. Any one of C1 & C2 will always provide wrong information to A and another will do exactly opposite. User A can only raise single query to C1 or C2 to find the right direction towards P. A graphical representation of the problem is given in Figure 1.

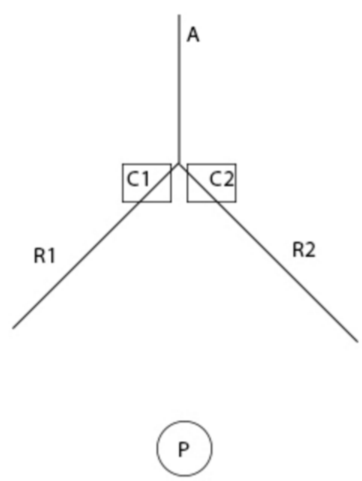

Figure 1: Graphical Representation of the problem.

Our task is to write an algorithm to solve this issue.

2 Problem Modeling

Let, R and C be two random variables referring to the roads and the checkpoints respectively. For the given problem, R can take values from $\{R1, R2\}$ and C can take values from $\{C1, C2\}$.

For the rest of the sections we shall adapt the following notations.

$$P(R) = \begin{cases} True & \text{if and only if road } R \text{ leads to location } P \\ False & \text{otherwise} \end{cases} \tag{1}$$

and,

$$T(C) = \begin{cases} True & \text{if and only if } C \text{ speaks the truth} \\ False & \text{otherwise} \end{cases} \tag{2}$$

For convenience, we shall use 1 for $True$ and 0 for $False$. Let us also eliminate the First Order Logic predicates by using Propositional Logic statements,

$$P(R = R_1) \leftrightarrow r_1 \tag{3}$$

$$P(R = R_2) \leftrightarrow r_2 \tag{4}$$

$$T(C = C_1) \leftrightarrow c_1 \tag{5}$$

$$T(C = C_2) \leftrightarrow c_2 \tag{6}$$

According to the problem statement, the Question Q may involve r1,r2,c1 and c2, i.e., Q may be regarded as a Boolean function depending on them.

$$Q(r1, r2, c1, c2)$$

Since there are 4 Boolean variables, there might be $2^4 = 16$ combinations. By careful observation we can see that some combinations are in disagreement with the given conditions.

A couple of such case is shown in Table 1.

Table 1: Some Contradictory cases

r1	r2	c1	c2
0	0	0	0
0	0	1	1

According to the conditions, one and only one of c1 and c2 must be True and one and only one of r1 and r2 must be True. The constraints may be written as in the equations 7, 8, 9 and 10.

$$c1 \leftrightarrow \overline{c2} \tag{7}$$

$$c2 \leftrightarrow \overline{c1} \tag{8}$$

$$r1 \leftrightarrow \overline{r2} \tag{9}$$

$$r2 \leftrightarrow \overline{r1} \tag{10}$$

Thus, without the loss of generality, we can eliminate any one of $\{c1, c2\}$ and any one of $\{r1, r2\}$.

Let, our question be only dependent on c1 and r1. So, the function Q's dependency is now reduced to,

$$Q(r1, c1)$$

In other words, we are going to ask C1 the question Q about the road R1. The function Q outputs the answer given by C1 about road R1.

3 Objective Formulation

As explained in the previous section, we need to find the right path R which leads to P by asking $Q(r1, c1)$.

Let us consider all of the 4 possible cases. Suppose,

$$Q(r1 = 0, c1 = 0) = A \tag{11}$$

Then, it must also be true that

$$Q(r1 = 0, c1 = 1) = A \tag{12}$$

Otherwise, we cannot differentiate and cannot make the right choice because the answer would not be consistent with the road. By extending this process we can infer that,

$$Q(r1 = 1, c1 = 0) = \overline{A} \tag{13}$$

Similarly,

$$Q(r1 = 1, c1 = 1) = \overline{A} \tag{14}$$

In other words, we must be able to make a decision regardless C1 speaks the truth or not.

The Truth Table for the objective is shown in Table 2.

Table 2: Objective of the question

c1	r1	Q
$\times$	0	A
$\times$	1	$\overline{A}$

Here, the '$\times$' refers to the don't care condition.

So, the objective of the question Q is to make the answer depend on only the roads, and not who is answering the question. The objective may be stated as another reduction in dependency as shown in equation 15.

$$Q(r1, c1) = Q(r1). \tag{15}$$

We can take the decision by observing the values of A for different cases.

4 Effect of Asking a Question

Let us now focus on the logical representation of what happens when we ask C1 a question. Suppose, Q_{actual} is the true response to the question Q, i.e., the answer given by C1 where C1 always speaks the truth (c1=1).

$$Q_{actual}(r1) = Q(r1, c1 = 1). \tag{16}$$

If $c1 = 0$, the answer is simply inverted, i.e.,

$$Q(r1, c1 = 0) = \overline{Q_{actual}(r1)}. \tag{17}$$

The complete Truth Table is shown in Table 3.

Table 3: Deriving Q from Q_{actual}

c1	Q_{actual}	Q
0	0	1
0	1	0
1	0	0
1	1	1

We can easily see that the relationship between the behavior of $C1$ and the fact Q_{actual} is the similarity function (XNOR), i.e.,

$$Q = \overline{Q_{actual} \oplus c1} \tag{18}$$

5 A General Approach to Solution

Let us go back and think about our objective, which was to remove dependency of c1 from Q. We know that exclusive OR holds the following relationship.

$$x \oplus x = 0$$

Now we have means to achieve our objective by simply putting,

$$Q_{actual} = \overline{r1 \oplus c1} \tag{19}$$

which yields,

$$Q = \overline{\overline{r1 \oplus c1} \oplus c1} = r1 \tag{20}$$

which is evident from the Truth Table in Table 4.

Table 4: Removing dependency of c1 from Q

c1	$r1$	$Q = \overline{\overline{r1 \oplus c1} \oplus c1}$
0	0	0
0	1	1
1	0	0
1	1	1

Thus, we have successfully eliminated the dependency of c1 from Q. This kind of solution can be achieved using,

$$Q(Q(r1, c1), c1) = r1. \tag{21}$$

In general terms, we have to ask C1 a question about if he would give answer yes to another question about R1, thus cancelling the effect of the behavior of C1.

6 A Sample Solution

Let us now give a concrete solution. We will ask C1 the following question:

"Will you say yes if you (C1) were asked if road R1 leads to location P?"

Let us think about all four possible cases one by one.

Case 1: C1 does not speak the truth and R1 does not lead to P (c1 = 0, r1 = 0).

In this case the answer would be **No** because C1 lies and would say **Yes** to the question *if road R1 leads to location P*. When asked if he would say yes, he lies yet again and tells **No**, which is the correct answer.

Case 2: C1 does not speak the truth and R1 leads to P (c1 = 0, r1 = 1).

In this case the answer would be **Yes** because C1 lies and would say no to the question *if road R1 leads to location P*. When asked about it he lies yet again and gives the correct answer.

Case 3: C1 speaks the truth and R1 does not lead to P (c1 = 1, r1 = 0).

In this case the answer would be **No** because C1 speaks the truth and would say **No** to the question *if road R1 leads to location P*. When asked if he would say yes, he speaks the truth again and sticks to his answer.

Case 4: C1 speaks the truth and R1 leads to P (c1 = 1, r1 = 1).

In this case the answer would be **Yes** because C1 speaks the truth and would say **Yes** to the question *if road R1 leads to location P*. When asked about it he speaks the truth again and gives correct answer.

The Truth Table is as given in Table 4. From Table 4,We see that when $Q = 1$, we have to chose R1 and when $Q = 0$, we have to chose R2.

7 Remarks

Since, we have dependencies between r1 and r2, and between c1 and c2 as given in equations 7, 8, 9 and 10, we may ask a set of other questions and make correct decisions by observing the answers. For example, using equation 7 and 8 we may ask C1,

"Will you say yes if C2 was asked if road R2 leads to location P?"

Using equation 9, we could ask C1,

"Will you say yes if you (C1) were asked if road R2 leads to location P?"

and so on. Thus, we have successfully been able to reach our destination P.