Performance Analysis of Evolutionary Optimization With Cumulative Step Length Adaptation

Dirk V. Arnold and Hans-Georg Beyer

***Abstract*—Iterative algorithms for continuous numerical optimization typically need to adapt their step lengths in the course of the search. While some strategies employ fixed schedules, others attempt to adapt dynamically in response to the outcome of trial steps or the history of the search process. Evolutionary algorithms are of the latter kind. A control strategy that is commonly used in evolution strategies is the cumulative step length adaptation approach. This paper presents a theoretical analysis of that adaptation strategy. The analysis includes the practically relevant case of noise interfering in the optimization process. Recommendations are made with respect to choosing appropriate population sizes.**

***Index Terms*—Evolution strategies, noise, optimization methods, step length adaptation.**

I. Introduction

A great number of strategies have been proposed for numerically obtaining solutions to optimization problems where no derivative information is available. Among them are stochastic approximation approaches [1], [2], implicit filtering [3], direct pattern search [4], simulated annealing [5], and a variety of evolutionary algorithms (EAs) [6]. All of those attempt to approach the optimum in a sequence of steps whose average length typically decreases in the course of the search. Some strategies, such as stochastic approximation methods, rely on fixed schedules that under certain mild conditions can guarantee convergence in the limit of infinitely many time steps. Others, such as EAs or implicit filtering, attempt to adapt step lengths dynamically.

Parameter control methods for EAs are surveyed in [7]. One of the methods that holds particular promise due to its ability to reliably adapt the entire mutation covariance matrix and that has been used successfully in industrial applications (see the references in [8]) is the cumulative step length adaptation mechanism by Hansen and Ostermeier [8], [9]. That mechanism adapts step lengths by analyzing information from the sequence of most recently taken steps. While recommendations with respect to the setting of some of the strategy's parameters have been made in [8], questions concerning the optimal choice of population sizes and the practically relevant issue of robustness in the presence of noise that was raised in [10] have been left unaddressed.

A common approach to studying the properties of evolution strategies—a type of EA often employed in real-valued search spaces—is to consider their dynamic behavior on classes of objective functions that possess symmetries that make the analysis mathematically tractable. An overview along with a number of important results can be found in [11]. Recommendations with regard to the setting of strategy parameters that have been made using that approach have proven to be valuable far beyond the relatively simple problems that they have been derived for.

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D. V. Arnold is with the Faculty of Computer Science, Dalhousie University, Halifax, NS B3H 1W5, Canada (e-mail: dirk@cs.dal.ca).

H.-G. Beyer is with the Department of Computer Science, University of Dortmund, 44221 Dortmund, Germany (e-mail: Hans-Georg.Beyer@cs.uni-dortmund.de).

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This note presents an analysis of the behavior of an evolution strategy with cumulative step length adaptation on a simple class of objective functions disturbed by noise. The algorithm and the class of objective functions are described in Sections II and III, respectively. In Section IV, the equations describing the dynamic behavior of the strategy are formulated. In Sections V–VII, qualitative results are obtained by making simplifications that afford a good understanding of how the performance of the strategy scales with the search space dimensionality and how it is affected by noise. In Section VIII, more exact results are obtained numerically and recommendations with regard to parameter settings are made. Section IX concludes with a brief summary.

II. $(\mu/\mu, \lambda)$-ES With Cumulative Step Length Adaptation

The $(\mu/\mu, \lambda)$-ES[1] is a particular type of evolution strategy that enjoys popularity both due to its proven good performance and to its amenability to mathematical analysis. In every time step, it generates λ new offspring candidate solutions $\mathbf{y}_j \in \mathbb{R}^N$, $j = 1, \ldots, \lambda$, from a population of μ parents $\mathbf{x}_i \in \mathbb{R}^N$, $i = 1, \ldots, \mu$, where $\lambda > \mu$. Subsequently, the parental population is replaced by the μ best of the offspring. Generation of an offspring candidate solution $\mathbf{y}_j$ consists of adding a vector $\sigma\mathbf{z}_j$, where $\mathbf{z}_j \in \mathbb{R}^N$ consists of independent, standard normally distributed components, to the centroid $\mathbf{x} = \sum_{i=1}^{\mu} \mathbf{x}_i/\mu$ of the parental population. The arithmetic averaging of the parents is referred to as intermediate recombination. Mutations are isotropic as the probability distribution of the offspring is spherically symmetric with the parental centroid as center.[2] The standard deviation σ of the components of vector $\sigma\mathbf{z}_j$ is referred to as the mutation strength, vector $\mathbf{z}_j$ as a mutation vector. The centroid of the population at the next time step is

$$\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} + \sigma^{(t)}\mathbf{z}^{(t)} \tag{1}$$

where $\mathbf{z}$ is the average of those mutation vectors $\mathbf{z}_j$ that correspond to offspring candidate solutions that are selected to form the population of the next time step and is referred to as the progress vector.

Clearly, the mutation strength determines the step length of the strategy. The cumulative step length adaptation mechanism relies on the conjecture that if the mutation strength σ is below its optimal value consecutive steps of the strategy tend to be parallel, and if the mutation strength is too high consecutive steps tend to be antiparallel. For optimally adapted mutation strength, the steps taken by the evolution strategy should be uncorrelated. This is instructive as several steps in one direction could better be replaced by a single, longer step, and as stepping back and forth suggests that a smaller step length should be used.

So, as to be able to reliably detect parallel or antiparallel correlations between successive steps, information from a number of time steps

[1]The $(\mu/\mu, \lambda)$-ES is an instance of the more general $(\mu/\rho \overset{+}{,} \lambda)$-ES, where the double appearance of the symbol μ indicates that recombination is global in that it involves the entire parental population. For a more thorough account of notational issues; see [12].

[2]It is important to note that the restriction to isotropic mutations has been made here only for the sake of mathematical tractability. For the class of objective functions to be introduced in Section III, this is not a serious limitation. However, most applications for efficiency reasons require mutation vectors that can be drawn from arbitrary normal distributions. In its full generality, the cumulative step length adaptation mechanism adapts the entire mutation covariance matrix. According to [8], it can be observed that mutation covariance matrices are adapted such that arbitrary convex quadratic objective functions are "rescaled into the sphere" to be introduced in Section III. The insights provided by the analysis presented here can thus be expected to have direct implications for the case of general mutations and general locally convex objective functions.

needs to be accumulated. For that purpose, the accumulated progress vector $\mathbf{s}$ is defined in [9] by $\mathbf{s}^{(0)} = \mathbf{0}$ and the recursive relationship

$$\mathbf{s}^{(t+1)} = (1-c)\mathbf{s}^{(t)} + \sqrt{\mu c(2-c)}\mathbf{z}^{(t)} \tag{2}$$

where $c \in (0,1)$ is a constant that determines how far back the "memory" of the accumulation process reaches and that is set to $1/\sqrt{N}$ according to a recommendation by [9]. The accumulated progress vector is a record of the directions of past steps taken by the strategy. It has been shown in [9] that if the candidate solutions to form the parental population are selected at random, the components of $\mathbf{z}$ are normally distributed with zero mean and with variance $1/\mu$. It was seen that as a consequence, the components of $\mathbf{s}$ are standard normally distributed for any choice of $c \in (0,1)$ after initialization effects have faded. Under random selection, the squared length of the accumulated progress vector is thus χ^2_N-distributed and, therefore, has mean N.

For nonrandom selection, consecutive progress vectors are not generally uncorrelated. According to [9], in the case of positive correlations, $\|\mathbf{s}\|^2$ tends to be greater than N; for negative correlations, $\|\mathbf{s}\|^2$ is usually less than N. Cumulative step length adaptation seeks to obtain uncorrelated steps even in the case of nonrandom selection by comparing the squared length of the accumulated progress vector with its expected squared length under random selection and updating the mutation strength according to[3]

$$\sigma^{(t+1)} = \sigma^{(t)} \exp\left(\frac{\|\mathbf{s}^{(t+1)}\|^2 - N}{2DN}\right). \tag{3}$$

Here, D denotes a damping constant that is set to $\sqrt{N}$ according to a recommendation made in [9].

III. Progress Rate Analysis of the $(\mu/\mu,\lambda)$-ES

EAs together with the objective functions they operate on form iterated dynamic systems. In order to be able to study the dynamics of those systems, particular classes of objective functions need to be considered. The most commonly considered class of objective functions assumes that the quality of a candidate solution $\mathbf{x}$ is determined by its distance $R = \|\hat{\mathbf{x}} - \mathbf{x}\|$ from some target $\hat{\mathbf{x}}$. That is, $f(\mathbf{x}) = g(R)$ for some monotonic function $g : \mathbb{R} \to \mathbb{R}$. Denoting the change in distance from the target by $\Delta_R^{(t)} = R^{(t)} - R^{(t+1)}$, progress is measured by the expected value of that quantity, the progress rate $\varphi = E[\Delta_R]$. Due to its spherical symmetries, this class of objective functions is referred to as the sphere model.

Noise is a common factor in real-world optimization problems. For theoretical analyses it is most commonly modeled as an additive, normally distributed term with mean zero. That is, when determining the objective function value of a candidate solution $\mathbf{x}$, it is not the true objective function value $f(\mathbf{x})$ that is obtained but rather a measured value that is drawn from a normal distribution with mean $f(\mathbf{x})$. The standard deviation of that distribution that may depend on the distance R from the candidate solution being evaluated to the target is referred to as the noise strength and is denoted by $\sigma_\epsilon(R)$.

Analyses of the behavior of evolution strategies on the sphere model rely on a decomposition of vectors that is illustrated in Fig. 1. A vector $\mathbf{z}$ originating at search space location $\mathbf{x}$ can be written as the sum of two vectors $\mathbf{z}_A$ and $\mathbf{z}_B$, where $\mathbf{z}_A$ is parallel to $\hat{\mathbf{x}} - \mathbf{x}$ and $\mathbf{z}_B$ is in the hyperplane perpendicular to that. In the present context, $\mathbf{z}$ can be either a mutation vector or a progress vector. The vectors $\mathbf{z}_A$ and $\mathbf{z}_B$ are referred to as the central and lateral components of vector $\mathbf{z}$, respectively. The signed length z_A of the central component of vector $\mathbf{z}$ is defined to equal $\|\mathbf{z}_A\|$ if $\mathbf{z}_A$ points toward the target and to equal $-\|\mathbf{z}_A\|$ if it points away from it.

[3]This is a minor deviation from [9] where the length $\|\mathbf{s}\|$ of the accumulated progress vector rather than its squared length is used as the basis on which adaptation is performed. The difference in performance appears to be insignificant and vanishes for $N \to \infty$. The argument in [9] that leads to recommendations with regard to the choice of c and D remains valid.

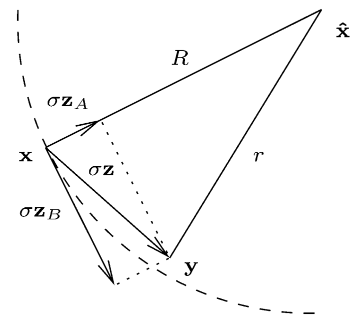

Fig. 1. Decomposition of a vector $\mathbf{z}$ into central component $\mathbf{z}_A$ and lateral component $\mathbf{z}_B$. Vector $\mathbf{z}_A$ is parallel to $\hat{\mathbf{x}} - \mathbf{x}$, vector $\mathbf{z}_B$ is in the hyperplane perpendicular to that. The starting and end points, $\mathbf{x}$ and $\mathbf{y} = \mathbf{x} + \sigma\mathbf{z}$, of vector $\sigma\mathbf{z}$ are at distances R and r from the target $\hat{\mathbf{x}}$, respectively.

As seen in [11], using normalized quantities

$$\sigma^* = \sigma\frac{N}{R} \quad \Delta_R^* = \Delta_R\frac{N}{R}, \text{ and } \sigma_\epsilon^* = \sigma_\epsilon\frac{N}{Rg'(R)} \tag{4}$$

it becomes possible to characterize the system behavior without direct reference to the distance from the target R. It has been shown in [13] that for the sphere model with fixed μ and λ, the relationships

$$\lim_{N\to\infty}\frac{E\left[\|\mathbf{z}\|^2\right]}{N} = \frac{1}{\mu} \text{ and } \lim_{N\to\infty} E[z_A] = \frac{c_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}} \tag{5}$$

where $\vartheta = \sigma_\epsilon^*/\sigma^*$, hold for any finite value of σ^*. In that same reference it was seen that σ^* equals the standard deviation of the normalized true objective function values of the offspring candidate solutions. As σ_ϵ^* is the standard deviation of the normalized noise term, the quotient ϑ is the noise-to-signal ratio of the system. The coefficient $c_{\mu/\mu,\lambda}$ is the expectation of the average of the μ highest order statistics of a random sample of λ independent, standard normally distributed random variables and has been computed in [11]. Finally, as shown in [13], the normalized progress rate is

$$\varphi^* = E[\Delta_R^*] = \frac{\sigma^* c_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}} - \frac{\sigma^{*2}}{2\mu}. \tag{6}$$

The first summand on the right-hand side of (6) is a nonnegative gain term that is due to the central component of the progress vector while the second term is a loss term that results from that vector's lateral component, the direction of which is entirely random in the plane defined by normal vector $\hat{\mathbf{x}} - \mathbf{x}$.

While all of those results are valid strictly only for fixed μ and λ in the limit $N \to \infty$, they can be used to make qualitative predictions with regard to the behavior of the $(\mu/\mu,\lambda)$-ES on the sphere for finite but sufficiently large search space dimensionality, provided that μ and λ are not too large. Improved estimates for the progress vector as well as the progress rate for moderate values of N have been derived in [14] and [15]. While the simple expressions quoted here form the basis for the calculations in Sections V–VII, those improved estimates will be used in Section VIII for numerically determining optimal population sizes and efficiencies.

IV. System Equations

In a generalization of our previous work [13] that did not consider step length adaptation, the accumulated progress vector just as mutation vectors and progress vectors can be written as the sum of its central and lateral components, $\mathbf{s}_A$ and $\mathbf{s}_B$. In analogy to what has been introduced above for progress vectors, s_A stands for the signed length of the central component of the accumulated progress vector. For symmetry reasons, the direction of the lateral component of the accumulated progress vector is random. The state of the strategy at time t is well described by the distance between the centroid of the population and the target, the squared length of the accumulated progress vector, the signed length of its central component, and the normalized mutation strength. State variables at time step $t+1$ can be expressed in terms of their values at time step t as follows.

- Using (4), the distance between the centroid of the population at time step $t+1$ and the target is

$$R^{(t+1)} = R^{(t)}\left(1 - \frac{\Delta_R^{*(t)}}{N}\right). \quad (7)$$

- Using (2), the squared length of the accumulated progress vector at time step $t+1$ is

$$\|\mathbf{s}^{(t+1)}\|^2 = (1-c)^2\|\mathbf{s}^{(t)}\|^2 + 2(1-c)\sqrt{\mu c(2-c)}\mathbf{s}^{(t)}\mathbf{z}^{(t)} + \mu c(2-c)\|\mathbf{z}^{(t)}\|^2. \quad (8)$$

- The signed length of the central component of the accumulated progress vector equals the inner product of the accumulated progress vector with a vector of length unity pointing from the centroid of the population to the target. Thus, using (1), (2), and (4), it follows that

$$s_A^{(t+1)} = \frac{R^{(t)}}{R^{(t+1)}}\left[(1-c)\left(s_A^{(t)} - \frac{\sigma^{*(t)}}{N}\mathbf{s}^{(t)}\mathbf{z}^{(t)}\right) + \sqrt{\mu c(2-c)}\left(z_A^{(t)} - \frac{\sigma^{*(t)}}{N}\|\mathbf{z}^{(t)}\|^2\right)\right]. \quad (9)$$

- From (3) and (4), it follows that the normalized mutation strength at time step $t+1$ is

$$\sigma^{*(t+1)} = \sigma^{*(t)}\frac{R^{(t)}}{R^{(t+1)}}\exp\left(\frac{\|\mathbf{s}^{(t+1)}\|^2 - N}{2DN}\right). \quad (10)$$

V. Determining the Accumulated Progress Vector

If the normalized noise strength σ_ϵ^* is independent of the location in search space, simple expressions can be obtained that describe the behavior of the strategy. For the particular case that $g(R) = k \cdot R^\alpha$ for some $k, \alpha > 0$, constant normalized noise strength implies that the standard deviation of the noise term decreases as the target is approached. This is of interest for example in connection with measurement devices that are accurate up to a certain percentage of the quantity they measure.

Under the assumption of constant normalized noise strength, the environment is scale invariant in that its response (in normalized variables) is independent of the location in search space. The $(\mu/\mu, \lambda)$-ES with cumulative step length adaptation thus approaches a state that is stationary in that the squared length of the accumulated progress vector, the signed length of that vector's central component, and the normalized mutation strength tend toward an invariant limit distribution; see [14] for details. The nonlinear character of the system (7)–(10) precludes determining that distribution exactly. However, the amount of fluctuations of the state variables relative to their expectations decreases with increasing search space dimensionality.[4] In a first-order approximation, fluctuations of the state variables can be ignored and it can be assumed that Δ_R^*, $\|\mathbf{s}\|^2$, s_A, and σ^* assume deterministic stationary values. Replacing the squared length of the progress vector and the signed length of its central component by their expected values given in (5) and modeling the approach of the target by the progress rate given in (6), those stationary values can be obtained from (8) and (9) by demanding stationarity. In particular, for the squared length of the accumulated progress vector, this amounts to demanding that $E[\|\mathbf{s}^{(t+1)}\|^2] \stackrel{!}{=} \|\mathbf{s}^{(t)}\|^2 \stackrel{!}{=} \|\mathbf{s}\|^2$ and, therefore, according to (8) to requiring that

$$\|\mathbf{s}\|^2 \stackrel{!}{=} (1-c)^2\|\mathbf{s}\|^2 + 2(1-c)\sqrt{\mu c(2-c)}s_A\frac{c_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}} + c(2-c)N \quad (11)$$

where $\mathbf{s}_A\mathbf{z}_B = \mathbf{s}_B\mathbf{z}_A = 0$ and the fact that $E[\mathbf{s}_B\mathbf{z}_B] = 0$ due to the randomness of $\mathbf{z}_B$ have been used.

Similarly, a stationarity condition can be formulated for the signed length of the central component of the accumulated progress vector. In order to keep things simple, terms in (9), the influence of which vanishes in the limit of infinite N, are disregarded. In particular, that is the case for the quotient $R^{(t)}/R^{(t+1)}$ that can be approximated by unity in the present context. Taylor expansion of (7) shows that the resulting error is of order $\mathcal{O}(\Delta_R^*/N)$. Moreover, as $s_A - \sigma^* s_A z_A/N$ approaches s_A as $N \to \infty$, it follows that the simplified stationarity demand resulting from (9) is

$$s_A \stackrel{!}{=} (1-c)s_A + \sqrt{\mu c(2-c)}\left(\frac{c_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}} - \frac{\sigma^*}{\mu}\right). \quad (12)$$

Solving for s_A yields

$$s_A = \frac{\sqrt{\mu c(2-c)}}{c}c_{\mu/\mu,\lambda}\left(\frac{1}{\sqrt{1+\vartheta^2}} - \frac{\sigma^*}{\mu c_{\mu/\mu,\lambda}}\right). \quad (13)$$

Using this result in (11), it follows:

$$\|\mathbf{s}\|^2 = N + \frac{2(1-c)}{c}\frac{\mu c_{\mu/\mu,\lambda}^2}{\sqrt{1+\vartheta^2}}\left(\frac{1}{\sqrt{1+\vartheta^2}} - \frac{\sigma^*}{\mu c_{\mu/\mu,\lambda}}\right) \quad (14)$$

for the squared length of the accumulated progress vector. While being inexact due to the simplifications made in their derivation—terms vanishing for $N \to \infty$ and fluctuations around the mean values have been ignored—for large N, (13) and (14) do provide a good basis for the understanding of the behavior of cumulative step length adaptation on the noisy sphere.

VI. Logarithmic Adaptation Response and Target Mutation Strength

Before considering the normalized mutation strength that cumulative step length adaptation realizes, it is instructive to first analyze the static behavior of the strategy. The logarithmic adaptation response

$$\Delta_\sigma^{(t)} = \log\left(\frac{\sigma^{(t+1)}}{\sigma^{(t)}}\right) \quad (15)$$

and its normalization $\Delta_\sigma^* = \Delta_\sigma cDN/(1-c)$ are useful quantities for describing the performance of the step length adaptation scheme. They

[4]It is seen in [15] that the major source of inaccuracies for finite N are fluctuations in the squared length of the lateral components of the mutation vectors, and that the quotient of standard deviation and expectation of that squared length scales as $N^{-1/2}$. Similar scaling properties can be observed for the additional variables that result from considering adaptation; however, a detailed analysis would go beyond what can be presented here.

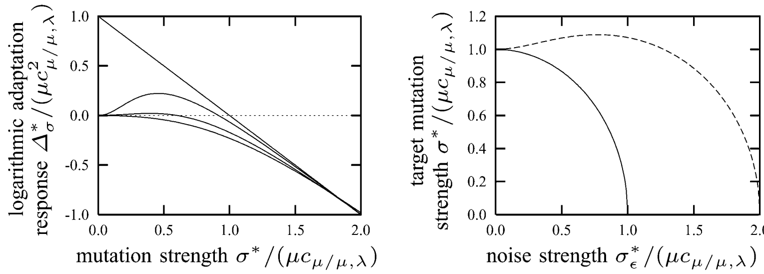

Fig. 2. Normalized logarithmic adaptation response Δ^*_σ as a function of normalized mutation strength σ^* and normalized target mutation strength σ^* as a function of normalized noise strength σ^*_ϵ. Note the scaling of the axes. The lines in the left-hand graph have been obtained from (16) for, from top to bottom, normalized noise strengths $\sigma^*_\epsilon/(\mu c_{\mu/\mu,\lambda}) = 0.0, 0.4, 0.8$, and 1.2. In the right-hand graph, the solid line corresponds to the target mutation strength given by (17), the dashed line to the optimal mutation strength that is obtained by numerically optimizing (6).

quantify how the strategy responds to an ill-adapted mutation strength. Positive logarithmic adaptation response indicates an increase in step length realized by the strategy, negative adaptation response indicates a decrease. Therefore, ideally, the logarithmic adaptation response is positive for mutation strengths that are too small and negative for mutation strengths that are too large. It has roots at $\sigma = 0$ (no progress) and at a unique positive mutation strength that we refer to as the target mutation strength of the strategy. From (3), (14), and (15) the estimate

$$\Delta^*_\sigma = \frac{\mu c^2_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}}\left(\frac{1}{\sqrt{1+\vartheta^2}} - \frac{\sigma^*}{\mu c_{\mu/\mu,\lambda}}\right) \tag{16}$$

for the normalized logarithmic adaptation response of cumulative step length adaptation on the noisy sphere is obtained. Substituting $\vartheta = \sigma^*_\epsilon/\sigma^*$ and subsequent root finding shows that the target mutation strength is

$$\sigma^* = \mu c_{\mu/\mu,\lambda}\sqrt{1 - \left(\frac{\sigma^*_\epsilon}{\mu c_{\mu/\mu,\lambda}}\right)^2}. \tag{17}$$

Fig. 2 shows the normalized logarithmic adaptation response given by (16) as a function of the normalized mutation strength and the normalized target mutation strength given by (17) as a function of the normalized noise strength. At least two things can be learned from the figure. First, the right-hand graph shows that for nonzero noise strength, the target mutation strength of cumulative step length adaptation is below the mutation strength that maximizes the progress rate of the strategy. For normalized noise strengths exceeding $\mu c_{\mu/\mu,\lambda}$, the target mutation strength is zero even though positive progress rates could be achieved with nonzero mutation strengths. Second, the left-hand graph shows that in the presence of noise, for mutation strengths significantly below their optimal values, the logarithmic adaptation response and, therefore, the tendency toward higher mutation strengths is very small. This is intuitively clear as cumulative step length adaptation attempts to achieve that consecutive progress vectors are uncorrelated. Small steps carry little information. In the presence of noise, that information is almost entirely hidden and correlations between consecutive progress vectors disappear as steps are increasingly random. The strategy, thus, sees no need to increase the mutation strength much, even though significantly higher mutation strengths would achieve a better noise-to-signal ratio and greater progress.[5]

[5]This insight sheds new light on the postulate suggested by Beyer and Deb [16] that in "flat" regions of the search space, i.e., in regions where the objective function values appear (nearly) constant, step length adaptation schemes should tend to systematically increase step lengths. In the presence of noise, this advice may be especially useful as regions in search space may appear to be flat due to a high noise-to-signal ratio, and as operating at higher mutation strengths might make more reliable information available to the strategy.

VII. Determining the Mutation Strength

The target mutation strength is not the mutation strength that is actually realized by the step length adaptation mechanism. As the distance R to the target continually changes, and as adaptation to the target mutation strength is not instantaneous, the mutation strength that is actually realized is always "behind." An estimate of that mutation strength can be obtained by solving (10) for σ^*. Expanding both the quotient $R^{(t)}/R^{(t+1)} = (1 - \Delta^*_R/N)^{-1}$ and the exponential function into Taylor series, replacing quantities by their expected values, and neglecting all terms that are without relevance in the limit $N \to \infty$ yields the stationarity demand

$$\sigma^* \stackrel{!}{=} \sigma^*\left(1 + \frac{\varphi^*}{N} + \frac{\|\mathbf{s}\|^2 - N}{2DN}\right) \tag{18}$$

where φ^* and $\|\mathbf{s}\|^2$ are given by (6) and (14), respectively. Using the fact that for the settings of c and D suggested in [9], i.e., $c = 1/\sqrt{N}$ and $D = \sqrt{N}$, the term $(1-c)/(cD)$ tends to unity as N increases, it is easily verified that (18) can be transformed into

$$0 \stackrel{!}{=} \frac{\sigma^* c_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}} - \frac{\sigma^{*2}}{2\mu} + \frac{\mu c^2_{\mu/\mu,\lambda}}{\sqrt{1+\vartheta^2}}\left(\frac{1}{\sqrt{1+\vartheta^2}} - \frac{\sigma^*}{\mu c_{\mu/\mu,\lambda}}\right).$$

Substituting $\vartheta = \sigma^*_\epsilon/\sigma^*$ and solving for the normalized mutation strength yields

$$\sigma^* = \mu c_{\mu/\mu,\lambda}\sqrt{2 - \left(\frac{\sigma^*_\epsilon}{\mu c_{\mu/\mu,\lambda}}\right)^2} \tag{19}$$

for the normalized mutation strength that is realized by cumulative step length adaptation on the noisy sphere. According to (6), the normalized progress rate achieved with this mutation strength is

$$\varphi^* = \frac{\sqrt{2}-1}{2}\mu c^2_{\mu/\mu,\lambda}\left(2 - \left(\frac{\sigma^*_\epsilon}{\mu c_{\mu/\mu,\lambda}}\right)^2\right). \tag{20}$$

Both the normalized mutation strength given by (19) and the normalized progress rate given by (20) are shown as functions of the normalized noise strength in Fig. 3. A comparison of Figs. 2 and 3 reveals that the mutation strength actually realized by the strategy differs from the target mutation strength as a consequence of the target being approached dynamically. The dynamics of the process lead to mutation strengths that are too large for normalized noise strengths of up to about $0.91\mu c_{\mu/\mu,\lambda}$ and too small for noise strengths above this value. For zero noise strength, the progress rate that is achieved with cumulative step length adaptation is about 83% of the progress rate that would be

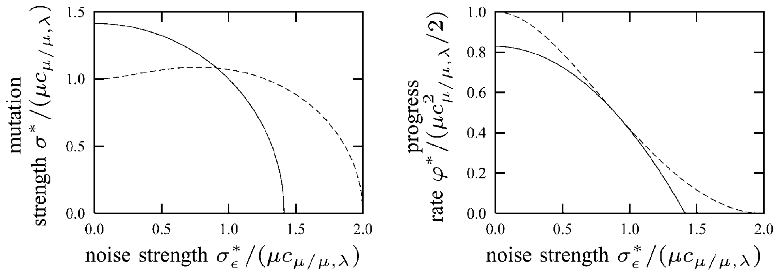

Fig. 3. Normalized mutation strength σ^* and normalized progress rate φ^* as functions of the normalized noise strength σ_ϵ^*. Note the scaling of the axes. The solid lines represent the values realized by the $(\mu/\mu,\lambda)$-ES with cumulative step length adaptation and have been obtained from (19) and (20). The dashed lines represent the optimal values obtained by numerically optimizing (6).

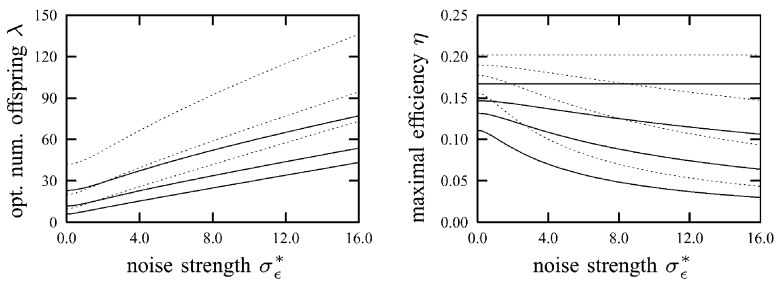

Fig. 4. Optimal number of offspring per time step λ and maximal efficiency η as functions of the normalized noise strength σ_ϵ^*. The curves correspond to, from bottom to top, search space dimensionalities $N = 40$, $N = 400$, and $N = 4000$. In the right-hand graph, the limiting case $N = \infty$ is included as well. The solid curves depict the results for cumulative step length adaptation. The dotted lines assume optimally adapted mutation strength.

achieved with optimally adapted mutation strength. Positive progress rates are achieved up to a normalized noise strength of $\sqrt{2}\mu c_{\mu/\mu,\lambda}$.

VIII. Population Sizing

While sufficient for obtaining a good qualitative understanding of the performance of cumulative step length adaptation in the presence of noise, the approximation considered thus far is too crude for addressing the problem of determining optimal population sizes. From the results obtained so far, it appears that by increasing the population size, the strategy can always be made to operate in the regime at the left-hand edge of the graphs in Fig. 3 and, thus, with maximum efficiency. However, it has been seen in [15] that the quality of the approximation given by (5) and (6) deteriorates with increasing population size. In that same reference, better estimates of the squared length of the progress vector, the signed length of its central component, and the progress rate have been derived. Using those estimates rather than (5) and (6) and not neglecting the N-dependent terms that had been neglected in the derivations of (12) and (19) yields a much improved approximation that extends the results obtained in [15] to include the case of cumulative step length adaptation.

On the basis of the improved estimates, optimal population size parameters and efficiencies can be determined. The efficiency of a strategy is defined in a way that takes not only the progress made but also the computational costs of the optimization into account. Assuming that those costs are dominated by the costs of evaluating the objective function and that other contributions such as those resulting from mutation and recombination can be neglected, the efficiency is commonly defined as the normalized progress rate per evaluation of the objective function

$$\eta = \frac{\varphi^*}{\lambda}. \tag{21}$$

Notice that the term λ in the denominator is the number of objective function evaluations per time step. Optimal parameter settings can be obtained numerically by optimizing (21).

Fig. 4 shows the optimal number of offspring per time step λ and the maximal efficiency η, i.e., the efficiency for optimally chosen population size parameter settings, as functions of the normalized noise strength σ_ϵ^*. Also shown are the corresponding values that would be obtained were the mutation strength continually adapted to the optimal values that have been derived in [15]. It can be seen that the efficiencies that cumulative step length adaptation is capable of realizing are—depending on the search space dimensionality and the noise strength—between 15% and 30% below the optimal values. Quite significantly, the right-hand graph in Fig. 4 shows that cumulative step length adaptation is robust in the sense that it fails to break down in the presence of noise at least for the range of noise strengths considered. The loss of efficiency that incurs in the presence of noise does not differ qualitatively from that in the absence of noise.

As for the population size parameter settings, it can be seen that using cumulative step length adaptation optimal population sizes are below the values computed in [15] for optimally adapted mutation strengths. In the absence of noise, optimal values for λ are 6, 12, and 23 for $N = 40$, 400, and 4000, respectively. In the presence of noise, larger values of λ need to be employed in order to achieve optimal efficiency. Overall, it can be said that the choice of a value for λ is not very critical provided that λ is chosen large enough to support positive progress for the given noise level, and that that choice becomes even less critical with increasing noise strength. For the range of noise strengths and search space dimensionalities considered, optimal values of μ are always in the range from 0.25λ to 0.30λ. Further numerical investigations show that for optimally chosen population size parameters, the strategies always operate in the regime in the left-hand half of the graphs in Fig. 3, i.e., that $\sigma_\epsilon^* < \mu c_{\mu/\mu,\lambda}$ for optimally chosen μ and λ.

IX. Conclusion

It has been seen that the target mutation strength that cumulative step length adaptation seeks to realize is optimal in the limit $N \to \infty$ in the absence of noise, but generally too small in its presence. However, the mutation strength that cumulative step length adaptation actually achieves differs from the target mutation strength as adaptation is not instantaneous. The mutation strength that is achieved is too large for low noise levels, and too small for high noise levels. The performance loss as compared to optimally adapted mutation strengths has been found to be below 20% in the idealized model from Section VII and below about 30% in the improved model from Section VIII. In the presence of (not too much) noise, the larger than optimal mutation strengths have the advantage of reducing the noise-to-signal ratio that the strategy operates under.

Of particular importance to the practitioner is the problem of choosing appropriate settings for the population size parameters. The investigation presented in this paper suggests that between 25% and 30% of the candidate solutions generated should be retained to serve as the population of the next time step. As for choosing how many candidate solutions to generate per time step, higher values buy additional robustness, i.e., the ability to proceed in the presence of higher levels of noise—at the price of decreased efficiency. Cumulative step length adaptation drives the mutation strength to zero if there is too much noise present. A useful course of action is therefore to start out with a relatively small number of candidate solutions to generate per time step, and to gradually increase that number if the strategy is observed to stall. The choice of values for μ and λ has been found to be rather uncritical.

Overall, the performance of the $(\mu/\mu, \lambda)$-ES with cumulative step length adaptation is robust in that it degrades gradually with increasing noise levels. An empirical evaluation of direct optimization strategies in the presence of noise in [17] has shown that this is not necessarily true for other commonly used approaches, thus making the evolution strategy a promising candidate for the optimization of noisy objective functions.

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Local Control Strategies for Groups of Mobile Autonomous Agents

Zhiyun Lin, Mireille Broucke, and Bruce Francis

***Abstract*—The problem of achieving a specified formation among a group of mobile autonomous agents by distributed control is studied. If convergence to a point is feasible, then more general formations are achievable too, so the focus is on convergence to a point (the agreement problem). Three formation strategies are studied and convergence is proved under certain conditions. Also, motivated by the question of whether collisions occur, formation evolution is studied.**

***Index Terms*—Distributed control, mobile autonomous agents.**

I. INTRODUCTION

In 1987, Reynolds [16] introduced a model and wrote a program called *boids* [17] that simulated a flock of birds in flight; they fly as a flock, with a common average heading, and they avoid colliding with each other. Each bird has a local control strategy, yet a desirable overall group behavior is achieved. The local strategy of each bird has three components: *separation*, steer to avoid crowding; *alignment*, steer toward the average heading of neighbors; *cohesion*, steer toward the average position of neighbors. Recently, Jadbabaie *et al.* [7] formulated a two-dimensional version of Reynolds' setup and studied one of the steering strategies. They proved that the alignment strategy leads, under a certain assumption (the graph representing which agents are neighbors of another always is connected, or at least periodically connected), to the result that all the agents' headings converge to a common heading. Besides being of interest in biology, Reynolds' ideas have relevance in the subject of multiple vehicle formations, e.g., [15], [19], and [20]. Generally, the objective is for a group of mobile agents (robot rovers, unmanned air vehicles, or unmanned underwater vehicles) either to achieve a formation, or to move while maintaining a formation, or to reconfigure from one formation to another.

Recently, several researchers have investigated issues in distributed algorithms for multiagent systems. In [19], a group of simulated robots form approximations to circles and simple polygons, using the scenario that each robot orients itself to, e.g., the furthest and nearest robot. In

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The authors are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: linz@control.utoronto.ca; broucke@control.utoronto.ca; francis@control.utoronto.ca).

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