Chapter 2: Finite-Dimensional Vector Spaces

Linear Algebra Done Right, by Sheldon Axler

A: Span and Linear Independence

Problem 1

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Proof. Let $w \in V$. Then there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

We wish to find $b_1, b_2, b_3, b_4 \in \mathbb{F}$ such that

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Simplifying the LHS, we have

$$b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Hence we may choose

$$\begin{aligned} b_1 &= a_1 \\ b_2 &= a_1 + a_2 \\ b_3 &= a_1 + a_2 + a_3 \\ b_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned}$$

so that w is given as a linear combination of the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$, and thus the list spans V as well. $\square$

Problem 3

Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

Proof. Let $t = 2$. Then

$$3(3,1,4) - 2(2,-3,5) = (5,9,2),$$

and hence the vectors are not linearly independent since one of the vectors can be written as a linear combination of the other two. $\square$

Problem 5

(a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1+i, 1-i)$ is linearly independent.

(b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

Proof. (a) Suppose

$$a(1+i) + b(1-i) = 0$$

for some $a, b \in \mathbb{R}$. Then

$$(a+b) + (a-b)i = 0.$$

Comparing imaginary parts, this implies $a - b = 0$ and hence $a = b$. But now substituting for b and comparing real parts, this implies $2a = 0$, and hence $a = b = 0$. Thus the vectors are linearly independent over $\mathbb{R}$.

(b) Note that

$$-i(1+i) = 1 - i,$$

so that $1 - i$ is a scalar multiple of $1 + i$ and hence the vectors are linearly dependent over $\mathbb{C}$. $\square$

Problem 7

Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Proof. Let $u = 5v_1 - 4v_2$. We claim the list $u, v_2, \ldots, v_m$ is linearly independent. To see this, suppose not. Then there exists some $j \in \{2, \ldots, m\}$ such that $v_j \in \operatorname{span}(u, v_2, \ldots, v_{j-1})$. But then v_j is also in $\operatorname{span}(v_1, v_2, \ldots, v_{j-1})$, since $u = 5v_1 - 4v_2$ is a linear combination of v_1 and v_2, a contradiction. $\square$

Problem 9

Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

Proof. The statement is false. To see this, let $w_k = -v_k$ for $k = 1, \ldots, m$. Then $w_1, \ldots, w_m$ are also linearly independent, but $v_1 + w_1 = \cdots = v_m + w_m = 0$. $\square$

Problem 11

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Proof. ($\Rightarrow$) First suppose $v_1, \ldots, v_m, w$ is linearly independent. If $w \in \operatorname{span}(v_1, \ldots, v_m)$, then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$w = a_1 v_1 + \cdots + a_m v_m.$$

But then

$$-w + a_1 v_1 + \cdots + a_m v_m = 0,$$

a contradiction. Therefore we must have $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

($\Leftarrow$) Now suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$ and consider the list $v_1, \ldots, v_m, w$. Suppose the list were linearly dependent. Then there exists a vector in the list which is in the span of its predecessors. Since this vector cannot be w by assumption, there exists some $j \in \{1, \ldots, m\}$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$, contradicting the hypothesis that $v_1, \ldots, v_m$ is linearly independent (and hence all sublists are). Thus $v_1, \ldots, v_m, w$ must be linearly independent. $\square$

Problem 13

Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$.

Proof. Note that the list $1, z, \ldots, z^4$ spans $\mathcal{P}_4(\mathbb{F})$, is linearly independent, and has length 5. Since the length of every spanning list must be at least as long as every linearly independent list, there exist no spanning lists of vectors in $\mathcal{P}(\mathbb{F})$ of length less than 5. $\square$

Problem 14

Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof. (⇒) First suppose V is infinite-dimensional. We will prove by induction that there exists a sequence $v_1, v_2, \ldots$ of vectors in V such that for every $m \in \mathbb{Z}^+$, the first m vectors are linearly independent.

Base Case: Since V is infinite-dimensional, V contains some nonzero vector v_1. The list containing only this vector is clearly linearly independent.

Inductive Step: Suppose the list of vectors $v_1, \ldots, v_k$ is linearly independent for some $k \in \mathbb{Z}^+$. Since V is infinite-dimensional, these vectors cannot span V, and hence there exists some $v_{k+1} \in V \setminus \operatorname{span}(v_1, \ldots, v_k)$. In particular, note that $v_{k+1} \neq 0$. But then $v_1, \ldots, v_k, v_{k+1}$ is linearly independent by the Linear Dependence Lemma (for if it were linearly dependent, the Lemma guarantees there would exist a vector in the list which could be written as a linear combination of its predecessors, which is impossible by our construction).

By induction, we have shown there exists a list $v_1, v_2, \ldots$ such that $v_1, \ldots, v_m$ is linearly independent for every $m \in \mathbb{Z}^+$.

(⇐) Now suppose there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every $m \in \mathbb{Z}^+$. If V were finite-dimensional, there would exist a list $v_1, \ldots, v_n$ for some $n \in \mathbb{Z}^+$ such that $V = \operatorname{span}(v_1, \ldots, v_n)$. But then, by our assumption, the list $v_1, \ldots, v_{n+1}$ is linearly independent. Since every linearly independent list must have length no longer than every spanning list, this is a contradiction. Thus V is infinite-dimensional. □

Problem 15

Prove that $\mathbb{F}^\infty$ is infinite-dimensional.

Proof. For each $k \in \mathbb{Z}$, define the vector e_k such that it has a 1 in coordinate k and 0 everywhere else. Then for the sequence $e_1, e_2, \ldots$, the list $e_1, \ldots, e_m$ is linearly independent for any choice of $m \in \mathbb{Z}^+$. By Problem 14, $\mathbb{F}^\infty$ must be infinite-dimensional. □

Problem 17

Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

Proof. Suppose it were. We will show that this implies $p_0, p_1, \ldots, p_m$ spans $\mathcal{P}_m(\mathbb{F})$ and that this in turn leads to a contradiction by explicitly constructing a polynomial that is not in this span.

Note that the list $1, z, \ldots, z^{m+1}$ spans $\mathcal{P}_m(\mathbb{F})$ and has length $m + 1$, hence every linearly independent list must have length $m + 1$ or less. If $\operatorname{span}(p_0, p_1, \ldots, p_m) \neq \mathcal{P}_m(\mathbb{F})$, there exists some $p \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$, and thus the list $p_0, p_1, \ldots, p_m, p$ is linearly independent and of length $m + 2$, a contradiction. And so we must have $\operatorname{span}(p_0, p_1, \ldots, p_m) = \mathcal{P}_m(\mathbb{F})$.

Now define the polynomial $q \equiv 1$. Then $q \in \operatorname{span}(p_0, p_1, \ldots, p_m)$, and hence there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$q = a_0 p_0 + a_1 p_1 + \cdots + a_m p_m,$$

which in turn implies

$$q(2) = a_0p_0(2) + a_1p_1(2) + \cdots + a_mp_m(2).$$

But this is absurd, since this implies $1 = 0$. Therefore $p_0, p_1, \ldots, p_m$ cannot be linearly independent, as desired. $\square$

B: Bases

Problem 1

Find all vector spaces that have exactly one basis.

Proof. We claim that only the trivial vector space has exactly one basis. We first consider finite-dimensional vector spaces. Let V be a nontrivial vector space with basis $v_1, \ldots, v_n$. We claim that for any $c \in \mathbb{F}^\times$, the list $cv_1, \ldots, cv_n$ is a basis as well. Clearly the list is still linearly independent, and to see that it still spans V, let $u \in V$. Then, since $v_1, \ldots, v_n$ spans V, there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$u = a_1v_1 + \cdots + a_nv_n.$$

But then we have

$$u = \frac{a_1}{c}(cv_1) + \cdots + \frac{a_n}{c}(cv_n)$$

and so $cv_1, \ldots, cv_n$ span V as well. Thus we have more than one basis for all finite-dimensional vector spaces.

Essentially the same proof shows the same thing for infinite-dimensional vector spaces. So let W be an infinite-dimensional vector space with basis $w_1, w_2, \ldots$. We claim that for any $c \in \mathbb{F}$, the list $cw_1, cw_2, \ldots$ is a basis as well. Clearly the list is again linearly independent, and to see that it still spans V, let $u \in V$. Then, since $w_1, w_2, \ldots$ spans W, there exist $a_1, a_2, \cdots \in \mathbb{F}$ such that

$$u = a_1w_1 + a_2w_2 + \ldots$$

But then we have

$$u = \frac{a_1}{c}(cw_1) + \frac{a_2}{c}(cw_2) + \ldots$$

and so $cw_1, cw_2, \ldots$ span W as well. Thus we have more than one basis for all infinite-dimensional vector spaces as well, proving our original claim. $\square$

Problem 3

(a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Proof. (a) We claim the list of vectors

$$(3,1,0,0,0),\ (0,0,7,1,0),\ (0,0,0,0,1)$$

is a basis of U. We first show they span U. So let $u \in U$. Then there exist $x_1, \ldots, x_5 \in \mathbb{R}$ such that

$$u = (x_1, x_2, x_3, x_4, x_5)$$

and such that $x_1 = 3x_2$ and $x_3 = 7x_4$. Substitution yields

$$u = (3x_2, x_2, 7x_4, x_4, x_5),$$

and hence we have

$$u = x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1)$$

and indeed they span U. Now suppose $a_1, a_2, a_3 \in \mathbb{R}$ are such that

$$a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) = 0.$$

Then we have

$$(3a_1, a_1, 0, 0, 0) + (0, 0, 7a_2, a_2, 0) + (0, 0, 0, 0, a_3) = 0$$

which clearly implies $a_1 = a_2 = a_3 = 0$. Thus they are also linearly independent, and hence a basis.

(b) We claim the list

$$v_1 = \begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix},\ v_2 = \begin{pmatrix}0\\0\\7\\1\\0\end{pmatrix},\ v_3 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}, v_4 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\ v_5 = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$$

is a basis of $\mathbb{R}^5$ expanding the basis from (a). To see that it spans $\mathbb{R}^5$, let $u = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5$. Notice

$$u_2v_1 + u_4v_2 + u_5v_3 + (u_1 - 2u_2)v_4 + (u_3 - 6u_4)v_5 = \begin{pmatrix}3u_2\\u_2\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\7u_4\\u_4\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\u_5\end{pmatrix} + \begin{pmatrix}u_1 - 2u_2\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\u_3 - 6u_4\\0\\0\end{pmatrix}.$$

Simplifying the RHS, we have

$$\begin{pmatrix}3u_2\\u_2\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\7u_4\\u_4\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\u_5\end{pmatrix} + \begin{pmatrix}u_1 - 2u_2\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\u_3 - 6u_4\\0\\0\end{pmatrix} = \begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{pmatrix},$$

and so indeed $v_1, \ldots, v_5$ span $\mathbb{R}^5$. To see that they are linearly independent, suppose $a_1, \ldots, a_5 \in \mathbb{R}$ are such that

$$a_1 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have the equivalent system of linear equations

$$\begin{aligned} 3a_1 + a_4 &= 0 \\ a_1 &= 0 \\ 7a_2 + a_5 &= 0 \\ a_2 &= 0 \\ a_3 &= 0, \end{aligned}$$

which clearly implies each of the a_k are 0. Hence $v_1, \ldots, v_5$ are linearly independent as well, and thus a basis.

(c) Let $W = \operatorname{span}(v_4, v_5)$, where v_4 and v_5 are defined as in (b). We claim $\mathbb{R}^5 = U \oplus W$. To see $\mathbb{R}^5 = U + W$, let $v \in \mathbb{R}^5$. Then, because we've already shown $v_1, \ldots, v_5$ span $\mathbb{R}^5$, there exist $a_1, \ldots, a_5 \in \mathbb{R}$ such that

$$u = (a_1 v_1 + a_2 v_2 + a_3 v_3) + (a_4 v_4 + a_5 v_5).$$

The first term in parentheses is an element of U, and the second is an element of W, and thus $V = U + W$.

To prove the sum is direct, it suffices to show $U \cap W = \{0\}$. So suppose $u \in U \cap W$. Then there exist $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}$ such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 = b_1 v_4 + b_2 v_5.$$

Thus

$$a_1 v_1 + a_2 v_2 + a_3 v_3 - b_1 v_4 - b_2 v_5 = 0.$$

Since $v_1, \ldots, v_5$ are linearly independent, this implies each of the a's and b's are 0, and so indeed $U \cap W = \{0\}$. Therefore the sum is direct, proving our claim that $\mathbb{R}^5 = U \oplus W$. □

Problem 5

Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. Consider the list

$$p_0 = 1, p_1 = X, p_2 = X^3 + X^2, p_3 = X^3$$

which contains no polynomial of degree 2. We claim this list is a basis. First we prove $\operatorname{span}(p_0, p_1, p_2, p_3) = \mathcal{P}_3(\mathbb{F})$. Let $q \in \mathcal{P}_3(\mathbb{F})$. Then there exist $a_0, \ldots, a_3 \in \mathbb{F}$ (some of which may be 0) such that

$$q = a_0 + a_1 X + a_2 X^2 + a_3 X^3.$$

But notice

$$\begin{aligned} a_0 p_0 + a_1 p_1 + a_2 p_2 + (a_3 - a_2) p_3 &= a_0 + a_1 X + a_2 (X^3 + X^2) + (a_3 - a_2) X^3 \\ &= a_0 + a_1 X + a_2 X^2 + a_3 X^3 \\ &= q, \end{aligned}$$

and so indeed p_0, p_1, p_2, p_3 spans $\mathcal{P}_3(\mathbb{F})$. To see the list is linearly independent, suppose $b_0, \ldots, b_3 \in \mathbb{F}$ are such that

$$b_0 p_0 + b_1 p_1 + b_2 p_2 + b_3 p_3 = 0.$$

It follows that

$$b_0 + b_1 X + b_2 X^2 + (b_2 + b_3) X^3 = 0$$

which is true iff all coefficients are zero. Hence we must have $b_0 = b_1 = b_2 = b_3 = 0$, and so $p_0, \ldots, p_3$ is linearly independent. Thus it is a basis, as claimed. $\square$

Problem 7

Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Proof. The statement is false. To see this, let $V = \mathbb{R}^4$ and let

$$v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0), v_4 = (0, 0, 0, 1).$$

Define

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 = x_4\}.$$

We have $v_1, v_2 \in U$ and $v_3, v_4 \notin U$. But since no linear combination of v_1, v_2 yields $(0, 0, 1, 1)$, v_1, v_2 do not span U, and hence they cannot form a basis. $\square$

C: Dimension

Problem 1

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let $n = \dim U = \dim V$, and let $u_1, \ldots, u_n$ be a basis for U. Since this list is linearly independent and has length equal to the dimension of V, it must be a basis for V as well (by Theorem 2.39). Clearly we have $U \subseteq V$, so it remains to show $V \subseteq U$. Let $v \in V$. Then there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \cdots + a_n u_n.$$

But now v is expressed as a linear combination of vectors in U and hence is in U as well. Thus $U = V$, as desired. □

Problem 3

Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^2$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

Proof. A subspace of $\mathbb{R}^3$ can have a basis of length $0, 1, 2$ or 3. We consider each in turn:

0: The only basis of length 0 is the empty basis, which generates $\{0\}$.

1: Any basis of length 1 contains a single $x \in \mathbb{R}^\times$. Notice $\operatorname{span}(x) = \{ax \in \mathbb{R} \mid a \in \mathbb{R}\}$, and hence bases of length 1 generate lines through the origin.

2: Any basis of length 2 consists of two linearly independent $x, y \in \mathbb{R}^\times$. Notice $\operatorname{span}(x, y) = \{ax + by \in \mathbb{R}^2 \mid a, b \in \mathbb{R}\}$, and hence bases of length 2 generate planes through the origin.

3: Any basis of length 3 is simply a basis of $\mathbb{R}^3$ and hence generates all of $\mathbb{R}^3$.

Since we've exhausted all possibilities, all subspaces of $\mathbb{R}^3$ have been classified as one of these four types. □

Problem 4

(a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

We first prove a helpful lemma that we will use repeatedly.

Lemma 1. *Any list of nonzero polynomials in $\mathcal{P}(\mathbb{F})$, no two of which have the same degree, is linearly independent.*

Proof of the lemma. Let $p_1, \ldots, p_n \in \mathcal{P}(\mathbb{F})$ be nonzero and each of unique degree, and without loss of generality suppose they are ordered from smallest degree to largest. Denote their degrees by $d_1, \ldots, d_n$. Now suppose $a_1, \ldots, a_n \in \mathbb{F}$ are such that

$$a_1 p_1 + \cdots + a_n p_n = 0.$$

Without explicitly expanding the LHS, we see that it must have an X^{d_n} term with a nonzero coefficient (since each polynomial is assumed to have unique degree). Since the RHS is identically 0, this implies $a_n = 0$. But now by repeating this same argument $n-1$ times, we see that in fact each of $a_1, \ldots, a_{n-1}$ must be zero as well, and hence the list is indeed linearly independent. $\square$

Proof. (a) We claim the list of polynomials

$$(X-6), (X-6)^2, (X-6)^3, (X-6)^4$$

is a basis of U. By Lemma 1, since each polynomial in the list has unique degree, the list is linearly independent. Thus $\dim U$ must be at least 4, since we've demonstrated a linearly independent list of length 4. Since U is a subspace of $\mathcal{P}_4(\mathbb{F})$, which has dimension 5, this implies $\dim U \in \{4, 5\}$. But notice U is a *proper* subset of $\mathcal{P}_4(\mathbb{F})$ since, in particular, it excludes the monomial X. Thus $\dim U$ cannot be 5, and we conclude $\dim U = 4$. Since our list is linearly independent and of length equal to $\dim U$, it must be a basis.

(b) We claim

$$1, (X-6), (X-6)^2, (X-6)^3, (X-6)^4$$

is an extension of our basis of U to $\mathcal{P}_4(\mathbb{F})$. Since this list is of length equal to $\dim \mathcal{P}_4(\mathbb{F})$, it suffices to show it is linearly independent. But this follows immediate by Lemma 1, since each polynomial in the list has unique degree.

(c) Let $W = \mathbb{F}$. We claim $\mathcal{P}_4(\mathbb{F}) = U \oplus W$. Label our basis from (*b*) as

$$p_0 = 1, p_1 = (X-6), p_2 = (X-6)^2, p_3 = (X-6)^3, p_4 = (X-6)^4.$$

In this notation, we have $W = \operatorname{span}(p_0)$ and $U = \operatorname{span}(p_1, \ldots, p_4)$. Clearly $\mathcal{P}_4(\mathbb{F}) = U + W$ since $p_0, \ldots, p_4$ is a basis of $\mathcal{P}_4(\mathbb{F})$, so it suffices to show $U \cap W = \{0\}$. Suppose $q \in U \cap W$. Then q must be a scalar by inclusion in W. If q were nonzero, there would exist $a_0, \ldots, a_3 \in \mathbb{F}$ such that

$$a_0(X-6) + a_1(X-6)^2 + a_2(X-6)^3 + a_3(X-6)^4 \neq 0$$

for all $X \in \mathbb{F}$. But this is absurd, since the LHS evaluates to 0 for $X = 6$. Thus q cannot be nonzero, and the sum is indeed direct. $\square$

Problem 7

(a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in part (*a*) to a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

Proof. (a) We claim the list of polynomials

$$1, (X-2)(X-5)(X-6), (X-2)(X-5)(X-6)^2 \tag{$\dagger$}$$

is a basis of U. Linear independence follows from Lemma 1, and so $\dim U$ must be at least 3. We will exhibit a proper subspace V of $\mathcal{P}_4(\mathbb{F})$ of dimension 4 such that U is a proper subspace of V. This will in turn imply that $3 \leq \dim U < 4$. Since all dimensions are of course integers, this will imply $\dim U = 3$. Since our list of polynomials is a linearly independent list of length equal to $\dim U$, this will prove it to be a basis. So consider the subspace

$$V = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(2) = p(5)\}$$

of $\mathcal{P}_4(\mathbb{F})$. Clearly U is a subspace of V, and moreover it is a proper subspace since $(X-2)(X-5)$ is in V but not in U. So it only remains to show $\dim V = 4$. Note that the list of polynomials

$$1, (X-2)(X-5), (X-2)^2(X-5), (X-2)^2(X-5)^2$$

is linearly independent in V (again by Lemma 1). Note also that V a proper subspace of $\mathcal{P}_4(\mathbb{F})$ since it does not contain the monomial X. Since this implies $4 \leq \dim V < 5$, we must have $\dim V = 4$, completing the proof that ($\dagger$) is indeed a basis of U.

(b) We claim

$$1, X, X^2, (X-2)(X-5)(X-6), (X-2)(X-5)(X-6)^2$$

is an extension of our basis of U to $\mathcal{P}_4(\mathbb{F})$. Since this list is of length equal to $\dim \mathcal{P}_4(\mathbb{F})$, it suffices to show it is linearly independent. But this follows immediately from Lemma 1.

(c) Label our basis from (b) as

$$\begin{aligned}
p_0 &= 1, \\
p_1 &= X, \\
p_2 &= X^2, \\
p_3 &= (X-2)(X-5)(X-6), \\
p_4 &= (X-2)(X-5)(X-6)^2
\end{aligned}$$

and let $W = \operatorname{span}(p_1, p_2)$. We claim $\mathcal{P}_4(\mathbb{F}) = U \oplus W$. That $\mathcal{P}_4(\mathbb{F}) = U + W$ follows from the fact that $p_0, \ldots, p_4$ is a basis of $\mathcal{P}_4(\mathbb{F})$ and since $U = \operatorname{span}(p_0, p_3, p_4)$. To prove the sum is direct, it suffices to show $U \cap W = \{0\}$. So suppose $q \in U \cap W$. Then there exist $a_0, a_1, b_0, b_1, b_2 \in \mathbb{F}$ such that

$$q = a_0 p_1 + a_1 p_2 = b_0 p_0 + b_1 p_3 + b_2 p_4.$$

But then

$$a_0 p_1 + a_1 p_2 - b_0 p_0 - b_1 p_3 - b_2 p_4 = 0,$$

and since the $p_0, \ldots, p_4$ are linearly independent, this implies each of the a's and b's are zero. Thus $q = 0$ and the sum is indeed direct. □

Problem 9

Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1.$$

Proof. Let $W = \operatorname{span}(v_1 + w, \dots, v_m + w)$, and consider the list

$$v_2 - v_1, v_3 - v_2, \dots, v_m - v_{m-1},$$

which has length $m - 1$. Note that $v_k - v_{k-1} = (v_k + w) - (v_{k-1} + w)$, so that each vector in this list is indeed in W. Since the dimension of W must be greater than the length of any linearly independent list, if we prove this list is linearly independent, we will have proved $\dim W \geq m - 1$. So suppose $a_1, \dots a_{m-1} \in \mathbb{F}$ are such that

$$a_1(v_2 - v_1) + \cdots + a_{m-1}(v_m - v_{m-1}) = 0.$$

Expanding, we see

$$(-a_1)v_1 + (a_1 - a_2)v_2 + \cdots + (a_{m-2} - a_m)v_{m-1} = 0.$$

But since $v_1, \dots, v_{m-1}$ is linearly independent by hypothesis, each of the coefficients must be zero. Thus $a_1 = 0$ and $a_{k-1} = a_k$ for $k = 2, \dots, m - 1$, and hence we must have $a_2 = \cdots = a_{m-1} = 0$ as well. Therefore, our list is linearly independent, and indeed $\dim W \geq m - 1$. $\square$

Problem 11

Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. We have

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W),$$

and thus since $U + W = \mathbb{R}^8$, $\dim U = 3$, and $\dim W = 5$, it follows

$$8 = 3 + 5 - \dim(U \cap W),$$

and hence $\dim(U \cap W) = 0$. Therefore we must have $U \cap W = \{0\}$, and hence $\mathbb{R}^8 = U \oplus W$. $\square$

Problem 13

Suppose U and W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Proof. Note that we view $\mathbb{C}^6$ as a vector space over $\mathbb{C}$. We have

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W),$$

and thus since $\dim U = \dim W = 4$, it follows

$$\dim(U+W) = 8 - \dim(U \cap W). \tag{1}$$

Since $U+W$ is a subspace of $\mathbb{C}^6$ and $\dim \mathbb{C}^6 = 6$, and since $\dim(U+W) \geq \max\{\dim U, \dim W\} = 4$, we have

$$4 \leq \dim(U+W) \leq 6. \tag{2}$$

Combining (1) and (2) yields

$$-4 \leq -\dim(U \cap W) \leq -2,$$

and hence

$$2 \leq \dim(U \cap W) \leq 4.$$

Thus $U \cap W$ has a basis of length at least two, and thus there exist two vectors in $U \cap W$ such that neither is a scalar multiple of the other (namely, two vectors in the basis). □

Problem 14

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspace of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Proof. For each $j = 1, \ldots, m$, choose a basis for U_j. Combine these bases to form a single list of vectors in V. Clearly this list spans $U_1 + \cdots + U_m$ by construction. Hence $U_1 + \cdots + U_m$ is finite-dimensional with dimension less than or equal to the number of vectors in this list, which is equal to $\dim U_1 + \cdots + \dim U_m$. That is,

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m,$$

as desired. □

Problem 15

Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Proof. Since $\dim V = n$, there exists a basis $v_1, \ldots, v_n$ of V. Let $U_k = \operatorname{span}(v_k)$ for $k = 1, \ldots, n$, so that each U_k has dimension 1. Clearly

$$V = U_1 + \cdots + U_n,$$

so it remains to show this sum is direct. If $u \in U_1 + \cdots + U_n$, there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$u = a_1 v_1 + \cdots + a_n v_n.$$

But since $v_1, \ldots, v_n$ is a basis, this representation of u as a linear combination of $v_1, \ldots, v_n$ is unique, and thus the sum is direct, as desired. $\square$

Problem 16

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Proof. For each $j = 1, \ldots, m$, choose a basis for U_j. Combine these bases to form a single list of vectors in V. Clearly this list spans $U_1 + \cdots + U_m$ by construction, so that $U_1 + \cdots + U_m$ is finite-dimensional. We claim this list must be linearly independent, hence it will be a basis of length $\dim U_1 + \cdots + \dim U_m$, and thus

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

So suppose some linear combination of the vectors in this list equals 0. For $k = 1, \ldots, m$, denote by u_k the sum of all terms in that linear combination which are formed from our chosen basis of U_k. Then we have

$$u_1 + \cdots + u_m = 0.$$

Since $U_1 + \cdots + U_m = U_1 \oplus \cdots \oplus U_m$, each u_k must equal 0. But then, since u_k is a linear combination of a basis of U_k, each of the coefficients in that linear combination must equal 0. Thus all coefficients in our original linear combination must be 0. That is, our basis is linearly independent, justifying our claim and completing the proof. $\square$

Problem 17

You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = &\dim U_1 + \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Proof. The statement is false. Consider

$$U_1 = \mathbb{R} \times \{0\},\ U_2 = \{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\},\ U_3 = \{0\} \times \mathbb{R}.$$

We have

$$\begin{aligned}
&\dim(U_1 + U_2 + U_3) = \dim \mathbb{R}^2 = 2 \\
&\dim U_1 = \dim U_2 = \dim U_3 = 1 \\
&\dim(U_1 \cap U_2) = \dim(U_2 \cap U_3) = 1 \\
&\dim(U_1 \cap U_3) = \dim(U_1 \cap U_2 \cap U_3) = 0,
\end{aligned}$$

and therefore

$$\begin{aligned}
\dim(U_1 + U_2 + U_3) \neq\, &\dim U_1 + \dim U_2 + \dim U_3 \\
&- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\
&+ \dim(U_1 \cap U_2 \cap U_3).
\end{aligned}$$

since the LHS is 2, whereas the RHS is 1 in this case. □