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Analyzing convergence of two-level deflation preconditioner

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What is Preconditioner

Linear system

$$Ax = b$$

Is this

$$P^{-1}Ax = P^{-1}b$$

equivalent ?
P is the preconditioner(matrix) What P should be ?

- cheap to perform $P^{-1}r$.
- better spectral properties of $P^{-1}A$
- can we choose $P = A^{-1}$?

Chosen Problem

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x_i) - k^2(x_i)\mathbf{u}(x_i) = \mathbf{g}(x_i) \ \text{ in } \Omega$$

$\mathbf{u}(x_i)$ is the pressure field

$\mathbf{k}(x_i)$ is the wave number

$\mathbf{g}(x_i)$ is the source function and

Ω is domain, bounded by absorbing boundary conditions

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is the unit normal vector pointing outwards on the boundary.

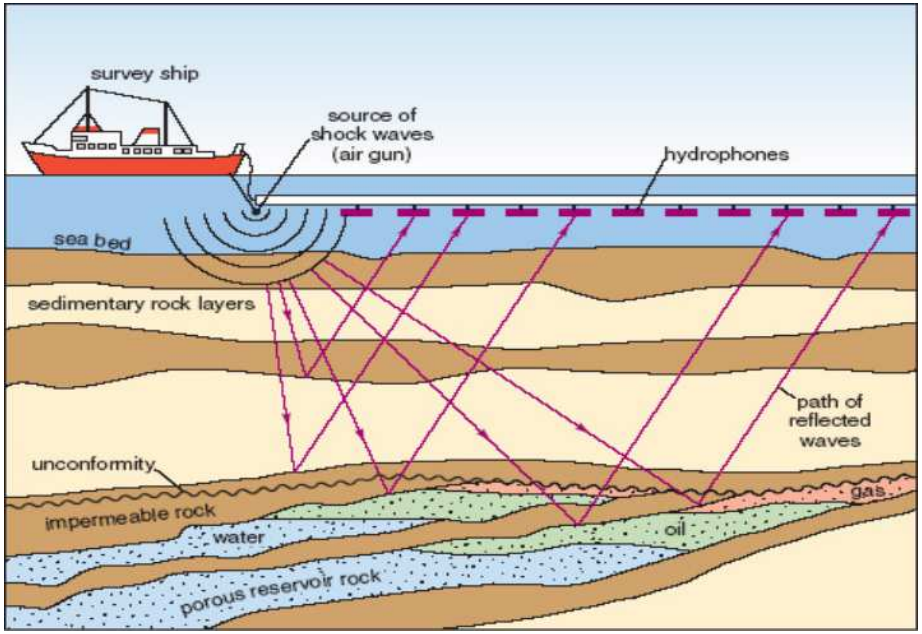
Why this problem?

People in following areas look for solution

- Acoustics (sound waves)
- Seismic imaging
- Optix and Electromagnetic waves
- Medical Imaging (ultrasound etc)
- Radars

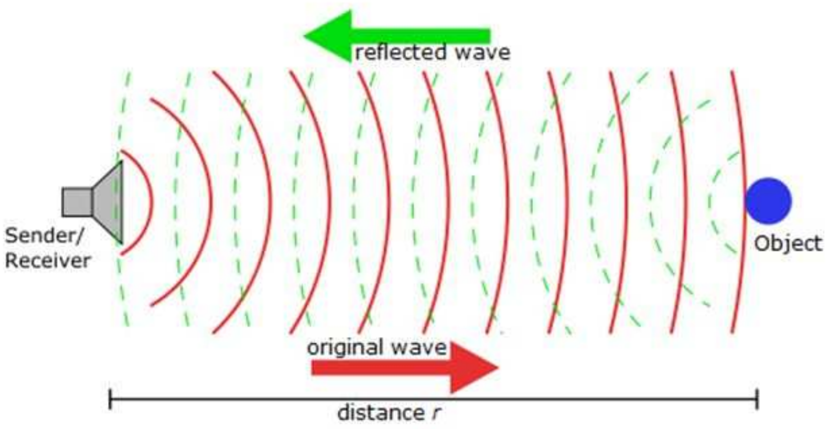
Application: Geophysical Survey

Seismic Imaging



Application: Radars

Radar system



Discretization

- Second order Finite Difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4-k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$
- Properties

  Sparse & complex valued

  Symmetric & Indefinite for large k
- High resolution requires very fine grid; fine grid $\Rightarrow$ extremely large linear system!
- Is traditionally solved by a Krylov subspace method, which exploits the sparsity.

Solvers

Brief survey

- Iterative Methods (BIM)

  They fail to converge for sufficient large system.

- Krylov subspace solvers
  - CGNR Paige and Saunders, 1975
  - Short recurrences

    Bi-CGSTAB van der Vorst, 1992

    IDR(s) Van Gijzen and Sonneveld, 2008
  - Minimal residual

    GMRES Saad and Schultz, 1986

    GCR Eisenstat, Elman and Schultz, 1983

Preconditioning

Equivalent linear system

$$P^{-1}Ax = P^{-1}b$$

where P is the preconditioning matrix

Requisite for a preconditioner

- cheap to perform $P^{-1}r$.
- better spectral properties of $P^{-1}A$

Preconditioning

ILU	Meijerink and van der Vorst, 1977
ILU(tol)	Saad, 2003
Multigrid	Lahaye, 2001
	Elman, Ernst and O' Leary, 2001
AILU	Gander and Nataf, 2001
	analytic parabolic factorization
ILU-SV	Plessix and Mulder, 2003
	separation of variables

Preconditioning

Laplace operator $M_0 := \Delta$ — Bayliss and Turkel, 1983

Definite Helmholtz $M_1 := \Delta + 1.I$ — Laird, 2000

Complex Shifted Laplace $M_\iota = \Delta - \iota I$ — Y.A. Erlangga et al, 2003

Complex Shifted Laplace preconditioner

$$M(\beta_1, \beta_2) \equiv -\Delta - (\beta_1 - \mathrm{i}\beta_2)k^2 I, \quad \beta_1, \beta_2 \in \mathbb{R} \quad .$$

What we gained?

Helmholtz equation with constant k in Ω

Iterative solver: Bi-CGSTAB

Preconditioner: Shifted-Laplace operator

k	ILU(0.01)	M_0	M_1	M_i
5	9	13	13	13
10	25	29	28	22
15	47	114	45	26
20	82	354	85	34
30	139	> 1000	150	52

What we gained ? (2)

Wedge problem; k in three layers.

k	10	20	40	50	100
grid	32^2	64^2	128^2	192^2	384^2
No-Prec	201(0.56)	1028(12)	5170(316)	–	–
ILU(A,0)	55(0.36)	348(9)	1484(131)	2344(498)	–
ILU(A,1)	26(0.14)	126(4)	577(62)	894(207)	–
ILU(M,0)	57(0.29)	213(8)	1289(122)	2072(451)	–
ILU(M,1)	28(0.28)	116(4)	443(48)	763(191)	2021(1875)
MG(V(1,1))	13(0.21)	38(3)	94(28)	115(82)	252(850)

CSL Preconditioning

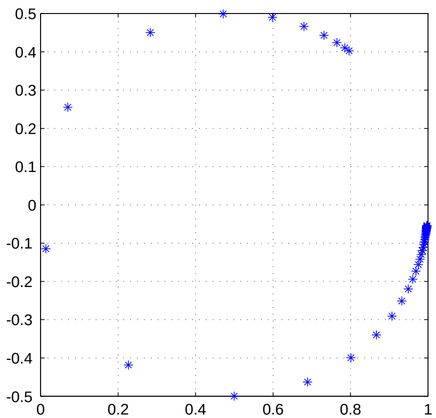
- How exactly CSLP doing with spectrum?

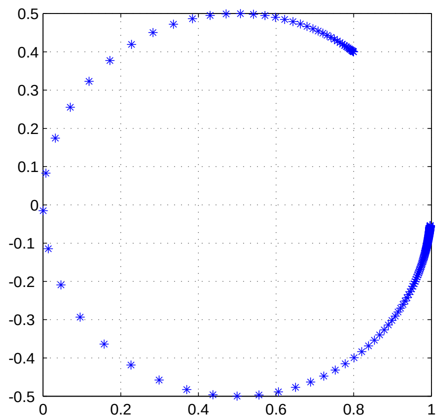
CSL Preconditioning

- Introduces damping; better Multigrid approximation
- Absolute eigenvalues are bounded by 1
- Small eigenvalues rush to zero, as k increases.

Spectrum of $M^{-1}(1, 0.5)A$ for

$k = 30$ and $k = 120$





CSLP Results

Number of Krylov iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**10**	17	28	44	70	14
$n = 64$	10	**17**	28	36	45	163
$n = 96$	10	17	**27**	35	43	97
$n = 128$	10	17	27	**35**	43	85
$n = 160$	10	17	27	35	**43**	82
$n = 320$	10	17	27	35	42	**80**

Number of iterations highly depend upon k.

Magic Numbers?

Number of Krylov iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

with / without deflation.

Magic Explained

That was Deflation Preconditioner.

Deflation

For any deflation subspace matrix

$$Z \in R^{n \times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rankZ = r$$

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \quad \text{and} \quad E = Z^T AZ$$

Solve $PAu = Pb$ preconditioned by M^{-1} or $M^{-1}PA = M^{-1}Pg$

For e.g. say,

$$\textbf{spec}(A) = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$$

and if Z is the matrix with columns the r eigenvectors then

$$\textbf{spec}(PA) = \{0, ..., 0, \lambda_{r+1}, ...\lambda_n\}$$

Deflation

Multigrid inter-grid transfer operator (Prolongation) as deflation matrix

Setting $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

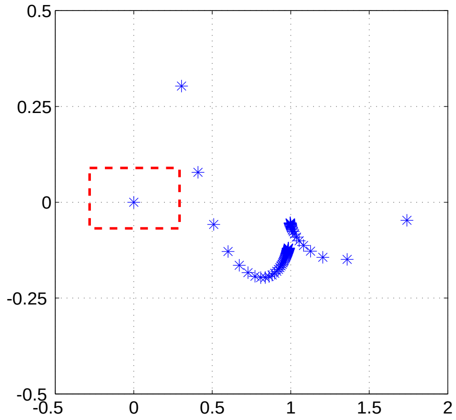
$$P_{h.2h} = I_h - A_h Q_h, \quad \text{with} \quad Q = I_h^{2h} A_{2h}^{-1} I_{2h}^h \quad \text{and } A_{2h} = I_{2h}^h A_h I_h^{2h}$$

where

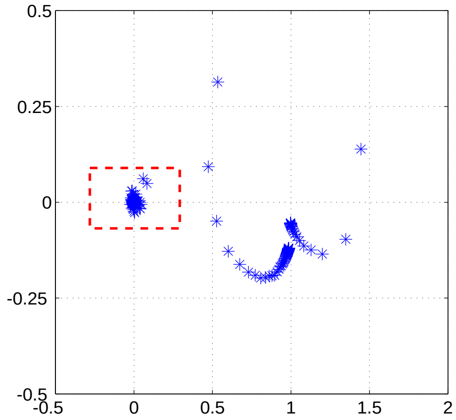
$P_{h,2h}$ can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

Deflation: Approximate solve A_{2h}^{-1}



Exact inversion of A_{2h}



In-exact inversion of A_{2h}

Deflation: Shifting Spectrum

Shift term

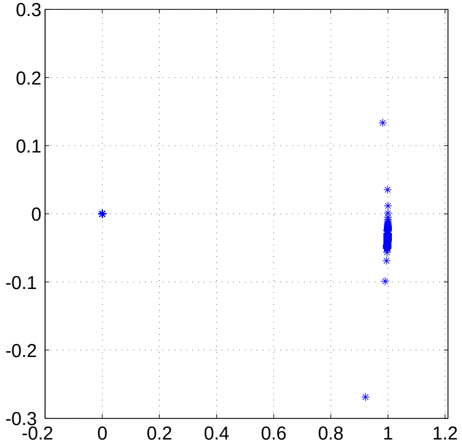
$$Q_h = I_h^{2h} A_{2h}^{-1} I_h^{2h}$$

Strategy: Solve A_{2h} iteratively to required accuracy on certain levels, and shift the deflated spectrum to λ_h^n by adding shift in deflation preconditoner, call it **ADEF1** preconditioner

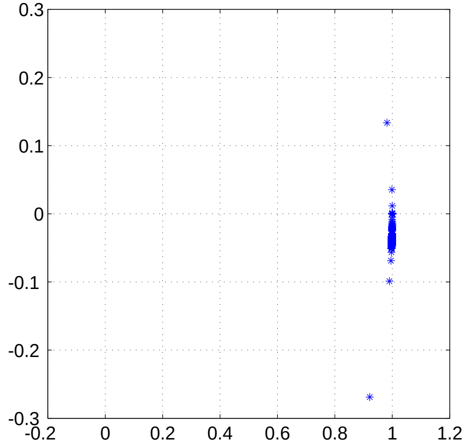
$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_h^n Q_h$$

It is theorotically proved that term Q_h shifts the spectrum to λ_h^n

Deflation: Shifting Spectrum



Without Shift Q_{2h}



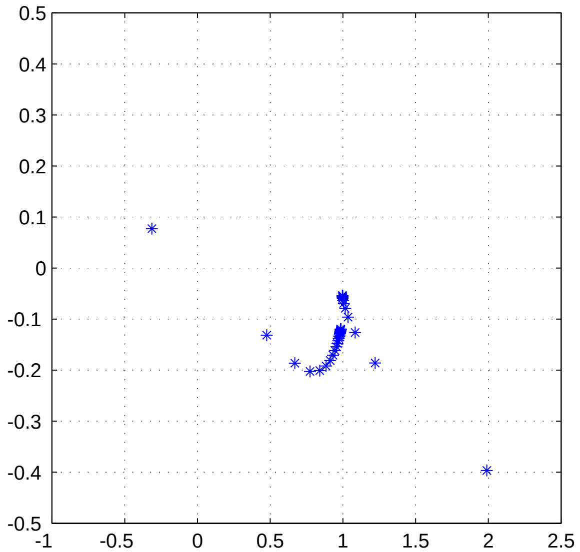
With Shift Q_{2h}

NEXT: $\lambda_h(B_{h,2h})$ where $B_{h,2h} = P_{(h,ADEF1)} M_h^{-1} A_h$

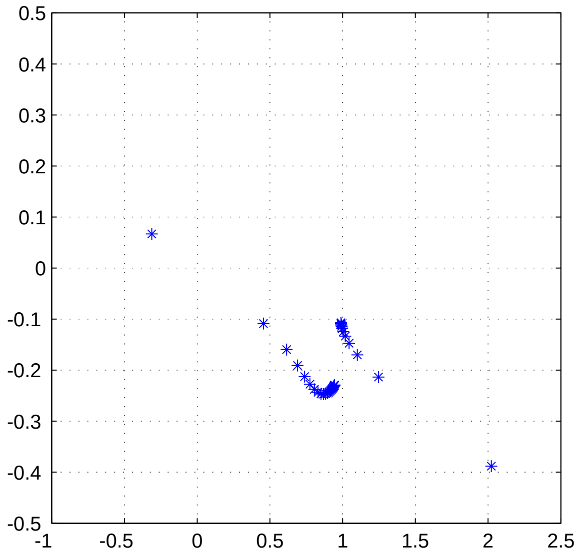
Deflation: Changing CSLP shifts

Analysis shows that an increase in the imaginary shift does not change the spectrum.

$(\beta_1, \beta_2) = (1, 0.5)$



$(\beta_1, \beta_2) = (1, 1)$



Numerical Results

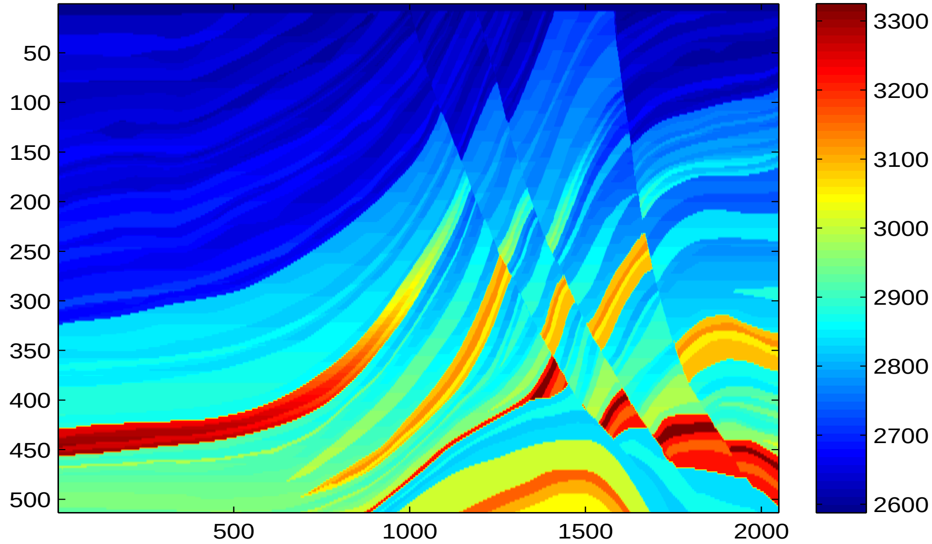
Number of Krylov iterations with/without deflation to solve a Wedge problem. Shifts in the preconditioner are $(1, 0.5)$

Grid	$freq = 10$	$freq = 20$	$freq = 30$	$freq = 40$	$freq = 50$
74×124	**7/33**	20/60	79/95	267/156	490/292
148×248	5/33	**9/57**	17/83	42/112	105/144
232×386	5/33	7/57	**10/81**	25/108	18/129
300×500	4/33	6/57	8/81	**12/105**	18/129
374×624	4/33	5/57	7/80	9/104	**13/128**

Adapted Marmousi Problem

Reduced velocity contrast: $2587 \leq c(x, y) \leq 3325$

Adapted geomegry convenient for geometric vectors.



Numerical Results

Mamousi Problem: Solve time and iterations; discretization20 gp/wl

Frequency f	Solve Time		Iterations	
	CSLP	**DEF**	**CSLP**	**DEF**
$f = 1$	1.23	5.08	13	7
$f = 10$	40.01	21.83	106	8
$f = 20$	280.08	131.30	177	12
$f = 40$	20232.6	3997.7	340	21

Numerical Results

Three Dimensional Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.3125$

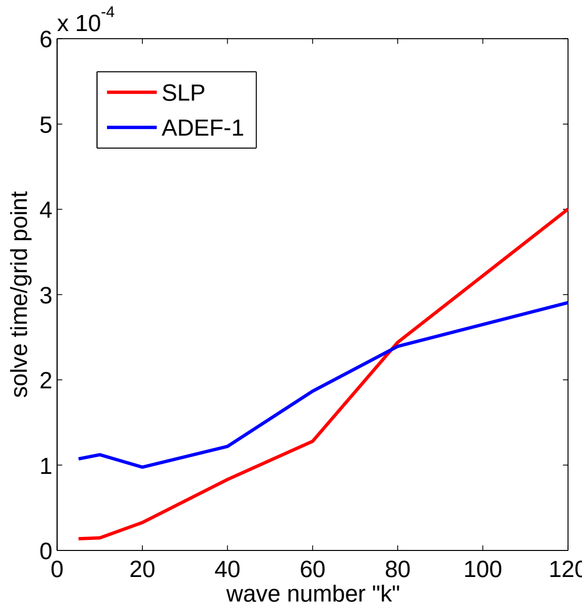
Wave number	Solve Time		Iterations	
k	CSLP	DEF	CSLP	DEF
5	0.04	0.32	7	8
10	0.48	2.32	9	9
20	8.14	17.28	20	9
40	228.29	155.52	70	10
60	1079.99	607.45	97	11

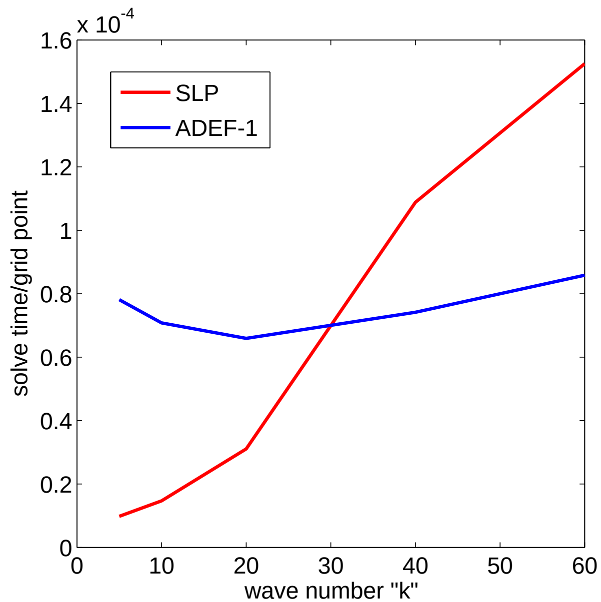
Numerical Results

Solve time per grid points .

10gp/wl

20gp/wl





Numerical Results

Three Dimensional Layered Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.625$

Wave number k	Solve Time		Iterations	
	CSLP	DEF	CSLP	DEF
5	0.09	0.24	9	11
10	1.07	1.94	15	12
20	16.70	18.89	32	16
30	73.82	78.04	43	21
40	1304.2	214.7	331	24
60	-	989.5	500+	34

Conclusions

- Parameter independent scheme
- Flexibility to increase imaginary shift, when deflation is combined with CSLP.
- Near null space modes appear, require testing
- Works with FEM too. (no results here in)

Thank you for your attention!

Please visit `http://ahsheikh.github.io`