Chapter 5:Linear functions, Applications

Recall that a linear function f involving one independent variable x and a dependent variable y has the general form

$$y = f(x) = a_1 x + a_0,$$

where $a \quad \neq 0$ and a_0 are constants.

Example

Consider the weekly salary function

$$y = f(x) = 300x + 2500$$

where y is defined as the weekly salary and x is the number of units sold per week.

Clearly, this is a weekly function in one independent variable x.

1. 2500 represents the base salary, i.e. when no units are sold per week and 300 is the commission of each unit sold.
2. The change in weekly salary is directly proportional to the change in the no. of units sold.
3. Slope of 3 indicates the increase in weekly salary associated with each additional unit sold.

In general, a linear function having the form $y = f(x) = a_1 x + a_0$ a change in the value of y is directly proportional to a change in the variable x.

Skip for Next Lecture.

Linear function in two independent variables

A linear function f involving two independent variables x_1 and x_2 and a dependent variable y has the general form

$$y = f(x_1, x_2) = a_1 x_1 + a_2 x_2 + a_0$$

where a_1 and a_2 are non-zero constants and a_0 is any constant.

1. This equation tells us that the variable y depends jointly on the values of x_1 and x_2.

2. The value of y is directly proportional to the changes in the values of x_1 and x_2.

Example

Assume that a salesperson salary depends on the number of units sold of each of two products, i.e. the salary function is given as

$$y = 300x_1 + 500\,x_2 + 2500$$

where y = weekly salary, x_1 is number of units sold of product 1 and x_2 is number of units sold of product 2. This salary function gives a base salary of 2500, commission of 300 on each unit sold of product 1 and 500 on each unit sold of product 2.

Linear cost function

The organizations are concerned with the costs as they reflect the money flowing out of the organisation. The total cost usually consists of two components: total variable cost and total fixed cost. These two components determine the total cost of the organisation.

$y = mx + b$ "x" is time
'y' cost

Example

A firm which produces a single product is interested in determining the functions that expresses annual total cost y as a function of the number of units produced x. Accountants indicate that the fixed expenditure each year are 50,000. They also have estimated that raw material costs for each unit produced are 5.50, labour costs per unit are 1.50 in the assembly department, 0.75 in the finishing room, and 1.25 in the packaging and shipping department. Find the total cost function.

Solution

Total cost function = total variable cost + total fixed cost
Total fixed cost = $50{,}000$
Total variable cost = total raw material cost + total labour cost

$$y = f(x) = 5.5\,x + (1.5\,x + 0.75x + 1.25x) + 50{,}000$$
$$= 9x + 50{,}000.$$

The 9 represents the combined variable cost per unit of $9. That is, for each additional unit produced, total cost will increase by $9.

Linear Revenue function

Revenue: The money which flows out into an organisation from either selling or providing services is often referred to as revenue.

Total Revenue = Price × Quantity sold

Suppose a firm sells product. Let p_i and x_i be the price of the product and number of units per product respectively. Then the revenue $R = p_1x_1 + \cdots + p_nx_n$.

Linear Profit function

Profit: The profit of an organisation is the difference between total revenue and total cost. In equation form if total revenue is denoted by $R(x)$ and Total cost is $C(x)$, where x is quantity produced and sold, then profit $P(x)$ is defined as

$$P(x) = R(x) - C(x).$$

1. If total revenue exceeds total cost the profit is positive
2. In such case, profit is referred as net gain or net profit
3. On the other hand the negative profit is referred to as a net loss or net deficit.

Example
A firm sells single product for $65 per unit. Variable costs per unit are $20 for materials and $27.5 for labour. Annual fixed costs are $100,000. Construct the profit function stated in terms of x, which is the number of units produced and sold. How much profit is earned if annual sales are 20,000 units.

Solution: Here $R(x) = 65x$ and total annual cost is made up for material costs, labour costs, and fixed cost.

$$C(x) = 20x + 27.5\,x + 100{,}000 = 47.5\,x + 100{,}000.$$

Thus

$$P(x) = R(x) - C(x) = 17.5x - 100{,}000.$$

As $x = 20{,}000$, so $P(20{,}000) = 250{,}000.$

Loss situation
- when profit is -ve

Let only 100 units are sold, then

Profit $= P(100) = 1750 - 100000$
$= -98250$

Straight Line Depreciation

When organizations purchase an item, usually cost is allocated for the item over the period the item is used.

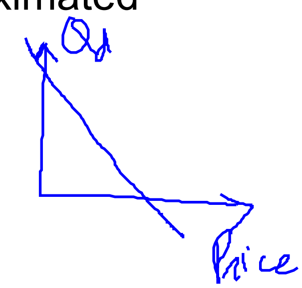
Example Exercise. Construct fn.
A company purchases a vehicle costing $20,000 having a useful life of 5 years, then accountants might allocate $4,000 per year as a cost of owning the vehicle. The cost allocated to any given period is called depreciation. The value of the truck at the limp of purchase is $20,000 but, after 1 year the price will be $20000-$4000 = $ 16000 and so forth. In this case, depreciation can also be thought of as an amount by which the book value of an asset has decreased over the period of time.
Thus, the book value declines as a linear function over time. If V

equals the book value of an asset and t equals time (in years) measured from the purchase date for the previously mentioned truck, then $V = f(t)$.

Linear demand function

A demand function is a mathematical relationship expressing the way in which the quantity demanded of an item varies with the price charged for it. The relationship between these two variables, quantity demanded and price per unit, is usually inversely proportional, i.e. a decrease in price results in increase in demand.
Most demand functions are nonlinear, but there are situations in which the demand relationship either is, or can be approximated by a linear function.

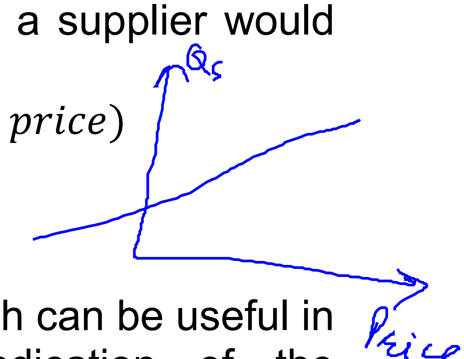
$$Quantity\ demanded = q_d = f(price\ per\ unit)$$



Linear Supply Function

A supply function relates market price to the quantities that suppliers are willing to produce or sell. The supply function implicates that what is brought to the market depends upon the price people are willing to pay.
In contrast to the demand function, the quantity which suppliers are willing to supply usually varies directly with the market price. The higher the market price, the more a supplier would like to produce and sell. The lower the price, the less a supplier would like to produce and sell.

$$Quantity\ supplied = q_s = f(market\ price)$$



Break-Even Models

Break-even model is a set of planning tools which can be useful in managing organizations. One significant indication of the

performance of a company is reflected by how much profit is earned. Break-even analysis focuses upon the profitability of a firm and identifies the level of operation or level of output that would result in a zero profit.
The level of operations or output is called the break-even point. The break-even point represents the level of operation at which total revenue equals total cost. Any changes from the level of operations will result in either a profit or a loss.

Break-even analysis is mostly used when:

1. Firms are offering new products or services.
2. Evaluating the pros and cons of starting a new business.

Assumption
Total cost function and total revenue function are linear.

Break-even Analysis

In break-even analysis the main goal is to determine the break-even point.
The break-even point may be expressed in terms of

i. Volume of output (or level of activity)
ii. Total sale in dollars
iii. Percentage of production capacity

e.g. a firm will break-even at 1000 units of output, when total sales equal 2 million dollars or when the firm is operating 60% of its plant capacity.

Method of performing break-even analysis

1. Formulate total cost as a function of x , the level of output.
2. Formulate total revenue as a function of x.
3. As break-even conditions exist when total revenue equals total cost, so we set $C(x)$ equals $R(x)$ and solve for x. The resulting value of x is the break-even level of output and

denoted by x_{BE}.

An alternate, to step 3 is to construct the profit function $P(x) = R(x) - C(x)$, set $P(x)$ equal to zero and solve to find x_{BE}.

Example

A Group of engineers is interested in forming a company to produce smoke detectors. They have developed a design and estimated that variable costs per unit, including materials, labor, and marketing costs are $22.50. Fixed costs associated with the formation, operation, management of the company and purchase of the machinery costs $250,000. They estimated that the selling price will be 30 dollars per detector.

a) Determine the number of smoke detectors which must be sold in order for the firm to break-even on the venture.
b) Preliminary marketing data indicate that the firm can expect to sell approximately 30,000 smoke detectors over the life of the project, if the detectors are sold at $30 per unit. Determine expected profits at this level of output.

Solution:

a) If x equals the number of smoke detectors produced and sold, the total revenue function $R(x) = 30x$. The total cost function is $C(x) = 22.5\,x + 250{,}000$. We put $R(x) = C(x)$ to get x_{BE} = 33333.3 units.
b) $P(x) = R(x) - C(x) = 7.5\,x - 250{,}000$. Now at $x = 30{,}000$, we have $P(30{,}000) = -25{,}000$. Thus the expected loss is $25,000.

<u>Market Equilibrium</u>

Given supply and demand functions of a product, market equilibrium exits if there is a price at which the quantity demanded equals the quantity supplied.

Example Left as Exercise

Suppose demand and supply functions have been estimated for two competing products.

$$q_{d_1} = 100 - 2p_1 + 3p_2 \text{(Demand, Product 1)}$$

$$q_{d_2} = 150 + 4p_1 - p_2 \text{(Demand, Product 2)}$$

$$q_{s_1} = 2p_1 - 4 \text{(Supply, Product 1)}$$

$$q_{s_2} = 3p_2 - 6 \text{(Supply, Product 2)}$$

Determine the price for which the market equilibrium would exist.

Solution: The demand and supply functions are linear. The quantity demanded of a given product depends on the price of the product and also on the price of the competing product and the quantity supplied of a product depends only on the price of that product.

Market equilibrium would exist in this two-product market place if prices existed and were offered such that $q_{d_1} = q_{s_1}$ and $q_{d_2} = q_{s_2}$, solving we get $p_1 = 221$ and $p_2 = 260$. Putting the values in above equations we get $q_{d_1} = q_{s_1} = 438$ and $q_{d_2} = q_{s_2} = 774$.