ITERATIVE TECHNIQUES FOR SOLVING

NON-LINEAR SYSTEMS (AND LINEAR SYSTEMS)

Jacobi Iterative Technique

Consider the following set of equations.

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 &= 6 \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\
2x_1 - x_2 + 10x_3 - x_4 &= -11 \\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}
$$

Convert the set Ax = b in the form of x = Tx + c.

$$x_1 = \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}$$

$$x_2 = \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}$$

$$x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}$$

$$x_4 = -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}$$

Start with an initial approximation of:

$x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$ **and** $x_4^{(0)} = 0$.

$$x_1^{(1)} = \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5}$$

$$x_2^{(1)} = \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11}$$

$$x_3^{(1)} = -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10}$$

$$x_4^{(1)} = -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8}$$

$$x_1^{(1)} = \frac{1}{10}(0) - \frac{1}{5}(0) + \frac{3}{5}$$

$$x_2^{(1)} = \frac{1}{11}(0) + \frac{1}{11}(0) - \frac{3}{11}(0) + \frac{25}{11}$$

$$x_3^{(1)} = -\frac{1}{5}(0) + \frac{1}{10}(0) + \frac{1}{10}(0) - \frac{11}{10}$$

$$x_4^{(1)} = -\frac{3}{8}(0) + \frac{1}{8}(0) + \frac{15}{8}$$

$x_1^{(1)} = 0.6000$, $x_2^{(1)} = 2.2727$,

$x_3^{(1)} = -1.1000$ **and** $x_4^{(1)} = 1.8750$

$$x_1^{(2)} = \frac{1}{10}x_2^{(1)} - \frac{1}{5}x_3^{(1)} + \frac{3}{5}$$

$$x_2^{(2)} = \frac{1}{11}x_1^{(1)} + \frac{1}{11}x_3^{(1)} - \frac{3}{11}x_4^{(1)} + \frac{25}{11}$$

$$x_3^{(2)} = -\frac{1}{5}x_1^{(1)} + \frac{1}{10}x_2^{(1)} + \frac{1}{10}x_4^{(1)} - \frac{11}{10}$$

$$x_4^{(2)} = -\frac{3}{8}x_2^{(1)} + \frac{1}{8}x_3^{(1)} + \frac{15}{8}$$

$$
\begin{aligned}
x_1^{(k)} &= \phantom{-\frac{1}{11}x_1^{(k-1)}} \quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k-1)} \phantom{+\frac{1}{10}x_2^{(k-1)}} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k-1)} + \frac{1}{10}x_2^{(k-1)} \phantom{+\frac{1}{11}x_3^{(k-1)}} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= \phantom{-\frac{1}{11}x_1^{(k-1)}} - \frac{3}{8}x_2^{(k-1)} + \frac{1}{8}x_3^{(k-1)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{15}{8}
\end{aligned}
$$

Results of Jacobi Iteration:

k	0	1	2	3
$x_1^{(k)}$	**0.0000**	**0.6000**	**1.0473**	**0.9326**
$x_2^{(k)}$	**0.0000**	**2.2727**	**1.7159**	**2.0530**
$x_3^{(k)}$	**0.0000**	**-1.1000**	**-0.8052**	**-1.0493**
$x_4^{(k)}$	**0.0000**	**1.8750**	**0.8852**	**1.1309**

Gauss-Seidel Iterative Technique

Consider the following set of equations.

$$
\begin{aligned}
10x_1 \quad - x_2 \quad + 2x_3 \qquad\qquad &= 6 \\
-x_1 + 11x_2 \quad - x_3 \quad + 3x_4 &= 25 \\
2x_1 \quad - x_2 + 10x_3 \quad - x_4 &= -11 \\
3x_2 \quad - x_3 \quad + 8x_4 &= 15
\end{aligned}
$$

$$
\begin{aligned}
x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}\boxed{x_1^{(k-1)}} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}\boxed{x_1^{(k-1)}} + \frac{1}{10}\boxed{x_2^{(k-1)}} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= -\frac{3}{8}\boxed{x_2^{(k-1)}} + \frac{1}{8}\boxed{x_3^{(k-1)}} + \frac{15}{8}
\end{aligned}
$$

$$
\begin{aligned}
x_1^{(k)} &= \phantom{-\frac{1}{5}x_1^{(k)}} \quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \phantom{+\frac{1}{10}x_2^{(k)}} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \phantom{+\frac{1}{11}x_3^{(k-1)}} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= \phantom{-\frac{1}{5}x_1^{(k)}} \quad -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{15}{8}
\end{aligned}
$$

Results of Gauss-Seidel Iteration:
(Blue numbers are for Jacobi iterations.)

k	0	1	2	3
$x_1^{(k)}$	**0.0000**	**0.6000**	**1.0300**	**1.0065**
		0.6000	**1.0473**	**0.9326**
$x_2^{(k)}$	**0.0000**	**2.3272**	**2.0370**	**2.0036**
		2.2727	**1.7159**	**2.0530**
$x_3^{(k)}$	**0.0000**	**-0.9873**	**-1.0140**	**-1.0025**
		-1.1000	**-0.8052**	**-1.0493**
$x_4^{(k)}$	**0.0000**	**0.8789**	**0.9844**	**0.9983**
		1.8750	**0.8852**	**1.1309**

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 1$

It required 15 iterations for Jacobi method and 7 iterations for Gauss-Seidel method to arrive at the solution with a tolerance of 0.00001.