Deflation type Preconditioners for Helmholtz Problem

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Overview

- Helmholtz and SLP
- Deflation preconditioning
- Variation in Deflation
- Analysis/Comparison
- Conclusions

The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x_i) - k^2(x_i)\mathbf{u}(x_i) = \mathbf{g}(x_i) \text{ in } \Omega$$

- Linear system $A_h u_h = g_h$ is:

  Sparse & complex valued, for certain boundary conditions

  Symmetric & Indefinite for large k

- For high resolution a very fine grid is required: $30 - 60$ gridpoints per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A_h$ is extremely large!
- Standard multigrid method does not work!
- Traditionally solved by a Krylov subspace method, which exploits the sparsity.

Complex Shifted Laplace Preconditioner

$M(\beta_1, \beta_2) := -\Delta - (\beta_1 - \iota\beta_2)k^2 I$

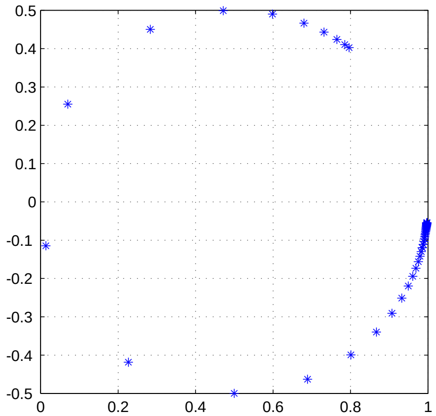
Advantage: Spectrum is bounded in circle.

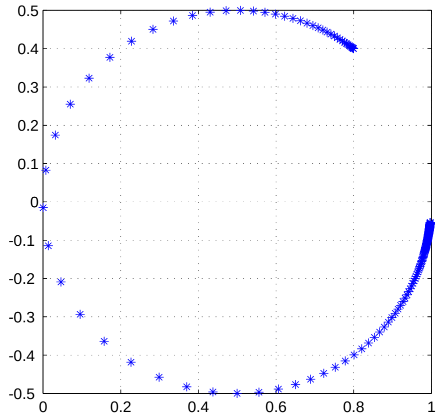
Disadvantage : That circle touches origin 0;

Spectrum encounters near-zero eigenvalues for large k.

Spectrum of CSLP preconditioned Helmholtz

$k = 30$ $k = 120$





Deflation

Deflation, a projection preconditioner

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \;\text{ and }\; E = Z^T AZ$$

where,

$$Z \in R^{n\times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank(Z) = r \leq n$$

Along with a traditional preconditioner M, deflated preconditioned system reads

$$PM^{-1}Au = PM^{-1}g.$$

The choice of deflation vectors: spectrum of matrix, physics of problem, etc

Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I_{2h}^{h}$ and $Z^T = I_h^{2h}$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I_{2h}^{h} A_{2h}^{-1} I_h^{2h} \;\text{ and }\; A_{2h} = I_h^{2h} A_h I_{2h}^{h}$$

where

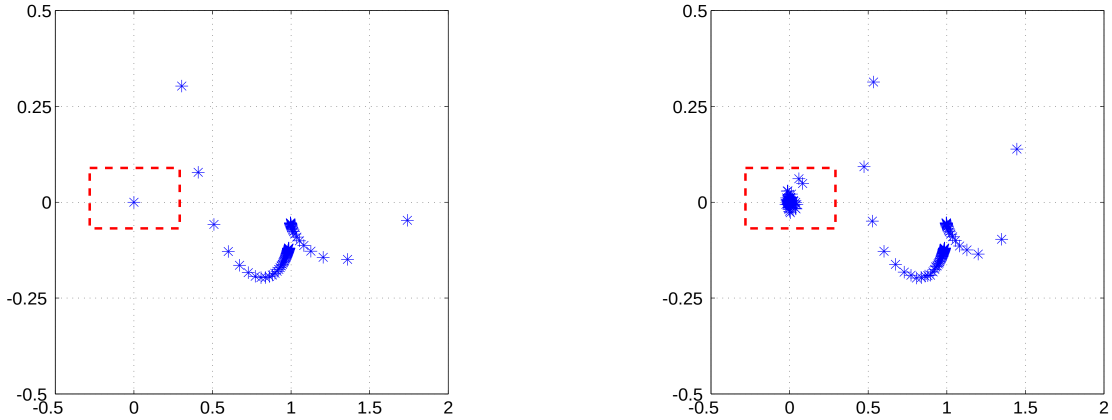
P_h can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

A_{2h}^{-1} How to solve this ? ?

MultiLevel approach; Krylov approximation of A_{2h}^{-1} preconditioned by CSLP and deflation again.

Deflation: Approximate solve A_{2h}^{-1}



Exact inversion of A_{2h} In-exact inversion of A_{2h}

Shifting Deflated-Spectrum

Shift term

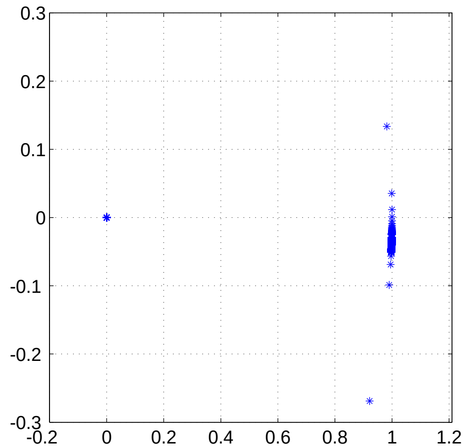
$$Q_h = I_h^{2h} A_{2h}^{-1} I_h^{2h}$$

Strategy: Solve A_{2h} iteratively to required accuracy on certain levels, and shift the deflated spectrum to λ_h^n by adding shift in deflation preconditoner, call it **ADEF1** preconditioner

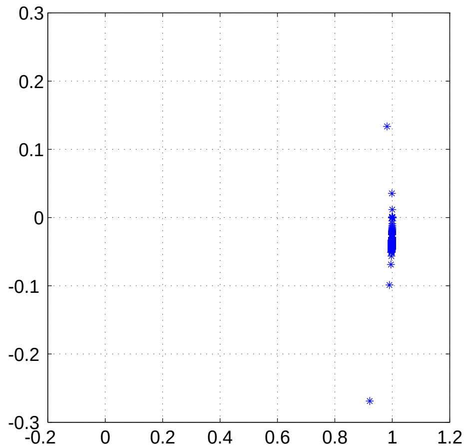
$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_h^n Q_h$$

It is theorotically proved that term Q_h shifts the spectrum to λ_h^n

Deflation: Shift to 1 ?



Without Shift Q_{2h}



With Shift Q_{2h}

NEXT: $\lambda_h(B_{h,2h})$ where $B_{h,2h} = P_{(h,ADEF1)} M_h^{-1} A_h$

Spectral formula

If $\mathsf{c}_\ell = cos(l\pi h)$, spectral formulae of $P_{h,ADEF}A_h$ is

$$\lambda_h\left(P_{h,ADEF}A_h\right) = -\frac{\left(\mathsf{c}_\ell{}^2+1\right)\kappa^4+\left(-4\ \mathsf{c}_\ell{}^2-4\right)\kappa^2-4\ \left(\mathsf{c}_\ell{}^4-1\right)}{\left(\left(\mathsf{c}_\ell{}^2+1\right)\kappa^2+2\left(\mathsf{c}_\ell{}^2-1\right)\right)h^2}$$

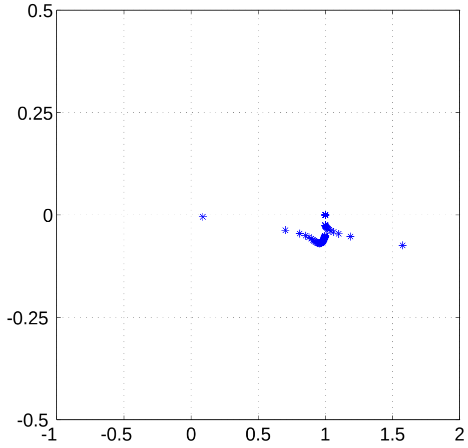
We also know, eigenvalues of Galerikin Helmholtz operator

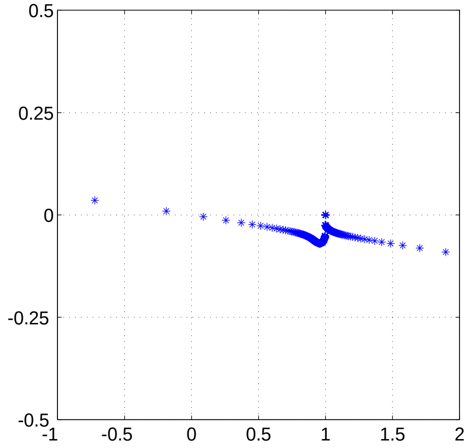
$$A_{2h} = (I_h^{2h})^\ell A_h^\ell (I_{2h}^h)^\ell = \frac{2(1-c_\ell^2)-\kappa^2(1+c_\ell^2)}{2h^2}$$

<u>Denominator in $\lambda_h(P_{h,ADEF1}A_h)$ is scaled formula of A_{2h}</u>

Spectrum insights: ADEF1

Plotting $\lambda_h(B_{h,2h})$

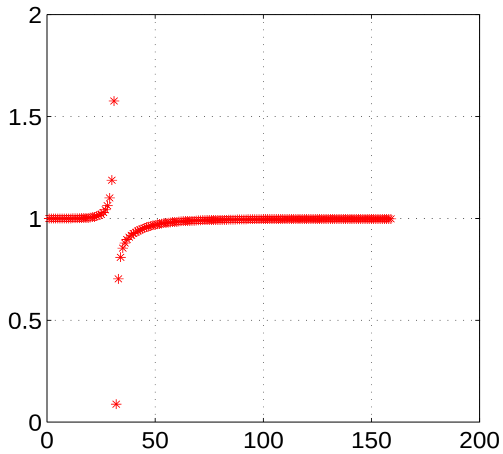


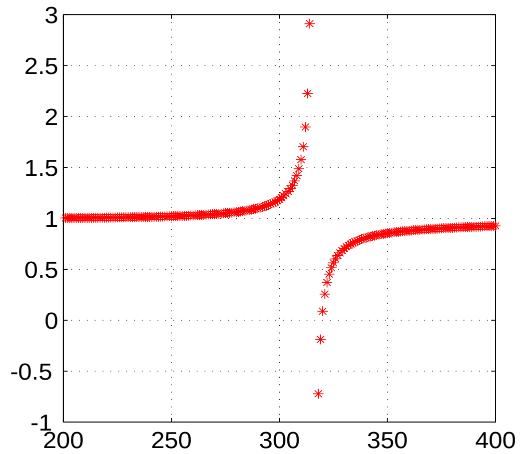


Spectrum of $B_{h,2h}$ for $k = 100$ and $k = 1000$, 20gp/wl

Spectrum insights: ADEF1

Plotting $Re(\lambda_h(P_{h,ADEF1}A_h))$

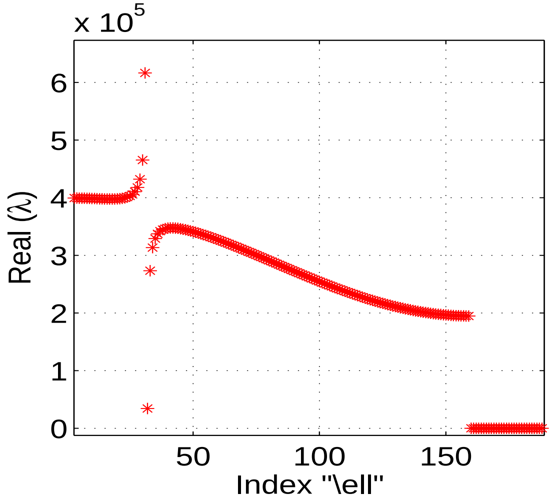




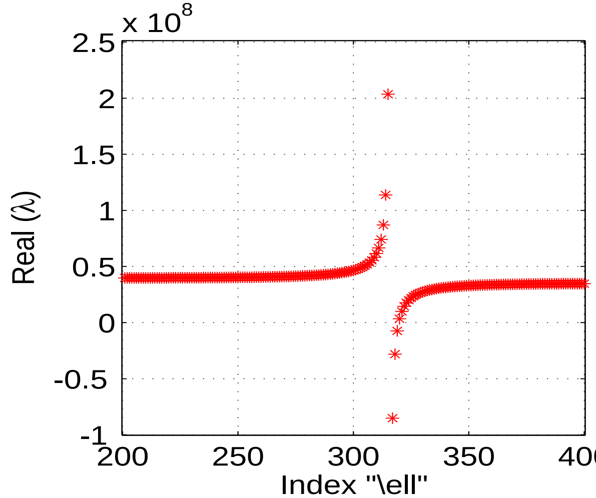
Real eigenvalues v/s index. $k = 100$ and $k = 1000$, 20gp/wl

Spectrum insights: ADEF1

$Re(\lambda_h(P_{h,ADEF}A_h))$



$Re(\lambda_h(P_{h,ADEF}A_h))$



Real eigenvalues v/s index. $k = 160$, $h = 320$

Deflation: TLKM

Two-Level Krylov Method [a], if $\hat{A}_h = M_h^{-1} A_h$ and $\hat{P}_h$ is based upon $\hat{A}_h$ (instead A_h)

$$\hat{P}_h = I_h - \hat{A}_h \hat{Q}_h,$$

where

$$\hat{Q}_h = I_{2h}^h \hat{A}_{2h}^{-1} I_h^{2h} \quad \text{and} \quad \hat{A}_{2h} = I_h^{2h} \hat{A}_h I_{2h}^h = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$$

Construction of coarse matrix A_{2h} at level $2h$ costs inversion of preconditioner at level h.

Approximate A_{2h} ?

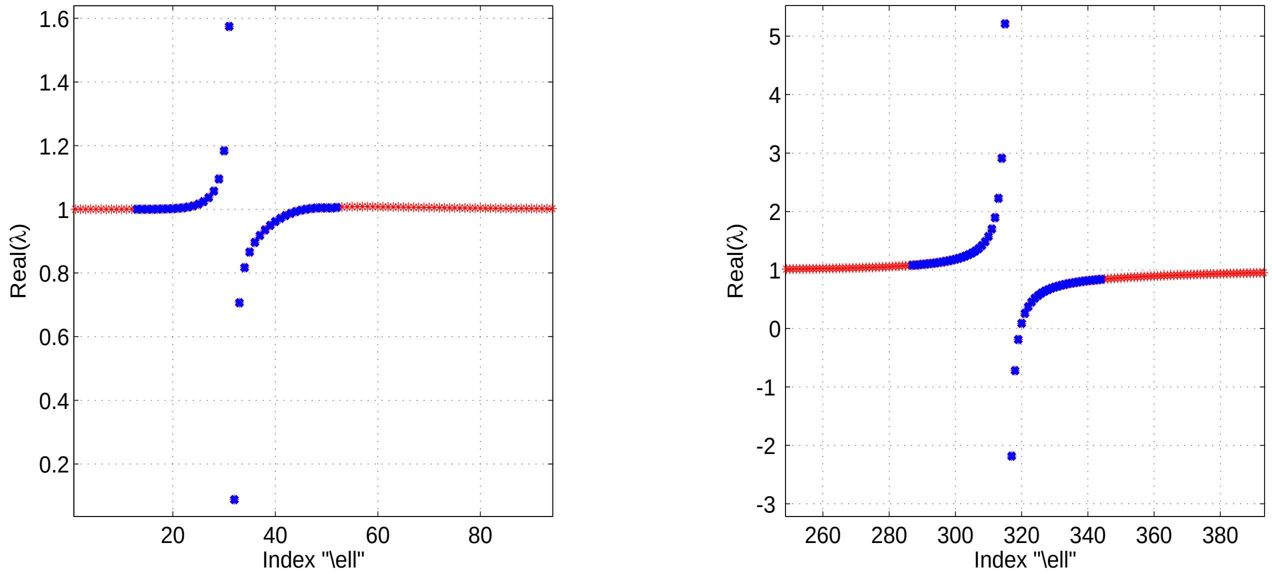
Ideal	**Practical**
$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$	$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$
	$A_{2h} \approx \Theta_h M_{2h}^{-1} A_{2h}, \;\; \Theta_h = I_h^{2h} I_{2h}^h$

[a] Erlangga, Y.A and Nabben R., ETNA 2008

Spectral insights: TLKM

Real part of spectrum of $\hat{B}_h$ where $\hat{B}_h = \hat{P}_h \hat{A}_h$

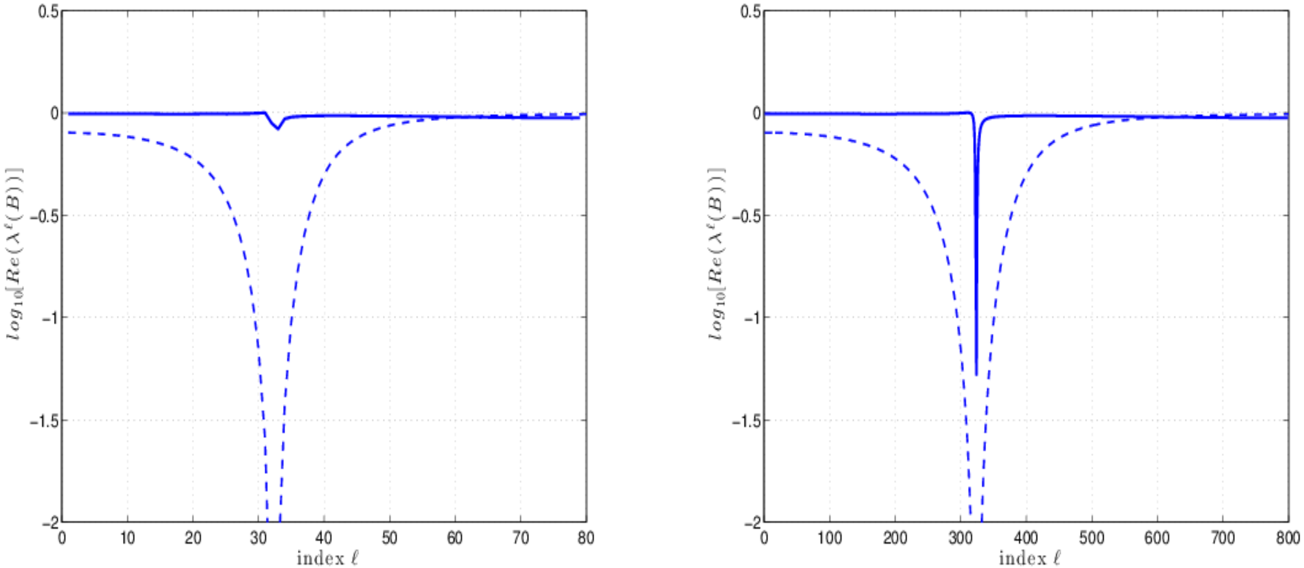
$$k = 100 \qquad\qquad k = 1000$$



Spectral insights: TLKM

Real part eigenvalues of $\hat{B}_h$ vs index. Also the Real part eigenvalues of $\hat{A}_h$;

$k = 100$ $k = 1000$



ADEF1 v TLKM

Differentiating ADEF1 and TLKM, assuming $\lambda_{max} = 1$ and left preconditioning

ADEF1	**MLKM***
$P_{(ADEF1)} = M_h^{-1}(I_h - A_h Q_h) + Q_h$	$P_{(MLKM)} = I_h - \hat{A}_h \hat{Q}_h + \hat{Q}_h$
Applocation on $Au = g$	Application on $\hat{A}u = \hat{g}$

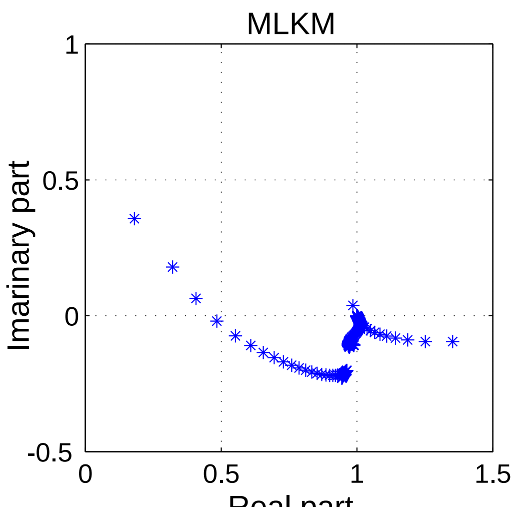
Fourier Analysis

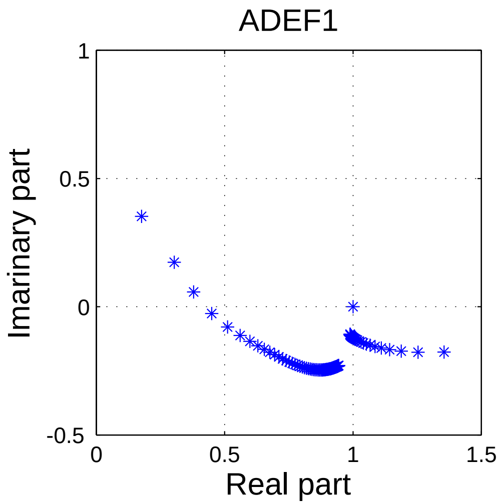
Spectrum of Helmholtz preconditioned by MLKM and ADEF1;

$k = 160$ and 10 gp/wl

TLKM

ADEF1

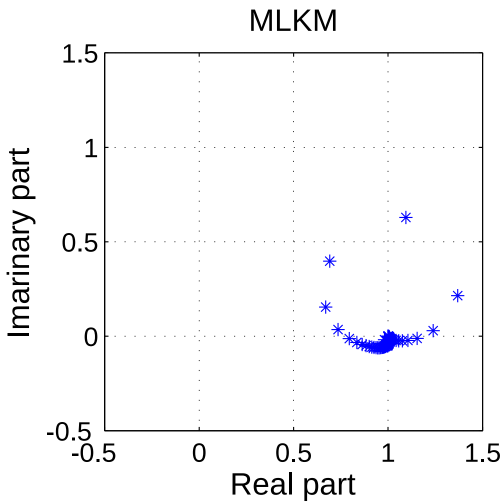




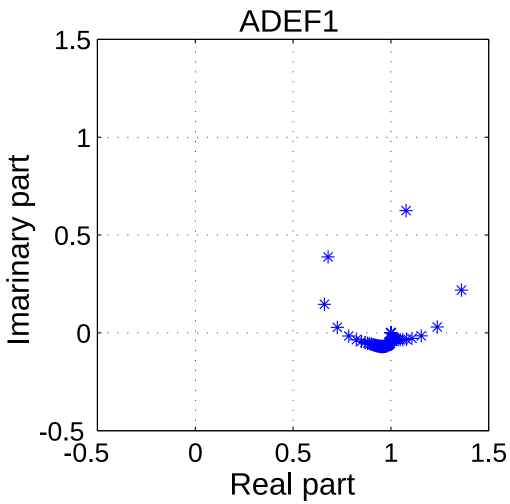
Fourier Analysis

Spectrum of Helmholtz preconditioned by TLKM and ADEF1;
$k = 160$ and 20 gp/wl

TLKM



ADEF1



Cost comparison

Application cost per iteration at two levels

For some vector v,

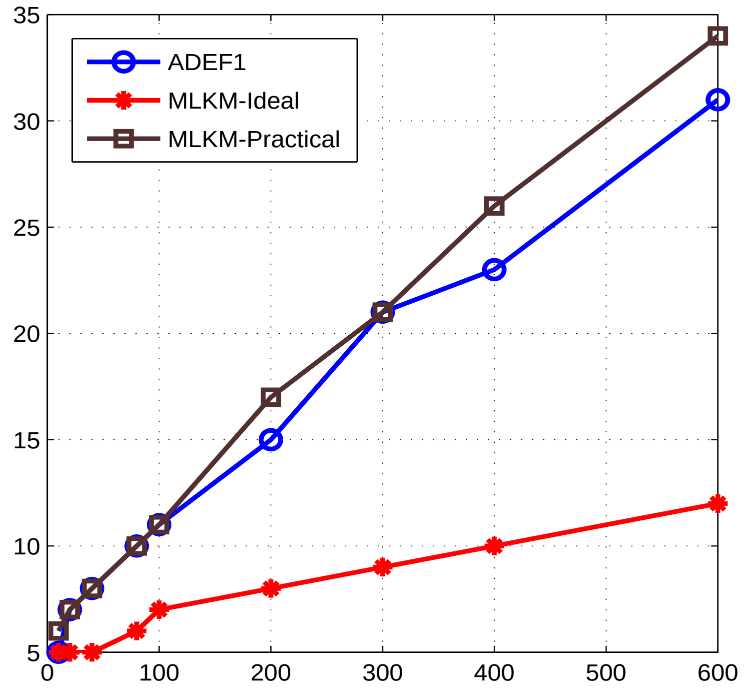
	ADEF1	TLMG
$A_h v$	1	1
$M_h^{-1} v$	1	2
$Q_h v$: $I_h^{2h} v$	1	1
$Q_h v$: $I_{2h}^{h} v$	1	1
$Q_h v$: $A_{2h}^{-1} v$	1	1
$Q_h v$: M_{2h}^{-1}	0	1
$\Theta_h v$	0	1

Numerical results

One Dimensional Helmholtz with Som. BCs.

Wave number against Krylov iterations

Two level solver

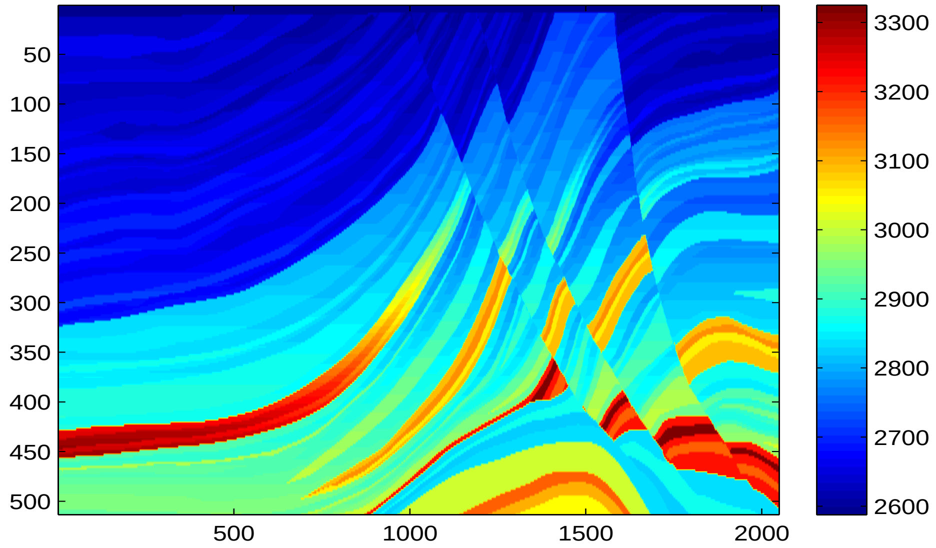


Comparison of number of iterations by ADEF1 and MLKM.

Adapted Marmousi Problem

Reduced velocity contrast: $2587 \leq c(x,y) \leq 3325$

Adapted geomegry convenient for geometric vectors.



Results

Mamousi Problem: Solve time and iterations

Frequency f	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
1	1.25	5.06	13	7
10	9.63	9.35	106	13
20	70.45	57.47	181	21
40	522.90	424.74	333	38

Results

Mamousi Problem: Solve time and iterations; discretization 20 gp/wl

Frequency f	Solve Time		Iterations	
	SLP-F	**ADEF1-F**	**SLP-F**	**ADEF1-F**
$f = 1$	1.23	5.08	13	7
$f = 10$	40.01	21.83	106	8
$f = 20$	280.08	131.30	177	12
$f = 40$	20232.6	3997.7	340	21

Numerical results

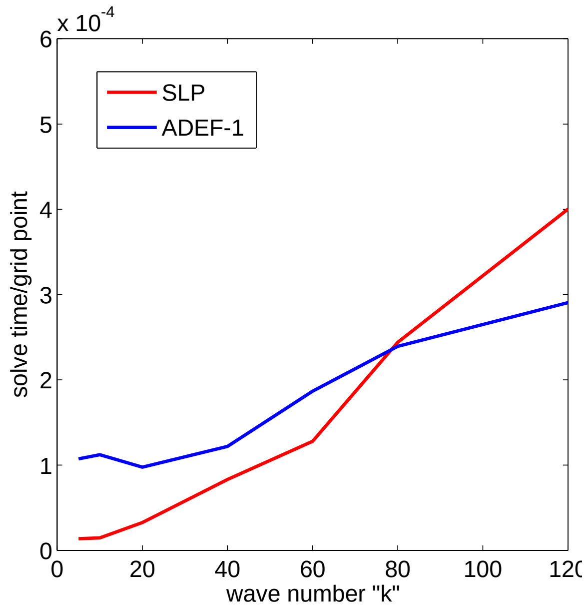
Three Dimensional Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.3125$

Wave number	Solve Time		Iterations	
k	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.04	0.32	7	8
10	0.48	2.32	9	9
20	8.14	17.28	20	9
40	228.29	155.52	70	10
60	1079.99	607.45	97	11

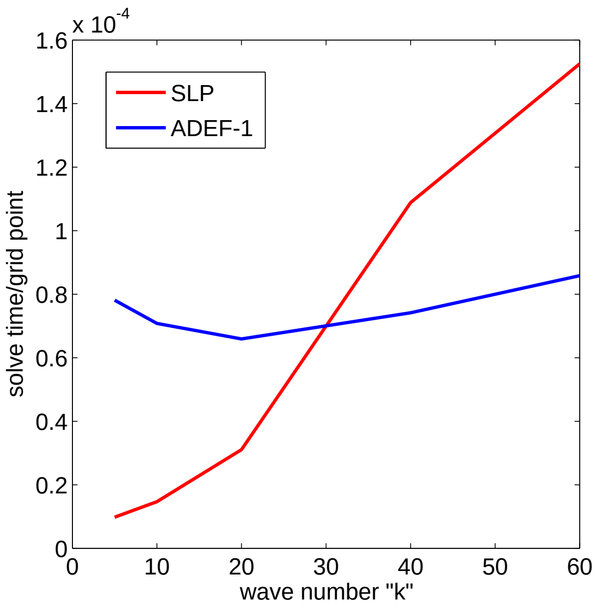
Results

Solve time per grid points .

10gp/wl



20gp/wl



Numerical results

Three Dimensional Layered Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.625$

Wave number k	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.09	0.24	9	11
10	1.07	1.94	15	12
20	16.70	18.89	32	16
30	73.82	78.04	43	21
40	1304.2	214.7	331	24
60	-	989.5	500+	34

Numerical results

Three Dimensional Layered Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces. Grid size h is such that $kh \approx 0.3125$

Wave number k	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.6	1.4	9	9
10	7.5	10.04	14	9
20	324.1	79.2	72	9
30	3810.9	361.7	285	11

Numerical results

Algebraic deflation vectors ?

- FEM regular mesh triangular element discretization.
- Algebraically constructed deflation; AMG cycle.
- ADEF1 preconditioner.
- Comparison with FDM.
- Algebraic vectors proceed the coarsening slower than geometric.
- Mesh is refined enough till satisfactory the wavelength resolution.

Numerical results

Solver	k=10	20	40	80	120	160	200
SLPD*	15(0.02)	30(0.07)	57(0.57)	108(5.8)	157(22.6)	204(59.6)	252(130.5
SLPF*	22(0.05)	43(0.16)	72(0.85)	128(6.33)	178(21.8)	232(55.7)	278(115.9
2Lev	7(0.00)	10(0.03)	14(0.27)	23(2.17)	37(8.8)	61(27.9)	87(67.8)
2Lev*	6(0.02)	8(0.05)	10(0.32)	15(2.46)	20(8.4)	26(21.4)	32(43.8)
MLV	16(0.25)	27(0.8)	58(3.6)	116(18.4)	177(50.3)	235(125.2)	292(233.1
MLV*	22(0.27)	40(1.27)	66(5.4)	118(32.8)	166(110.8)	214(240.6)	258(447.0
MLF	10(0.6)	11(1.6)	15(4.5)	24(15.7)	32(28.2)	41(70.1)	51(103.9)
MLF*	7(0.25)	8(0.85)	10(2.4)	16(15.2)	19(38.3)	24(81.4)	27(144.5)
MLD	7(0.05)	10(.2)	14(1.26)	21(9.04)	29(31.6)	36(76.3)	43(149.8)
MLD*	6(0.07)	8(0.5)	10(2.9)	15(23.7)	19(80.4)	24(191.8)	27(387.3)

Conclusion and Discussion

- Near null space modes in A_h persist. Same time extraordinary gain in Krylov iterations. Ritz testing in progress.
- How to treat near-null space modes in coarser operators ?
- FEM discretization and algebraic deflation vectors in 3-dimension failed. Mass matrix NOT diagonal, it has negative entries off-diagonal.
- Flexible in choosing larger imaginary shift in CSLP. Reported!
- Adapted coarse grid operator. Work in progress!
- Different shifts in SLP at different levels. Future!

References

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- M.B. van Gijzen, Y.A. Erlangga and C. Vuik. Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian. SIAM J.of Sc. Comp. 2007.
- J.M. Tang. Two level preconditioned Conjugate Gradient methods with applications to bubbly flow problems. PhD Thesis, DIAM TU Delft 2008.
- A.H. Sheikh, D. Lahaye and C. Vuik. On the convergence of shifted Laplace preconditioner combined with multi-grid deflation. NLAA Volume 20, Issue 4, pages 645-662, August 2013

Thank you!