Math 319 MATLAB Homework 3

In this homework we will use Euler's method and Improved Euler's method to solve the ODE

$$y'(t) = t^2 + y$$

with initial data $y(0) = 1$ in the interval $t \in [0, 2]$. This ODE is an first order linear equation, therefore one can explicitly solve it. The exact solution to this IVP is

$$y = 3e^t - t^2 - 2t - 2.$$

1 Euler's Method

Step 1. Let us plot the exact solution in MATLAB first. In the command window, type

```
time = 0:0.001:2;
y_exact =3*exp(time) - time.^2 - 2*time - 2;
plot(time, y_exact);
```

At this point a graph for y versus t should show up. And the exact solution at $t = 2$ is

```
y_exact_final = 3*exp(2)- 2^2 - 2*2 - 2
```

Step 2. Now we are ready to compute the approximate solution using Euler's method. We will first partition the interval $[0, 2]$ into $N = 10$ intervals of equal length, and you will be asked to change N later. We do some initializations as below:

```
N=10;
t_init=0; t_final=2;
h=(t_final-t_init)/N;
y_init = 1;  t=[]; y=[];
t(1)=t_init; y(1)=y_init;
```

The sentence "t=[]; y=[];" above basically set t and y to be empty at this point, since we want to have a fresh start. Then we set $t(1)$ and $y(1)$ to be their corresponding initial values, since an array in MATLAB start with index 1. (While an array in many other programming languages start with index 0).

Now we are ready to use Euler's method to compute. I'm sure that you do not want to write N identical sentences with different indexes, so we will use a loop to save some labor. Note that partitioning [0,2] into N intervals means that there are a total of $N + 1$ endpoints, and since the first endpoint has index 1, the last one will have index $N + 1$. Therefore the loop is written as following:

```
for i=2:N+1
    t(i) = t(i-1) + h;
    y(i) = y(i-1) + h* (t(i-1)^2+y(i-1));
end
```

Now we want to plot our approximate solution and compare it with the real solution. In order to plot more than one curves into the same graph, type

```
hold on;
```

So that every time you plot something, it will be added to the current graph, instead of in a new window.

Now we will plot the approximate solution on the same graph of the exact solution, so that we can compare them. Also, we want to compare the error between the exact solution and approximate solution at $t = 2$.

```
plot(t, y,'--');
error = abs(y(N+1)-y_exact_final)
```

The extra '$--$' in the plot sentence means we want to plot it in dashed line, so that we will be able to tell which curve is which. Now if you go back to your figure, you should see that a dashed curve (the approximate solution) showed up in the same graph as the solid curve (the exact solution). Also, in the command line you should see the error is shown to you, **write down that number into the answersheet in the last page.**

Step 3. Repeat Step 2 for $N = 20, 40, 80$. (Tips: You don't need to type everything again – actually you just need to type the first line in step 2 for the new N. You can copy all the rest of the commands from this file into any text editor, then select them all, and paste them into the command window all at once.)

Now print your graph (If you typed "hold on" before, all the curves should be on one graph, so it should contain one solid curve and 4 dashed curves), and fill in the blanks for the Euler's method in the answersheet (which is the last page of this file).

2 Improved Euler's Method

In this section you'll learn a new numerical method solving the IVP, called the Improved Euler's method. It is defined as following: (For more details, see Sec 8.2, or google it)

Let $t_1 = t_0 + h$. Let's first make an attempt forward step at time t_1:
$\tilde{y}_1 = y_0 + f(t_0, y_0)h$
Then make a real forward step with slope $\frac{1}{2}(f(t_0, y_0) + f(t_1, \tilde{y}_1))$:
$y_1 = y_0 + \frac{1}{2}(f(t_0, y_0) + f(t_1, \tilde{y}_1))h$.
Then we repeat this procedure to get $y_2, y_3, ...$

Now we will try to program this. Let's have a fresh start for this new method: Type "clear;" to clear all variables:

```
clear;
```

Also, **if your graph for the Euler's method is still there, close it.** (We want to have two seperate graphs for the two methods.)

To implement the Improved Euler's Method with $N = 10$, we can almost **repeat step 1 and 2 for Euler's method**; the only thing need to be changed is the loop part, which is the first yellow box in Page 2. **The loop should be changed into**

```
y_tilde = [];
for i=2:N+1
    t(i) = t(i-1) + h;
    y_tilde(i) = y(i-1) + h* (t(i-1)^2+y(i-1));
    slope = 1/2 * (t(i-1)^2+y(i-1) + t(i)^2+y_tilde(i));
    y(i) = y(i-1) + h* slope;
end
```

After you successfully run all the code, you should see a graph containing both the exact solution and the approximate solution, and in the command window you should see the error for $N = 10$ at $t = 2$. Write down this number into the answersheet in the last page. Then **redo the process for $N = 20, 40, 80$. Finally, print your graph of the exact solution and the 4 approximate solutions.** (It is fine if the curves almost coincide and you cannot tell them apart.) So you should print a total of 2 graphs, one is Euler's method and one is the Improved Euler's Method; write down on your graph about which is which.

MATLAB Homework 3 Answer Sheet

Fill in the blanks below:

Euler's Method

- For $N = 10$, (i.e. $h = 0.2$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 20$, (i.e. $h = 0.1$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 40$, (i.e. $h = 0.05$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 80$, (i.e. $h = 0.025$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.

From the numbers above, we arrive at the conclusion that every time we halve h, the error will roughly decreases to ____________ of the error before. (Write down a ratio at the blank line.)

This means that Euler's method should have the accuracy of ____________ order. (Here's the non-rigorous definition: If everytime we halve h, the error roughly decreases to $1/2^n$ of the error before, then we say this numerical scheme has accuracy of n-th order. Usually n should be an integer.)

Improved Euler's Method

- For $N = 10$, (i.e. $h = 0.2$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 20$, (i.e. $h = 0.1$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 40$, (i.e. $h = 0.05$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.
- For $N = 80$, (i.e. $h = 0.025$), the error between the exact solution and the approximate solution at $t = 2$ is ____________.

From the numbers above, we arrive at the conclusion that every time we halve h, the error will roughly decreases to ____________ of the error before. (Write down a ratio at the blank line.)

This means the Improved Euler's method should have the accuracy of ____________ order. (Here's the non-rigorous definition: If everytime we halve h, the error roughly decreases to $1/2^n$ of the error before, then we say this numerical scheme has accuracy of n-th order. Usually n should be an integer.)