Fibonacci numbers

$$F_{n+2} = F_{n+1} + F_n \quad , F_0 = 0, \; F_1 = 1$$

$$\Rightarrow F_n = F_{n-1} + F_{n-2}$$

1. $$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

2. $$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \quad n \geq 1$$

3. $$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

4. $$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}, \quad n \geq 0$$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{\sqrt{5}} (\alpha^k - \beta^k) \right)$$

$$= \frac{1}{\sqrt{5}} \left[\sum_{k=0}^{n} \binom{n}{k} \alpha^k - \sum_{k=0}^{n} \binom{n}{k} \beta^k \right]$$

$$= \frac{1}{\sqrt{5}} \left[(1+\alpha)^n - (1+\beta)^n \right]$$

$$\Rightarrow 1+\alpha = \alpha^2 \Rightarrow 1 + \frac{1+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$1+\beta = \beta^2 \Rightarrow 1 + \frac{1-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$\Rightarrow \frac{1}{\sqrt{5}} \left[(\alpha^2)^n - (\beta^2)^n \right] = \frac{1}{\sqrt{5}} (\alpha^{2n} - \beta^{2n})$$

$$= F_{2n}.$$