QR分解（QR-factorization）

Suppose $A \in F^{m\times n}$，$\text{rank}(A) = n$，

則 $A = QR$，Q 為 orthonormal set，$Q \in F^{m\times n}$，

R 為上三角矩陣，且可逆。

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pf. 假設 $A = [a_1\ a_2 \cdots a_n]$，因為 $\text{rank}(A) = n$，所以 $a_1, a_2, \ldots, a_n$ 為線性獨立。

對 $a_1, a_2, \ldots, a_n$ 做 Gram-Schmidt process，且單位化，

得 $q_1, q_2, \ldots, q_n$

$\Rightarrow$ for some $r_{1k}, r_{2k}, \ldots, r_{kk} \in F$，$k = 1, 2, \ldots, n$

$$
\begin{cases}
a_1 = r_{11} q_1 \\
a_2 = r_{12} q_1 + r_{22} q_2 \\
\vdots \\
a_n = r_{1n} q_1 + r_{2n} q_2 + \cdots + r_{nn} q_n
\end{cases}
$$

$$
\Rightarrow A = [a_1\ a_2 \cdots a_n] = \underbrace{[q_1\ q_2 \cdots q_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}}_{R}
$$

欲證 R 可逆，可用 contradiction 去證明

R 的對角都不為零，因為一旦有零，會產生 dependent，就不能 inverse 了。

ex. Find the QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

where the matrix Q has orthonormal column vectors and R is an invertible upper triangular matrix.

Ans.

let. $V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$，對 V_1, V_2, V_3 做 Gram-Schmidt process

(1) $u_1 = V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\langle u_1, u_1 \rangle = 2$, $\|u_1\| = \sqrt{2}$

(2) $u_2 = V_2 - \dfrac{\langle V_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 3 \end{bmatrix}$, $\langle u_2, u_2 \rangle = \dfrac{19}{2}$, $\|u_2\| = \sqrt{\dfrac{19}{2}}$

(3) $u_3 = V_3 - \dfrac{\langle V_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \dfrac{\langle V_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \dfrac{1}{19} \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$, $\langle u_3, u_3 \rangle = \dfrac{1}{19}$

$\|u_3\| = \sqrt{\dfrac{1}{19}}$

$$\Rightarrow \begin{cases} u_1 = V_1 \\ u_2 = V_2 - \frac{3}{2} u_1 \\ u_3 = V_3 - \frac{6}{19} u_2 - u_1 \end{cases} \Rightarrow \begin{cases} V_1 = u_1 \\ V_2 = \frac{3}{2} u_1 + u_2 \\ V_3 = u_1 + \frac{6}{19} u_2 + u_3 \end{cases}$$

$$\boxed{R = \begin{bmatrix} 1 \cdot \|u_1\| & \frac{3}{2} \cdot \|u_1\| & 1 \cdot \|u_1\| \\ 0 & 1 \cdot \|u_2\| & \frac{6}{19} \cdot \|u_2\| \\ 0 & 0 & 1 \cdot \|u_3\| \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} & 1 \\ 0 & 1 & \frac{6}{19} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix} \cdot R$$