Change the basis $\{1, x, x^2\}$ to an orthonormal basis.

Inner product defined as

$$\langle p(x), q(x) \rangle = \sum_{i=1}^{3} p(x_i) q(x_i)$$

where $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

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令 $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$

(1) $h_1(x) = f_1(x) = 1$, $\langle h_1(x), h_1(x) \rangle = h_1(-1)h_1(-1) + h_1(0)h_1(0) + h_1(1)h_1(1)$

$= 3$

(2) $h_2(x) = f_2(x) - \dfrac{\langle f_2, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1(x) = x - \dfrac{f_2(-1)h_1(-1) + f_2(0)h_1(0) + f_2(1)h_1(1)}{3} \cdot 1$

$= x$,

$\langle h_2(x), h_2(x) \rangle = h_2(-1)^2 + h_2(0)^2 + h_2(1)^2$

$= 2$

(3) $h_3(x) = f_3(x) - \dfrac{\langle f_3, h_2 \rangle}{\langle h_2, h_2 \rangle} h_2(x) - \dfrac{\langle f_3, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1(x) = x^2 - \dfrac{2}{3}$,

$\langle h_3(x), h_3(x) \rangle = \dfrac{2}{3}$

令 $k_1(x) = \dfrac{h_1(x)}{\|h_1(x)\|} = \dfrac{1}{\sqrt{3}}$, $k_2(x) = \dfrac{h_2(x)}{\|h_2(x)\|}$, $k_3(x) = \dfrac{h_3(x)}{\|h_3(x)\|}$

$= \dfrac{x}{\sqrt{2}}$ $\qquad = \dfrac{x^2 - \frac{2}{3}}{\sqrt{\frac{2}{3}}}$

則 $\{k_1(x), k_2(x), k_3(x)\}$ 為一組 orthonormal basis。

Representing $1+x$ in terms of the orthonormal basis.

令 $g(x) = 1+x$,

則

$$g(x) = \langle g, k_1 \rangle k_1(x) + \langle g, k_2 \rangle k_2(x) + \langle g, k_3 \rangle k_3(x)$$

計算:

$$\langle g, k_1 \rangle = g(-1)k_1(-1) + g(0)k_1(0) + g(1)k_1(1) = \sqrt{3}$$

$$\langle g, k_2 \rangle = g(-1)k_2(-1) + g(0)k_2(0) + g(1)k_2(1) = \sqrt{2}$$

$$\langle g, k_3 \rangle = g(-1)k_3(-1) + g(0)k_3(0) + g(1)k_3(1) = 0$$

$$\Rightarrow g(x) = \sqrt{3}\, k_1(x) + \sqrt{2}\, k_2(x) + 0\, k_3(x) \quad \#$$