線性獨立集內的向量運算後還是線性獨立集

證明，85 台大資工

Let $\beta = \{x_1, x_2, \cdots, x_n\}$ be a subset of V, and let $\beta' = \{x_1+2x_2, x_2+2x_3, \cdots, x_n+2x_1\}$, where $n \geq 3$. Then, β is linearly independent if and only if β' is linearly independent.

Ans.

(1) β is linearly independent $\rightarrow$ β' is linearly independent.

$\exists\, \alpha_1, \alpha_2, \cdots, \alpha_n \in F$.

$$\alpha_1(x_1+2x_2)+\alpha_2(x_2+2x_3)+\cdots+\alpha_n(x_n+2x_1)=0$$

$$\Rightarrow \alpha_1 x_1 + 2\alpha_1 x_2 + \alpha_2 x_2 + 2\alpha_2 x_3 + \cdots + \alpha_n x_n + 2\alpha_n x_1 = 0$$

$$\Rightarrow (\alpha_1+2\alpha_n)x_1 + (2\alpha_1+\alpha_2)x_2 + \cdots (2\alpha_{n-1}+\alpha_n)x_n = 0$$

因為 $\beta = \{x_1, x_2, \cdots, x_n\}$ 為線性獨立集

$\Rightarrow$ $0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 0$ 為唯一解

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_n = 0 \\ 2\alpha_1 + \alpha_2 = 0 \\ \vdots \\ 2\alpha_{n-1} + \alpha_n = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 2 \\ 2 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

利用 Gaussian elimination 運算

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 2 \\ 2 & 1 & \cdots & \cdots & 0 \\ 0 & 2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 2 \;\; 1 & \vdots \\ 0 & \cdots & \cdots & 2 & 1 \end{bmatrix} \xrightarrow{R_{12}^{(-2)}} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 2 \\ 0 & 1 & \cdots & \cdots & -2\times 2 \\ \vdots & 2 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 2 \;\; 1 & \vdots \\ 0 & & & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_{23}^{(-2)}} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 2 \\ 0 & 1 & \cdots & \cdots & -2\times 2 \\ 0 & 0 & 1 & \cdots & 2\times(-2)^2 \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \cdots & \cdots & 2 \;\; 1 & \vdots \\ 0 & \cdots & \cdots & 2 & 1 \end{bmatrix}$$

經過兩次運算可推斷算到第n列為

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 2 \\ 0 & 1 & \cdots & \cdots & (-1)^{2-1}\cdot 2^2 \\ 0 & 0 & 1 & \cdots & (-1)^{3-1}\cdot 2^3 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 1+(-1)^{n-1}\cdot 2^n \end{bmatrix} \Rightarrow \begin{cases} \alpha_1 + \quad 2\cdot\alpha_n = 0 \\ \alpha_2 + (-1)^{2-1}\cdot 2^2\alpha_n = 0 \\ \quad\vdots \\ \quad\vdots \\ \left(1+(-1)^{n-1}\cdot 2^n\right)\alpha_n = 0 \end{cases}$$

$\Rightarrow$ 解得 $\alpha_n = 0$，則 $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$

因此，β' 為線性獨立集。

(2)

β' is linearly independent $\longrightarrow$ β is linearly independent

$\beta' = \{x_1+2x_2, x_2+2x_3, \cdots, x_n+2x_1\}$

$\beta = \{x_1, x_2, \cdots, x_n\}$

令 $W = span(\beta')$，因 β' 為 linearly independent set.

所以 β' 為 W 的一組基底。

$\Rightarrow dim(span(\beta')) = dim(W) = n$

$\Rightarrow \forall v \in W, \exists \alpha_1, \alpha_2, \cdots, \alpha_n \in F$

$\Rightarrow v = \alpha_1(x_1+2x_2) + \alpha_2(x_2+2x_3) + \cdots + \alpha_n(x_n+2x_1)$

$\Rightarrow v = (\alpha_1+2\alpha_n)\underline{x_1} + (2\alpha_1+\alpha_2)\underline{x_2} + \cdots + (2\alpha_{n-1}+\alpha_n)\underline{x_n}$

$\Rightarrow v \in span(\beta)$

$\Rightarrow W \subseteq span(\beta)$

因為 β 有 n 個向量，所以 $dim(span(\beta))$ 最多就

n 次元。

$$\Rightarrow \dim(W) \leq \dim(\operatorname{span}(\beta))$$

$$\dim(W) = n, \quad \dim(\operatorname{span}(\beta)) \leq n$$

$$\Rightarrow n = \dim(W) \leq \dim(\operatorname{span}(\beta)) \leq n$$

$$\Rightarrow n \leq \dim(\operatorname{span}(\beta)) \leq n$$

$$\Rightarrow \dim(\operatorname{span}(\beta)) = n = \dim(W)$$

$$\Rightarrow W = \operatorname{span}(\beta)$$

$$\Rightarrow |\beta| = n = \dim(W)$$

所以 β 為 W 的一組基底，β 為 linearly independent。