假設 $A \in F^{m \times n}$，$\text{rank}(A) = n$，A 可做 QR-factorization，

$Q \in F^{m \times n}$ 且 orthonormal set，$R \in F^{n \times n}$。

而 $\text{rank}(A) < n$，也可以做 QR-factorization，

但 $\text{rank}(Q) < n$，R 不能可逆。

$\Rightarrow$ 任意矩陣皆能 QR-factorization

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ex、 Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 4 & 6 \end{bmatrix}。$$

Ans.

let $V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$，$V_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$，$V_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$，對 V_1, V_2, V_3 做 Gram-Schmidt process

(1) $u_1 = V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$，$\langle u_1, u_1 \rangle = 2$，$\|u_1\| = \sqrt{2}$

(2) $u_2 = V_2 - \dfrac{\langle V_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \dfrac{6}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$，$\langle u_2, u_2 \rangle = 3$，$\|u_2\| = \sqrt{3}$

(3) $u_3 = V_3 - \dfrac{\langle V_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \dfrac{\langle V_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \dfrac{3}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{10}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\langle u_3, u_3 \rangle = 0$，$\|u_3\| = 0$

$$\begin{cases} u_1 = v_1 \\ u_2 = v_2 - 3u_1 \\ u_3 = v_3 - u_2 - 5u_1 \end{cases} \Rightarrow \begin{cases} v_1 = u_1 \\ v_2 = 3u_1 + u_2 \\ v_3 = 5u_1 + u_2 + u_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\because u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \end{bmatrix} \begin{bmatrix} \|u_1\| & 3 \cdot \|u_1\| & 5 \cdot \|u_1\| \\ 0 & \|u_2\| & \|u_2\| \end{bmatrix}$$

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