遞迴特徵方程式題目

Question:

For x, y belonging to the set of positive real numbers, consider the determinant D_n of the n by n matrix $A_{n\times n}$.

$$A_{n\times n}=\begin{bmatrix} y & x & & \cdots & 0 \\ x & y & x & & \vdots \\ & x & y & x & \\ & & x & y & x \\ & & & x & \ddots & x \\ \vdots & & & & x & y & x \\ 0 & \cdots & & & & x & y \end{bmatrix}.$$

$$\begin{matrix} 0 & x & & \\ & y & x & \\ & x & y & x \\ & & x & y & \ddots \end{matrix}$$

(a) Find the recurrent relation for the value of D_n.

Ans. 找 $A_{n\times n}$ 的 determine D_n 的遞迴關係式.

$$D_1=|y|=y,\quad D_2=\begin{vmatrix} y & x \\ x & y\end{vmatrix}=y^2-x^2,\quad D_3=\begin{vmatrix} y & x & 0 \\ x & y & x \\ 0 & x & y\end{vmatrix}=y^3-2x^2y$$

用餘因子找 determine

$$D_n=y\begin{vmatrix} y & x & \\ x & y & x \\ & x & \ddots \end{vmatrix}_{(n-1)\times(n-1)}-x\begin{vmatrix} x & x & \\ & y & x \\ & x & y & \cdots \end{vmatrix}_{(n-1)\times(n-1)}$$

$$=y\begin{vmatrix} y & x & \\ x & y & x \\ & x & \ddots \end{vmatrix}_{(n-1)\times(n-1)}-x\underbrace{\begin{vmatrix} x & 0 & \\ & y & x \\ & x & y & \ddots \end{vmatrix}_{(n-1)\times(n-1)}}$$

determine 特性是 column 間倍數相加, determine 不變。

$$= y\begin{vmatrix} y & x & \\ x & y & \\ & & \ddots \end{vmatrix}_{(n-1)\times(n-1)} - x\left(x\begin{vmatrix} y & x & \\ x & y & \\ & & \ddots \end{vmatrix}_{(n-2)\times(n-2)} - 0\begin{vmatrix} 0 & x & \\ y & x & \\ x & y & \\ & & \ddots \end{vmatrix}_{(n-2)\times(n-2)} \right)$$

$$= y\begin{vmatrix} y & x & \\ x & y & \\ & & \ddots \end{vmatrix}_{(n-1)\times(n-1)} - x^2\begin{vmatrix} y & x & \\ x & y & \\ & & \ddots \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= y D_{n-1} - x^2 D_{n-2}$$

$$\therefore \begin{cases} D_1 = y \ , \ D_2 = y^2 - x^2 \\ D_n = y D_{n-1} - x^2 D_{n-2} \ , \ n \geq 2 \end{cases}$$

(b) Find the value of D_n as a function of n, when $y = x$

Ans. $D_1 = x$, $D_2 = 0$, $D_n = x D_{n-1} - x^2 D_{n-2}$

$$\Rightarrow \alpha^2 - x\alpha + x^2 = 0$$

$$\Rightarrow \alpha = \frac{x \pm \sqrt{x^2 - 4x^2}}{2} = \frac{x \pm \sqrt{-3x^2}}{2} = \frac{x \pm x\sqrt{-3}}{2} = \frac{x \pm x\sqrt{3}\,i}{2}$$

$$= x \cdot \frac{1 \pm \sqrt{3}\,i}{2}$$

$$\boxed{\text{hint: } \frac{1+\sqrt{3}\,i}{2} = \cos\frac{\pi}{3} \ , \ \frac{1-\sqrt{3}\,i}{2} = \sin\frac{\pi}{3}}$$

$$\Rightarrow D_n = C_1 \cdot x^n \cdot \cos\frac{n\pi}{3} + C_2 \cdot x^n \cdot \sin\frac{n\pi}{3}$$

$$\Rightarrow \begin{cases} D_1 = x = C_1 \cdot x \cdot \frac{1}{2} + C_2 \cdot x \cdot \frac{\sqrt{3}}{2} \\ D_2 = 0 = C_1 \cdot x^2 \cdot \left(-\frac{1}{2}\right) + C_2 \cdot x^2 \cdot \frac{\sqrt{3}}{2} \end{cases}$$

$$\begin{cases} \frac{1}{2}x C_1 + \frac{\sqrt{3}}{2}x C_2 = x \\ -\frac{1}{2}x^2 C_1 + \frac{\sqrt{3}}{2}x^2 C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = \frac{1}{\sqrt{3}} \end{cases}$$

$$\Rightarrow D_n = 1 \cdot x^n \cdot \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}} \cdot x^n \cdot \sin\frac{n\pi}{3}$$

$$= x^n \left(\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin\frac{n\pi}{3} \right), \forall n \geq 1 \ \#$$

(c) Find the value of D_n as a function of n, when $x = 2y$.

Ans.

$$\begin{cases} D_1 = 2x, \ D_2 = 4x^2 - x^2 = 3x^2 \\ D_n = 2x D_{n-1} - x^2 D_{n-2}, \ n > 2 \end{cases}$$

$$\alpha^2 - 2x\alpha + x^2 = 0, \quad \alpha = \frac{2x \pm \sqrt{4x^2 - 4x^2}}{2} = x$$

$$D_n = C_1 \cdot x^n + C_2 \cdot n x^n$$

?? → $\boxed{\text{hint：兩根相同時，代 } a_n = c_1 \lambda_1^n + c_2 n \lambda_2^n}$

$$\Rightarrow \begin{cases} D_1 = x C_1 + x C_2 = 2x \\ D_2 = x^2 C_1 + 2x^2 C_2 = 3x^2 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 1 \end{cases}$$

$$\therefore D_n = x^n + n x^n = x^n (1 + n), \forall n \geq 1 \ \#$$