利用單位向量的題目： 92 交大資料

Suppose the complete solution to the equation

$Ax=\begin{bmatrix}3\\6\\9\end{bmatrix}$ is $x=\begin{bmatrix}3\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\0\\1\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}$. Find A.

Ans.

當 $s=0$，$t=0$ 時， $A\begin{bmatrix}3\\0\\0\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix} \Rightarrow A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

當 $s=0$，$t=1$ 時， $A\begin{bmatrix}3\\0\\0\end{bmatrix}+A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix} \Rightarrow A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

當 $s=-3$，$t=0$ 時， $A\begin{bmatrix}3\\0\\0\end{bmatrix}+A\begin{bmatrix}-3\\0\\-3\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix} \Rightarrow A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\-2\\-3\end{bmatrix}$

$\therefore A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&-1\\2&0&-2\\3&0&-3\end{bmatrix}=A$ #

判斷線性系統的條件

85 中正資工

Let $A=\begin{bmatrix}1&2&0&3\\0&0&0&0\\2&4&0&1\end{bmatrix}$ and $b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

(a) Under what conditions on b so that $Ax=b$ has a solution.

(b) Find the general solution to $Ax=b$, where a solution exists.

Ans.

(a)

$$[A|b]\xrightarrow{r_{13}^{(-2)}}\left[\begin{array}{cccc|c}1&2&0&3&b_1\\0&0&0&0&b_2\\0&0&0&-5&b_3-2b_1\end{array}\right]$$

，因為 $rank(A)=rank([A|b])$ 時，$Ax=b$ 有解，所以 $b_2=0$ 時 $Ax=b$ 有解。

(b)

假設 $A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=b \Rightarrow \begin{cases}x_1+2x_2+0x_3+3x_4=b_1\\0x_1+0x_2+0x_3+0x_4=b_2\\0x_1+0x_3+0x_3-5x_4=b_3-2b_1\end{cases}$，

$\Rightarrow x_2, x_3$ 為 A 的自由變數。

general solution = 特解 + 齊次解

$\Rightarrow$ 計算特解：$x_2=x_3=0$

$$\Rightarrow\begin{cases}x_4=\dfrac{2b_1-b_3}{5}\\x_1=b_1-3x_4=\dfrac{5b_1}{5}+\dfrac{-6b_1+3b_3}{5}=\dfrac{-b_1+3b_3}{5}\end{cases}$$

$\Rightarrow$ 特解集為 $\left\{\begin{bmatrix}\frac{-b_1+3b_3}{5}\\0\\0\\\frac{2b_1-b_3}{5}\end{bmatrix}\right\}$

計算齊次解：$b_1 = b_2 = b_3 = 0$

$$\Rightarrow \begin{array}{l} x_4 = 0 \\ x_1 = -2x_2 - x_3 \end{array} \Rightarrow \text{齊次解集為} \left\{ \begin{bmatrix} -2s-t \\ s \\ t \\ 0 \end{bmatrix} \middle| s,t \in F \right\}$$

$$\Rightarrow \left\{ s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \middle| s,t \in F \right\}$$

因此通解集為

$$\left\{ \begin{bmatrix} \frac{-b_1+3b_3}{5} \\ 0 \\ 0 \\ \frac{2b_1-b_3}{5} \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \middle| s,t \in F \right\}$$

88 中正資工

Consider the following augmented matrix of a linear system in $\mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & a & a+3 \end{array}\right]$$

Determine all the possible value of a for the following cases:

(a) This linear system has infinite solutions.

(b) This linear system has no solution.

(c) This linear system has a unique solution.

Ans.

★盡量把a消掉

$$R \xrightarrow{r_{12}^{(-1)},\, r_{13}^{(-1)}} \left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 0 & 2-a & -1 & 1 \\ 0 & 3-a & a-3 & a+1 \end{array}\right] \xrightarrow{r_{23}^{(-1)}} \left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 0 & 2-a & -1 & 1 \\ 0 & 1 & a-2 & a \end{array}\right]$$

$$\xrightarrow{r_{23}} \left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 0 & 1 & a-2 & a \\ 0 & 2-a & -1 & 1 \end{array}\right] \xrightarrow{r_{23}^{(a-2)}} \left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 0 & 1 & a-2 & a \\ 0 & 0 & (a-2)^2-1 & a(a-2)+1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 0 & 1 & a-2 & a \\ 0 & 0 & (a-3)(a-1) & (a-1)^2 \end{array}\right]$$

(a)

當 $\begin{cases}(a-1)(a-3)=0\\(a-1)^2=0\end{cases}$ 時，即 $a=1$ 時具無限多解

(b)

當 $\begin{cases}(a-1)(a-3)\neq 0\\(a-1)^2=0\end{cases}$ 時，即 $\begin{cases}a\neq 1\\a=3\end{cases}$ 時無解。

(c)

當 $(a-1)(a-3)\neq 0$ 時，即 $a\notin\{1,3\}$ 時，有唯一解。

91 高雄第一科大電通

Assume that there are two lines whose parametric descriptions are, respectively, $(1,-2,1)t+(2,4,5)$ and $(2,4,4)t+(2,0,4)$.

Will these two lines intersect? Explain.

Ans.

因兩條線相交，假設相交於 (x,y,z)，存在 t_1, t_2，

使得 $(1,-2,1)t_1+(2,4,5)=(2,4,4)t_2+(2,0,4)$

$$\Rightarrow \begin{cases} t_1+2=2t_2+2 \\ -2t_1+4=4t_2 \\ t_1+5=4t_2+4 \end{cases} \Rightarrow \begin{cases} t_1-2t_2=0 \\ -2t_1-4t_2=-4 \\ t_1-4t_2=-1 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & -4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}=\begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & -4 & -4 \\ 1 & -4 & -1 \end{array}\right] \xrightarrow{R_{12}^{(2)}\ R_{13}^{(-1)}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & -2 & -1 \end{array}\right] \xrightarrow{R_{2}^{(-4)}\ R_{23}^{(1)}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_{21}^{(1)}\ R_{2}^{(\frac{1}{2})}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

因此此系統有解，$t_1=1, t_2=\frac{1}{2}$

兩線相交於 $(1,-2,1)+(2,4,5)$

$=(3,2,6)$。