University of Warsaw

UW1

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2024-11-05

1 Contest 1
2 Mathematics 1
3 Data structures 2
4 Numerical 4
5 Number theory 8
6 Combinatorial 11
7 Graph 12
8 Geometry 18
9 Strings 23
10 Various 25

Contest (1)

sol.cpp 27 lines

```
#include <bits/stdc++.h>
using namespace std;

#define rep(i, a, b) for (int i = (a); i < (b); i++)
#define all(x) begin(x), end(x)
#define sz(x) int((x).size())
using ll = long long;
using pii = pair<int, int>;
using vi = vector<int>;

#ifdef LOCAL
auto operator<<(auto& o, auto x) -> decltype(x.first, o);
auto operator<<(auto& o, auto x) -> decltype(x.end(), o) {
  o << "{";
  for (int i = 0; auto y : x) o << ", " + !i++ * 2 << y;
  return o << "}"; }
auto operator<<(auto& o, auto x) -> decltype(x.first, o) {
  return o << "(" << x.first << ", " << x.second << ")"; }
void __print(auto... x) { ((cerr << x << " "), ...) << endl; }
#define debug(x...) __print("[" #x "]:", x)
#else
#define debug(...) 2137
#endif

int main() {
  cin.tie(0)->sync_with_stdio(0);
}
```

.vimrc 8 lines

```
set nu et ts=2 sw=2
filetype indent on
syntax on
colorscheme habamax
hi MatchParen ctermfg=66 ctermbg=234 cterm=underline
nnoremap ; :
nnoremap : ;
inoremap {<cr> {<cr>}<esc>O <bs>
```

.bashrc 8 lines

```
c() {
  g++ -std=c++20 -fsanitize=address,undefined -g \
  -DLOCAL -Wall -Wextra -Wshadow $1.cpp -o $1;
}
nc() { g++ -std=c++20 -O2 $1.cpp -o $1; }
alias rm='trash'
alias mv='mv -i'
alias cp='cp -i'
```

hash.sh 3 lines

```
# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]'| md5sum |cut -c-6
```

test.sh 5 lines

```
for((i=1;i>0;i++)) do
  echo "$i"
  echo "$i" | ./gen > int
  diff -w <(./sol < int) <(./slow < int) || break
done
```

Mathematics (2)

2.1 Trigonometry

$$\sin(v+w) = \sin v\cos w + \cos v\sin w$$
$$\cos(v+w) = \cos v\cos w - \sin v\sin w$$

$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v\tan w}$$
$$\sin v + \sin w = 2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$
$$\cos v + \cos w = 2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$
$$(V+W)\tan(v-w)/2 = (V-W)\tan(v+w)/2$$

where V, W are lengths of sides opposite angles v, w.

$$a\cos x + b\sin x = r\cos(x-\phi)$$
$$a\sin x + b\cos x = r\sin(x+\phi)$$

where $r = \sqrt{a^2+b^2}, \phi = \text{atan2}(b, a)$.

2.2 Geometry

2.2.1 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \frac{a+b+c}{2}$

Area: $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius: $R = \frac{abc}{4A}$

Inradius: $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two): $s_a = \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}$

Law of sines: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Law of tangents: $\frac{a+b}{a-b} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$

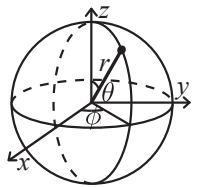
2.2.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle θ, area A and magic flux $F = b^2+d^2-a^2-c^2$:

$$4A = 2ef\cdot\sin\theta = F\tan\theta = \sqrt{4e^2f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is $180°$, $ef = ac+bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

2.2.3 Spherical coordinates



$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\phi \qquad \theta = \text{acos}(z/\sqrt{x^2+y^2+z^2})$$
$$z = r\cos\theta \qquad \phi = \text{atan2}(y, x)$$

2.3 Derivatives/Integrals

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x = 1+\tan^2 x \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.4 Sums

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1}-c^a}{c-1}, c \neq 1$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$
$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(2n+1)(n+1)}{6}$$
$$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$
$$1^4+2^4+3^4+\cdots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

2.5 Series

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots, (-\infty < x < \infty)$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{2x^3}{32}-\frac{5x^4}{128}+\ldots, (-1 \leq x \leq 1)$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots, (-\infty < x < \infty)$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots, (-\infty < x < \infty)$$

2.6 Optimization

2.6.1 Lagrange multipliers

To optimize $f(x_1,\ldots,x_n)$ subject to the constraints $g_k(x_1,\ldots,x_n) = 0$, a necessary condition for $(x_1,\ldots,x_n)$ to be a local extremum is that the gradient $\nabla f(x_1,\ldots x_n)$ must be a linear combination of the gradients $\nabla g_k(x_1,\ldots,x_n)$.

Data structures (3)

OrderStatisticTree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change `null_type`.
Time: $\mathcal{O}(\log N)$ b9b97b, 17 lines

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
                  tree_order_statistics_node_update>;

void example() {
  Tree<int> t, t2; t.insert(8);
  auto it = t.insert(10).first;
  assert(it == t.lower_bound(9));
  assert(t.order_of_key(10) == 1);
  assert(t.order_of_key(11) == 2);
  assert(*t.find_by_order(0) == 8);
  t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
}
```

HashMap.h

Description: Hash map with mostly the same API as unordered_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided). 1489f7, 8 lines

```
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
  const uint64_t C = ll(4e18 * acos(0)) | 71;
  ll operator()(ll x) const { return __builtin_bswap64(x*C); }
};
gp_hash_table<ll, ll, chash> h({},{},{},{},{1<<16});
```

LazySegtree.h

Description: Basic segment tree template with lazy propagation. Can be easily extended with advanced functionality.
Time: $\mathcal{O}(\log n)$ 0eef98, 27 lines

```
struct ST {
  struct Node {};
  int n;
  vector<Node> t;
  ST(int _n) : n(_n) { t.resize(2 * n); }
  Node join(const Node& a, const Node& b) {}
  void push(int u, int len) {} // push to u + 1, u + (len & -2)
  void rec(int u,int lo,int hi,int l,int r,bool mod,auto f) {
    if (l <= lo && hi <= r) return f(u, lo, hi);
    push(u, hi - lo);
    int mid = (lo + hi) / 2;
    if (mid > l) rec(u + 1, lo, mid, l, r, mod, f);
    if (mid < r) rec(u + (mid-lo) * 2, mid, hi, l, r, mod, f);
    if (mod) t[u] = join(t[u + 1], t[u + (mid - lo) * 2]);
  }
  Node get(int l, int r) {
    bool in = 0;
    Node res;
    rec(0, 0, n, l, r, 0, [&](int u, int, int) {
      res = in ? join(res, t[u]) : t[u], in = 1;
    });
    return res;
  }
  void modify(int l, int r) {
    rec(0, 0, n, l, r, 1, [&](int u, int lo, int hi) {});
  }
};
```

UnionFindRollback.h

Description: Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().
Usage: `int t = uf.time(); ...; uf.rollback(t);`
Time: $\mathcal{O}(\log(N))$ de4ad0, 21 lines

```
struct RollbackUF {
  vi e; vector<pii> st;
  RollbackUF(int n) : e(n, -1) {}
  int size(int x) { return -e[find(x)]; }
  int find(int x) { return e[x] < 0 ? x : find(e[x]); }
  int time() { return sz(st); }
  void rollback(int t) {
    for (int i = time(); i --> t;)
      e[st[i].first] = st[i].second;
    st.resize(t);
  }
  bool join(int a, int b) {
    a = find(a), b = find(b);
    if (a == b) return false;
    if (e[a] > e[b]) swap(a, b);
    st.push_back({a, e[a]});
    st.push_back({b, e[b]});
    e[a] += e[b]; e[b] = a;
    return true;
  }
};
```

LineContainer.h

Description: Container where you can add lines of the form kx+m, and query maximum values at points x. Useful for dynamic programming ("convex hull trick").
Time: $\mathcal{O}(\log N)$ 8ec1c7, 30 lines

```
struct Line {
  mutable ll k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(ll x) const { return p < x; }
};

struct LineContainer : multiset<Line, less<>> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  static const ll inf = LLONG_MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) return x->p = inf, 0;
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(ll k, ll m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
```

```
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  ll query(ll x) {
    assert(!empty());
    auto l = *lower_bound(x);
    return l.k * x + l.m;
  }
};
```

Treap.h

Description: Treap with lazy propagation and parent information. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.
Time: $\mathcal{O}(\log n)$ f18144, 48 lines

```
mt19937 rng(2137);
struct Node {
  Node *l = 0, *r = 0, *p = 0;
  int val, pr, c = 1;
  Node(int x) : val(x), pr(rng()) {}
  void pull();
  void push();
};

int cnt(Node* n) { return n ? n->c : 0; }
void Node::pull() { c = cnt(l) + cnt(r) + 1; }
void Node::push() {}

pair<Node*, Node*> split(Node* n, int k) {
  if (!n) return {};
  n->push();
  if (cnt(n->l) >= k) { // "n->val >= k" for lower_bound(k)
    auto pa = split(n->l, k);
    n->l = pa.second;
    n->p = 0;
    if (n->l) n->l->p = n;
    n->pull();
    return {pa.first, n};
  } else {
    auto pa = split(n->r, k - cnt(n->l) - 1); // and just "k"
    n->r = pa.first;
    n->p = 0;
    if (n->r) n->r->p = n;
    n->pull();
    return {n, pa.second};
  }
}
Node* merge(Node* l, Node* r) {
  if (!l || !r) return l ?: r;
  if (l->pr > r->pr) {
    l->push();
    l->r = merge(l->r, r);
    l->r->p = l;
    l->pull();
    return l;
  } else {
    r->push();
    r->l = merge(l, r->l);
    r->l->p = r;
    r->pull();
    return r;
  }
}
```

FenwickTree.h

Description: Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new value.
Time: Both operations are $\mathcal{O}(\log N)$. e62fac, 22 lines

```
struct FT {
  vector<ll> s;
  FT(int n) : s(n) {}
  void update(int pos, ll dif) { // a[pos] += dif
    for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
  }
  ll query(int pos) { // sum of values in [0, pos)
    ll res = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
  }
  int lower_bound(ll sum) {// min pos st sum of [0, pos] >= sum
    // Returns n if no sum is >= sum, or -1 if empty sum is.
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1) {
      if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
        pos += pw, sum -= s[pos-1];
    }
    return pos;
  }
};
```

FenwickTree2d.h

Description: Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).
Time: $\mathcal{O}\left(\log^2 N\right)$. (Use persistent segment trees for $\mathcal{O}(\log N)$.)
"FenwickTree.h" 157f07, 22 lines

```
struct FT2 {
  vector<vi> ys; vector<FT> ft;
  FT2(int limx) : ys(limx) {}
  void fakeUpdate(int x, int y) {
    for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
  }
  void init() {
    for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
  }
  int ind(int x, int y) {
    return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()); }
  void update(int x, int y, ll dif) {
    for (; x < sz(ys); x |= x + 1)
      ft[x].update(ind(x, y), dif);
  }
  ll query(int x, int y) {
    ll sum = 0;
    for (; x; x &= x - 1)
      sum += ft[x-1].query(ind(x-1, y));
    return sum;
  }
};
```

RMQ.h

Description: Range Minimum Queries on an array. Returns min(V[a], V[a + 1], ... V[b - 1]) in constant time.
Usage: `RMQ rmq(values);`
`rmq.query(inclusive, exclusive);`
Time: $\mathcal{O}(|V| \log |V| + Q)$ 510c32, 16 lines

```
template<class T>
struct RMQ {
  vector<vector<T>> jmp;
  RMQ(const vector<T>& V) : jmp(1, V) {
    for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
      jmp.emplace_back(sz(V) - pw * 2 + 1);
      rep(j,0,sz(jmp[k]))
        jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
    }
  }
  T query(int a, int b) {
    assert(a < b); // or return inf if a == b
    int dep = 31 - __builtin_clz(b - a);
    return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
  }
};
```

MoQueries.h

Description: Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change `step` to add/remove the edge (a, c) and remove the initial `add` call (but keep `in`).
Time: $\mathcal{O}\left(N\sqrt{Q}\right)$ a12ef4, 49 lines

```
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
  int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
  vi s(sz(Q)), res = s;
#define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))
  iota(all(s), 0);
  sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
  for (int qi : s) {
    pii q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);
    res[qi] = calc();
  }
  return res;
}

vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root=0){
  int N = sz(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
  vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
  add(0, 0), in[0] = 1;
  auto dfs = [&](int x, int p, int dep, auto& f) -> void {
    par[x] = p;
    L[x] = N;
    if (dep) I[x] = N++;
    for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
    if (!dep) I[x] = N++;
    R[x] = N;
  };
  dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
  iota(all(s), 0);
  sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
  for (int qi : s) rep(end,0,2) {
    int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
                  else { add(c, end); in[c] = 1; } a = c; }
    while (!(L[b] <= L[a] && R[a] <= R[b]))
      I[i++] = b, b = par[b];
    while (a != b) step(par[a]);
    while (i--) step(I[i]);
    if (end) res[qi] = calc();
  }
  return res;
}
```

LinkCutTree.h

Description: Link-cut tree with path and subtree queries. Path operations can be arbitrary, but subtree operations need to be reversible. Current implementation supports subtree addition and sum.
Time: $\mathcal{O}(\log n)$

6758b4, 105 lines

```
struct Node {
  Node *p, *c[2];
  Node() { p = c[0] = c[1] = 0; }
  // Vertex data (flip is required)
  bool rev = 0;
  ll v, s, vs = 0;
  int sz, vsz = 0;
  ll d = 0, vd = 0, cc = 0;
  void add(ll x) {
    d += x, vd += x;
    v += x, s += sz * x, vs += vsz * x;
  }
  void flip() {
    swap(c[0], c[1]), rev ^= 1;
  }
  // BST operations
  void push() {
    if (rev) {
      rep(i, 0, 2) if (c[i]) c[i]->flip();
      rev = 0;
    }
    if (d) {
      rep(i, 0, 2) if (c[i]) c[i]->add(d);
      d = 0;
    }
  }
  void pull() {
    s = v + vs;
    sz = 1 + vsz;
    if (c[0]) s += c[0]->s, sz += c[0]->sz;
    if (c[1]) s += c[1]->s, sz += c[1]->sz;
  }
  // Virtual operations (vd = virtual delta, cc = cancel)
  void vadd(Node* x) { // Add x, x.cc = vd
    vs += x->s;
    vsz += x->sz;
    x->cc = vd;
  }
  void vdel(Node* x) { // Push vd - x.cc, delete x, x.cc = 0
    x->add(vd - x->cc);
    vs -= x->s;
    vsz -= x->sz;
    x->cc = 0;
  }
  void vswap(Node* x, Node* y) { // Swap x.cc, y.cc
    swap(x->cc, y->cc);
  }
  // Splay operations
  int up() {
    if (!p) return -2;
    rep(i, 0, 2) if (p->c[i] == this) return i;
    return -1;
  }
  bool isRoot() { return up() < 0; }
  friend void setLink(Node* x, Node* y, int d) {
    if (y) y->p = x;
    if (d >= 0) x->c[d] = y;
  }
  void rot() {
    int x = up(); Node* pp = p;
    setLink(pp->p, this, pp->up());
    setLink(pp, c[x ^ 1], x); setLink(this, pp, x ^ 1);
    if (pp->p) pp->p->vswap(pp, this);
    pp->pull();
  }
  void fix() { if (!isRoot()) p->fix(); push(); }
  void splay() {
    for (fix(); !isRoot();) {
      if (p->isRoot()) rot();
      else if (up() == p->up()) p->rot(), rot();
      else rot(), rot();
    }
    pull();
  }
};
struct LinkCut {
  vector<Node> t;
  LinkCut(int n) : t(n) {}
  void link(int u, int v) { // link u -> v
    makeRoot(&t[v]); access(&t[u]);
    setLink(&t[v], &t[u], 0); t[v].pull();
  }
  void cut(int u, int v) { // cut u -> v
    makeRoot(&t[u]); access(&t[v]);
    t[v].c[0] = t[u].p = 0; t[v].pull();
  }
  bool connected(int u, int v) {
    return lca(&t[u], &t[v]);
  }
  Node* lca(Node* u, Node* v) {
    if (u == v) return u;
    access(u); access(v); if (!u->p) return 0;
    u->splay(); return u->p ?: u;
  }
  void access(Node* u) {
    for (Node* x = u, *y = 0; x; x = x->p) {
      x->splay();
      if (y) x->vdel(y);
      if (x->c[1]) x->vadd(x->c[1]);
      x->c[1] = y; x->pull(); y = x;
    }
    u->splay();
  }
  void makeRoot(Node* u) { access(u), u->flip(), u->push(); }
};
```

Numerical (4)

4.1 Polynomials and recurrences

Polynomial.h

c9b7b0, 17 lines

```
struct Poly {
  vector<double> a;
  double operator()(double x) const {
    double val = 0;
    for (int i = sz(a); i--;) (val *= x) += a[i];
    return val;
  }
  void diff() {
    rep(i,1,sz(a)) a[i-1] = i*a[i];
    a.pop_back();
  }
  void divroot(double x0) {
    double b = a.back(), c; a.back() = 0;
    for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
    a.pop_back();
  }
};
```

PolyRoots.h

Description: Finds the real roots to a polynomial.
Usage: `polyRoots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0`
Time: $\mathcal{O}\left(n^2 \log(1/\epsilon)\right)$

"Polynomial.h" b00bfe, 23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax) {
  if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
  vector<double> ret;
  Poly der = p;
  der.diff();
  auto dr = polyRoots(der, xmin, xmax);
  dr.push_back(xmin-1);
  dr.push_back(xmax+1);
  sort(all(dr));
  rep(i,0,sz(dr)-1) {
    double l = dr[i], h = dr[i+1];
    bool sign = p(l) > 0;
    if (sign ^ (p(h) > 0)) {
      rep(it,0,60) { // while (h - l > 1e-8)
        double m = (l + h) / 2, f = p(m);
        if ((f <= 0) ^ sign) l = m;
        else h = m;
      }
      ret.push_back((l + h) / 2);
    }
  }
  return ret;
}
```

PolyInterpolate.h

Description: Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them: $p(x) = a[0] * x^0 + ... + a[n-1] * x^{n-1}$. For numerical precision, pick $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \ldots n-1$.
Time: $\mathcal{O}\left(n^2\right)$

08bf48, 13 lines

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
  vd res(n), temp(n);
  rep(k,0,n-1) rep(i,k+1,n)
    y[i] = (y[i] - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;
  rep(k,0,n) rep(i,0,n) {
    res[i] += y[k] * temp[i];
    swap(last, temp[i]);
    temp[i] -= last * x[k];
  }
  return res;
}
```

BerlekampMassey.h

Description: Recovers any n-order linear recurrence relation from the first $2n$ terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size $\leq n$.
Usage: `berlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}`
Time: $\mathcal{O}\left(N^2\right)$

"../number-theory/ModInt.h" 810031, 20 lines

```
vector<mint> berlekampMassey(vector<mint> s) {
  int n = sz(s), L = 0, m = 0;
  vector<mint> C(n), B(n), T;
  C[0] = B[0] = 1;

  mint b = 1;
  rep(i,0,n) { ++m;
    mint d = s[i];
    rep(j,1,L+1) d += C[j] * s[i - j];
    if (d == 0) continue;
    T = C; mint coef = d / b;
```

```
    rep(j,m,n) C[j] -= coef * B[j - m];
    if (2 * L > i) continue;
    L = i + 1 - L; B = T; b = d; m = 0;
  }

  C.resize(L + 1); C.erase(C.begin());
  for (mint& x : C) x *= -1;
  return C;
}
```

LinearRecurrence.h

Description: Generates the k'th term of an n-order linear recurrence $S[i] = \sum_j S[i-j-1]tr[j]$, given $S[0 \dots \geq n-1]$ and $tr[0 \dots n-1]$. Faster than matrix multiplication. Useful together with Berlekamp–Massey.
Usage: `linearRec({0, 1}, {1, 1}, k) // k'th Fibonacci number`
Time: $\mathcal{O}\left(n^2 \log k\right)$ f4e444, 26 lines

```
typedef vector<ll> Poly;
ll linearRec(Poly S, Poly tr, ll k) {
  int n = sz(tr);

  auto combine = [&](Poly a, Poly b) {
    Poly res(n * 2 + 1);
    rep(i,0,n+1) rep(j,0,n+1)
      res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
    for (int i = 2 * n; i > n; --i) rep(j,0,n)
      res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
    res.resize(n + 1);
    return res;
  };

  Poly pol(n + 1), e(pol);
  pol[0] = e[1] = 1;

  for (++k; k; k /= 2) {
    if (k % 2) pol = combine(pol, e);
    e = combine(e, e);
  }

  ll res = 0;
  rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
  return res;
}
```

4.2 Optimization

GoldenSectionSearch.h

Description: Finds the argument minimizing the function f in the interval $[a, b]$ assuming f is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is eps. Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version.
Usage: `double func(double x) { return 4+x+.3*x*x; }`
`double xmin = gss(-1000,1000,func);`
Time: $\mathcal{O}\left(\log((b-a)/\epsilon)\right)$ 31d45b, 14 lines

```
double gss(double a, double b, double (*f)(double)) {
  double r = (sqrt(5)-1)/2, eps = 1e-7;
  double x1 = b - r*(b-a), x2 = a + r*(b-a);
  double f1 = f(x1), f2 = f(x2);
  while (b-a > eps)
    if (f1 < f2) { //change to > to find maximum
      b = x2; x2 = x1; f2 = f1;
      x1 = b - r*(b-a); f1 = f(x1);
    } else {
      a = x1; x1 = x2; f1 = f2;
      x2 = a + r*(b-a); f2 = f(x2);
    }
  return a;
}
```

HillClimbing.h

Description: Poor man's optimization for unimodal functions. 8eeeaf, 14 lines

```
typedef array<double, 2> P;

template<class F> pair<double, P> hillClimb(P start, F f) {
  pair<double, P> cur(f(start), start);
  for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
    rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
      P p = cur.second;
      p[0] += dx*jmp;
      p[1] += dy*jmp;
      cur = min(cur, make_pair(f(p), p));
    }
  }
  return cur;
}
```

Integrate.h

Description: Simple integration of a function over an interval using Simpson's rule. The error should be proportional to h^4, although in practice you will want to verify that the result is stable to desired precision when epsilon changes. 4756fc, 7 lines

```
template<class F>
double quad(double a, double b, F f, const int n = 1000) {
  double h = (b - a) / 2 / n, v = f(a) + f(b);
  rep(i,1,n*2)
    v += f(a + i*h) * (i&1 ? 4 : 2);
  return v * h / 3;
}
```

IntegrateAdaptive.h

Description: Fast integration using an adaptive Simpson's rule.
Usage: `double sphereVolume = quad(-1, 1, [](double x) {`
`return quad(-1, 1, [&](double y) {`
`return quad(-1, 1, [&](double z) {`
`return x*x + y*y + z*z < 1; });});});` 92dd79, 15 lines

```
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template <class F>
d rec(F& f, d a, d b, d eps, d S) {
  d c = (a + b) / 2;
  d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
  if (abs(T - S) <= 15 * eps || b - a < 1e-10)
    return T + (T - S) / 15;
  return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
  return rec(f, a, b, eps, S(a, b));
}
```

Simplex.h

Description: Solves a general linear maximization problem: maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$. Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of $c^T x$ otherwise. The input vector is set to an optimal x (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that $x = 0$ is viable.
Usage: `vvd A = {{1,-1}, {-1,1}, {-1,-2}};`
`vd b = {1,1,-4}, c = {-1,-1}, x;`
`T val = LPSolver(A, b, c).solve(x);`
Time: $\mathcal{O}(NM * \#pivots)$, where a pivot may be e.g. an edge relaxation. $\mathcal{O}(2^n)$ in the general case. aa8530, 68 lines

```
typedef double T; // long double, Rational, double + mod<P>...
typedef vector<T> vd;
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s])) s=j

struct LPSolver {
  int m, n;
  vi N, B;
  vvd D;

  LPSolver(const vvd& A, const vd& b, const vd& c) :
    m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
      rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
      rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i];}
      rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
      N[n] = -1; D[m+1][n] = 1;
    }

  void pivot(int r, int s) {
    T *a = D[r].data(), inv = 1 / a[s];
    rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
      T *b = D[i].data(), inv2 = b[s] * inv;
      rep(j,0,n+2) b[j] -= a[j] * inv2;
      b[s] = a[s] * inv2;
    }
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
    rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
    D[r][s] = inv;
    swap(B[r], N[s]);
  }

  bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
      int s = -1;
      rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
      if (D[x][s] >= -eps) return true;
      int r = -1;
      rep(i,0,m) {
        if (D[i][s] <= eps) continue;
        if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                     < MP(D[r][n+1] / D[r][s], B[r])) r = i;
      }
      if (r == -1) return false;
      pivot(r, s);
    }
  }

  T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
      pivot(r, n);
      if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
      rep(i,0,m) if (B[i] == -1) {
        int s = 0;
        rep(j,1,n+1) ltj(D[i]);
        pivot(i, s);
      }
    }
    bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
```

```
  }
};
```

4.3 Matrices

Determinant.h

Description: Calculates determinant of a matrix. Destroys the matrix.
Time: $\mathcal{O}(N^3)$

4583fb, 16 lines

```
template<class T>
T det(vector<vector<T>>& a) {
  int n = sz(a); T res = 1;
  rep(i,0,n) {
    int b = i;
    rep(j,i+1,n) if (abs(a[j][i]) > abs(a[b][i])) b = j;
    if (i != b) swap(a[i], a[b]), res *= -1;
    res *= a[i][i];
    if (res == 0) return 0;
    rep(j,i+1,n) {
      T v = a[j][i] / a[i][i];
      if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
    }
  }
  return res;
}
```

SolveLinear.h

Description: Solves $Ax = b$. If no solutions exist, returns -1. Otherwise, returns the rank of A and transforms it s.t. $\{A'_1, A'_2, \dots\}$ is a basis of the kernel of A.
Time: $\mathcal{O}(n^2 m)$

8a34d1, 40 lines

```
const double eps = 1e-12;

template<class T>
int solveLinear(auto& A, vector<T>& b, vector<T>& x) {
  int n = sz(A), m = sz(x), rank = 0, br, bc;
  if (n) assert(sz(A[0]) == m);
  vi col(m); iota(all(col), 0);
  rep(i,0,n) {
    T v, bv = 0;
    rep(r,i,n) rep(c,i,m)
      if ((v = abs(A[r][c])) > bv)
        br = r, bc = c, bv = v;
    if (bv <= eps) {
      rep(j,i,n) if (abs(b[j]) > eps) return -1;
      break;
    }
    swap(A[i], A[br]);
    swap(b[i], b[br]);
    swap(col[i], col[bc]);
    rep(j,0,n) swap(A[j][i], A[j][bc]);
    bv = 1/A[i][i];
    rep(j,0,n) if (j != i) {
      T fac = A[j][i] * bv;
      b[j] -= fac * b[i];
      rep(k,i+1,m) A[j][k] -= fac*A[i][k];
    }
    rank++;
  }
  x.assign(m, 0);
  for (int i = rank - 1; i >= 0; i--) {
    b[i] /= A[i][i];
    x[col[i]] = b[i];
  }
  vector<vector<T>> ker(m - rank, vector<T>(m));
  rep(i, rank, m) {
    ker[i - rank][col[i]] = 1;
    rep(j, 0, rank) ker[i - rank][col[j]] -= A[j][i] / A[j][j];
  }
  return A = ker, rank;
}
```

SolveLinearBinary.h

Description: Solves $Ax = b$ over $\mathbb{F}_2$. If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys A and b.
Time: $\mathcal{O}(n^2 m)$

fa2d7a, 34 lines

```
typedef bitset<1000> bs;

int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
  int n = sz(A), rank = 0, br;
  assert(m <= sz(x));
  vi col(m); iota(all(col), 0);
  rep(i,0,n) {
    for (br=i; br<n; ++br) if (A[br].any()) break;
    if (br == n) {
      rep(j,i,n) if(b[j]) return -1;
      break;
    }
    int bc = (int)A[br]._Find_next(i-1);
    swap(A[i], A[br]);
    swap(b[i], b[br]);
    swap(col[i], col[bc]);
    rep(j,0,n) if (A[j][i] != A[j][bc]) {
      A[j].flip(i); A[j].flip(bc);
    }
    rep(j,i+1,n) if (A[j][i]) {
      b[j] ^= b[i];
      A[j] ^= A[i];
    }
    rank++;
  }

  x = bs();
  for (int i = rank; i--;) {
    if (!b[i]) continue;
    x[col[i]] = 1;
    rep(j,0,i) b[j] ^= A[j][i];
  }
  return rank; // (multiple solutions if rank < m)
}
```

MatrixInverse.h

Description: Invert matrix A. Returns rank; result is stored in A unless singular (rank < n). For prime powers, repeatedly set $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$ where A^{-1} starts as the inverse of A mod p, and k is doubled in each step.
Time: $\mathcal{O}(n^3)$

d43579, 33 lines

```
template<class T>
int matInv(vector<vector<T>>& A) {
  int n = sz(A); vi col(n);
  vector<vector<T>> tmp(n, vector<T>(n));
  rep(i,0,n) tmp[i][i] = 1, col[i] = i;
  rep(i,0,n) {
    int r = i, c = i;
    rep(j,i,n) rep(k,i,n)
      if (abs(A[j][k]) > abs(A[r][c]))
        r = j, c = k;
    if (abs(A[r][c]) < 1e-12) return i;
    A[i].swap(A[r]); tmp[i].swap(tmp[r]);
    rep(j,0,n)
      swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
    swap(col[i], col[c]);
    T v = A[i][i];
    rep(j,i+1,n) {
      T f = A[j][i] / v;
      A[j][i] = 0;
      rep(k,i+1,n) A[j][k] -= f*A[i][k];
      rep(k,0,n) tmp[j][k] -= f*tmp[i][k];
    }
    rep(j,i+1,n) A[i][j] /= v;
    rep(j,0,n) tmp[i][j] /= v;
    A[i][i] = 1;
  }
  for (int i = n-1; i > 0; --i) rep(j,0,i) {
    T v = A[j][i];
    rep(k,0,n) tmp[j][k] -= v*tmp[i][k];
  }
  rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
  return n;
}
```

Tridiagonal.h

Description: $x = \text{tridiagonal}(d, p, q, b)$ solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i,\ 1 \le i \le n,$$

where a_0, a_{n+1}, b_i, c_i and d_i are known. a can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, ..., -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.
If $|d_i| > |p_i| + |q_{i-1}|$ for all i, or $|d_i| > |p_{i-1}| + |q_i|$, or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.
Time: $\mathcal{O}(N)$

8f9fa8, 26 lines

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T>& super,
    const vector<T>& sub, vector<T> b) {
  int n = sz(b); vi tr(n);
  rep(i,0,n-1) {
    if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
      b[i+1] -= b[i] * diag[i+1] / super[i];
      if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
      diag[i+1] = sub[i]; tr[++i] = 1;
    } else {
      diag[i+1] -= super[i]*sub[i]/diag[i];
      b[i+1] -= b[i]*sub[i]/diag[i];
    }
  }
  for (int i = n; i--;) {
    if (tr[i]) {
      swap(b[i], b[i-1]);
      diag[i-1] = diag[i];
      b[i] /= super[i-1];
    } else {
      b[i] /= diag[i];
      if (i) b[i-1] -= b[i]*super[i-1];
    }
  }
  return b;
}
```

BlackBoxDet.h

Description: Black-box algorithm for the determinant of a matrix. f should be a function s.t. $f(v) = Mv$. Can add multiple iterations if order of recurrence is less than n.
Time: $\mathcal{O}\left(n^2 + 2n \text{ calls to } f\right)$

"BerlekampMassey.h" b316d2, 20 lines

```
mt19937_64 rng(2137);
mint det(int n, auto f) {
  auto rnd = [&]() {
    vector<mint> v(n);
    rep(i, 0, n) v[i] = rng() % mint::MOD;
    return v;
  };
  auto a = rnd(), b = rnd(), c = rnd();
  vector<mint> s(2 * n);
  rep(i, 0, 2 * n) {
    rep(j, 0, n) s[i] += a[j] * b[j];
    rep(j, 0, n) b[j] *= c[j];
    b = f(move(b));
  }
  auto v = berlekampMassey(s);
  if (sz(v) != n) return 0;
  mint p = 1;
  rep(i, 0, n) p *= c[i];
  return v[n - 1] / p * (n % 2 ? 1 : -1);
}
```

4.4 Fourier transforms

FFT.h

Description: Multiply polynomials for any modulus. Works for $n+m \leq 2^{24}$ and $c_k \leq 5 \cdot 10^{25}$.
Time: $\mathcal{O}\left((n+m)\log(n+m)\right)$

"../number-theory/ModInt.h" e993f5, 44 lines

```
template<class T>
void ntt(vector<T>& a, bool inv) {
  int n = sz(a); vector<T> b(n);
  for (int i = n / 2; i; i /= 2, swap(a, b)) {
    T w = T(T::ROOT).pow((T::MOD - 1) / n * i), m = 1;
    for (int j = 0; j < n; j += 2 * i, m *= w) rep(k, 0, i) {
      T u = a[j + k], v = a[j + k + i] * m;
      b[j / 2 + k] = u + v, b[j / 2 + k + n / 2] = u - v;
    }
  }
  if (inv) {
    reverse(1 + all(a));
    T z = T(n).inv(); rep(i, 0, n) a[i] *= z;
  }
}
template<class T>
vector<T> conv(vector<T> a, vector<T> b) {
  int s = sz(a) + sz(b) - 1, n = 1 << __lg(2 * s - 1);
  a.resize(n); ntt(a, 0); b.resize(n); ntt(b, 0);
  rep(i, 0, n) a[i] *= b[i];
  ntt(a, 1); a.resize(s);
  return a;
}
template<class T>
vector<T> mconv(const auto& x, const auto& y) {
  auto con = [&](const auto& v) {
    vector<T> w(sz(v)); rep(i, 0, sz(v)) w[i] = v[i].x;
    return w; };
  return conv(con(x), con(y));
}
template<class T>
vector<T> conv3(const vector<T>& a, const vector<T>& b) {
  using m0 = Mod<754974721, 11>; auto c0 = mconv<m0>(a, b);
  using m1 = Mod<167772161, 3>;  auto c1 = mconv<m1>(a, b);
  using m2 = Mod<469762049, 3>;  auto c2 = mconv<m2>(a, b);
  int n = sz(c0); vector<T> d(n); m1 r01 = m1(m0::MOD).inv();
  m2 r02 = m2(m0::MOD).inv(), r12 = m2(m1::MOD).inv();
  rep(i, 0, n) {
    int x = c0[i].x, y = ((c1[i] - x) * r01).x,
        z = (((c2[i] - x) * r02 - y) * r12).x;
    d[i] = (T(z) * m1::MOD + y) * m0::MOD + x;
  }
  return d;
}
```

FFTPoly.h

Description: Fast operations on polynomials.
Time: $\mathcal{O}\left(n \log n\right)$, eval and interp are $\mathcal{O}\left(n \log^2 n\right)$

"FFT.h" ed5af1, 117 lines

```
using Poly = vector<mint>;
Poly& operator+=(Poly& a, const Poly& b) {
  a.resize(max(sz(a), sz(b)));
  rep(i, 0, sz(b)) a[i] += b[i];
  return a;
}
Poly& operator-=(Poly& a, const Poly& b) {
  a.resize(max(sz(a), sz(b)));
  rep(i, 0, sz(b)) a[i] -= b[i];
  return a;
}
Poly& operator*=(Poly& a, const Poly& b) {
  if (min(sz(a), sz(b)) < 50) {
    Poly c(sz(a) + sz(b) - 1);
    rep(i, 0, sz(a)) rep(j, 0, sz(b)) c[i + j] += a[i] * b[j];
    return a = c;
  }
  return a = conv(move(a), b);
}
Poly operator+(Poly a, const Poly& b) { return a += b; }
Poly operator-(Poly a, const Poly& b) { return a -= b; }
Poly operator*(Poly a, const Poly& b) { return a *= b; }
Poly modK(Poly a, int k) { return a.resize(min(sz(a), k)), a; }
Poly inv(const Poly& a) { // a[0] != 0
  Poly b = {1 / a[0]};
  while (sz(b) < sz(a))
    b = modK(b * (Poly{2} - modK(a, 2 * sz(b)) * b), 2*sz(b));
  return modK(b, sz(a));
}
Poly deriv(Poly a) {
  if (!sz(a)) return {};
  rep(i, 1, sz(a)) a[i - 1] = a[i] * i;
  return a.pop_back(), a;
}
Poly integr(const Poly& a) {
  if (!sz(a)) return {};
  Poly b(sz(a) + 1);
  rep(i, 1, sz(b)) b[i] = a[i - 1] / i;
  return b;
}
Poly log(const Poly& a) { // a[0] = 1
  Poly b = integr(deriv(a) * inv(a));
  return b.resize(sz(a)), b;
}
Poly exp(const Poly& a) { // a[0] = 0
  Poly b = {1};
  if (!sz(a)) return b;
  while (sz(b) < sz(a)) {
    b.resize(sz(b) * 2);
    b *= Poly{1} + modK(a, sz(b)) - log(b);
    b.resize(sz(b) / 2 + 1);
  }
  return modK(b, sz(a));
}
Poly pow(Poly a, ll m) {
  int p = 0, n = sz(a);
  if (!m) { a.assign(n, 0); a[0] = 1; return a; }
  while (p < n && !a[p].x) p++;
  if (p >= (n + m - 1) / m) return Poly(n);
  mint j = a[p];
  a = Poly(p + all(a)) * Poly{1 / j};
  a.resize(n);
  Poly res = exp(log(a) * Poly{m}) * Poly{j.pow(m)};
  res.insert(res.begin(), p * m, 0);
  return modK(res, n);
}
Poly& operator/=(Poly& a, Poly b) {
  if (sz(a) < sz(b)) return a = {};
  int s = sz(a) - sz(b) + 1;
  reverse(all(a)), reverse(all(b));
  a.resize(s), b.resize(s);
  a *= inv(b);
  a.resize(s), reverse(all(a));
  return a;
}
Poly operator/(Poly a, Poly b) { return a /= b; }
Poly& operator%=(Poly& a, const Poly& b) {
  if (sz(a) < sz(b)) return a;
  return a = modK(a - (a / b) * b, sz(b) - 1);
}
Poly operator%(Poly a, const Poly& b) { return a %= b; }
vector<mint> eval(const Poly& a, const vector<mint>& x) {
  int n = sz(x);
  if (!n) return {};
  vector<Poly> up(2 * n);
  rep(i, 0, n) up[i + n] = Poly{0 - x[i], 1};
  for (int i = n - 1; i > 0; i--)
    up[i] = up[2 * i] * up[2 * i + 1];
  vector<Poly> down(2 * n);
  down[1] = a % up[1];
  rep(i, 2, 2 * n) down[i] = down[i / 2] % up[i];
  vector<mint> y(n);
  rep(i, 0, n) y[i] = down[i + n][0];
  return y;
}
Poly interp(vector<mint> x, vector<mint> y) {
  int n = sz(x);
  vector<Poly> up(2 * n);
  rep(i, 0, n) up[i + n] = Poly{0 - x[i], 1};
  for (int i = n - 1; i > 0; i--)
    up[i] = up[2 * i] * up[2 * i + 1];
  vector<mint> a = eval(deriv(up[1]), x);
  vector<Poly> down(2 * n);
  rep(i, 0, n) down[i + n] = Poly{y[i] / a[i]};
  for(int i = n - 1; i > 0; i--)
    down[i] = down[2*i] * up[2*i+1] + down[2*i+1] * up[2*i];
  return down[1];
}
Poly subsetSum(Poly a) { // a[0] = 0
  int n = sz(a);
  Poly b(n);
  rep(i, 1, n) b[i] = mint(i).inv() * (i % 2 ? 1 : -1);
  for (int i = n - 2; i > 0; i--)
    for (int j = 2; i * j < n; j++)
      a[i * j] += b[j] * a[i];
  return exp(a);
}
```

FastFourierTransform.h

Description: fft(a) computes $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$ for all k. N must be a power of 2. Useful for convolution: conv(a, b) = c, where $c[x] = \sum a[i]b[x-i]$. For convolution of complex numbers or more than two

vectors: FFT, multiply pointwise, divide by n, reverse(start+1, end), FFT back. Rounding is safe if $(\sum a_i^2 + \sum b_i^2)\log_2 N < 9 \cdot 10^{14}$ (in practice 10^{16}; higher for random inputs). Otherwise, use NTT/FFTMod.
Time: $\mathcal{O}(N \log N)$ with $N = |A| + |B|$ ($\sim 1s$ for $N = 2^{22}$)
00ced6, 35 lines

```
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
  int n = sz(a), L = 31 - __builtin_clz(n);
  static vector<complex<long double>> R(2, 1);
  static vector<C> rt(2, 1);  // (^ 10% faster if double)
  for (static int k = 2; k < n; k *= 2) {
    R.resize(n); rt.resize(n);
    auto x = polar(1.0L, acos(-1.0L) / k);
    rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
  }
  vi rev(n);
  rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
  rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2)
    for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
      C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-rolled)
      a[i + j + k] = a[i + j] - z;
      a[i + j] += z;
    }
}
vd conv(const vd& a, const vd& b) {
  if (a.empty() || b.empty()) return {};
  vd res(sz(a) + sz(b) - 1);
  int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
  vector<C> in(n), out(n);
  copy(all(a), begin(in));
  rep(i,0,sz(b)) in[i].imag(b[i]);
  fft(in);
  for (C& x : in) x *= x;
  rep(i,0,n) out[i] = in[-i & (n - 1)] - conj(in[i]);
  fft(out);
  rep(i,0,sz(res)) res[i] = imag(out[i]) / (4 * n);
  return res;
}
```

FastFourierTransformMod.h

Description: Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$ (in practice 10^{16} or higher). Inputs must be in $[0, \text{mod})$.
Time: $\mathcal{O}(N \log N)$, where $N = |A| + |B|$ (twice as slow as NTT or FFT)
"FastFourierTransform.h" b82773, 22 lines

```
typedef vector<ll> vl;
template<int M> vl convMod(const vl &a, const vl &b) {
  if (a.empty() || b.empty()) return {};
  vl res(sz(a) + sz(b) - 1);
  int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M));
  vector<C> L(n), R(n), outs(n), outl(n);
  rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);
  rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);
  fft(L), fft(R);
  rep(i,0,n) {
    int j = -i & (n - 1);
    outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
    outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
  }
  fft(outl), fft(outs);
  rep(i,0,sz(res)) {
    ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);
    ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);
    res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
  }
  return res;
}
```

FastSubsetTransform.h

Description: Transform to a basis with fast convolutions of the form $c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$, where $\oplus$ is one of AND, OR, XOR. The size of a must be a power of two.
Time: $\mathcal{O}(N \log N)$
8fd8c5, 18 lines

```
template<class T>
void FST(vector<T>& a, bool inv) {
  for (int n = sz(a), step = 1; step < n; step *= 2) {
    for (int i = 0; i < n; i += 2 * step) rep(j,i,i+step) {
      T &u = a[j], &v = a[j + step]; tie(u, v) =
        inv ? pair(v - u, u) : pair(v, u + v); // AND
        inv ? pair(v, u - v) : pair(u + v, u); // OR
        pair(u + v, u - v);                    // XOR
    }
  }
  if (inv) for (T& x : a) x /= sz(a); // XOR only
}
template<class T>
vector<T> conv(vector<T> a, vector<T> b) {
  FST(a, 0); FST(b, 0);
  rep(i, 0, sz(a)) a[i] *= b[i];
  FST(a, 1); return a;
}
```

Number theory (5)

5.1 Modular arithmetic

ModInt.h

Description: Operators for modular arithmetic.
a902ca, 28 lines

```
template<int M, int R>
struct Mod {
  static const int MOD = M, ROOT = R;
  int x;
  Mod(ll y = 0) : x(y % M) { x += (x < 0) * M; }
  Mod& operator+=(Mod o) {
    if ((x += o.x) >= M) x -= M;
    return *this; }
  Mod& operator-=(Mod o) {
    if ((x -= o.x) < 0) x += M;
    return *this; }
  Mod& operator*=(Mod o) {
    x = 1ll * x * o.x % M;
    return *this; }
  Mod& operator/=(Mod o) { return *this *= o.inv(); }
  friend Mod operator+(Mod a, Mod b) { return a += b; }
  friend Mod operator-(Mod a, Mod b) { return a -= b; }
  friend Mod operator*(Mod a, Mod b) { return a *= b; }
  friend Mod operator/(Mod a, Mod b) { return a /= b; }
  auto operator<=>(const Mod&) const = default;
  Mod pow(ll n) const {
    Mod a = x, b = 1;
    for (; n; n /= 2, a *= a) if (n & 1) b *= a;
    return b;
  }
  Mod inv() const { assert(x != 0); return pow(M - 2); }
};
using mint = Mod<998244353, 3>;
```

ModInverse.h

Description: Pre-computation of modular inverses. Assumes LIM $\leq$ mod and that mod is a prime.
6f684f, 3 lines

```
const ll mod = 1000000007, LIM = 200000;
ll* inv = new ll[LIM] - 1; inv[1] = 1;
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

ModLog.h

Description: Returns the smallest $x > 0$ s.t. $a^x = b \pmod m$, or -1 if no such x exists. modLog(a,1,m) can be used to calculate the order of a.
Time: $\mathcal{O}\left(\sqrt{m}\right)$
c040b8, 11 lines

```
ll modLog(ll a, ll b, ll m) {
  ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
  unordered_map<ll, ll> A;
  while (j <= n && (e = f = e * a % m) != b % m)
    A[e * b % m] = j++;
  if (e == b % m) return j;
  if (__gcd(m, e) == __gcd(m, b))
    rep(i,2,n+2) if (A.count(e = e * f % m))
      return n * i - A[e];
  return -1;
}
```

ModSum.h

Description: Sums of mod'ed arithmetic progressions.
modsum(to, c, k, m) $= \sum_{i=0}^{\text{to}-1} (ki + c)\%m$. divsum is similar but for floored division.
Time: $\log(m)$, with a large constant.
9c796e, 15 lines

```
ll sumsq(ll to) { return to / 2 * ((to-1) | 1); }

ll divsum(ll to, ll c, ll k, ll m) {
  ll res = k / m * sumsq(to) + c / m * to;
  k %= m; c %= m;
  if (!k) return res;
  ll to2 = (to * k + c) / m;
  return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}

ll modsum(ll to, ll c, ll k, ll m) {
  c = ((c % m) + m) % m;
  k = ((k % m) + m) % m;
  return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

ModMulLL.h

Description: Calculate $a \cdot b \bmod c$ (or $a^b \bmod c$).
Time: $\mathcal{O}(1)$ for modmul, $\mathcal{O}(\log b)$ for modpow
02ea06, 9 lines

```
ll modmul(ll a, ll b, ll M) {
  return (__int128)a * b % M;
}
ll modpow(ll b, ll e, ll mod) {
  ll ans = 1;
  for (; e; b = modmul(b, b, mod), e /= 2)
    if (e & 1) ans = modmul(ans, b, mod);
  return ans;
}
```

ModSqrt.h

Description: Tonelli-Shanks algorithm for modular square roots. Finds x s.t. $x^2 = a \pmod p$ ($-x$ gives the other solution).
Time: $\mathcal{O}\left(\log^2 p\right)$ worst case, $\mathcal{O}(\log p)$ for most p
"ModMulLL.h" b7cab4, 24 lines

```
ll sqrt(ll a, ll p) {
  a %= p; if (a < 0) a += p;
  if (a == 0) return 0;
  if (modpow(a, (p-1)/2, p) != 1) return -1; // no solution
  if (p % 4 == 3) return modpow(a, (p+1)/4, p);
  // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
  ll s = p - 1, n = 2;
  int r = 0, m;
```

```
  while (s % 2 == 0)
    ++r, s /= 2;
  while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;
  ll x = modpow(a, (s + 1) / 2, p);
  ll b = modpow(a, s, p), g = modpow(n, s, p);
  for (;; r = m) {
    ll t = b;
    for (m = 0; m < r && t != 1; ++m)
      t = t * t % p;
    if (m == 0) return x;
    ll gs = modpow(g, 1LL << (r - m - 1), p);
    g = gs * gs % p;
    x = x * gs % p;
    b = b * g % p;
  }
}
```

ModGen.h

Description: Finds a primitive root modulo p.

"Factor.h", "ModMulLL.h" ff3110, 8 lines

```
mt19937_64 rng(2137);
ll modGen(ll n) {
  map<ll, int> f; factor(n - 1, f); rep:
  ll g = rng() % (n - 1) + 1;
  for (auto [p, _] : f)
    if (modpow(g, (n - 1) / p, n) == 1) goto rep;
  return g;
}
```

ModArith.h

Description: Statistics on a mod'ed arithmetic sequence.

Time: $\mathcal{O}(\log m)$

"Euclid.h" 45f202, 32 lines

```
ll cdiv(ll x, ll y) { return x / y + ((x ^ y) > 0 && x % y); }

// min (ax + b) % m for 0 <= x <= n
ll minRemainder(ll a, ll b, ll m, ll n) {
  assert(a >= 0 && m > 0 && b >= 0 && n >= 0);
  a %= m, b %= m; n = min(n, m - 1);
  if (a == 0) return b;
  if (b >= a) {
    ll ad = cdiv(m - b, a);
    n -= ad; if (n < 0) return b;
    b += ad * a - m;
  }
  ll q = m / a, m2 = m % a;
  if (m2 == 0) return b;
  if (b / m2 > n / q) return b - n / q * m2;
  n -= b / m2 * q; b %= m2;
  ll y2 = (n * a + b) / m;
  ll x2 = cdiv(m2 * y2 - b, a);
  if (x2 * a - m2 * y2 + b >= m2) --x2;
  return minRemainder(a, b, m2, x2);
}

// min x >= 0 s.t. l <= (ax + b) % m <= r
ll minBetween(ll a, ll b, ll m, ll l, ll r) {
  ll x, y, g = euclid(a, m, x, y);
  if (g > 1)
    return minBetween(a/g,b/g,m/g,l/g+(l%g>b%g),r/g-(r%g<b%g));
  if (l > r) return -1; // no solution
  if ((x %= m) < 0) x += m;
  ll b2 = (l - b) * x % m;
  return minRemainder(x, b2 < 0 ? b2 + m : b2, m, r - l);
}
```

5.2 Primality

FastEratosthenes.h

Description: Prime sieve for generating all primes smaller than LIM.

Time: LIM=1e9 $\approx$ 1.5s

6b2912, 20 lines

```
const int LIM = 1e6;
bitset<LIM> isPrime;
vi eratosthenes() {
  const int S = (int)round(sqrt(LIM)), R = LIM / 2;
  vi pr = {2}, sieve(S+1); pr.reserve(int(LIM/log(LIM)*1.1));
  vector<pii> cp;
  for (int i = 3; i <= S; i += 2) if (!sieve[i]) {
    cp.push_back({i, i * i / 2});
    for (int j = i * i; j <= S; j += 2 * i) sieve[j] = 1;
  }
  for (int L = 1; L <= R; L += S) {
    array<bool, S> block{};
    for (auto &[p, idx] : cp)
      for (int i=idx; i < S+L; idx = (i+=p)) block[i-L] = 1;
    rep(i,0,min(S, R - L))
      if (!block[i]) pr.push_back((L + i) * 2 + 1);
  }
  for (int i : pr) isPrime[i] = 1;
  return pr;
}
```

MillerRabin.h

Description: Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to $7 \cdot 10^{18}$; for larger numbers, use Python and extend A randomly.

Time: 7 times the complexity of $a^b \mod c$.

"ModMulLL.h" 418a8d, 12 lines

```
bool isPrime(ll n) {
  if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
  ll A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},
     s = __builtin_ctzll(n-1), d = n >> s;
  for (ll a : A) {   // ^ count trailing zeroes
    ll p = modpow(a%n, d, n), i = s;
    while (p != 1 && p != n - 1 && a % n && i--)
      p = modmul(p, p, n);
    if (p != n-1 && i != s) return 0;
  }
  return 1;
}
```

Factor.h

Description: Pollard-rho randomized factorization algorithm.

Time: $\mathcal{O}\left(n^{1/4}\right)$, less for numbers with small factors.

"ModMulLL.h", "MillerRabin.h" 0750d1, 16 lines

```
ll pollard(ll n) {
  ll x = 0, y = 0, t = 30, prd = 2, i = 1, q;
  auto f = [&](ll k) { return modmul(k, k, n) + i; };
  while (t++ % 40 || __gcd(prd, n) == 1) {
    if (x == y) x = ++i, y = f(x);
    if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
    x = f(x), y = f(f(y));
  }
  return __gcd(prd, n);
}
void factor(ll n, map<ll, int>& cnt) {
  if (n == 1) return;
  if (isPrime(n)) { cnt[n]++; return; }
  ll x = pollard(n);
  factor(x, cnt); factor(n / x, cnt);
}
```

Min25.h

Description: Fast multiplicative function prefix sums. Requires `isPrime` calculated up to $\sqrt{n}$.

Time: $\mathcal{O}\left(\frac{n^{3/4}}{\log n}\right)$

"FastEratosthenes.h" c0b399, 47 lines

```
template<class T> struct Min25 {
  ll n, sq, s; vi p;
  Min25(ll _n) : n(_n) {
    sq = sqrtl(n) + 5;
    while (sq * sq > n) sq--;
    ll hls = quo(n, sq);
    while (hls != 1 && quo(n, hls - 1) == sq) hls--;
    s = hls + sq;
    rep(i, 2, sq + 1) if (isPrime[i]) p.push_back(i);
  }
  vector<T> sieve(auto f) {
    ll hls = s - sq;
    vector<T> h(s);
    rep(i, 1, hls) h[i] = f(quo(n, i)) - 1;
    rep(i, 1, sq + 1) h[s - i] = f(i) - 1;
    for (int x : p) {
      T xt = f(x) - f(x - 1), pi = h[s - x + 1];
      ll x2 = 1ll * x * x, imax = min(hls, quo(n, x2) + 1);
      for (ll i = 1, ix = x; i < imax; i++, ix += x)
        h[i] -= ((ix < hls ? h[ix] : h[s-quo(n, ix)])-pi) * xt;
      for (int i = sq; i >= x2; i--)
        h[s - i] -= (h[s - quo(i, x)] - pi) * xt;
    }
    return h;
  }
  vector<T> unsieve(const vector<T>& fp, auto f) {
    vector<ll> ns = {0};
    rep(i, 1, s - sq) ns.push_back(quo(n, i));
    rep(i, 0, sq) ns.push_back(sq - i);
    auto F = fp, G = fp;
    for (ll P : p | views::reverse) {
      for (ll pk = P, k = 1; quo(n, P) >= pk; k++, pk *= P) {
        T x = fp[idx(P)], y = f(P, k, pk), z = f(P, k+1, pk*P);
        rep(i, 1, s) {
          ll m = ns[i];
          if (P * pk > m) break;
          G[i] += y * (F[idx(quo(m, pk))] - x) + z;
        }
      }
      copy_n(G.begin(), min((int)s, idx(P*P) + 1), F.begin());
    }
    rep(i, 1, sz(ns)) F[i] += 1;
    return F;
  }
  ll quo(ll a, ll b) { return (double)a / b; }
  int idx(ll a) { return a <= sq ? s - a : quo(n, a); }
};
```

5.3 Divisibility

Euclid.h

Description: Finds two integers x and y, such that $ax + by = \gcd(a, b)$. If you just need gcd, use the built in __gcd instead. If a and b are coprime, then x is the inverse of $a \pmod b$.

33ba8f, 5 lines

```
ll euclid(ll a, ll b, ll &x, ll &y) {
  if (!b) return x = 1, y = 0, a;
  ll d = euclid(b, a % b, y, x);
  return y -= a/b * x, d;
}
```

CRT.h

Description: Chinese Remainder Theorem.
`crt(a, m, b, n)` computes x such that $x \equiv a \pmod m$, $x \equiv b \pmod n$. If $|a| < m$ and $|b| < n$, x will obey $0 \le x < \text{lcm}(m, n)$. Assumes $mn < 2^{62}$.
Time: $\log(n)$

"Euclid.h" 04d93a, 7 lines

```
ll crt(ll a, ll m, ll b, ll n) {
  if (n > m) swap(a, b), swap(m, n);
  ll x, y, g = euclid(m, n, x, y);
  assert((a - b) % g == 0); // else no solution
  x = (b - a) % n * x % n / g * m + a;
  return x < 0 ? x + m*n/g : x;
}
```

SameDiv.h

Description: Divides the interval $[1, \infty)$ into constant division intervals. For a significant speedup, get rid of v and do the calculations directly in the loop.
Time: $\mathcal{O}\left(\sqrt{n}\right)$

24617c, 13 lines

```
vector<ll> sameFloor(ll n) {
  vector<ll> v;
  for (ll i = 1; i <= n; i = n / (n / i) + 1) v.push_back(i);
  return v.push_back(n + 1), v;
}
vector<ll> sameCeil(ll n) {
  vector<ll> v;
  for (ll i = 1, j; i < n; i = (n + j - 2) / (j - 1)) {
    j = (n + i - 1) / i;
    v.push_back(i);
  }
  return v.push_back(n), v;
}
```

5.4 Fractions

ContinuedFractions.h

Description: Given N and a real number $x \ge 0$, finds the closest rational approximation p/q with $p, q \le N$. It will obey $|p/q - x| \le 1/qN$.
For consecutive convergents, $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$. ($p_k/q_k$ alternates between $> x$ and $< x$.) If x is rational, y eventually becomes ∞; if x is the root of a degree 2 polynomial the a's eventually become cyclic.
Time: $\mathcal{O}\left(\log N\right)$

dd6c5e, 21 lines

```
typedef double d; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(d x, ll N) {
  ll LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x;
  for (;;) {
    ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
       a = (ll)floor(y), b = min(a, lim),
       NP = b*P + LP, NQ = b*Q + LQ;
    if (a > b) {
      // If b > a/2, we have a semi-convergent that gives us a
      // better approximation; if b = a/2, we *may* have one.
      // Return {P, Q} here for a more canonical approximation.
      return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)) ?
        make_pair(NP, NQ) : make_pair(P, Q);
    }
    if (abs(y = 1/(y - (d)a)) > 3*N) {
      return {NP, NQ};
    }
    LP = P; P = NP;
    LQ = Q; Q = NQ;
  }
}
```

FracBinarySearch.h

Description: Given f and N, finds the smallest fraction $p/q \in [0, 1]$ such that $f(p/q)$ is true, and $p, q \le N$. You may want to throw an exception from f if it finds an exact solution, in which case N can be removed.
Usage: `fracBS([](Frac f) { return f.p>=3*f.q; }, 10); // {1,3}`
Time: $\mathcal{O}\left(\log(N)\right)$

27ab3e, 25 lines

```
struct Frac { ll p, q; };

template<class F>
Frac fracBS(F f, ll N) {
  bool dir = 1, A = 1, B = 1;
  Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
  if (f(lo)) return lo;
  assert(f(hi));
  while (A || B) {
    ll adv = 0, step = 1; // move hi if dir, else lo
    for (int si = 0; step; (step *= 2) >>= si) {
      adv += step;
      Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
      if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
        adv -= step; si = 2;
      }
    }
    hi.p += lo.p * adv;
    hi.q += lo.q * adv;
    dir = !dir;
    swap(lo, hi);
    A = B; B = !!adv;
  }
  return dir ? hi : lo;
}
```

Fraction.h

Description: Arithmetic on rational numbers. Consider using python or doubles.

52e650, 26 lines

```
template<class T>
struct Frac {
  typedef Frac F;
  T p, q;
  Frac(T x = 0) : Frac(x, 1) {}
  Frac(T x, T y) : p(x), q(y) {
    T g = __gcd(p, q); p /= g, q /= g;
    if (q < 0) p *= -1, q *= -1;
  }
  friend F abs(F f) { return F(abs(f.p), f.q); }
  F operator-() const { return F(-p, q); }
  friend F operator+(F a, F b){
    return F(a.p * b.q + b.p * a.q, a.q * b.q); }
  friend F operator-(F a, F b) {
    return F(a.p * b.q - b.p * a.q, a.q * b.q); }
  friend F operator*(F a, F b) { return F(a.p*b.p, a.q*b.q); }
  friend F operator/(F a, F b) { return F(a.p*b.q, a.q*b.p); }
  friend F& operator+=(F& a, F b) { return a = a + b; }
  friend F& operator-=(F& a, F b) { return a = a - b; }
  friend F& operator*=(F& a, F b) { return a = a * b; }
  friend F& operator/=(F& a, F b) { return a = a / b; }
  auto operator<=>(const F& f) const {
    return p * f.q - f.p * q <=> 0; }
  bool operator==(const F& f) const {
    return p == f.p && q == f.q; }
};
```

5.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2),\ \ b = k \cdot (2mn),\ \ c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

5.6 Primes

$p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

5.7 Estimates

$\sum_{d|n} d = O(n \log \log n)$.

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

5.8 Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$

$g(n) = \sum_{1 \le m \le n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \le m \le n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$

Combinatorial (6)

6.1 Permutations

6.1.1 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
n	11	12	13	14	15	16	17			
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
n	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

IntPerm.h
Description: Permutation -> integer conversion. (Not order preserving.) Integer -> permutation can use a lookup table.
Time: $\mathcal{O}(n)$
044568, 6 lines

```
int permToInt(vi& v) {
  int use = 0, i = 0, r = 0;
  for(int x:v) r = r * ++i + __builtin_popcount(use & -(1<<x)),
    use |= 1 << x;                    // (note: minus, not ~!)
  return r;
}
```

6.1.2 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

6.1.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor \frac{n!}{e} \right\rceil$$

6.1.4 Burnside's lemma

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).

If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k).$$

6.2 Partitions and subsets

6.2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1,\ p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1} p(n-k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$$\left(\sum_{n=0}^{\infty} p(n)x^n\right)^{-1} = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{n(3n+1)/2} + x^{n(3n-1)/2}\right)$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	∼2e5	∼2e8

6.2.2 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_k p^k + ... + n_1 p + n_0$ and $m = m_k p^k + ... + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod p$.

6.3 General purpose numbers

6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able). $B[0,\ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

6.3.2 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k),\ c(0,0) = 1$$
$$\textstyle\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\ldots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$

6.3.3 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n$$

6.3.4 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

6.3.5 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod p$$

6.3.6 Labeled unrooted trees

on n vertices: n^{n-2}
on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
with degrees d_i: $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$

6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2}C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

6.3.8 LGV lemma

Let $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$ be subsets of vertices of a DAG. By $\omega(P)$ denote a path weight, the product of edge weights in that path. Let $M_{i,j}$ be the sum of path weights over all possible paths from a_i to b_j (when unit weights, note this is the number of paths).

$$\det(M) = \sum_{(P_1,\ldots,P_n)\in S_\pi(A,B)} \operatorname{sgn}(\pi) \prod_{i=1}^{n} \omega(P_i)$$

Where $S_\pi(A, B)$ is the set of n-tuples of vertex disjoint paths (including endpoints) where the k-th path is from a_k to $b_{\pi(k)}$. Particularly useful when only identity permutation is possible.

6.4 Other

NimProduct.h

Description: Nim product.
Time: 64^2 xors per multiplication.

9bba25, 16 lines

```
using ull = uint64_t;
ull _nimProd2[64][64];
ull nimProd2(int i, int j) {
  if (_nimProd2[i][j]) return _nimProd2[i][j];
  if ((i & j) == 0) return _nimProd2[i][j] = 1ull << (i|j);
  int a = (i&j) & -(i&j);
  return _nimProd2[i][j] =
    nimProd2(i^a,j) ^ nimProd2((i^a)|(a-1),(j^a)|(i&(a-1)));
}
ull nimProd(ull x, ull y) {
  ull res = 0;
  for (int i = 0; (x >> i) && i < 64; i++) if ((x >> i) & 1)
    for (int j = 0; (y >> j) && j < 64; j++) if ((y >> j) & 1)
      res ^= nimProd2(i, j);
  return res;
}
```

MatroidIntersection.h

Description: Given two matroids, finds the largest common independent set. Pass the matroid with more expensive add/clear operations to M1.
Time: R^2N (M2.add + M1.check + M2.check) + R^3 M1.add + R^2 M1.clear + RN M2.clear, where R is the size of the largest independent set.

"../data-structures/UnionFind.h" 9812a7, 60 lines

```
struct ColorMat {
  vi cnt, clr;
  ColorMat(int n, vector<int> clr) : cnt(n), clr(clr) {}
  bool check(int x) { return !cnt[clr[x]]; }
  void add(int x) { cnt[clr[x]]++; }
  void clear() { fill(all(cnt), 0); }
};
struct GraphMat {
  UF uf;
  vector<array<int, 2>> e;
  GraphMat(int n, vector<array<int, 2>> e) : uf(n), e(e) {}
  bool check(int x) { return !uf.sameSet(e[x][0], e[x][1]); }
  void add(int x) { uf.join(e[x][0], e[x][1]); }
  void clear() { uf = UF(sz(uf.e)); }
};
template <class M1, class M2> struct MatroidIsect {
  int n;
  vector<char> iset;
  M1 m1; M2 m2;
  MatroidIsect(M1 m1, M2 m2, int n) : n(n), iset(n + 1), m1(m1)
      , m2(m2) {}
  vi solve() {
    rep(i,0,n) if (m1.check(i) && m2.check(i))
      iset[i] = true, m1.add(i), m2.add(i);
    while (augment());
    vi ans;
    rep(i,0,n) if (iset[i]) ans.push_back(i);
    return ans;
  }
  bool augment() {
    vector<int> frm(n, -1);
    queue<int> q({n}); // starts at dummy node
    auto fwdE = [&](int a) {
      vi ans;
      m1.clear();
      rep(v, 0, n) if (iset[v] && v != a) m1.add(v);
      rep(b, 0, n) if (!iset[b] && frm[b] == -1 && m1.check(b))
        ans.push_back(b), frm[b] = a;
      return ans;
    };
    auto backE = [&](int b) {
      m2.clear();
      rep(cas, 0, 2) rep(v, 0, n)
        if ((v == b || iset[v]) && (frm[v] == -1) == cas) {
          if (!m2.check(v))
            return cas ? q.push(v), frm[v] = b, v : -1;
          m2.add(v);
        }
      return n;
    };
    while (!q.empty()) {
      int a = q.front(), c; q.pop();
      for (int b : fwdE(a))
        while((c = backE(b)) >= 0) if (c == n) {
          while (b != n) iset[b] ^= 1, b = frm[b];
          return true;
        }
    }
    return false;
  }
};
```

DeBruijnSeq.h

Description: Constructs a cyclic string from the alphabet $[0, k)$ of length k^n that contains every length n string as a substring.

ae52d9, 13 lines

```
vi deBruijnSeq(int k, int n) {
  if (k == 1) return {0};
  vi seq, aux(n + 1);
  auto rec = [&](auto f, int t, int p) -> void {
    if (t > n) {
      if (n % p == 0) rep(i, 1, p + 1) seq.push_back(aux[i]);
    } else {
      aux[t] = aux[t - p]; f(f, t + 1, p);
      while (++aux[t] < k) f(f, t + 1, t);
    }
  };
  return rec(rec, 1, 1), seq;
}
```

Graph (7)

7.1 Network flow

Dinic.h

Description: Flow algorithm with complexity $O(VE \log U)$ where $U = \max|\text{cap}|$. $O(\min(E^{1/2}, V^{2/3})E)$ if $U = 1$; $O(\sqrt{V}E)$ for bipartite matching.

d7f0f1, 42 lines

```
struct Dinic {
  struct Edge {
    int to, rev;
    ll c, oc;
    ll flow() { return max(oc - c, 0LL); } // if you need flows
  };
  vi lvl, ptr, q;
  vector<vector<Edge>> adj;
  Dinic(int n) : lvl(n), ptr(n), q(n), adj(n) {}
  void addEdge(int a, int b, ll c, ll rcap = 0) {
    adj[a].push_back({b, sz(adj[b]), c, c});
    adj[b].push_back({a, sz(adj[a]) - 1, rcap, rcap});
  }
  ll dfs(int v, int t, ll f) {
    if (v == t || !f) return f;
    for (int& i = ptr[v]; i < sz(adj[v]); i++) {
      Edge& e = adj[v][i];
      if (lvl[e.to] == lvl[v] + 1)
        if (ll p = dfs(e.to, t, min(f, e.c))) {
          e.c -= p, adj[e.to][e.rev].c += p;
          return p;
        }
    }
    return 0;
  }
  ll calc(int s, int t) {
    ll flow = 0; q[0] = s;
    rep(L,0,31) do { // 'int L=30' maybe faster for random data
      lvl = ptr = vi(sz(q));
      int qi = 0, qe = lvl[s] = 1;
      while (qi < qe && !lvl[t]) {
        int v = q[qi++];
        for (Edge e : adj[v])
          if (!lvl[e.to] && e.c >> (30 - L))
            q[qe++] = e.to, lvl[e.to] = lvl[v] + 1;
      }
      while (ll p = dfs(s, t, LLONG_MAX)) flow += p;
    } while (lvl[t]);
    return flow;
  }
  bool leftOfMinCut(int a) { return lvl[a] != 0; }
};
```

PushRelabel.h

Description: Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only.
Time: $\mathcal{O}\left(V^2\sqrt{E}\right)$

0ae1d4, 48 lines

```
struct PushRelabel {
  struct Edge {
    int dest, back;
    ll f, c;
  };
  vector<vector<Edge>> g;
  vector<ll> ec;
  vector<Edge*> cur;
  vector<vi> hs; vi H;
```

```
  PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}

  void addEdge(int s, int t, ll cap, ll rcap=0) {
    if (s == t) return;
    g[s].push_back({t, sz(g[t]), 0, cap});
    g[t].push_back({s, sz(g[s])-1, 0, rcap});
  }

  void addFlow(Edge& e, ll f) {
    Edge &back = g[e.dest][e.back];
    if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
    e.f += f; e.c -= f; ec[e.dest] += f;
    back.f -= f; back.c += f; ec[back.dest] -= f;
  }
  ll calc(int s, int t) {
    int v = sz(g); H[s] = v; ec[t] = 1;
    vi co(2*v); co[0] = v-1;
    rep(i,0,v) cur[i] = g[i].data();
    for (Edge& e : g[s]) addFlow(e, e.c);

    for (int hi = 0;;) {
      while (hs[hi].empty()) if (!hi--) return -ec[s];
      int u = hs[hi].back(); hs[hi].pop_back();
      while (ec[u] > 0)  // discharge u
        if (cur[u] == g[u].data() + sz(g[u])) {
          H[u] = 1e9;
          for (Edge& e : g[u]) if (e.c && H[u] > H[e.dest]+1)
            H[u] = H[e.dest]+1, cur[u] = &e;
          if (++co[H[u]], !--co[hi] && hi < v)
            rep(i,0,v) if (hi < H[i] && H[i] < v)
              --co[H[i]], H[i] = v + 1;
          hi = H[u];
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
          addFlow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
    }
  }
  bool leftOfMinCut(int a) { return H[a] >= sz(g); }
};
```

MinCostMaxFlow.h

Description: Min-cost max-flow. If costs can be negative, call setpi before maxflow, but note that negative cost cycles are not supported. To obtain the actual flow, look at positive values only.
Time: $\mathcal{O}(FE\log(V))$ where F is max flow. $\mathcal{O}(VE)$ for setpi. f8c012, 79 lines

```
#include <ext/pb_ds/priority_queue.hpp>

const ll INF = numeric_limits<ll>::max() / 4;

struct MCMF {
  struct edge {
    int from, to, rev;
    ll cap, cost, flow;
  };
  int N;
  vector<vector<edge>> ed;
  vi seen;
  vector<ll> dist, pi;
  vector<edge*> par;

  MCMF(int N) : N(N), ed(N), seen(N), dist(N), pi(N), par(N) {}

  void addEdge(int from, int to, ll cap, ll cost) {
    if (from == to) return;
    ed[from].push_back(edge{ from,to,sz(ed[to]),cap,cost,0 });
    ed[to].push_back(edge{ to,from,sz(ed[from])-1,0,-cost,0 });
  }

  void path(int s) {
    fill(all(seen), 0);
    fill(all(dist), INF);
    dist[s] = 0; ll di;

    __gnu_pbds::priority_queue<pair<ll, int>> q;
    vector<decltype(q)::point_iterator> its(N);
    q.push({ 0, s });

    while (!q.empty()) {
      s = q.top().second; q.pop();
      seen[s] = 1; di = dist[s] + pi[s];
      for (edge& e : ed[s]) if (!seen[e.to]) {
        ll val = di - pi[e.to] + e.cost;
        if (e.cap - e.flow > 0 && val < dist[e.to]) {
          dist[e.to] = val;
          par[e.to] = &e;
          if (its[e.to] == q.end())
            its[e.to] = q.push({ -dist[e.to], e.to });
          else
            q.modify(its[e.to], { -dist[e.to], e.to });
        }
      }
    }
    rep(i,0,N) pi[i] = min(pi[i] + dist[i], INF);
  }

  pair<ll, ll> maxflow(int s, int t) {
    ll totflow = 0, totcost = 0;
    while (path(s), seen[t]) {
      ll fl = INF;
      for (edge* x = par[t]; x; x = par[x->from])
        fl = min(fl, x->cap - x->flow);

      totflow += fl;
      for (edge* x = par[t]; x; x = par[x->from]) {
        x->flow += fl;
        ed[x->to][x->rev].flow -= fl;
      }
    }
    rep(i,0,N) for(edge& e : ed[i]) totcost += e.cost * e.flow;
    return {totflow, totcost/2};
  }

  // If some costs can be negative, call this before maxflow:
  void setpi(int s) { // (otherwise, leave this out)
    fill(all(pi), INF); pi[s] = 0;
    int it = N, ch = 1; ll v;
    while (ch-- && it--)
      rep(i,0,N) if (pi[i] != INF)
        for (edge& e : ed[i]) if (e.cap)
          if ((v = pi[i] + e.cost) < pi[e.to])
            pi[e.to] = v, ch = 1;
    assert(it >= 0); // negative cost cycle
  }
};
```

MinCut.h

Description: After running max-flow, the left side of a min-cut from s to t is given by all vertices reachable from s, only traversing edges with positive residual capacity.

GlobalMinCut.h

Description: Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.
Time: $\mathcal{O}(V^3)$ 8b0e19, 21 lines

```
pair<int, vi> globalMinCut(vector<vi> mat) {
  pair<int, vi> best = {INT_MAX, {}};
  int n = sz(mat);
  vector<vi> co(n);
  rep(i,0,n) co[i] = {i};
  rep(ph,1,n) {
    vi w = mat[0];
    size_t s = 0, t = 0;
    rep(it,0,n-ph) { // O(V^2) -> O(E log V) with prio. queue
      w[t] = INT_MIN;
      s = t, t = max_element(all(w)) - w.begin();
      rep(i,0,n) w[i] += mat[t][i];
    }
    best = min(best, {w[t] - mat[t][t], co[t]});
    co[s].insert(co[s].end(), all(co[t]));
    rep(i,0,n) mat[s][i] += mat[t][i];
    rep(i,0,n) mat[i][s] = mat[s][i];
    mat[0][t] = INT_MIN;
  }
  return best;
}
```

GomoryHu.h

Description: Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.
Time: $\mathcal{O}(V)$ Flow Computations
"PushRelabel.h" 0418b3, 13 lines

```
typedef array<ll, 3> Edge;
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
  vector<Edge> tree;
  vi par(N);
  rep(i,1,N) {
    PushRelabel D(N); // Dinic also works
    for (Edge t : ed) D.addEdge(t[0], t[1], t[2], t[2]);
    tree.push_back({i, par[i], D.calc(i, par[i])});
    rep(j,i+1,N)
      if (par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i;
  }
  return tree;
}
```

7.2 Matching

Matching.h

Description: Fast biparite matching algorithm. Graph g should be a list of neighbors of the left partition. Returns the match for every left vertex.
Time: $\mathcal{O}\left(E\sqrt{V}\right)$ 797692, 19 lines

```
vi match(int n, int m, vector<vi>& g) {
  vi l(n, -1), r(m, -1), q(n), d(n);
  auto dfs = [&](auto f, int u) -> bool {
    int t = exchange(d[u], 0) + 1;
    for (int v : g[u])
      if (r[v] == -1 || (d[r[v]] == t && f(f, r[v])))
        return l[u] = v, r[v] = u, 1;
    return 0;
  };
  for (int t = 0, f = 0;; t = f = 0, d.assign(n, 0)) {
    rep(i, 0, n) if (l[i] == -1) q[t++] = i, d[i] = 1;
    rep(i, 0, t) for (int v : g[q[i]]) {
      if (r[v] == -1) f = 1;
      else if (!d[r[v]]) d[r[v]] = d[q[i]] + 1, q[t++] = r[v];
    }
    if (!f) return l;
    rep(i, 0, n) if (l[i] == -1) dfs(dfs, i);
  }
}
```

MinimumVertexCover.h

Description: Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

"DFSMatching.h" da4196, 20 lines

```
vi cover(vector<vi>& g, int n, int m) {
  vi match(m, -1);
  int res = dfsMatching(g, match);
  vector<bool> lfound(n, true), seen(m);
  for (int it : match) if (it != -1) lfound[it] = false;
  vi q, cover;
  rep(i,0,n) if (lfound[i]) q.push_back(i);
  while (!q.empty()) {
    int i = q.back(); q.pop_back();
    lfound[i] = 1;
    for (int e : g[i]) if (!seen[e] && match[e] != -1) {
      seen[e] = true;
      q.push_back(match[e]);
    }
  }
  rep(i,0,n) if (!lfound[i]) cover.push_back(i);
  rep(i,0,m) if (seen[i]) cover.push_back(n+i);
  assert(sz(cover) == res);
  return cover;
}
```

WeightedMatching.h

Description: Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost. Requires $N \leq M$.

Time: $\mathcal{O}\left(N^2M\right)$

b35132, 31 lines

```
pair<ll, vi> hungarian(const vector<vi> &a) {
  if (a.empty()) return {0, {}};
  int n = sz(a) + 1, m = sz(a[0]) + 1;
  vi p(m), ans(n - 1); vector<ll> u(n), v(m);
  rep(i,1,n) {
    p[0] = i;
    int j0 = 0; // add "dummy" worker 0
    vi pre(m, -1); vector<ll> dist(m, LLONG_MAX);
    vector<bool> done(m + 1);
    do { // dijkstra
      done[j0] = true;
      int i0 = p[j0], j1; ll delta = LLONG_MAX;
      rep(j,1,m) if (!done[j]) {
        ll cur = a[i0 - 1][j - 1] - u[i0] - v[j];
        if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
        if (dist[j] < delta) delta = dist[j], j1 = j;
      }
      rep(j,0,m) {
        if (done[j]) u[p[j]] += delta, v[j] -= delta;
        else dist[j] -= delta;
      }
      j0 = j1;
    } while (p[j0]);
    while (j0) { // update alternating path
      int j1 = pre[j0];
      p[j0] = p[j1], j0 = j1;
    }
  }
  rep(j,1,m) if (p[j]) ans[p[j] - 1] = j - 1;
  return {-v[0], ans}; // min cost
}
```

Blossom.h

Description: Matching for general graphs.

Time: $\mathcal{O}\left(nm\right)$, fast in practice

4e943d, 46 lines

```
vi blossom(vector<vi>& g) {
  int n = sz(g), t = -1;
  vi m(n, -1), l(n), p(n), o(n), b(n, -1), q;
  auto lca = [&](int x, int y) {
    for (t++; ; swap(x, y)) {
      if (x == -1) continue;
      if (b[x] == t) return x;
      b[x] = t;
      x = (m[x] == -1 ? -1 : o[p[m[x]]]);
    }
  };
  auto blossom = [&](int v, int w, int a) {
    while (o[v] != a) {
      p[v] = w; w = m[v];
      if (l[w] == 1) l[w] = 0, q.push_back(w);
      o[v] = o[w] = a; v = p[w];
    }
  };
  auto augment = [&](int v) {
    while (v != -1) {
      int pv = p[v], nv = m[pv];
      m[v] = pv; m[pv] = v; v = nv;
    }
  };
  auto bfs = [&](int r) {
    fill(all(l), -1); iota(all(o), 0); q.clear();
    l[r] = 0; q.push_back(r);
    rep(i, 0, sz(q)) {
      int v = q[i];
      for (auto x : g[v]) {
        if (l[x] == -1) {
          l[x] = 1; p[x] = v;
          if (m[x] == -1) return augment(x), 1;
          l[m[x]] = 0; q.push_back(m[x]);
        } else if (l[x] == 0 && o[v] != o[x]) {
          int a = lca(o[v], o[x]);
          blossom(x, v, a); blossom(v, x, a);
        }
      }
    }
    return 0;
  };
  // Time halves if you start with (any) maximal matching.
  rep(i, 0, n) if (m[i] == -1) bfs(i);
  return m;
}
```

WeightedBlossom.h

Description: General max weight matching. Edge weights must be positive.

Time: $\mathcal{O}\left(n^3\right)$, faster in practice

812025, 147 lines

```
template<int N> struct WeightedBlossom {
  struct Edge { int u,v,w; } g[N*2][N*2];
  int n,m,lab[N*2],match[N*2],slack[N*2],st[N*2];
  int par[N*2],floFrom[N*2][N],s[N*2],aux[N*2];
  vi flo[N*2]; queue<int> q;
  void init(int _n) { n = _n;
    rep(u,1,n+1) rep(v,1,n+1) g[u][v] = {u,v,0}; }
  void ae(int u, int v, int w) { u++, v++;
    g[u][v].w = g[v][u].w = max(g[u][v].w, w); }
  int eDelta(Edge e) {
    return lab[e.u]+lab[e.v]-g[e.u][e.v].w*2; }
  void updSlack(int u, int x) {
    if (!slack[x] || eDelta(g[u][x]) < eDelta(g[slack[x]][x]))
      slack[x] = u; }
  void setSlack(int x) {
    slack[x] = 0; rep(u,1,n+1) if (g[u][x].w > 0
     && st[u] != x && s[st[u]] == 0) updSlack(u,x); }
  void qPush(int x) {
    if (x <= n) q.push(x);
    else for (int t : flo[x]) qPush(t); }
  void setSt(int x, int b) {
    st[x] = b; if (x > n) for (int t : flo[x]) setSt(t,b); }
  int getPr(int b, int xr) {
    int pr = find(all(flo[b]),xr)-begin(flo[b]);
    if (pr&1) { reverse(1+all(flo[b])); return sz(flo[b])-pr; }
    return pr; }
  void setMatch(int u, int v) {
    Edge e = g[u][v]; match[u] = e.v; if (u <= n) return;
    int xr = floFrom[u][e.u], pr = getPr(u,xr);
    rep(i, 0, pr) setMatch(flo[u][i],flo[u][i^1]);
    setMatch(xr,v); rotate(begin(flo[u]),pr+all(flo[u])); }
  void augment(int u, int v) {
    while (1) {
      int xnv = st[match[u]]; setMatch(u,v);
      if (!xnv) return;
      setMatch(xnv,st[par[xnv]]);
      u = st[par[xnv]], v = xnv;
    }
  }
  int lca(int u, int v) {
    static int t = 0;
    for (++t;u||v;swap(u,v)) {
      if (!u) continue;
      if (aux[u] == t) return u;
      aux[u] = t; u = st[match[u]];
      if (u) u = st[par[u]];
    }
    return 0;
  }
  void addBlossom(int u, int anc, int v) {
    int b = n+1; while (b <= m && st[b]) ++b;
    if (b > m) ++m;
    lab[b] = s[b] = 0; match[b] = match[anc]; flo[b] = {anc};
    auto blossom = [&](int x) {
      for (int y; x != anc; x = st[par[y]])  {
        flo[b].push_back(x), flo[b].push_back(y=st[match[x]]);
        qPush(y);
      }
    };
    blossom(u); reverse(1+all(flo[b])); blossom(v); setSt(b,b);
    rep(x,1,m+1) g[b][x].w = g[x][b].w = 0;
    rep(x,1,n+1) floFrom[b][x] = 0;
    for(int xs : flo[b]) {
      rep(x,1,m+1) if (g[b][x].w == 0 || eDelta(g[xs][x]) <
        eDelta(g[b][x])) g[b][x]=g[xs][x], g[x][b]=g[x][xs];
      rep(x,1,n+1) if (floFrom[xs][x]) floFrom[b][x] = xs;
    }
    setSlack(b);
  }
  void expandBlossom(int b) {
    for (int t : flo[b]) setSt(t,t);
    int xr = floFrom[b][g[b][par[b]].u], pr = getPr(b,xr);
    for(int i = 0; i < pr; i += 2) {
      int xs = flo[b][i], xns = flo[b][i+1];
      par[xs] = g[xns][xs].u; s[xs] = 1;
      s[xns] = slack[xs] = slack[xns] = 0; qPush(xns);
    }
    s[xr] = 1, par[xr] = par[b];
    rep(i,pr+1,sz(flo[b])) {
      int xs = flo[b][i]; s[xs] = -1, setSlack(xs); }
    st[b] = 0;
  }
  bool onFoundEdge(Edge e) {
    int u = st[e.u], v = st[e.v];
```

```
    if (s[v] == -1) {
      par[v] = e.u, s[v] = 1; slack[v] = 0;
      int nu = st[match[v]]; s[nu] = slack[nu] = 0; qPush(nu);
    } else if (s[v] == 0) {
      int anc = lca(u,v);
      if (!anc) return augment(u,v),augment(v,u),1;
      addBlossom(u,anc,v);
    }
    return 0;
  }
  bool matching() {
    q = queue<int>();
    rep(x,1,m+1) {
      s[x] = -1, slack[x] = 0;
      if (st[x] == x && !match[x]) par[x] = s[x] = 0, qPush(x);
    }
    if (!sz(q)) return 0;
    while (1) {
      while (sz(q)) {
        int u=q.front(); q.pop(); if (s[st[u]] == 1) continue;
        rep(v,1,n+1) if (g[u][v].w > 0 && st[u] != st[v]) {
          if (eDelta(g[u][v]) == 0) {
            if (onFoundEdge(g[u][v])) return 1;
          } else updSlack(u,st[v]);
        }
      }
      int d = INT_MAX;
      rep(b,n+1,m+1) if (st[b] == b && s[b] == 1)
        d = min(d,lab[b]/2);
      rep(x,1,m+1) if (st[x] == x && slack[x]) {
        if (s[x] == -1) d=min(d,eDelta(g[slack[x]][x]));
        else if (s[x] == 0) d=min(d,eDelta(g[slack[x]][x])/2);
      }
      rep(u,1,n+1) {
        if (s[st[u]] == 0) {
          if (lab[u] <= d) return 0;
          lab[u] -= d;
        } else if (s[st[u]] == 1) lab[u] += d;
      }
      rep(b,n+1,m+1) if (st[b] == b && s[b] != -1)
        lab[b] += (s[b] == 0 ? 1 : -1)*d*2;
      q = queue<int>();
      rep(x,1,m+1) if (st[x]==x && slack[x]
        && st[slack[x]] != x && eDelta(g[slack[x]][x]) == 0)
          if (onFoundEdge(g[slack[x]][x])) return 1;
      rep(b,n+1,m+1) if (st[b]==b && s[b]==1 && lab[b]==0)
        expandBlossom(b);
    }
    return 0;
  }
  pair<ll, vi> calc() {
    m = n; st[0] = 0; rep(i,1,2*n+1) aux[i] = 0;
    rep(i,1,n+1) match[i] = 0, st[i] = i, flo[i].clear();
    int wMax = 0;
    rep(u,1,n+1) rep(v,1,n+1)
      floFrom[u][v] = (u==v ? u : 0), wMax=max(wMax,g[u][v].w);
    rep(u,1,n+1) lab[u] = wMax;
    ll w = 0; vi mt(n, -1); while (matching());
    rep(i,1,n+1) if (match[i])
      w += g[i][match[i]].w, mt[i-1] = match[i]-1;
    return {w/2,mt};
  }
};
```

7.3 DFS algorithms

SCC.h

Description: Finds strongly connected components in a directed graph. If vertices u, v belong to the same component, we can reach u from v and vice versa.
Usage: `scc(graph, [&](vi& v) { ... })` visits all components in reverse topological order. `comp[i]` holds the component index of a node (a component only has edges to components with lower index). `ncomps` will contain the number of components.
Time: $\mathcal{O}(E+V)$ 76b5c9, 24 lines

```
vi val, comp, z, cont;
int Time, ncomps;
template<class G, class F> int dfs(int j, G& g, F& f) {
  int low = val[j] = ++Time, x; z.push_back(j);
  for (auto e : g[j]) if (comp[e] < 0)
    low = min(low, val[e] ?: dfs(e,g,f));

  if (low == val[j]) {
    do {
      x = z.back(); z.pop_back();
      comp[x] = ncomps;
      cont.push_back(x);
    } while (x != j);
    f(cont); cont.clear();
    ncomps++;
  }
  return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
  int n = sz(g);
  val.assign(n, 0); comp.assign(n, -1);
  Time = ncomps = 0;
  rep(i,0,n) if (comp[i] < 0) dfs(i, g, f);
}
```

BiconnectedComponents.h

Description: Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.
Usage: `int eid = 0; ed.resize(N);`
`for each edge (a,b) {`
`ed[a].emplace_back(b, eid);`
`ed[b].emplace_back(a, eid++); }`
`bicomps([&](const vi& edgelist) {...});`
Time: $\mathcal{O}(E+V)$ c6b7c7, 32 lines

```
vi num, st;
vector<vector<pii>> ed;
int Time;
template<class F>
int dfs(int at, int par, F& f) {
  int me = num[at] = ++Time, top = me;
  for (auto [y, e] : ed[at]) if (e != par) {
    if (num[y]) {
      top = min(top, num[y]);
      if (num[y] < me)
        st.push_back(e);
    } else {
      int si = sz(st);
      int up = dfs(y, e, f);
      top = min(top, up);
      if (up == me) {
        st.push_back(e);
        f(vi(st.begin() + si, st.end()));
        st.resize(si);
      }
      else if (up < me) st.push_back(e);
      else { /* e is a bridge */ }
    }
  }
  return top;
}

template<class F>
void bicomps(F f) {
  num.assign(sz(ed), 0);
  rep(i,0,sz(ed)) if (!num[i]) dfs(i, -1, f);
}
```

2sat.h

Description: Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type $(a||b)\&\&(!a||c)\&\&(d||!b)\&\&...$ becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions (~x).
Usage:
```
TwoSat ts(number of boolean variables);
ts.either(0, ~3); // Var 0 is true or var 3 is false
ts.setValue(2); // Var 2 is true
ts.atMostOne({0,~1,2}); // <= 1 of vars 0, ~1 and 2 are true
ts.solve(); // Returns true iff it is solvable
ts.values[0..N-1] holds the assigned values to the vars
```
Time: $\mathcal{O}(N+E)$, where N is the number of boolean variables, and E is the number of clauses. 5f9706, 56 lines

```
struct TwoSat {
  int N;
  vector<vi> gr;
  vi values; // 0 = false, 1 = true

  TwoSat(int n = 0) : N(n), gr(2*n) {}

  int addVar() { // (optional)
    gr.emplace_back();
    gr.emplace_back();
    return N++;
  }

  void either(int f, int j) {
    f = max(2*f, -1-2*f);
    j = max(2*j, -1-2*j);
    gr[f].push_back(j^1);
    gr[j].push_back(f^1);
  }
  void setValue(int x) { either(x, x); }

  void atMostOne(const vi& li) { // (optional)
    if (sz(li) <= 1) return;
    int cur = ~li[0];
    rep(i,2,sz(li)) {
      int next = addVar();
      either(cur, ~li[i]);
      either(cur, next);
      either(~li[i], next);
      cur = ~next;
    }
    either(cur, ~li[1]);
  }

  vi val, comp, z; int time = 0;
  int dfs(int i) {
    int low = val[i] = ++time, x; z.push_back(i);
    for(int e : gr[i]) if (!comp[e])
      low = min(low, val[e] ?: dfs(e));
    if (low == val[i]) do {
      x = z.back(); z.pop_back();
```

```
      comp[x] = low;
      if (values[x>>1] == -1)
        values[x>>1] = x&1;
    } while (x != i);
    return val[i] = low;
  }

  bool solve() {
    values.assign(N, -1);
    val.assign(2*N, 0); comp = val;
    rep(i,0,2*N) if (!comp[i]) dfs(i);
    rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
    return 1;
  }
};
```

EulerWalk.h

Description: Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of pairs (node, edge) in the Eulerian path/cycle with `src` at the start, or empty list if no cycle/path exists.
Time: $\mathcal{O}(V+E)$

428fec, 16 lines

```
vector<pii> eulerWalk(vector<vector<pii>>&g,int m,int src=0) {
  int n = sz(g);
  vi D(n), its(n), eu(m); vector<pii> ret, s = {{src, -1}};
  D[src]++; // to allow Euler paths, not just cycles
  while (!s.empty()) {
    int x = s.back().first, y, e, &it = its[x], end = sz(g[x]);
    if(it == end) {
      ret.push_back(s.back()); s.pop_back(); continue; }
    tie(y, e) = g[x][it++];
    if (!eu[e]) {
      D[x]--, D[y]++;
      eu[e] = 1; s.push_back({y, e});
    }}
  for (int x : D) if (x < 0 || sz(ret) != m+1) return {};
  return {ret.rbegin(), ret.rend()};
}
```

PlanarFaces.h

Description: Finds the faces of a simple planar graph and returns the vertex indices for each face in either clockwise (inner) or counterclockwise (outer) order. Disconnected graphs may have multiple outer faces and require careful handling.
Time: $\mathcal{O}(n \log n)$

"../geometry/Point.h", "../geometry/AngleCmp.h" 2c9685, 23 lines

```
template<class P>
vector<vi> planarFaces(vector<vi>& g, vector<P>& p) {
  int n = sz(g); P o;
  auto cmp = [&](int x,int y){return angleCmp(p[x]-o,p[y]-o);};
  vector<vi> vis(n);
  rep(i, 0, n) {
    o = p[i], sort(all(g[i]), cmp);
    vis[i].resize(sz(g[i]));
  }
  vector<vi> f;
  rep(i, 0, n) rep(j, 0, sz(adj[i])) {
    if (vis[i][j]) continue;
    vi s; int u = i, k = j;
    while (!vis[u][k]) {
      vis[u][k] = 1; s.push_back(u);
      int v = adj[u][k]; o = p[v];
      int kk = lower_bound(all(g[v]), u, cmp) - g[v].begin();
      u = v, k = (kk + 1) % sz(adj[u]);
    }
    f.push_back(s);
  }
  return f;
}
```

7.4 Coloring

EdgeColoring.h

Description: Given a simple, undirected graph with max degree D, computes a $(D+1)$-coloring of the edges such that no neighboring edges share a color. (D-coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)
Time: $\mathcal{O}(NM)$

e210e2, 31 lines

```
vi edgeColoring(int N, vector<pii> eds) {
  vi cc(N + 1), ret(sz(eds)), fan(N), free(N), loc;
  for (pii e : eds) ++cc[e.first], ++cc[e.second];
  int u, v, ncols = *max_element(all(cc)) + 1;
  vector<vi> adj(N, vi(ncols, -1));
  for (pii e : eds) {
    tie(u, v) = e;
    fan[0] = v;
    loc.assign(ncols, 0);
    int at = u, end = u, d, c = free[u], ind = 0, i = 0;
    while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
      loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
    cc[loc[d]] = c;
    for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
      swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
    while (adj[fan[i]][d] != -1) {
      int left = fan[i], right = fan[++i], e = cc[i];
      adj[u][e] = left;
      adj[left][e] = u;
      adj[right][e] = -1;
      free[right] = e;
    }
    adj[u][d] = fan[i];
    adj[fan[i]][d] = u;
    for (int y : {fan[0], u, end})
      for (int& z = free[y] = 0; adj[y][z] != -1; z++);
  }
  rep(i,0,sz(eds))
    for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
  return ret;
}
```

ChromaticNumber.h

Description: Fast computation of chromatic number.
Time: $\mathcal{O}(n2^n)$

b21806, 19 lines

```
const int MOD = 1034865179; // random large prime
int chromaticNumber(vi& g) {
  int n = sz(g);
  vi dp(1 << n), f(n); dp[0] = 1;
  rep(i, 1, 1 << n) {
    int ctz = __builtin_ctz(i), j = i - (1 << ctz);
    dp[i] = dp[j] + dp[j & ~g[ctz]];
    if (dp[i] >= MOD) dp[i] -= MOD;
  }
  rep(i, 0, 1 << n) {
    ll x = (n - __builtin_parity(i)) & 1 ? MOD - 1 : 1;
    rep(j, 0, n) {
      if ((f[j] += x) >= MOD) f[j] -= MOD;
      x = x * dp[i] % MOD;
    }
  }
  rep(i, 0, n) if (f[i]) return i;
  return n;
}
```

7.5 Heuristics

MaximalCliques.h

Description: Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.
Time: $\mathcal{O}\left(3^{n/3}\right)$, much faster for sparse graphs

b0d5b1, 12 lines

```
typedef bitset<128> B;
template<class F>
void cliques(vector<B>& eds, F f, B P = ~B(), B X={}, B R={}) {
  if (!P.any()) { if (!X.any()) f(R); return; }
  auto q = (P | X)._Find_first();
  auto cands = P & ~eds[q];
  rep(i,0,sz(eds)) if (cands[i]) {
    R[i] = 1;
    cliques(eds, f, P & eds[i], X & eds[i], R);
    R[i] = P[i] = 0; X[i] = 1;
  }
}
```

MaximumClique.h

Description: Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.
Time: Runs in about 1s for n=155 and worst case random graphs (p=.90). Runs faster for sparse graphs.

f7c0bc, 49 lines

```
typedef vector<bitset<200>> vb;
struct Maxclique {
  double limit=0.025, pk=0;
  struct Vertex { int i, d=0; };
  typedef vector<Vertex> vv;
  vb e;
  vv V;
  vector<vi> C;
  vi qmax, q, S, old;
  void init(vv& r) {
    for (auto& v : r) v.d = 0;
    for (auto& v : r) for (auto j : r) v.d += e[v.i][j.i];
    sort(all(r), [](auto a, auto b) { return a.d > b.d; });
    int mxD = r[0].d;
    rep(i,0,sz(r)) r[i].d = min(i, mxD) + 1;
  }
  void expand(vv& R, int lev = 1) {
    S[lev] += S[lev - 1] - old[lev];
    old[lev] = S[lev - 1];
    while (sz(R)) {
      if (sz(q) + R.back().d <= sz(qmax)) return;
      q.push_back(R.back().i);
      vv T;
      for(auto v:R) if (e[R.back().i][v.i]) T.push_back({v.i});
      if (sz(T)) {
        if (S[lev]++ / ++pk < limit) init(T);
        int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1, 1);
        C[1].clear(), C[2].clear();
        for (auto v : T) {
          int k = 1;
          auto f = [&](int i) { return e[v.i][i]; };
          while (any_of(all(C[k]), f)) k++;
          if (k > mxk) mxk = k, C[mxk + 1].clear();
          if (k < mnk) T[j++].i = v.i;
          C[k].push_back(v.i);
        }
        if (j > 0) T[j - 1].d = 0;
        rep(k,mnk,mxk + 1) for (int i : C[k])
          T[j].i = i, T[j++].d = k;
        expand(T, lev + 1);
      } else if (sz(q) > sz(qmax)) qmax = q;
```

```
      q.pop_back(), R.pop_back();
    }
  }
  vi maxClique() { init(V), expand(V); return qmax; }
  Maxclique(vb conn) : e(conn), C(sz(e)+1), S(sz(C)), old(S) {
    rep(i,0,sz(e)) V.push_back({i});
  }
};
```

MaximumIndependentSet.h

Description: To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertexCover.

7.6 Trees

HLD.h

Description: Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most log(n) light edges. Code does additive modifications and max queries, but can support commutative segtree modifications/queries on paths and subtrees. Takes as input the full adjacency list. VALS_EDGES being true means that values are stored in the edges, as opposed to the nodes. All values initialized to the segtree default. Root must be 0.

Time: $\mathcal{O}\left((\log N)^2\right)$

"../data-structures/LazySegmentTree.h" 03139d, 46 lines

```
template <bool VALS_EDGES> struct HLD {
  int N, tim = 0;
  vector<vi> adj;
  vi par, siz, rt, pos;
  Node *tree;
  HLD(vector<vi> adj_)
    : N(sz(adj_)), adj(adj_), par(N, -1), siz(N, 1),
      rt(N),pos(N),tree(new Node(0, N)){ dfsSz(0); dfsHld(0); }
  void dfsSz(int v) {
    if (par[v] != -1) adj[v].erase(find(all(adj[v]), par[v]));
    for (int& u : adj[v]) {
      par[u] = v;
      dfsSz(u);
      siz[v] += siz[u];
      if (siz[u] > siz[adj[v][0]]) swap(u, adj[v][0]);
    }
  }
  void dfsHld(int v) {
    pos[v] = tim++;
    for (int u : adj[v]) {
      rt[u] = (u == adj[v][0] ? rt[v] : u);
      dfsHld(u);
    }
  }
  template <class B> void process(int u, int v, B op) {
    for (; rt[u] != rt[v]; v = par[rt[v]]) {
      if (pos[rt[u]] > pos[rt[v]]) swap(u, v);
      op(pos[rt[v]], pos[v] + 1);
    }
    if (pos[u] > pos[v]) swap(u, v);
    op(pos[u] + VALS_EDGES, pos[v] + 1);
  }
  void modifyPath(int u, int v, int val) {
    process(u, v, [&](int l, int r) { tree->add(l, r, val); });
  }
  int queryPath(int u, int v) { // Modify depending on problem
    int res = -1e9;
    process(u, v, [&](int l, int r) {
        res = max(res, tree->query(l, r));
    });
    return res;
  }
  int querySubtree(int v) { // modifySubtree is similar
    return tree->query(pos[v] + VALS_EDGES, pos[v] + siz[v]);
  }
};
```

RerootDP.h

Description: Calculates a DP from every root in a tree.

Time: $\mathcal{O}\left(\sum d \log d\right)$

c0a8b6, 40 lines

```
struct S {
  void init(int u) {}
  void join(int u, int i, const S& c) {}
  void push(int u, int i) {} // i = -1 if root
};
vector<S> reroot(vector<vi>& g) {
  int n = sz(g), t = 1;
  vi q(n), p(n);
  for (int u : q) for (int v : g[u]) if (p[u] != v) {
    p[v] = u, q[t++] = v;
  }
  vector<S> dp(n), rdp(n), ans(n);
  for (int i = n - 1; i >= 0; i--) {
    int u = q[i], k = -1;
    dp[u].init(u);
    rep(j, 0, sz(g[u])) {
      if (g[u][j] != p[u]) dp[u].join(u, j, dp[g[u][j]]);
      else k = j;
    }
    ans[u] = dp[u], dp[u].push(u, k);
  }
  if (n == 1) return dp;
  for (int u : q) {
    int d = sz(g[u]); vector<S> e(d);
    rep(i, 0, d) e[i].init(u);
    for (int b = __lg(d); b >= 0; b--) {
      for (int i = d - 1; i >= 0; i--) e[i] = e[i / 2];
      rep(i, 0, d - (d & !b)) {
        S& s = g[u][i] != p[u] ? dp[g[u][i]] : rdp[u];
        e[(i >> b) ^ 1].join(u, i, s);
      }
    }
    rep(i, 0, sz(g[u])) {
      if (p[u] != g[u][i]) (rdp[g[u][i]] = e[i]).push(u, i);
      else ans[u].join(u, i, rdp[u]);
    }
    ans[u].push(u, -1);
  }
  return ans;
}
```

7.7 Advanced

DirectedMST.h

Description: Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

Time: $\mathcal{O}\left(E \log V\right)$

"../data-structures/UnionFindRollback.h" 39e620, 60 lines

```
struct Edge { int a, b; ll w; };
struct Node {
  Edge key;
  Node *l, *r;
  ll delta;
  void prop() {
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }
  Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
  if (!a || !b) return a ?: b;
  a->prop(), b->prop();
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }

pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
  RollbackUF uf(n);
  vector<Node*> heap(n);
  for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node{e});
  ll res = 0;
  vi seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, {-1,-1}), comp;
  deque<tuple<int, int, vector<Edge>>> cycs;
  rep(s,0,n) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      if (!heap[u]) return {-1,{}};
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
        Node* cyc = 0;
        int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.join(u, w));
        u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cycs.push_front({u, time, {&Q[qi], &Q[end]}});
      }
    }
    rep(i,0,qi) in[uf.find(Q[i].b)] = Q[i];
  }

  for (auto& [u,t,comp] : cycs) { // restore sol (optional)
    uf.rollback(t);
    Edge inEdge = in[u];
    for (auto& e : comp) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;
  }
  rep(i,0,n) par[i] = in[i].a;
  return {res, par};
}
```

KthWalk.h

Description: Eppstein's algorithm for the k-th shortest walk in a directed graph with non-negative edge weights.

Memory: $\mathcal{O}\left((n+m)\log n + k\right)$

Time: $\mathcal{O}\left((n+m)\log n + k \log k\right)$

f7b9b0, 53 lines

```
struct KthWalk {
  using Edge = pair<int, ll>;
  struct Node { // persistent leftist heap node
    Node *l = 0, *r = 0;
    int s; Edge e;
    Node(Edge _e) : e(_e) {}
  };
  ll d0;
  priority_queue<pair<ll, Node*>> q;
  vector<Node*> h;
  KthWalk(vector<vector<Edge>>& g, int s, int t) {
```

```
    int n = sz(g); vector<vector<Edge>> r(n);
    rep(i, 0, n) for (auto [j,w] : g[i]) r[j].push_back({i,w});
    vector<ll> d(n, LLONG_MAX);
    vi ord, p(n, -1);
    priority_queue<pair<ll, int>> pq;
    pq.push({d[t] = 0, t});
    while (sz(pq)) {
      auto [dd, u] = pq.top(); pq.pop();
      if (d[u] != -dd) continue;
      ord.push_back(u);
      for (auto [v, w] : r[u]) if (d[u] + w < d[v])
        pq.push({-(d[v] = d[u] + w), v}), p[v] = u;
    }
    if ((d0 = d[s]) == LLONG_MAX) return;
    h.resize(n);
    for (int u : ord) {
      int pp = p[u]; if (pp != -1) h[u] = h[pp];
      for (auto [v, w] : g[u]) if (d[v] != LLONG_MAX) {
        ll x = w + d[v] - d[u];
        if (x || v != pp) h[u] = merge(h[u], new Node({v, x}));
        else pp = -1;
      }
    }
    q.push({0, new Node({s, 0})});
  }
  Node* merge(Node* a, Node* b) {
    if (!a || !b) return a ?: b;
    if(a->e.second > b->e.second) swap(a, b);
    Node* c = new Node(*a); c->r = merge(c->r, b);
    if (!c->l || c->l->s < c->r->s) swap(c->l, c->r);
    c->s = (c->r ? c->r->s : 0) + 1; return c;
  }
  ll next() { // -1 if no path
    if (!sz(q)) return -1;
    auto [d, a] = q.top(); q.pop();
    if (a->l) q.push({d - a->l->e.second + a->e.second, a->l});
    if (a->r) q.push({d - a->r->e.second + a->e.second, a->r});
    Node* t = h[a->e.first];
    if (t) q.push({d - t->e.second, t});
    return d0 - d;
  }
};
```

DominatorTree.h

Description: Finds the parent of each vertex in the dominator tree of g. Vertex a dominates vertex b iff every path from *src* to b passes through a.
Time: $\mathcal{O}(m \log n)$

08d90c, 35 lines

```
vi dominatorTree(vector<vi>& g, int src) {
  int n = sz(g), tt = 0;
  vi ans(n, -1), t(n, -1), rt(n), s(n), p(n), d(n), b(n);
  vector<vi> c(n), r(n), sc(n);
  auto get = [&](auto f, int u) -> int {
    if (p[u] != u) {
      int v = f(f, p[u]); p[u] = p[p[u]];
      if (s[v] < s[b[u]]) b[u] = v;
    }
    return b[u];
  };
  auto dfs = [&](auto f, int u) -> void {
    t[u] = tt, rt[tt] = u;
    s[tt] = p[tt] = b[tt] = tt; tt++;
    for (int v : g[u]) {
      if (t[v] == -1) f(f, v), c[t[u]].push_back(t[v]);
      r[t[v]].push_back(t[u]);
    }
  };
  dfs(dfs, src);
  for (int i = tt - 1; i >= 0; i--) {
    for (int j : r[i]) s[i] = min(s[i], s[get(get, j)]);
    if (i) sc[s[i]].push_back(i);
    for (int j : sc[i]) {
      int k = get(get, j);
      d[j] = s[j] == s[k] ? s[j] : k;
    }
    for (int j : c[i]) p[j] = i;
  }
  rep(i, 1, tt) {
    if (d[i] != s[i]) d[i] = d[d[i]];
    ans[rt[i]] = rt[d[i]];
  }
  return ans;
}
```

7.8 Math

7.8.1 Matrix tree theorem

Create an $N \times N$ matrix `mat`, and for each edge $a \to b \in G$, do `mat[a][b]--`, `mat[b][b]++` (and `mat[b][a]--`, `mat[a][a]++` if G is undirected). Remove the ith row and column and take the determinant; this yields the number of directed spanning trees rooted at i (if G is undirected, remove any row/-column).

7.8.2 Erdős-Gallai theorem

A simple graph with node degrees $d_1 \geq \cdots \geq d_n$ exists iff $d_1 + \cdots + d_n$ is even and for every $k = 1 \ldots n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

7.8.3 Gale-Ryser theorem

A simple bipartite graph with degree sequences $a_1 \geq \cdots \geq a_n$ and $b_1, \ldots, b_m$ exists iff $\sum a_i = \sum b_i$ and for every $1 \leq k \leq n$

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{m} \min(b_i, k).$$

7.8.4 BEST theorem

The number of Eulerian circuits on an Eulerian graph equals

$$t(v) \prod_{u} (\deg(u) - 1)!$$

where $t(v)$ is the number of spanning trees directed towards an arbitrary root v, and deg(u) is the outdegree of vertex u.

Geometry (8)

8.1 Geometric primitives

Point.h

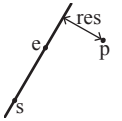
Description: Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

3e64f3, 26 lines

```
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
  typedef Point P;
  T x, y;
  auto operator<=>(const P&) const = default;
  P operator+(P p) const { return P(x+p.x, y+p.y); }
  P operator-(P p) const { return P(x-p.x, y-p.y); }
  P operator*(T d) const { return P(x*d, y*d); }
  P operator/(T d) const { return P(x/d, y/d); }
  T dot(P p) const { return x*p.x + y*p.y; }
  T cross(P p) const { return x*p.y - y*p.x; }
  T cross(P a, P b) const { return (a-*this).cross(b-*this); }
  T dist2() const { return x*x + y*y; }
  double dist() const { return sqrt((double)dist2()); }
  // angle to x-axis in interval [-pi, pi]
  double angle() const { return atan2(y, x); }
  P unit() const { return *this/dist(); } // makes dist()=1
  P perp() const { return P(-y, x); } // rotates +90 degrees
  P normal() const { return perp().unit(); }
  // returns point rotated 'a' radians ccw around the origin
  P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
  friend ostream& operator<<(ostream& os, P p) {
    return os << "(" << p.x << "," << p.y << ")"; }
};
```

LineDist.h

Description:
Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.

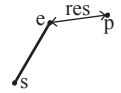


"Point.h" f6bf6b, 4 lines

```
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
  return (double)(b-a).cross(p-a)/(b-a).dist();
}
```

SegDist.h

Description:
Returns the shortest distance between point p and the line segment from point s to e.
Usage: `Point<double> a, b(2,2), p(1,1);`
`bool onSegment = segDist(a,b,p) < 1e-10;`

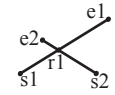


"Point.h" 5c88f4, 6 lines

```
typedef Point<double> P;
double segDist(P& s, P& e, P& p) {
  if (s==e) return (p-s).dist();
  auto d = (e-s).dist2(), t = min(d,max(.0,(p-s).dot(e-s)));
  return ((p-s)*d-(e-s)*t).dist()/d;
}
```

SegInter.h

Description:
If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.



Usage: `vector<P> inter = segInter(s1,e1,s2,e2);`

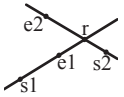
```
if (sz(inter)==1)
cout << "segments intersect at " << inter[0] << endl;
```

"Point.h", "OnSegment.h" 9d57f2, 13 lines

```
template<class P> vector<P> segInter(P a, P b, P c, P d) {
  auto oa = c.cross(d, a), ob = c.cross(d, b),
       oc = a.cross(b, c), od = a.cross(b, d);
  // Checks if intersection is single non-endpoint point.
  if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
    return {(a * ob - b * oa) / (ob - oa)};
  set<P> s;
  if (onSegment(c, d, a)) s.insert(a);
  if (onSegment(c, d, b)) s.insert(b);
  if (onSegment(a, b, c)) s.insert(c);
  if (onSegment(a, b, d)) s.insert(d);
  return {all(s)};
}
```

LineInter.h

Description:
If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.



Usage: `auto res = lineInter(s1,e1,s2,e2);`

```
if (res.first == 1)
cout << "intersection point at " << res.second << endl;
```

"Point.h" a01f81, 8 lines

```
template<class P>
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
  auto d = (e1 - s1).cross(e2 - s2);
  if (d == 0) // if parallel
    return {-(s1.cross(e1, s2) == 0), P(0, 0)};
  auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
  return {1, (s1 * p + e1 * q) / d};
}
```

SideOf.h

Description: Returns where p is as seen from s towards e. 1/0/-1 $\Leftrightarrow$ left/on line/right. If the optional argument *eps* is given 0 is returned if p is within distance *eps* from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.

Usage: `bool left = sideOf(p1,p2,q)==1;`

"Point.h" 3af81c, 9 lines

```
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }

template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
  auto a = (e-s).cross(p-s);
  double l = (e-s).dist()*eps;
  return (a > l) - (a < -l);
}
```

OnSegment.h

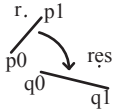
Description: Returns true iff p lies on the line segment from s to e. Use `(segDist(s,e,p)<=epsilon)` instead when using Point<double>.

"Point.h" c597e8, 3 lines

```
template<class P> bool onSegment(P s, P e, P p) {
  return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
```

LinearTransformation.h

Description:
Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.



"Point.h" 03a306, 6 lines

```
typedef Point<double> P;
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
  P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
  return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```

LineProj.h

Description: Projects point p onto line ab. Set refl=true to get reflection of point p across line ab instead. The wrong point will be returned if P is an integer point and the desired point doesn't have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow.

"Point.h" b5562d, 5 lines

```
template<class P>
P lineProj(P a, P b, P p, bool refl=false) {
  P v = b - a;
  return p - v.perp()*(1+refl)*v.cross(p-a)/v.dist2();
}
```

AngleCmp.h

Description: Sorts points in ascending order by angle within the interval $(-\pi, \pi]$. The point $(0, 0)$ has an angle of 0.

"Point.h" 496e9b, 6 lines

```
template<class P>
bool angleCmp(P a, P b) {
  auto half = [](P p) { return sgn(p.y) ?: -sgn(p.x); };
  int A = half(a), B = half(b);
  return A == B ? a.cross(b) > 0 : A < B;
}
```

HalfPlane.h

Description: Computes the intersection of a set of half-planes. Input is given as a set of planes, facing left. Output is the convex polygon representing the intersection. The points may have duplicates and be collinear. Will not fail catastrophically if 'eps > sqrt(2)(line intersection error)'. Likely to work for more ranges if 3 half planes are never guaranteed to intersect at the same point.

Time: $\mathcal{O}(n \log n)$

"Point.h", "SideOf.h", "LineInter.h" eda44b, 31 lines

```
typedef Point<double> P;
typedef array<P, 2> Line;
#define sp(a) a[0], a[1]
#define ang(a) (a[1] - a[0]).angle()

int angDiff(Line a, Line b) { return sgn(ang(a) - ang(b)); }
bool cmp(Line a, Line b) {
  int s = angDiff(a, b);
  return (s ? s : sideOf(sp(a), b[0])) < 0;
}
vector<P> halfPlaneIntersection(vector<Line> vs) {
  const double EPS = sqrt(2) * 1e-8;
  sort(all(vs), cmp);
  vector<Line> deq(sz(vs) + 5);
  vector<P> ans(sz(vs) + 5);
  deq[0] = vs[0];
  int ah = 0, at = 0, n = sz(vs);
  rep(i,1,n+1) {
    if (i == n) vs.push_back(deq[ah]);
    if (angDiff(vs[i], vs[i - 1]) == 0) continue;
    while (ah<at && sideOf(sp(vs[i]), ans[at-1], EPS) < 0)
      at--;
    while (i!=n && ah<at && sideOf(sp(vs[i]),ans[ah],EPS)<0)
      ah++;
    auto res = lineInter(sp(vs[i]), sp(deq[at]));
    if (res.first != 1) continue;
    ans[at++] = res.second, deq[at] = vs[i];
  }
  if (at - ah <= 2) return {};
  return {ans.begin() + ah, ans.begin() + at};
}
```

8.2 Circles

CircleInter.h

Description: Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

"Point.h" 84d6d3, 11 lines

```
typedef Point<double> P;
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
  if (a == b) { assert(r1 != r2); return false; }
  P vec = b - a;
  double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
         p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
  if (sum*sum < d2 || dif*dif > d2) return false;
  P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
  *out = {mid + per, mid - per};
  return true;
}
```

CircleTangents.h

Description: Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h" b0153d, 13 lines

```
template<class P>
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
  P d = c2 - c1;
  double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
  if (d2 == 0 || h2 < 0)  return {};
  vector<pair<P, P>> out;
  for (double sign : {-1, 1}) {
    P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
    out.push_back({c1 + v * r1, c2 + v * r2});
  }
  if (h2 == 0) out.pop_back();
  return out;
}
```

CircleLine.h

Description: Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<double>.

"Point.h" e0cfba, 9 lines

```
template<class P>
```

```
vector<P> circleLine(P c, double r, P a, P b) {
  P ab = b - a, p = a + ab * (c-a).dot(ab) / ab.dist2();
  double s = a.cross(b, c), h2 = r*r - s*s / ab.dist2();
  if (h2 < 0) return {};
  if (h2 == 0) return {p};
  P h = ab.unit() * sqrt(h2);
  return {p - h, p + h};
}
```

CirclePoly.h

Description: Returns the area of the intersection of a circle with a ccw polygon.
Time: $\mathcal{O}(n)$

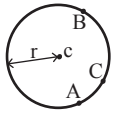
"Point.h" a1ee63, 19 lines

```
typedef Point<double> P;
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
  auto tri = [&](P p, P q) {
    auto r2 = r * r / 2;
    P d = q - p;
    auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
    auto det = a * a - b;
    if (det <= 0) return arg(p, q) * r2;
    auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
    if (t < 0 || 1 <= s) return arg(p, q) * r2;
    P u = p + d * s, v = p + d * t;
    return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
  };
  auto sum = 0.0;
  rep(i,0,sz(ps))
    sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);
  return sum;
}
```

Circumcircle.h

Description:

The circumcirle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.



"Point.h" 1caa3a, 9 lines

```
typedef Point<double> P;
double ccRadius(const P& A, const P& B, const P& C) {
  return (B-A).dist()*(C-B).dist()*(A-C).dist()/
      abs((B-A).cross(C-A))/2;
}
P ccCenter(const P& A, const P& B, const P& C) {
  P b = C-A, c = B-A;
  return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
```

MinEnclosingCircle.h

Description: Computes the minimum circle that encloses a set of points.
Time: expected $\mathcal{O}(n)$

"Circumcircle.h" 09dd0a, 17 lines

```
pair<P, double> mec(vector<P> ps) {
  shuffle(all(ps), mt19937(time(0)));
  P o = ps[0];
  double r = 0, EPS = 1 + 1e-8;
  rep(i,0,sz(ps)) if ((o - ps[i]).dist() > r * EPS) {
    o = ps[i], r = 0;
    rep(j,0,i) if ((o - ps[j]).dist() > r * EPS) {
      o = (ps[i] + ps[j]) / 2;
      r = (o - ps[i]).dist();
      rep(k,0,j) if ((o - ps[k]).dist() > r * EPS) {
        o = ccCenter(ps[i], ps[j], ps[k]);
        r = (o - ps[i]).dist();
      }
    }
  }
  return {o, r};
}
```

8.3 Polygons

InsidePolygon.h

Description: Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.
Usage: `vector<P> v = {P{4,4}, P{1,2}, P{2,1}};`
`bool in = inPolygon(v, P{3, 3}, false);`
Time: $\mathcal{O}(n)$

"Point.h", "OnSegment.h", "SegDist.h" 2bf504, 11 lines

```
template<class P>
bool inPolygon(vector<P> &p, P a, bool strict = true) {
  int cnt = 0, n = sz(p);
  rep(i,0,n) {
    P q = p[(i + 1) % n];
    if (onSegment(p[i], q, a)) return !strict;
    //or: if (segDist(p[i], q, a) <= eps) return !strict;
    cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) > 0;
  }
  return cnt;
}
```

PolygonArea.h

Description: Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h" f12300, 6 lines

```
template<class T>
T polygonArea2(vector<Point<T>>& v) {
  T a = v.back().cross(v[0]);
  rep(i,0,sz(v)-1) a += v[i].cross(v[i+1]);
  return a;
}
```

PolygonCenter.h

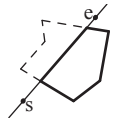
Description: Returns the center of mass for a polygon.
Time: $\mathcal{O}(n)$

"Point.h" 9706dc, 9 lines

```
typedef Point<double> P;
P polygonCenter(const vector<P>& v) {
  P res(0, 0); double A = 0;
  for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);
  }
  return res / A / 3;
}
```

PolygonCut.h

Description:
Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.
Usage: `vector<P> p = ...;`
`p = polygonCut(p, P(0,0), P(1,0));`



"Point.h", "LineInter.h" f2b7d4, 13 lines

```
typedef Point<double> P;
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
  vector<P> res;
  rep(i,0,sz(poly)) {
    P cur = poly[i], prev = i ? poly[i-1] : poly.back();
    bool side = s.cross(e, cur) < 0;
    if (side != (s.cross(e, prev) < 0))
      res.push_back(lineInter(s, e, cur, prev).second);
    if (side)
      res.push_back(cur);
  }
  return res;
}
```

PolygonUnion.h

Description: Calculates the area of the union of n polygons (not necessarily convex). The points within each polygon must be given in CCW order. (Epsilon checks may optionally be added to sideOf/sgn, but shouldn't be needed.)
Time: $\mathcal{O}(N^2)$, where N is the total number of points

"Point.h", "SideOf.h" 3931c6, 33 lines

```
typedef Point<double> P;
double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y; }
double polyUnion(vector<vector<P>>& poly) {
  double ret = 0;
  rep(i,0,sz(poly)) rep(v,0,sz(poly[i])) {
    P A = poly[i][v], B = poly[i][(v + 1) % sz(poly[i])];
    vector<pair<double, int>> segs = {{0, 0}, {1, 0}};
    rep(j,0,sz(poly)) if (i != j) {
      rep(u,0,sz(poly[j])) {
        P C = poly[j][u], D = poly[j][(u + 1) % sz(poly[j])];
        int sc = sideOf(A, B, C), sd = sideOf(A, B, D);
        if (sc != sd) {
          double sa = C.cross(D, A), sb = C.cross(D, B);
          if (min(sc, sd) < 0)
            segs.emplace_back(sa / (sa - sb), sgn(sc - sd));
        } else if (!sc && !sd && j<i && sgn((B-A).dot(D-C))>0){
          segs.emplace_back(rat(C - A, B - A), 1);
          segs.emplace_back(rat(D - A, B - A), -1);
        }
      }
    }
    sort(all(segs));
    for (auto& s : segs) s.first = min(max(s.first, 0.0), 1.0);
    double sum = 0;
    int cnt = segs[0].second;
    rep(j,1,sz(segs)) {
      if (!cnt) sum += segs[j].first - segs[j - 1].first;
      cnt += segs[j].second;
    }
    ret += A.cross(B) * sum;
  }
  return ret / 2;
}
```

ConvexHull.h

Description:
Returns a vector of the points of the convex hull in counterclockwise order. Points on the edge of the hull between two other points are not considered part of the hull.
Time: $\mathcal{O}(n \log n)$

"Point.h" 310954, 13 lines

```
typedef Point<ll> P;
vector<P> convexHull(vector<P> pts) {
  if (sz(pts) <= 1) return pts;
  sort(all(pts));
  vector<P> h(sz(pts)+1);
  int s = 0, t = 0;
  for (int it = 2; it--; s = --t, reverse(all(pts)))
    for (P p : pts) {
      while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
      h[t++] = p;
    }
```

```
  return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}
```

HullDiameter.h

Description: Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).
Time: $\mathcal{O}(n)$

"Point.h" c571b8, 12 lines

```
typedef Point<ll> P;
array<P, 2> hullDiameter(vector<P> S) {
  int n = sz(S), j = n < 2 ? 0 : 1;
  pair<ll, array<P, 2>> res({0, {S[0], S[0]}});
  rep(i,0,j)
    for (;; j = (j + 1) % n) {
      res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
      if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
        break;
    }
  return res.second;
}
```

PointInsideHull.h

Description: Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.
Time: $\mathcal{O}(\log N)$

"Point.h", "SideOf.h", "OnSegment.h" 71446b, 14 lines

```
typedef Point<ll> P;

bool inHull(const vector<P>& l, P p, bool strict = true) {
  int a = 1, b = sz(l) - 1, r = !strict;
  if (sz(l) < 3) return r && onSegment(l[0], l.back(), p);
  if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
  if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p)<= -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
  }
  return sgn(l[a].cross(l[b], p)) < r;
}
```

LineHullIntersection.h

Description: Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon: $\bullet$ $(-1,-1)$ if no collision, $\bullet$ $(i,-1)$ if touching the corner i, $\bullet$ (i,i) if along side $(i,i+1)$, $\bullet$ (i,j) if crossing sides $(i,i+1)$ and $(j,j+1)$. In the last case, if a corner i is crossed, this is treated as happening on side $(i,i+1)$. The points are returned in the same order as the line hits the polygon. `extrVertex` returns the point of a hull with the max projection onto a line.
Time: $\mathcal{O}(\log n)$

"Point.h" 7cf45b, 39 lines

```
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
  int n = sz(poly), lo = 0, hi = n;
  if (extr(0)) return 0;
  while (lo + 1 < hi) {
    int m = (lo + hi) / 2;
    if (extr(m)) return m;
    int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
    (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
  }
  return lo;
}
```

```
#define cmpL(i) sgn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
  int endA = extrVertex(poly, (a - b).perp());
  int endB = extrVertex(poly, (b - a).perp());
  if (cmpL(endA) < 0 || cmpL(endB) > 0)
    return {-1, -1};
  array<int, 2> res;
  rep(i,0,2) {
    int lo = endB, hi = endA, n = sz(poly);
    while ((lo + 1) % n != hi) {
      int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
      (cmpL(m) == cmpL(endB) ? lo : hi) = m;
    }
    res[i] = (lo + !cmpL(hi)) % n;
    swap(endA, endB);
  }
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
      case 0: return {res[0], res[0]};
      case 2: return {res[1], res[1]};
    }
  return res;
}
```

HullTangents.h

Description: Finds the left and right tangent vertices of a convex polygon relative to point a. The polygon must have at least 3 vertices, be CCW-ordered, and no collinear points. Returns the vertex indices.
Time: $\mathcal{O}(\log n)$

"Point.h" 249823, 27 lines

```
template<class P>
pii hullTangents(const vector<P>& p, P a) {
  int n = sz(p), t[2];
  rep(it, 0, 2) {
    auto dir = [&](int i) {
      P u = p[i] - a, v = p[(i + 1) % n] - a;
      auto c = u.cross(v);
      if (c != 0) return c < 0;
      if (u.dot(v) <= 0) return true;
      return u.dist2() > v.dist2();
    };
    auto idir = [&](int i) { return dir(i) ^ it; };
    if (idir(0) && !idir(n - 1)) { t[it] = 0; continue; }
    int s[2] = {0, n - 1};
    while (s[1] - s[0] > 2) {
      int mid = (s[0] + s[1]) / 2, x = idir(mid);
      if (idir(s[x ^ 1]) == (x ^ 1)) {
        s[x] = mid;
      } else {
        bool b = a.cross(p[mid], p[s[1]]) < 0;
        s[b ^ x ^ it ^ 1] = mid;
      }
    }
    t[it] = s[0] + 1 + (idir(s[0] + 1) == 0);
  }
  return {t[0], t[1]};
}
```

8.4 Misc. Point Set Problems

ClosestPair.h

Description: Finds the closest pair of points.
Time: $\mathcal{O}(n \log n)$

"Point.h" ac41a6, 17 lines

```
typedef Point<ll> P;
pair<P, P> closest(vector<P> v) {
  assert(sz(v) > 1);
  set<P> S;
  sort(all(v), [](P a, P b) { return a.y < b.y; });
  pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
  int j = 0;
  for (P p : v) {
    P d{1 + (ll)sqrt(ret.first), 0};
    while (v[j].y <= p.y - d.x) S.erase(v[j++]);
    auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
    for (; lo != hi; ++lo)
      ret = min(ret, {(*lo - p).dist2(), {*lo, p}});
    S.insert(p);
  }
  return ret.second;
}
```

ManhattanMST.h

Description: Given N points, returns up to 4*N edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights $w(p,q) = |p.x - q.x| + |p.y - q.y|$. Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.
Time: $\mathcal{O}(N \log N)$

"Point.h" df6f59, 23 lines

```
typedef Point<int> P;
vector<array<int, 3>> manhattanMST(vector<P> ps) {
  vi id(sz(ps));
  iota(all(id), 0);
  vector<array<int, 3>> edges;
  rep(k,0,4) {
    sort(all(id), [&](int i, int j) {
         return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
    map<int, int> sweep;
    for (int i : id) {
      for (auto it = sweep.lower_bound(-ps[i].y);
                it != sweep.end(); sweep.erase(it++)) {
        int j = it->second;
        P d = ps[i] - ps[j];
        if (d.y > d.x) break;
        edges.push_back({d.y + d.x, i, j});
      }
      sweep[-ps[i].y] = i;
    }
    for (P& p : ps) if (k & 1) p.x = -p.x; else swap(p.x, p.y);
  }
  return edges;
}
```

kdTree.h

Description: KD-tree (2d, can be extended to 3d)

"Point.h" bac5b0, 63 lines

```
typedef long long T;
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }

struct Node {
  P pt; // if this is a leaf, the single point in it
  T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
  Node *first = 0, *second = 0;

  T distance(const P& p) { // min squared distance to a point
    T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
    T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
    return (P(x,y) - p).dist2();
  }
```

```
  Node(vector<P>&& vp) : pt(vp[0]) {
    for (P p : vp) {
      x0 = min(x0, p.x); x1 = max(x1, p.x);
      y0 = min(y0, p.y); y1 = max(y1, p.y);
    }
    if (vp.size() > 1) {
      // split on x if width >= height (not ideal...)
      sort(all(vp), x1 - x0 >= y1 - y0 ? on_x : on_y);
      // divide by taking half the array for each child (not
      // best performance with many duplicates in the middle)
      int half = sz(vp)/2;
      first = new Node({vp.begin(), vp.begin() + half});
      second = new Node({vp.begin() + half, vp.end()});
    }
  }
};

struct KDTree {
  Node* root;
  KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {}

  pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {
      // uncomment if we should not find the point itself:
      // if (p == node->pt) return {INF, P()};
      return make_pair((p - node->pt).dist2(), node->pt);
    }

    Node *f = node->first, *s = node->second;
    T bfirst = f->distance(p), bsec = s->distance(p);
    if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);

    // search closest side first, other side if needed
    auto best = search(f, p);
    if (bsec < best.first)
      best = min(best, search(s, p));
    return best;
  }

  // find nearest point to a point, and its squared distance
  // (requires an arbitrary operator< for Point)
  pair<T, P> nearest(const P& p) {
    return search(root, p);
  }
};
```

FastDelaunay.h

Description: Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.
Time: $\mathcal{O}(n \log n)$

"Point.h" eefdf5, 88 lines

```
typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point

struct Quad {
  Q rot, o; P p = arb; bool mark;
  P& F() { return r()->p; }
  Q& r() { return rot->rot; }
  Q prev() { return rot->o->rot; }
  Q next() { return r()->prev(); }
} *H;

bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
  lll p2 = p.dist2(), A = a.dist2()-p2,
      B = b.dist2()-p2, C = c.dist2()-p2;
  return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}
Q makeEdge(P orig, P dest) {
  Q r = H ? H : new Quad{new Quad{new Quad{new Quad{0}}}};
  H = r->o; r->r()->r() = r;
  rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->r();
  r->p = orig; r->F() = dest;
  return r;
}
void splice(Q a, Q b) {
  swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}
Q connect(Q a, Q b) {
  Q q = makeEdge(a->F(), b->p);
  splice(q, a->next());
  splice(q->r(), b);
  return q;
}

pair<Q,Q> rec(const vector<P>& s) {
  if (sz(s) <= 3) {
    Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
    if (sz(s) == 2) return { a, a->r() };
    splice(a->r(), b);
    auto side = s[0].cross(s[1], s[2]);
    Q c = side ? connect(b, a) : 0;
    return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
  }

#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
  Q A, B, ra, rb;
  int half = sz(s) / 2;
  tie(ra, A) = rec({all(s) - half});
  tie(B, rb) = rec({sz(s) - half + all(s)});
  while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
         (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
  Q base = connect(B->r(), A);
  if (A->p == ra->p) ra = base->r();
  if (B->p == rb->p) rb = base;

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
      Q t = e->dir; \
      splice(e, e->prev()); \
      splice(e->r(), e->r()->prev()); \
      e->o = H; H = e; e = t; \
    }
  for (;;) {
    DEL(LC, base->r(), o);  DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
      base = connect(RC, base->r());
    else
      base = connect(base->r(), LC->r());
  }
  return { ra, rb };
}

vector<P> triangulate(vector<P> pts) {
  sort(all(pts));  assert(unique(all(pts)) == pts.end());
  if (sz(pts) < 2) return {};
  Q e = rec(pts).first;
  vector<Q> q = {e};
  int qi = 0;
  while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
    q.push_back(c->r()); c = c->next(); } while (c != e); }
  ADD; pts.clear();
  while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
  return pts;
}
```

8.5 3D

PolyhedronVolume.h

Description: Magic formula for the volume of a polyhedron. Faces should point outwards.

3058c3, 6 lines

```
template<class V, class L>
double signedPolyVolume(const V& p, const L& trilist) {
  double v = 0;
  for (auto i : trilist) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
  return v / 6;
}
```

Point3D.h

Description: Class to handle points in 3D space. T can be e.g. double or long long.

8058ae, 32 lines

```
template<class T> struct Point3D {
  typedef Point3D P;
  typedef const P& R;
  T x, y, z;
  explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
  bool operator<(R p) const {
    return tie(x, y, z) < tie(p.x, p.y, p.z); }
  bool operator==(R p) const {
    return tie(x, y, z) == tie(p.x, p.y, p.z); }
  P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
  P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
  P operator*(T d) const { return P(x*d, y*d, z*d); }
  P operator/(T d) const { return P(x/d, y/d, z/d); }
  T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
  P cross(R p) const {
    return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
  }
  T dist2() const { return x*x + y*y + z*z; }
  double dist() const { return sqrt((double)dist2()); }
  //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
  double phi() const { return atan2(y, x); }
  //Zenith angle (latitude) to the z-axis in interval [0, pi]
  double theta() const { return atan2(sqrt(x*x+y*y),z); }
  P unit() const { return *this/(T)dist(); } //makes dist()=1
  //returns unit vector normal to *this and p
  P normal(P p) const { return cross(p).unit(); }
  //returns point rotated 'angle' radians ccw around axis
  P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
  }
};
```

3dHull.h

Description: Computes all faces of the 3-dimension hull of a point set. *No four points must be coplanar*, or else random results will be returned. All faces will point outwards.
Time: $\mathcal{O}(n^2)$

"Point3D.h" 5b45fc, 49 lines

```
typedef Point3D<double> P3;

struct PR {
  void ins(int x) { (a == -1 ? a : b) = x; }
  void rem(int x) { (a == x ? a : b) = -1; }
```

```
  int cnt() { return (a != -1) + (b != -1); }
  int a, b;
};

struct F { P3 q; int a, b, c; };

vector<F> hull3d(const vector<P3>& A) {
  assert(sz(A) >= 4);
  vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
#define E(x,y) E[f.x][f.y]
  vector<F> FS;
  auto mf = [&](int i, int j, int k, int l) {
    P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
    if (q.dot(A[l]) > q.dot(A[i]))
      q = q * -1;
    F f{q, i, j, k};
    E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
    FS.push_back(f);
  };
  rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
    mf(i, j, k, 6 - i - j - k);

  rep(i,4,sz(A)) {
    rep(j,0,sz(FS)) {
      F f = FS[j];
      if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
        E(a,b).rem(f.c);
        E(a,c).rem(f.b);
        E(b,c).rem(f.a);
        swap(FS[j--], FS.back());
        FS.pop_back();
      }
    }
    int nw = sz(FS);
    rep(j,0,nw) {
      F f = FS[j];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
      C(a, b, c); C(a, c, b); C(b, c, a);
    }
  }
  for (F& it : FS) if ((A[it.b] - A[it.a]).cross(
    A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
  return FS;
};
```

sphericalDistance.h

Description: Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 (ϕ_1) and f2 (ϕ_2) from x axis and zenith angles (latitude) t1 (θ_1) and t2 (θ_2) from z axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx*radius is then the difference between the two points in the x direction and d*radius is the total distance between the points.

611f07, 8 lines

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
  double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
  double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
  double dz = cos(t2) - cos(t1);
  double d = sqrt(dx*dx + dy*dy + dz*dz);
  return radius*2*asin(d/2);
}
```

Strings (9)

KMP.h

Description: pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.
Time: $\mathcal{O}(n)$

d4375c, 16 lines

```
vi pi(const string& s) {
  vi p(sz(s));
  rep(i,1,sz(s)) {
    int g = p[i-1];
    while (g && s[i] != s[g]) g = p[g-1];
    p[i] = g + (s[i] == s[g]);
  }
  return p;
}

vi match(const string& s, const string& pat) {
  vi p = pi(pat + '\0' + s), res;
  rep(i,sz(p)-sz(s),sz(p))
    if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
  return res;
}
```

Zfunc.h

Description: z[i] computes the length of the longest common prefix of s[i:] and s.
Time: $\mathcal{O}(n)$

584523, 13 lines

```
vi Z(const string& S) {
  vi z(sz(S));
  int l = -1, r = -1;
  rep(i,1,sz(S)) {
    z[i] = i >= r ? 0 : min(r - i, z[i - l]);
    while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
      z[i]++;
    if (i + z[i] > r)
      l = i, r = i + z[i];
  }
  if (sz(S)) z[0] = sz(S);
  return z;
}
```

Manacher.h

Description: For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).
Time: $\mathcal{O}(N)$

e7ad79, 13 lines

```
array<vi, 2> manacher(const string& s) {
  int n = sz(s);
  array<vi,2> p = {vi(n+1), vi(n)};
  rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
    int t = r-i+!z;
    if (i<r) p[z][i] = min(t, p[z][l+t]);
    int L = i-p[z][i], R = i+p[z][i]-!z;
    while (L>=1 && R+1<n && s[L-1] == s[R+1])
      p[z][i]++, L--, R++;
    if (R>r) l=L, r=R;
  }
  return p;
}
```

MinRotation.h

Description: Finds the lexicographically smallest rotation of a string.
Usage: rotate(v.begin(), v.begin()+minRotation(v), v.end());
Time: $\mathcal{O}(N)$

d07a42, 8 lines

```
int minRotation(string s) {
  int a=0, N=sz(s); s += s;
  rep(b,0,N) rep(k,0,N) {
    if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
    if (s[a+k] > s[b+k]) { a = b; break; }
  }
  return a;
}
```

SuffixArray.h

Description: Builds suffix array for a string. sa[i] is the starting index of the suffix which is i'th in the sorted suffix array. The returned vector is of size $n + 1$, and sa[0] = n. The lcp array contains longest common prefixes for neighbouring strings in the suffix array: lcp[i] = lcp(sa[i], sa[i-1]), lcp[0] = 0. The input string must not contain any zero bytes.
Time: $\mathcal{O}(n \log n)$

38db9f, 23 lines

```
struct SuffixArray {
  vi sa, lcp;
  SuffixArray(string& s, int lim=256) { // or basic_string<int>
    int n = sz(s) + 1, k = 0, a, b;
    vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
    sa = lcp = y, iota(all(sa), 0);
    for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
      p = j, iota(all(y), n - j);
      rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
      fill(all(ws), 0);
      rep(i,0,n) ws[x[i]]++;
      rep(i,1,lim) ws[i] += ws[i - 1];
      for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
      swap(x, y), p = 1, x[sa[0]] = 0;
      rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
        (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
    }
    rep(i,1,n) rank[sa[i]] = i;
    for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
      for (k && k--, j = sa[rank[i] - 1];
          s[i + k] == s[j + k]; k++);
  }
};
```

SuffixTree.h

Description: Ukkonen's algorithm for online suffix tree construction. Each node contains indices [l, r) into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining [l, r) substrings. The root is 0 (has l = -1, r = 0), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).
Time: $\mathcal{O}(26N)$

aae0b8, 50 lines

```
struct SuffixTree {
  enum { N = 200010, ALPHA = 26 }; // N ~ 2*maxlen+10
  int toi(char c) { return c - 'a'; }
  string a; // v = cur node, q = cur position
  int t[N][ALPHA],l[N],r[N],p[N],s[N],v=0,q=0,m=2;

  void ukkadd(int i, int c) { suff:
    if (r[v]<=q) {
      if (t[v][c]==-1) { t[v][c]=m;  l[m]=i;
        p[m++]=v; v=s[v]; q=r[v];  goto suff; }
      v=t[v][c]; q=l[v];
    }
    if (q==-1 || c==toi(a[q])) q++; else {
      l[m+1]=i;  p[m+1]=m;  l[m]=l[v];  r[m]=q;
      p[m]=p[v];  t[m][c]=m+1;  t[m][toi(a[q])]=v;
      l[v]=q;  p[v]=m;  t[p[m]][toi(a[l[m]])]=m;
      v=s[p[m]];  q=l[m];
      while (q<r[m]) { v=t[v][toi(a[q])];  q+=r[v]-l[v]; }
      if (q==r[m])  s[m]=v;  else s[m]=m+2;
```

```
      q=r[v]-(q-r[m]);  m+=2;  goto suff;
    }
  }

  SuffixTree(string a) : a(a) {
    fill(r,r+N,sz(a));
    memset(s, 0, sizeof s);
    memset(t, -1, sizeof t);
    fill(t[1],t[1]+ALPHA,0);
    s[0] = 1; l[0] = l[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
    rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
  }

  // example: find longest common substring (uses ALPHA = 28)
  pii best;
  int lcs(int node, int i1, int i2, int olen) {
    if (l[node] <= i1 && i1 < r[node]) return 1;
    if (l[node] <= i2 && i2 < r[node]) return 2;
    int mask = 0, len = node ? olen + (r[node] - l[node]) : 0;
    rep(c,0,ALPHA) if (t[node][c] != -1)
      mask |= lcs(t[node][c], i1, i2, len);
    if (mask == 3)
      best = max(best, {len, r[node] - len});
    return mask;
  }
  static pii LCS(string s, string t) {
    SuffixTree st(s + (char)('z' + 1) + t + (char)('z' + 2));
    st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
    return st.best;
  }
};
```

AhoCorasick.h

Description: Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(−, word) finds all words (up to $N\sqrt{N}$ many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries.
Time: construction takes $\mathcal{O}(26N)$, where N = sum of length of patterns. find(x) is $\mathcal{O}(N)$, where N = length of x. findAll is $\mathcal{O}(NM)$. f35677, 66 lines

```
struct AhoCorasick {
  enum {alpha = 26, first = 'A'}; // change this!
  struct Node {
    // (nmatches is optional)
    int back, next[alpha], start = -1, end = -1, nmatches = 0;
    Node(int v) { memset(next, v, sizeof(next)); }
  };
  vector<Node> N;
  vi backp;
  void insert(string& s, int j) {
    assert(!s.empty());
    int n = 0;
    for (char c : s) {
      int& m = N[n].next[c - first];
      if (m == -1) { n = m = sz(N); N.emplace_back(-1); }
      else n = m;
    }
    if (N[n].end == -1) N[n].start = j;
    backp.push_back(N[n].end);
    N[n].end = j;
    N[n].nmatches++;
  }
  AhoCorasick(vector<string>& pat) : N(1, -1) {
    rep(i,0,sz(pat)) insert(pat[i], i);
    N[0].back = sz(N);
    N.emplace_back(0);

    queue<int> q;
    for (q.push(0); !q.empty(); q.pop()) {
      int n = q.front(), prev = N[n].back;
      rep(i,0,alpha) {
        int &ed = N[n].next[i], y = N[prev].next[i];
        if (ed == -1) ed = y;
        else {
          N[ed].back = y;
          (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
            = N[y].end;
          N[ed].nmatches += N[y].nmatches;
          q.push(ed);
        }
      }
    }
  }
  vi find(string word) {
    int n = 0;
    vi res; // ll count = 0;
    for (char c : word) {
      n = N[n].next[c - first];
      res.push_back(N[n].end);
      // count += N[n].nmatches;
    }
    return res;
  }
  vector<vi> findAll(vector<string>& pat, string word) {
    vi r = find(word);
    vector<vi> res(sz(word));
    rep(i,0,sz(word)) {
      int ind = r[i];
      while (ind != -1) {
        res[i - sz(pat[ind]) + 1].push_back(ind);
        ind = backp[ind];
      }
    }
    return res;
  }
};
```

Duval.h

Description: Constructs Lyndon factorization of s. A word is called simple iff it is strictly smaller than any of its nontrivial suffixes. The Lyndon factorization of a string is the unique division into non-increasing simple words.
Time: $\mathcal{O}(n)$ cab391, 12 lines

```
vi duval(const string& s) {
  int n = sz(s); vi f;
  for (int i = 0; i < n;) {
    int j = i + 1, k = i;
    for (; j < n && s[k] <= s[j]; j++) {
      if (s[k] < s[j]) k = i;
      else ++k;
    }
    for (; i <= k; i += j - k) f.push_back(i);
  }
  return f.push_back(n), f;
}
```

WildcardMatching.h

Description: Finds all occurrences of T in S over an alphabet with wildcards. Requires $|T| \leq |S|$.
Time: $\mathcal{O}(|S| \log |S|)$
"../numerical/FFT.h" 639ceb, 20 lines

```
mt19937 rng(2137);
vector<bool> match(string s, string t, char w = '*') {
  int n = sz(s), m = sz(t); mint d = rng();
  vector<mint> f1(n), f2(n), f3(n), g1(m), g2(m), g3(m);
  rep(i, 0, n) f1[i] = s[i] == w ? 0 : s[i] + d;
  rep(i, 0, n) f2[i] = f1[i] * f1[i], f3[i] = f2[i] * f1[i];
  rep(i, 0, m) g1[i] = t[i] == w ? 0 : t[i] + d;
  rep(i, 0, m) g2[i] = g1[i] * g1[i], g3[i] = g2[i] * g1[i];
  auto mul = [&](auto a, auto b) {
    int sz = 1 << __lg(2 * n - 1); reverse(all(b));
    a.resize(sz); ntt(a, 0); b.resize(sz); ntt(b, 0);
    rep(i, 0, sz) a[i] *= b[i];
    ntt(a, 1); a.erase(a.begin(), a.begin() + m - 1);
    return a;
  };
  auto a = mul(f1, g3), b = mul(f2, g2), c = mul(f3, g1);
  vector<bool> ans(n - m + 1);
  rep(i, 0, n - m + 1) ans[i] = a[i] - b[i] - b[i] + c[i] == 0;
  return ans;
}
```

Hash61.h

Description: Arithmetic for fast hashing modulo $2^{61} - 1$ (prime).
Time: About 30% faster than naive modulo. 51cf65, 7 lines

```
const ll MOD = (1ll << 61) - 1;
ll add(ll a, ll b) { return a+b >= MOD ? a+b - MOD : a+b; }
ll sub(ll a, ll b) { return add(a, MOD - b); }
ll mul(ll a, ll b) {
  auto c = (__int128)a * b;
  return add(c & MOD, c >> 61);
}
```

PalTree.h

Description: Palindrome tree. Can be used for counting number of occurrences, just add 1 to suffix link path. Replace array with map if ML is tight.
Time: $\mathcal{O}(26N)$, split is $\mathcal{O}(n \log n)$ 2b16ac, 43 lines

```
const int A = 26;
struct PalTree {
  int last = 0;
  vi len = {0, -1}, link = {1, 0}, s = {-1};
  vector<array<int, A>> to = {{}, {}};
  int find(int u) {
    while (s.back() != s[sz(s) - len[u] - 2]) u = link[u];
    return u;
  }
  int add(int x) { // x in [0, A)
    s.push_back(x); last = find(last);
    if (!to[last][x]) {
      to.push_back({});
      len.push_back(len[last] + 2);
      link.push_back(to[find(link[last])][x]);
      to[last][x] = sz(to) - 1;
    }
    return last = to[last][x];
  }
};
// dp[i] = min even/odd palindromic split of prefix of size i
const int INF = 1e9;
struct F { int e, o; };
F op(F x, F y) { return {min(x.e, y.e), min(x.o, y.o)}; }
vector<F> split(vi v) {
  PalTree t;
  vector<F> s(2), ans(sz(v) + 1, {INF, INF});
  vi go(2), d(2); ans[0] = s[0] = s[1] = {0, INF};
  rep(i, 0, sz(v)) {
    int x = t.add(v[i]), y = t.link[x];
    if (x >= sz(go)) {
      d.push_back(t.len[x] - t.len[y]);
      go.push_back(d[x] == d[y] ? go[y] : y);
```

```
      s.push_back(ans[0]);
    }
    for (int u = x; t.len[u] > 0; u = go[u]) {
      s[u] = ans[i + 1 - t.len[go[u]] - d[u]];
      if (d[u] == d[t.link[u]]) s[u] = op(s[u], s[t.link[u]]);
      ans[i + 1] = op(ans[i + 1], {s[u].o + 1, s[u].e + 1});
    }
  }
  return ans;
}
```

Squares.h

Description: Main-Lorentz algorithm for finding all squares in a string. Returns triples (l, r, t) signifying that for each $i \in [l, r)$ there is a square at position i of size $2t$. Triples with the same t don't intersect or touch.

Time: $\mathcal{O}(n \log n)$

"Zfunc.h" 51e311, 23 lines

```
vector<array<int, 3>> squares(const string& s) {
  vector<array<int, 3>> ans; vi pos(sz(s) / 2 + 2, -1);
  rep(m, 1, sz(s)) {
    int p = m & ~(m - 1), o = m - p, e = min(m + p, sz(s));
    auto a = s.substr(o, p), b = s.substr(m, e - m);
    auto ra = a, rb = b; reverse(all(ra)); reverse(all(rb));
    rep(j, 0, 2) {
      vi z1 = Z(ra), z2 = Z(b + '\0' + a);
      z1.push_back(0); z2.push_back(0);
      rep(c, 0, sz(a)) {
        int l = sz(a) - c;
        int x=c-min(l-1, z1[l]), y=c-max(l-z2[sz(b)+c+1], j);
        if (x > y) continue;
        int sb = j ? e-y-1*2 : o+x, se = j ? e-x-1*2+1 : o+y+1;
        int& k = pos[l];
        if (k != -1 && ans[k][1] == sb) ans[k][1] = se;
        else k = sz(ans), ans.push_back({sb, se, l});
      }
      swap(a, rb); swap(b, ra);
    }
  }
  return ans;
}
```

Various (10)

10.1 Misc. algorithms

TernarySearch.h

Description: Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \ldots < f(i) \geq \cdots \geq f(b)$. To reverse which of the sides allows non-strict inequalities, change the $<$ marked with (A) to $<=$, and reverse the loop at (B). To minimize f, change it to $>$, also at (B).

Usage: `int ind = ternSearch(0,n-1,[&](int i){return a[i];});`

Time: $\mathcal{O}(\log(b - a))$

9155b4, 11 lines

```
template<class F>
int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
    int mid = (a + b) / 2;
    if (f(mid) < f(mid+1)) a = mid; // (A)
    else b = mid+1;
  }
  rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
  return a;
}
```

LIS.h

Description: Compute indices for the longest increasing subsequence.

Time: $\mathcal{O}(N \log N)$

2932a0, 17 lines

```
template<class I> vi lis(const vector<I>& S) {
  if (S.empty()) return {};
  vi prev(sz(S));
  typedef pair<I, int> p;
  vector<p> res;
  rep(i,0,sz(S)) {
    // change 0 -> i for longest non-decreasing subsequence
    auto it = lower_bound(all(res), p{S[i], 0});
    if (it == res.end()) res.emplace_back(), it = res.end()-1;
    *it = {S[i], i};
    prev[i] = it == res.begin() ? 0 : (it-1)->second;
  }
  int L = sz(res), cur = res.back().second;
  vi ans(L);
  while (L--) ans[L] = cur, cur = prev[cur];
  return ans;
}
```

FastKnapsack.h

Description: Given N non-negative integer weights w and a non-negative target t, computes the maximum S <= t such that S is the sum of some subset of the weights.

Time: $\mathcal{O}(N \max(w_i))$

b20ccc, 16 lines

```
int knapsack(vi w, int t) {
  int a = 0, b = 0, x;
  while (b < sz(w) && a + w[b] <= t) a += w[b++];
  if (b == sz(w)) return a;
  int m = *max_element(all(w));
  vi u, v(2*m, -1);
  v[a+m-t] = b;
  rep(i,b,sz(w)) {
    u = v;
    rep(x,0,m) v[x+w[i]] = max(v[x+w[i]], u[x]);
    for (x = 2*m; --x > m;) rep(j, max(0,u[x]), v[x])
      v[x-w[j]] = max(v[x-w[j]], j);
  }
  for (a = t; v[a+m-t] < 0; a--) ;
  return a;
}
```

10.2 Dynamic programming

KnuthDP.h

Description: When doing DP on intervals: $a[i][j] = \min_{i<k<j}(a[i][k] + a[k][j]) + f(i, j)$, where the (minimal) optimal k increases with both i and j, one can solve intervals in increasing order of length, and search $k = p[i][j]$ for $a[i][j]$ only between $p[i][j-1]$ and $p[i+1][j]$. This is known as Knuth DP. Sufficient criteria for this are if $f(b, c) \leq f(a, d)$ and $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$ for all $a \leq b \leq c \leq d$. Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

Time: $\mathcal{O}(N^2)$

DivideAndConquerDP.h

Description: Given $a[i] = \min_{lo(i)\leq k<hi(i)}(f(i, k))$ where the (minimal) optimal k increases with i, computes $a[i]$ for $i = L..R-1$.

Time: $\mathcal{O}((N + (hi - lo)) \log N)$

d38d2b, 18 lines

```
struct DP { // Modify at will:
  int lo(int ind) { return 0; }
  int hi(int ind) { return ind; }
  ll f(int ind, int k) { return dp[ind][k]; }
  void store(int ind, int k, ll v) { res[ind] = pii(k, v); }
  void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<ll, int> best(LLONG_MAX, LO);
    rep(k, max(LO,lo(mid)), min(HI,hi(mid)))
      best = min(best, make_pair(f(mid, k), k));
    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second+1);
    rec(mid+1, R, best.second, HI);
  }
  void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
};
```

10.3 Optimization tricks

FastMod.h

Description: Compute $a\%b$ about 5 times faster than usual, where b is constant but not known at compile time. Returns a value congruent to a (mod b) in the range $[0, 2b)$.

751a02, 8 lines

```
typedef unsigned long long ull;
struct FastMod {
  ull b, m;
  FastMod(ull b) : b(b), m(-1ULL / b) {}
  ull reduce(ull a) { // a % b + (0 or b)
    return a - (ull)((__uint128_t(m) * a) >> 64) * b;
  }
};
```

FastInput.h

Description: Read an integer from stdin. Usage requires your program to pipe in input from file.

Usage: `./a.out < input.txt`

Time: About 5x as fast as cin/scanf.

7b3c70, 17 lines

```
inline char gc() { // like getchar()
  static char buf[1 << 16];
  static size_t bc, be;
  if (bc >= be) {
    buf[0] = 0, bc = 0;
    be = fread(buf, 1, sizeof(buf), stdin);
  }
  return buf[bc++]; // returns 0 on EOF
}

int readInt() {
  int a, c;
  while ((a = gc()) < 40);
  if (a == '-') return -readInt();
  while ((c = gc()) >= 48) a = a * 10 + c - 480;
  return a - 48;
}
```