On the Probabilistic Completeness of the Sampling-based Feedback Motion Planners in Belief Space

Ali-akbar Agha-mohammadi, Suman Chakravorty, Nancy M. Amato

***Abstract*— This paper extends the concept of "probabilistic completeness" defined for motion planners in state space (or configuration space) to the concept of "probabilistic completeness under uncertainty" for motion planners in belief space. Accordingly, an approach is proposed to verify the probabilistic completeness of the sampling-based planners in belief space. Finally, through the proposed approach, it is shown that under mild conditions the sampling-based methods constructed based on the abstract framework of FIRM (Feedback-based Information Roadmap Method) are probabilistically complete under uncertainty.**

I. Introduction

Motion planning for a moving object in the presence of obstacles is one of the main challenges in robotics, and has attracted immense attention over the last two decades [1], [2], [3]. Sampling-based methods are one of the successful strategies for solving many planning problems. Sampling-based methods were initially developed for motion planning in the absence of noise, (e.g., [4], [5], and [6]), and later were generalized to motion planning methods in the presence of the uncertainty (e.g., [7], [8], [9], [10]).

Due to the success of sampling-based methods in many practical planning problems, many researchers have investigated the theoretical basis for this success. However, almost all of these investigations have been done for algorithms that are designed for planning in the absence of uncertainty. The literature in this direction falls into two categories: path isolation-based methods and space covering-based methods.

Path isolation-based analysis: In this approach, one path is chosen, and it is tiled with sets such as ϵ-balls in [11] or sets with arbitrary shapes but strictly positive measure in [12]. Then the success probability is analyzed by investigating the probability of sampling in each of the sets that tile the given path in the obstacle-free space. Methods in [11], [12], [13], and [14] are among the approaches that perform path isolation-based analysis of planning algorithm.

Space Covering-based analysis: In the space covering-based analysis approach, an adequate number of sampled points to find a successful path is expressed in terms of a parameter ϵ, which is a property of the environment. A space is ϵ-good, if every point in the state space can be connected to at least an ϵ fraction of the space using local planners. Kavraki *et al.* [15] and Hsu [16] have used these approaches.

These methods were developed for the situation where the desired result from the planning algorithm is a path. However, in the presence of uncertainty, the concept of "successful path" is no longer meaningful, because on a given path, different policies may result in different success probabilities, some interpreted as successful, some not.

This paper extends these concepts to probabilistic spaces, i.e., to sampling-based methods concerning planning under uncertainty. In this case, since the planning algorithm returns a policy instead of a path, success has to be defined for a policy. We define the concept of successful policy and the concept of globally successful policy and formulate them rigorously.

Under uncertainty, the belief or information-state encodes the robot's knowledge about its state (or configuration). In the probabilistic setting, belief is the probability distribution function (pdf) over all possible robot states. Belief space is the space of all possible beliefs. Indeed, we generalize the conventional concept of "probabilistic completeness" defined for motion planning methods in the absence of uncertainty to the concept of "probabilistic completeness under uncertainty," defined for planners in the belief space. According to this definition, we prove that planning algorithms derived from the abstract FIRM (Feedback-based Information Roadmap Method) framework [7], are probabilistically complete algorithms. That is, the FIRM-based methods are those that guarantee node reachability in belief space and generate a roadmap in belief space whose edges are independent of each other. The procedure introduced in this proof also provides useful results in analyzing local planners and sampling-based planners in belief space.

In the next section, we first review the general formulation for the planning problem under uncertainty, and briefly explain the abstract FIRM framework [7]. In Section III we extend the concept of probabilistic completeness and define the concept of "probabilistic completeness under uncertainty." In Section IV, we analyze the success probability of local planners in belief space and in Section V, it is proved that, under mild conditions, FIRM-based methods are probabilistically complete under uncertainty. We conclude the paper in section VI.

II. Motion Planning Under Uncertainty

Mainly, uncertainty in planning originates from a lack of exact knowledge about the robot's motion model, the robot's sensing model, and the environment model, which

A. Agha-mohammadi and N. Amato are with the Dept. of Computer Science and Engineering and S. Chakravorty is with the Dept. of Aerospace Engineering, Texas A&M University, TX 77843, USA. Emails: {aliagha,schakrav,amato}@tamu.edu

The work of Agha-mohammadi and Chakravorty is supported in part by AFOSR Grant FA9550-08-1-0038 and the work of Amato is supported in part by NSF awards CRI-0551685, CCF-0833199, CCF-0830753, IIS-096053, IIS-0917266 by THECB NHARP award 000512-0097-2009, and by Chevron, IBM, Intel, Oracle/Sun.

are referred to as motion uncertainty, sensing uncertainty, and map uncertainty, respectively. In this paper, we focus on motion and sensing uncertainties, but some of the concepts are extensible to problems with map uncertainty. The Markov Decision Process (MDP) problem and the Partially Observable MDP (POMDP) are the most general formulations, respectively, for the planning problem under motion uncertainty and for the planning problem under both motion and sensing uncertainty.

While in the deterministic setting we seek an optimal path as a solution of the motion planning problem, in the stochastic setting we seek an optimal feedback (mapping) π as a solution of the motion planning problem. π as a solution of an MDP, is a mapping from the state space to the control space and π as a solution of a POMDP is a mapping from the belief space to the control space. In the rest of paper, we focus on POMDPs, as they are more general. However, all statements can be rephrased for MDPs.

A. POMDP

In solving POMDP problems, we deal with the following components:

- $X \in \mathbb{X}$ denotes the robot state (generalization of the robot's configuration) and $\mathbb{X}$ is the state space. $u \in \mathbb{U}$ is the control input and $\mathbb{U}$ is the control space. $z \in \mathbb{Z}$ is the observation and $\mathbb{Z}$ is the observation space.
- b_k is the belief at the k-th step, which is the pdf of the system state conditioned on the obtained measurements up to the k-th time step, $b_k = p(X_k|z_{0:k})$. Space $\mathbb{B}$ is the belief space of the problem, containing all possible beliefs, $b \in \mathbb{B}$.
- $p(X'|X,u)$ and $p(b'|b,u)$ are the state and belief transition pdf's, respectively. Also, $p(z|X)$ is the observation pdf conditioned on the system's state.
- $c(b,u)$ is the step cost, i.e., the cost of taking control u at belief b.
- $\pi(\cdot) : \mathbb{B} \to \mathbb{U}$ is the solution of the POMDP, which is a mapping (feedback) that assigns a control action for every belief in belief space. It is well known that the infinite horizon POMDP problem can be cast as a belief MDP problem [3], [17], whose solution is obtained by solving the following stationary Dynamic Programming (DP) equation for all b on the belief space $\mathbb{B}$ [3], [17]:

$$J(b) = \min_u \{c(b,u) + \int_{\mathbb{B}} p(b'|b,u)J(b')db'\}, \tag{1a}$$

$$\pi(b) = \arg\min_u \{c(b,u) + \int_{\mathbb{B}} p(b'|b,u)J(b')db'\}. \tag{1b}$$

- $J(\cdot) : \mathbb{B} \to \mathbb{R}$ is called the cost-to-go (or value) function, that assigns a cost-to-go for every belief in belief space.
- Π is the set of admissible policies. The mapping π lives in the function space and can have extremely complex formats. Thus, often, in solving POMDPs the set of admissible policies, i.e., Π is chosen as a rich subset of this huge space, on which the optimization in (1) can be carried out.

B. FIRM

Solving the belief MDP in (1) over continuous state, control, observation, and belief spaces, and finding the best feedback $\pi \in \Pi$ is a challenge, in particular in the presence of state constraints such as obstacles in the environments.

Inspired by the sampling-based methods, FIRM [7] samples local controllers, whose concatenation results in a feedback controller $\pi \in \Pi$ that approximates the solution of (1), over continuous spaces in the presence of obstacles. FIRM represents a family of algorithms that utilize feedback controllers to reach pre-defined nodes in belief space and thus construct a true roadmap (i.e., a roadmap with independent local planners and edge costs) in the belief space. Any choice of filter and controller that can fulfil the "belief reachability" in belief space, can introduce an instantiation of the general FIRM template. The LQG-based FIRM is introduced in [7] as an example instantiation of the general FIRM framework.

The FIRM graph is a generalization of the roadmap graph in Probabilistic Roadmap Method (PRM), whose nodes are small subsets of belief space and whose edges are Markov chains induced by feedback controllers. As a result, planning on FIRM is a Markov Decision Process (MDP) in the belief space, which is defined on FIRM nodes (a finite set), and thus it can be solved using standard Dynamic Programming (DP) techniques [17].

In the following, we briefly explain the FIRM framework and the elements used in its construction. First, let us denote the obstacles by F and the rest of state space by $\mathbb{X}_{free}$, i.e., $F \cap \mathbb{X}_{free} = \emptyset$ and $F \cup \mathbb{X}_{free} = \mathbb{X}$.

Underlying PRM: FIRM-based methods are constructed (or, parametrized) by an underlying PRM. This PRM can be considered as an input of the FIRM construction algorithm. We denote the nodes and edges of this underlying PRM by $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^{N_v}$ and $\mathcal{E} = \{\mathbf{e}_{ij}\}$.

FIRM nodes and local planners: Corresponding to every PRM node $\mathbf{v}_j$, there exists a FIRM node B_j. FIRM nodes are disjoint subsets of the belief space, i.e., $B_j \subset \mathbb{B}$ or $B_j \in \mathfrak{B}_{\mathbb{B}}$, where $\mathfrak{B}_{\mathbb{B}}$ denotes the sigma-algebra on the belief space $\mathbb{B}$. Corresponding to every PRM edge $\mathbf{e}_{ij}$, there exists a belief-space local planners (or controller) $\mu^{ij}(\cdot;\mathbf{v}_j) : \mathbb{B} \to \mathbb{U}$ in FIRM that drives the robot belief from the node B_i to the node B_j. Local controller μ^{ij} is parametrized by the j-th PRM node $\mathbf{v}_j$. It generates control inputs until reaching B_j or hitting an obstacle. Set $\mathbb{V} = \{B_i\}_{i=1}^{N_v}$ denotes the set of all FIRM nodes and $\mathbb{M} = \{\mu^{ij}\}$ is the set of all local controllers in FIRM. $\mathbb{M}(i) = \{\mu^{ij} \in \mathbb{M} | \exists \mathbf{e}_{ij} \in \mathcal{E}\}$ denotes the set of local controllers that can be invoked from the node B_i.

Induced pdf's and measures: Each local controller μ induces one-step transition pdf's over the state and belief spaces, respectively denoted by $p^{\mu}(X'|X)$ and $p^{\mu}(b'|b)$. Similarly, $\mathbb{P}_n(\cdot|b,\mu) : \mathfrak{B}_{\mathbb{B}} \to [0,1]$ is the probability measure over the belief space, induced by the local controller μ after n steps, starting from the belief b. Therefore, $\mathbb{P}(B_j|b,\mu^{ij}) := \sum_{n=0}^{\infty} \mathbb{P}_n(B_j|b,\mu^{ij})$ is the probability of landing in B_j before hitting obstacles F, and $\mathbb{P}(F|b,\mu^{ij})$ is the probability of colliding with obstacles F before landing in B_j, both under the controller μ^{ij} taken at b.

Main condition on FIRM: The main condition on the FIRM nodes and local controllers is that in the absence of obstacles, the B_j has to be reachable under μ^{ij}, for all i, j, i.e., $\mathbb{P}(B_j|b,\mu^{ij})=1$, if $F=\emptyset$.

FIRM MDP: Similar to the PRM, where the desired result is a path made by concatenation of the roadmap edges, in FIRM the desired solution (π in (5)) is a policy made by concatenation of the local planners. The law of this concatenation is encoded in the optimal mapping $\pi^g(\cdot):\mathbb{V}\to\mathbb{M}$ that decides which local controller has to be taken at every FIRM node. In [7], it is shown that the π^g is the solution of an (N_v+1)-state MDP, called FIRM MDP, whose corresponding DP equation is:

$$
\begin{aligned}
J^g(B_i) = \min_{\mathbb{M}(i)}\{&C^g(B_i,\mu^{ij}) + J^g(F)\mathbb{P}^g(F|B_i,\mu^{ij}) \\
&+ J^g(B_j)\mathbb{P}^g(B_j|B_i,\mu^{ij})\}, \quad (2a)\\
\pi^g(B_i) = \arg\min_{\mathbb{M}(i)}\{&C^g(B_i,\mu^{ij}) + J^g(F)\mathbb{P}^g(F|B_i,\mu^{ij}) \\
&+ J^g(B_j)\mathbb{P}^g(B_j|B_i,\mu^{ij})\}, \quad (2b)
\end{aligned}
$$

where $J^g:\{\mathbb{V},F\}\to\mathbb{R}$ is the cost-to-go function over the FIRM nodes, that assigns a cost-to-go for every FIRM node B_i. $J^g(F)$ is some suitable user-defined cost for hitting obstacles. Mapping $C^g:\mathbb{V}\times\mathbb{M}\to\mathbb{R}$ computes the cost of taking a controller in a FIRM node, and $\mathbb{P}^g:\{\mathbb{V},F\}\times\mathbb{V}\times\mathbb{M}\to[0,1]$, encodes the transition probabilities between FIRM nodes, which are all defined as:

$$
\forall b\in B_i, \forall i,j \quad
\begin{cases}
J^g(B_i) := J(b_c^i) \approx J(b),\\
C^g(B_i,\mu^{ij}) := C(b_c^i,\mu^{ij}) \approx C(b,\mu^{ij}),\\
\mathbb{P}^g(\cdot|B_i,\mu^{ij}) := \mathbb{P}(\cdot|b_c^i,\mu^{ij}) \approx \mathbb{P}(\cdot|b,\mu^{ij}),
\end{cases}
$$

in which, b_c^i is a point in B_i (usually its center if B_i is a ball). For a sufficiently small B_i, any point in B_i can be considered as b_c^i and hence above approximation follows for smooth cost and transition probability functions.

$C(b,\mu^{ij})$ represents the expected cost of invoking local controller μ^{ij} starting at belief state b until the local controller stops executing, i.e.,

$$C(b,\mu^{ij}) = \sum_{t=0}^{\mathcal{T}^{ij}} c(b_t,\mu^{ij}(b_t)|b_0=b), \quad (3)$$

$$\mathcal{T}^{ij}(b) = \inf_t\{t|b_t\in B_j, b_0=b\}, \quad (4)$$

where $\mathcal{T}^{ij}$, is a random stopping time denoting the time at which the belief state enters the FIRM node B_j under the controller μ^{ij}. Also, "inf" in (4) returns the infimum of the set.

FIRM policy: π is the overall feedback generated using FIRM, which is constructed by concatenation of the local controllers μ^{ij}, based on the policy π^g on the graph. Suppose at k-th time step the active local controller is shown by $\mu_k\in\mathbb{M}$. Then, this local controller remains same at the future time steps, i.e., $\mu_{k+1}=\mu_k$, and keeps generating control signals based on the belief at each time step, until the belief reaches the corresponding stopping region, denoted by $B(\mu)$. For example, if the controller μ^{ij} is chosen, the stopping region is B_j, i.e., $B_j=B(\mu^{ij})$. Once belief enters the stopping region $B(\mu)$, the higher level decision making is performed over graph by π^g that chooses the next local controller, i.e., $\mu_{k+1}=\pi^g(B(\mu_k))$. Thus, this hybrid policy is stated as follows:

$$\pi:\mathbb{B}\to\mathbb{U}, \quad (5)$$

$$u_k=\pi(b_k)=\begin{cases}\mu_k(b_k),\ \mu_k=\pi^g(B(\mu_{k-1})), & \text{if } b_k\in B(\mu_{k-1})\\ \mu_k(b_k),\ \mu_k=\mu_{k-1}, & \text{if } b_k\notin B(\mu_{k-1})\end{cases}$$

Initial controller: Given the initial belief b_0 is in one of the FIRM nodes, we choose the best local controller using π^g. However, if the b_0 is not in any of the FIRM nodes, we first compute the mean state $\mathbb{E}[x_0]$ based on b_0. Then, we add the $\mathbb{E}[x_0]$ to the PRM graph using a set of edges denoted by $\mathcal{E}(0)$. Afterwards, for every $e_{ij}\in\mathcal{E}(0)$, we design local controllers μ^{ij} and compute the transition cost $C(b_0,\mu^{ij})$ and probabilities $\mathbb{P}(B_j|b_0,\mu^{ij})$ and $\mathbb{P}(F|b_0,\mu^{ij})$ of taking these local controllers at b_0. We denote the set of newly added local controllers by $\mathbb{M}(0)$. Accordingly, we compute the μ_0:

$$
\mu_0(\cdot)=\begin{cases}\arg\min\limits_{\mu^{ij}\in\mathbb{M}(0)}\{C(b_0,\mu^{ij})+\mathbb{P}(B_j|b_0,\mu^{ij})J^g(B_j)\\ \qquad +\mathbb{P}(F|b_0,\mu^{ij})J^g(F)\}, & \text{if } \nexists l, b_0\in B_l\\ \pi^g(B_l), & \text{if } \exists l, b_0\in B_l\end{cases} \quad (6)
$$

Admissible policies: Π is the set of admissible policies. It is worth noting the mapping π is parametrized by the PRM nodes, i.e., $\mathcal{V}$. More rigorously it can be written as $\pi(\cdot;\mathcal{V})$. For a given environment, there are infinite possible PRM graphs (and thus $\mathcal{V}$'s), any of which gives rise to a FIRM policy π. The set of all these possible FIRM policies are referred to as "admissible policies" and is denoted by Π.

Algorithms: The generic algorithms for offline construction of FIRM is presented in Algorithm 1. The concrete instantiation of this algorithm for LQG-based FIRM is given in [7].

Algorithm 1: Offline Construction of FIRM

1 Construct a PRM with nodes $\mathcal{V}=\{\mathbf{v}_j\}$ and edges $\mathcal{E}=\{\mathbf{e}_{ij}\}$;
2 For each edge-node pair $(\mathbf{e}_{ij},\mathbf{v}_j)$ in PRM, design a proper controller-node pair (μ^{ij},B_j) in FIRM;
3 For each B_i and $\mu^{ij}\in\mathbb{M}(i)$, compute the transition cost and probabilities $C^g(B_i,\mu^{ij})$, $\mathbb{P}^g(B_j|B_i,\mu^{ij})$, and $\mathbb{P}^g(F|B_i,\mu^{ij})$;
4 Solve the FIRM MDP in (2) to compute feedback π^g over FIRM nodes and compute π accordingly using (5).

In the online phase of planning, we only need to compute and invoke the controller $\mu_0(\cdot)$ in (6), to absorb the robot into some FIRM node B_i, and then let feedback π in (5) generate the controls until absorption to the goal node B_{goal} or failure.

Success probability: We would also like to quantify the quality of the resulted planner π. Let us assume $B_{goal}=$

B_{N_v} without loss of generality. In the view of the fact that FIRM MDP is defined on the space $\{\mathbb{V}, F\}$, which has two absorbing states B_{goal} and F, we define the matrix $\mathcal{Q}$, to describe the transition probabilities, between transient states, whose ij-th entry is $\mathcal{Q}[i,j] = \mathbb{P}(B_i|B_j, \pi^g(B_j))$, for $i,j \in \{1,2,\cdots,N_v-1\}$. Similarly, we define the $(N_v-1)\times 1$ vector $\mathcal{R}_g$ that represent the probability of getting absorbed into the goal node, whose j-th entry is $\mathcal{R}_g[j] = \mathbb{P}(B_{goal}|B_j, \pi^g(B_j))$. Then, based on discrete absorbing Markov chain theory [18] the success probability from any node $B_i \in \mathbb{V}\setminus B_{goal}$ is computed as:

$$\mathbb{P}(\text{success}|B_i, \pi^g) = \Gamma_i^T(I-\mathcal{Q})^{-1}\mathcal{R}_g, \tag{7}$$

where Γ_i is a column vector with all elements equal to zero but the i-th element, which is one. Thus, the success probability under π from initial belief b_0 is given by:

$$\mathbb{P}(\text{success}|b_0,\pi)=\mathbb{P}(B(\mu_0)|b_0,\mu_0)\mathbb{P}(\text{success}|B(\mu_0),\pi^g), \tag{8}$$

where μ_0 is given by (6) and $B(\mu_0)$ is its corresponding stopping region.

III. Probabilistic completeness under uncertainty

We start by reviewing the definition of success and probabilistic completeness in the deterministic case, and then we extend these definitions to the stochastic case.

Success in the deterministic case: In the deterministic case, such as conventional PRM, the outcome of the planning algorithm is a path. Thus, success is defined for paths: For a given initial and goal point, a successful path is a path connecting the start point to the goal point, which entirely lies in the obstacle-free space.

Probabilistic completeness in the deterministic case: In the absence of uncertainty, a sampling-based motion planning algorithm is probabilistically complete if by increasing the number of samples, the probability of finding a successful path, if one exists, asymptotically approaches to one.

Difference between deterministic and probabilistic case: In the presence of uncertainty, success cannot be defined for a path and it has to be defined for a policy. Indeed, on a given path, different policies may result in different success probabilities. Moreover, under uncertainty, one can only assign the probability for reaching goal. Thus, to define a success for a policy we consider a threshold p_{min} and decide about success or failure accordingly.

Successful policy: In the presence of uncertainty, the solution of the planning algorithm is a policy (feedback) within the class of admissible policies. Therefore, success is defined for policies: For a given initial belief b_0 and goal region B_{goal}, a successful policy is a policy within the class of admissible policies under which the probability of reaching goal from the given initial point is greater than some predefined threshold p_{min}. In other words, $\pi \in \Pi$ is successful if $\mathbb{P}(\text{success}|b_0,\pi) := \mathbb{P}(B_{goal}|b_0,\pi) > p_{min}$.

Feasible initial belief: A belief $b_0 \in \mathbb{B}$ is a feasible initial belief for class Π, if there exists a policy $\pi \in \Pi$ such that $\mathbb{P}(\text{success}|b_0,\pi) > p_{min}$. The set of all feasible initial beliefs corresponding to a class Π is denoted by $\mathbb{B}_\Pi$. It is worth noting that the richer the set of admissible policies Π, the greater the set of feasible initial beliefs $\mathbb{B}_\Pi$. For example, in obstacle-free FIRM with stationary Linear Quadratic Gaussian (LQG) controllers as the local controllers, the set of all Gaussian beliefs is a subset of $\mathbb{B}_\Pi$ [7].

Globally successful policy: Instead of a single initial belief, we can also define the concept of successful policy for $\mathbb{B}_\Pi$. For a given goal region, policy $\pi \in \Pi$ is called globally successful, if the probability of reaching goal from any belief in $\mathbb{B}_\Pi$ is greater than p_{min}. In other words, $\pi \in \Pi$ is globally successful if $\mathbb{P}(\text{success}|b_0,\pi) = \mathbb{P}(B_{goal}|b_0,\pi) > p_{min},\ \forall b_0 \in \mathbb{B}_\Pi$.

Probabilistic completeness under uncertainty: Probabilistic completeness can be defined based on either one of the definitions for the successful policy. Suppose there exists a (globally) successful policy $\pi \in \Pi$. Then, a sampling-based motion planning algorithm is probabilistically complete under uncertainty, if by increasing the number of samples without bound, the probability of finding a (globally) successful policy is one. In other words, if there exists a globally successful policy $\pi \in \Pi$, we have following property:

$$\lim_{N_v\to\infty}\mathbb{P}(B_{goal}|b_0,\pi(\cdot;\mathcal{V})) > p_{min} \quad \forall b_0 \in \mathbb{B}_\Pi, \tag{9}$$

where $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^{N_v}$.

IV. Analysis of local planners in belief space

Before stating the results on the probabilistic completeness of the sampling-based methods under uncertainty, in this section, we analyze the success probability of the local planners in belief space and prove a continuity result on the success probability of local planners.

Notation: The norm $\|\cdot\|$ is the supremum norm, when it is applied to functions. The norm $\|\cdot\|_{op}$ is applied on operators and it stands for the operator norm [19]. It is worth noting that, in this paper, by the word “continuous”, we mean “Lipschitz continuous”.

h-state space: $\mathbb{X}_h = \mathbb{X}\times\mathbb{B}$ is referred to as hyper-state (or h-state) space that contain all possible h-states (state-belief pairs), $\mathcal{X} = (X,b) \in \mathbb{X}_h$. The $p^\mu(\mathcal{X}'|\mathcal{X})$ denotes the one-step transition pdf induced by the local planner μ, over the h-state space. Also, $\mathbb{P}_n(\cdot|\mathcal{X},\mu) : \mathfrak{B}_{\mathbb{X}_h} \to [0,1]$ is the probability measure over the h-state space, induced by the local controller μ after n steps, starting from the h-state $\mathcal{X}$. Set $\mathfrak{B}_{\mathbb{X}_h}$ is the sigma-algebra of the h-state space $\mathbb{X}_h$.

Local planner and extended stopping region: The role of the (i,j)-th local planner (or local controller) i.e., μ^{ij}, in the belief space, is to drive the belief from the belief node B_i to the belief node B_j in belief space. Therefore B_j is the stopping region of the local planner μ^{ij}. In the presence of obstacles, we extend the concept of stopping region to include the obstacles also. The stopping regions in the belief space $\{B_j\}$ and the stopping region in the state space F, both can be extended to the h-state space, respectively denoted by $\{\mathcal{B}_j\}$ and $\mathcal{F}$, where $\mathcal{B}_j \subset \mathbb{X}_h$ and $\mathcal{F} \subset \mathbb{X}_h$:

$$\mathcal{B}_j = \{(X,b) | X \in \mathbb{X}_{free}, b \in B_j\}, \tag{10}$$

$$\mathcal{F} = \{(X,b) | X \in F, b \in \mathbb{B}\}, \tag{11}$$

$$\mathcal{S}_j = \mathcal{B}_j \cup \mathcal{F}, \quad \overline{\mathcal{S}}_j = \mathbb{X}_h \setminus \mathcal{S}_j, \tag{12}$$

where $\mathcal{S}_j$ and $\overline{\mathcal{S}}_j$, respectively, denote the entire stopping region and transient region under the local controller μ^{ij}.

Absorption probability of local planners: If under the dynamics induced by the local planner, the system reaches the target node $\mathcal{B}_j$, the local planner is considered to be successful and if the system hits an obstacle, the local planner is considered to be failed. The success probability of the local planners or the absorption probability into the FIRM nodes are computed through solving the following integral equation that comes from the law of total probability:

$$\mathbb{P}(\mathcal{B}_j|\mathcal{X}, \mu^{ij}) = \int_{\mathbb{X}_h} p^{\mu^{ij}}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}_j|\mathcal{X}', \mu^{ij})d\mathcal{X}' \tag{13}$$
$$= \int_{\mathcal{B}_j} p^{\mu^{ij}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' + \int_{\overline{\mathcal{S}_j}} p^{\mu^{ij}}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}_j|\mathcal{X}', \mu^{ij})d\mathcal{X}',$$

where the second equality in (13) follows from substituting following conditions, inherited from FIRM construction, into the first integral:

$$\mathbb{P}(\mathcal{B}_j|\mathcal{X}, \mu^{ij}) = \begin{cases} 1, \text{ if } \mathcal{X} \in \mathcal{B}_j \\ 0, \text{ if } \mathcal{X} \in \mathcal{F} \end{cases}. \tag{14}$$

Henceforth, we drop the indices i and j to unclutter the expressions. Thus, we can write:

$$\mathbb{P}(\mathcal{B}|\mathcal{X}, \mu) = \int_{\mathcal{B}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}' + \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})\mathbb{P}(\mathcal{B}|\mathcal{X}', \mu)d\mathcal{X}'$$
$$= R(\mathcal{X}) + \mathbf{T}_{\mathcal{S}}\left[\mathbb{P}(\mathcal{B}|\cdot, \mu)\right](\mathcal{X}), \tag{15}$$

where the operator $\mathbf{T}_{\mathcal{S}}$ and the function $R(\mathcal{X})$ are defined as:

$$\mathbf{T}_{\mathcal{S}}\left[f(\cdot)\right](\mathcal{X}) := \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X})f(\mathcal{X}')d\mathcal{X}', \tag{16}$$

$$R(\mathcal{X}) := \int_{\mathcal{B}} p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'. \tag{17}$$

The solution of the integral equation in (15) is expressed in the following as a Liouville-Neumann series [19], similar to the solution of inhomogeneous Fredholm equation of second type [19].

$$\mathbb{P}(\mathcal{B}|\mathcal{X}, \mu) = \sum_{n=1}^{\infty} \mathbf{T}^n_{\mathcal{S}}\left[R(\cdot)\right](\mathcal{X}). \tag{18}$$

We show that the series in (18) is a convergent series, by resorting to the following assumption, which is a weaker version of the aforementioned FIRM condition on the design of node-local controller pair.

Assumption 1: We assume that after sufficient number of steps, say $N < \infty$, the probability of arriving into $\mathcal{S}_j$, the stopping region of μ^{ij}, is greater than arbitrary number $\beta > 0$ regardless of the starting point, i.e., we assume for each μ^{ij}, $\mathbb{P}_n(\mathcal{S}_j|\mathcal{X}, \mu^{ij}) \geq \beta > 0$, for all $n > N$ and $\mathcal{X}$.
This is a reasonable assumption for a properly designed controller as it rephrases the role of controller, in a probabilistic setting, in driving the system toward the target region[1].

Lemma 1: Given Assumption 1, we have:

$$\begin{cases} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq 1, & n < N \\ \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq 1 - \beta < 1, & n \geq N \\ \sum_{n=0}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq c < \infty. \end{cases} \tag{19}$$

Proof: See Appendix I. ∎

Corollary 1: Series $\sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}}[R]$ is a convergent series, and therefore, we can define the resolvent operator $(I - \mathbf{T}_{\mathcal{S}})^{-1}[R] = \sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}}[R]$, where $\|(I - \mathbf{T}_{\mathcal{S}})^{-1}\|_{op} \leq c < \infty$.

Proof: See Appendix II. ∎

According to Corollary 1, the success probability can be written using the defined resolvent operators as:

$$\mathbb{P}(\mathcal{B}|\mathcal{X}, \mu) = (I - \mathbf{T}_{\mathcal{S}})^{-1}[R(\cdot)](\mathcal{X}). \tag{20}$$

As the main result of this section (Proposition 1), we aim to show that this absorption probability varies continuously with respect to changes in the parameters of the local planner. However, we first state two more assumptions.

Assumption 2: We assume the local planning law and induced transition probabilities are smooth, i.e.,

- Local control laws are continuous in their parameters, i.e., for the ij-th local controller, mapping $\mu^{ij}(\cdot; \mathbf{v}_j) : \mathbb{B} \to \mathbb{U}$ is a continuous function in its parameter $\mathbf{v}_j$.
- The transition pdf on h-state, i.e., $p(\mathcal{X}'|\mathcal{X}, u)$ is a continuous function of control u, i.e., there exists a $c_1 < \infty$, such that $\|p(\mathcal{X}'|\mathcal{X}, u) - p(\mathcal{X}'|\mathcal{X}, \check{u})\| \leq c_1\|u - \check{u}\|$.

Finally, we state the following assumption, in which we emphasize the fact that as $\mathbf{v} \to \check{\mathbf{v}}$, the probability measure induced by the local controller $\mu(\cdot; \mathbf{v})$ over the sets $\mathcal{B}$ and $\check{\mathcal{B}}$ have to converge, which is a reasonable assumption for an smooth control law.

Assumption 3: Consider the controllers $\mu(\cdot; \mathbf{v})$, and $\check{\mu}(\cdot; \check{\mathbf{v}})$, whose corresponding extended absorption regions are denoted by $\mathcal{B}$ and $\check{\mathcal{B}}$, respectively. We assume that there exist real numbers $r > 0$ and $c' < \infty$, such that for $\|\mathbf{v} - \check{\mathbf{v}}\| \leq r$, we have:

$$\|\mathbb{P}_1(\mathcal{B} \ominus \check{\mathcal{B}}|\mathcal{X}, \mu)\| \leq c'\|\mathbf{v} - \check{\mathbf{v}}\|, \tag{21}$$

where $\ominus$ is the symmetric difference operator, i.e., $\mathcal{B} \ominus \check{\mathcal{B}} = (\mathcal{B} \setminus \check{\mathcal{B}}) \cup (\check{\mathcal{B}} \setminus \mathcal{B})$.

Now, we state the main result of this section:

Proposition 1: (Continuity of absorption probabilities): Given Assumptions 1, 2, and 3, the absorption probability $\mathbb{P}(B_j|b, \mu^{ij})$ is continuous in parameter $\mathbf{v}_j$, for all i, j, and b.

Proof: See Appendix III. ∎

[1]The condition $\forall n > N$ can be replaced by a weaker statement of "infinitely often", defined in probability theory [20].

V. Probabilistic Completeness of FIRM

In this section, we present the main result on the probabilistic completeness of the FIRM-based methods (Theorem 1). The result is stated for the more general case of globally successful policies. However, we first start with the following proposition that concludes the continuity of the success probability of π (overall planner) given the continuity of the success probability of the individual local planners (μ^{ij}s).

Proposition 2: (Continuity of success probability of π): The success probability $\mathbb{P}(success|b_0,\pi)$ is continuous in $\mathcal{V}$, if the absorption probability $\mathbb{P}(B_j|b,\mu^{ij})$ is continuous in $\mathbf{v}_j$, for all i, j, and b.

Proof: Given that $\mathbb{P}(B_j|b,\mu^{ij})$ is continuous in $\mathbf{v}_j$, for all i, j and noticing that the elements of matrices $\mathcal{Q}$ and $\mathcal{R}_g$ in (7) are of the form $\mathbb{P}(B'|B,\mu)$, the $\mathbb{P}(\text{success}|B_j,\pi^g)$ is continuous in $\mathcal{V}$ based on (7). Moreover, $\mathbb{P}(B(\mu_0)|b_0,\mu_0)$, in (8), is continuous in $\mathcal{V}$, and therefore, based on (8), $\mathbb{P}(\text{success}|b_0,\pi)$ is continuous in $\mathcal{V}$. The more detailed proof can be found in the Appendix IV in [21]. ■

Now, we are ready to state the main theorem:

Theorem 1: Given Assumptions 1, 2, and 3, any planning algorithm under uncertainty that is generated based on FIRM framework (i.e. guarantees belief node reachability and induces a roadmap in the belief space with independent local planners) is probabilistically complete under uncertainty.

Proof: Based on the definition of probabilistic completeness under uncertainty, if there exists a globally successful policy $\breve{\pi}$, FIRM has to find a globally successful policy π as the number of FIRM nodes increases unboundedly. Thus, we start by assuming that there exists a globally successful policy $\breve{\pi}\in\Pi$. Since each policy in Π is parametrized by a PRM graph, there exists a PRM with nodes $\check{\mathcal{V}}=\{\check{\mathbf{v}}_i\}_{i=1}^N$ that parametrizes the policy $\breve{\pi}$. Since $\breve{\pi}$ is a globally successful policy, we know $\mathbb{P}(\text{success}|b_0,\breve{\pi})>p_{min}$ for all $b_0\in\mathbb{B}_\Pi$. Thus, we can define $\epsilon^*=\mathbb{P}(\text{success}|b_0,\breve{\pi})-p_{min}>0$.

Given Assumptions 1, 2, and 3, and based on Propositions 1 and 2, we know that $\mathbb{P}(\text{success}|b_0,\pi)$ is continuous with respect to the parameters of the local planners, i.e., for any $\epsilon>0$, there exists a $\delta>0$, such that if $\|\mathcal{V}-\check{\mathcal{V}}\|<\delta$, then $|\mathbb{P}(\text{success}|b_0,\pi(\cdot;\mathcal{V}))-\mathbb{P}(\text{success}|b_0,\breve{\pi}(\cdot;\check{\mathcal{V}}))|<\epsilon$. The notation $\|\mathcal{V}-\check{\mathcal{V}}\|<\delta$ means that $\|\mathbf{v}_i-\check{\mathbf{v}}_i\|<\delta$, for all i, or equivalently, $\mathbf{v}_i\in\check{\Omega}_i$, for all i, where $\check{\Omega}_i$ is a ball with radius δ, centred at $\check{\mathbf{v}}_i$.

Therefore, for ϵ^*, there exists a δ^* and corresponding regions $\{\check{\Omega}_i\}_{i=1}^N$, such that if we have a PRM whose nodes (or a subset of nodes[2]) satisfy the condition $\mathbf{v}_i^*\in\check{\Omega}_i$, for all $i=1,\cdots,N$, then the planner π parametrized by this PRM has a success probability greater than p_{min}, i.e., $\mathbb{P}(\text{success}|b_0,\pi(\cdot;\mathcal{V}))>p_{min}$, and therefore π is successful.

Since $\delta>0$, the regions $\check{\Omega}_i$ have a nonempty interior. Consider a PRM with a sampling algorithm, under which there is positive probability of sampling in $\check{\Omega}_i$, such as uniform sampling. Thus, starting with any PRM, if we increase the number of nodes, a PRM node will eventually be chosen at every $\check{\Omega}_i$, with probability one. Therefore the policy constructed based on these nodes will have a success probability greater than p_{min}, i.e., we eventually get a successful policy if one exists. Thus, FIRM is probabilistically complete. ■

[2]A subset of nodes is enough, because the success probability is a non-decreasing function in terms of the number of nodes.

VI. Conclusion

In this paper, we reformulated the sampling-based motion planning problem under uncertainty, in terms of the local planners and their interactions. Inspired by the concept of the probabilistic completeness in the deterministic situation, we introduced the concept of the probabilistic completeness under uncertainty. We analyzed the success probability of local planners in the belief space and accordingly, we proposed a way to verify the probabilistic completeness of the sampling-based motion planners in the belief space. Finally, we proved that, under mild conditions, FIRM-based motion planners in belief space are probabilistically complete under uncertainty.

References

[1] H. Choset, K. M. Lynch, S. Hutchinson, G. Kantor, W. Burgard, L. E. Kavraki, and S. Thrun, *Principles of robot motion: theory, algorithms, and implementations*. MIT Press, 2005.

[2] S. Lavalle, *Planning algorithms*. Cambridge University Press, 2006.

[3] S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics*. MIT Press, 2005.

[4] L. Kavraki, P. Švestka, J. Latombe, and M. Overmars, “Probabilistic roadmaps for path planning in high-dimensional configuration spaces,” *IEEE Transactions on Robotics and Automation*, vol. 12, no. 4, pp. 566–580, 1996.

[5] N. Amato, B. Bayazit, L. Dale, C. Jones, and D. Vallejo, “OBPRM: An Obstacle-Basaed PRM for 3D workspaces,” in *International Workshop on the Algorithmic Foundations of Robotics*, 1998, pp. 155–168.

[6] S. Lavalle and J. Kuffner, “Randomized kinodynamic planning,” *International Journal of Robotics Research*, vol. 20, no. 378-400, 2001.

[7] A. Agha-mohammadi, S. Chakravorty, and N. Amato, “FIRM: Feedback controller-based Information-state RoadMap -a framework for motion planning under uncertainty-,” in *International Conference on Intelligent Robots and Systems (IROS)*, 2011.

[8] S. Prentice and N. Roy, “The belief roadmap: Efficient planning in belief space by factoring the covariance,” *International Journal of Robotics Research*, vol. 28, no. 11-12, October 2009.

[9] R. Alterovitz, T. Siméon, and K. Goldberg, “The stochastic motion roadmap: A sampling framework for planning with markov motion uncertainty,” in *Proceedings of Robotics: Science and Systems (RSS)*, June 2007.

[10] S. Chakravorty and S. Kumar, “Generalized sampling-based motion planners,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 41, no. 3, pp. 855–866, 2011.

[11] L. Kavraki, M. Kolountzakis, and J. Latombe, “Analysis of probabilistic roadmaps for path planning,” *IEEE Transactions on Robotics and Automation*, vol. 14, pp. 166–171, February 1998.

[12] A. Ladd and L. Kavraki, “Measure theoretic analysis of probabilistic path planning,” *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 229–242, April 2004.

[13] P. Švestka and M. Overmars, “Motion planning for car-like robots using a probabilistic learning approach,” *International Journal of Robotics Research*, vol. 16, no. 2, pp. 119–143, 1997.

[14] R. Bohlin, “Robot path planning,” Ph.D. dissertation, Chalmers University of Technology, Goteborg, Sweden, 2002.

[15] L. Kavraki, J. Latombe, R. Motwani, and P. Raghavan, “Randomized query processing in robot motion planning,” in *Proc. ACM Symp. Theory of Computing*, 1995, pp. 353–362.

[16] D. Hsu, “Randomized single-query motion planning in expansive spaces,” Ph.D. dissertation, Department of Computer Science, Stanford University, Stanford, CA, 2000.

[17] D. Bertsekas, *Dynamic Programming and Optimal Control: 3rd Ed.* Athena Scientific, 2007.
[18] J. R. Norris, *Markov Chains*. Cambridge Univeristy Press, 1997.
[19] J. P. Keener, *Principles of Applied Mathematics: Transformation and Approximation, 2nd Edition*. Westview Press, 2000.
[20] S. I. Resnick, *A probability path*. Birkhauser, 1999.
[21] A. Agha-mohammadi, S. Chakravorty, and N. Amato, "Probabilistic completeness of the belief space motion planners," *Technical Report: TR11-006, Parasol Lab., CSE Dept., Texas A&M University*, 2011.

Appendix I
Proof of Lemma 1

Before proving Lemma 1, we state and prove following lemma:

Lemma 2: Consider the bounded function $0 \leq f(\mathcal{X}) \leq 1$, *and kernel* $k(\mathcal{X}', \mathcal{X}) \geq 0$. *Then, for any set* $\mathcal{A}$, *we have:*

$$\| \int_{\mathcal{A}} k(\mathcal{X}', \mathcal{X}) f(\mathcal{X}') d\mathcal{X}' \| \leq \| \int_{\mathcal{A}} k(\mathcal{X}', \mathcal{X}) d\mathcal{X}' \|. \quad (22)$$

Proof: Given the properties of $f(\cdot)$ and $k(\cdot,\cdot)$, we have $k(\mathcal{X}', \mathcal{X}) f(\mathcal{X}') \leq k(\mathcal{X}', \mathcal{X})$, for all $\mathcal{X}$ and $\mathcal{X}'$. Taking the integral from both sides with respect to $\mathcal{X}'$ and then taking the supremum norm with respect to $\mathcal{X}$, the result follows. ■

Now, we prove Lemma 1.

Proof: If we denote the domain of operator $\mathbf{T}_{\mathcal{S}}$ by $\mathcal{D}$, we know that for all $f \in \mathcal{D}$, we have $0 \leq f(\mathcal{X}) \leq 1$, because $f(\mathcal{X})$ is the probability of some given set $\mathcal{S}$ under some given controller invoked at point $\mathcal{X}$. Thus, it cannot be negative or greater than one and based on Lemma 2, we have:

$$\begin{aligned} \|\mathbf{T}_{\mathcal{S}}\,[f]\,\| &= \| \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X}) f(\mathcal{X}') d\mathcal{X}' \| \leq \| \int_{\overline{\mathcal{S}}} p^{\mu}(\mathcal{X}'|\mathcal{X}) d\mathcal{X}' \| \\ &= \|\mathbb{P}_1(\overline{\mathcal{S}}|\mathcal{X}, \mu)\| \leq 1. \quad (23) \end{aligned}$$

Therefore, based on the definition of operator norm, we have:

$$\|\mathbf{T}_{\mathcal{S}}\|_{op} = \sup_{f(\cdot)} \{ \|\mathbf{T}_{\mathcal{S}}[f]\| \ : \ \forall f \in \mathcal{D}, \|f\| \leq 1 \} \leq 1. \quad (24)$$

According to Assumption 1, there exists a finite number N, such that:

$$\inf_{\mathcal{X}} \mathbb{P}_n(\mathcal{S}|\mathcal{X}, \mu) = \beta > 0 \ \ \forall n > N, \quad (25)$$

where "inf" and "sup" denote the infimum and supremum, respectively. Thus, we have

$$\begin{aligned} \|\mathbb{P}_n(\overline{\mathcal{S}}|\mathcal{X}, \mu)\| &= \sup_{\mathcal{X}} (1 - \mathbb{P}_n(\mathcal{S}|\mathcal{X}, \mu)) = 1 - \inf_{\mathcal{X}} \mathbb{P}_n(\mathcal{S}|\mathcal{X}, \mu) \\ &= 1 - \beta < 1 \quad \forall n > N. \quad (26) \end{aligned}$$

Let us denote the n-th iterated kernel of $\mathbf{T}_{\mathcal{S}}$ as $p_n(\mathcal{X}'|\mathcal{X}, \mu)$. Since this iterated kernel is a pdf, we have $p_n(\mathcal{X}'|\mathcal{X}, \mu) \geq 0, \ \forall \mathcal{X}, \forall \mathcal{X}', \forall n$. We can write:

$$\|\mathbf{T}^N_{\mathcal{S}}\,[f]\,\| = \| \int_{\overline{\mathcal{S}}} p_N(\mathcal{X}'|\mathcal{X}, \mu) f(\mathcal{X}') d\mathcal{X}' \|$$

$$\leq \| \int_{\overline{\mathcal{S}}} p_N(\mathcal{X}'|\mathcal{X}, \mu) d\mathcal{X}' \| = \|\mathbb{P}_N(\overline{\mathcal{S}}|\mathcal{X}, \mu)\| \leq \alpha < 1, \quad (27)$$

where $\alpha = 1 - \beta$, and similar to (24), we get $\|\mathbf{T}^N_{\mathcal{S}}\|_{op} \leq \alpha < 1$. From the operator norm properties, we have:

$$\|\mathbf{T}^{N+1}_{\mathcal{S}}\|_{op} \leq \|\mathbf{T}^N_{\mathcal{S}}\|_{op} \|\mathbf{T}_{\mathcal{S}}\|_{op} \leq \alpha < 1$$

and similarly for all $n \geq N$, we have:

$$\|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq \alpha < 1 \ \ \forall n \geq N.$$

Now, consider the series: $\sum_{i=1}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op}$. We can split the sum to smaller pieces as follows:

$$\sum_{n=1}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} = \sum_{n=1}^{N} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} + \sum_{i=1}^{\infty} \sum_{n=iN+1}^{(i+1)N} \|\mathbf{T}^n_{\mathcal{S}}\|_{op}.$$

But because $\|\mathbf{T}^{n+1}_{\mathcal{S}}\|_{op} \leq \|\mathbf{T}^n_{\mathcal{S}}\|_{op}$ for all $n \geq N$, we have

$$\sum_{n=iN+1}^{(i+1)N} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq N \|\mathbf{T}^{iN}_{\mathcal{S}}\|_{op}.$$

Also, we know

$$\|\mathbf{T}^{iN}_{\mathcal{S}}\|_{op} \leq \|\mathbf{T}^N_{\mathcal{S}}\|^i_{op} \leq \alpha^i$$

and thus, we have:

$$\begin{aligned} \sum_{n=1}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} &= \underbrace{\sum_{n=1}^{N} \|\mathbf{T}^n_{\mathcal{S}}\|_{op}}_{\leq N} + \sum_{i=1}^{\infty} \sum_{n=iN+1}^{(i+1)N} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \\ &\leq N + \sum_{i=1}^{\infty} N\alpha^i = N + \frac{N}{1-\alpha} = c < \infty. \end{aligned}$$

■

Appendix II
Proof of Corollary 1

Proof: We know $\|R\| \leq 1$, and thus we can write:

$$\| \sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}}[R] \| \leq \sum_{n=0}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \|R\| \leq \sum_{n=0}^{\infty} \|\mathbf{T}^n_{\mathcal{S}}\|_{op} \leq c < \infty.$$

Thus, series $\sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}}[R]$ is a convergent series and we can define the operator $(I - \mathbf{T}_{\mathcal{S}})^{-1}[R] = \sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}}[R]$. We have

$$\|(I - \mathbf{T}_{\mathcal{S}})^{-1}\|_{op} = \| \sum_{n=0}^{\infty} \mathbf{T}^n_{\mathcal{S}} \|_{op} \leq c < \infty. \quad (28)$$

■

Appendix III
Proof of Proposition 1

Before proving Proposition 1, we state and prove a lemma on the continuity of the transition probability induced by the local controllers in its parameter.

Lemma 3: Given Assumption 2, there exists a $c_2 < \infty$ *such that*

$$\|p(\mathcal{X}'|\mathcal{X}, \mu(b; \mathbf{v})) - p(\mathcal{X}'|\mathcal{X}, \check{\mu}(b; \check{\mathbf{v}}))\| \leq c_2 \|\mathbf{v} - \check{\mathbf{v}}\|. \quad (29)$$

Proof: The result directly follows by combining the two parts of Assumption 2. ■

Now, we are ready to prove the Proposition 1.

Proof: To show $\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)$ is continuous w.r.t. $\mathbf{v}$, we perturb $\mathbf{v}$ to some $\check{\mathbf{v}}$, such that $\|\mathbf{v}-\check{\mathbf{v}}\|<r$. The local controller associated with node $\check{\mathbf{v}}$ is referred to as $\check{\mu}$, whose successful absorption region is denoted by $\check{\mathcal{B}}$ and stopping region is $\check{\mathcal{S}}$. Similarly, the corresponding transient operator and recurrent function are referred to as $\check{\mathbf{T}}_{\check{\mathcal{S}}}$ and $\check{R}$. Finally, the success probability associated with the perturbed node $\check{\mathbf{v}}$ is $\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu})$. To shorten the statements, we refer to $\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)$ and $\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu})$, respectively, by $\mathfrak{P}(\mathcal{X})$ and $\check{\mathfrak{P}}(\mathcal{X})$. As a result of node perturbation, the success probability is perturbed as:

$$\begin{aligned}&\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)-\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu}):=\mathfrak{P}-\check{\mathfrak{P}}=R+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}]-\check{R}-\check{\mathbf{T}}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\\&=R-\check{R}+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}]-\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}]+\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}]-\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]+\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]-\check{\mathbf{T}}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\\&=(R-\check{R})+\mathbf{T}_{\mathcal{S}}[\mathfrak{P}-\check{\mathfrak{P}}]+(\mathbf{T}_{\mathcal{S}}-\mathbf{T}_{\check{\mathcal{S}}})[\check{\mathfrak{P}}]+(\mathbf{T}_{\check{\mathcal{S}}}-\check{\mathbf{T}}_{\check{\mathcal{S}}})[\check{\mathfrak{P}}],\end{aligned}$$

where

$$\mathbf{T}_{\check{\mathcal{S}}}\left[f(\cdot)\right](\mathcal{X}):=\int_{\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})f(\mathcal{X}')d\mathcal{X}'. \tag{30}$$

Let us define the operators $\mathbf{T}_{\Delta\mathcal{S}}:=(\mathbf{T}_{\mathcal{S}}-\mathbf{T}_{\check{\mathcal{S}}})$ and $\Delta\mathbf{T}_{\check{\mathcal{S}}}:=(\mathbf{T}_{\check{\mathcal{S}}}-\check{\mathbf{T}}_{\check{\mathcal{S}}})$. Now, based on Corollary 1, we can write:

$$\mathfrak{P}-\check{\mathfrak{P}}=(I-\mathbf{T}_{\mathcal{S}})^{-1}\left[R-\check{R}+\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]+\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\right], \tag{31}$$

and the following inequality holds on the supremum norm of the perturbation of the absorption probability:

$$\begin{aligned}&\|\mathfrak{P}-\check{\mathfrak{P}}\|\\&\le\|(I-\mathbf{T}_{\mathcal{S}})^{-1}\|_{op}\left(\|R-\check{R}\|+\|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\|+\|\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\right)\\&\le c\left(\|R-\check{R}\|+\|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\|+\|\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\right)\\&=c\left(\|K_1(\mathcal{X})\|+\|K_2(\mathcal{X})\|+\|K_3(\mathcal{X})\|\right),\end{aligned} \tag{32}$$

where $K_1(\mathcal{X}):=R(\mathcal{X})-\check{R}(\mathcal{X})$, $K_2(\mathcal{X}):=\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}(\cdot)](\mathcal{X})$, and $K_3(\mathcal{X}):=\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}(\cdot)](\mathcal{X})$. In the following, we bound K_1, K_2, and K_3, and thus bound $\|\mathfrak{P}-\check{\mathfrak{P}}\|$, accordingly.

1) Bound for $K_1(\mathcal{X})$: The supremum norm of $K_1(\mathcal{X})$ is:

$$\begin{aligned}&\|K_1(\mathcal{X})\|=\|R(\mathcal{X})-\check{R}(\mathcal{X})\|\\&=\|\int_{\mathcal{B}}p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'-\int_{\check{\mathcal{B}}}p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\|\\&=\|\int_{\mathcal{B}\cap\check{\mathcal{B}}}[p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})]d\mathcal{X}'\\&+\int_{\mathcal{B}-\check{\mathcal{B}}}p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'-\int_{\check{\mathcal{B}}-\mathcal{B}}p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\|\\&\le\int_{\mathcal{B}\cap\check{\mathcal{B}}}\|p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})\|d\mathcal{X}'\\&+\|\int_{\mathcal{B}-\check{\mathcal{B}}}p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'+\int_{\check{\mathcal{B}}-\mathcal{B}}p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\|\\&\overset{\text{from (29)}}{\le}\int_{\mathcal{B}\cap\check{\mathcal{B}}}c_2\|\mathbf{v}-\check{\mathbf{v}}\|d\mathcal{X}'+\|\mathbb{P}_1(\mathcal{B}\ominus\check{\mathcal{B}}|\mathcal{X},\mu)\|\\&\qquad+\|\mathbb{P}_1(\check{\mathcal{B}}\ominus\mathcal{B}|\mathcal{X},\check{\mu})\|\\&\overset{\text{from (21)}}{\le}c_2'\|\mathbf{v}-\check{\mathbf{v}}\|+2c'\|\mathbf{v}-\check{\mathbf{v}}\|=\gamma_1\|\mathbf{v}-\check{\mathbf{v}}\|,\end{aligned} \tag{33}$$

where $c_2'<\infty$ and $\gamma_1=c_2'+2c'<\infty$. In the penultimate inequality, we also used the fact that $\mathbb{P}_1(\check{\mathcal{B}}-\mathcal{B}|\mathcal{X},\check{\mu})\le\mathbb{P}_1(\check{\mathcal{B}}\ominus\mathcal{B}|\mathcal{X},\check{\mu})$ and $\mathbb{P}_1(\mathcal{B}-\check{\mathcal{B}}|\mathcal{X},\mu)\le\mathbb{P}_1(\mathcal{B}\ominus\check{\mathcal{B}}|\mathcal{X},\mu)$ because $\check{\mathcal{B}}-\mathcal{B}\subseteq\check{\mathcal{B}}\ominus\mathcal{B}$ and $\mathcal{B}-\check{\mathcal{B}}\subseteq\mathcal{B}\ominus\check{\mathcal{B}}$.

2) Bound for $K_2(\mathcal{X})$: We have:

$$\begin{aligned}&\|K_2(\mathcal{X})\|=\|\mathbf{T}_{\Delta\mathcal{S}}[\check{\mathfrak{P}}]\|=\|\mathbf{T}_{\mathcal{S}}[\check{\mathfrak{P}}]-\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\\&=\|\int_{\overline{\mathcal{S}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'-\int_{\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\\&=\|\int_{\overline{\mathcal{S}}-\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'-\int_{\overline{\check{\mathcal{S}}}-\overline{\mathcal{S}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\\&\le\|\int_{\overline{\mathcal{S}}-\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'+\int_{\overline{\check{\mathcal{S}}}-\overline{\mathcal{S}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\\&=\|\int_{\overline{\mathcal{S}}\ominus\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\overset{\text{from (22)}}{\le}\|\int_{\overline{\mathcal{S}}\ominus\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})d\mathcal{X}'\|\\&=\|\mathbb{P}_1(\overline{\mathcal{S}}\ominus\overline{\check{\mathcal{S}}}|\mathcal{X},\mu)\|\le\|\mathbb{P}_1(\overline{\mathcal{B}}\ominus\overline{\check{\mathcal{B}}}|\mathcal{X},\mu)\|\\&=\|\mathbb{P}_1(\mathcal{B}\ominus\check{\mathcal{B}}|\mathcal{X},\mu)\|\overset{\text{from (21)}}{\le}\gamma_2\|\mathbf{v}-\check{\mathbf{v}}\|,\end{aligned} \tag{34}$$

where $\gamma_2=c'<\infty$. The penultimate inequality and equality follow from the relations $\overline{\mathcal{S}}\ominus\overline{\mathcal{S}'}\subseteq\overline{\mathcal{B}}\ominus\overline{\mathcal{B}'}$ and $\overline{\mathcal{B}}\ominus\overline{\mathcal{B}'}=\mathcal{B}\ominus\mathcal{B}'$, respectively.

3) Bound for $K_3(\mathcal{X})$: We have:

$$\begin{aligned}&\|K_3(\mathcal{X})\|=\|\Delta\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|=\|\mathbf{T}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]-\check{\mathbf{T}}_{\check{\mathcal{S}}}[\check{\mathfrak{P}}]\|\\&=\|\int_{\overline{\check{\mathcal{S}}}}p^{\mu}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'-\int_{\overline{\check{\mathcal{S}}}}p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\\&=\|\int_{\overline{\check{\mathcal{S}}}}\left(p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})\right)\check{\mathfrak{P}}(\mathcal{X}')d\mathcal{X}'\|\\&\le\int_{\overline{\check{\mathcal{S}}}}\|p^{\mu}(\mathcal{X}'|\mathcal{X})-p^{\check{\mu}}(\mathcal{X}'|\mathcal{X})\|\|\check{\mathfrak{P}}(\mathcal{X}')\|d\mathcal{X}'\\&\overset{\text{from (29)}}{\le}\int_{\overline{\check{\mathcal{S}}}}c_2\|\mathbf{v}-\check{\mathbf{v}}\|d\mathcal{X}'=\gamma_3\|\mathbf{v}-\check{\mathbf{v}}\|,\end{aligned} \tag{35}$$

where $\gamma_3<\infty$.

Therefore, based on (33), (34), (35), and (32), we can conclude that:

$$\|\mathbb{P}(\mathcal{B}|\mathcal{X},\mu)-\mathbb{P}(\check{\mathcal{B}}|\mathcal{X},\check{\mu})\|\le\gamma\|\mathbf{v}-\check{\mathbf{v}}\|, \tag{36}$$

where $\gamma=c(\gamma_1+\gamma_2+\gamma_3)<\infty$, which completes the proof that the absorption probability under the controller μ is continuous in the PRM node $\mathbf{v}$. ■

Appendix IV
Proof of Proposition 2
Continuity of success probability

Proof: Given that $\mathbb{P}(B_j|b,\mu^{ij})$ is continuous wrt $\mathbf{v}_j$, for all i,j, we want to show that $\mathbb{P}(\text{success}|\pi,b_0)$ is continuous wrt all $\mathbf{v}_j$. First, let us look at the structure of the success probability.

$$\mathbb{P}(\text{success}|b_0,\pi){=}\mathbb{P}(B(\mu_0)|b_0,\mu_0)\mathbb{P}(\text{success}|B(\mu_0),\pi^g), \tag{37}$$

where μ_0 is computed using (6). The term $\mathbb{P}(B(\mu_0)|b_0,\mu_0)$ in the right hand side of (37) is continuous because the continuity of $\mathbb{P}(B_j|b,\mu^{ij})$ for all i,j is assumed in this lemma. Thus, we only need to show the continuity of the second term in (37). Without loss of generality we can consider $B_i = B(\mu_0)$. Then, it is desired to show that $\mathbb{P}(\text{success}|B_i,\pi^g)$ is continuous wrt $\mathbf{v}_i$ for all i.

As we saw in Section II-B, the probability of success from the i-th FIRM node is as follows:

$$\mathbb{P}(\text{success}|B_i,\pi^g) = \Gamma_i^T(I-\mathcal{Q})^{-1}\mathcal{R}_g, \tag{38}$$

Moreover, we can consider $B_{goal} = B_N$ without loss of generality; then, the (i,j)-th element of matrix $\mathcal{Q}$ is $\mathcal{Q}[i,j] = \mathbb{P}(B_i|B_j,\pi^g(B_j))$, and the j-th element of vector $\mathcal{R}_g$ is $\mathcal{R}_g[j] = \mathbb{P}(B_N|B_j,\pi^g(B_j))$. Since we considered the B_j as the stopping region of the local controller μ^{ij}, we have:

$$\mathbb{P}(B_j|B_i,\mu^{il}) = 0,\ \text{if } l \neq j. \tag{39}$$

Therefore, all the non-zero elements in the matrices $\mathcal{R}_g$ and $\mathcal{Q}$ are of the form $\mathbb{P}(B_j|B_i,\mu^{ij})$. Thus, Given the continuity of $\mathbb{P}(B_j|b,\mu^{ij})$, the transition probability $\mathbb{P}(B_j|B_i,\mu^{ij})$ are continuous and the matrices $\mathcal{R}_g$ and $\mathcal{Q}$ are continuous. Therefore, $\mathbb{P}(\text{success}|B_i,\pi^g)$ and thus $\mathbb{P}(\text{success}|b_0,\pi)$ are continuous wrt underlying PRM nodes. ∎