# What is a Dirac Operator?

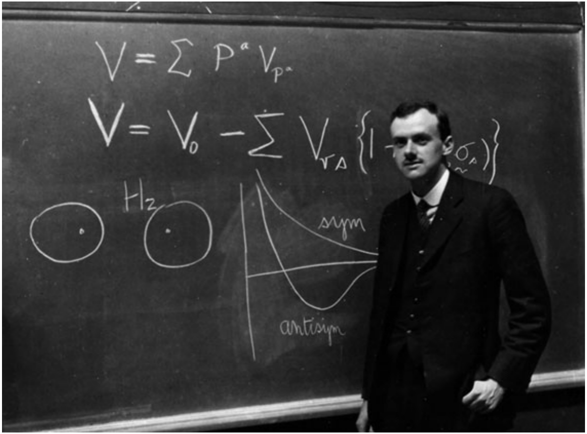
Juan Orduz

Berlin Mathematical School

Berlin Mathematical School

# Relativistic Quantum Mechanics?

The origin of the Dirac operator can be found in the attempt to find a relativistic wave equation for a free electron.



# Relativistic Quantum Free Electron?

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- A first ansatz for a relativistic wave equation is

$$(\partial_0^2 - m^2c^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\psi = 0,$$

# Dirac Equation

- We seek for a wave equation which is linear in $\partial_0$, so we look for coefficients $\alpha_j$ and $\beta$ such that the wave equation has the form

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- Moreover, we want a ”compatibility” condition:

$$(\partial_0 + \alpha_1\partial_1 - \alpha_2\partial_2 - \alpha_3\partial_3 - \beta)(\partial_0 - \alpha_1\partial_1 - \alpha_2\partial_2 - \alpha_3\partial_3 - \beta)$$
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If we set $\beta = \alpha_0 mc^2$ the we must have the following relations:

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = -2\delta_{\mu\nu}, \text{ for } \mu, \nu = 0, 1, 2, 3.$$

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# Clifford Algebra

Let $(V, g)$ be a finite dimensional inner product vector space. We define its **Clifford algebra** by

$$Cl(V, g) := \left( \bigoplus_{n \geq 0} V^{\otimes n} \right) \Big/ \mathcal{I}(V, g)$$

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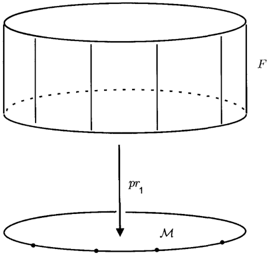
## Examples

- $Cl(\mathbb{R}, \langle , \rangle) = \mathbb{C}$
- $Cl(\mathbb{R}^2, \langle , \rangle) = \mathbb{H}$ (Quaternions)

# Vector Bundles & Connections



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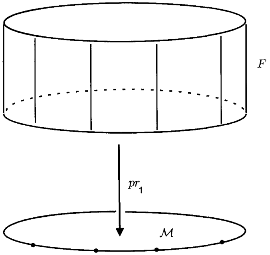


A **connection** $\nabla$ over a vector bundle $\mathcal{S} \longrightarrow M$ is a linear map

$$\nabla : \Gamma(\mathcal{S}) \longrightarrow \Gamma(T^*M) \otimes \Gamma(\mathcal{S})$$

which satisfies a Leibniz rule: $\nabla_X(fs) = X(f)s + f\nabla_X s$.

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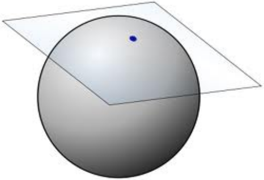
# Clifford Modules

Let $(M, g)$ be a closed (compact+without boundary) Riemannian smooth manifold.



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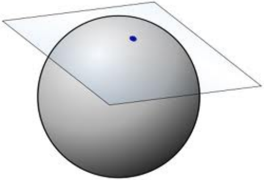

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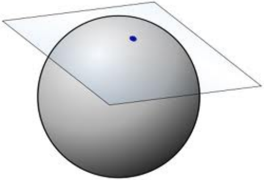

- The tangent bundle of a Riemannian manifold $(M, g)$ has a unique metric and torsion free connection $\nabla^{LC}$.
- We define $Cl(TM)$ to be the vector bundle whose fibers are $Cl(T_pM, g_p)$ for $p \in M$.

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- Each fiber $\mathcal{S}_p$ is a left module over $Cl(T_pM, g_p) \otimes \mathbb{C}$.
- There is a connection (**Clifford connection**) $\nabla$ on $\mathcal{S}$ which satisfies $\forall X, Y \in \Gamma(TM)$ and $s \in \Gamma(\mathcal{S})$,

$$\nabla_X(Y \cdot s) = (\nabla_X^{LC} Y) \cdot s + Y \cdot (\nabla_X s)$$

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  1. $X[h(s_1, s_2)] = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2)$.
  2. $h(X \cdot s_1, s_2) + h(s_1, X \cdot s_2) = 0$.

# Dirac Operator

We define the Dirac operator $D : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S})$ via the Clifford action and the Clifford connection

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M) \otimes \Gamma(\mathcal{S}) \xrightarrow{g} \Gamma(TM) \otimes \Gamma(\mathcal{S}) \xrightarrow{\cdot} \Gamma(\mathcal{S})$$

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## Example

The Dirac operator in $\mathbb{R}^n$ on its tangent bundle is given by

$$D = \sum_j e_j \cdot \partial_{e_j} \Rightarrow D^2 = -\Delta$$

# Some Properties of the Dirac Operator

We consider the Hilbert space $L^2(M, \mathcal{S})$ of square-integrable sections of $S$ with inner product

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- $D$ is a closable unbounded operator and $\bar{D}$ is a self-adjoint operator with $\mathrm{dom}(\bar{D}) = H^1(S)$.
- $D$ is a first order elliptic differential operator.

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# Example: Index-type theorem

- Let $S = \bigwedge^\bullet T^*M$ with Clifford action

$$v \cdot \omega = v^\flat \wedge \omega + \imath_v \omega, \quad \text{where} \quad v^\flat = g(v, \cdot)$$

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## Gauß-Bonnet Theorem

$$ind(D^+) = \int_M eu(M) = \chi(M)$$