#1 Space Complexity

Definition 1. *let M be a deterministic, deciding TM. The space complexity of M is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of tape cells that M scans on any input of length n. For nondeterministic N, $f(n)$ is the same but for any branch of computation for any length of input n.*

#2 Example

We will show that SAT runs in linear space. Recall that SAT can be decided by the machine M_1 on input $\langle \phi \rangle$, where ϕ is a Boolean formula:

1. Enumerate all 0-1 strings of length m, for variables $x_1, x_2, ..., x_m$ in ϕ. Treat each of these strings as a "truth assignment."
2. Evaluate ϕ on each truth assignment. For evaluation:
   (a) write a copy of ϕ on some segment of the tape;
   (b) replace each variable with its truth assignment;
   (c) evaluate the formula ϕ to 0 or 1.
3. if ϕ ever evaluates to 1, accept; otherwise, reject;

Think: how much space do we really need to run this? If $|w| = n$, then we may assume that $m \leq n$ (we have less variables than the length of the formula ϕ, due to overhead). So the space cost of M_1 is $m + |w|$, where $|w|$ is our copy of ϕ used for evaluation. This is $\mathcal{O}(n)$.

#3 And so...

$$\text{SPACE}(f(n)) = \{L \mid L \text{ decided by } \mathcal{O}(f(n)) \text{ space DTM } \}$$
$$\text{NSPACE}(f(n)) = \{L \mid L \text{ decided by } \mathcal{O}(f(n)) \text{ space NTM } \}$$

and, of course,

$$\mathsf{PSPACE} = \bigcup_k \text{SPACE}(n^k)$$
$$\mathsf{NPSPACE} = \bigcup_k \text{NSPACE}(n^k)$$

And so we just have already proved this claim above:

Claim 1. SAT $\in \mathsf{PSPACE}$

because SAT runs in linear-space.

#4 Savitch's Theorem

Here is an interesting result.

Theorem 1 (Savitch's Theorem)**.** *For any function $f : \mathbb{N} \to \mathbb{N}$, where $f(n) \geq n$,*

$$\textsc{nspace}(f(n)) \subseteq \textsc{space}(f^2(n))$$

In other words, any NTM that uses $f(n)$ of space can be converted to a DTM that uses only $f^2(n)$ space.
Questions:

1. The language $ALL_{\text{NFA}} = \{\langle N \rangle \mid N \text{ is an NFA and } L(A) = \Sigma^*\}$ runs in $\mathcal{O}(n)$ nondeterministic space. What space does it run in deterministically?
2. What about if A has nondeterministic space complexity $\mathcal{O}(n^3)$? $\mathcal{O}(2^n)$?

#5 Implications

Is the following claim true?

Claim 2. $\mathsf{PSPACE} = \mathsf{NPSPACE}$*?*

Proof. (Let the class give an answer.) The gist is simple:

1. $\mathsf{PSPACE} \subseteq \mathsf{NPSPACE}$ trivially.
2. $\mathsf{NPSPACE} \subseteq \mathsf{PSPACE}$ because of Savitch's theorem.

□

And this one:

Claim 3. $\mathsf{P} \subseteq \mathsf{PSPACE}$*?*

Proof. Time bounds space. □

similarly: $\mathsf{NP} \subseteq \mathsf{NPSPACE}$. How about this one?

Claim 4. $\mathsf{NP} \subseteq \mathsf{PSPACE}$*?*

Proof. see your lab. □

In summary, what we know now is that:

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXPTIME}$$