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Quiz #1A (Solutions)

This quiz is designed to familiarize me with the degree of your mathematical literacy. Full credit is given upon participation. Because it will be submitted anonymously, I will instruct you on how to receive participatory credit on the day it is given. (You are allowed to put your name in the top-right corner if you wish to de-anonymize your submission.)

This quiz is *not* designed to trick, judge, or challenge you. It just helps me avoid either (i) over-explaining content you understand well or (ii) under-explaining content you do not understand well. For this reason, I ask that you answer questions on which you are unsure according to the scale below.

The *Not-Clear (NC)* Scale. In the case that you are unsure on an answer, please do not guess. Instead, write the code, defined below, that most appropriately describes in what way you are confused by the problem.

- **NC0**—I just have no clue.
- **NC1**—This looks familiar but I couldn't really tell right from wrong.
- **NC2**—This looks familiar and I know I would have gotten this right in the past, but can't recall now.

If you find yourself on the fence between an NC code and a guess, please write both. In the case that this quiz is timed, please leave answers you did not get to blank.

#0 Definitions

Definition 1 (Set Difference)**.** *The* difference *of sets A and B, denoted $A \setminus B$, is the set of elements in A that are not in B. For example,*

$$\{0, 1, 2\} \setminus \{2\} = \{0, 1\}$$

#1 Sets and Set Operations

Define the sets R, S, and T as follows. Note that S is a set of functions.

$$\begin{aligned} R =& \{0, 1, 2\} \\ S =& \{n^2, 2^n, n!\} \\ T =& \{a, 42, R\} \end{aligned}$$

Label each of the following claims as either `T` (True) or `F` (False).

- __F__ $42 \in R$. *The symbol $\in$ means "in". 42 is not "in" R, as R is the set containing (only) 0, 1, and 2.*
- __F__ $42 \notin T$. *The symbol $\notin$ means "not in". 42 is "in" T, and so $42 \notin T$ is false.*
- __T__ $\{0, 1\} \subseteq R$. *The symbol $\subseteq$ means "subset"; refer to Sipser §0.2 for a definition. Here, it is true that $\{0, 1\}$ is a subset of R because both 0 and 1 are elements of R.*

- F $\{0,1\} \supset R$. *It is not true that R is a* proper subset *of $\{0,1\}$ because 2 is in R but not in $\{0,1\}$. Recall the definition of subset. Refer to Sipser §0.2 for defn of proper subset.*

- T The function f, defined by $f(x) = x!$, is an element of S. *This is true. I gave this question to show to you that functions are equivalent modulo renaming of their variables, and also for you to see that sets can really contain whatever we want to put in them.*

- F $R \cap S = \{0, 1, 2\}$. *This is false; the sets R and S have no elements in common. See definition of* set intersection *in Sipser §0.2.*

- T $R \cup S = \{0, 1, 2, n^2, 2^n, n!\}$. *This is true. See definition of* set union *in Sipser §0.2.*

- T $|R| = |S| = |T|$. *This is true because each set has exactly three elements. See definition of* cardinality *in §0.2.*

- F The difference between sets R and S (written $R \setminus S$) is the empty set. *This is false; the difference of R and S is just R. See def of set difference above.*

- T The power set of R, denoted $\mathcal{P}(R)$, is the set of all subsets of R. *This is simply the definition of a* powerset.

Write below the notation that describes "the ordered product of sets A, B, and C".

We write $A \times B \times C$ to denote the set of tuples of the form (a, b, c) where $a \in A$, $b \in B$, and $c \in C$. Refer to the subsection "Sequences and Tuples" in Sipser §0.2.

#2 Important Sets

Please draw arrows between the following sets and their descriptions. We will follow Sipser in letting the set of natural numbers ($\mathbb{N}$) start at 1. If unsure, write your NC code close to the left or right of the thing that confuses you.

$\emptyset$	$\mathbb{Q} \cup \{z \mid z \text{ is irrational}\}$	(1)
$\mathbb{N}$	$\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$	(2)
$\mathbb{R}$	$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$	(3)
$\mathbb{Q}$	$\{1, 2, 3, ...\}$	(4)
$\mathbb{Z}$	the empty set	(5)

In order:

1. $\emptyset$ is the empty set;
2. $\mathbb{N}$ is $\{1, 2, 3, ...\}$;
3. $\mathbb{R}$ is $\{z \mid z \text{ is irrational}\}$;
4. $\mathbb{Q}$ is $\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$;
5. $\mathbb{Z}$ is $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

As we are doing (largely) discrete mathematics, the most important sets for this course are $\mathbb{N}$ and $\mathbb{Z}$. I recommend learning each of these notations, still.

#3 Logical Quantification

What do $\forall$ and $\exists$ stand for?

Forall and *there exists*, respectively.

Translate the following mathematical statement to plain English.

$$\forall x \in \mathbb{Z}. \exists y \in \mathbb{Z}. x + y = 0$$

for all integers x there exists y such that $x + y = 0$, or: every integer in $\mathbb{Z}$ has an additive inverse. Sipser does not have much on this notation, opting instead to write the quantification of variables in plain English. I will try to do the same. Nevertheless, it is important that you know how to interpret this language, with or without symbols.

#4 Properties of Functions

Please fill in the blanks and answer the question below.

- A function $f : X \to Y$ is called surjective if, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- A function $g : X \to Y$ is called injective if, for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Surjective is also called "onto" and injective also called "one to one". Refer to the subsection titled "Function and Relations" in Sipser §0.2.

A function $f : X \to Y$ is called bijective if ...

It's both injective and surjective.

An equivalent definition of bijective is that there exists an inverse, $f^{-1} : Y \to X$, such that for all $x \in X$ and $y \in Y$ we have

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(y)) = y$$

We will touch on this later in the course when we it comes up.

#5 Properties of Relations

The binary relations below are defined below so that exactly one of each is *reflexive*, *symmetric*, and *transitive*.

$$\begin{aligned} A =& \{(0,1),(1,0),(1,1)\} \\ B =& \{(a,a),(b,b),(a,b)\} \\ C =& \{(x,y),(y,z),(x,z)\} \end{aligned}$$

- Which relation is *reflexive*? *B, because a and b are the only elements in the relation and they each relate to themselves.*
- Which relation is *symmetric*? *A, because 0 relates to 1 and 1 relates to 0, symmetrically. Further, 1 relates to 1, which is symmetrically just 1 relates to 1.*
- Which relation is *transitive*? *C, as x relates to y, y relates to z, and thus x relates to z.*

Refer to the end of "function and relations" in Sipser §0.2. These three properties of relations are very important! Please reach out on Teams if this is not clicking for you.

#6 Boolean Logic

Let the notation

- $\wedge$ denote Boolean conunction (`AND`);
- $\vee$ denote Boolean disjunction (`OR`);
- $\neg$ denote Boolean negation (`NOT`);
- $\oplus$ denote Boolean Exclusive-Or (`XOR`); and
- $\Rightarrow$ denote Boolean implication.

Label each of the following boolean statements as either `T` (True) or `F` (False).

- <u>T</u> $\neg T \vee T$. *By the law of excluded middle, this is true not just for T but any boolean expression.*
- <u>T</u> $\neg(T \wedge F)$. *The subexpression $(T \wedge F)$ evaluates to false, and so $\neg F$ evaluates to true.*
- <u>T</u> $T \oplus F$. *$\oplus$ requires exactly one of the operands to be true; in this case, T on the left is true.*
- <u>T</u> $F \Rightarrow T$. *This is actually called "the principle of explosion": from false comes anything.*

See the subsection titled "Boolean Logic" in Sipser §0.2.