#1 A note to my readers

I am trying to get in the habit of having shorter lectures with a stronger emphasis on active learning during lab time. These notes cover less and are more abridged. I believe this is a better way to teach[1], but it may feel more uncomfortable.

#2 PSpace Completeness

Definition 1. *A language B is* PSPACE*-complete if it satisfies two conditions:*

1. $B \in \mathsf{PSPACE}$,
2. B *is* PSPACE*-hard: for all* $A \in \mathsf{PSPACE}$, $A \leq_p B$.

Note that we use polynomial-time reducibility, even though this chapter is about space.
Just as with NP-completeness, we have the following theorem.

Theorem 1. *If A is* PSPACE*-complete and $A \leq_p B$, where $B \in \mathsf{PSPACE}$, then B is* PSPACE*-complete.*

#3 TQBF Problem

A quantifier is either the symbol "forall" $\forall$, or "there exists" $\exists$.

1. Quantifiers bind boolean variables in boolean formulae. E.g., the expression $\forall x.\exists y.x \vee y$ has variables x and y bound in the body $x \vee y$.
2. A formula in which all variables in the body of the expression are quantified is called fully quantified. The expression $\forall y.x \wedge z$ is not fully quantified.
3. A quantified boolean formula is said to be in prenex-normal form if all of its quantifiers are at the front. Example above is in PNF; the expression $\forall x.(x \vee (\exists y.y))$ is not. **All formulae have a prenex-normal form**, so we consider only formulae in PNF.

Finally:

$$\textsc{tqbf} = \{\langle\phi\rangle \mid \phi \text{ is a true, prenex-normal form, fully quantified boolean formula }\}$$

For example,

- $\forall x.\exists y.x \vee y$ is true and in TQBF, but
- $\forall x.\forall y.x \vee y$ is false and not in TQBF.

[1] All pedagogical research I know of suggests this is the case.

Next,

Claim 1. *TQBF is PSPACE-complete.*

Proof. I will show just that TQBF $\in$ PSPACE. See the text for proof of completeness. (TQBF is to PSPACE-completeness as SATis to NP-completeness—i.e., our "first complete language" has a painful completeness proof.)
Let T decide TQBF, on input $\langle\phi\rangle$, according to the following:

1. if ϕ contains no quantifiers, then it is an expression with only constants, so evaluate ϕ and accept if 1, reject if 0.

2. If ϕ has the form $\exists x.\psi$, recursively call T on ψ, first with 0 substituted for x and then with 1. If either accepts, accept; otherwise, reject.

3. if ϕ has the form $\forall x.\psi$, recursively call T on ψ, first with 0 for x and then with 1 for x. If both accept, accept; otherwise, reject.

Think to yourself: how much space does this really need? □

#4 The Formula Game

Let ϕ be a quantified boolean formula in prenex form. Then two players, Player A and Player E, can play a game:

1. For each $\forall x.$ quantifier, Player A chooses if $x = 1$ or $x = 0$;

2. For each $\exists y.$ quantifier, Player E chooses if $y = 1$ or $y = 0$;

3. Players pick in order of the quantifiers left-to-right appearance (as ϕ is in PNF, they are all ordered at the front);

4. Player E wins if the formula ends up equal to 1; Player A wins if it ends up equal to 0.

We say that a player has a winning strategy if they will win with "optimal player". For example, Player A has a winning strategy if, no matter what player E picks in their choices, Player A can win with the correct choices.
(Side note: A game is called solved if the first (or second) player always has a winning strategy. Tic-tac-toe is solved! We don't know about chess.)
Finally,

$$\text{FORMULA-GAME} = \{\langle\phi\rangle \mid \text{Player E has a winning strategy in the formula-game of } \phi\}$$

Theorem 2. *FORMULA-GAME is PSPACE-complete.*

Proof. Actually, TQBF = FORMULA-GAME. A quantified boolean formula is true iff Player E has a winning strategy. □

#5 Other games and reductions

I expect you to be able to "learn by doing" the following.

1. Read "Generalized Geography" (Sipser pp. 343) for another *game* and an example reduction from TQBF.
2. Your lab will have you do a reduction from FORMULA-GAME to the language PUZZLE-GAME to prove that PUZZLE-GAME is PSPACE-complete.
3. See Sipser Example 8.10, pp. 342 for an example game played on the formula $\exists x_1 \forall x_2 \exists x_3 [(x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)]$.