Exam #3 (Complexity)

Due: May 14, 2024

Academic Honesty & Collaboration. I expect you to work alone, and for your work to represent wholly your own efforts. You are, of course, allowed to ask your classmates (and me) general questions about the material. Please include the following information at the top of your submission, along with your name.

- Written sources used: (Include textbook(s), complete citations for web or other written sources. Write <u>none</u> if no sources used)
- Help obtained: (Include names of anyone other than the instructor.)

Extensions. I am happy to offer extensions, but I ask that you ask *in advance* and at least prior to *Friday, May 10.* Further, I expect you to be able to show either (i) proof of effort and some progress on the exam, or (ii) some minor proof of conflict in schedule. (Exceptions will be made, of course, in cases of sickness, injury, or emergency.) Finally, please talk to me *as soon as possible* if you know you have a conflict in schedule (e.g., another take-home exam in a separate course) and you need some help meeting this deadline.

The latest extension I *would like* to give is to Thursday, May 16. The latest extension I *can* give is to Friday, May 17th. Unlike the rest of my deadlines, this is a <u>hard one</u>: as per college policy, the course is done, and I cannot ask you to keep doing work past Friday. <u>Without exception</u>, I will not be accepting work past this date.

Learning outcomes. As per the syllabus, this exam is designed to test your mastery of the following course learning outcomes.

1. Prove that a problem is in a particular time complexity class.
2. Prove that a problem is NP-complete by way of a reduction.
3. Explain the practical ramifications of the P vs. NP problem.
4. Prove that a problem is in a particular space complexity class.
5. Describe the essential characteristics of problems belonging to each of the major complexity classes.
6. Describe the practial relationships between various time and space complexity classes.

#1 (50 pts)

Define coP, the complement of P, as

$$\mathsf{coP} = \{L \mid \overline{L} \in \mathsf{P}\}.$$

In this question, we will prove (and think about) the following claim.

Claim 1. $\mathsf{P} = \mathsf{coP}$.

Specifically:

1. (30 pts). Prove that $\mathsf{P} \subseteq \mathsf{coP}$ and $\mathsf{coP} \subseteq \mathsf{P}$. *Hint: both directions should be similar.*
2. (10 pts). You have just proven an important property of languages in P. What is that property? (Fill in the blanks in this sentence: *Languages in P are* ______ *under* ______.)
3. (10 pts). Define coNP similarly to coP as

$$\mathsf{coNP} = \{L \mid \overline{L} \in \mathsf{NP}\}.$$

   Argue (in 1-3 sentences) why your proof that $\mathsf{P} = \mathsf{coP}$ cannot be replicated or modified to show that $\mathsf{NP} = \mathsf{coNP}$.

#2 (50 pts)

Prove that, if every NP-hard language is also PSPACE-hard, then $\mathsf{PSPACE} = \mathsf{NP}$.

Hint 1: We know already that $\mathsf{NP} \subseteq \mathsf{PSPACE}$. So your obligation is to prove the other direction.

Hint 2: Let A be arbitrary such that $A \leq_p \mathrm{SAT}$. Is $A \in \mathsf{NP}$?

#3 (50 pts)

Define the language ADD as

$$\textsc{add} = \{a\#b\#c \mid a, b, c \text{ are binary natural numbers and } a + b = c\}$$

Prove that $\textsc{add} \in \mathsf{L}$. Specifically, describe (at a very-high level) how a TM M can decide ADD and argue that the amount of space it needs on its work tape is sublinear.[1]

#4 (50 pts)

The following claim is true, and follows from the Time Hierarchy Theorem, which we did not have time to cover.

Claim 2. *There exists a language $A \in \mathsf{EXPTIME}$ that is not in P.*

Using Claim 2, prove that, if $\mathsf{NP} = \mathsf{EXPTIME}$, then $\mathsf{P} \neq \mathsf{NP}$.

[1]For our purposes, sublinear means that the number of cells used on the work-tape are $\mathcal{O}(\log n)$ or $\mathcal{O}(1)$ on an input length of n.

#5 (30 pts)

Mark each answer `true`, `false`, or `unknown`.

(This question is placed on the exam to make up the points lost from omitting Lab 14B, Quiz 14, and HW14.)

1. (5 pts). $\mathsf{NP} = \mathsf{EXPTIME}$.
2. (5 pts). SAT $\in \mathsf{P}$.
3. (5 pts). If 3SAT $\in \mathsf{P}$ then $\mathsf{P} = \mathsf{NP}$.
4. (5 pts). $\mathsf{HALT}_{\mathrm{TM}} \in \mathsf{P}$.
5. (5 pts). SAT is undecidable.
6. (5 pts). If TQBF $\leq_p$ CLIQUE, then CLIQUE is PSPACE-complete.