#1 Sublinear Space Complexity

We want to consider TMs that use sublinear-space—which does not make sense under our previous definition of space complexity, as we assume these machines will also want to read their inputs.

Redefine space complexity as follows:

Definition 1. *let M be a deterministic, deciding TM such that:*

1. *M has a read-only input tape, and*
2. *M has a read/write work tape.*

The space complexity of M is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of tape cells on the work-tape that M scans on any input of length n. For nondeterministic N, $f(n)$ is the same but for any branch of computation for any length of input n.

So we only count what M does on its work tape—this lets us consider "sublinear" languages which use less-than-linear space on their work tape.

Next:

Redefine SPACE and NSPACE according to the new definition of space complexity. Then:

$$\mathsf{L} = \text{SPACE}(\log n)$$
$$\mathsf{NL} = \text{NSPACE}(\log n)$$

#2 Example

Here is a neat example:

Recall this language that you are surely bored of by now.

$$A = \{0^n 1^n \mid n \geq 0\}$$

A few questions:

1. We've solved this two ways before. What were they?
2. Are these two ways logarithmic in space complexity?
3. There is a third way: keep track of how many 0's and 1's there are *in binary*. Is this logarithmic?

#3 Complexity Classes and their Complements

A *very important concept that did not turn up as much as I thought it would* is that of co-complexity classes.
Consider a few examples.

$$\mathsf{coP} = \{A \mid \overline{A} \in \mathsf{P}\}$$
$$\mathsf{coNP} = \{A \mid \overline{A} \in \mathsf{NP}\}$$
$$\mathsf{coNL} = \{A \mid \overline{A} \in \mathsf{NL}\}$$

(Hopefully we see the pattern.)
Another open problem in CS: Does $\mathsf{NP} = \mathsf{coNP}$? You will think about this in your lab.

#4 Interesting things about NL that I didn't have time to teach

A few facts.

- $\mathsf{NL} \subseteq P$. See Sipser §8.5.
- PATH is NL-complete. See Sipser §8.5.
- $\mathsf{L} = \mathsf{NL}$. See Sipser §8.6.

#5 Where we stop

We know the following:

$$\mathsf{L} \subseteq \mathsf{NL} = \mathsf{coNL} \subseteq P \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXPTIME}$$

I will expect you to be able to explain most of these (except those about NL and coNL).