HW #9

Due: Apr 9 2024

Instructions. You are allowed to collaborate with others, however you should write up solutions independently. Copying an answer from another source (e.g. the Web) or from another student may yield few or zero points. Write solutions neatly and legibly, or type your solutions in LaTeX. Be sure to number each problem, and indicate a final solution (if relevant). Answers to problems should include justification (show your work).

Acknowledgments. Problems from this homework come from published sources. The specific sources are withheld due to the nature of this assignment.

Academic Honesty. Include the following information at the top of your submission, along with your name.

- Written sources used: (Include textbook(s), complete citations for web or other written sources. Write <u>none</u> if no sources used)
- Help obtained: (Include names of anyone other than the instructor.)

#1 (5 pts)

Let A and B be languages such that A reduces to B ($A \leq_m B$) by computable function $f : \Sigma^* \to \Sigma^*$. If B is a regular language, does that imply that A is a regular language? If not, exhibit a a counter-example. If yes, prove so.

#2 (5 pts)

A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states.

$$\mathsf{USELESS} = \{\, \langle M \rangle \mid M \text{ is a TM with at least one useless state} \,\}.$$

Prove that USELESS is undecidable by a reduction from A_{TM}.

N.b. see attached hint page for guidance.

#3 (5 pts)

Prove that the Post-Correspondence Problem is decidable over the unary alphabet $\Sigma = \{1\}$. So, given an (encoded) input of dominos

$$\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_n}{b_n}\right]\}$$

where each $t_i, b_i \in \{1\}^*$, describe a TM that decides if there exists a combination (with repetitions allowed) of dominos such that the top row equals the bottom row. For example, if our input is

$$\{\left[\frac{11}{111}\right], \left[\frac{111}{11}\right]\}$$

Then a solution exists, which is precisely the input set with no repetitions. So your TM would accept this input. Your TM should reject inputs for which no solution exists.

#4 (5 pts)

Prove that the following language is undecidable by exhibiting a mapping reduction f from A_{TM} to L_1.

$$L_1 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is an infinite language}\}$$

Specifically, build a function $f : \Sigma^* \to \Sigma^*$ such that

1. $\forall x \in \Sigma^*$, if $x \in A_{\text{TM}}$ then $f(x) \in L_1$, and
2. $\forall x \in \Sigma^*$, if $x \notin A_{\text{TM}}$ then $f(x) \notin L_1$

f should be a computable function—so it should be decided by some TM M_f. Make sure in your solution to describe how M_f behaves. (This gives the construction part of your proof.) Next, argue for correctness—that is, argue why the above two bullet points hold for all $x \in \Sigma^*$.

Hints

Here is a guide (taken from a lab of Prof. Osera) to solving the USELESS *problem. Please feel free to use it to help you get comfortable with reduction proofs.* ***Please*** *do not feel obligated to structure your answer as is instructed here. Also, do not hand in the answer to each subquestion below. Just hand in what you feel is a valid reduction and proof of correctness.*

#2 Guide (USELESS)

We're going to use a proof by contradiction to prove that determining whether TM's have a useless state is undecidable.

1. Now we're going to show that $\mathsf{USELESS}$ is undecidable by a mapping reduction from A_{TM}. First let's establish what our mapping function should look like and how it should behave.

   The heart of the mapping reduction is a mapping from inputs to A_{TM} to $\mathsf{USELESS}$. What does the mapping function f take as input and produce as output?

2. Once we know the types of the mapping reduction, we now need to reason about the correctness condition linking acceptance between A_{TM} and $\mathsf{USELESS}$.

   Let $\langle M, w\rangle$ be inputs to a decider for A_{TM} and $\langle M'\rangle$ the input to a decider for $\mathsf{USELESS}$. Give the correctness condition for f based on these inputs.

3. We have now established our correctness condition for our construction. Based on this condition, we need to figure out a way to have our machine M' conditionally have the property of $\mathsf{USELESS}$: M' has a useless state. Recall that a state is useless if no input string causes M' to enter that state. Because the transitions of a Turing machine are dependent on the positions of the tape head and contents of the tape at that position, we need to know how a Turing machine executes in order to determine if it contains a useless state. This is ultimately why A_{TM} can be reduced to $\mathsf{USELESS}$.

   For the purposes of our construction, however, we want to conditionally make M' have a useless state in such a way that it is painfully obvious that this is the case. This will simplify our reasoning when we go to make sure that M' obeys the correct condition outlined above. Two ways that we might go about this are:

   - Give M' a state u that has no physical transition to it from any other state in M'. It is obvious that u is useless because there is no way to reach it in this situation.
   - Give M' a state u that is connected from some other state in the Turing machine, but make that transition involve reading a tape symbol that provably never appears on the tape, e.g., by ensuring that it is a alphabet symbol and it is not mentioned in the read position of any transition of the Turing Machine.

   The first option does not not work for our purposes because the transitions are fixed when M' is constructed; we can't condition the existence of a transition based on whether M accepts w. However, the second option is controllable at runtime—we can condition the appearance of the symbol that will allow us to transition to our useless state on whether M accepts w.

   Putting all of this together, we have the following skeleton for the construction of our mapping function and the Turing machine M' it produces:

Assume for the sake of contradiction that $\mathsf{USELESS}$ is decidable. Let D decide it. Define a decider for A_{TM} as follows:

A = "On input $\langle M, w\rangle$:

1. Define a TM M'

M' = On input x:

+ Run M on w.
- If M accepts: <u>M' should have a useless state</u>.
- If M rejects: <u>M' should not have a useless state</u>.

2. Run D on M', ...

——

Note that also in the case where M loops on w, we want to have the same behavior as the rejection case.

Use this information to complete the construction of M' and A as specified above.

4. Finally, we need to verify that M' does the right thing. Rather than proving our correctness condition instead (which is a biconditional), I find it easier to reason about the three cases that might occur as a result of M's execution of w (convince yourself these three cases cover the two cases of the biconditional you derived above):

   - M accept w
   - M rejects w
   - M loops on w

   Prove that M' has the appropriate behavior for the three cases outlined above with respect to your correctness condition for the mapping function.