#1 Countability

1. Prove that the language $L = \Sigma^*$ where $\Sigma = \{0, 1\}$ is countably infinite.
   (Hint: simply declaring that the mapping from Σ^* to $\mathbb{N}$ is the binary interpretation of $w \in \Sigma^*$ is insufficient since 0 and 000 are distinct strings but both represent the value zero. The resulting mapping would, therefore, not be injective!)

2. Generalize your construction to any Σ of finite size.

3. Use this fact to argue that the set of possible Java programs is countably infinite.

#2 $\mathbb{N}$ is uncountable?!

Consider the following (false) proof that $\mathbb{N}$ is uncountable.

Proof. Assume that $\mathbb{N}$ is countable. Then there is a bijection f that covers every natural number in $\mathbb{N}$. Construct the natural number n where the ith digit of n is the ith digit of the ith natural number in the bijection (i.e., $f(i)$) plus one mod 10 (so that it is a decimal digit). That is, if k is the ith digit of the ith natural number, then the ith digit of n is given by $(k + 1) \mod 10$. n is a valid natural number and by construction, n differs from every natural number in the bijection by one digit. Therefore, n cannot be in the bijection and therefore our assumption that such a bijection exists is incorrect. Thus, $\mathbb{N}$ is uncountable. □

Of course, we already know that $\mathbb{N}$ is countable (the bijection is the identity function). What is wrong with this proof?

(Hint: think about the assumptions latent in Cantor's diagonalization argument that the reals are uncountable. What specifically is different in this case?)