
LUNAICY NUMERICS OF ODES AND PDES

1 Mathematical Derivation of the Main Problem

Consider a one dimensional slice of the Ice sheet of Jupiters moon Europa. This is our state space $\Omega := [0, d] \subset \mathbb{R}$ where d denotes the depth od the ice sheet. The average microstructure $y(x, t) := (r_g(x, t), r_b(x, t))^T$, where r_g is the average grain radius and r_b is the average bond radius is modelled by the system of ODEs for every $x \in \Omega$ as

$$\frac{d}{dt}y(x, t) = f(y(x, t), T(x, t)), t > 0, \quad y(x, 0) = y_0(x). \quad (0.1)$$

The average temperature at $x \in \Omega$ is modelled via the heat eqaution with Neuman boundary data

$$\rho(x, t)c_p(x, t)\frac{\partial}{\partial t}T(x, t) = \frac{\partial}{\partial x}k(x, t)\frac{\partial}{\partial x}T(x, t), \quad x \in \Omega, t > 0 \quad (0.2)$$

$$k(0, t)\frac{\partial}{\partial x}T(0, t) = -F_{solar}(t) + \epsilon\sigma_{SB}T(0, t)^4 \text{ and } k(d, t)\frac{\partial}{\partial x}T(d, t) = 0, \quad t > 0 \quad (0.3)$$

$$T(x, 0) = T_0(x), x \in \Omega, \quad (0.4)$$

where ρ, c_p and k are model specific functions that depend on $y(x, t)$, coupling the ODE to the PDE. F_{solar} is the energy that comes from the sun and σ_{SB} and ϵ are model specific constants.

To numerically solve this problem we first discretize the state space by $\Omega_h \subset \Omega$ and thus get a system of ODEs as (0.1) and approximating (0.2) by the methods of lines. We integrate both sides from $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and divide by the length of the intervall h :

$$\begin{aligned} \rho(x, t)c_p(x, t)\frac{\partial}{\partial t}T(x, t) &= \frac{\partial}{\partial x}k(x, t)\frac{\partial}{\partial x}T(x, t) \\ \iff \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t)c_p(x, t)\frac{\partial}{\partial t}T(x, t)dx &= \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x}k(x, t)\frac{\partial}{\partial x}T(x, t)dx \\ \iff \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t)c_p(x, t)\frac{\partial}{\partial t}T(x, t)dx &= \frac{1}{h} \left(k(x_{i+\frac{1}{2}}, t)\frac{\partial}{\partial x}T(x_{i+\frac{1}{2}}, t) - k(x_{i-\frac{1}{2}}, t)\frac{\partial}{\partial x}T(x_{i-\frac{1}{2}}, t) \right) \\ \iff \frac{\partial}{\partial t} \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t)c_p(x, t)T(x, t)dx & \\ = \frac{1}{h} \left(k(x_{i+\frac{1}{2}}, t)\frac{T(x_{i+1}, t) - T(x_i, t)}{h} - k(x_{i-\frac{1}{2}}, t)\frac{T(x_i, t) - T(x_{i-1}, t)}{h} + \mathcal{O}(h^2) \right). \end{aligned}$$

Thus for every x_i we get the approximate ODE (second order error)

$$\frac{\partial}{\partial t} \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x, t)c_p(x, t)T(x, t)dx \approx k(x_{i+\frac{1}{2}}, t)\frac{T(x_{i+1}, t) - T(x_i, t)}{h^2} - k(x_{i-\frac{1}{2}}, t)\frac{T(x_i, t) - T(x_{i-1}, t)}{h^2}.$$

We will approximate the left hand side integral by its center point (mean value theorem) and thus get the three point method

$$\frac{\partial}{\partial t} \frac{1}{h} \rho(x_i, t) c_p(x_i, t) T(x_i, t) h \approx k(x_{i+\frac{1}{2}}, t) \frac{T(x_{i+1}, t) - T(x_i, t)}{h^2} - k(x_{i-\frac{1}{2}}, t) \frac{T(x_i, t) - T(x_{i-1}, t)}{h^2}.$$

This approximation only holds for the interior. Because we have Neumann boundary data (0.3) on both sides we need to introduce ghost cells. Approximating the boundary data by symmetric differences, i.e. also second order, we get

$$\begin{aligned} k(x_0, t) \frac{T(x_1, t) - T(x_{-1}, t)}{2h} + \mathcal{O}(h^2) &= -F_{solar}(t) + \epsilon \sigma_{SB} T(x_0, t)^4 \\ k(x_d, t) \frac{T(x_{d+1}, t) - T(x_{d-1}, t)}{2h} + \mathcal{O}(h^2) &= 0. \end{aligned}$$

Thus approximately it holds

$$\begin{aligned} T(x_{-1}, t) &\approx \frac{2h}{k(x_0, t)} (F_{solar}(t) - \epsilon \sigma_{SB} T(x_0, t)^4) + T(x_1, t) =: T_{-1}(t) \\ T(x_{d+1}, t) &= T(x_{d-1}, t) \end{aligned}$$

Thus in total the following system of ODEs needs to be solved

$$\frac{d}{dt} \begin{pmatrix} y(x_0, t) \\ \rho(x_0, t) c_p(x_0, t) T(x_0, t) \\ y(x_1, t) \\ \rho(x_1, t) c_p(x_1, t) T(x_1, t) \\ \vdots \\ y(x_{d-1}, t) \\ \rho(x_{d-1}, t) c_p(x_{d-1}, t) T(x_{d-1}, t) \\ y(x_d, t) \\ \rho(x_d, t) c_p(x_d, t) T(x_d, t) \end{pmatrix} = \begin{pmatrix} f(y(x_0, t), T(x_0, t)) \\ k(x_{\frac{1}{2}}, t) \frac{T(x_1, t) - T(x_0, t)}{h^2} - k(x_{-\frac{1}{2}}, t) \frac{T(x_0, t) - T_{-1}(t)}{h^2} \\ f(y(x_1, t), T(x_1, t)) \\ k(x_{1+\frac{1}{2}}, t) \frac{T(x_2, t) - T(x_1, t)}{h^2} - k(x_{1-\frac{1}{2}}, t) \frac{T(x_1, t) - T(x_0, t)}{h^2} \\ \vdots \\ f(y(x_{d-1}, t), T(x_{d-1}, t)) \\ k(x_{d-\frac{1}{2}}, t) \frac{T(x_d, t) - T(x_{d-1}, t)}{h^2} - k(x_{d-\frac{3}{2}}, t) \frac{T(x_{d-1}, t) - T(x_{d-2}, t)}{h^2} \\ f(y(x_d, t), T(x_d, t)) \\ k(x_{d+\frac{1}{2}}, t) \frac{T(x_{d-1}, t) - T(x_d, t)}{h^2} - k(x_{d-\frac{1}{2}}, t) \frac{T(x_d, t) - T(x_{d-1}, t)}{h^2} \end{pmatrix}$$

When considering the right hand side the product rule yields

$$\frac{d}{dt} \rho(x_i, t) c_p(x_i, t) T(x_i, t) = \rho(x_i, t) c_p(x_i, t) \frac{d}{dt} (T(x_i, t)) + T(x_i, t) \frac{d}{dt} (\rho(x_i, t) c_p(x_i, t)).$$

The second summand is neglegable due to the fact that it describes the rate at which the material properties change. One estimates the age of the Ice sheet to be 30 Million years old (R. T. Pappalardo et al. 1998). This begs the question wether Europa's ice microstructure changed in a shorter time than its ice crust age?

For simplifications, we answer this questions with no, for now. This simplifies the system of ODEs to:

$$\frac{d}{dt} \begin{pmatrix} y(x_0, t) \\ T(x_0, t) \\ y(x_1, t) \\ T(x_1, t) \\ \vdots \\ y(x_{d-1}, t) \\ T(x_{d-1}, t) \\ y(x_d, t) \\ T(x_d, t) \end{pmatrix} = \begin{pmatrix} f(y(x_0, t), T(x_0, t)) \\ \frac{1}{\rho(x_0, t)c_p(x_0, t)} \left(k(x_{\frac{1}{2}}, t) \frac{T(x_1, t) - T(x_0, t)}{h^2} - k(x_{-\frac{1}{2}}, t) \frac{T(x_0, t) - T_{-1}(t)}{h^2} \right) \\ f(y(x_1, t), T(x_1, t)) \\ \frac{1}{\rho(x_1, t)c_p(x_1, t)} \left(k(x_{1+\frac{1}{2}}, t) \frac{T(x_2, t) - T(x_1, t)}{h^2} - k(x_{1-\frac{1}{2}}, t) \frac{T(x_1, t) - T(x_0, t)}{h^2} \right) \\ \vdots \\ f(y(x_{d-1}, t), T(x_{d-1}, t)) \\ \frac{1}{\rho(x_{d-1}, t)c_p(x_{d-1}, t)} \left(k(x_{d-\frac{1}{2}}, t) \frac{T(x_d, t) - T(x_{d-1}, t)}{h^2} - k(x_{d-\frac{3}{2}}, t) \frac{T(x_{d-1}, t) - T(x_{d-2}, t)}{h^2} \right) \\ f(y(x_d, t), T(x_d, t)) \\ \frac{1}{\rho(x_d, t)c_p(x_d, t)} \left(k(x_{d+\frac{1}{2}}, t) \frac{T(x_{d+1}, t) - T(x_d, t)}{h^2} - k(x_{d-\frac{1}{2}}, t) \frac{T(x_d, t) - T(x_{d-1}, t)}{h^2} \right) \end{pmatrix}$$

2 Introduction into the Physics Motivated Equations

Here the volumes of the grains and bonds will be determined, as they are essential in the dynamics of the model. Then follows the derivation of the ODE that describes the dynamics of the grain and bonds with the water vapor.

2.1 Grain Volume

The set

$$V^{r_g} := \{x \in \mathbb{R}^3 \mid |x| < r_g, 0 \leq x_1 < \hat{x}^*, 0 \leq x_2\}.$$

Further, for $u : \mathbb{R}^3 \rightarrow \mathbb{R}, x \mapsto \sqrt{x_1^2 + x_2^2 + x_3^2} = |x|$ the level set u^{-1} describes the set of all points on the sphere S_r^2 with radius r . Clearly for all $x \in u^{-1}(r)$ the gradient is $\nabla u = (\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r})$ satisfies $|\nabla u| = 1$. With the Co-area formula it holds

$$\int_{V^{r_g}} |\nabla u| d^3x = \int_0^{r_g} \int_{V^{r_g} \cap u^{-1}(r)} d\sigma(z) dr.$$

Define the parametrization $\phi_r : U_r \rightarrow V^{r_g} \cap u^{-1}(r), (\theta, \vartheta) \mapsto (r \sin(\vartheta) \cos(\theta), r \sin(\vartheta) \sin(\theta), r \cos(\theta))$, where

$$U_r := \{(\theta, \vartheta) \in [0, \frac{\pi}{2}] \times [0, \pi] \mid r \sin(\vartheta) \cos(\theta) < \hat{x}^*\}.$$

Its partial derivatives are

$$\begin{aligned} \frac{\partial}{\partial \theta} \phi_r(\theta, \vartheta) &= (-r \sin(\vartheta) \sin(\theta), r \cos(\theta) \sin(\vartheta), 0) \\ \frac{\partial}{\partial \vartheta} \phi_r(\theta, \vartheta) &= (r \cos(\vartheta) \cos(\theta), r \sin(\theta) \cos(\vartheta), -r \sin(\vartheta)). \end{aligned}$$

Then it holds that

$$\det \begin{pmatrix} \left(\left(\frac{\partial}{\partial \theta} \phi_r \right)^T \frac{\partial}{\partial \theta} \phi_r \quad \left(\frac{\partial}{\partial \theta} \phi_r \right)^T \frac{\partial}{\partial \vartheta} \phi_r \right) \\ \left(\left(\frac{\partial}{\partial \vartheta} \phi_r \right)^T \frac{\partial}{\partial \theta} \phi_r \quad \left(\frac{\partial}{\partial \vartheta} \phi_r \right)^T \frac{\partial}{\partial \vartheta} \phi_r \right) \end{pmatrix} = \det \begin{pmatrix} r^2 \sin(\vartheta)^2 & 0 \\ 0 & r^2 \end{pmatrix} = r^4 \sin(\vartheta)^2.$$

Thus the manifold integral is given by

$$\begin{aligned} \int_0^{r_g} \int_{V^{r_g} \cap u^{-1}(r)} d\sigma(z) dr &= \int_0^{r_g} \int_{U_r} 1 \circ \phi_r(\theta) \sqrt{\det((\phi')^T \phi')} d(\theta, \vartheta) dr \\ &= \int_0^{r_g} \int_{U_r} 1 \cdot r^2 \sin(\vartheta) d(\theta, \vartheta) dr \end{aligned}$$

For $\hat{x}^* > r_g$ this integral is trivial equal to $\int_0^{r_g} \pi r^2 dr = \pi \frac{r_g^3}{3}$. More interesting in this application is the opposite case. In order to solve it, we use Fubini (where $U_r^\theta := \{\vartheta \in [0, \pi] \mid (\theta, \vartheta) \in U_r\}$)

$$\int_0^{r_g} r^2 \int_0^{\pi/2} \int_{U_r^\theta} \sin(\vartheta) d\vartheta d\theta dr$$

We now need to take a closer look at the inner most integral. The condition $\sin(\vartheta) < \frac{\hat{x}^*}{r \cos(\theta)}$ is satisfied if we choose $\theta = \arccos(\frac{\hat{x}^*}{r})$ or larger, because

$$\sin(\vartheta) \leq 1 = \frac{\hat{x}^*}{r \cos \arccos(\frac{\hat{x}^*}{r})} \text{ is always satisfied.}$$

Thus for $\theta \geq \arccos(\frac{\hat{x}^*}{r})$ we need no restriction. For $\theta < \arccos(\frac{\hat{x}^*}{r})$ our condition is only satisfied, if

$$\vartheta = \underbrace{\arcsin\left(\frac{\hat{x}^*}{r \cos(\theta)}\right)}_{=:B}$$

or smaller. It is also satisfied during this choice of $\theta < \arccos(\frac{\hat{x}^*}{r})$, if

$$\pi - B < \vartheta \leq \pi, \text{ because } 0 < \sin(\pi - \arcsin(\frac{\hat{x}^*}{r \cos(\theta)})) < \frac{\hat{x}^*}{r \cos(\arccos(\frac{\hat{x}^*}{r}))} = 1$$

and on that proposed interval $\sin(\vartheta)$ is monotonically decreasing.??? Thus in total we get for $r < r_g$:

$$\begin{aligned} \int_0^{\pi/2} \int_{U_r^\theta} \sin(\vartheta) d\vartheta d\theta &= \int_0^{\arccos(\frac{\hat{x}^*}{r})} \left(\int_0^B \sin(\vartheta) d\vartheta + \int_{\pi-B}^\pi \sin(\vartheta) d\vartheta \right) d\theta + \int_{\arccos(\frac{\hat{x}^*}{r})}^{\pi/2} \int_0^\pi \sin(\vartheta) d\vartheta d\theta \\ &= \int_0^{\arccos(\frac{\hat{x}^*}{r})} (\cos(0) - \cos(B) + \cos(\pi - B) - \cos(\pi)) d\theta + \int_{\arccos(\frac{\hat{x}^*}{r})}^{\pi/2} \cos(0) - \cos(\pi) d\theta \\ &= \int_0^{\arccos(\frac{\hat{x}^*}{r})} (1 - \cos(B) + \cos(\pi - B) + 1) d\theta + \int_{\arccos(\frac{\hat{x}^*}{r})}^{\pi/2} 2 d\theta \\ &= \int_0^{\arccos(\frac{\hat{x}^*}{r})} (2 - \cos(B) - \cos(\pi - B)) d\theta + 2 \left(\frac{\pi}{2} - \arccos\left(\frac{\hat{x}^*}{r}\right) \right) \\ &= 2 \arccos\left(\frac{\hat{x}^*}{r}\right) - 2 \int_0^{\arccos(\frac{\hat{x}^*}{r})} \cos(B) d\theta + \pi - 2 \arccos\left(\frac{\hat{x}^*}{r}\right) \\ &= \pi - 2 \int_0^{\arccos(\frac{\hat{x}^*}{r})} \cos(\arcsin(\frac{\hat{x}^*}{r \cos(\theta)})) d\theta \\ &= \pi - 2 \int_0^{\arccos(\frac{\hat{x}^*}{r})} \sqrt{1 - \left(\frac{\hat{x}^*}{r \cos(\theta)}\right)^2} d\theta \end{aligned}$$

Thus in total it holds for $\hat{x}^* \leq r_g$ that

$$\begin{aligned} \int_{V^{r_g}} d^3x &= \int_0^{r_g} r^2 1_{0 \leq r < \hat{x}^*} (\pi - 2 \cdot 0) + r^2 \left(\pi - 2 \int_0^{\arccos(\frac{\hat{x}^*}{r})} \sqrt{1 - \left(\frac{\hat{x}^*}{r \cos(\theta)} \right)^2} d\theta \right) 1_{\hat{x}^* < r \leq r_g} dr \\ &= \pi \frac{(\hat{x}^*)^3}{3} + \pi \left(\frac{r_g^3}{3} - \frac{(\hat{x}^*)^3}{3} \right) - 2 \int_{\hat{x}^*}^{r_g} r^2 \underbrace{\int_0^{\arccos(\frac{\hat{x}^*}{r})} \sqrt{1 - \left(\frac{\hat{x}^*}{r \cos(\theta)} \right)^2} d\theta dr}_{=: A(\hat{x}^*, r)}. \end{aligned}$$

Note that $A(\hat{x}^*, r)$ does not have an analytical solution and needs to be solved numerically.

2.2 Bond Volume

...

2.3 Surface Area and Curvature

The Surface area is in this model of crucial importance, as it models the exchange of water vapor mass. For the grain and the bond we can easily calculate it from our derivation above. The grains surface area is given by

$$S_g := r_g^2 \cdot \left(\pi - 2 \int_0^{\arccos(\frac{\hat{x}^*}{r_g})} \sqrt{1 - \left(\frac{\hat{x}^*}{r_g \cos(\theta)} \right)^2} d\theta \right)$$

and the surface area of the grain is ...

The curvature of the grain is $K_g = \frac{2}{r_g}$

2.4 Physics behind the ODE

We denote by $P_{gas}(t)$ the pressure of the surrounding gas/water vapor at time t . $P_{K_j}(t)$ denotes the equilibrium vapor pressure of a curved surface j with average curvature K_j . Note that a result in ??? is that

convex surface pressure > flat surface pressure > concave surface pressure.

The Flux of a grain or a bond is then given by

$$J_i(t) := \alpha(P_{K_i}(t) - P_{gas}(t)) \sqrt{\frac{M}{2\pi RT(x, t)}}, \quad i = g, b,$$

where $\alpha, R, M \in \mathbb{R}$ are constants and $T(x, t)$ denotes the temperature of the grain at point $x \in [0, d]$ at time $t \geq 0$. The curved surface pressure of K_j is given by

$$P_{K_j}(t) := P_{sat}(T(x, t)) \left(1 + \frac{\gamma M}{RT(x, t)\rho_0} K_j(t) \right), \quad j = g, b,$$

where $\gamma, \rho_0, R, M \in \mathbb{R}$ are constants and $P_s(T(x, t)) = P_0 e^{-\frac{Q_{sub}}{RT(x, t)}}$ is the saturated water pressure of a flat surface (???). Here $P_0, Q_{sub}, R \in \mathbb{R}$ are constants. Furhter,

$$P_{gas}(T(x, t)) := \frac{m_g(t)RT(x, t)}{MV_{pore}} = \frac{(1-\phi)m_g(t)\rho_0RT(x, t)}{MV_g\phi}, \quad V_{pore} := V_g \frac{\phi}{1-\phi},$$

where V_g denotes the volume of the grain, ...???

2.5 Deriving the ODE

In the introduction we defined an ODE over the radii of the average grain and bond of the mircrostructure. This paper models the change of mass of the grain and the bond using the interactions with the adjacent water vapor and then connect that to the change of the radii. We consider the ODE

$$\frac{d}{dt} \begin{pmatrix} m_g(t) \\ m_b(t) \end{pmatrix} = \begin{pmatrix} J_{v,g}(t)S_{v,g}(t) + J_{b,g}(t)S_{b,g}(t) \\ J_{v,b}(t)S_{v,b}(t) + J_{g,b}(t)S_{g,b}(t) \end{pmatrix}$$

which stems from the Hertz-Knudsen formula. Here $S_{v,g}(t)$ denotes the area of the grain with the outward gas and $J_{v,g}$ is the repective flux, $S_{v,b}(t)$ the area of the bond with the gas, $S_{g,b}$ and $S_{b,g}$ the area of the grain with the bond.

Due to mass conservation, i.e. $J_{b,g}(t)S_{b,g}(t) = -J_{g,b}(t)S_{g,b}(t)$, it follows the total mass is

$$\frac{d}{dt}m_g(t) + \frac{d}{dt}m_b(t) = J_{v,g}(t)S_{v,g}(t) + J_{b,g}(t)S_{b,g}(t) + J_{v,b}(t)S_{v,b}(t) + J_{g,b}(t)S_{g,b}(t), = J_{v,g}(t)S_{v,g}(t) + J_{v,b}(t)S_{v,b}(t).$$

Plugging this into the ODE yields

$$\begin{aligned} \frac{d}{dt}m_g(t) &= J_{v,g}(t)S_{v,g}(t) + J_{v,b}(t)S_{v,b}(t) \\ &= S_g\alpha\sqrt{\frac{M}{2\pi RT(x,t)}} \left(P_s(T(x,t)) \left(1 + \frac{\gamma M}{RT(x,t)\rho_0} K_g \right) - P_{gas}(T(x,t)) \right) \\ &\quad + S_b\alpha\sqrt{\frac{M}{2\pi RT(x,t)}} \left(P_s(T(x,t)) \left(1 + \frac{\gamma M}{RT(x,t)\rho_0} K_b \right) - P_{gas}(T(x,t)) \right) \\ &= \alpha\sqrt{\frac{M}{2\pi RT(x,t)}} \left((s_g + S_b)P_s(T(x,t)) + S_gP_s(T(x,t))\frac{\gamma M}{RT(x,t)\rho_0}K_g + S_bP_s(T(x,t))\frac{\gamma M}{RT(x,t)\rho_0}K_b \right) \\ &\quad - \alpha\sqrt{\frac{M}{2\pi RT(x,t)}}P_{gas}(T(x,t))(S_g + S_b) \\ &= P_s(T(x,t))\alpha\sqrt{\frac{M}{2\pi RT(x,t)}} \left(S_g + S_b + \frac{\gamma M}{RT(x,t)\rho_0}(S_gK_g + S_bK_b) \right) - P_{gas}(T(x,t))\alpha\sqrt{\frac{M}{2\pi RT(x,t)}}(S_g + S_b) \\ &= \alpha\sqrt{\frac{M}{2\pi RT(x,t)}} \left(P_0 e^{-\frac{Q_{sub}}{RT(x,t)}} \left(S_g + S_b + \frac{\gamma M}{RT(x,t)\rho_0}(S_gK_g + S_bK_b) \right) - \frac{(1-\phi)m_g(t)RT(x,t)}{MV_g\phi}(S_g + S_b) \right) \end{aligned}$$

2.6 Physics behind the PDE

A crucial term in the PDE is the thermal conductivity given by

$$k(\phi, r_b, r_g) = k_0(T(x,t))(1-\phi)\frac{r_b}{r_g}.$$

This is not only interesting as it links the PDE to the ODE, but also as makes sure, that not only second derivatives of temperature are in the PDE but also first.

2.7 The Pipeline

We solve the ODE via implicit Euler with step size $\tau > 0$

$$m_g(t + \tau) = m_g(t) + \tau (J_{v,g}(t + \tau)S_{v,g}(t + \tau) + J_{b,g}(t + \tau)S_{b,g}(t + \tau)).$$

and for m_b analogously. After every step the calculation of the volume of the grain and bond is done using

$$V_g(t + \tau) = \frac{m_g(t + \tau)}{\rho_0}, \quad V_b(t + \tau) = \frac{m_b(t + \tau)}{\rho_0},$$

where ρ_0 is ??. Using this the Radii are calculated as

$$\begin{pmatrix} r_g(t + \tau) \\ r_b(t + \tau) \end{pmatrix} = \arg \min_{r_g, r_b} \left(\begin{pmatrix} |V_g(t + \tau) - v_g(r_g, r_b)| \\ |V_b(t + \tau) - v_b(r_g, r_b)| \end{pmatrix} \right)$$

Finally, we do one time step of the PDE and update the temperature.

2.8 Stiffness of the Problem

3 Simulations