SIT718 Real World Analytics

Assessment Task 3

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Q1)

a) A linear programming model is used in real world case scenarios when we want to maximize the profit or minimize the loss subject to requirements and constraints which are represented by linear relationships. In the case study, the food factory would like to minimize the total cost of producing the beverages while satisfying all constraints and requirements. The constraints and requirements such as composition of lime, orange and mango in the beverage, the amount of beverage required by the customer per week and the total cost of producing the beverage can be represented by linear relationships. The food factory would like to minimize the total cost of producing the beverage by making efficient use of the available resources. Hence, a linear programming model would be a good fit for this study (Dr Ye Zhu, SIT718 Real World Analytics, Deakin University, Lecture 7, 29 August 2020).

b) **Linear programming (LP) model for food factory:**

Decision variables:

x: number of litres of product A required to mix

y: number of litres of product B required to mix

Objective function:

Minimize the total cost of producing the beverage

Total cost of producing beverage = cost of product A + cost of product B

$$\boldsymbol{Min\ z = 3x + 10y}$$

Constraints:

1. $\frac{3x+8y}{x+y} \leq 6$

   $-3x + 2y \leq 0$
2. $\frac{6x+4y}{x+y} \geq 4.5$

   $1.5x - 0.5y \geq 0$
3. $\frac{4x+6y}{x+y} \geq 5$

   $y - x \geq 0$
4. $x + y \geq 100$
5. $x, y \geq 0$

c)

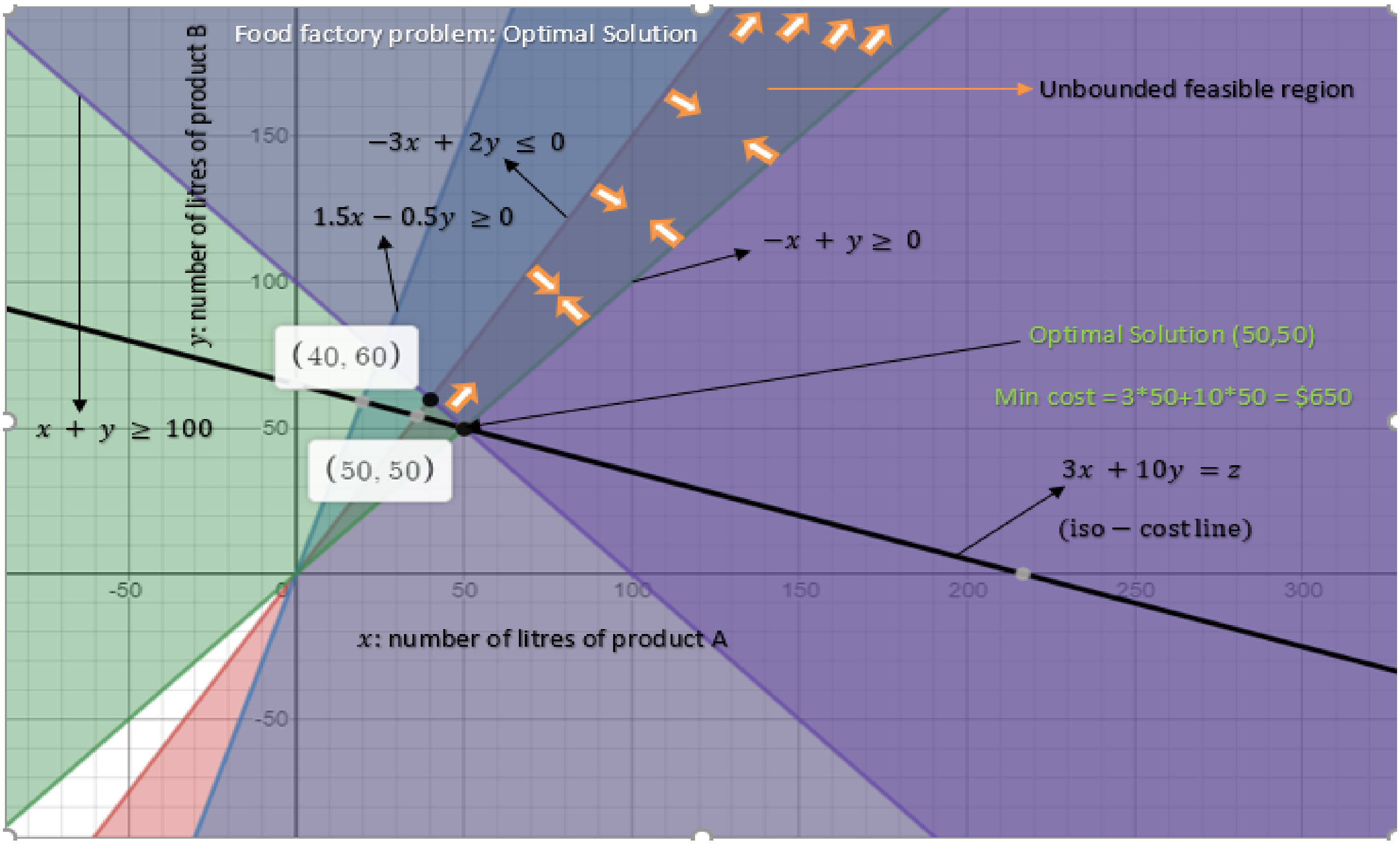

Figure 1

Based on the graph solution by using *Desmos* (Eli Luberoff 2011), the minimum cost of producing the beverage while satisfying all the constraints is $650 (refer figure 1).

d) Based on the graph solution, we can infer that the current solution of minimizing the total cost of beverage production by mixing 50 litres of product A and 50 litres of product B will remain optimal if the coefficient of x (i.e. cost ($/L) of product A) is between 0 to 10. If we go beyond 10, then the optimal solution will change and we would have to adjust our beverage production from mixing 50 litres of product A and 50 litres of product B, to mixing 40 litres of product A and 60 litres of product B instead to minimize the total cost of producing the beverage (refer figure 1).

Hence, the cost ($/L) of product A can vary from $0 to $10 without affecting the optimal solution of beverage production by mixing 50 litres of product A and 50 litres of product B

Q2)

a) Decision variables:

Let $x_{ij} \geq 0$ be a decision variable that denotes number of tonnes of products j for j ∈ {1 = Spring, 2 = Autumn, 3 = Winter} to be produced from materials i ∈ {C = Cotton, W = Wool, S = Silk}

Hence, total 9 decision variables are used in LP model.

Objective function:

Using the decision variables mentioned above and details provided on the sales price, production cost and purchase cost per ton of products and materials respectively, we will formulate our objective function.

Number of tons of each type of material used:

1. Cotton: x_{C1}+ x_{C2}+ x_{C3}
2. Wool: x_{W1}+ x_{W2}+ x_{W3}
3. Silk: x_{S1}+ x_{S2}+ x_{S3}

Number of tons of each type of product produced:

1. Spring: $x_{C1}+x_{W1}+x_{S1}$
2. Autumn: $x_{C2}+x_{W2}+x_{S2}$
3. Winter: $x_{C3}+x_{W3}+x_{S3}$

Revenue from product sales:

$60(x_{C1}+x_{W1}+x_{S1}) + 55(x_{C2}+x_{W2}+x_{S2}) + 60(x_{C3}+x_{W3}+x_{S3})$

Production cost of products:

$5(x_{C1}+x_{W1}+x_{S1}) + 4(x_{C2}+x_{W2}+x_{S2}) + 5(x_{C3}+x_{W3}+x_{S3})$

Purchase cost of materials:

$30(x_{C1}+x_{C2}+x_{C3}) + 45(x_{W1}+x_{W2}+x_{W3}) + 50(x_{S1}+x_{S2}+x_{S3})$

Combining all these together for our objective function:

Profit = Revenue – Production cost – Purchase cost

$$\boldsymbol{Max\ z = 25x_{C1}+ 10x_{W1}+ 5x_{S1} + 21x_{C2}+ 6x_{W2}+ x_{S2} + 25x_{C3}+10x_{W3}+5x_{S3}}$$

Constraints:

Product demand constraints:

1. $x_{C1}+x_{W1}+x_{S1} \leq 3800$
2. $x_{C2}+x_{W2}+x_{S2} \leq 3200$
3. $x_{C3}+x_{W3}+x_{S3} \leq 3500$

Material Proportion constraints:

1. $x_{C1} / (x_{C1}+x_{W1}+x_{S1}) \geq 0.55$
2. $x_{C2} / (x_{C2}+x_{W2}+x_{S2}) \geq 0.45$
3. $x_{C3} / (x_{C3}+x_{W3}+x_{S3}) \geq 0.30$
4. $x_{W1} / (x_{C1}+x_{W1}+x_{S1}) \geq 0.30$
5. $x_{W2} / (x_{C2}+x_{W2}+x_{S2}) \geq 0.40$
6. $x_{W3} / (x_{C3}+x_{W3}+x_{S3}) \geq 0.50$

Hence, total 9 constraints are used in LP model.

b) The optimal profit of the factory by using *RStudio* (RStudio, PBC 2019) while satisfying the demand and the cotton and wool production constraints by using the LP model is $187150 and the optimal values of the decision variables for achieving this optimal profit are:

$x_{C1} = 2660$ tonnes, $x_{C2} = 1920$ tonnes, $x_{C3} = 1750$ tonnes, $x_{W1} = 1140$ tonnes, $x_{W2} = 1280$ tonnes, $x_{W3} = 1750$ tonnes, $x_{S1} = 0$ tonnes, $x_{S2} = 0$ tonnes, $x_{S3} = 0$ tonnes where $x_{ij} \geq 0$ is a decision variable that denotes number of tonnes of products j for j ∈ {1 = Spring, 2 = Autumn, 3 = Winter} to be produced from materials i ∈ {C = Cotton, W = Wool, S = Silk}

Also, for achieving this optimal profit, the factory should produce:

3800 tonnes of Spring product
3200 tonnes of Autumn product
3500 tonnes of Winter product
And,
6330 tonnes of Cotton material will be used
4170 tonnes of Wool material will be used
0 tonnes of Silk material will be used

Q3)

a) In game theory, two or more players simultaneously choose a strategy and the strategies chosen by each player affects the final rewards earned by the other players. The final reward depends upon the combination of strategies selected by the players. The two player zero sum game is a game which involves two players and each player has a number of strategies to select from and for each strategy pair selected by the players, the total reward is zero i.e. reward to player 1 = -reward to player 2 (Dr Ye Zhu, SIT718 Real World Analytics, Deakin University, Lecture 9, 12 September 2020).

In the given scenario, there are two players Helen and David. Helen has got 6 chips to put in the two piles P1 and P2. As a result, she has total 7 strategies to select from whereas David has 4 chips and therefore he has total 5 strategies to select from. The final reward depends on the combination of strategies selected by them.
It is done by comparing the number of chips put in each pile P1 and P2 by both the players and the player with larger number of chips in each pile is rewarded 5 points and -5 points for player with less number of chips which makes the total reward zero. The final score of the game is determined by adding all the rewards for the 4 comparisons between Helen's and David's chosen chips to be put in pile P1 and P2.

Hence, the given scenario of the game player between Helen and David completely satisfies the conditions of the two player zero sum game described earlier. Thus, it can be described as a two player zero sum game.

b) Payoff matrix for the given two player sum game between Helen and David:
David's point of view:

			Helen (Player 2)								
			Helen's Strategies H(j) where j ≤ n								
David (Player 1)	David's Strategies D(i) where i ≤ m		H1 (1,5)	H2 (5,1)	H3 (2,4)	H4 (4,2)	H5 (3,3)	H6 (6,0)	H7 (0,6)	s(i)	
		D1 (1,3)	-5	-5	-10	-10	-10	0	0	-10	
		D2 (3,1)	-5	-5	-10	-10	-10	0	0	-10	
		D3 (2,2)	0	0	-10	-10	-20	0	0	-20	L = -10
		D4 (4,0)	-10	-10	-5	-5	0	-5	-5	-10	
		D5 (0,4)	-10	-10	-5	-5	0	-5	-5	-10	
		t(j)	0	0	-5	-5	0	0	0	L<U	
						U = -5					

Figure 2

We have considered David as Player 1 and Helen as Player 2 where:
m = number of strategies for David (Player 1) = 5
n = number of strategies for Helen (Player 2) = 7
s(i) = maximum security level for David associated with strategy D(i)
t(j) = maximum security level for Helen associated with strategy H(j)
L = largest security level for David (as we use MinMax)
U = smallest security level for Helen (as we use MaxMin)
$\boldsymbol{v}_{ij}$ = Payoff to David when he selects strategy D(i) and if Helen selects strategy H(j)
Also, as per two player sum game,

Payoff to David (5 or 0) = — Payoff to Helen (5 or 0)

c) An optimal solution to the two-player sum game is achieved when both the players do not find an incentive to change their strategy for a better payoff. This stage is also called as equilibrium.
If U=L, then this equilibrium stage is achieved by pure strategy {D(i), H(j)} and we call this solution as saddle point and the corresponding $\boldsymbol{v}$ as value of the game. But for this game we have L<U (refer figure 2), so a pure strategy will not result in equilibrium and hence, the given game does not have a saddle point. Both Helen and David must resort to mixed strategies for maximizing their payoff. The resulting value of the game, $\boldsymbol{v}$, is within the range [L, U] (Dr Ye Zhu, SIT718 Real World Analytics, Deakin University, Lecture 9, 12 September 2020).

d) **Linear Programming for David (Player 1):**
Let David choose the mixed strategy (x_1, x_2, x_3, x_4, x_5) with x_1, x_2, x_3, x_4, x_5 are the decision variables denoting the probability of using the corresponding strategy D(i).
Now, Helen would want to minimize David's payoff, so she will choose a strategy H(j) that makes David's payoff equal to min($-5x_1-5x_2-10x_4-10x_5$, $-10x_1-10x_2-10x_3-5x_4-5x_5$, $-10x_1-10x_2-20x_3$, $-5x_4-5x_5$). Then David would choose x_1, x_2, x_3, x_4, x_5 to make min($-5x_1-5x_2-10x_4-10x_5$, $-10x_1-10x_2-10x_3-5x_4-5x_5$, $-10x_1-10x_2-20x_3$, $-5x_4-5x_5$) as large as possible (refer figure 2)

Hence, David's game can be written as:

$$\boldsymbol{Max\,z = v}$$

s.t. $v - (-5x_1-5x_2-10x_4-10x_5) \leq 0$

$v - (-10x_1-10x_2-10x_3-5x_4-5x_5) \leq 0$

$v - (-10x_1-10x_2-20x_3) \leq 0$

$v - (-5x_4-5x_5) \leq 0$

$x_1+ x_2+ x_3+ x_4+ x_5=1$, this is to ensure that all probabilities are adding up to 1.

$x_i \geq 0, \forall$ i = 1,2,3,4,5

v u. r. s. (means - unrestricted sign)

Linear Programming for Helen (Player 2):
Let Helen choose the mixed strategy (y_1, y_2, y_3, y_4, y_5, y_6, y_7) with y_1, y_2, y_3, y_4, y_5, y_6, y_7 are the decision variables denoting the probability of using the corresponding strategy H(j).

Now, David would choose a strategy D(i) to obtain an expected reward of max($-5y_1$-$5y_2$-$10y_3$-$10y_4$-$10y_5$, -$10y_3$-$10y_4$-$20y_5$, -$10y_1$-$10y_2$-$5y_3$-$5y_4$-$5y_6$ -$5y_7$). Then Helen would choose y_1, y_2, y_3, y_4, y_5, y_6, y_7 to make max(-$5y_1$-$5y_2$-$10y_3$-$10y_4$-$10y_5$, -$10y_3$-$10y_4$-$20y_5$, -$10y_1$-$10y_2$-$5y_3$-$5y_4$-$5y_6$ -$5y_7$) as small as possible (refer figure 2)

Hence, Helen's game can be written as:

$$\boldsymbol{Min\, w} = \boldsymbol{v}$$

s.t. $v - (-5y_1 - 5y_2 - 10y_3 - 10y_4 - 10y_5) \geq 0$

$v - (-10y_3 - 10y_4 - 20y_5) \geq 0$

$v - (-10y_1 - 10y_2 - 5y_3 - 5y_4 - 5y_6 - 5y_7) \geq 0$

$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 = 1$, ensuring that all probabilities are adding up to 1.

$y_i \geq 0, \forall\, i = 1,2,3,4,5,6,7$

v u. r. s. (means - unrestricted sign)

e) **Player 1's game (David) code** by using *RStudio* (RStudio, PBC 2019)

```
library(lpSolveAPI)
davidgame<-make.lp(0,6)  #initializing 0 constraints and 6 decision variables
lp.control(davidgame, sense="maximize")  #setting control parameters
set.objfn(davidgame, c(0,0,0,0,0,1)) #x1,x2,x3,x4,x5,v
add.constraint(davidgame, c(5, 5, 0, 10, 10, 1), "<=", 0)
add.constraint(davidgame, c(10, 10, 10, 5, 5, 1), "<=", 0)
add.constraint(davidgame, c(10, 10, 20, 0, 0, 1), "<=", 0)
add.constraint(davidgame, c(0, 0, 0, 5, 5, 1), "<=", 0)
add.constraint(davidgame, c(1 , 1, 1, 1, 1, 0), "=", 1)
set.bounds(davidgame, lower = c(0, 0, 0, 0, 0, -Inf))
RowNames <- c("Constraint1", "Constraint2", "Constraint3","Constraint4", "Constraint5")
ColNames <- c("x1", "x2", "x3", "x4", "x5", "v")
dimnames(davidgame) <- list(RowNames, ColNames)
davidgame #Display the LP model
solve(davidgame)  # http://lpsolve.sourceforge.net/5.5/solve.htm
valueofgame<-get.objective(davidgame)
valueofgame   #value of the game
optimal_solution<-get.variables(davidgame)
optimal_solution  #optimal strategies for David
```

Player 2's game (Helen) code by using *RStudio* (RStudio, PBC 2019)

```
library(lpSolveAPI)
helengame<-make.lp(0,8)  #initializing 0 constraints and 6 decision variables
lp.control(helengame, sense="minimize")  #setting control parameters
set.objfn(helengame, c(0,0,0,0,0,0,0,1)) #y1,y2,y3,y4,y5,y6,y7,v
add.constraint(helengame, c(5, 5, 10, 10, 10, 0, 0, 1), ">=", 0)
add.constraint(helengame, c(0, 0, 10, 10, 20, 0, 0, 1), ">=", 0)
add.constraint(helengame, c(10, 10, 5, 5, 0, 5, 5, 1), ">=", 0)
```

```
add.constraint(helengame, c(1, 1, 1, 1, 1, 1, 1, 0), "=", 1)
set.bounds(helengame, lower = c(0, 0, 0, 0, 0, 0, 0, -Inf))
RowNames <- c("Constraint1", "Constraint2", "Constraint3","Constraint4")
ColNames <- c("y1", "y2", "y3", "y4", "y5", "y6", "y7", "v")
dimnames(helengame) <- list(RowNames, ColNames)
helengame #Display the LP model
solve(helengame) # http://lpsolve.sourceforge.net/5.5/solve.htm
valueofgame<-get.objective(helengame)
valueofgame   #value of the game
optimal_solution<-get.variables(helengame)
optimal_solution  #optimal strategies for Helen
```

f) Value of the game, v, is -6.67 for both the players which lies in the range [L, U] i.e. [-10, -5]. Hence, the maximum payoff for David and Helen both will be -6.67 while playing this game i.e. in the least worst case, they both will lose get a negative score of -6.67.

The optimal solution for David is coming out to be:

$x_1 = 0$, $x_2 = 0$, $x_3 = 0.3333$, $x_4 = 0.6667$, $x_5 = 0$.

Therefore, for David, in order to achieve this score, the strategy for placing 1 chip in pile 1 and 3 chips in pile 2 interchangeably will not be used for all the times game is played and the same can be said for the strategy of placing 0 chip in pile 1 and 4 chips in pile 2. Whereas the strategy for placing 2 chips in pile 1 and 2 chips in pile 2 will be used 33.33% of the time and the strategy for placing 4 chips in pile 1 and 0 chips in pile 2 will be used 66.67% of the time game is played (Dr Ye Zhu, SIT718 Real World Analytics, Deakin University, Lecture 10, 19 September 2020).

Q4) a) Payoff matrix for the three players displaying the profits of each player under all strategy combinations is given below:

	If Player 3 contributes $0 (C1)			
		Player 2		
		B1 ($0)	**B2 ($3)**	**B3 ($6)**
Player 1	**A1 ($0)**	(0,0,0)	(2,-1,0)	(4,-2,4)
	A2 ($3)	(-1,2,2)	(1,1,4)	(3,0,6)
	A3 ($6)	(-2,4,4)	(0,3,6)	(2,2,8)

Figure 3

	If Player 3 contributes $3 (C2)			
		Player 2		
		B1 ($0)	**B2 ($3)**	**B3 ($6)**
Player 1	**A1 ($0)**	(2,2,-1)	(4,1,1)	(6,0,3)
	A2 ($3)	(1,4,1)	(3,3,3)	(5,2,5)
	A3 ($6)	(0,6,3)	(2,5,5)	(4,4,7)

Figure 4

	If Player 3 contributes $6 (C3)			
		Player 2		
		B1 ($0)	**B2 ($3)**	**B3 ($6)**
Player 1	**A1 ($0)**	(4,4,-2)	(6,3,0)	(8,2,2)
	A2 ($3)	(3,6,0)	(5,5,2)	(7,4,4)
	A3 ($6)	(2,8,2)	(4,7,4)	(6,6,6)

Figure 5

In the above figures, A(i) where i ∀ 1,2,3 denotes the strategies of player 1 on contributing $0, $3 and $6 respectively. In the same way, B(j) where j ∀ 1,2,3 denotes the strategies available to player 2. There are three payoff tables based on the amount contributed by player 3 i.e. $0, $3 and $6.

b) The solution (0,0,0) is the Nash equilibrium of this game as for this strategy profile, none of the players can get a better payoff by unilaterally changing their strategy from this profile. It is evident from the first payoff table that player 1's and player 2's payoff is decreasing when shifting to other strategy (figure 3) and the same can be said for player 3 based on other 2 payoff tables (i.e. -1 and -2 in figure 4 and 5 respectively). Hence, {A1($0), B1($0), C1($0)} is the optimal solution for this game. The optimal profits at this equilibrium is $0 for all the three players which can be achieved by the pure strategy (A1, B1, C1).

c) If the players can co-operate, the Nash equilibrium (A1, B1, C1) of this game can provide the best strategy profile (Pareto optimal solution) for all the players. As, in this case, if the players co-operate i.e. share their payoff, the current Nash equilibrium (A1, B1, C1) becomes unstable and shifts to (A1, B3, C1) because player 1 and 3 both can contribute $2 each to player 2 and hence, the final payoff for all the players will be $2 each which is better than the current payoff of $0 each (figure 3). Pareto optimal solution of the game is a strategy profile from where any attempt in changing the strategy will result in none of the players getting a better payoff without making at least one player's payoff worse whereas Nash equilibrium is a stable strategy profile in which none of the players can get a better payoff by unilaterally changing their strategy from the stable profile. Thus, in this case, Nash equilibrium becomes pareto optimal solution by shifting from (A1, B1, C1) to (A1, B3, C1) resulting in payoff of player 2 getting decreased whereas payoffs of player 1 and 3 got increased (figure 3). But if the players co-operate, this pareto optimal solution becomes the best strategy solution for everyone (Pareto efficiency n.d.).

d) A real-life example implicating the importance of co-operation would be a case of two cars coming along different roads towards an intersection. As they approach the intersection, they observe each other coming towards it. Both have two choices namely "Go" and "Stop". Let us draw a hypothetical payoff matrix for the corresponding scenario.

		Car 2	
		Go	**Stop**
Car 1	**Go**	(-5,-5)	(2,-1)
	Stop	(-1,2)	(-3,-3)

Figure 6

Figure 6 illustrates that suppose if both the cars stop, there is a moderate negative payoff of -3 as they are making the other cars behind them wait. If both the cars go then there will be an accident, implicating a heavy negative payoff of -5 to both the cars. And, if one stops and other goes, payoff of -1 and 1 respectively is awarded. Now, in this case, consider moving from the strategy pair (Car1Go, Car2Go) i.e. (-5, -5) to (Car1Stop, Car2Go), the car 1 which is stopping to let car 2 cross the intersection is still getting a negative payoff of -1 points which is not certainly fair. Hence, if both the cars co-operate to share their payoff, they both can benefit i.e. if car 2 agrees to contribute 1 point out of 2 points to car 1 then the car 1 would now have non-negative payoff of 0 better than current negative payoff of -1 point for stopping and car 2 will still have a positive payoff of 1 points which is better than the previous negative payoff of -5 points. Hence, co-operation will result in optimal profit for both the players in this game (Sachin Joglekar 2015).

References:

- Eli Luberoff 2011, *Desmos,* computer program
- RStudio, PBC 2019, *RStudio,* computer program
- Pareto efficiency n.d., *Pareto efficiency,* 1 September, retrieved 9 September 2020
- Sachin Joglekar 2015, *Traffic Lights and Nash Equilibrium,* wordpress.com, retrieved 9 September 2020